1 First order ODE

1.1 Separable

Formulation: $g(y)y' = f(x)$
Solution: $\int g(y)dy = \int f(x)dx + c$

1.2 Homogeneous of degree n

Definition: $f(x, y)$ is homogeneous of degree $n \implies f(tx, ty) = t^n f(x, y)$
Formulation 1: $M(x, y) + N(x, y)y' = 0$, where M and N are homogeneous of degree n. $y' = f(x, y) = \frac{-M(x,y)}{N(x,y)}$, where $f(x, y)$ is homogeneous of degree 0.
Solution 1: Substitution $y = zx$, then $y' = z + xz'$, then $z + xz' = f(x, zx) = x^0 f(1, z) = f(1, z)$, the equation is now separable: $\frac{dz}{f(1,z)-z} = \frac{dx}{x}$
Formulation 2: $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$
Solution 2-1: If $a_1b_2 \neq a_2b_1$, consider $x = z+h, y = w + k$, where $a_1h + b_1k + c_1 = 0, a_2h + b_2k + c_2 = 0$, the equation is transformed to $\frac{dw}{dz} = \frac{a_1z+b_1w}{a_2z+b_2w}$, back to formulation 1.
Solution 2-2: If $a_1b_2 = a_2b_1$, consider $r = \frac{a_1}{b_1} = \frac{a_2}{b_2}$, take $z = rx + y$, the equation is transformed to $\frac{b_2z+c_2}{b_1z+c_1+r(b_2z+c_2)} z' = 1$, which is separable.

1.3 Exact

Formulation: $M(x, y)dx+N(x, y)dy = 0$, and there exists $u(x, y)$ such that $M(x, y)dx + N(x, y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$
Solution: $u(x, y) = c$
Theorem: Assume M, N and their first partial derivatives are continuous in the rectangle $S : |x - x_0| < a, |y - y_0| < b$. A necessary and sufficient condition for the equation to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in S$, then $u(x, y) = \int_{x_0}^{x} M(s, y)ds + \int_{y_0}^{y} N(x_0, t)dt$

1.4 Integrating factor

Definition: A non-zero function $\mu(x, y)$ is an integrating factor of the formulation above if $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$ is exact. In this case, $(\mu M)_y = (\mu N)_x \implies N\mu_x - M\mu_y = \mu(M_y - N_x)$.
One may look for an integrating factor of the form $\mu = \mu(v)$, where v is a known function of x and y, then $\mu_x = \frac{d\mu}{dv}v_x$ and $\mu_y = \frac{d\mu}{dv}v_y$, and by substitution we have $\frac{1}{\mu}\frac{d\mu}{dv} = \frac{M_y-N_x}{Nv_x-Mv_y}$, if RHS is a function of v alone, say $\phi(v)$, then $\mu = e^{\int^v \phi(v)dv}$ is an integrating factor.
Common choices of v: If $v = x$, check $\frac{M_y-N_x}{N}$ is a function of x. If $v = y$, check $-\frac{M_y-N_x}{M}$ is a function of y. If $v = xy$, check $\frac{M_y-N_x}{yN-xM}$ is a function of xy.

1.5 Homogeneous linear equations

Formulation: $y' + p(x)y = 0$
Solution: Take integrating factor $e^{P(x)}$, where $P(x) = \int_a^x p(s)ds$, then the general solution is $y(x) = ce^{-P(x)}$.

1.6 Non-homogeneous linear equations

Formulation:$y' + p(x)y = q(x)$
Solution: $y(x) = e^{-P(x)}[\int_a^x e^{P(t)}q(t)dt + c]$, where $P(x) = \int_a^x p(s)ds$

1.7 Bernoulli equation

Formulation: $y' + p(x)y = q(x)y^n$
Solution: Consider substitution $u = y^{1-n}$, the equation is transformed into $u' + (1 - n)p(x)u = (1 - n)q(x)$, which is first order linear.

1.8 Riccati equation

Formulation: $y' = P(x) + Q(x)y + R(x)y^2$
Theorem: Let $y = y_0(x)$ be a particular solution of the Riccati equation. Set $H(x) = \int_{x_0}^{x}[Q(t) + 2R(t)y_0(t)]dt$, $Z(x) = e^{-H(x)}[c - \int_{x_0}^{x} e^{H(t)}R(t)dt]$, where c is an arbitrary constant, the the general solution is given by $y = y_0 + \frac{1}{Z(x)}$
The general solution of the Riccati equation can be written as $y(x) = \frac{cF(x)+G(x)}{cf(x)+g(x)}$, where $f(x) = e^{-H(x)}, g(x) = -e^{-H(x)}\int_{x_0}^{x} e^{H(t)}R(t)dt$, $F(x) = y_0(x)f(x), G(x) = y_0g(x) + 1$
Given four distinct functions $p(x), q(x), r(x), s(x)$, we define cross-ratio $\frac{(p-q)(r-s)}{(p-s)(r-q)}$. The cross ratio of four distinct particular solutions of a Riccati equation is independent of x. As a consequence, suppose y_1, y_2, y_3 are three distinct particular solutions of a Riccati equation, then the general solution is given by $\frac{(y_1-y_2)(y_3-y)}{(y_1-y)(y_3-y_2)} = c$, where c is an arbitrary constant.
Suppose y_1, y_2 are two distinct particular solutions of a Riccati equation, then the general solution is given by $\ln|\frac{y-y_1}{y-y_2}| = \int R(x)(y_1(x) - y_2(x))dx + c$, where c is an arbitrary constant.

1.9 Method of differentiation

Formulation: $y = f(x, y')$
Solution: Let $p = y'$. Differentiating $y' = f(x, p)$ we get $[f_x(x, p) - p]dx + f_p(x, p)dp = 0$, which is a first order explicit equation in x and p. If it is solvable to give $p = \phi(x)$, then the original equation has a solution $y = f(x, \phi(x))$.

1.10 Clairaut's equation

Formulation: $y = xy' + f(y')$
Solution: Let $p = y'$. We have $y = xp + f(p)$. Differentiating we get $(x+f'(p))p' = 0$. When $p' = 0$ we have $y = cx + f(c)$. When $x + f'(p) = 0$ we have parameterized solution $x = -f'(p), y = -pf'(p)+f(p)$.

1.11 Method of parameterization

Used to solve equations where either x or y is missing. Suppose $F(y, y') = 0$, let $p = y'$, then $F(y, p) = 0$. It determines a family of curves in yp−plane. Let $y = g(t), p = h(t)$ be one of the curves $F(g(t), h(t)) = 0$, since $dx = \frac{dy}{y'} = \frac{dy}{p} = \frac{g'(t)dt}{h(t)}$, we have $x = \int_{t_0}^{t} \frac{g'(t)}{h(t)}dt + c, y = g(t)$ This method can also be applied to equations $F(x, y') = 0$ where y is missing.

1.12 Reduction of order

Formulation: $F(x, y', y'') = 0$
Solution: Let $p = y'$, then $F(x, p, p') = 0$, if $p = \phi(x, c_1)$ is a general solution of the new equation, then the general solution of the original equation is $y = \int_{x_0}^{x} \phi(t, c_1)dt + c_2$.
Formulation: $F(y, y', y'') = 0$.
Solution: Let $p = y'$, then $F(y, p, p\frac{dp}{dy}) = 0$. If $p = \psi(y, c_1)$ is a general solution of the new equation, then the general solution of the original equation is given by solving $y' = \psi(y, c_1)$.

2 Linear ODE

2.1 General formulation

Definition: n-th order linear ODE is defined as $y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_n(x)y = f(x)$. If $f(x) \not\equiv 0$, it is non-homogeneous, otherwise, it is homogeneous. The initial value problem is defined by the equation together with n conditions $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

Theorem 1 (Existence and uniqueness theorem)**.** Assume that $a_i(x)$ and $f(x)$ are continuous functions defined on interval (a, b). Then for any $x_0 \in (a, b)$ and for any numbers $y_0, \ldots, y_{n-1}$, the initial value problem has a unique solution defined on (a, b).

Theorem 2. If $f(x) \equiv 0$, and if there exists $x_0 \in (a, b)$ such that $y(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0$, then $y(x) \equiv 0$ on (a, b).

Definition: We define an operator L by $L[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$. L is a linear operator, since $L[cy] = cL[y]$ and $L[u + v] = L[u] + L[v]$.

Theorem 3. If y_1, y_2 are solutions of the homogeneous equation in an interval (a, b), then for any constant c_1, c_2, $y = c_1y_1+c_2y_2$ is also a solution in interval (a, b). If y_p is a solution of the non-homogeneous equation and y_h is a solution of the homogeneous equation, then $y = y_p + y_h$ is also a solution of the non-homogeneous equation.

Definition: Functions $\phi_1(x), \ldots, \phi_k(x)$ are linearly dependent on (a, b) if there exists constant $c_1, \ldots, c_k$ not all zero such that $c_1\phi_1(x) + \cdots + c_k\phi_k(x) \equiv 0$ on (a, b). They are linearly independent otherwise. Similar definitions apply to vector valued functions.

Theorem 4. Functions $\phi_i(x)$ are linearly dependent on $(a, b) \iff$ the following vector-valued functions $(\phi_i(x), \phi_i'(x), \ldots, \phi_i^{(n-1)}(x))$ are linearly dependent.

Definition: The Wronskian of n functions $\phi_1(x), \ldots, \phi_n(x)$ is defined by $W(\phi_1, \ldots, \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \ldots & \phi_n(x) \\ \ldots & \ldots & \ldots \\ \phi_1^{(n-1)}(x) & \ldots & \phi_n^{(n-1)}(x) \end{vmatrix}$

Theorem 5. Let $y_1(x), \ldots, y_n(x)$ be n solutions of the homogeneous equation and let $W(x)$ be their Wronskian. They are linearly dependent on $(a, b) \iff W(x) \equiv 0$ on $(a, b) \iff W(x) = 0$ for some $x \in (a, b)$. They are linearly independent $\iff W(x)$ is never zero on (a, b).

Theorem 6. The Wronskian of n solutions of the homogeneous equation is either identically zero or nowhere zero. n solutions $y_1, \ldots, y_n$ are linearly independent on $(a, b) \iff$ vectors $(y_i(x_0), \ldots, y_i^{(n-1)}(x_0))$ are linearly independent for some $x_0 \in (a.b)$.

Theorem 7 (Abel's theorem)**.** Assume y_1, y_2 are solutions to the equation $y'' + p(x)y' + q(x)y = 0$ on interval $[a, b]$, then their Wronskian satisfies $W(y_1, y_2)(x) = ce^{-\int p(x)dx}$

Theorem 8. Let $a_i(x)$ and $f(x)$ be continuous on (a, b). The homogeneous equation has n linearly independent solutions on (a, b). Let $y_1, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation. The general solution is given by $y(x) = \sum c_iy_i(x)$ where c_i are arbitrary constants. Any such linearly independent set of solutions is called a fundamental set of solutions.

Theorem 9. Let y_p be a particular solution of the non-homogeneous equation. The general solution is given by $y = y_p + \sum c_iy_i$, where the latter is the general solution of the associated homogeneous solution.

2.2 Linear equations with constant coefficients

Formulation: $y'' + ay' + by = 0$ where a, b are constants.
Solution: We look for solutions of form $e^{\lambda x}$. $e^{\lambda x}$ is a solution $\iff \lambda^2 + a\lambda + b = 0$. This is the characteristic equation. The roots are characteristic values: $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$.
If $a^2 - 4b > 0$, we have two distinct real characteristic values λ_1, λ_2, the general solution is given by $y = c_1e^{\lambda_1x} + c_2e^{\lambda_2x}$. If $a^2 - 4b = 0$, we have a repeated real characteristic value λ, the general solution is given by $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$. If $a^2 - 4b < 0$, we have two complex characteristic values $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. The general solution is given by $y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$
Formulation: $y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny = 0$, where a_i are real constants.
Solution: The characteristic equation is $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$. We first find all characteristic values. Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues and $m_1, \ldots, m_s$ the corresponding multiplicity. We have that $e^{\lambda x}$ is a solution. If $m > 1$, then for any positive integer $1 \le k \le m - 1$, $x^ke^{\lambda x}$ is a solution. If $\lambda = \alpha + i\beta$, then $x^ke^{\alpha x}\cos\beta x, x^ke^{\alpha x}\sin\beta x$ are solutions for $0 \le k \le m - 1$.

Theorem 10. Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues for the equation, with multiplicity $m_1, \ldots, m_s$ respectively. Then a fundamental set of solutions is $e^{\lambda_ix}, xe^{\lambda_ix}, \ldots, x^{m_i-1}e^{\lambda_ix}$

2.3 Non-homogeneous equation

Formulation: $y'' + P(x)y' + Q(x)y = f(x)$, the associated homogeneous equation is $y'' + P(x)y' + Q(x)y = 0$. We shall look for a particular solution.

2.3.1 Variation of parameters

Let y_1, y_2 be two linearly independent solutions of the associated homogeneous solution and $W(x)$ their Wronskian. We look for a particular solution of the non-homogeneous equation with the form $y_p = u_1y_1 + u_2y_2$.
By direct substitution and differentiation, we have $y_p'' + P(x)y_p' + Q(x)y_p = (u_1'y_1 + u_2'y_2)' + (u_1'y_1' + u_2'y_2') + P(u_1'y_1 + u_2'y_2)$. Set $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f$, we solve that $u_1' = -\frac{y_2}{W}f, u_2' = \frac{y_1}{W}f$, thus $u_1(x) = -\int_{x_0}^{x}\frac{y_2(t)}{W(t)}f(t)dt, u_2(x) = \int_{x_0}^{x}\frac{y_1(t)}{W(t)}f(t)dt$.
In addition, if z is a known solution of the homogeneous equation. We assume $y = vz$ is a solution, then we have $v = \int z^{-2}e^{-\int Pdx}dx$

2.3.2 Undetermined coefficient

Remark: Only applicable to $y''+ay'+by=f(x)$, and $f(x)=P_n(x)e^{\alpha x}$ or $f(x)=P_n(x)e^{\alpha x}\cos\beta x$ or $f(x)=P_n(x)e^{\alpha x}\sin\beta x$ where P is a polynomial of degree n.
When $f(x)=P_n(x)e^{\alpha x}$, we look for a particular solution of the form $y=Q(x)e^{\alpha x}$, where Q is a polynomial. By substitution we have $Q''+(2\alpha+a)Q'+(\alpha^2+a\alpha+b)Q=P_n(x)$. If $\alpha^2+a\alpha+b\neq 0$, we choose $Q=R_n$, a polynomial of degree n, and solve for R_n by comparing coefficients. If $\alpha^2+a\alpha+b=0$ but $2\alpha+a\neq 0$, then $Q''+(2\alpha+a)Q'=P_n$. We choose $Q=xR_n$ and solve for coefficients. If $\alpha^2+a\alpha+b=0$ and $2\alpha+a=0$, we have $Q''=P_n$, we choose $Q=x^2R_n$.
When $f(x)=P_n(x)e^{\alpha x}\cos\beta x$ or $f(x)=P_n(x)e^{\alpha x}\sin\beta x$. We first look for a solution of $y''+ay'+by=P_n(x)e^{(\alpha+i\beta)x}$. By previous case, we obtain a complex-valued solution $z(x)=u(x)+iv(x)$, and we have u is a solution of $y''+ay'+by=P_n(x)e^{\alpha x}\cos\beta x$, and v is a solution of $y''+ay'+by=P_n(x)e^{\alpha x}\sin\beta x$. Alternatively, directly try a solution of the form $Q_n(x)e^{\alpha x}\cos\beta x+R_n(x)e^{\alpha x}\sin\beta x$ when $a+i\beta$ is not a root of $\lambda^2+a\lambda+b=0$ and $xQ_n(x)e^{\alpha x}\cos\beta x+xR_n(x)e^{\alpha x}\sin\beta x$ otherwise.

Theorem 11. Let y_1, y_2 be particular solutions of the equations $y''+ay'+by=f_1, y''+ay'+by=f_2$, then y_1+y_2 is a particular solution of $y''+ay'+by=f_1+f_2$.

2.3.3 Operator method

We define a differential operator $L(D)y=\sum_{j=0}^{n}a_jD^jy$.
Formulation: $L(D)y=f(x)$.
Let $y=L(D)^{-1}f$ denote any solution, we have $DD^{-1}=D^{-1}D=D^0$ and $L(D)^{-1}L(D)=L(D)L(D)^{-1}=D^0$.

Theorem 12. More generally, we have

1. $D^{-1}f(x)=\int f(x)dx+C$
2. $(D-a)^{-1}f(x)=Ce^{ax}+e^{ax}\int e^{-ax}f(x)dx$
3. $L(D)(e^{ax}f(x))=e^{ax}L(D+a)f(x)$
4. $L(D)^{-1}(e^{ax}f(x))=e^{ax}L(D+a)^{-1}f(x)$

To find a particular solution, we can ignore arbitrary constants.
If $L(x)=\prod_{i=1}^{n}(x-r_i)$, then $y=L(D)^{-1}f=(D-r_1)^{-1}\dots(D-r_n)^{-1}f$, we could either obtain solution by successive integration, or if the roots are all distinct, consider partial fraction $\frac{1}{L(x)}=\sum_{i=1}^{n}\frac{A_i}{x-r_i}$, and thus $y=[A_1(D-r_1)^{-1}+\dots+A_n(D-r_n)^{-1}]f$
Furthermore, if f is a polynomial, then $(1-D)(1+D+D^2+\dots)f=f$ by power series, thus $(1-D)^{-1}f=(1+D+D^2+\dots)f$. We may formally expand $(D-r)^{-1}$ into power series of D and apply it to f, it is only necessary to expand up to degree of f, since further derivatives evaluate to zero.

Theorem 13. Common power series expansion:

1. $(1-D)^{-1}f=(1+D+D^2+\dots)f$
2. $(1-D)^{-2}f=(1+2D+3D^2+4D^3+\dots)f$
3. $(1-2D^2+D^3)^{-1}f=(1+2D^2-D^3+4D^4-4D^5+\dots)f$

3 Second order linear ODE

Formulation: $p_0(x)y''+p_1(x)y'+p_2(x)y=f(x)$

3.1 Exact

The equation can be written as $(p_0y'-p_0'y)'+(p_1y)'+(p_0''-p_1'+p_2)y=f(x)$. It is exact if $p_0''-p_1'+p_2=0$. If exact, integrate both sides to get $p_0(x)y'-p_0'(x)y+p_1(x)y=\int f(x)dx+C_1$.

3.2 Integrating factor

If the equation is not exact but becomes exact by multiplying a function $v(x)$, then v is an integrating factor, that is $(p_0v)''-(p_1v)'+p_2v=0$. This is a differential equation for v and more explicitly $p_0v''+(2p_0'-p_1)v'+(p_0''-p_1'+p_2)v=0$. This equation is called the adjoint of the original second order linear ODE. v is an integrating factor of the original equation $\iff$ v is a solution of the adjoint equation. In this case, integrating both sides to get $v(x)p_0(x)y'-(v(x)p_0(x))'y+v(x)p_1(x)y=\int v(x)f(x)dx+C_1$

Theorem 14 (Lagrange's identity)**.** Let $L[y]\equiv p_0(x)y''+p_1(x)y'+p_2(x)y$, the formal adjoint of L is the differential operator defined by $L^+[y]=(p_0(x)y)''-(p_1(x)y)'+p_2(x)y$, where p_0'',p_1',p_2 are continuous on an interval $[a,b]$, let u,v be continuous on $[a,b]$. we have $L[u]v-uL^+[v]=\frac{d}{dx}[P(u,v)]$, where $P(u,v)=up_1v-u(p_0v)'+u'p_0v$.

Theorem 15 (Green's formula)**.** $\int_a^b(L[u]v-uL^+[v])dx=P(u,v)(x)|_a^b$.

We define an inner product for continuous real-valued function on $[a,b]$ by $(f,g)=\int_a^b f(x)g(x)dx$. Green's formula becomes $(L[u],v)=(u,L^+[v])+P(u,v)(x)|_a^b$.
Note that if we restrict L and L^+ so that $(L[u],v)=(u,L^+[v])$. In this case, L^+ is the adjoint operator of L, and if further $L^+=L$, we say that L is self-adjoint.
For Sturm-Liouville equation, $L[y]=(p(x)y')'+q(x)y$, L is self-adjoint, then Lagrange's identity reduces to $L[u]v-uL[v]=-\frac{d}{dx}[pW(u,v)]$, and Green's formula reduces to $(L[u],v)-(u,L[v])=-pW(u,v)(x)|_a^b$.

3.3 Two-point boundary value problem

Formulation: Solve $y''+p(x)y'+q(x)y=f(x), x\in(a,b)$ with boundary conditions $a_{11}y(a)+a_{12}y'(a)+b_{11}y(b)+b_{12}y'(b)=d_1, a_{21}y(a)+a_{22}y'(a)+b_{21}y(b)+b_{22}y'(b)=d_2$. We say the boundary conditions are homogeneous if $d_1=d_2=0$.
If the equation is homogeneous and the boundary conditions are homogeneous, one could verify: If $\phi(x)$ is a non-trivial solution, so is $c\phi(x)$ for any constant c, so there is a one-parameter family of solutions. If ϕ_1,ϕ_2 are two linearly independent solutions, then $c_1\phi_1+c_2\phi_2$ is also a solution, thus there is a two-parameter family of solutions. Else, the trivial solution is the unique solution. If the equation is non-homogeneous, it is also possible that the problem has no solution.

3.4 Regular Sturm-Liouville boundary value problem

Formulation: $L[y]=(p(x)y')'+q(x)y$, $L[y]+\lambda r(x)y=0, x\in(a,b)$, $a_1y(a)+a_2y'(a)=0, b_1y(b)+b_2y'(b)=0$. p,p',q,r are continuous on $[a,b]$ and $p(x)>0, r(x)>0$ on $[a,b]$, and a_1,a_2 are not both zero, b_1,b_2 are not both zero.
Let u,v be functions with continuous second derivatives on $[a,b]$ and satisfy the boundary conditions. It implies that $W(u,v)(b)=W(u,v)(a)=0$. By Green's formula, $(L[u],v)=(u,L[v])$.
The objective is to determine for which values of λ, the equation has non-trivial solutions satisfying the given boundary conditions. The non-trivial solutions are called eigenfunctions, and the corresponding λ an eigenvalue. If all eigenfunctions associated with a particular eigenvalue are just scalar multiples of each other, then the eigenvalue is simple.

Theorem 16. All eigenvalues of the regular Sturm-Liouville boundary value problem are real, simple with real-valued eigenfunctions.

Definition: Two real-valued function f,g defined on $[a,b]$ are orthogonal with respect to a positive weight function r if $\int_a^b f(x)g(x)r(x)dx=0$.

Theorem 17. Eigenfunctions that correspond to distinct eigenvalues of the regular Sturm-Liouville boundary value problem are orthogonal with repsect to r on $[a,b]$.

Theorem 18. The eigenvalues of the regular Sturm-Liouville boundary value problem form a countable and increasing sequence with $\lim_{n\to+\infty}\lambda_n=+\infty$

For $\lambda\le 0$, the problem has only the trivial solution. When $\lambda>0$, the general solution of the equation is $y=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$, the boundary conditions imply that $\lambda=n^2$ with corresponding eigenfunctions $\phi_n(x)=B_n\sin nx$. We can use these functions to form an orthonormal system. For any piecewise continuous function f on $[a,b]$, we can form an orthogonal expansion.

Theorem 19. Let $\{\phi_n\}$ be an orthonormal system of eigenfunctions for the regular Sturm-Liouville boundary value problem. Let f be a continuous function on $[a,b]$ such that f' is piecewise continuous on $[a,b]$ and f satisfies the boundary conditions, then $f(x)=\sum_{n=1}^{+\infty}c_n\phi_n(x), x\in[a,b]$, where $c_n=\int_a^b f(x)\phi_n(x)r(x)dx$. The series converges uniformly on $[a,b]$.

3.5 Non-homogeneous regular Sturm-Liouville bondary value problem

Formulation: $L[y]=f(x)$, where f is continuous on $[a,b]$. Same boundary conditions. We let $L[y]=0$ be the associated homogeneous problem.

Theorem 20. The non-homogeneous problem has a unique solution $\iff$ the associated homogeneous problem has only the trivial solution.

If the associated homogeneous problem has only the trivial solution, we construct a solution of the non-homogeneous solution. Let y_1,y_2 be nontrivial solutions to the equation $L[y]=0$ satisfying only the first and the second boundary condition respectively. By variation of parameters, we construct a particular solution for $L[y]=f \iff y''+\frac{p'}{p}y+\frac{q}{p}y=\frac{f}{p}$, where $y=u_1y_1+u_2y_2$ and $u_1=-\int_b^x\frac{y_2(t)f(t)}{W(t)p(t)}dt, u_2=\int_a^x\frac{y_1(t)f(t)}{W(t)p(t)}dt$. We write $y=\int_a^b G(x,t)f(t)$, where $G(x,t)=\frac{y_1(t)y_2(x)}{W(t)p(t)}, a\le t\le x, \frac{y_1(x)y_2(t)}{W(t)p(t)}, x\le t\le b$, since y_1,y_2 satisfy $L[y]=0$, by Lagrange's identity, $W(x)p(x)=C$, a constant. G is called the Green's function for the non-homogeneous problem. y is a solution to the non-homogeneous problem.

Theorem 21 (Fredholm alternative)**.** If the homogeneous problem has non-trivial solutions, then the non-homogeneous problem has a solution $\iff$ $\int_a^b f(t)y(t)dt=0$ for all non-trivial solutions y of the homogeneous problem.

Theorem 22. Any equation $p_0(x)y''+p_1(x)y'+p_2(x)y=0$ can be made self-adjoint by multiplying by $\frac{1}{p_0}e^{\int\frac{p_1}{p_0}dx}$

Definition: If a function has an infinite number of zeros in an interval $[a,\infty)$, we say that the function is oscillatory.

Theorem 23 (Sturm separation theorem)**.** If y_1,y_2 are two linearly independent solutions of $y''+P(x)y'+Q(x)y=0$, then the zeros of these functions are distinct and occur alternatively in the sense that y_1 vanishes exactly once between any two successive zeros of y_2, and vice versa.

Theorem 24. Suppose one nontrivial solution to the equation above is oscillatory on $[a,\infty)$, then all solutions are oscillatory.

Theorem 25. Let y be a non-trivial solution of the equation above on a closed interval $[a,b]$, then y has at most a finite number of zeros in this interval.

$y''+P(x)y'+Q(x)y=0$ can be written as $u''+q(x)u=0$ by $y=uv$, where $v=e^{-\frac{1}{2}\int Pdx}$ and $q(x)=Q(x)-\frac{1}{4}P(x)^2-\frac{1}{2}P'(x)$. We call the first standard form and the second normal form of a homogeneous second order linear ODE. Since $v(x)>0$, the transformation has no effect on distribution of zeros and leaves unaltered the oscillation behavior.

Theorem 26 (Sturm commparison theorem)**.** Let y_1 be a non-trivial solution to $y''+q_1(x)y=0$ and y_2 a non-trivial solution to $y''+q_2(x)y=0, x\in(a,b)$. Assume $q_2(x)\ge q_1(x)$ on (a,b). If x_1,x_2 are two consecutive zeros of y_1, then there exists a zero of y_2 in (x_1,x_2), unless $q_2=q_1$, in which case y_1,y_2 are linearly dependent.

Theorem 27. Suppose $q(x)<0$ on $[a,b]$, if y is a nontrivial solution of $y''+q(x)y=0$, then y has at most one zero.