1 Phase plane method

$$\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \ddot{x} \text{ (chain rule)}$$

Intuition: we can use the trick to transform a second order equation into a first order one. Although the equation is still hard to solve as it is not linear, we can often interpret $\dot{x}$ as a function of x, which is called the phase plane. We can still perform useful analysis on the phase plane, such as observing the stability of equilibrium.

2 Malthus model of population

2.1 Terminologies

B: birth rate per capita
D: death rate per capita
N: population
N_0: initial population

2.2 Assumption

Malthus assumed a socially static society in which human reproductive behavior never changes. B and D are constant.

2.3 Model

$$\frac{dN}{dt} = (B - D)N = kN$$

Solving the differential equation, we get

$$N(t) = N_0 e^{kt}$$

The model implies that if $k < 0$, the population will collapse. If $k = 0$, the population remains static. If $k > 0$, the population will explode.

3 Improvement on Malthus model

3.1 Idea

Naturally, we should observe that in a world of finite resources, overcrowding will lead to a higher death rate. Therefore, we use logistic assumption $D = sN$ to replace the constant D. We could interpret s as the extent to favour crowding. If a species favour crowding, s will be low, which means D is low with large N, and vice versa.

3.2 Model

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

We could observe that if N starts to be very low, then $\frac{dN}{dt}$ will be positive to lead to increasing N. If N starts to be very large, then $\frac{dN}{dt}$ will be negative to lead to decreasing N. Equilibrium occurs when $\frac{dN}{dt} = 0$, in this case $N = \frac{B}{s}$, which is the logistic equilibrium.
The equation is a Bernouli equation. We could solve it and obtain

$$N = \frac{\frac{B}{s}}{1 + e^{-Bt}(\frac{B}{N_0 s - 1})}$$

This is the logistic curve. Note that equilibrium solution is $\frac{B}{s}$.

4 Harvesting

4.1 Idea

Suppose a fish species behave logistically, and now we want to harvest them at a certain rate. The logistic model is changed to

$$\frac{dN}{dt} = (B - sN)N - E = -sN^2 + BN - E$$

where E is the harvesting rate.
Note that the first order equation could also be interpreted as a phase plane. We derive further analysis from now on.

4.2 Analysis

We could observe that the phase diagram is a concave parabola. We therefore can split into three cases.

4.2.1 Over-fishing

When the quadratic has no real roots, which means that $B^2 - 4sE < 0 \implies E > \frac{B^2}{4s}$, $\frac{dN}{dt} < 0 \; \forall N$, which means that no matter what the current population is, the population is always decreasing. It will lead to extinction because over-fishing occurs.

4.2.2 Controlled fishing

Suppose we control our fishing under the threshold $\frac{B^2}{4s}$, then the quadratic has two real roots, which is

$$\beta_{1,2} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s}$$

Note that for the smaller equilibrium β_1, $\frac{dN}{dt} < 0$ when N is slightly reduced from β_1, and $\frac{dN}{dt} > 0$ when N is slightly increased from β_1. Therefore β_1 is an unstable equilibrium since it will run away when a small perturbation is given. We could check that β_2 is a stable equilibrium.

4.2.3 Threshold fishing

Suppose now $E = \frac{B^2}{4s}$, then the quadratic has two repeated roots. Note that $\frac{dN}{dt} < 0$ when N is perturbed from β in either direction. It implies that the population will fall back to equilibrium if it is increased a bit, but will collapse if it is decreased a bit. It is still an unstable equilibrium. (Stability must hold for both sides.)

4.3 Implication

The harvesting model is an impressive improvement from the logistic model, because now it allows for extinction and thus provides a better description to the real world.
Since the equation is separable, we could compute the extinction time by

$$\int_0^T dt = T = \int_{N_0}^0 \frac{dN}{N(B - sN) - E}$$

5 Steady growth

Suppose we want to control the growth rate, which is to control the growth rate when the population is rising rapidly. We modify B as

$$B = B_0 - \alpha\frac{dN}{dt}$$

so now we have

$$\frac{dN}{dt} = (B_0 - \alpha\frac{dN}{dt})N - DN$$

$$(\frac{1}{N} + \alpha)\frac{dN}{dt} = B_0 - D$$

When N becomes large, then approximately

$$\frac{dN}{dt} = \frac{B_0 - D}{\alpha}$$

The population now grows at a constant rate rather than exponential rate.