1 First order ODE

1.1 Homogeneous of degree n

Definition: $f(x, y)$ is homogeneous of degree $n \implies f(tx, ty) = t^n f(x, y)$

Formulation 1: $M(x, y) + N(x, y)y' = 0$, where M and N are homogeneous of degree n. $y' = f(x, y) = \frac{-M(x,y)}{N(x,y)}$, where $f(x, y)$ is homogeneous of degree 0.

Solution 1: Substitution $y = zx$, then $y' = z + xz'$, then $z + xz' = f(x, zx) = x^0 f(1, z) = f(1, z)$, the equation is now separable: $\frac{dz}{f(1,z)-z} = \frac{dx}{x}$

Formulation 2: $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Solution 2-1: If $a_1b_2 \neq a_2b_1$, consider $x = z+h, y = w + k$, where $a_1h + b_1k + c_1 = 0, a_2h + b_2k + c_2 = 0$, the equation is transformed to $\frac{dw}{dz} = \frac{a_1z+b_1w}{a_2z+b_2w}$, back to formulation 1.

Solution 2-2: If $a_1b_2 = a_2b_1$, consider $r = \frac{a_1}{b_1} = \frac{a_2}{b_2}$, take $z = rx + y$, the equation is transformed to $\frac{b_2z+c_2}{b_1z+c_1+r(b_2z+c_2)} z' = 1$, which is separable.

1.2 Exact

Formulation: $M(x, y)dx+N(x, y)dy = 0$, and there exists $u(x, y)$ such that $M(x, y)dx + N(x, y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

Solution: $u(x, y) = c$

Theorem: Assume M, N and their first partial derivatives are continuous in the rectangle $S : |x - x_0| < a, |y - y_0| < b$. A necessary and sufficient condition for the equation to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in S$.

1.3 Integrating factor

Definition: A non-zero function $\mu(x, y)$ is an integrating factor of the formulation above if $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$ is exact.

One may look for an integrating factor of the form $\mu = \mu(v)$, where v is a known function of x and y, then we have $\frac{1}{\mu}\frac{d\mu}{dv} = \frac{M_y-N_x}{Nv_x-Mv_y}$, if RHS is a function of v alone, say $\phi(v)$, then $\mu = e^{\int^v \phi(v)dv}$ is an integrating factor.

Common choices of v: If $v = x$, check $\frac{M_y-N_x}{N}$ is a function of x. If $v = y$, check $-\frac{M_y-N_x}{M}$ is a function of y. If $v = xy$, check $\frac{M_y-N_x}{yN-xM}$ is a function of xy.

1.4 Homogeneous linear equations

Formulation: $y' + p(x)y = 0$

Solution: Take integrating factor $e^{P(x)}$, where $P(x) = \int_a^x p(s)ds$, then the general solution is $y(x) = ce^{-P(x)}$.

1.5 Non-homogeneous linear equations

Formulation:$y' + p(x)y = q(x)$

Solution: $y(x) = e^{-P(x)}[\int_a^x e^{P(t)}q(t)dt + c]$, where $P(x) = \int_a^x p(s)ds$

1.6 Bernoulli equation

Formulation: $y' + p(x)y = q(x)y^n$

Solution: Consider substitution $u = y^{1-n}$, the equation is transformed into $u' + (1 - n)p(x)u = (1 - n)q(x)$, which is first order linear.

1.7 Riccati equation

Formulation: $y' = P(x) + Q(x)y + R(x)y^2$

Theorem: Let $y = y_0(x)$ be a particular solution of the Riccati equation. Set $H(x) = \int_{x_0}^x [Q(t) + 2R(t)y_0(t)]dt$, $Z(x) = e^{-H(x)}[c - \int_{x_0}^x e^{H(t)}R(t)dt]$, where c is an arbitrary constant, the the general solution is given by $y = y_0 + \frac{1}{Z(x)}$

Given four distinct functions $p(x), q(x), r(x), s(x)$, we define cross-ratio $\frac{(p-q)(r-s)}{(p-s)(r-q)}$. Suppose y_1, y_2, y_3 are three distinct particular solutions of a Riccati equation, then the general solution is given by $\frac{(y_1-y_2)(y_3-y)}{(y_1-y)(y_3-y_2)} = c$, where c is an arbitrary constant.

Suppose y_1, y_2 are two distinct particular solutions of a Riccati equation, then the general solution is given by $\ln|\frac{y-y_1}{y-y_2}| = \int R(x)(y_1(x) - y_2(x))dx + c$, where c is an arbitrary constant.

1.8 Euler-Cauchy equation

Formulation: $x^2y''+Pxy'+Qy = 0$, where P, Q are constants. Let r_1, r_2 be roots of $r(r-1)+Pr+Q = 0$. $y_1 = x^{r_1}, y_2 = x^{r_2}$ are solutions. If $r_1 = r_2$, then $y_2 = x^{r_1} \ln x$

2 Linear ODE

2.1 General formulation

Theorem:[Existence and uniqueness theorem] Assume that $a_i(x)$ and $f(x)$ are continuous functions defined on interval (a, b). Then for any $x_0 \in (a, b)$ and for any numbers $y_0, \dots, y_{n-1}$, the initial value problem has a unique solution defined on (a, b).

Definition: The Wronskian of n functions $\phi_1(x), \dots, \phi_n(x)$ is defined by $W(\phi_1, \dots, \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \dots & \phi_n(x) \\ \dots & \dots & \dots \\ \phi_1^{(n-1)}(x) & \dots & \phi_n^{(n-1)}(x) \end{vmatrix}$

Theorem: Let $y_1(x), \dots, y_n(x)$ be n solutions of the homogeneous equation and let $W(x)$ be their Wronskian. They are linearly dependent on $(a, b) \iff W(x) \equiv 0$ on $(a, b) \iff W(x) = 0$ for some $x \in (a, b)$. They are linearly independent $\iff W(x)$ is never zero on (a, b).

Theorem: The Wronskian of n solutions of the homogeneous equation is either identically zero or nowhere zero. n solutions $y_1, \dots, y_n$ are linearly independent on $(a, b) \iff$ vectors $(y_i(x_0), \dots, y_i^{(n-1)}(x_0))$ are linearly independent for some $x_0 \in (a.b)$.

Theorem:[Abel's theorem] Assume y_1, y_2 are solutions to the equation $y'' + p(x)y' + q(x)y = 0$ on interval $[a, b]$, then their Wronskian satisfies $W(y_1, y_2)(x) = ce^{-\int p(x)dx}$

2.2 Linear equations with constant coefficients

Formulation: $y'' + ay' + by = 0$ where a, b are constants.

Solution: We look for solutions of form $e^{\lambda x}$. $e^{\lambda x}$ is a solution $\iff \lambda^2 + a\lambda + b = 0$. This is the characteristic equation. The roots are characteristic values: $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$.

If $a^2 - 4b > 0$, we have two distinct real characteristic values λ_1, λ_2, the general solution is given by $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$. If $a^2 - 4b = 0$, we have a repeated real characteristic value λ, the general solution is given by $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$. If $a^2 - 4b < 0$, we have two complex characteristic values $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. The general solution is given by $y = c_1e^{\alpha x} \cos \beta x + c_2e^{\alpha x} \sin \beta x$

Formulation: $y^{(n)}+a_1y^{(n-1)}+\dots+a_{n-1}y'+a_ny = 0$, where a_i are real constants.

Solution: The characteristic equation is $\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$. We first find all characteristic values. Let $\lambda_1, \dots, \lambda_s$ be the distinct eigenvalues and $m_1, \dots, m_s$ the corresponding multiplicity. We have that $e^{\lambda x}$ is a solution. If $m > 1$, then for any positive integer $1 \leq k \leq m - 1$, $x^ke^{\lambda x}$ is a solution. If $\lambda = \alpha + i\beta$, then $x^ke^{\alpha x} \cos \beta x, x^ke^{\alpha x} \sin \beta x$ are solutions for $0 \leq k \leq m - 1$.

Theorem: Let $\lambda_1, \dots, \lambda_s$ be the distinct eigenvalues for the equation, with multiplicity $m_1, \dots, m_s$ respectively. Then a fundamental set of solutions is $e^{\lambda_i x}, xe^{\lambda_i x}, \dots, x^{m_i-1}e^{\lambda_i x}$

2.3 Non-homogeneous equation

Formulation: $y'' + P(x)y' + Q(x)y = f(x)$

2.4 Variation of parameters

Let y_1, y_2 be two linearly independent solutions of the associated homogeneous solution and $W(x)$ their Wronskian. We look for a particular solution with the form $y_p = u_1y_1 + u_2y_2$.

We have $u_1(x) = -\int_{x_0}^x \frac{y_2(t)}{W(t)} f(t)dt, u_2(x) = \int_{x_0}^x \frac{y_1(t)}{W(t)} f(t)dt$.

In addition, if z is a known solution of the homogeneous equation. We assume $y = vz$ is another solution, then we have $v = \int z^{-2}e^{-\int P dx}dx$

2.5 Undetermined coefficient

Remark: Only applicable to $y'' + ay' + by = f(x)$, and $f(x) = P_n(x)e^{\alpha x}$ or $f(x) = P_n(x)e^{\alpha x} \cos \beta x$ or $f(x) = P_n(x)e^{\alpha x} \sin \beta x$ where P is a polynomial of degree n.

When $f(x) = P_n(x)e^{\alpha x}$, we look for a particular solution of the form $y = Q(x)e^{\alpha x}$, where Q is a polynomial. By substitution we have $Q'' + (2\alpha + a)Q' + (\alpha^2 + a\alpha + b)Q = P_n(x)$. If $\alpha^2 + a\alpha + b \neq 0$, we choose $Q = R_n$, a polynomial of degree n, and solve for R_n by comparing coefficients. If $\alpha^2 + a\alpha + b = 0$ but $2\alpha + a \neq 0$, then $Q'' + (2\alpha + a)Q' = P_n$. We choose $Q = xR_n$ and solve for coefficients. If $\alpha^2 + a\alpha + b = 0$ and $2\alpha + a = 0$, we have $Q'' = P_n$, we choose $Q = x^2R_n$. When $f(x) = P_n(x)e^{\alpha x} \cos \beta x$ or $f(x) = P_n(x)e^{\alpha x} \sin \beta x$. We first look for a solution of $y'' + ay' + by = P_n(x)e^{(\alpha+i\beta)x}$. By previous case, we obtain a complex-valued solution $z(x) = u(x) + iv(x)$, and we have u is a solution of $y'' + ay' + by = P_n(x)e^{\alpha x} \cos \beta x$, and v is a solution of $y'' + ay' + by = P_n(x)e^{\alpha x} \sin \beta x$.

2.6 Operator method

We define a differential operator $L(D)y = \sum_{j=0}^n a_jD^jy$. **Formulation**: $L(D)y = f(x)$.

Theorem: More generally, we have $(1)D^{-1}f(x) = \int f(x)dx + C$; $(2)(D - a)^{-1}f(x) = Ce^{ax} + e^{ax} \int e^{-ax}f(x)dx$; $(3)L(D)(e^{ax}f(x)) = e^{ax}L(D + a)f(x)$; $(4)L(D)^{-1}(e^{ax}f(x)) = e^{ax}L(D + a)^{-1}f(x)$

To find a particular solution, we can ignore arbitrary constants.

If $L(x) = \prod_{i=1}^n (x - r_i)$, then $y = L(D)^{-1}f = (D-r_1)^{-1} \dots (D-r_n)^{-1}f$, we could either obtain solution by successive integration, or if the roots are all distinct, consider partial fraction $\frac{1}{L(x)} = \sum_{i=1}^n \frac{A_i}{x-r_i}$, and thus $y = [A_1(D - r_1)^{-1} + \dots + A_n(D - r_n)^{-1}]f$

Furthermore, if f is a polynomial, then $(1 - D)(1 + D + D^2 + \dots)f = f$ by power series, thus $(1 - D)^{-1}f = (1 + D + D^2 + \dots)f$. We may formally expand $(D - r)^{-1}$ into power series of D and apply it to f, it is only necessary to expand up to degree of f, since further derivatives evaluate to zero.

Theorem: Common power series expansion: $(1)(1-D)^{-1}f = (1 + D + D^2 + \dots)f$; $(2)(1 - D)^{-2}f = (1 + 2D + 3D^2 + 4D^3 + \dots)f$

3 Second order linear ODE

Formulation: $p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x)$

3.1 Exact

The equation can be written as $(p_0y' - p_0'y)' + (p_1y)' + (p_0'' - p_1' + p_2)y = f(x)$. It is exact if $p_0'' - p_1' + p_2 = 0$. If exact, integrate both sides to get $p_0(x)y' - p_0'(x)y + p_1(x)y = \int f(x)dx + C_1$.

3.2 Two-point boundary value problem

Formulation: Solve $y'' + p(x)y' + q(x)y = f(x), x \in (a, b)$ with boundary conditions $a_{11}y(a) + a_{12}y'(a) + b_{11}y(b)+b_{12}y'(b) = d_1, a_{21}y(a)+a_{22}y'(a)+b_{21}y(b)+ b_{22}y'(b) = d_2$.

3.3 Regular Sturm-Liouville boundary value problem

Formulation: $L[y] = (p(x)y')' + q(x)y$, $L[y] + \lambda r(x)y = 0, x \in (a, b)$, $a_1y(a) + a_2y'(a) = 0, b_1y(b) + b_2y'(b) = 0$. p, p', q, r are continuous on $[a, b]$ and $p(x) > 0, r(x) > 0$ on $[a, b]$, and a_1, a_2 are not both zero, b_1, b_2 are not both zero.

3.4 Non-homogeneous regular Sturm-Liouville bondary value problem

Formulation: $L[y] = f(x)$, where f is continuous on $[a, b]$. Same boundary conditions. We let $L[y] = 0$ be the associated homogeneous problem.

If the associated homogeneous problem has only the trivial solution, we construct a solution of the non-homogeneous solution. Let y_1, y_2 be nontrivial solutions to the equation $L[y] = 0$ satisfying only the first and the second boundary condition respectively. We write $y = \int_a^b G(x, t)f(t)$, where $G(x, t) = \frac{y_1(t)y_2(x)}{W(t)p(t)}, a \leq t \leq x$, $\frac{y_1(x)y_2(t)}{W(t)p(t)}, x \leq t \leq b$.

Definition: If a function has an infinite number of zeros in an interval $[a, \infty)$, we say that the function is oscillatory.

Theorem:[Sturm separation theorem] If y_1, y_2 are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then the zeros of these functions are distinct and occur alternatively in the sense that y_1 vanishes exactly once between any two successive zeros of y_2, and vice versa.

Theorem: Suppose one nontrivial solution to the equation above is oscillatory on $[a, \infty)$, then all solutions are oscillatory.

Theorem: Let y be a non-trivial solution of the equation above on a closed interval $[a, b]$, then y has at most a finite number of zeros in this interval.

Theorem:[Sturm comparison theorem] Let y_1 be a non-trivial solution to $y'' + q_1(x)y = 0$ and y_2 a non-trivial solution to $y''+q_2(x)y = 0, x \in (a, b)$. Assume $q_2(x) \geq q_1(x)$ on (a, b). If x_1, x_2 are two consecutive zeros of y_1, then there exists a zero of y_2 in (x_1, x_2), unless $q_2 = q_1$, in which case y_1, y_2 are linearly dependent.

Theorem: Suppose $q(x) < 0$ on $[a, b]$, if y is a non-trivial solution of $y'' + q(x)y = 0$, then y has at most one zero.

4 Linear system

4.1 General homogeneous and non-homogeneous system

Formulation: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$. Homogeneous if $\mathbf{g} = \mathbf{0}$. Together with an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we form an IVP. EU theorem holds for the system IVP as well.

Theorem: A set of solutions $\mathbf{x}_i(t)$, $i = 1, \ldots, r$ of the system are linearly dependent on (a, b) $\iff$ they are linearly dependent for any fixed $t_0 \in (a, b)$.

Definition: The Wronskian of n vector-valued functions $\mathbf{x}_i(t) = (x_{1i}(t) \cdots x_{ni}(t))$ is the determinant $W(\mathbf{x}_1, \ldots, \mathbf{x}_n)(t) = \det W$, where $W_{ij}(t) = x_{ij}(t)$

Theorem: The Wronskian of n solutions of the system is either identically zero or nowhere zero in (a, b). n solutions are linearly dependent in (a, b) $\iff$ Wronskian is identically zero in (a, b).

Theorem: A set of n linearly independent solutions is called a fundamental set/basis of solutions. The matrix-valued function $\phi(t) = (\mathbf{x}_1(t) \cdots \mathbf{x}_n(t))$ is called a fundamental matrix. The general solution is given by $\mathbf{x}(t) = \phi(t)\mathbf{x}$ where $\mathbf{c}$ is an arbitrary constant vector.

Variation of parameter: Let ϕ be a fundamental matrix of the associated homogeneous system. We look for a particular solution of the non-homogeneous system in the form $\mathbf{x} = \phi\mathbf{u}$, we have $\phi\mathbf{u}' = \mathbf{g} \to \mathbf{u}' = \phi^{-1}\mathbf{g} \to \mathbf{u} = \int_{t_0}^{t} \phi^{-1}(s)\mathbf{g}(s)ds$. The general solution of non-homogeneous system is given by $\mathbf{x}(t) = \phi(t)\mathbf{c} + \phi(t)\int_{t_0}^{t} \phi^{-1}(s)\mathbf{g}(s)ds$

4.2 Homogeneous system with constant coefficients

We recall the concepts of eigenvalues and eigenvectors. We define quasi-simple if geometric multiplicity is equal to algebraic multiplicity. We define simple if they are 1.

Theorem: If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{k}$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{k}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If λ is complex, then the real and imaginary parts of $e^{\lambda t}\mathbf{k}$ are two linearly independent solutions.

Theorem: If $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{k}_i$ associated with eigenvalues λ_i, then $\phi(t) = (e^{\lambda_1 t}\mathbf{k}_1, \ldots, e^{\lambda_n t}\mathbf{k}_n)$ is a fundamental matrix.

Theorem: Assume λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$, then the system $(\mathbf{A} - \lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}$ has exactly m linearly independent solutions.

Theorem: Assume λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$. Let $\mathbf{v}_0 \neq \mathbf{0}$ be a solution of the system above, define $\mathbf{v}_l = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{l-1}, 1 \leq l \leq m-1$, and let $\mathbf{x}(t) = e^{\lambda t}[\mathbf{v}_0 + t\mathbf{v}_1 + \cdots + \frac{t^{m-1}}{(m-1)!}\mathbf{v}_{m-1}]$, then $\mathbf{x}$ is a solution of the original homogeneous system.

Alternative algorithm to reduce the number of constant vectors: Consider an eigenvalue λ of $\mathbf{A}$ with algebraic multiplicity m. Start with $r = m$. Let $\mathbf{v}_0$ be a vector such that $(\mathbf{A} - \lambda\mathbf{I})^r\mathbf{v}_0 = \mathbf{0}$ while $(\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v}_0 \neq \mathbf{0}$. $\mathbf{v}_0$ is called a generalized eigenvector of rank r associated with λ. If no such $\mathbf{v}_0$ exists, reduce r by 1. We have $\mathbf{v}_i = (\mathbf{A} - \lambda\mathbf{I})^i\mathbf{v}_0, 0 \leq i \leq r-1$ form a chain of linearly independent solutions of the system in theorem 34 with $\mathbf{v}_{r-1}$ being the base eigenvector associated with λ. This gives r independent solutions of the original system. If $r < m$, repeat the algorithm by finding another choice of $\mathbf{v}_0$ which is not in the previous chain.

4.3 Autonomous system

Consider a system $x'(t) = f(x, y), y'(t) = g(x, y)$, we have $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ by chain rule. We sketch the phase plane by considering the behavior from $t = -\infty$ to $t = \infty$. We summarize the cases divided by eigenvalues: **distinct and positive**, improper node, unstable; **distinct and negative**, improper node, asymptotically stable; **opposite sign**, saddle, unstable; **equal and positive**, proper or improper node, unstable; **equal and negative**, proper or improper node, asymptotically stable; **complex with positive real part**, spiral, unstable; **complex with negative real part**, spiral, asymptotically stable; **purely imaginary**, center, stable

5 Power series

Definition: Consider homogeneous second order linear ODE $y'' + P(x)y' + Q(x)y = 0$. x_0 is an ordinary point if P, Q are analytic at x_0. If P or Q is not analytic at x_0, then x_0 is a singular point. A singular point at which the functions $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic is called a regular singular point, otherwise an irregular singular point. For the point at infinity, substitute $x = \frac{1}{t}$ and study the behavior of t approaching 0.

Theorem: Let x_0 be an ordinary point and let a_0, a_1 be arbitrary constants. There exists a unique solution y that is analytic at x_0, and satisfies the initial conditions $y(x_0) = a_0, y'(x_0) = a_1$. Furthermore, if the power series expansions of P, Q are valid on an interval $|x - x_0| < R$, then the power series expansion of the solution is also valid on this interval.

5.1 Legendre's equation

Formulation: $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$, where p is a constant called the order of Legendre's equation. $x = 0$ is an ordinary point.

The recursion formula for series coefficients is given by $a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}a_n, n \geq 0$. The entire series is defined by choice of a_0 and a_1. We could take one to be 0 and another to be 1 to obtain two linearly independent solutions. When $p = n$ is a non-negative integer, one of the series could terminate and become a polynomial of degree n in x. The coefficients in the series solution are called Legendre functions. We could choose the arbitrary constants a_0 or a_1 so that the coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n(n!)^2}$ so that $P_n(1) = 1$, then $P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$. $P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$. Rodrigue's formula given by $P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2 - 1)^n$ gives a particular solution of Legendre's equation of order n.

5.2 Hermite's equation

Formulation: $y'' - 2xy' + 2py = 0$ The general solution is given by $y = a_0y_1 + a_1y_2$, where $y_1(x) = 1 - \frac{2p}{2!}x^2 + \frac{2^2p(p-2)}{4!}x^4 - \frac{2^3p(p-2)(p-4)}{6!}x^6 \ldots, y_2(x) = x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 \ldots$. The Hermite polynomial of degree n denoted by $H_n(x)$ is the polynomial of degree n that is a solution, multiplied by suitable constant such that the coefficient of x^n is 2^n. $H_0 = 1, H_1 = 2x, H_2 = 4x_2 - 2, H_3 = 8x^3 - 12x, H_5 = 32x^5 - 160x^3 + 120x. H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$

5.3 Method of Frobenius

If $x = 0$ is a regular singular point of $y'' + P(x)y' + Q(x)y = 0$, then $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$. We let $p(x) = xP(x)$ and $q(x) = x^2Q(x)$ and write the equation as $x^2y'' + xp(x)y' + q(x)y = 0$. We now that p, q has Taylor series expansion $p(x) = \sum_{n=0}^{\infty} p_nx^n, q(x) = \sum_{n=0}^{\infty} q_nx^n$. Suppose there exists a series solution of the form $y = x^r\sum_{n=0}^{+\infty} a_nx^n = \sum_{n=0}^{+\infty} a_nx^{n+r}$, substitute y, y', y'' into the equation, then LHS is polynomial, thus all coefficients must vanish. The coefficient of x^r is $r(r-1)a_0 + p_0ra_0 + q_0a_0 = 0$. As $a_0 \neq 0$, r satisfies the equation $r(r-1) + p_0r + q_0 = 0$. This is the indicial equation of the DE and the two roots are the exponents of the DE at regular singular point $x = 0$. If $r_1 \neq r_2$, then we have two possible linearly independent Frobenius solutions. If $r_1 = r_2$, there is only one Frobenius solution. The second one cannot be a Frobenius series and must be found by other means. If r_1, r_2 are complex conjugates, we always get two linearly independent solutions. If $x < 0$, we substitute $x = -t$ and study for t. We now consider r_1, r_2 are real and $x > 0$.

After substitution of y, y', y'', we get the recurrence relation $a_n[(r+n)(r+n-1) + (r+n)p_0 + q_0] + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0$. When $n = 0$, the summation term vanished and we recover the indicial equation. a_0 is an arbitrary constant. a_n can be recursively determined as long as $(r+n)(r+n-1) + (r+n)p_0 + q_0 \neq 0$. This is the case if r_1, r_2 do not differ by an integer. Otherwise, suppose $r_1 > r_2$, only the Forbenius series solution with exponent r_1 is guaranteed, the other one may not be a Frobenius series or fail to exist.

5.4 Bessel's equation

Formulation: $x^2y'' + xy' + (x^2 - p^2)y = 0$

The general solution is $y = c_1J_p(x) + c_2Y_p(x)$, where J_p is the Bessel function of order p of first kind and Y_p is of second kind. $x = 0$ is a regular singular point. The exponents are $\pm p$. We consider a series solution $y = \sum_{m=0}^{+\infty} a_mx^{m+r}$ and by substitution and check that a_0 is arbitrary, $a_1 = 0$ and $[(m+r)^2 - p^2]a_m + a_{m-2} = 0$ for $m \geq 2$.

When $r = p > 0$, we have $a_m = -\frac{a_{m-2}}{m(2p+m)}$, since $a_1 = 0$, then $a_m = 0$ if m is odd. $a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!(p+1)(p+2)\ldots(p+m)}$.

When $r = -p < 0$, there is a Frobenius series solution if p is not an integer., we have $m(m-2p)a_m + a_{m-2}$ for $m \geq 2$. The result is the same except we replace p by $-p$.

6 EUT

Definition: Let G be a subset of $\mathbb{R}^2$. $f(t, x) : G \to \mathbb{R}$ is said to satisfy a Lipschitz condition with respect to x in G if there exists a constant $L > 0$ such that for any $(t, x_1), (t, x_2) \in G$, we have $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$. L is called a Lipschitz constant.

Theorem: Let $f(t, x)$ be continuous on the rectangle $R : |t - t_0| \leq a, |x - x_0| \leq b$ and let $|f(t, x)| \leq M$ for all $(t, x) \in R$. Furthermore, f satisfies a Lipschitz condition with constant L in R, then there is a unique solution to IVP $\frac{dx}{dt} = f(t, x), x(t_0) = x_0$ on the interval $I = [t_0 - \alpha, t_0 + \alpha]$, where $\alpha = \min(a, \frac{b}{M})$.

Theorem: Suppose $f(t, x)$ has a continuous partial derivative f_x on a closed rectangle R in the tx-plane, then f satisfies a Lipschitz condition on R.

Definition: Let $\{f_n\}$ be a sequence of functions on $[a, b]$. It is said to converge uniformly to f if for every $\epsilon > 0$, there exists a positive integer N such that $|f_n - f| < \epsilon$ for all $n > N$.

Definition: The sequence is said to converge to f pointwise if for each $x \in [a, b]$, $\lim_{n\to\infty} f_n(x) = f(x)$. If $\{f_n\}$ converges uniformly to f and each f_n is continuous, then f is continuous, and $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$

Uniform convergence of series means uniform convergence of sequence of partial sum.

Theorem:[Weisertrass M-test] Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on $[a, b]$. Let $\{M_n\}$ be a sequence of non-negative numbers such that $0 \leq |f_n(x)| \leq M_n$ for all $x \in [a, b]$ and for all n. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Theorem: A function ϕ is a solution of the IVP $\frac{dx}{dt} = f(t, x), x(t_0) = x_0$ on an interval I $\iff$ it is a solution of the integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$

To find the solution, we consider iterative approximations $\phi_0(t) = x_0, \phi_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_k(s))ds$, we might expect ϕ_k converges to ϕ.

Theorem: Suppose $|f(t, x)| \leq M$ for all $(t, x) \in R$, then the successive approximations ϕ_k exist as continuous functions on $I : |t - t_0| \leq \alpha = \min(a, \frac{b}{M})$ and $(t, \phi_k(t))$ is in R for $t \in I$ and satisfy $|\phi_k(t) - x_0| \leq M|t - t_0|$ for all $t \in I$.

Theorem: Let $f(t, x)$ be a continuous function on the strip $S = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \leq a\}$, where $a > 0$, and f satisfies the Lipschitz condition with respect to S, then IVP $\frac{dx}{dt} = f(t, x), x(t_0) = x_0$ where $(t_0, x_0) \in S$ has a unique solution on the entire interval $[t_0 - a, t_0 + a]$.

Theorem: Let $f(t, x)$ be a continuous function defined on $\mathbb{R}^2$. Let $(t_0, x_0) \in \mathbb{R}^2$. Suppose that for any $a > 0$, f satisfies the Lipschitz condition with respect to $S = \{(t, x) \in \mathbb{R}^2 : |t| \leq a\}$, then IVP has a unique solution on entire $\mathbb{R}$.

Theorem: Let f, g, h be continuous nonnegative functions defined for $t \geq t_0$, if $f(t) \leq h(t) + \int_{t_0}^{t} g(s)f(s)ds, t \geq t_0$, then $f(t) \leq h(t) + \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)du}ds, t \geq t_0$

Theorem:[Gronwall's inequality] Let f, g be continuous nonnegative functions for $t \geq t_0$, let k be any nonnegative constant, if $f(t) \leq k + \int_{t_0}^{t} g(s)f(s)ds$ for $t \geq t_0$, then $f(t) \leq ke^{\int_{t_0}^{t} g(s)ds}$, for $t \geq t_0$.

Theorem: Let f be a continuous nonnegative function for $t \geq t_0$ and $k \geq 0$, if $f(t) \leq k\int_{t_0}^{t} f(s)ds$ for all $t \geq t_0$, then $f(t) \equiv 0$ for $t \geq t_0$.

Theorem: Let $f(t, x)$ be a continuous function which satisfies a Lipshitz condition on R with a constant L, where R is either a rectangle or a strip. If ϕ and ψ are two solutions of IVP, on an interval I containing t_0, then $\phi(t) = \psi(t)$ for all $t \in I$.

Theorem:[Peano] Assume G is an open set of $\mathbb{R}^2$ containing (t_0, x_0) and $f(t, x)$ is continuous on G, then there exists $a > 0$ such that IVP has at least one solution on the interval $[t_0 - a, t_0 + a]$.