1 Static games of complete information

1.1 Pure strategy profile

Definition: The normal-form representation of an n-player game specifies the players' strategy spaces S_i and their payoff functions u_i. We denote this game by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$. Let $(s_1, \ldots, s_n)$ be a combination of strategies, then $u_i(s_1, \ldots, s_n)$ is the payoff to player i if for each $1 \le j \le n$, player j chooses strategy j.

Definition: In a normal form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy S_i' is strictly dominated by strategy s_i'' if $u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$.

Definition: In the n-player normal-form game $G = \{S_i, u_i\}$, the best response for player i to a combination of other players' strategies $s_{-i} \in S_{-i}$ is $R_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$

Definition: In the n-player normal-form game $G = \{S_i, u_i\}$, the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium if $s_i^* \in R_i(s_{-i}^*)$ for all $1 \le i \le n$. Equivalently, $u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$ for all $1 \le i \le n$.

Theorem 1. If the strategy profile $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium in an n-player normal-form game G, then each s_i^* cannot be eliminated in iterated elimination of strictly dominated strategies. Furthermore, if iterated elimination of strictly dominated strategies eliminates all but one strategy profile, then it is the unique Nash equilibrium.

Theorem 2. In the n-player normal-form game $G = \{S_i, u_i\}$, where S_i is finite, if iterated elimination of strictly dominated strategies eliminates all but one strategy profile, then that profile is the unique Nash equilibrium.

Proposition 1. If the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium in an n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, then each s_i^* cannot be eliminated in iterated elimination of strictly dominated strategies.

Proposition 2. In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ where $S_1, \ldots, S_n$ are finite sets, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, \ldots, s_n^*)$, then these strategies are the unique Nash equilibrium of the game.

If the strategy space is **finite**, draw the game matrix and find Nash equilibrium by indicating best response for each player and each opponent strategy and find intersections.

If the strategy space is **infinite**, draw the graph of best response(found pointwise by either direct observation or differentiation) and find intersections.

Cournot model of duopoly: Suppose two firms produce decide the amount to produce by q_1, q_2. Since the products are identical, they must sell at same marking clearing price $P(q_1, q_2) = \max(a - q_1 - q_2, 0)$, otherwise losing all markets. Suppose unit cost is c, thus the profit(ignoring fixed cost) is $\pi_i = P(q_1,q_2)q_i - cq_i = q_i(a - q_i - q_j - c)$ if $q_i + q_j < a$ and $-cq_i$ otherwise for $i = 1, 2$.

Fix q_j and by differentiation, we find that optimal q_i is $\frac{1}{2}(a - q_j - c)$ if $q_j \le a - c$ and 0 otherwise. The intersection is $q_1 = q_2 = \frac{1}{3}(a - c)$.

Bertrand model of duopoly: Suppose now two firms produce **differentiated** products, thus they decide on price p_1, p_2 rather than quantity. The quantity demand is defined as $q_i(p_i, p_j) = a - p_i + bp_j$. The revenue is thus $\pi_i = (a - p_i + bp_j)(p_i - c)$. By differentiation, the best response graph is $p_i^* = \frac{1}{2}(a + bp_j + c)$ for $i = 1, 2$. Taking intersection of two graphs, we find Nash equilibrium at $(\frac{a+c}{2-b}, \frac{a+c}{2-b})$.

If we analyze Bertrand model with **homogeneous** products, then $q_i(p_i, p_j) = a - p_i$ when $p_i < p_j$, 0 when $p_i > p_j$, and $\frac{1}{2}(a - p_i)$ when $p_i = p_j$. That is, the producer with lower price will take the whole market, and the market will be equally divided if both firms offer same price. In this case, $\pi_i(p_i, p_j) = (a - p_i)(p_i - c)$ if $p_i < p_j$, and $\frac{1}{2}(a - p_i)(p_i - c)$ if $p_i = p_j$, and 0 if $p_i > p_j$. We consider strategy space $S_i = [c, a]$ to ensure positive payoff.

An important observation is that the graph of $f(x) = (a - x)(x - c)$ attains maximum when $x = \frac{a+c}{2}$. Therefore, if $p_j \in (\frac{a+c}{2}, a]$, then $p_i = \frac{a+c}{2}$ will be the best response. If $p_j \in (c, \frac{a+c}{2}]$, since the supremum is not attainable, we do not have any optimal p_i. If $p_j = c$, then $\pi_i = 0$ for any $p_i \in [c, a]$, thus any feasible p_i is a supremum. The case for another firm is symmetric. Draw the best response graphs and take intersections, we have Nash equilibrium at (c, c).

Problem of commons: Suppose n players choose goats to own g_i. The cost of maintaining a goat is c. Let $G = \sum g_i$. The value of grazing a goat is $v(G)$ is a graph that $v'(G) < 0$ and $v''(G) < 0$ when $G < G_{max}$, and $v(G) = 0$ if $G \ge G_{max}$. For each farmer, the feasible strategy space is thus $[0, G_{max})$. The payoff $\pi_i = g_i v(G) - cg_i$. By observing symmetry and differentiation and solving the system of equation for intersection, we find NE. For example, if $v(G) = a - G^2$, the NE is $g_i^* = \sqrt{\frac{a-c}{n^2+2n}}$.

In contrast, suppose the whole village decides G as one, then the social optimum $G' = \arg\max_{G \in [0, G_{max})]} Gv(G) - cG$. $G' < G^*$. In this example, $G^* = \frac{\sqrt{a-c}}{\sqrt{1+\frac{2}{n}}}$ and $G' = \frac{\sqrt{a-c}}{\sqrt{3}}$, when $n \to \infty$, we have $G^* \approx \sqrt{3}G'$.

Final-Offer Arbitration

Firm and union submit wage offers w_f and w_u ($w_f < w_u$). The arbitrator, selects the offer closer to x. The settlement follows the decision rule: w_f is chosen if $x < (w_f + w_u)/2$, otherwise w_u is selected. The variable x follows a cumulative distribution $F(x)$ with density function $f(x)$.

The expected wage settlement is: $\phi(w_f, w_u) = w_f F((w_f + w_u)/2) + w_u[1 - F((w_f + w_u)/2)]$.

x_m is the median of x. At equilibrium, $\frac{w_f + w_u}{2} = x_m$, $w_f^* = x_m - \frac{1}{2f(x_m)}$, $w_u^* = x_m + \frac{1}{2f(x_m)}$.

The wage gap at equilibrium is: $w_u - w_f = 1/f(x_m)$.

1.2 Mixed strategy profile

Definition: In the normal-form game $G = \{S_i, u_i\}$. Suppose $S_i = \{s_{i,1}, \ldots, s_{i,k}\}$, then each strategy $s_{i,k} \in S_i$ is called a pure strategy for player i. A mixed strategy for player i is a probability distribution on S_i.

Definition: In the two-player normal-form game $G = \{S_1, S_2, u_1, u_2\}$, the mixed strategy profile (p_1^*, p_2^*) is a Nash equilibrium if each player's mixed strategy is a best response in terms of expected return to other player's mixed strategy. $v_1(p_1^*, p_2^*) \ge v_1(p_1, p_2^*)$ and $v_2(p_1^*, p_2^*) \ge v_2(p_1^*, p_2)$ for all distribution p_1 on S_1 and p_2 on S_2. v is the expected return given mixed strategy profile.

Note that suppose that player 2 has mixed strategy p_2, when player 1 chooses to play $s_{1,j}$, the expected return is $v_1(s_{1,j}, p_2) = \sum_{i=1}^{k} u_1(s_{1,j}, s_{2,i}) p_{2,i}$, thus $v_1(p_1, p_2) = \sum_{j=1}^{k} p_{1,j} v_1(s_{1,j}, p_2) = \sum_{j=1}^{k} \sum_{i=1}^{k} p_{1,j} p_{2,i} u_1(s_{1,j}, s_{2,i})$

Definition: In the two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a **Nash equilibrium** if each player's mixed strategy is a best response to the other player's mixed strategy: $v_1(p_1^*, p_2^*) \ge v_1(p_1, p_2^*)$ and $v_2(p_1^*, p_2^*) \ge v_2(p_1^*, p_2)$ for all probability distributions p_1 and p_2 on S_1 and S_2 respectively.

Finding Nash Equilibria: Let player 1 have mixed strategy $p_1 = (r, 1 - r)$ and given player 2 has mixed strategy $p_2 = (q, 1 - q)$, we have that $v_1(p_1, p_2) = rv_1(s_{1,1}, p_2) + (1 - r)v_1(s_{1,2}, p_2)$. For each given $p_2(q)$, we could compute the best response $r^*(q) = \arg\max_{r \in [0,1]} v_1(p_1, p_2)$. Observing $v_1(p_1, p_2)$, we have $r^*(q) = 1$ if $v_1(s_{1,1}, p_2) > v_1(s_{1,2}, p_2)$, 0 if $v_1(s_{1,1}, p_2) < v_1(s_{1,2}, p_2)$, and $[0, 1]$ if $v_1(s_{1,1}, p_2) = v_1(s_{1,2}, p_2)$. Similarly, let player 2 have mixed strategy $p_2 = (q, 1 - q)$ and given player 1 has mixed strategy $p_1 = (r, 1 - r)$, we have that $v_2(p_1, p_2) = qv_2(p_1, s_{2,1}) + (1 - q)v_2(p_1, q_{2,2})$, which implies that $q^*(r) = 1$ if $v_2(p_1, s_{2,1}) > v_2(p_1, q_{2,2})$ and 0 if $v_2(p_1, s_{2,1}) < v_2(p_1, q_{2,2})$ and $[0, 1]$ if $v_2(p_1, s_{2,1}) = v_2(p_1, q_{2,2})$.

Matching pennies: Two players simultaneously choose head or tail. Player 1 wins both if the decisions match, otherwise player 2 wins both. By the discussion above, we have $r^*(q) = 1$ if $q \in [0, \frac{1}{2})$ and 0 if $q \in (\frac{1}{2}, 1]$ and $[0, 1]$ if $q = \frac{1}{2}$. Similarly, we have $q^*(r) = 0$ if $r \in [0, \frac{1}{2})$ and 1 if $r \in (\frac{1}{2}, 1]$ and $[0, 1]$ if $r = \frac{1}{2}$. The only intersection is $(\frac{1}{2}, \frac{1}{2})$, which means $p_1^* = p_2^* = (\frac{1}{2}, \frac{1}{2})$. The best strategy is to play randomly.

An remark is that a player will play a strictly dominated strategy with zero probability, thus we could eliminate them to simplify the game. If the probability distribution is deterministic, then the NE is in fact a pure NE.

1.3 Existence of NE

Theorem 3. In the n-player normal-form game $G = \{S_i, u_i\}$, if n is finite and each S_i is finite, then there exists at least one Nash equilibrium, possibly a mixed one.

2 Dynamic games of complete information

Definition: A *dynamic game of complete and perfect information* is a game where players move in sequence.

Perfect information: all previous moves are observed before the next move is chosen. *Complete information*: payoffs are common knowledge.

Backwards Induction

At the second stage, player 2 observes player 1's action a_1 and chooses a_2 to maximize $u_2(a_1, a_2)$. The optimal response is $R_2(a_1)$, meaning player 2's best response to a_1 is $R_2(L) = R'$, $R_2(R) = L'$.

Knowing this, player 1 chooses a_1 to maximize $u_1(a_1, R_2(a_1))$, leading to the optimal solution a_1^*, where $a_1^* = R$ and $R_2(a_1^*) = L'$.

The outcome $(a_1^*, R_2(a_1^*))$ is called the **backwards-induction outcome** of the game.

Stackelberg model of duopoly: Firm 1 chooses a quantity $q_1 > 0$. Firm 2 observes q_1 then chooses a quantity $q_2 > 0$. The price is then decided by $P(q_1, q_2) = \max(a - q_1 - q_2, 0)$, and profit $\pi_i = P(q_1, q_2)q_i - cq_i$. Suppose firm 2 has observed firm 1 choosing q_1, then knows that if $q_1 + q_2 < a$, then $\pi_2(q_1, q_2) = q_2(a - q_1 - q_2 - c)$, otherwise $\pi_2(q_1, q_2) = -cq_2$. Therefore, if $q_1 < a - c$, we go to first case and maximize to have $R_2(q_1) = \frac{a - q_1 - c}{2}$, otherwise price is guaranteed zero, so just choose $R_2(q_1) = 0$. Knowing the action of firm 2, firm 1 knows that if $q_1 < a - c$, then $\pi_1(q_1, q_2) = \pi_1(q_1, \frac{a - c - q_1}{2}) = q_1(a - q_1 - \frac{a - q_1 - c}{2} - c)$, for which $q_1^* = \frac{a-c}{2}$. For $q_1 > a - c$, $\pi_1 < 0$, thus we do not consider. Therefore, the backward induction outcome is $(\frac{a-c}{2}, \frac{a-c}{4})$.

Two stage games of complete and imperfect information: At first stage, Player 1 and 2 simultaneously choose actions (a_1, a_2). At second stage, Player 3 and 4 observe the outcome of the first stage and simultaneously choose actions (a_3, a_4). Each player receives payoff $u_i(a_1, a_2, a_3, a_4)$.

We perform backward analysis. Suppose (a_1, a_2) is chosen, player 3 and 4 find NE in stage 2 by choosing $(a_3(a_1, a_2), a_4(a_1, a_2))$. Therefore, from the perspective of player 1 and 2, if they choose (a_1, a_2), the game will end off with payoff $u_i(a_1, a_2, a_3(a_1, a_2), a_4(a_1, a_2))$. Hence, player 1 and 2 will choose NE based on the predicted payoff and lead to the subgame-perfect outcome $(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$.

Tariff game: Two countries has a firm that produces output h_i for domestic market and e_i for export market. The total quantity on market in country i is $Q_i = h_i + e_j$. The market price is $P_i(Q_i) = a - Q_i$. Each government can impose a tariff rate t_i to let firm i pay tariff $e_i t_j$ to government j. At stage 1, the governments simultaneously choose tariff (t_i, t_j). At stage 2, the firms observe the tariff rates and simultaneously choose $((h_1, e_1), (h_2, e_2))$. The payoff to firm i will be profit $\pi_i = [a - (h_i + e_j)]h_i + [a - (h_j + e_i)]e_i - c(h_i + e_i) - t_j e_i$. The payoff to government i is welfare defined by $w_i = \frac{1}{2}Q_i^2 + \pi_i + t_i e_j$.

We perform backward analysis. Suppose tariff (t_1, t_2) is chosen. By differentiation and setting $\frac{\partial \pi_i}{\pi h_i} = \frac{\partial \pi_i}{\partial e_i} = 0$, we obtain the best response of (h_i, e_i) given (h_j, e_j) as $h_i = \frac{1}{2}(a - e_j - c); e_i = \frac{1}{2}(a - h_j - c - t_j)$. Swap i, j to get another 2 equations, solve the system of 4 equations for $h_i^*, e_i^*, h_j^*, e_j^*$ we have $h_i^* = \frac{a - c + t_i}{3}, e_i^* = \frac{a - c - 2t_j}{3}$. We go back to stage 1. We know what the quantities are when a tariff is imposed, thus we substitute and compute welfare $w_i(t_1, t_2) = \frac{1}{2}(h_i^* + e_j^*)^2 + \pi_i + t_i e_j = \frac{1}{2}[\frac{2(a-c) - t_i}{3}]^2 + [\frac{a - c + t_i}{3}]^2 + [\frac{a - c - 2t_j}{3}]^2 + t_i[\frac{a - c - 2t_i}{3}]$. By differentiation we have that $t_i^* = \frac{a-c}{3}$. By substitution we have that $h_i^* = \frac{4(a-c)}{9}, e_i^* = \frac{a-c}{9}$.

If both governments impose zero tariff, indeed $w_1(t_1, t_2) + w_2(t_2, t_1)$ is maximized to achieve social optimality.

Definition: The extensive-form representation of a game specifies: the players in the game; when each player has the move; what each player can do at each of his or her opportunity to move; what each player knows at each of his or her opportunity to move; the payoff received by each player for each combination of moves that could be chosen by the players.

A dynamic game of complete and perfect information is a game in which the player moves in sequence, all previous moves are observed before the next move is chosen, and payoffs are common knowledge. When the information is not perfect, some previous moves are not observed by the player with the current move.

Definition: An information set for a player is a collection of decision nodes satisfying: the player needs to move at every node in the information set; when the play of the game reached a node in the information set, the player with the move does not know which node in the set has, or has not, been reached. A game is said to have imperfect information if some of its information sets are not singletons.

Definition: A strategy for a player is a complete plan of actions. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

The normal-form representation of a dynamic game specifies payoffs for each combination of strategies to derive NE.

Definition: A subgame in an extensive-form game begins at a decision node n that is a singleton information set but not the first decision node of the game, and includes all the decision and terminal nodes following the the node n in the game tree, and does not cut any information sets(i.e.if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in the subgame.)

Definition: A Nash equilibrium is subgame-perfect if the players' strategies constitute a Nash equilibrium in every subgame.

Three-period bargaining: Player 1 and 2 bargaining for one dollar. At stage 1, player 1 proposes $(s_1(1), 1 - s_1(1))$. If player 2 accepts, the game ends. If player 2 rejects, the game goes to stage 2. At stage 2, player 2 proposes $(1 - s_2(2), s_2(2))$. If player 1 accepts, the payoff is factored by δ. The game ends. If player 1 rejects, the game goes to stage 3. In stage 3, the game is forced to end by the proposal (s'_1, s'_2), factored by δ^2.

Suppose it is known that player 1 and 2 will receive (u_1, u_2) at stage $t + 1$, where $(u_1, u_2) \geq 0$ and $u_1 + u_2 \leq 1$. In period t, we claim that player i has best strategy to propose $s_j(t) = \delta u_j$ to player j and player j will accept the offer. If player i's offer in period t is rejected, the players will receive $\delta^t(u_1, u_2)$ in round $t+1$, therefore, if player i wants player j to accept the offer at round t by proposing $(s_i(t)), s_j(t))$, which gives actual payoff $\delta^{t-1}(s_i(t), s_j(t))$, then the payoff at round t must be at least the payoff at round $t + 1$, thus $\delta^{t-1}s_j(t) \geq \delta^t u_j \implies s_j(t) \geq \delta u_j$. Therefore, player i at round t has choice to propose $s_j(t) = \delta u_j$ or $s_j(t) < \delta u_j$. If he propose $(1 - \delta u_j, \delta u_j)$, then the game ends and they receive payoff $\delta^{t-1}(1 - \delta u_j, \delta u_j)$ respectively. However, if he chooses $s_j(t) < \delta u_j$ and let the game go to next round, he will receive $\delta^t u_i \leq \delta^t(1 - u_j) = \delta^t - \delta^t u_j = \delta^{t-1}(\delta - \delta u_j) < \delta^{t-1}(1 - \delta u_j)$, which is worse than the previous round. Hence, player i should propose $s_j(t) = \delta u_j$. If we move back to previous stage, the players swap their roles, and the same strategy follows to let the game end.

We discuss the backward induction outcome as follows: In round 2, player 2 is to propose. Because the payoff in round 3 is (u_1, u_2), so player 2 should propose $s_1(2) = \delta u_1$ and $s_2(2) = 1 - \delta u_1$. In round 1, player 1 is to propose, because the optimal proposal in round 2 is $(s_1(2), s_2(2))$, he should propose $s_2(1) = \delta s_2(2) = \delta(1 - \delta u_1)$ and $s_1(1) = 1 - s_2(1) = 1 - \delta(1 - \delta u_1)$ and player 2 will accept that offer to let the game end at round 1.

Infinite-period bargaining: The game beginning is round 3 is identical to the game beginning in round 1, because it is an infinitely repeated game and both start from player 1's proposal. Define (u_1, u_2) as the optimal payoffs players can receive in the backward induction outcome of the game. We interpret the game as follows: At round 1, player 1 makes a proposal. If player 2 rejects, the game go to round 2. At round 2, player 2 makes a proposal. If player 1 rejects, the game go to round 3. At round 3, based on previous insights, we interpret as if we go over to the infinite-period game and finally receive optimal payoff (u_1, u_2). In the three-period game, player 1 should propose $s_1(1) = 1 - \delta(1 - \delta u_1), s_2(1) = \delta(1 - \delta u_1)$ to let player 2 accept and end the game. Therefore, the optimal payoff of the infinite-period game received by player 1 is $1 - \delta(1 - \delta u_1)$. We have defined u_1 as the optimal payoff of this game, thus $u_1 = 1 - \delta(1 - \delta u_1)$, which gives $u_1 = \frac{1}{1+\delta}, u_2 = \frac{\delta}{1+\delta}$.