1 Model of Ants

1.1 Idea

Let n be the number of ants in the colony and let $x(t)$ be the number of ants on a particular trail at time t. Initially, the ants start to join the trail at a rate α, when the trail attracts more ants, the amount of pheromone increases to attract ants more effectively. However, when most of the ants are on the trail, then the rate will certainly decrease. Furthermore, pheromone will evaporate to cause ants to get lost, which we model by $\frac{s}{r+x}$ as the rate per capita.

1.2 Model

$$\frac{dx}{dt} = (\alpha + \beta x)(n - x) - \frac{sx}{r + x}$$

If we choose $\alpha = 0.0045, \beta = 0.00015, s = 10, r = 10$, when $n = 450$, there is only one stable equilibrium but extremely small. When $n = 600$, there are two stable equilibria, one very small and one large, and an unstable equilibrium at $x = 80$, which means if one can get more than 80 ants on the trail, then a large number of ants will be on the trail.

2 Finance

2.1 Idea

Let u denote the value of the company, and $\frac{du}{dt}$ is the rate of generating profit, thus we can define profitability $P = \frac{du}{dt}/u$. If profitability is constant in time, we recover $\frac{du}{dt} = Pu$. We further consider the case that we only invest a fraction k of profit back and pay $(1-k)$ as dividends, then $\frac{du}{dt} = kPu$. Let $w(t)$ be the total dividends paid out from the foundation of the company up to time t, then $\frac{dw}{dt} = (1-k)Pu$. Let $u(0) = U$, then $u = Ue^{kPt}$ and so $\frac{dw}{dt} = (1-k)PUe^{kPt}$. If $k = 0$, then $w = PUt$. If $k \neq 0$, then $w = \frac{1}{k}(1-k)U[e^{kPt} - 1]$.

2.2 Strategy

Suppose one plans to be in business for a fixed time T before quitting. Given P and T, our goal is to optimize $w(T)$ by choosing k, the total earning generated before pulling out. If $k = 0$, $w(T) = PUT$. If $k \neq 0$, $w(T) = \frac{1}{k}(1-k)U[e^{kPT} - 1]$. For ease of notation, let $x = kPT, y = \frac{w(T)}{U}$, then we have $y = (PT - x)\frac{e^x - 1}{x}$, and we aim to maximise y by choosing x.

2.3 Conclusion

The choice of x, or equivalently k, depends on PT: If $PT > 2$, the slope is positive at $x = 0$, and the curve attains a maximum for a non-trivial x, solve by $\frac{dy}{dx} = 0$. If $PT \leq 2$, the slope is negative, and the curve always goes down, so we should choose $x = 0 \iff k = 0$, which is to invest nothing back into the business.

3 Solving linear systems of ODE

3.1 Problem formulation

We solve the system

$$\frac{dx}{dt} = ax + by; \frac{dy}{dt} = cx + dy$$

Consider the coefficient matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \iff \frac{d\mathbf{u}}{dt} = B\mathbf{u}$$

Steps to solve:

1. Compute eigenvalues of B, the coefficient matrix. Fast way to compute is that $r_{1,2} = \frac{1}{2}[Tr(B) \pm \sqrt{(Tr[B])^2 - 4Det[B]}$
2. Compute two eigenvectors, $\mathbf{u}_1, \mathbf{u}_2$, associated with r_1, r_2
3. The general solution is composed as $\mathbf{u} = C_1 e^{r_1 t}\mathbf{u}_1 + C_2 e^{r_2 t}\mathbf{u}_2$
4. If the eigenvalues are complex, say $r_{1,2} = p \pm qi$, then we only consider one of them, say $r_1 = p + qi$, and find an eigenvector $\mathbf{u}_1$, then the part of solution corresponding to r_1 is $e^{r_1}\mathbf{u}_1$, which is
$$e^{p+qi}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$
5. Write the imaginary part of the exponential as a product of sine and cosine, and split the general solution into the form $e^{pt}\mathbf{u}(t) + ie^{pt}\mathbf{v}(t)$.
6. The general solution of the system is then $\mathbf{u} = C_1 e^{pt}\mathbf{u}(t) + C_2 e^{pt}\mathbf{v}(t)$

3.2 Classification of phase plane

We can classify shape of phase planes based on two eigenvalues:

1. If both real and positive, nodal source.
2. If both real and negative, nodal sink.
3. If both real and opposite signs, saddle.
4. If both complex with positive real parts, spiral source.
5. If both complex with negative real parts, spiral sink.
6. If both complex with zero real parts, centre.

In cases when eigenvalues are real, the phase plane consists of all curves except two straight lines representing either C_1 or C_2 is 0, and the flow goes in the direction of the eigenvector. All curves should start from the direction one eigenvector starting from $t = -\infty$ and converge to another direction as $t \to +\infty$, depending on the dominance of the exponential term.

4 Predators and Prey

4.1 Idea

The interaction of zebras and lions is proposed to satisfy the following features:

1. Equilibrium $(0, 0)$ should exist, since neither lion nor zebra can come into existence out of nothing.
2. The birth rate per capita of lion depends on zebra, since zebra is their food source. The death rate per capita of lion is assumed constant, since they are strong and can only die naturally.
3. The birth rate per capita of zebra is assumed constant, since they eat grass, which is of assumed infinite supply. The death rate per capita of zebra depends on lion, since lion kills them.

4.2 Lotka-Volterra model

$$\frac{dL}{dt} = uZL - D_L L$$

$$\frac{dZ}{dt} = B_Z Z - sLZ$$

We have $(L, Z) = (\frac{B_Z}{s}, \frac{D_L}{u})$ and $(0, 0)$ as equilibrium.

Multiply $\frac{dL}{dt}$ by $\frac{B_Z}{L} - s$ and $\frac{dZ}{dt}$ by $\frac{D_L}{z} - u$ and add the two equations together, we will have

$$(\frac{B_Z}{L} - s)\frac{dL}{dt} + (\frac{D_L}{z} - u)\frac{dZ}{dt} = 0$$

which integrates to

$$B_Z ln(L) - sL + D_L ln(Z) - uZ = C$$

By this exact relation, we can see that all trajectories in the phase plan are periodic if we graph them.

4.3 Linearization

The model is not linear. However, near equilibrium point, we can approximate the system linearly, as in Talyor expansions around the equilibrium, quadratic terms and beyond will become negligible.

If the system takes the form

$$\frac{dx}{dt} = f(x, y); \frac{dy}{dt} = g(x, y)$$

We compute the Jacobian matrix, $J(x, y)$, as

$$J(x, y) = \begin{bmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{bmatrix}$$

Then near an equilibrium (a, b), $J(a, b)$ will give the linearized approximation of the system, which allows us to see what the phase plane looks like near the equilibrium. For Lotka-Volterra model, there is a saddle around $(0, 0)$ and a center at $(\frac{B_Z}{s}, \frac{D_L}{u})$

However, we still want to improve the model, since the current model suggests everlasting periodic interaction near the non-trivial equilibrium. In reality, the system should forget about initial perturbations after a while.

4.4 Logistic Lotka-Volterra model

As the old model suggests population explosion of zebra in absence of lions, we now add a logistic constraint on zebra, and the model becomes

$$\frac{dL}{dt} = uZL - D_L L$$

$$\frac{dZ}{dt} = B_Z Z - pZ^2 - sLZ$$

Now even in the absence of lions, the population of zebra will reach a logistic equilibrium, which is $\frac{B_Z}{p}$, since zebra is quite tolerant to crowding, we should deem p as a small value.

There are now three equilibria: $(0, 0)$ the trivial equilibrium, and $(0, \frac{B_Z}{p})$ the logistic equilibrium of zebra, and $(\frac{B_z - \frac{pD_L}{u}}{s}, \frac{D_L}{u})$

There is a saddle around $(0, 0)$ and $(0, \frac{B_Z}{p})$. In the last equilibrium, there is a spiral sink, which means that any perturbation from the third equilibrium will ultimately die out.

Note that in this case, if B_z or u is too small,or p or D_L is too large, the third equilibrium fails to exist in the first quadrant, and the lions are going to extinct.

5 Competing species

5.1 Idea

Thylacines and Dingoes compete for food in the same environment. In absence of Dingo, we assume a logistic model for Thylacine. The arrival of Dingo will deprive Thylacine of food, reducing their effective birth rate. Since the two species are competing, the presence of Thylacine will have a similar effect on Dingo.

5.2 Model

$$\frac{dT}{dt} = (a - kD)T - bT^2$$

$$\frac{dD}{dt} = (c - \sigma T)D - dD^2$$

We assume $a \approx c, b \approx d \implies \frac{a}{b} \approx \frac{c}{d}$. The ratio is their logistic equilibrium in absence of each other, and we call it N.

The Dingoes are less affected by the Thylacines than vice versa. It means that k is large and σ is small. This implies

$$\frac{a}{k} < \frac{c}{d} \approx N \approx \frac{a}{b} < \frac{c}{\sigma}$$

We have $(0, 0)$ the trivial equilibrium and $(0, \frac{c}{d})$, $(\frac{a}{b}, 0)$ the two logistic equilibria for Dingo and Thylacine respectively.

There may be a fourth equilibrium, if $(a - kD) - bT = 0$ and $(c - \sigma T) - dD = 0$ intersects on the first quadrant. Note that if $\frac{a}{k} < \frac{c}{d}$ and $\frac{a}{b} < \frac{c}{\sigma}$, the two lines will not cross, and in the case of Dingo and Thylacine it unfortunately happens.

If we look at the three remaining equilibria, we can see that there is nodal source around $(0, 0)$ and saddle around $(\frac{a}{b}, 0)$, and nodal sink around $(0, \frac{c}{d})$. It implies that as long as Dingo is introduced, Thylacine will totally lose the competition.

Suppose we add some significant differences between two species: say Dingo suffers from a high death rate, this makes $\frac{c}{d} < \frac{a}{k}$, and then the fourth equilibrium occurs. In this case, $(0, 0)$ is still a nodal source, but the other two equilibria become saddle and the fourth equilibrium is a nodal sink. Two competing species can co-exist if they are very different in some senses.

6 Space exploration

6.1 Idea

Probes are dispatched to conquer the space. They are self-replicating. Mutation occurs within probes within a certain rate, and the mutants hunt original probes.

6.2 Model

$$\frac{dM}{dt} = uPM - D_M M + mP$$

$$\frac{dP}{dt} = B_P P - sMP - mP$$

We have $(0, 0)$ the trivial equilibrium and

$$(M, P) = (\frac{B_P - m}{s}, \frac{D_M}{u + \frac{ms}{B_P - m}})$$

We in this case assume $m < B_P$. If $m = 0$, we recover the usual Lotka-Volterra equilibrium. Note that if $m > 0$, the equilibrium values are both smaller, and we have a spiral sink.

7 Far from equilibrium

Note that the technique of linearization assumes that we are looking near the equilibrium point, and it could not guarantee that by patching various observations give the full picture. We could observe phenomenon like limiting cycles.

8 Non-linear second order system

8.1 Idea

For a second order non-linear ODE $\frac{d^2r}{dt^2} = f(r)$, we can consider the trick of formulating a system of shape $\frac{dr}{dt} = R; \frac{dR}{dt} = f(r)$, and therefore we can obtain phase plane of r and $R = \frac{dr}{dt}$, which is already useful for analysis.

8.2 Motion of planet

Let $r(t)$ denote the distance from the Sun to Earth. It is governed by

$$\frac{d^2r}{dt^2} = -\frac{M}{r^2} + \frac{L^2}{r^3}$$

By the trick, we have the system

$$\frac{dr}{dt} = R$$

$$\frac{dR}{dt} = -\frac{M}{r^2} + \frac{L^2}{r^3}$$

The system has equilibrium $(\frac{L^2}{M}, 0)$, and this corresponds to a circular orbit. It is a center. It means that a small perturbation from this equilibrium will still make r vary periodically, which means that it is bounded. If we change the inverse square law to power of more than three, the equilibrium will become a saddle, which means that a small perturbation will cause the Earth to either fly into the Sun or leave the Sun forever. Too bad in either case.

9 Partial differential equations

9.1 Separation of variable

Suppose $u(x, y) = X(x)Y(y)$, then $u_x(x, y) = X'(x)Y(y); u_y = X(x)Y'(y); u_{xx}(x, y) = X''(x)Y(y); u_{yy}(x, y) = X(x)Y''(y); u_{xy}(x, y) = X'(x)Y'(y)$

Consider a PDE of the form $u_x = f(x)g(y)u_y$, suppose we assume a solution of the form described above exists,then we have

$$X'(x)Y(y) = f(x)g(y)X(x)Y'(y)$$

$$\frac{1}{f(x)}\frac{X'(x)}{X(x)} = g(y)\frac{Y'(y)}{Y(y)}$$

Since LHS is a function of x and RHS is a function of y, we conclude LHS = RHS = k.
We obtain two ODE

$$X'(x) = kf(x)X(x)$$

$$Y'(y) = \frac{k}{g(y)}Y(y)$$

Solve the two equations and combine to form u.

9.2 Wave equation

9.2.1 Idea

Suppose we have a string stretched tightly along x axis and has its ends at $x = 0$ and $x = \pi$, then we pull it in y direction so it is stationary and has some specified shape, $y = f(x)$ at time $t = 0$, so that $f(0) = f(\pi) = 0$. If we let the string go, the string will start to move, and the y-coordinate of any point on the string will become a function of time and position, so we model it as $y(t, x)$.

9.2.2 Initial conditions

We gather conditions required to meet:

1. $y(t, 0) = y(t, \pi) = 0$, the two ends are fixed.
2. $y(0, x) = f(x)$, when the string is pulled, it observes a defined shape.
3. $\frac{\delta y}{\delta t}(0, x) = 0$, the string is initially stationary.

9.2.3 Model

$$c^2\frac{\delta^2 y}{\delta x^2} = \frac{\delta^2 y}{\delta t^2}$$

The solution of the wave equation takes the form

$$y(t, x) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

which, once we define an odd extension, are periodic in either x or t.
Solve by separation of variables, our complete solution is

$$y(t, x) = b_n sin(nx)cos(nct)$$

where b_n is an arbitrary constant and n is any integer. It satisfies all conditions except $y(0, x) = f(x)$.

9.3 Fourier series

We could approximate any piecewise continuous function $g(x)$ on $[0, \pi]$ by the series

$$g(x) = \sum_{n=1}^{\infty} b_n sin(nx)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} g(x)sin(nx)dx$.
This is the Fourier series of $g(x)$.
Return to the case of wave equation. We could express $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} b_n sin(nx)$$

Consider the series

$$y(t, x) = \sum_{n=1}^{\infty} b_n sin(nx)cos(nct)$$

When $t = 0$, it gives the Fourier series of $f(x)$. Since wave equation is linear, each term in the series is a solution, making the whole series as a solution as well. Therefore, this is the general solution of wave equation.
If the interval changes from $[0, \pi]$ to a general $[0, L]$, we just replace $sin(nx)$ by $sin(\frac{n\pi x}{L})$, and $b_n = \frac{2}{L}\int_0^L g(x)sin(\frac{n\pi x}{L})dx$, and $cos(nct)$ by $cos(\frac{nc\pi t}{L})$

9.4 Tsunami equation

A model to describe a surface wave in shallow water is by the Korteweg-de Vries equation

$$\partial_t \eta + \sqrt{gh}\partial_x \eta + \frac{3}{2}\sqrt{\frac{g}{h}}\eta\partial_x\eta + \frac{1}{6}h^2\sqrt{gh}\partial_x^3\eta = 0$$

where η, h, g are constants. We want a wave solution $E(x - ct)$, which is a wave of fixed shape moving to the right at speed c. A typical solution is a multiple of the function $sech^2$. Note that linear wave equation can represent wave of any shape by Fourier analysis. If in this case the equation is not linear, it dictates the shape of the wave.

9.5 Heat equation

9.5.1 Idea

Consider the temperature in a long, thin and homogeneous bar. Assume heat only flows in x direction. Temperature u then depends on x and t.

9.5.2 Initial conditions

We gather conditions required to meet:

1. $u(t, 0) = u(t, L) = 0$, assume both ends kept at zero temperature.
2. $u(0, x) = f(x)$, assume initial temperture follows a pattern.

9.5.3 Model

$$u_t = c^2 u_{xx}$$

Heat equation says that the second spatial derivative of u is related to its time derivative. If the graph of u is convex in x, then u will increase and vice versa. The effect is to reduce the curvature of the graph, the heat wave becomes 'straightened' in terms of time, which agrees with intuition. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})exp(-\frac{\pi^2 n^2 c^2}{L^2}t)$$

where b_n are the Fourier sine coefficients of $f(x)$. Note that the solutions of heat equation depend on the direction of time. It is useful to model situation involving irreversible time evolution.

9.6 Fisher's equation

9.6.1 Idea

Plants mutate to gain the ability from living in water to living on land. Note that this mutation is irreversible in time. Let $u(t, x)$ denote the fraction of the mutated plants at any given time and space.

9.6.2 Model

$$u_t = \alpha u_{xx} + \beta u(1 - u)$$

Again, we want a wave solution moving to the right, $u(t, x) = U(x - ct)$. Let $s = x - ct$, then $u_{xx} = U'', u_t = -cU'$. In this case Fisher's equation reduces to ODE $\alpha U'' + cU' + \beta U - \beta U^2 = 0$, turn it into a system of U and $V = U'$, we have $(0, 0)$ and $(1, 0)$ as equilibria. If $U(s) = 0\ \forall s$, it means set any $t = t_0$, $u(x, t_0) = 0\ \forall x$, it means mutation does not occur. Similarly, the other case means that mutation completely takes over.
At $(1, 0)$ it is always a saddle, while at $(0, 0)$, it is a spiral sink if $c < 2\sqrt{\alpha\beta}$, but a spiral sink at origin means U could be negative, which does not make sense. We thus insist $c > 2\sqrt{\alpha\beta}$, which imposes a lower bound on wave propagation rate. In this case, $(0, 0)$ is a nodal sink. Note that in the diagram the time factor is s, which means that when $s \to \infty \implies t \to -\infty$, we are at node $(0, 0)$, and when time goes by, $s \to -\infty \implies t \to \infty$, we are at saddle $(1, 0)$. The mutation propagates from the initial spot until it takes over the whole shoreline.
We can modify the equation as

$$u_t = \alpha u_{xx} + \beta u(\gamma - u)$$

which suggests that the mutants are less successful and only constitutes a fraction γ of the total population at each point. By a trick of $w = \frac{u}{\gamma}$, we can transform it into

$$w_t = \alpha w_{xx} + \beta\gamma w(1 - w)$$

We recover the shape of original Fisher's equation, the wave now moves at a minimum speed of $2\sqrt{\alpha\beta\gamma}$, more slowly by a factor of $\sqrt{\gamma}$.

9.7 Diffusion of lions

9.7.1 Idea

We study the distribution of lions in a square space, where x, y ranges in $[0, \pi]$, with three sides blocked by electric fence and one side by a river. Dominant lions distribute at the river at a certain distribution.
We model the density of lions as a function of time and two dimensions of space as $u(t, x, y)$.

9.7.2 Model

$$u_t = c^2(u_{xx} + u_{yy})$$

We focus on the situation where the distribution is steady, thus irrelevant of time, we consider $u(x, y)$:

$$c^2(u_{xx} + u_{yy}) = 0$$

9.7.3 Initial conditions

We gather conditions required to meet:

1. $u(x, 0) = u(0, y) = u(\pi, y) = 0$, fences.
2. $u(x, \pi) = f(x)$, distribution by river.

9.8 Conclusion

Solve by separation of variables, the general solution takes the form $c_n sin(nx)sinh(ny)$, since the equation is linear, it gives us

$$u(x, y) = \sum_{n=1}^{\infty} c_n sin(nx)sinh(ny)$$

Consider the condition $u(x, \pi) = f(x)$, we have

$$f(x) = u(x, \pi) = \sum_{n=1}^{\infty} c_n sin(nx)sinh(n\pi)$$

then $c_n sinh(nx)$ gives the Fourier sine coefficients of $f(x)$, we then can compute

$$c_n = \frac{\frac{2}{\pi}\int_0^{pi} f(x)sin(nx)dx}{sinh(n\pi)}$$