1 Complex numbers

1.1 Preliminaries

Notation: $z = x + iy$
Modulus: $\|z\| = \sqrt{x^2 + y^2}$
Two-square identity: $\|z_1 z_2\| = \|z_1\|\|z_2\|$
Circle centered at z_0: $\|z - z_0\| = r$
Conjugate: $\bar{z} = x - iy$
Conjugate basic properties: $\Re(z) = \frac{1}{2}(z+\bar{z}); \Im(z) = \frac{1}{2i}(z - \bar{z}); \overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}; \overline{z_1 z_2} = \bar{z_1}\bar{z_2}; \overline{(\frac{z_1}{z_2})} = \frac{\bar{z_1}}{\bar{z_2}}; z\bar{z} = \|z\|^2$
Triangle inequality: $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$

1.2 Polar form

$z = x + iy = r(\cos\theta + i\sin\theta)$
$r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}; x = r\cos\theta, y = r\sin\theta$
Definition: If $\theta \in \mathbb{R}$ such that $z = r(\cos\theta + i\sin\theta)$, we call θ an *argument* of z. The set of all possible arguments of z is denoted by z, $\arg z = \{\theta \in \mathbb{R} : z = r(\cos\theta + i\sin\theta)\}$. If θ_0 is an argument of z, then $\arg z = \{\theta \in \mathbb{R}, n \in \mathbb{Z} : \theta = \theta_0 + 2n\pi\}$. We can always fix a unique argument ϑ such that $\vartheta \in (-\pi, \pi]$, we call ϑ the principal argument of z and write $\mathrm{Arg}z = \vartheta$

1.3 Exponential form

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$
Circle centered at z_0: $\|z - z_0\| = r \implies z - z_0 = re^{i\theta} \implies z = z_0 + re^{it}, t \in [0, 2\pi]$
Product and quotient:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}; \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2); \arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$$

de Moivre's theorem: $(e^{i\theta})^n = e^{in\theta} \iff (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, n \in \mathbb{Z}$
We could use the theorem to find roots of z.
Suppose $z^n = z_0$, by de Moivre's, we have $r^n e^{in\theta} = r_0 e^{i\theta_0} \implies r^n = r_0$ & $n\theta = \theta_0 + 2k\pi \implies r = \sqrt[n]{r_0}$ & $\theta = \frac{\theta_0 + 2k\pi}{n}, k = 0, \ldots, n-1$

2 Topology preliminaries

Let $S \subseteq \mathbb{C}$, then: z_1 is an interior point of $S \implies$ there is an open ball $B_1 = B(z_1, r_1)$ such that $B_1 \subseteq S$. z_2 is an exterior point of $S \implies$ there is an open ball $B_2 = B(z_2, r_2)$ such that $B_2 \cap S = \varnothing$. z_3 is a boundary point of $S \implies B(z_3, r) \cap S \neq \varnothing$ & $B(z_3, r) \cap S^c \neq \varnothing \forall r > 0$. The boundary of S is ∂S =the set of all boundary points of S.
Definition: S is open if $\partial S \cap S = \varnothing$. S is closed if $\partial S \subseteq S$
Theorem: S is open $\iff S^c$ is closed.
Definition: The closure of S is $\bar{S} = S \cup \partial S$
Definition: An open set S is called connected if $\forall z_1, z_2 \in S$ we can join them by a polygonal line entirely in S.
Definition: An open connected set is called a domain. All open balls are domains.
Definition: A set S is bounded if there exists $R > 0$ such that $S \subseteq B(0, R)$. A set is called unbounded if it is not bounded.
Definition: A set is called compact if it is closed and bounded. All closed balls are compact.

3 Analytic functions

3.1 Complex functions

Let $S \subseteq \mathbb{C}$, then $f : S \to \mathbb{C}$ is a complex-valued function of a complex variable. Suppose $z = x + iy, w = u + iv, f(z) = w$,then $f(z) = f(x + iy) = u(x,y) + iv(x,y)$, where $u, v : S \to \mathbb{R}$ are real-valued functions.
If $f(z) = \frac{p(z)}{q(z)}$ where p, q are polynomials in z, we call f a rational function, which is well-defined as long as $q(z) \neq 0$

3.2 Limits

Definition: Let f be defined in some deleted open ball $B(z_0, r) - \{z_0\}$ of z_0. We say that w_0 is the limit of f as z approaches z_0 if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|z - z_0\| < \delta \implies \|f(z) - w_0\| < \epsilon$$

that is

$$z \in B(z_0, \delta) - \{z_0\} \implies f(z) \in B(w_0, \epsilon)$$

We write $\lim_{z \to z_0} f(z) = w_0$.
Theorem: If $f(z) = u(x,y) + iv(x,y), z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$, then $\lim_{z \to z_0} = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ & $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$
Theorem: Suppose that $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1. $\lim_{z \to z_0} f(z) \pm g(z) = A \pm B$
2. $\lim_{z \to z_0} f(z)g(z) = AB$
3. If $B \neq 0$,$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$
4. If f is polynomial/rational, then $\lim_{z \to z_0} f(z) = f(z_0)$

We could introduce ∞ and call $\mathbb{C} \cup \{\infty\}$ the extended complex plane.
Definition:

1. $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f(\frac{1}{z}) = w_0$

3.3 Continuity

Definition: The function f is continuous at z_0 if $f(z_0)$ is defined, and $\lim_{z \to z_0} f(z)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$. We say f is continuous at set S if f is continuous at every point of S.
We see that every polynomial and rational function is continuous at its domain.
Theorem: Composition of continuous function is continuous.

3.4 Derivatives

Definition: The derivative of f at z_0 is defined as

$$\frac{d}{dz} f(z)|_{z=z_0} = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided that the limit exists. If so, we say f is differentiable at z_0.
Basic properties

1. $\frac{d}{dz}[f(z) \pm g(z)]|_{z=z_0} = f'(z_0) \pm g'(z_0)$
2. If c is constant, then $\frac{d}{dz} cf(z)|_{z=z_0} = cf'(z_0)$
3. $\frac{d}{dz}[f(z)g(z)]|_{z=z_0} = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
4. If $g(z_0) \neq 0$,
$$\frac{d}{dz}[\frac{f(z)}{g(z)}]|_{z=z_0} = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$$
5. $\frac{du}{dz} = \frac{du}{dw}\frac{dw}{dz}$, or $\frac{d}{dz} f[g(z)]|_{z=z_0} = f'(g(z_0))g'(z_0)$

3.5 Cauchy-Riemann equations

Suppose f is differentiable at z_0, we approach z_0 horizontally and vertically, and we reach same limit and therefore obtain Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0); \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, then u and v satisfies Cauchy-Riemann equations at (x_0, y_0), and

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined in a neighbourhood $B(z_0, \epsilon)$ of point $z_0 = x_0 + iy_0$. Suppose the first order partial derivatives of u, v exist in $B(z_0, \epsilon)$ and: they satisfy CR equations at (x_0, y_0), they are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

3.6 Analytic functions

Definition: A function f is said to be analytic in an open set S if $f'(z)$ exists at every $z \in S$. If E is not an open set, then we say f is analytic in E if f is analytic in an open set containing E. We say f is analytic at a point z_0 if $f'(z)$ exists for every $z \in B(z_0, r)$ for $r > 0$. If f is analytic at every point in $\mathbb{C}$, then we call f an entire function.
Theorem: If f is analytic in a domain D and if $f'(z) = 0$ everywhere is D, then f is constant in D.

3.7 Harmonic functions

Definition: Let D be a domain. A function $u : D \to \mathbb{R}$ is harmonic in D if u has continuous first and second partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u, v are harmonic in D. In this case, we call v a harmonic conjugate of u in D.

4 Elementary functions

4.1 Exponential function

Definition: $f(z) = e^z = e^x(\cos y + i\sin y)\ \forall z \in \mathbb{C}$.
All first order partial derivatives of u and v are continuous everywhere and CR equations are satisfied everywhere, hence f is entire and $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = e^x \cos y + ie^x \sin y = f(z)$.
We have some immediate results: $\|e^z\| = e^x = e^{\Re(z)}$, $\arg e^z = y + 2k\pi, k \in \mathbb{Z}$
If $\theta \in \mathbb{R}$, by definition of exponential function, we have $e^{i\theta} = e^{0+i\theta} = e^0(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$, we recover Euler's formula, thus $e^z = e^{x+iy} = e^x e^{iy}\ \forall z = x + iy \in \mathbb{C}$.
We have several properties for exponential functions: For all complex numbers z_1, z_2, we have $e^{z_1}e^{z_2} = e^{z_1 + z_2}$, $e^{-z} = \frac{1}{e^z}$, $e^{nz} = (e^z)^n (n \in \mathbb{Z})$, $e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$
Theorem: $e^z = 1 \iff z = 2k\pi i, k \in \mathbb{Z}$, $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i, k \in \mathbb{Z}$. (The second property implies that different from real case, exponential function in complex case is no longer into.) In fact, exponential function in complex case is also not onto, the range is $\mathbb{C} - \{0\}$

4.2 Logarithmic function

Suppose $w = re^{i\theta} \neq 0$, then $e^z = w \implies e^x e^{iy} = re^{i\theta} \implies x = \ln r, y = \theta + 2k\pi \implies z = \ln|w| + i\theta', \theta' \in \arg w$. Therefore, we could define logarithmic functions.
Definition: $w \neq 0$, $\log w = \ln|w| + i\arg w$.
Note that $\arg w$ is not a single value but a set of values, thus logarithmic function in complex case is a multi-valued function. We also have $e^{\log z} = z\ \forall z \neq 0$, converse is not true, since multiple values are returned.
Definition: $w \neq 0$, $\mathrm{Log}w = \ln w + i\mathrm{Arg}w$. This is a single-valued function and we call it the principle value of $\log w$.
Note that Log is not onto, but if we restrict codomain to $\{z \in \mathbb{C} : -\pi < \Im(Z) \leq \pi\}$, then it is a bijection.
Theorem: For all $z_1, z_2 \neq 0$, we have $\log z_1 z_2 = \log z_1 + \log z_2$, $\log\frac{z_1}{z_2} = \log z_1 - \log z_2$. Not true for Log.
Theorem: The function $\mathrm{Log}z$ is analytic on $\mathbb{C} - (-\infty, 0]$ and $\frac{d}{dz}\mathrm{Log}z = \frac{1}{z}\ \forall z \in \mathbb{C} - (-\infty, 0]$
Definition: $F(z)$ is said to be a branch of a multi-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and analytic on D, and for each $z \in D$, $F(z)$ is one of the values of $f(z)$.
Note that $\mathrm{Log}z$ is a branch of $\log z$ in $D = \mathbb{C} - (-\infty, 0]$, called the principal branch of $\log z$.
We could define other branches of $\log z$: For any $\alpha \in \mathbb{R}$, we let $\mathbb{C}_\alpha = \mathbb{C} - \{\text{the ray:}0\ \&\ \theta = \alpha\}$. For each $z \in \mathbb{C}_\alpha$, we could define $L_\alpha(z) = \ln z + i\theta$, where θ is the unique value of $\arg z$ in $(\alpha, \alpha + 2\pi]$, therefore L_α is analytic in $\mathbb{C}_\alpha$, and $\frac{d}{dz}L_\alpha(z) = \frac{1}{z}\ \forall z \in \mathbb{C}_\alpha$

4.3 Complex exponents

Definition: For z, c with $z \neq 0$, we define $z^c = e^{c\log z}$. Note that since $\log z$ is multi-valued, so as z^c.
Definition: The principal branch of z^c is defined by $\mathbf{Pr}(z^c) = e^{c\mathrm{Log}z}, z \in \mathbb{C} - (-\infty, 0]$
Theorem: $\mathbf{Pr}(z^c)$ is analytic on $\mathbb{C} - (-\infty, 0]$, and $\frac{d}{dz}\mathbf{Pr}(z^c) = c\mathbf{Pr}(z^{c-1})\ \forall z \in \mathbb{C} - (-\infty, 0]$.
Definition: More generally, for each $\alpha \in \mathbb{R}$, the function defined by $F_{\alpha,c}(z) = e^{cL_\alpha(z)}, \forall z \in \mathbb{C}_\alpha$ is a branch of z^c, and $\frac{d}{dz}F_{\alpha,c}(z) = cF_{\alpha,c-1}(z), \forall z \in \mathbb{C}_\alpha$

4.4 Trigonometric function

Definition: for $z \in \mathbb{C}$, we define

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

Definition: for $z \in \mathbb{C}$, we define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Theorem: $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sin z = \cos z$

Theorem: we have some trigonometric identities consistent in both real and complex cases:

1. $\sin^2 z + \cos^2 z = 1$
2. $\sin z_1 \pm z_2 = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$
3. $\cos z_1 \pm z_2 = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

Definition: we define some other trigonometric functions: $\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{1}{\tan z}, \sec z = \frac{1}{\cos z}, \csc z = \frac{1}{\sin z}$

4.5 Hyperbolic functions

Definition: we define some hyperbolic functions: $\sinh z = \frac{1}{2}(e^z - e^{-z}), \cosh z = \frac{1}{2}(e^z + e^{-z}), \tanh z = \frac{\sinh z}{\cosh z}$

Theorem: $\frac{d}{dz}\sinh z = \cosh z$, $\frac{d}{dz}\cosh z = \sinh z$

Theorem: We state some hyperbolic identities:

1. $\cosh^2 z - \sinh^2 z = 1$
2. $\sinh z_1 + z_2 = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
3. $\cosh z_1 + z_2 = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$
4. $\sinh x + iy = \sinh x \cos y + i \cosh x \sin y$
5. $\cosh x + iy = \cosh x \cos y + i \sinh x \sin y$

5 Integrals

5.1 Definition

Consider a complex-valued function of a real variable:

$$w : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$$
$$w(t) = u(t) + iv(t)$$

If u and v are piecewise continuous on $[a, b]$, then we define

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

If u' and v' both exist, then we define

$$\frac{d}{dt}w(t) = u'(t) + iv'(t)$$

Theorem: Let $w_1, w_2 : [a, b] \to \mathbb{C}$ and $z_0 \in \mathbb{C}$

1. $\frac{d}{dt}[w_1(t) + w_2(t)] = w_1'(t) + w_2'(t)$
2. $\frac{d}{dt}[z_0 w_1(t)] = z_0 w_1'(t)$
3. $\int_a^b [w_1(t) + w_2(t)]dt = \int_a^b w_1(t)dt + \int_a^b w_2(t)dt$
4. $\int_a^b z_0 w_1(t)dt = z_0 \int_a^b w_1(t)dt$

Theorem: If $w : [a, b] \to \mathbb{C}$, then

$$|\int_a^b w(t)dt| \leq \int_a^b |w(t)|dt$$

Theorem: Suppose $F(t) = A(t) + iB(t)$ and $f(t) = a(t) + ib(t)$ such that $F'(t) = f(t)$, then $\int_a^b f(t)dt = F(b) - F(a)$

5.2 Curve in complex plane

Definition: A curve in the complex plane is a continuous function $\gamma : [a, b] \to \mathbb{C}$. Equivalently, if we write $\gamma(t) = x(t) + iy(t)$, then x and y are both continuous on $[a, b]$

Definition: γ is simple if $t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2) \iff \gamma$ does not cross itself.

Definition: γ is closed if $\gamma(a) = \gamma(b)$.

Definition: γ is a simple closed curve if both simple and closed.

Definition: γ is called smooth if $\gamma'(t) = x'(t) + iy'(t)$ exists and is continuous on $[a, b]$ and $\gamma'(t) \neq 0\ \forall t \in [a, b]$

Definition: The length of a smooth curve γ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)|dt$$

Definition: Let S be an open set in $\mathbb{C}$ and let γ be a smooth curve in S. If f is continuous, then the integral of f along γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Theorem: We have following identities:

1. $\int_\gamma [f(z) + g(z)]dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$
2. $\int_\gamma z_0 f(z)dz = z_0 \int_\gamma f(z)dz$

Theorem: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve. Let $\phi : [c, d] \to [a, b]$ such that ϕ' exists and continuous on $[c, d]$, and $\phi(c) = a, \phi(d) = b$. Define $\alpha(t) = \gamma(\phi(t)), t \in [c, d]$, then for any function f continuous on the curve, we have $\int_\gamma f(t)dt = \int_\alpha f(t)dt$.

Definition: Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Define the opposite of γ by $(-\gamma)(t) = \gamma(-t), t \in [-b, -a]$

Theorem: For any smooth curve γ, $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$

Definition: A contour γ is a sequence of smooth curves $\{\gamma_1, \ldots, \gamma_n\}$ such that the terminal points of γ_k coincides with the initial point of γ_{k+1}. We write $\gamma = \sum_{i=1}^n \gamma_i$

Definition: If f is continuous on γ, we define

$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_{\gamma_i} f(z)dz$$

Theorem:(ML-inequality): Suppose that f is continuous on an open set containing the track of a contour γ, and $|f(z)| \leq M\ \forall z \in \{\gamma\}$, then

$$|\int_\gamma f(z)dz| \leq ML$$

where L is the length of γ

6 Useful references

1. $\Re(z) \leq |\Re(z)| \leq |z|$
2. $\Im(z) \leq |\Im(z)| \leq |z|$
3. **Corollary of triangle inequality**: $||z_1| - |z_2|| \leq |z_1 - z_2|$
4. **Mean value theorem**: If g is differentiable on (a, b) and continuous on $[a, b]$, then

   $$\frac{g(a) - g(b)}{a - b} = g'(c)$$

   for some $c \in (a, b)$
5. **Several conditions to deduce an analytic $f(z)$ is constant in domain D**: $\Re(f(z))$ is constant in D; $f(z)$ is real in D; $\overline{f(z)}$ is analytic in D; $|f(z)|$ is constant in D; $\mathbf{Arg}[f(z)]$ is constant in D
6. $\sin x + iy = \sin x \cosh y + i \cos x \sinh y$
7. A curve is positively oriented if the interior domain lies to the left of an observer tracing out the points in order.