EIETL LAB, INGLIS BUILDING EXPERIMENT 3F2-A

PENDULUM CONTROLLER EXPERIMENT

Objectives

This experiment uses the control of a crane/inverted pendulum to illustrate

- state-space dynamic models
- state-feedback and pole-placement
- limit cycles and describing functions.

The experiment is primarily a demonstration but some questions are asked at various times in the text, which should be answered in your report, together with a description of your results and necessary graphs, calculations and discussion.

Appendix A contains relevant theory. It should be read and understood before writing your report — ideally before doing the experiment!

************ WARNING ************

THIS APPARATUS CONTAINS MECHANISMS OF REASONABLE MASS, MOVING AT POTENTIALLY VERY HIGH SPEED AND SOMETIMES QUITE UNPREDICTABLY. KEEP WELL CLEAR OF ALL MOVING PARTS AND IF IT IS NECESSARY TO TOUCH THE PENDULUM DO SO WITH DUE CARE. DO NOT MAKE ANY ADJUSTMENTS TO THE APPARATUS, INCLUDING POWERING UP, WHILE ANOTHER PERSON IS CLOSE TO THE PENDULUM.

****

Note that the experimental results should be collected first, and then analyzed on the teaching system (either from the EIETL or the DPO as directed by your demonstrator). The analysis uses a Python Jupyter notebook.

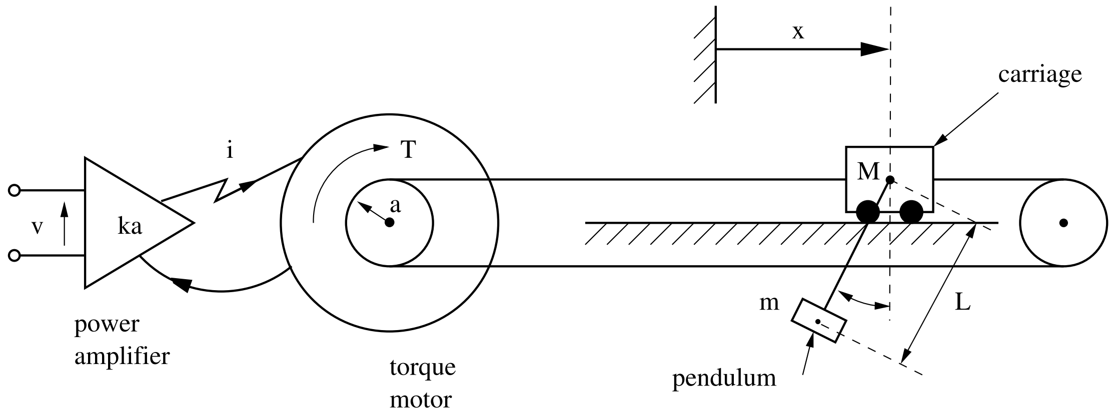

Figure 1: Apparatus

1 Apparatus

1.1 The System

The apparatus to be controlled is illustrated in Figure 1. It consists of a pendulum pivoted on a carriage which can be pulled up and down a track by a wire attached to a torque motor.

The system input is the voltage, v, to the power amplifier, which then produces a current, i, in the torque motor.
The observations are

- carriage position, CP, and carriage velocity, CV, from an optical encoder on the motor shaft.

- pendulum angular position, PP, and angular velocity, PV, from an optical encoder on the shaft on the carriage.

- motor current, MC.

Scale factors for these observations (all measured in volts) are as follows,

x	$=$	-0.08 $\times$ CP	m	
$\dot{x}$	$=$	-0.43 $\times$ CV	m/s	
θ	$=$	0.314 $\times$ PP	radians	$-\pi/2 < \theta < \pi/2$.
		π - 0.314 $\times$ PP	radians	$\pi/2 < \theta < 3\pi/2$.
$\dot{\theta}$	$=$	1.5 $\times$ PV	rads/s	
i	$=$	MC/0.146	A	

Other physical constants of the apparatus are:
Distance from the pendulum's centre of mass to the pivot, $L = 125$ mm
Radius of the pulley, $a = 16$ mm

Mass of the pendulum, $m = 0.320$ kg
Mass of the carriage, $M = 0.700$ kg
Moment of inertia on motor shaft, $I \cong 80 \times 10^{-6}$ kg-m^2
Torque-motor constant, $k_m = 0.080$ N-m/A
Amplifier constant, $k_a =$ -0.50 A/V
Frictional force on carriage $= F$ sgn $\dot{x}$ N
(F to be measured)

1.2 The Controller

The control box contains the power amplifier, signal conditioning and scaling for CP, CV, PP, PV and MC and various controller options. A circuit diagram of the controller is given in Figure 2.

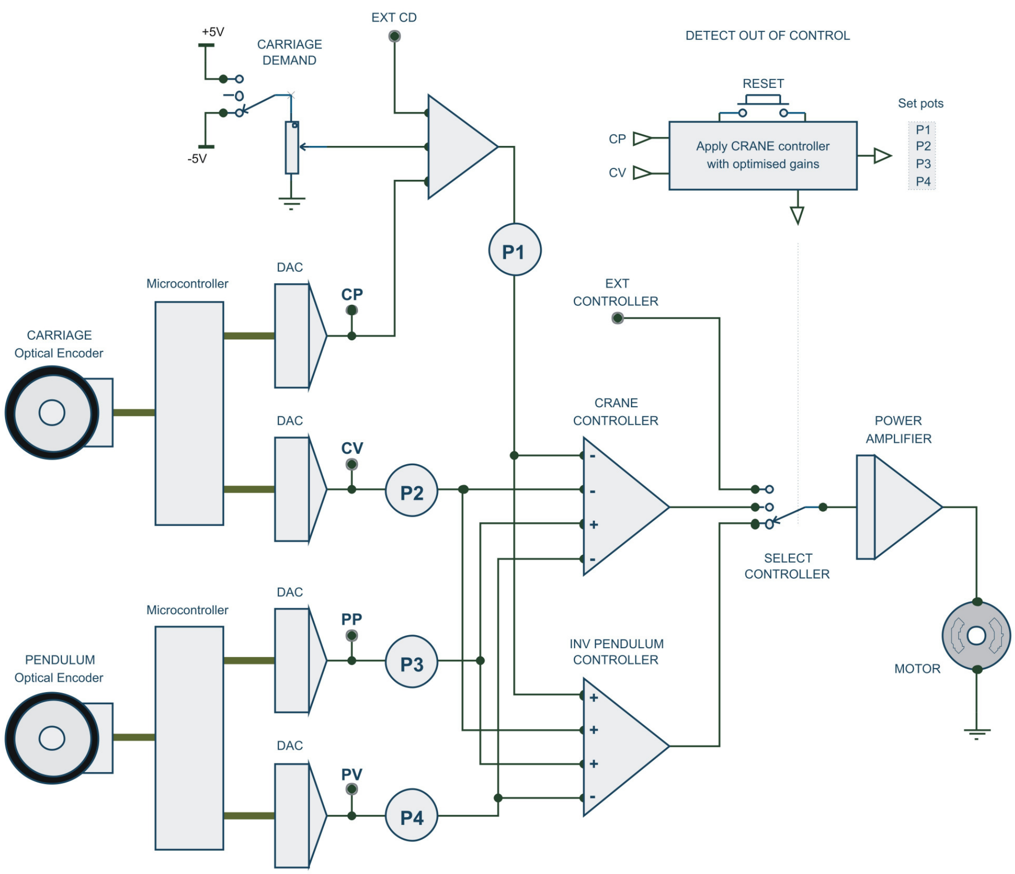

Figure 2: Controller Circuit

A circuit detects when the system is out of control and then switches in a controller that controls the carriage back to the centre with the pendulum vertically down. This latch remains set (with the red RESET light lit), until the RESET button is pressed. For this experiment external controllers will *not* be used, so the EXT CONT light should be off and either the CRANE (for section 4) or INV PEND (for section 5) control mode

selected.
The CRANE and INV PEND controllers differ in the sign and magnitudes of the gains associated with the variables.

For the CRANE controller,

$$v = -20 \times p_1 \times (CP - CD) - 30 \times p_2 \times CV + 20 \times p_3 \times PP - 10 \times p_4 \times PV$$

For the INV PEND controller,

$$v = 10 \times p_1 \times (CP - CD) + 20 \times p_2 \times CV + 30 \times p_3 \times PP - 20 \times p_4 \times PV,$$

where CD = "EXT + OFFSET" is the carriage position demand signal. By adjusting the OFFSET potentiometer, different sizes of step inputs can be produced. The feedback gains p_1, p_2, p_3 and p_4 are the settings **(in the range 0.000 to 1.000)** of the potentiometers labelled CP, CV, PP and PV respectively on this front panel **(do not confuse these feedback gains with the state measurements CP, CV, PP, and PV)**.

2 Using the computer

2.1 Data Logging

To start the data-logging program, double-click on the shortcut to *Pendulum Datalogger* on the windows desktop. Or alternatively navigate to the folder '*c:\3F2*' and double click on '*inverted pendulum.exe*'.

The Pendulum Datalogger will log and display data from the apparatus, sampled at 400 Hz. Data may be logged by clicking on the *Start Logging* button. Spacebar or enter may be used to start and stop data acquisition. The plot of the logged data is immediately displayed in the main window but the axes are unchanged. If the axes do not cover the most suitable ranges, press the *Auto* button.
The plot defaults to displaying all five signals; however each signal may be hidden by removing the *check* in the *show* box besides its name. Signals may also be *inverted*.

The *Save* and *Open* buttons may be used to store and retrieve plots. The plots are automatically named `Plot` n`.csv`, where the number n is incremented with each plot, and are stored in a folder called *Exp_3F2* on the Desktop. A screenshot with, extension `png`, of the plots is also saved and named in the same way as the `csv` file.

2.2 Introduction to the Jupyter notebook and Anaconda

Download the Jupyter Notebook file and the Jupyter engine launcher [*3F2_Jupyter_Launcher.bat*] via *vle.cam.ac.uk* to the folder *Exp_3F2* on the desktop. If such folder does not exist, start the inverted pendulum demonstration as per the previous section. Run the file *3f2_Jupyter_Launcher.bat* to start the Jupyter Engine and navigate to the notebook you have downloaded, i.e *3F2-Exp-2023.ipynb*.

The provided notebook has a number of cells, that follow the numbering of the paragraphs of this document. Only the paragraphs that require an answer to a question are numbered. The first 3 cells load some physical constants and define some helper functions. A number of cells plot the data allowing to select the values of `t0` and `tf` for each simulation.
The helper functions are the following:

- `loadlogdata(n,ScaleF)` This function loads the data from file `Plot00` n`.csv`. The first argument refers to the file name to load, i.e. it refers to the n-th saved plot with file name `Plot00` n`.csv`. The second argument is a matrix containing the scaling factors to load the logged data correctly.

- `solver_crane(x0,tsim,tdem,xdem,p)|solver_pendulum(x0,tsim,tdem,xdem,p)` These function integrate the equations of motion respectively for the crane and pendulum configuration. The parameter `x0` is the initial condition, `tsim` is the time base associated with the simulation, `xdem` is the demand signal, `p` is the vector of feedback gains.

- `plot_function_crane|plot_function_pendulum(t0,tf,t,xdata,tsim,x_sim,p)`

  These functions plot the simulated data and the logged data in a single plot for the crane and pendulum configuration respectively. `t0` and `tf` are the initial and final time to be considered, `t` is the time base associated with logged data, `tsim` is the time base associated with the simulation, `x_sim` is the simulated data and `p` is the vector of feedback gains.

2.3 Pole calculation in the Jupyter notebook

To calculate the closed loop poles of the linear model run the appropriate cells of the notebook in order to load the physical constants, and input the potentiometer settings, e.g.

```
P = np.array([[0.35, 0.15, 0.0, 0.0]])
```

and to obtain the poles for the crane position type

```
print(np.linalg.eig(Ac-B@P@Cc)[0])
```

or for the inverted pendulum position type

```
print(np.linalg.eig(Ap-B@P@Cc)[0])
```

3 Crane Control: Experimental Procedure

Throughout the experiment a large number of step responses are requested; however it is only necessary to plot them when the instructions explicitly say so, and not when verbs such as note and observe are used. A step response can be obtained by setting the carriage demand to a value different from 0 volts and moving the switch below the potentiometer to either left or right.

3.1 Friction measurement

Note the apparatus number. Make sure the potentiometer values are set to 0.
Now measure the moving frictional force, F, on the carriage with a spring balance. With the same method measure also the static friction force. Select the CRANE controller.

3.2 Carriage Controller

Design a carriage position servo by setting $p_1 = 0.350, p_3 = p_4 = 0$ and varying p_2 to give a critically-damped response in the carriage (ignoring the pendulum), say $p_2 \simeq 0.150$. Note the step response.

3.3 Empirical synthesis of the p_3 and p_4 controller

With p_1 and p_2 as in 3.2, increase p_3 from zero and note the step responses. Now spend 5 minutes varying p_3 and p_4 to minimize the oscillations in the pendulum. Log your best effort and obtain the theoretical closed-loop poles using the appropriate cell of the provided Jupyter notebook (see section 2.3 above). Record your optimal values of p_3 and p_4, and the theoretical closed-loop pole positions.

Comment on the transient response and pole positions.

3.4 Pole-placement

(a) Use example 1 of section A.5 together with the scale factors of section A.7 to place all the closed-loop poles at $-\omega_1 = -\sqrt{78.5}$. Record your calculated values of p_1–p_4, and note the step response. Check the theoretical poles using the provided Jupyter notebook, and comment on the consistency with the target pole position of $-\omega_1 = -\sqrt{78.5}$.

(b) Now increase the speed of response by placing the closed-loop poles at $-\alpha, -\beta, -\omega \pm j\omega$ for suitable values of α, β and ω.

Use the appropriate cell of the Jupyter notebook to obtain the corresponding potentiometer settings. Record your choice (of α, β and ω) and the corresponding potentiometer settings. Log the step response and comment.
[It is not expected that you will choose the same values as any other students!].

3.5 Variation of p_2

With the design 3.4(b) vary p_2 until instability just occurs. Log the step response just prior to the onset of oscillations, and record the value of p_2. Use the linear model in appendix A to predict the gain k_2 at which oscillation will occur, and predict the resonant frequency $\hat{\omega}$ (see section A.5). Compare these with your experimental results (note: $k_2 = 165p_2$ — see A.7(a))

4 Inverted Pendulum: Experimental Procedure

4.1

Stand well clear of the pendulum and select the INV. PENDULUM controller. The carriage will probably be driven to the end of the track and the cut-out operated. To start controllers in future, hold the pendulum upright in the centre of the track and press RESET at the same time as releasing the pendulum.

4.2 No carriage feedback

Set $p_1 = p_2 = 0$ and p_3 and p_4 to stabilize the pendulum dynamics, say $p_3 = 0.500$ and $p_4 = 0.110$. Now hold the pendulum upright and press RESET. Manually note the force, on the pendulum, required to move the carriage - TAKE DUE CARE. Let go of the pendulum and explain the subsequent behaviour.

4.3 Pole Placement

Using example 2 in section A.5 and the data in section A.7 calculate p_i to place the closed-loop poles at $-\omega_1 = -\sqrt{78.5}$, and record your calculated values. Log the response.

4.4 Limit Cycles

Set the potentiometers to

$$p_1 = 0.23 \quad p_2 = 0.50 \quad p_3 = 0.63 \quad p_4 = 0.40$$

which should give a reasonably stable response. Now reduce p_2 until large oscillations occur (i.e. the carriage nearly hits the end stops). Record your value of p_2, and log the response. Now increase p_2 until the system is almost unstable. Record your value of p_2, and log the response to a small step.

4.5 No pendulum feedback

If the design of Section 4.3 is implemented except with $p_3 = p_4 = 0$ what would happen, and why?

5 Analysis of results

When you save a plot a folder called *Exp_3F2* will be created on the Desktop (if it does not exist). In the *Exp_3F2* folder all plots are saved, this allows to access the data from any Teaching system PC. Note that in order to analyse the data with the Jupyter engine on the EITEL PC's you have to copy the logged csv files to the folder *c:\Users\crsid* as per Section 2.2.

Using the Jupyter notebook you can perform a variety of comparisons between the linear theory, the nonlinear model, and these experimental results.

> On each of your six plots from sections 3 and 4, comment on the consistency between pole positions of the linear model and the experiment response.

A nonlinear simulation model is available in the notebook and can be run using the one the `solver_crane|solver_pendulum` functions described in Section 2.2. The simulation is done using the Verlet[1] integration method and is implemented in the helper functions `solver_crane|solver_pendulum` discussed in Section 2.2. The methods assume a one degree of freedom mechanical system system of the form

$$\frac{\mathrm{d}v^{\mu}}{\mathrm{d}t} = \frac{1}{m_{\mu}} F_{\mu}(x^{\mu}, v^{\mu})$$
$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = v^{\mu}$$

where v^{μ} denotes the velocity of the considered body, x^{μ} the position, m_{μ} the mass and F_{μ} denotes the force.

The Verlet algorithm successively calculates for a time step of length h

$$\begin{aligned}
a_1^{\mu} &= F_{\mu}\left(x_i^{\mu},\, v_i^{\mu}\right)/m_{\mu} \\
x_{i+1}^{\mu} &= x_i^{\mu} + \left(v_i + \frac{1}{2}\cdot a_1^{\mu}\cdot h\right)\cdot h \\
v_{i+1}^{\mu} &= v_i^{\mu} + \frac{1}{2}\cdot a_1^{\mu}\cdot h \\
a_2^{\mu} &= F_{\mu}\left(x_{i+1}^{\mu},\, v_{i+1}^{\mu}\right)/m_{\mu} \\
v_{i+1}^{\mu} &= v_i^{\mu} + \frac{1}{2}\cdot\left(a_2^{\mu} - a_1^{\mu}\right)\cdot h
\end{aligned}$$

A particular version has been written to enable the static and dynamic function to be simulated, with two different equation sets dependent on whether the carriage is stopped or moving. A empirical method is use to determine if the cart is moving or is stopped at the beginning of the simulation `t0`.

> Plot a comparison of the experimental and simulation results for each of the six cases. On each plot, comment on the consistency between your experimental and simulation results.

> * Compare the describing function predictions of the limit cycle amplitude and frequency with those observed in the results of section 4.4. You will need to work through the theory of section A.6 using your measured dynamic friction term.

[1] The method in its original form is a numerical method used to integrate Newton's equations of motion. In the notebook and in the following equations it is adapted to allow the force to depend upon position and velocity.

6 Writing up

6.1 Laboratory Report

Do not write a report in the standard format (i.e. one with introduction, aims and objectives etc). *All* that is required is a pdf copy of the Jupyter notebook, with the plots requested in the various section, and with brief comments provided the appropriate cells. In order to generate a pdf in the Jupyter notebook click '*File*', the click on '*Print preview*'. Once the page opens click on '*print pdf*' and finally select '*Save as pdf*' and save the file with an appropriate name.

Before submitting your report you must verify that:

1. The first cell is filled appropriately with your *name, surname, crsid and apparatus number.*

2. A pdf copy of the the notebook must be submitted after a "restart and run all" showing all graphs and the required information.

6.2 Full Technical Report

Guidance on the preparation of Full Technical Reports is provided both in Appendix I of the General Instructions document and in the CUED booklet A Guide to Report Writing, with which you were issued in the first year.

If you are offering a Full Technical Report on this experiment, address the starred questions contained in the analysis Section 5 and in the theory appendix A. In the appendix include a pdf copy of the Jupyter notebook lab report you have worked on. Your FTR should be in pdf format.

Denote clearly the name and number of the paragraphs associated with the question you are answering. In preparing your report, you may find [1] useful on matters relating to linear state space systems, and [2] useful on nonlinear systems and the describing function method.

K. Glover,	October 2000
Revised: J. Paxman and J. Maciejowski,	March 2003
Revised: P. Goulart,	January 2007
Corrected: J. Maciejowski,	January 2011
Revised: R. Pates,	January 2015
Revised: T. Hughes,	January 2017
Revised: G. Vinnicombe,	January 2022
Revised: G. Pugliese Carratelli,	January 2023

References

[1] Franklin, G. F., Powell, J. D., and Emami-Naeni, A., *Feedback Control of Dynamic Systems*, Addison-Wesley, 2nd edition, 1991.

[2] Khalil, H. C., *Nonlinear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 2nd edition, 1996.

A Theory

The following sections describe some theoretical results which will aid the analysis and understanding of the experiment. You should ensure you have read and understood this material before completing your laboratory report.

A.1 State-space model

Analysis of the dynamics of Figure 1, (assuming that the moments of inertia of the pendulum about its centre of mass is negligible) gives the equations

$$L\ddot{\theta} = \cos\theta.\ddot{x} - g\sin\theta \tag{1}$$

$$\left(1 + \frac{M}{m} + \frac{I}{ma^2}\right)\ddot{x} = \frac{T}{ma} - \left(\frac{F}{m}\right)\ \mathrm{sgn}\ (\dot{x}) + L\cos\theta.\ddot{\theta} - L\sin\theta.\dot{\theta}^2 \tag{2}$$

where

$$\mathrm{sgn}\ (\dot{x}) = \begin{cases} 1 & \dot{x} > 0 \\ -1 & \dot{x} < 0 \\ \text{undefined for } \dot{x} = 0. & \end{cases}$$

*	<u>Derive</u> (1) and (2) using figure 1, and appropriate free body diagrams.

The sgn $(\dot{x})$ term cannot be sensibly linearized but the others can to give approximately,

$$L\ddot{\theta} = \ddot{x} - g\theta \tag{3}$$

$$\left(1 + \frac{M}{m} + \frac{I}{ma^2}\right)\ddot{x} = \frac{T}{ma} - \left(\frac{F}{m}\right)\ \mathrm{sgn}\ (\dot{x}) + L\ddot{\theta} \tag{4}$$

These two simultaneous 2nd-order o.d.e.'s are equivalent to a 4-th order o.d.e. in either x or θ, but can also be written as a *first-order vector o.d.e., as follows. Let the* state-vector be the 4×1 vector,

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ L\theta \\ L\dot{\theta} \end{bmatrix} \tag{5}$$

We want to write $\dot{\underline{x}}$ in terms of $\underline{x}$ and T. Equations (3) and (4) can be solved for $\ddot{x}$ and $L\ddot{\theta}$ to give

$$\ddot{x} = -(\omega_0^2 - \omega_1^2)L\theta + u - f \tag{6}$$

$$L\ddot{\theta} = -\omega_0^2 L\theta + u - f \tag{7}$$

where

$$\omega_1^2 = \frac{g}{L} \tag{8}$$

$$\omega_0^2 = \omega_1^2 \left(1 + \frac{m}{(M + I/a^2)}\right) \tag{9}$$

$$u = \frac{T}{a\,(M + I/a^2)} \tag{10}$$

$$f = \left(\frac{F}{M + I/a^2}\right) \ \mathrm{sgn}\ (\dot{x}) \tag{11}$$

Note that ω_1 = natural frequency of the pendulum with the carriage fixed and ω_0 = natural frequency of the pendulum with the carriage free to move (assuming no friction).

Equations (6) and (7) can be written as a first order vector o.d.e. as follows

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ L\theta \\ L\dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_1^2 - \omega_0^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_0^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \\ L\theta \\ L\dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\underline{B}} (u - f) \tag{12}$$

Ignoring friction, this is a particular case of the standard form,

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \tag{13}$$

which will be studied extensively in the Module 3F2 lectures, and some basic results will now be summarized.

A.2 Inverted pendulum model

Equations (1) and (2) still hold for the inverted case but the linearization is now about $\theta = \pi$. Let $\phi = \pi - \theta$ gives

$$-L\ddot{\phi} = -\cos(\phi)\ddot{x} - g\sin(\phi) \tag{14}$$

$$\left(1 + \frac{M}{m} + \frac{I}{ma^2}\right)\ddot{x} = \frac{T}{ma} - \left(\frac{F}{m}\right) \ \mathrm{sgn}\ (\dot{x}) + L\cos(\phi)\ddot{\phi} - L\sin(\phi)\dot{\phi}^2 \tag{15}$$

which linearizes to,

$$L\ddot{\phi} = \ddot{x} + g\phi \tag{16}$$

$$\left(1+\frac{M}{m}+\frac{I}{ma^2}\right)\ddot{x}=\frac{T}{ma}-\left(\frac{F}{m}\right)\ \mathrm{sgn}\ (\dot{x})+L\ddot{\phi} \tag{17}$$

which are identical to (3) and (4) replacing θ by ϕ and changing the sign of the $g\phi$ term. Hence (12) becomes

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ L\phi \\ L\dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_0^2-\omega_1^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ L\phi \\ L\dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}(u-f) \tag{18}$$

*	Verify equation (18), assuming that equations (14), and (15) are correct.

A.3 System Poles

Consider equation (13) in the general case when dimension of $\underline{x}$ is n. Assume $\underline{x}(0)=\underline{0}$ and take Laplace transforms to give,

$$s\underline{X}(s)=A\underline{X}(s)+\underline{B}U(s) \tag{19}$$

$\Rightarrow$

$$(sI-A)\underline{X}(s)=\underline{B}U(s) \qquad (I=n\times n \text{ identity matrix}) \tag{20}$$

$\Rightarrow$

$$\underline{X}(s)=(sI-A)^{-1}\underline{B}U(s) \tag{21}$$

Now $(sI-A)^{-1}$ can be written as the matrix of cofactors of $(sI-A)$ divided by the determinant of $(sI-A)$, say

$$(sI-A)^{-1}=N(s)/d(s) \tag{22}$$

where

$$d(s)=\det(sI-A) \tag{23}$$

and $N(s)$ is an $n\times n$ matrix of polynomials in s. $d(s)$ is called the characteristic polynomial of A. Hence

$$\underline{X}(s)=\frac{N(s)\underline{B}}{d(s)}\cdot U(s) \tag{24}$$

Now if (λ_i) is such that $d(\lambda_i)=0$ then λ_i is an *eigenvalue* of A by (23) and a system pole by (24). Hence,

<u>the system poles = eigenvalues of A</u>

Recall from your Part IB Linear Systems and Control lectures that the pole positions determine the stability and speed of response of a system. Briefly, if a system has a pole at the real value λ, then the transient response includes a term $e^{\lambda t}$. Similarly if a system has a pair of complex poles at $\sigma \underline{+} j\omega$ then the transient response includes terms like

$$Ae^{j\phi}e^{(\sigma+j\omega)t} + Ae^{-j\phi}e^{(\sigma-j\omega)t} = 2Ae^{\sigma t}\cos(\omega t + \phi) \tag{25}$$

which is an exponentially increasing ($\sigma > 0$) or decreasing ($\sigma < 0$) sinusoid of frequency ω rad/s.

In the present experiment the system is fourth order with its response made up from combinations of responses from the four poles.

A.4 State-feedback

Suppose that in (13) all the states are measured and can be used for control, what sort of performance can be achieved? Let

$$\begin{aligned} u &= w - k_1 x_1 - k_2 x_2 \cdots, -k_n x_n \\ &= w - \underline{K}\ \underline{x} \end{aligned} \tag{26}$$

where $\underline{K}$ is a $1 \times n$ matrix of feedback gains. (13) now becomes

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + \underline{B}(w - \underline{K}\underline{x}) \\ &= (A - \underline{B}\underline{K})\underline{x} + \underline{B}w \end{aligned} \tag{27}$$

This means that the "A-matrix" is now $(A - \underline{BK})$ and its new eigenvalues give the new closed-loop poles.

The *pole-placement theorem* states that if a system is "controllable" then its poles can be arbitrarily assigned using state-feedback.

* <u>Define</u> controllability. Determine the values of ω_0 and ω_1 for which the linear crane equation (12) describes a controllable system.

Hence we can theoretically obtain any set of closed-loop poles by suitable choice of the state-feedback gains, $\underline{K}$. Fast and stable poles are generally desired but it is sometimes difficult to know which combinations of poles are easy to achieve and which are hard.

A.5 Closed-loop characteristic equation

For the crane and inverted pendulum systems (still assuming $f = 0$), state feedback will give the closed-loop characteristic polynomials

$$\begin{aligned} d_c(s) &= \det(sI - A + \underline{BK}) \\ &= s^4 + (k_2 + k_4)s^3 + (k_1 + k_3 + \omega_0^2)s^2 + k_2\omega_1^2 s + k_1\omega_1^2 \end{aligned} \tag{28}$$

for the crane model, and

$$d_p(s) = s^4 + (k_2 + k_4)s^3 + (k_1 + k_3 - \omega_0^2)s^2 - k_2\omega_1^2 s - k_1\omega_1^2, \tag{29}$$

for the inverted pendulum model.

It is clear that $\underline{K}$ can be chosen to make $d_c(s)$ or $d_p(s)$ arbitrary polynomials with any set of prescribed poles.

> *
>
> Use the Routh Hurwitz test (see Electrical and Information Data Book) to verify that the crane linear model will be stable if $k_1 > 0$, $k_2 > 0$, $k_2 + k_4 > 0$, $k_1 + k_3 + w_0^2 > 0$ and
>
> $$k_2^2(k_3 + \omega_0^2 - \omega_1^2) + k_2k_4(k_3 + \omega_0^2 - k_1) > k_1k_4^2, \tag{30}$$
>
> Furthermore if the last inequality becomes an equality <u>show</u> that there will be an oscillation of frequency
>
> $$\hat{\omega} = \sqrt{\frac{k_2}{k_2 + k_4}}\omega_1 \tag{31}$$
>
> *<u>Derive</u>* similar conditions for the inverted pendulum.

Example 1

For the crane set all the closed-loop poles to $-\omega_1 = -\sqrt{g/L}$. Then

$$\begin{aligned} d_c(s) &= (s + \omega_1)^4 = s^4 + 4\omega_1 s^3 + 6\omega_1^2 s^2 + 4\omega_1^3 s + \omega_1^4 \\ \Rightarrow k_1 &= \omega_1^2,\ k_2 = 4\omega_1,\ k_3 = 5\omega_1^2 - \omega_0^2,\ k_4 = 0 \end{aligned} \tag{32}$$

Example 2

For the inverted pendulum set all the closed-loop poles to $-\omega_1$. Then from (29)

$$k_1 = -\omega_1^2,\ k_2 = -4\omega_1,\ k_3 = \omega_0^2 + 7\omega_1^2,\ k_4 = 8\omega_1. \tag{33}$$

Eigenvalue Sensitivity

The roots of a polynomial can be very sensitive to the coefficients, particularly when the roots are nearly repeated. For example suppose $d(s) = (s+k)^4 + \varepsilon k^4$ where ε is the proportional error in the constant term.

> * Find the roots of $d(s)$ for $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$ and plot on the Argand diagram (Hint: consider the 4th roots of $-\varepsilon$). Comment on this sensitivity in the light of your results on pole placement.

A.6 Friction and Limit Cycles

So far we have only considered the linear theory. Unfortunately in this experiment there is a significant frictional force on the carriage. One approximate method of analysis for sinusoidal responses or stability determination is the *describing function method.* Suppose that $\dot{x} = E \ \sin \ \omega t$ then $\text{sgn}(\dot{x})$ will be a square wave and the Fourier series expansion of $\text{sgn}(\dot{x})$ gives

$$\text{sgn}\ (\dot{x}) = \frac{4}{\pi}\sin\omega t + \frac{4}{3\pi}\sin 3\omega t + \dots \tag{34}$$

The describing function method "describes" the nonlinearity by the first term only, assuming that the higher harmonics are filtered out by the system. The gain of the $\text{sgn}\ (\dot{x})$ nonlinearity is hence

$$N(E) = \frac{4}{\pi E} \tag{35}$$

and increases quickly as the amplitude of $\dot{x}, E$,decreases. Therefore if an oscillation is present the friction term looks like an additional feedback gain from $\dot{x}$, with equivalent gain (see (11))

$$\frac{4F}{\pi E(M + I/a^2)} \tag{36}$$

For the crane problem this implies that for small amplitudes of oscillation the system has a large additional damping term from the friction and will be stable; whereas for large amplitudes of oscillation the additional damping will be smaller and hence a large initial disturbance may excite large oscillations in certain circumstances.

For the inverted pendulum the friction is the cause of limit cycling (i.e. oscillations of well-defined amplitude) about the equilibrium. Notice that in this case the k_2 term must be negative for closed-loop stability (see Section A.5), and this term will be opposed by the effective damping term due to friction. The amplitude of the oscillation, E, will then adjust so that the additional damping term $N(E)$ would be just sufficient to cause instability in the linear model. The amplitude of the resulting limit cycle at a particular value of k_2 can be estimated as follows.

(i) From the linear model and section A.5 determine the reduction in gain k_2 that will just induce instability, call this Δk_2, and note the frequency, $\widehat{\omega}$.

(ii) The amplitude of the limit cycle for $\dot{x}, E$, then satisfies

$$E = \frac{4F}{\pi(M + I/a^2)\Delta k_2} \tag{37}$$

The corresponding limit cycle in x will be of amplitude $E/\widehat{\omega}$.

Therefore if the linear design has Δk_2 sufficiently large the amplitude of the limit cycle in $\dot{x}$ can be made acceptably small.

A.7 Physical Constants in the Models

The data of section 1.1 and 1.2 can now be incorporated into the state-space model and controller. Firstly (8) and (9) imply

$$\begin{aligned} \omega_1^2 &= 78.5 \quad (\text{rad/s})^2 \\ \omega_0^2 &= 103.3 \quad (\text{rad/s})^2 \end{aligned}$$

and hence

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \pm 24.8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \pm 103.3 & 0 \end{bmatrix}$$

(the data of section 1.1 and 1.2 is accurate to only 10% in some cases so the values given in this section will be rounded to 2 or 3 significant digits).

From (10) we obtain

$$\begin{aligned} u &= \frac{T}{a\,(M + I/a^2)} \\ &= \frac{k_a k_m v}{a\,(M + I/a^2)} \end{aligned} \tag{38}$$

and substituting the physical values of section 1.1 gives

$$u = -2.47v \tag{39}$$

(a) <u>Crane controller</u>

The equation for v in section 1.2 together with the scale factors on the measurements give

$$\begin{aligned} u &= -2.47v \\ &= -617p_1(x - x_d) - 165p_2\dot{x} - 1256p_3L\theta + 126p_4L\dot{\theta} \end{aligned} \tag{40}$$

Hence

$$\begin{aligned} k_1 &= 617p_1 \\ k_2 &= 165p_2 \\ k_3 &= 1256p_3 \\ k_4 &= -126p_4 \end{aligned} \tag{41}$$

(b) Inverted pendulum controller

Similarly in this case, noting that $\phi = 0.314$ PP and $\dot{\phi} = -1.5$ PV, we obtain

$$\begin{aligned} k_1 &= -309p_1 \\ k_2 &= -110p_2 \\ k_3 &= 1884p_3 \\ k_4 &= 253p_4 \end{aligned} \tag{42}$$