1 Complex numbers

1.1 Preliminaries

Circle centered at z_0: $\|z - z_0\| = r$
Triangle inequality: $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$
Corollary of triangle inequality: $||z_1| - |z_2|| \leq |z_1 - z_2|$

1.2 Polar form

$z = x + iy = r(\cos\theta + i\sin\theta)$
$r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}; x = r\cos\theta, y = r\sin\theta$
Definition: If θ_0 is an argument of z, then $\arg z = \{\theta \in \mathbb{R}, n \in \mathbb{Z} : \theta = \theta_0 + 2n\pi\}$. We can always fix a unique argument ϑ such that $\vartheta \in (-\pi, \pi]$, we call ϑ the principal argument of z and write $\text{Arg}z = \vartheta$

1.3 Exponential form

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$
Circle centered at z_0: $\|z - z_0\| = r \implies z - z_0 = re^{i\theta} \implies z = z_0 + re^{it}, t \in [0, 2\pi]$
de Moivre's theorem: $(e^{i\theta})^n = e^{in\theta} \iff (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, n \in \mathbb{Z}$
We could use the theorem to find roots of z.
Suppose $z^n = z_0$, by de Moivre's, we have $r^n e^{in\theta} = r_0 e^{i\theta_0} \implies r^n = r_0$ & $n\theta = \theta_0 + 2k\pi \implies r = \sqrt[n]{r_0}$ & $\theta = \frac{\theta_0 + 2k\pi}{n}, k = 0, \ldots, n-1$

2 Analytic functions

2.1 Limits

Theorem: If $f(z) = u(x,y) + iv(x,y), z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$, then $\lim_{z\to z_0} = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ & $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$
We could introduce ∞ and call $\mathbb{C}\cup\{\infty\}$ the extended complex plane.
Definition: $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0$; $\lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f(\frac{1}{z}) = w_0$

2.2 Continuity

Definition: The function f is continuous at z_0 if $f(z_0)$ is defined, and $\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$. We say f is continuous at set S if f is continuous at every point of S.
We see that every polynomial and rational function is continuous at its domain.
Theorem: Composition of continuous function is continuous.

2.3 Derivatives

Definition: The derivative of f at z_0 is defined as $\frac{d}{dz}f(z)|_{z=z_0} = f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ provided that the limit exists. If so, we say f is differentiable at z_0.
Basic properties

1. $\frac{d}{dz}[f(z) \pm g(z)]|_{z=z_0} = f'(z_0) \pm g'(z_0)$
2. If c is constant, then $\frac{d}{dz}cf(z)|_{z=z_0} = cf'(z_0)$
3. $\frac{d}{dz}[f(z)g(z)]|_{z=z_0} = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
4. If $g(z_0) \neq 0$,
$$\frac{d}{dz}[\frac{f(z)}{g(z)}]|_{z=z_0} = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$$
5. $\frac{d}{dz}f[g(z)]|_{z=z_0} = f'(g(z_0))g'(z_0)$

2.4 Cauchy-Riemann equations

Cauchy-Riemann equations: $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$; $\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$
Theorem: If $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, then u and v satisfies Cauchy-Riemann equations at (x_0, y_0), and $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$
Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined in a neighbourhood $B(z_0, \epsilon)$ of point $z_0 = x_0 + iy_0$. Suppose the first order partial derivatives of u, v exist in $B(z_0, \epsilon)$ and: they satisfy CR equations at (x_0, y_0), they are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

2.5 Analytic functions

Theorem: If f is analytic in a domain D and if $f'(z) = 0$ everywhere is D, then f is constant in D.

2.6 Harmonic functions

Definition: Let D be a domain. A function $u : D \to \mathbb{R}$ is harmonic in D if u has continuous first and second partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Theorem: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u, v are harmonic in D. In this case, we call v a harmonic conjugate of u in D.

3 Elementary functions

3.1 Exponential function

Definition: $f(z) = e^z = e^x(\cos y + i\sin y)\ \forall z \in \mathbb{C}$.
All first order partial derivatives of u and v are continuous everywhere and CR equations are satisfied everywhere, hence f is entire and $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = e^x\cos y + ie^x\sin y = f(z)$.
We have some immediate results: $\|e^z\| = e^x = e^{\Re(z)}$, $\arg e^z = y + 2k\pi, k \in \mathbb{Z}$
If $\theta \in \mathbb{R}$, by definition of exponential function, we have $e^{i\theta} = e^{0+i\theta} = e^0(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$, we recover Euler's formula, thus $e^z = e^{x+iy} = e^x e^{iy}\ \forall z = x + iy \in \mathbb{C}$.
Theorem: $e^z = 1 \iff z = 2k\pi i, k \in \mathbb{Z}$, $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i, k \in \mathbb{Z}$. (The second property implies that different from real case, exponential function in complex case is no longer into.) In fact, exponential function in complex case is also not onto, the range is $\mathbb{C} - \{0\}$

3.2 Logarithmic function

Suppose $w = re^{i\theta} \neq 0$, then $e^z = w \implies e^x e^{iy} = re^{i\theta} \implies x = \ln r, y = \theta + 2k\pi \implies z = \ln|w| + i\theta', \theta' \in \arg w$. Therefore, we could define logarithmic functions.
Definition: $w \neq 0$, $\log w = \ln|w| + i\arg w$.
Note that $\arg w$ is not a single value but a set of values, thus logarithmic function in complex case is a multi-valued function. We also have $e^{\log z} = z\ \forall z \neq 0$, converse is not true, since multiple values are returned.
Definition: $w \neq 0$, $\text{Log}w = \ln w + i\text{Arg}w$. This is a single-valued function and we call it the principle value of $\log w$.
Note that Log is not onto, but if we restrict codomain to $\{z \in \mathbb{C} : -\pi < \Im(Z) \leq \pi\}$, then it is a bijection.
Theorem: For all $z_1, z_2 \neq 0$, we have $\log z_1 z_2 = \log z_1 + \log z_2$, $\log\frac{z_1}{z_2} = \log z_1 - \log z_2$. Not true for Log.
Theorem: The function $\text{Log}z$ is analytic on $\mathbb{C} - (-\infty, 0]$ and $\frac{d}{dz}\text{Log}z = \frac{1}{z}\ \forall z \in \mathbb{C} - (-\infty, 0]$
Definition: $F(z)$ is said to be a branch of a multi-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and analytic on D, and for each $z \in D$, $F(z)$ is one of the values of $f(z)$.
Note that $\text{Log}z$ is a branch of $\log z$ in $D = \mathbb{C} - (-\infty, 0]$, called the principal branch of $\log z$.
We could define other branches of $\log z$: For any $\alpha \in \mathbb{R}$, we let $\mathbb{C}_\alpha = \mathbb{C} - \{\text{the ray:}0\ \&\ \theta = \alpha\}$. For each $z \in \mathbb{C}_\alpha$, we could define $L_\alpha(z) = \ln z + i\theta$, where θ is the unique value of $\arg z$ in $(\alpha, \alpha + 2\pi]$, therefore L_α is analytic in $\mathbb{C}_\alpha$, and $\frac{d}{dz}L_\alpha(z) = \frac{1}{z}\ \forall z \in \mathbb{C}_\alpha$

3.3 Complex exponents

Definition: For z, c with $z \neq 0$, we define $z^c = e^{c\log z}$. Note that since $\log z$ is multi-valued, so as z^c.
Definition: The principal branch of z^c is defined by $\mathbf{Pr}(z^c) = e^{c\text{Log}z}, z \in \mathbb{C} - (-\infty, 0]$
Theorem: $\mathbf{Pr}(z^c)$ is analytic on $\mathbb{C} - (-\infty, 0]$, and $\frac{d}{dz}\mathbf{Pr}(z^c) = c\mathbf{Pr}(z^{c-1})\ \forall z \in \mathbb{C} - (-\infty, 0]$.
Definition: More generally, for each $\alpha \in \mathbb{R}$, the function defined by $F_{\alpha,c}(z) = e^{cL_\alpha(z)}, \forall z \in \mathbb{C}_\alpha$ is a branch of z^c, and $\frac{d}{dz}F_{\alpha,c}(z) = cF_{\alpha,c-1}(z), \forall z \in \mathbb{C}_\alpha$

3.4 Trigonometric function

Definition: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
Definition: $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
Theorem: $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sin z = \cos z$

3.5 Hyperbolic functions

Definition: we define some hyperbolic functions: $\sinh z = \frac{1}{2}(e^z - e^{-z}), \cosh z = \frac{1}{2}(e^z + e^{-z})$
Theorem: $\frac{d}{dz}\sinh z = \cosh z$, $\frac{d}{dz}\cosh z = \sinh z$

4 Integrals

4.1 Definition

Consider a complex-valued function of a real variable: $w : [a,b] \subseteq \mathbb{R} \to \mathbb{C}$; $w(t) = u(t) + iv(t)$
If u and v are piecewise continuous on $[a,b]$, then we define $\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$
If u' and v' both exist, then we define $\frac{d}{dt}w(t) = u'(t) + iv'(t)$
Theorem: Let $w_1, w_2 : [a,b] \to \mathbb{C}$ and $z_0 \in \mathbb{C}$

1. $\frac{d}{dt}[w_1(t) + w_2(t)] = w_1'(t) + w_2'(t)$
2. $\frac{d}{dt}[z_0 w_1(t)] = z_0 w_1'(t)$
3. $\int_a^b [w_1(t) + w_2(t)]dt = \int_a^b w_1(t)dt + \int_a^b w_2(t)dt$
4. $\int_a^b z_0 w_1(t)dt = z_0\int_a^b w_1(t)dt$

Theorem: If $w : [a,b] \to \mathbb{C}$, then $|\int_a^b w(t)dt| \leq \int_a^b |w(t)|dt$
Theorem: Suppose $F(t) = A(t) + iB(t)$ and $f(t) = a(t) + ib(t)$ such that $F'(t) = f(t)$, then $\int_a^b f(t)dt = F(b) - F(a)$

4.2 Curve in complex plane

Definition: A curve in the complex plane is a continuous function $\gamma : [a,b] \to \mathbb{C}$. Equivalently, if we write $\gamma(t) = x(t) + iy(t)$, then x and y are both continuous on $[a,b]$
Definition: γ is simple if $t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2) \iff \gamma$ does not cross itself.
Definition: γ is closed if $\gamma(a) = \gamma(b)$.
Definition: The length of a smooth curve γ is defined by $L(\gamma) = \int_a^b |\gamma'(t)|dt$
Definition: Let S be an open set in $\mathbb{C}$ and let γ be a smooth curve in S. If f is continuous, then the integral of f along γ is defined by $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$
Theorem: We have following identities:
$\int_\gamma [f(z) + g(z)]dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$; $\int_\gamma z_0 f(z)dz = z_0\int_\gamma f(z)dz$
Theorem: Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve. Let $\phi : [c,d] \to [a,b]$ such that ϕ' exists and continuous on $[c,d]$, and $\phi(c) = a, \phi(d) = b$. Define $\alpha(t) = \gamma(\phi(t)), t \in [c,d]$, then for any function f continuous on the curve, we have $\int_\gamma f(t)dt = \int_\alpha f(t)dt$.
Definition: Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Define the opposite of γ by $(-\gamma)(t) = \gamma(-t), t \in [-b, -a]$
Theorem: For any smooth curve γ, $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$
Definition: A contour γ is a sequence of smooth curves $\{\gamma_1, \ldots, \gamma_n\}$ such that the terminal points of γ_k coincides with the initial point of γ_{k+1}. We write $\gamma = \sum_{i=1}^n \gamma_i$
Definition: If f is continuous on γ, we define $\int_\gamma f(z)dz = \sum_{i=1}^n \int_{\gamma_i} f(z)dz$
Theorem:(ML-inequality): Suppose that f is continuous on an open set containing the track of a contour γ, and $|f(z)| \leq M\ \forall z \in \{\gamma\}$, then $|\int_\gamma f(z)dz| \leq ML$ where L is the length of γ.

4.3 Antiderivatives

Definition: Let f be a continuous function on a domain D, a function F such that $F'(z) = f(z)\forall z \in D$ is called an antiderivative of f in D.
Theorem: Suppose f has an antiderivative F on a domain D. If $z_1, z_2 \in D$ and γ is a contour in D joining z_1, z_2, then $\int_\gamma f(z)dz = F(z_2) - F(z_1)$. In particular, if γ is closed in D, then the integral evaluates to 0.
Theorem: Let f be continuous on a domain D, then the following are equivalent: f has an antiderivative in D; for any closed contour γ in D, $\int_\gamma f(z)dz = 0$; the contour integrals of f are independent of paths in D, that is, if z_1, z_2 in D and γ_1, γ_2 are two different contours in D joining z_1, z_2, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$
Theorem(Cauchy-Goursat): (a)If a function f is analytic at all points interior to and on a simple closed contour γ, then $\int_\gamma f(z)dz = 0$. (b)If f is analytic in a simply connected domain D, then $\int_\gamma f(z)dz = 0$ for every closed contour γ in D.
Corollary: (a)Let γ_1, γ_2 be positively oriented simple closed contours with γ_2 interior to γ_1. If f is analytic on closed region containing γ_1 and γ_2 and

points between them, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.
(b)If f is analytic in a simply connected domain D, then it has an antiderivative in D.
Theorem(Cauchy integral formula): Let γ be a positively oriented simple closed contour and let f be analytic everywhere within and on γ, then for any z_0 interior to γ, $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz$
Theorem: Let γ be a positively oriented simple closed contour and let f be analytic within and on γ, then for any point z_0 interior to γ: (1)$f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}dz$ (2)$\frac{f''(z_0)}{2!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^3}dz$
(3)$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$
Corollary: If f is analytic in a domain D, then all its derivatives f', f'', f''' exist and are analytic in D. In particular, if $f(z) = u(x,y) + iv(x,y)$, then u, v have continuous partial derivatives of all orders in D.
Theorem(Morera): If f is continuous on a domain D, and $\int_\gamma f(z)dz = 0$ for every closed contour γ in D, then f is analytic in D.
Theorem(Cauchy inequality): Let $z_0 \in \mathbb{C}$, consider $B = B(z_0, R)$. If f is analytic within and on B, then for $n \in \mathbb{Z}$, we have $\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}dz$, let $M_R = \max_{x\in\gamma_R}|f(z)|$, then for $z \in \gamma_R$, $|\frac{f(z)}{(z-z_0)^{n+1}}| \le \frac{M_R}{R^{n+1}}$, which by ML inequality leads to $|f^{(n)}| \le \frac{n!M_R}{R^n}$
Theorem(Liouville): If an entire function f is bounded, then it must be a constant function.

5 Series representation

5.1 Complex sequences

Definition: We say that the sequence $\{z_n\}$ has a limit z if for every $\epsilon > 0$, $\exists N \in \mathbb{Z}^{>0}$ such that $n \ge N \implies |z_n - z| < \epsilon$. If a limit does not exist, we say the sequence diverges.
Remark: $\lim f(z) = 0 \iff \lim|f(z)| = 0 \iff \lim \overline{f(z)} = 0$
Theorem: A convergent sequence has a unique limit. Any subsequence of a convergent sequence has the same limit.
Theorem: If $z \in \mathbb{C}$ and $|z| < 1$, then $\lim z^n = 0$
Theorem: If $z_n = x_n + iy_n$ and $z = x + iy$, then $z_n \to z \iff x_n \to x$ and $y_n \to y$.
Theorem(Properties of convergent sequences): If $z_n \to z$ and $w_n \to w$, then $z_n + w_n \to z + w$, $z_n w_n \to zw$, $\frac{z_n}{w_n} \to \frac{z}{w}$(if $w \neq 0$ and $w_n \neq 0 \forall n$)
Definition: A sequence $\{z_n\}$ is called a Cauchy sequence if $\forall\epsilon > 0, \exists N \in \mathbb{Z}^{>0}$ such that $n, m > N \implies |z_n - z_m| < \epsilon$.
Theorem(Cauchy criterion): $\{z_n\}$ is convergent $\iff$ it is Cauchy.

5.2 Complex series

Definition: Given a sequence $\{z_n\}$, form the sequence of partial sums $S_n = \sum_{i=1}^n z_i$, if $S_n \to S$, we say the series converges to S. If a limit does not exist, we say that the series diverges.
Theorem: If $z_n = x_n + iy_n$ and $S = \alpha + i\beta$, then $S_n \to S \iff \sum x_n \to \alpha$ and $\sum y_n \to \beta$
Theorem: If S_n converges, then $z_n \to 0$.
Theorem(Cauchy criterion): S_n converges $\iff \forall\epsilon > 0, \exists N \in \mathbb{Z}^{>0}$ such that $m > n \ge N \implies |z_{n+1} + z_{n+2} + \cdots + z_m| < \epsilon$
Definition: If $\sum_{n=1}^{+\infty}|z_n|$ converges, we say $\sum_{n=1}^{+\infty} z_n$ converges absolutely.
Theorem: If S_n converges absolutely, it converges.
Theorem(Comparison test): If $|z_n| \le a_n$ and $\sum a_n$ converges, then $\sum z_n$ converges absolutely.
Theorem(Ratio test): If $\lim_{n\to\infty}|\frac{z_{n+1}}{z_n}| = L$, then $L < 1 \implies \sum z_n$ converges absolutely. $L > 1 \implies \sum z_n$ diverges. No conclusion if $L = 1$.
Theorem(Root test): If $\limsup|z_n|^{\frac{1}{n}} = C$, then $C < 1 \implies \sum z_n$ converges. $C > 1 \implies \sum z_n$ diverges.

5.3 Power series

Definition(Geometric series): If $z \in B(0,1)$, then $\sum z^n = \frac{1}{1-z}$
Definition: A series of the form $\sum_{n=0}^\infty a_n(z-z_0)^n$ is called a power series.
Theorem: If $\sum a_n(z-z_0)^n$ converges at $z = z_1 \neq z_0$, then it converges absolutely for all z such that $|z - z_0| < R$, where $R = |z_1 - z_0|$
Theorem: For any power series $\sum_{n=0}^\infty a_n(z-z_0)^n$, there is a unique $0 \le R \le \infty$ such that (1)the series converges absolutely for all $z \in B(z_0, R)$ (2)the series diverges for all z such that $|z - z_0| > R$ (3)no conclusion if $|z - z_0| = R$
Definition: R is called the radius of convergence of the series.
Theorem(Ratio test): If $\lim_{n\to\infty}|\frac{z_{n+1}}{z_n}| = L$, then $R = \frac{1}{L}$

5.4 Pointwise and uniform convergence

Definition: Let $\{f_n\}$ be a sequence of functions defined on a subset D of the complex plane $\mathbb{C}$. Suppose that for all $z \in D$, the sequence $\{f_n(z)\}$ of complex number converges, then we define a function $f : D \to \mathbb{C}$ by $f(z) = \lim f_n(z)$ and we say that $\{f_n\}$ converges to f pointwise on D.
Definition: We say that $\{f_n\}$ converges to f uniformly on D if $\forall\epsilon > 0, \exists N \in \mathbb{Z}^{>0}$ such that $n \ge N \implies |f_n - f(z)| < \epsilon\ \forall z \in D$ (same N works for all points $z \in D$)
Remark: Suppose that $f_n \to f$ uniformly on a set S and $g(z)$ is bounded on S, then (product) $gf_n \to gf$ uniformly on S.
Theorem: If $f_n \to f$ uniformly on D and each f_n is continuous on D, then f is also continuous on D.
Theorem: Let γ be a contour and let $\{f_n\}$ be a sequence of functions continuous on γ. If $f_n \to f$ uniformly on γ, then $\lim_{n\to\infty}\int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \int_\gamma f(z)dz$.
Theorem: Let $\{f_n\}$ be a sequence of analytic functions on a domain D. If $f_n \to f$ uniformly on D, then f is analytic on D.
Definition: We say that the series of functions $\sum_{n=1}^\infty f_n(z)$ converges to $S(z)$ uniformly on D if the sequence of partial sums $S_n(z) = \sum_{k=1}^n f_k(z)$ converges to $S(z)$ uniformly on D.
Theorem: If the series converges uniformly on γ, then we could interchange the summation with the integration: $\int_\gamma \sum_{n=1}^\infty f_n(z)dz = \sum_{n=1}^\infty \int_\gamma f_n(z)dz$
Theorem(Weierstrass M-test): Let $\sum_{n=1}^\infty M_n$ be a convergent series of positive numbers. Suppose that $|f_n(z)| \le M_n \forall z \in \mathbb{Z}^{>0}$, then $\sum_{n=1}^\infty f_n(z)$ converges uniformly and absolutely on D.
Theorem: Let R be the radius of convergence of $\sum_{n=0}^\infty a_n(z-z_0)^n$. For each $0 < R_1 < R$, the series converges uniformly and absolutely on $\overline{B(z_0, R_1)}$(closure of open ball).
Theorem: Let R be the radius of convergence of $\sum_{n=0}^\infty a_n(z-z_0)^n$, then (a)$S(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$ is analytic on $B(z_0, R)$ (b)If γ is a contour in $B(z_0, R)$ and g is continuous on γ, then $\int_\gamma g(z)S(z)dz = \int_\gamma g(z)\sum_{n=0}^\infty a_n(z-z_0)^n dz = \sum_{n=0}^\infty a_n \int_\gamma g(z)(z-z_0)^n dz$. In particular, if $g(z) \equiv 1$, then $\int_\gamma \sum_{n=0}^\infty a_n(z-z_0)^n dz = \sum_{n=0}^\infty a_n \int_\gamma (z-z_0)^n dz$ (c)$S'(z) = \frac{d}{dz}\sum_{n=0}^\infty a_n(z-z_0)^n = \sum_{n=1}^\infty na_n(z-z_0)^{n-1}$

5.5 Taylor series

Theorem(Taylor): If f is analytic in an open ball $B(z_0, r)$, then $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\ \forall z \in B(z_0, r)$, this series is called the Taylor series of f at z_0.
Remark: If f is analytic in $B(0, r)$, then $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}z^n \forall z \in B(0, r)$ is the Maclaurin series of f.
Theorem: If $f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n \forall z \in B(z_0, r)$, then the series is the Taylor series of f at z_0.

5.6 Laurent series

Theorem(Laurent): Let f be analytic in the annulus $A = \{z : R_1 < |z - z_0| < R_2\}$, then f is represented by a Laurent series

$$f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n + \sum_{n=1}^\infty \frac{b_n}{(z-z_0)^n}, z \in A$$

where $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z_0)^{n+1}}ds, n = 0, 1, 2\ldots$
$b_n = \frac{1}{2\pi i}\int_\gamma f(s)(s-z_0)^{n-1}ds, n = 1, 2\ldots$
and γ is any positively oriented simple closed contour around z_0 lying in A.
Remark: We could take $R_1 = 0$, which corresponds to punctured ball domain.

6 Residues and poles

6.1 Singularities and residues

Definition: A point z_0 is a singular point of a function f is f is not analytic at z_0 but analytic at some point in $B(z_0, \epsilon)$ for all $\epsilon > 0$.
Definition: A singular point z_0 of f is isolated if there exists $R > 0$ such that f is analytic in $B(z_0, R) - \{z_0\}$.
Definition: The residue of f at an isolated singularity point z_0 is defined by $Res_{z=z_0}f(z) = b_1 =$coefficient of $\frac{1}{z-z_0}$ in the Laurent expansion of f at z_0.
Theorem: If f is analytic in $B(z_0, R) - \{z_0\}$, then $\int_\gamma f(z)dz = 2\pi i Res_{z=z_0}f(z)$, where γ is a positively oriented simple closed contour around z_0 in the domain.
Definition: Let f have an isolated singular point at z_0, then $f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n + \sum_{n=1}^\infty \frac{b_n}{(z-z_0)^n}, 0 < |z - z_0 < R$.
(1)If $b_n = 0$ for all n, then we say that z_0 is a removable singularity of f. In this case, the Laurent series of f reduces to a power series. Since the Laurent series has no principal part, $Res_{z=z_0}f(z) = 0$.
(2)If $b_n \neq 0$ for infinitely many n, then we say that z_0 is an essential singularity.
(3)If there exists $m \in \mathbb{Z}^+$ such that $b_m \neq 0$ and $b_n = 0$ for all $n > m$, then the principal part is up to power m, we say z_0 is a pole of order m for f. Simple pole if $m = 1$. Double pole if $m = 2$.
Theorem: f has a pole of order m at z_0 if and only if there exists $R > 0$ such that $f(z) = \frac{\phi(z)}{(z-z_0)^m}, 0 < |z - z_0| < R$, where ϕ is analytic at z_0 and $\phi(z_0) = 0$.
Corollary: If f has a pole at $z = z_0$, then $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0}\frac{1}{f(z)} = 0$.
Theorem: Suppose f has an isolated singularity at $z = z_0$. Then z_0 is a removable singularity for f if and only if f is bounded in a deleted neighbourhood around z_0.
Theorem(Picard): If f has an essential singularity at z_0, then in any open neighbourhood of z_0, f assumes every finite value with one possible exception for an infinite number of times.
Theorem: If z_0 is a simple pole of f, then $Res_{z=z_0}f(z) = \lim_{z\to z_0}(z - z_0)f(z)$.
Theorem: If z_0 is a pole of order m where $m > 1$, then $Res_{z=z_0}f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m f(z)$.
Theorem(Cauchy residue theorem): If γ is a positively oriented simple closed contour and f is analytic inside and on γ except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside γ, then $\int_\gamma f(z)dz = 2\pi i\sum_{j=1}^n Res_{z=z_j}f(z)$.

6.2 Poles and zeros

Definition: A point z_0 is called a zero of f if $f(z_0) = 0$. If $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 but f^{(m)}(z_0) \neq 0$, then we say f has a zero of order m at z_0. Simple zero if $m = 1$.
Theorem: Let f be analytic at z_0, then f has a zero of order m at z_0 if and only if $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
Theorem: Let p and q be analytic at z_0 and $p(z_0) \neq 0$. If q has a zero of order m at z_0, then $f(z) = \frac{p(z)}{q(z)}$ has a pole of order m at z_0.
Corollary: If p, q are analytic at z_0 and $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$(simple zero), then $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at z_0, and $Res_{z=z_0}f(z) = \frac{p(z_0)}{q'(z_0)}$.

6.3 Applications

Definition: Let $f : [0, \infty) \to \mathbb{R}$, the improper integral of f over $[0.\infty)$ is defined by $\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$
Definition: $\int_{-\infty}^{+\infty} f(x)dx = \lim_{R_1\to-\infty}\int_{R_1}^0 f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$
Definition: The Cauchy principal value of $\int_{-\infty}^{+\infty} f(x)dx$ is defined as $\lim_{R\to\infty}\int_{-R}^R f(x)dx$.
Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be an even function, then if Cauchy principal value converges, so is the improper integral.