50003 - Models of Computation - (Dr Raad) Lecture 1

Oliver Killane

14/10/21

Lecture Recording

Lecture recording is available here

Course Admin

Part 1 - Azalea Raad (Week 2 - Week 6)

- **8 Virtual Lectures - Teams & Recorded** Monday (11:00 - 12:00) Friday (16:00 - 17:00)
- **4 Hybrid tutorials - Teams & In Person** Friday (17:00 - 18:00)
- **1 Bonus Hour - Teams & Recorded** Monday 8/11/2012

We will cover:

- Operational Semantics of a small while language
- Denotational semantics of a small while language (notes only)
- Register machines, universal register machine, Halting problem.
- Turing machines and Turing computable functions, primitive and partial recursive functions
- Lambda calculus and equivalence results.

Part 2 - Herbert Wiklicky (Week 6 - Week 10)

- **Hybrid Lectures - Teams & In Person & Recorded** Friday (16:00 - 18:00)
- **Virtual Tutorials - Teams** Monday (11:00 - 12:00)

Algorithms

Examples of Algorithms

- **Euclid's Algorithm ≈300 B.C.** Algorithm to find the greatest common divisor.

```haskell
-- Euclid's algorithm:
-- continually take the modulus and compare until the modulus is zero
euclidGCD :: Int -> Int -> Int
euclidGCD a b
  | b == 0 = a
  | otherwise = euclidGCD b (a `mod` b)
```

- **Sieve of Eratosthenes ≈200 B.C.** Used to find the prime numbers within a limit. Done by starting from the 2, adding the number to the primes, then marking all multiples, then repeating progressing to the next non-marked number (a prime), marking all multiples and repeating.

```haskell
-- Sieve of Eratosthenes
eraSieve :: Int -> [Int]
eraSieve lim = eraSieveHelper [2..lim]
  where
    eraSieveHelper :: [Int] -> [Int]
    eraSieveHelper (x:xs) = x:eraSieveHelper (filter (\n -> n `mod` x /= 0) xs)
    eraSieveHelper [] = []
```

- **Well-Known Rules for $+ - \div \times$ ≈900 A.D.** Al-Khwarizmi, a persian mathematician who was appointed astronomer and head of the library in the Bagdad House of Wisdom.
- **Simple Machines using punchcards. Mainly 19th Century** Weaving looms, pianola, census tabulating machine.
- **Analytical Engine** A proposed multi-purpose calculating machine designed by Charles Babbage. Using a simple ALU, with conditional branching and integrated memory. As a result the first Turing complete computer.

  First ever program was written by Ada Lovelace to calculate the Bernoulli numbers.

Decision Problems

Given:

- A set S of finite data structures of some kind (e.g formulae in first order logic).
- A property P of elements of S (e.g the porperty of a formula that it has a proof).

Formulas

Well formed logical statements that are a sequence of symbols form a given formal language. e.g $(p \vee q) \wedge i$ is a formula, but $) \vee \wedge ji$ is not.

The associated decision procedure is:

Find an algorithm such that for any $s \in S$, if s has property P the algorithm terminates with 1, otherwise with 0.

Hilbert's Entscheidungsproblem

Is there an algorithm which can take any statement in first-order logic, and determine in a finite number of steps if the statement is provable?

First Order Logic

Also called predicate logic, is an extension of propositional logic that includes quantifiers $(\forall, \exists)$, equality, function symbols (e.g $\times, \div, +, -$) and structured formulas (predicate functions).

This problem was originally presented in a more ambiguous form, using a logic system more powerful than first-order.

'*Entscheidungsproblem*' means 'decision problem'

Many tired to solve the porblem, without success. One strategy was to try and disprove that such an algorithm can exist. In order to answer this question properly a formal definiotion of algorithm was required.

Algorithms Informally

Common features of Algorithms:

- **Finite** Description of the procedure in terms of elementary operations.
- **Deterministic** If there is a next step, it is uniquely determined - that is on the same data, the same steps will be made.
- **Maybe Terminate** procedure may not terminate on some input data, however we can recognise when it terminates and what the result is.

In 1935/35, Alan Turing (Cambridge) and Church (Princeton) independently gave negative soltuions to Hilberts Entscheidungsproblem (showed such an algorithm could not exist).

1. They gave concrete/precise definitions of what algorithms are (Turing Machines & Lambda Calculus).
2. They regarded algorithms as data, on which other algorithms could act.
3. They reduced the problem to the **Halting problem**.

This work led to the Church-Turing Thesis, that shows everything computable is computed by a Turing Machine. Church's Thesis extended this to show that General Recurisve Functions were the same type as those expressed by lambda calculus, and Turning showed that lambda calculus and the turning machine were equivalent.

Algorithms Formalised

Any formal definition of an algorithm should be:

- **Precise** No ambiguities, no implicit assumptions, Should be phrased mathematically.
- **Simple** No unnecessary details, only the few axioms required. Makes it easier to reason about.
- **General** So all algorithms and types of algorithms are covered.

The Halting Problem

The **Halting problem** is a decision problem with:

- The set of all pairs (A, D) such that A is an algorithm, and D is some input datum on which the algorithm operates.
- The property $A(D) \downarrow$ holds for $(A, D) \in S$ if algorithm A when applied to D eventually produces a result (halts).

Turning and Church showed that there is no algorithm such that:

$$\forall (A, D) \in S \begin{bmatrix} H(A, D) & = & 1 & A(D) \downarrow \\ & & 0 & otherwise \end{bmatrix}$$

The final step for Turing/Church's proof was to construct an algorithm encoding instances (A, D) of the halting problem as statements such that:

$$\Phi_{A,D} \textit{ is provable} \leftrightarrow A(D) \downarrow$$

Algorithms as Functions

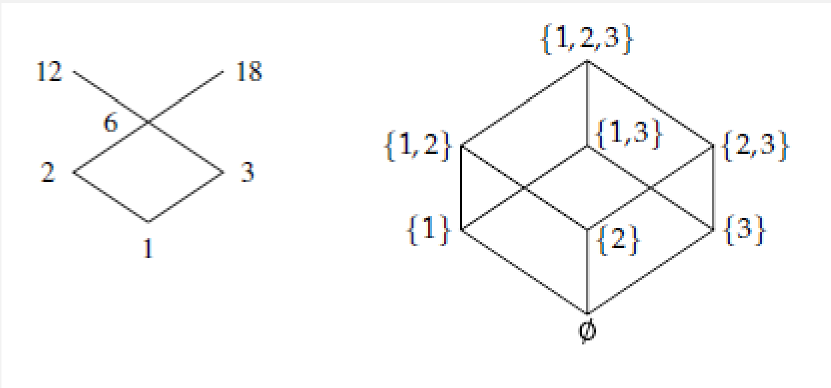
It is possible to give a mathematical description of a computable function as a special function between special sets.

In the 1960s Strachey & Scott (Oxford) introduced **denotational semantics**, which describes tje meaning (denotation) of an algorithm as a function that maps input to output.

Domains

Domains are special kinds of partially ordered sets. Partial orders meaning there is an order of elements in the set, but not every element is comparable.

Partial orders are reflexive, transitive and anti-symmetric. You can easily represent them on a Hasse Diagram.



Scott solved the most difficult part, considering recursively defined algorithms as continuous functions between domains.

Haskell Programs

Example using a basic implementation of power.

```haskell
-- Precondition: n >= 0
power :: Integer -> Integer -> Integer
power x 0 = 1
power x n = x * power x (n-1)

-- Precondition: n >= 0
```

```haskell
power' :: Integer -> Integer -> Integer
power' x 0 = 1
power' x n
  | even n = k2
  | odd n  = x * k2
  where
    k  = power' x (n `div` 2)
    k2 = k * k
```

O(n)

power 7 5
⇝ 7 * (power 7 4)
⇝ 7 * (7 * (power 7 3))
⇝ 7 * (7 * (7 * (power 7 2)))
⇝ 7 * (7 * (7 * (7 * (power 7 1))))
⇝ 7 * (7 * (7 * (7 * (7 * (power 7 0)))))
⇝ 7 * (7 * (7 * (7 * (7 * 1))))
⇝ 16807

O(log(n)) steps

power’ 7 5
⇝ 7 * (power’ 7 2)2̂
⇝ 7 * ((power’ 7 1)2̂)2̂
⇝ 7 * ((7 * (power’ 7 0)2̂)2̂)2̂
⇝ 7 * ((7 * (1)2̂)2̂)2̂
⇝ 16807

These two functions are equivalent in result however operate differently (one much faster than the other).

Program Semantics

Denotational Semantics:

- A program’s meaning is described compositionally using denotations (mathematical objects)
- A denotation of a program phrase is built from its subphrases.

Operational Semantics:

- Program’s meaning is given in terms of the steps taken to make it run.

There is also **axiomatic semantics** and **declarative semantics** but we will not cover them.

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Syntax of a while Language

We can define a simple while language (if, else, while loops) to build programs from & to analyse.

$$\begin{array}{rcl} B \in Bool & ::= & true|false|E = E|E < E|B\&B|\neg B \dots \\ E \in Exp & ::= & x|n|E + E|E \times E| \dots \\ C \in Com & ::= & x := E|if\ B\ then\ C\ else\ C|C;C|skip|while\ B\ do\ C \end{array}$$

Where $x \in Var$ ranges over variable identifiers, and $n \in \mathbb{N}$ ranges over natural numbers.

We can also define simple expressions (**SimpleExp**) to work on:

$$E \in SimpleExp\ ::=\ n|E + E|E \times E| \dots$$

Operational Semantics for SimpleExp

- **Small-Step** Also called structural, gives a method for evaluating an expression step-by-step.
- **Big-Step** Also called Natural, ignores intermediate steps and gives result immediately.

Big Step Semantics of SimpleExp

The properties OF $\Downarrow$ are:

- **Determinacy** For all E, n_1 and n_2 if $E \Downarrow n_1$ and $E \Downarrow n_2$ then $n_1 = n_2$
- **Totality** For all E there exists an n such that $E \Downarrow n$.

  We can break this with loops in matching, e.g

  $$\text{(B-NON-TOTAL)}\frac{}{true \Downarrow true}$$

  As a result, on hitting true will not stop.

Small Step Semantics of SimpleExp

Given a realtion $\rightarrow$ we can define a new relation $\leftarrow *$ such that:

$E \leftarrow *E'$ holds if and only if $E = E'$ or there is some finite sequence $E \rightarrow E_1 \rightarrow E_3 \rightarrow \dots \rightarrow E_k \rightarrow E'$

- **Normal Form** E is in its normal form (irreducable) if there is no E' such that $E \rightarrow E'$

  In **SimpleExp** the normal form is the natural numbers.
- **Determinacy** For all E, E_1, E_2 if $E \rightarrow E_1$ and $E \rightarrow E_2$ then $E_1 = E_2$.

  There is at most one next step.

- **Confluence** For all E, E_1, E_2 if $E \rightarrow *E_1$ and $E \rightarrow *E_2$ then there exists some E' such that $E_1 \rightarrow *E'$ and $E_2 \rightarrow *E'$.

  Determinate $\rightarrow$ Confluent.

  There are several evaluations paths, but they all get the same end result.

- **(Strong) Normalisation** There are no infinite sequences of expressions $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \dots$ such that for all i, $E_i \rightarrow E_{i+1}$.

  Every evaluation path eventually reaches a normal form.

Theorem: for all E, n_1, n_2, if $E \rightarrow *n_1$ and $E \rightarrow *n_2$ then $n_1 = n_2$.

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24/10/21

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Syntax of While

We can define a simple **While** language (if, else, while loops) to build programs from & to analyse.

$$\begin{array}{rcl} B \in Bool & ::= & true|false|E = E|E < E|B\&B|\neg B \dots \\ E \in Exp & ::= & x|n|E + E|E \times E| \dots \\ C \in Com & ::= & x := E|if\ B\ then\ C\ else\ C|C;C|skip|while\ B\ do\ C \end{array}$$

Where $x \in Var$ ranges over variable identifiers, and $n \in \mathbb{N}$ ranges over natural numbers.

States

A **state** is a partial function from variables to numbers. For state s, and variable x, $s(x)$ is defined, e.g:

$$s = (x \mapsto 2, y \mapsto 200, z \mapsto 20)$$

(In the current state, $x = 2, y = 200, z = 20$).

Partial Functions

A partial function is a mapping of every member of its domain, to at most one member of its codomain.

A state is a partial function as it is only defined for some variables.

For example:

$$\begin{array}{rcll} s[x \mapsto 7](u) & = & 7 & \text{if } u = x \\ & = & s(u) & \text{otherwise} \end{array}$$

The small step semantics of **While** are defined using **configurations** of form:

$$\langle E, s \rangle, \langle B, s \rangle, \langle C, s \rangle$$

(Evaluating E, B, or C with respect to state s)

We can create a new state, where variable x equals value a, from an existing state s:

$$s'(u) \triangleq \alpha(x) = \begin{cases} a & u = x \\ s(u) & otherwise \end{cases}$$

$$s' = s[x \mapsto u] \text{ is equivalent to } dom(s') = dom(s) \land \forall y.[y \neq x \to s(y) = s'(y) \land s'(x) = a]$$

(s' equals s where x maps to a)

Expressions

$$(\text{W-EXP.LEFT})\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 + E_2, s\rangle \rightarrow_e \langle E_1' + E_2, s'\rangle}$$

$$(\text{W-EXP.RIGHT})\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n + E, s\rangle \rightarrow_e \langle n + E', s'\rangle}$$

$$(\text{W-EXP.VAR})\frac{}{\langle x, s\rangle \rightarrow_e \langle n, s\rangle}\ s(x) = n$$

$$(\text{W-EXP.ADD})\frac{}{\langle n_1 + n_2, s\rangle}\ \langle n_3, s\rangle n_3 = n_1 + n_2$$

These rules allow for side effects, despite the While language being side effect free in expression evaluation. We show this by changing state $s \rightarrow_e s'$.

We can show inductively (from the base cases W-EXP.VAR and W-EXP.ADD) that expression evaluation is side effect free.

Booleans

(Based on expressions, one can create the same for booleans) ($b \in \{true, false\}$)

AND

$$(\text{W-BOOL.AND.LEFT})\frac{\langle B_1, s\rangle \rightarrow_b \langle B_1', s'\rangle}{\langle B_1 \& B_2, s\rangle \rightarrow_b \langle B_1' \& B_2, s'\rangle}$$

$$(\text{W-BOOL.AND.RIGHT})\frac{\langle B, s\rangle \rightarrow_b \langle B', s'\rangle}{\langle b \& B_2, s\rangle \rightarrow_b \langle b \& B', s'\rangle}$$

$$(\text{W-BOOL.AND.TRUE})\frac{}{\langle true \& true, s\rangle \rightarrow_b \langle true, s\rangle}$$

$$(\text{W-BOOL.AND.FALSE})\frac{}{\langle false \& b, s\rangle \rightarrow_b \langle true, s\rangle}$$

(Notice we do not short circuit, as the right arm may change the state. In a side effect free language, we could.)

EQUAL

$$(\text{W-BOOL.EQUAL.LEFT})\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 = E_2, s\rangle \rightarrow_b \langle E_1' = E_2, s'\rangle}$$

$$(\text{W-BOOL.EQUAL.RIGHT})\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n = E, s\rangle \rightarrow_b \langle n = E, s'\rangle}$$

$$(\text{W-BOOL.EQUAL.TRUE})\frac{}{\langle n_1 = n_2, s\rangle \rightarrow_b \langle true, s\rangle}\ n_1 = n_2$$

$$(\text{W-BOOL.EQUAL.FALSE})\frac{}{\langle n_1 = n_2, s\rangle \rightarrow_b \langle false, s\rangle}\ n_1 \neq n_2$$

LESS

$$\text{(W-BOOL.LESS.LEFT)}\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 < E_2, s\rangle \rightarrow_b \langle E_1' < E_2, s'\rangle}$$

$$\text{(W-BOOL.LESS.RIGHT)}\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n < E, s\rangle \rightarrow_b \langle n < E, s'\rangle}$$

$$\text{(W-BOOL.LESS.TRUE)}\frac{}{\langle n_1 < n_2, s\rangle \rightarrow_b \langle true, s\rangle}\ n_1 < n_2$$

$$\text{(W-BOOL.EQUAL.FALSE)}\frac{}{\langle n_1 < n_2, s\rangle \rightarrow_b \langle false, s\rangle}\ n_1 \geq n_2$$

NOT

$$\text{(W-BOOL.NOT)}\frac{}{\langle \neg true, s\rangle \rightarrow_b \langle false, s\rangle}$$

$$\text{(W-BOOL.NOT)}\frac{}{\langle \neg false, s\rangle \rightarrow_b \langle true, s\rangle}$$

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Factorial Program

$$C = y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$$

We can attempt to evaluate this for a given input, for example:

$$s = [x \mapsto 3, y \mapsto 17, z \mapsto 42]$$

The evaluation path is as follows:

Start

$$\langle y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), [x \mapsto 3, y \mapsto 17, z \mapsto 42]\rangle$$

Get x variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$(\text{W-SEQ.LEFT})\dfrac{(\text{W-ASS.EXP})\dfrac{(\text{W-EXP.VAR})\dfrac{}{\langle x, s\rangle \rightarrow_e \langle 3, s\rangle}}{\langle y := x, s\rangle \rightarrow_c \langle y := 3, s\rangle}}{\langle y := x; C, s\rangle \rightarrow_c \langle y := 3; C, s\rangle}$$

Result:

$$\langle y := 3; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 17, z \mapsto 42)\rangle$$

Assign to y variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$(\text{W-SEQ.LEFT})\dfrac{(\text{W-ASS.NUM})\dfrac{}{\langle y := 3, s\rangle \rightarrow_c \langle skip, s[y \mapsto 3]\rangle}}{\langle y := 3; C, s\rangle \rightarrow_c \langle skip; C, s[y \mapsto 3]\rangle}$$

Result:

$$\langle skip; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42)\rangle$$

Eliminate skip

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$(\text{W-SEQ.SKIP})\dfrac{}{\langle skip; C, s\rangle \rightarrow_c \langle C, s\rangle}$$

Result:

$$\langle a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42)\rangle$$

Assign a

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$(\text{W-SEQ.LEFT})\dfrac{(\text{W-ASS.NUM})\dfrac{}{\langle a := 1, s\rangle \rightarrow_c \langle skip, s[a \mapsto 1]\rangle}}{\langle a := 1; C, s\rangle \rightarrow_c \langle skip; C, s[a \mapsto 1]\rangle}$$

Result:

$$\langle skip; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \rightarrow_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Expand while

where $C = (a := a \times y; y := y - 1)$, $B = 0 < y$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-WHILE)}\frac{}{\langle \text{while } B \text{ do } C, s\rangle \rightarrow_c \langle \text{if } B \text{ then } (C; \text{while } B \text{ do } C) \text{ else } skip, s\rangle}$$

Result:

$$\langle \text{if } 0 < y \text{ then } (a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } a := a \times y; y := y - 1) \text{ else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Get y variable

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.BEXP)}\frac{\text{(W-BOOL.LESS.RIGHT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \rightarrow \langle 3, s\rangle}}{\langle 0 < y, s\rangle \rightarrow_b \langle 0 < 3, s\rangle}}{\langle \text{if } 0 < y \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \rightarrow_c \langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip,}$$

Result:

$$\langle \text{if } 0 < 3 \text{ then } (a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } a := a \times y; y := y - 1);\ \text{else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Complete if boolean

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.EXP)}\frac{\text{(W-BOOl.LESS.TRUE)}\dfrac{}{\langle 0 < 3, s\rangle \rightarrow_b \langle true, s\rangle}}{\langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \rightarrow_c \langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle}$$

Result:

$$\langle \text{if } true \text{ then } (a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } a := a \times y; y := y - 1);\ \text{else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate if

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.TRUE)}\frac{}{\langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \rightarrow_c \langle C; \text{while } 0 < y \text{ do } C, s\rangle}$$

Result:

$$\langle a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression a

where $C = y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\dfrac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle a, s\rangle \rightarrow \langle 1, s\rangle}}{\langle a \times y, s\rangle \rightarrow_e \langle 1 \times y, s\rangle}}{\langle a := a \times y, s\rangle \rightarrow_c \langle a := 1 \times y, s\rangle}}{\langle a := a \times y; C, s\rangle \rightarrow_c \langle a := 1 \times y; C, s\rangle}$$

Result:

$$\langle a := 1 \times y; y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression y

where $C = y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\dfrac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.RIGHT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \rightarrow_e \langle 3, s\rangle}}{\langle 1 \times y, s\rangle \rightarrow_e \langle 1 \times 3, s\rangle}}{\langle a := 1 \times y, s\rangle \rightarrow_c \langle a := 1 \times 3, s\rangle}}{\langle a := 1 \times y; C, s\rangle \rightarrow \langle a := 1 \times 3; C, s\rangle}$$

Result:

$$\langle a := 1 \times 3; y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Multiply

where $C = y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\dfrac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL)}\dfrac{}{\langle 1 \times 3, s\rangle \rightarrow_e \langle 3, s\rangle}}{\langle a := 1 \times 3, s\rangle \rightarrow_c \langle a := 3, s\rangle}}{\langle a := 1 \times 3; C, s\rangle \rightarrow_c \langle a := 3; C, s\rangle}$$

Result:

$$\langle a := 3; y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Assign 3 to a

where $C = y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\dfrac{\text{(W-ASS.NUM)}\dfrac{}{\langle a := 3, s\rangle \rightarrow_c \langle \mathit{skip}, s[a \mapsto 3]\rangle}}{\langle a := 3; C, s\rangle \rightarrow_c \langle \mathit{skip}; C, s[a \mapsto 3]\rangle}$$

Result:

$$\langle \mathit{skip}; y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate Skip

where $C = y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle y := y-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 3 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \to \langle 3, s\rangle}}{\langle y-1, s\rangle \to_e \langle 3-1, s\rangle}}{\langle y := y-1, s\rangle \to_c \langle y := 3-1, s\rangle}}{\langle y := y-1; C, s\rangle \to_c \langle y := 3-1, s\rangle}$$

Result:

$$\langle y := 3-1; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Evaluate Subtraction

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB)}\dfrac{}{\langle 3-1, s\rangle \to_e \langle 2, s\rangle}}{\langle y := 3-1, s\rangle \to_c \langle y := 2, s\rangle}}{\langle y := 3-1; C, s\rangle \to_c \langle y := 2; C, s\rangle}$$

Result:

$$\langle y := 2; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 2 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle y := 2, s\rangle \to_c \langle skip, s[y \mapsto 2]\rangle}}{\langle y := 2; C, s\rangle \to_c \langle skip; C, s[y \mapsto 2]\rangle}$$

Result:

$$\langle skip; \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y-1)$ and $s = (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y-1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$

50003 - Models of Computation - (Dr Raad) Lecture 4

Oliver Killane

24/10/21

Lecture Recording

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Small Step semantics

Expressions

$$(\text{W-EXP.LEFT})\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 + E_2, s\rangle \rightarrow_e \langle E_1' + E_2, s'\rangle}$$

$$(\text{W-EXP.RIGHT})\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n + E, s\rangle \rightarrow_e \langle n + E', s'\rangle}$$

$$(\text{W-EXP.VAR})\frac{}{\langle x, s\rangle \rightarrow_e \langle n, s\rangle}\ s(x) = n$$

$$(\text{W-EXP.ADD})\frac{}{\langle n_1 + n_2, s\rangle \rightarrow_e \langle n_3, s\rangle}\ n_3 = n_1 + n_2$$

Assignment

$$(\text{W-ASS.EXP})\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle x := E, s\rangle \rightarrow_c \langle x := E', s'\rangle}$$

$$(\text{W-ASS.NUM})\frac{}{\langle x := n, s\rangle \rightarrow_c \langle skip, s[x \mapsto n]\rangle}$$

Sequential Composition

$$(\text{W-SEQ.LEFT})\frac{\langle C_1, s\rangle \rightarrow_c \langle C_1', s'\rangle}{\langle C_1; C_2, s\rangle \rightarrow_c \langle C_1'; C_2, s'\rangle}$$

$$(\text{W-SEQ.SKIP})\frac{}{\langle skip; C, s\rangle \rightarrow_c \langle C, s\rangle}$$

Conditional

$$(\text{W-COND.TRUE})\frac{}{\langle \text{if } true \text{ then } C_1 \text{ else } C_2, s\rangle \rightarrow_c \langle C_1, s\rangle}$$

$$(\text{W-COND.FALSE})\frac{}{\langle \text{if } false \text{ then } C_1 \text{ else } C_2, s\rangle \rightarrow_c \langle C_2, s\rangle}$$

$$(\text{W-COND.BEXP})\frac{\langle B, s\rangle \rightarrow_b \langle B', s'\rangle}{\langle \text{if } B \text{ then } C_1 \text{ else } C_2, s\rangle \rightarrow_c \langle \text{if } B' \text{ then } C_1 \text{ else } C_2, s'\rangle}$$

While

$$(\text{W-WHILE})\frac{}{\langle \text{while } B \text{ do } C, s\rangle \rightarrow_c \langle \text{if } B \text{ then } (C; \text{while } B \text{ do } C) \text{ else } skip, s\rangle}$$

Determinacy and Confluence

The execution relation ($\rightarrow_c$) is deterministic.

$$\forall C, C_1, C_2 \in Com \forall s, s_1, s_2.[\langle C, s\rangle \rightarrow_c \langle C_1, s_1\rangle \wedge \langle C, s\rangle \rightarrow_c \langle C_2, s_2\rangle \rightarrow \langle C_1, s_1\rangle = \langle C_2, s_2\rangle]$$

Hence the relation is also confluent:

$$\forall C, C_1, C_2 \in Com \forall s, s_1, s_2.[\langle C, s\rangle \rightarrow_c \langle C_1, s_1\rangle \wedge \langle C, s\rangle \rightarrow_c \langle C_2, s_2\rangle \rightarrow$$

$$\exists C' \in Com, s'.[\langle C_1, s_1\rangle \rightarrow_c \langle C', s'\rangle \wedge \langle C_2, s_2\rangle \rightarrow_c \langle C', s'\rangle]]$$

Both also hold for $\rightarrow_e$ and $\rightarrow_b$.

Answer Configuration

A configuration $\langle skip, s\rangle$ is an **answer configuration**. As there is no rule to execute skip, it is a normal form.

$$\neg\exists C \in Com, s, s'.[\langle skip, s\rangle \rightarrow_c \langle C, s'\rangle]$$

For booleans $\langle true, s\rangle$ and $\langle false, s\rangle$ are answer configurations, and for expressions $\langle n, s\rangle$.

Stuck Configurations

A configuration that cannot be evaluated to a normal form is called a **suck configuration**.

$$\langle y, (x \mapsto 3)\rangle$$

Note that a configuration that leads to a **stuck configuration** is not itself stuck.

$$\langle 5 < y, (x \mapsto 2)\rangle$$

(Not stuck, but reduces to a stuck state)

Normalising

The relations $\rightarrow_b$ and $\rightarrow_e$ are normalising, but $\rightarrow_c$ is not as it may not have a normal form.

$$\text{while } true \text{ do } skip$$

$$\langle \text{while } true \text{ do } skip, s\rangle \rightarrow_c^3 \langle \text{while } true \text{ do } skip, s\rangle$$

($\rightarrow_c^3$ means 3 steps, as we have gone through more than one to get the same configuration, it is an infinite loop)

Side Effecting Expressions

If we allow programs such as:

$$\text{do } x := x + 1 \ return \ x$$

$$(\text{do } x := x + 1 \ return \ x) + (\text{do } x := x \times 1 \ return \ x)$$

(value depends on evaluation order)

Short Circuit Semantics

$$\frac{B_1 \to_b B_1'}{B_1 \& B_2 \to_b B_1' \& B_2}$$

$$\frac{}{false \& B \to_b false}$$

$$\frac{}{true \& B \to_b B}$$

Strictness

An operation is **strict** when arguments must be evaluated before the operation is evaluated. Addition is struct as both expressions must be evaluated (left, then right).

Due to short circuiting, & is left strict as it is possible for the operation to be evaluated without evaluating the right (**non-strict** in right argument).

Factorial Program

$$C = y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$$

We can attempt to evaluate this for a given input, for example:

$$s = [x \mapsto 3, y \mapsto 17, z \mapsto 42]$$

The evaluation path is as follows:

Start

$$\langle y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), [x \mapsto 3, y \mapsto 17, z \mapsto 42]\rangle$$

Get x variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle x, s\rangle \to_e \langle 3, s\rangle}}{\langle y := x, s\rangle \to_c \langle y := 3, s\rangle}}{\langle y := x; C, s\rangle \to_c \langle y := 3; C, s\rangle}$$

Result:

$$\langle y := 3; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 17, z \mapsto 42)\rangle$$

Assign to y variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle y := 3, s\rangle \to_c \langle skip, s[y \mapsto 3]\rangle}}{\langle y := 3; C, s\rangle \to_c \langle skip; C, s[y \mapsto 3]\rangle}$$

Result:

$$\langle skip; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42)\rangle$$

Eliminate skip

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$\text{(W-SEQ.SKIP)}\,\frac{}{\langle \mathit{skip}; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42)\rangle$$

Assign a

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)}\,\frac{\text{(W-ASS.NUM)}\,\dfrac{}{\langle a := 1, s\rangle \to_c \langle \mathit{skip}, s[a \mapsto 1]\rangle}}{\langle a := 1; C, s\rangle \to_c \langle \mathit{skip}; C, s[a \mapsto 1]\rangle}$$

Result:

$$\langle \mathit{skip}; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$

$$\text{(W-SEQ.SKIP)}\,\frac{}{\langle \mathit{skip}; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Expand while

where $C = (a := a \times y; y := y - 1)$, $B = 0 < y$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-WHILE)}\,\frac{}{\langle \text{while } B \text{ do } C, s\rangle \to_c \langle \text{if } B \text{ then } (C; \text{while } B \text{ do } C) \text{ else } \mathit{skip}, s\rangle}$$

Result:

$$\langle \text{if } 0 < y \text{ then } (a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } a := a \times y; y := y - 1) \text{ else } \mathit{skip}, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Get y variable

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.BEXP)}\,\frac{\text{(W-BOOL.LESS.RIGHT)}\,\dfrac{\text{(W-EXP.VAR)}\,\dfrac{}{\langle y, s\rangle \to \langle 3, s\rangle}}{\langle 0 < y, s\rangle \to_b \langle 0 < 3, s\rangle}}{\langle \text{if } 0 < y \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } \mathit{skip}, s\rangle \to_c \langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } \mathit{skip},}$$

Result:

$$\langle \text{if } 0 < 3 \text{ then } (a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } a := a \times y; y := y - 1);\ \text{else } \mathit{skip}, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Complete if boolean

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.EXP)}\frac{\text{(W-BOOl.LESS.TRUE)}\dfrac{}{\langle 0 < 3, s\rangle \rightarrow_b \langle true, s\rangle}}{\langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \rightarrow_c \langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle}$$

Result:

$$\langle \text{if } true \text{ then } (a := a\times y; y := y-1; \text{while } 0 < y \text{ do } a := a\times y; y := y-1);\ \text{else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate if

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.TRUE)}\frac{}{\langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \rightarrow_c \langle C; \text{while } 0 < y \text{ do } C, s\rangle}$$

Result:

$$\langle a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression a

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle a, s\rangle \rightarrow \langle 1, s\rangle}}{\langle a \times y, s\rangle \rightarrow_e \langle 1 \times y, s\rangle}}{\langle a := a \times y, s\rangle \rightarrow_c \langle a := 1 \times y, s\rangle}}{\langle a := a \times y; C, s\rangle \rightarrow_c \langle a := 1 \times y; C, s\rangle}$$

Result:

$$\langle a := 1 \times y; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression y

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.RIGHT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \rightarrow_e \langle 3, s\rangle}}{\langle 1 \times y, s\rangle \rightarrow_e \langle 1 \times 3, s\rangle}}{\langle a := 1 \times y, s\rangle \rightarrow_c \langle a := 1 \times 3, s\rangle}}{\langle a := 1 \times y; C, s\rangle \rightarrow \langle a := 1 \times 3; C, s\rangle}$$

Result:

$$\langle a := 1 \times 3; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Multiply

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL)}\dfrac{}{\langle 1 \times 3, s\rangle \rightarrow_e \langle 3, s\rangle}}{\langle a := 1 \times 3, s\rangle \rightarrow_c \langle a := 3, s\rangle}}{\langle a := 1 \times 3; C, s\rangle \rightarrow_c \langle a := 3; C, s\rangle}$$

Result:

$$\langle a := 3; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Assign 3 to a

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle a := 3, s\rangle \rightarrow_c \langle skip, s[a \mapsto 3]\rangle}}{\langle a := 3; C, s\rangle \rightarrow_c \langle skip; C, s[a \mapsto 3]\rangle}$$

Result:

$$\langle skip; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate Skip

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \rightarrow_c \langle C, s\rangle}$$

Result:

$$\langle y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 3 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \rightarrow \langle 3, s\rangle}}{\langle y - 1, s\rangle \rightarrow_e \langle 3 - 1, s\rangle}}{\langle y := y - 1, s\rangle \rightarrow_c \langle y := 3 - 1, s\rangle}}{\langle y := y - 1; C, s\rangle \rightarrow_c \langle y := 3 - 1, s\rangle}$$

Result:

$$\langle y := 3 - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Evaluate Subtraction

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB)}\dfrac{}{\langle 3-1, s\rangle \to_e \langle 2, s\rangle}}{\langle y := 3-1, s\rangle \to_c \langle y := 2, s\rangle}}{\langle y := 3-1; C, s\rangle \to_c \langle y := 2; C, s\rangle}$$

Result:

$$\langle y := 2; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 2 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle y := 2, s\rangle \to_c \langle skip, s[y \mapsto 2]\rangle}}{\langle y := 2; C, s\rangle \to_c \langle skip; C, s[y \mapsto 2]\rangle}$$

Result:

$$\langle skip; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$

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Oliver Killane

25/10/21

Lecture Recording
Lecture recording is available here

Structural Induction

Structural induction is used for reasoning about collections of objects, which are:

- structured in a well defined way
- finite but can be arbitrarily large and complex

We can use this is reason about:

- natural numbers
- data structures (lists, trees, etc)
- programs (can be large, but are finite)
- derivations of assertions like $E \Downarrow 4$ (finite trees of axioms and rules)

Structural Induction over Natural Numbers

$$\mathbb{N} \in Nat ::= zero | succ(\mathbb{N})$$

To prove a property $P(\mathbb{N})$ holds, for every number $N \in Nat$ by induction on structure $\mathbb{N}$:

- **Base Case** Prove $P(zero)$
- **Inductive Case** Inductive Case is $P(Succ(K))$ where $P(K)$ holds

For example, we can prove the property:

$$plus(\mathbb{N}, zero) = \mathbb{N}$$

- **Base Case**
  Show $plus(zero, zero) = zero$

$$\begin{array}{llll} (1) & \text{LHS} & = & plus(zero, zero) & \\ (2) & & = & zero & \text{(By definition of } plus\text{)} \\ (3) & & = & \text{RHS} & \text{(As Required)} \end{array}$$

- **Inductive Case**
  $N = succ(K)$
  Inductive Hypothesis $plus(K, zero) = K$
  Show $plus(succ(K), zero) = succ(K)$

$$\begin{array}{lllll} (1) & \text{LHS} & = & plus(succ(K), zero) & \\ (2) & & = & succ(plus(K, zero)) & \text{(By definition of } plus\text{)} \\ (3) & & = & succ(K) & \text{(By Inductive Hypothesis)} \\ (4) & & = & \text{RHS} & \text{(As Required)} \end{array}$$

Mathematics induction is a special case of structural induction:

$$P(0) \wedge [\forall k \in \mathbb{N}.P(k) \Rightarrow P(k+1)]$$

In the exam you may use $P(0)$ and $P(K+1)$ rather than $P(zero)$ and $P(succ(k))$ to save time.

Binary Tree Example

$$bTree \in BinaryTree ::= Node|Branch(bTree, bTree)$$

We can define a function *leaves*:

$$leaves(Node) = 1$$

$$leaves(Branch(T_1, T_2)) = leaves(T_1) + leaves(T_2)$$

Or *branches*:

$$branches(Node) = 0$$

$$branches(Branch(T_1, T_2)) = branches(T_1) + branches(T_2) + 1$$

Exercise

Prove By induction that $leaves(T) = branches(T) + 1$

Induction over SimpleExp

$$E \in SimpleExp ::= n|E + E|E \times E|\ldots$$

where $n \in N$.

Properties of $\Downarrow$

- **Determinacy**
  A simple expression can only evaluate to one answer.

$$E \Downarrow n_1 \wedge E \Downarrow n_2 \rightarrow n_1 = n_2$$

- **Totality**
  A simple expression evaluates to at least one answer.

$$\forall E \in SimpleExp.\exists n \in \mathbb{N}.[E \Downarrow n]$$

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Oliver Killane

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Lecture Recording

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Definition by Induction for SimpleExp

To define a function on all expressions in **SimpleExp**:

- define $f(n)$ directly, for each number n.
- define $f(E_1 + E_2)$ in terms of $f(E_1)$ and $f(E_2)$.
- define $f(E_1 \times E_2)$ in terms of $f(E_1)$ and $f(E_2)$.

For example, we can do this with *den*:

$$den(E) = n \leftrightarrow E \Downarrow n$$

Evaluation

Many Steps of Evaluation

Given $\to$ we can define a new relation $\to^*$ as:

$$E \to^* E' \leftrightarrow (E = E' \lor E \to E_1 \to E_2 \to \cdots \to E_k \to E')$$

For expressions, the final answer is n if $E \to^* n$.

Multi-Step Reductions

The relation $E \to^n E'$ is defined using mathematics induction by:

- **Base Case**
  $E \to^0 E$ for all $E \in \mathit{SimpleExp}$
- **Inductive Case**
  For every $E, E' \in \mathit{SimpleExp}$, $E \to^{k+1} E'$ if and only if there is some E'' such that:
  $$E \to^k E'' \land E'' \to E'$$
- **Definition**
  $\to^*$ - there are some number of steps to evaluate to E'.
  $$E \to^* E' \Leftrightarrow \exists n.[E \to^n E']$$

Properties of $\to$

- **Determinacy** If $E \to E_1$ and $E \to E_2$ then $E_1 = E_2$.
- **Confluence** If $E \to^* E_1$ and $E \to^* E_2$ then there exists E' such that $E_1 \to^* E'$ and $E_2 \to^* E'$.

- **Unique answer** If $E \rightarrow^* n_1$ and $E \rightarrow^* n_2$ then $n_1 = n_2$.
- **Normal Forms** Normal form is numbers ($\mathbb{N}$) for any E, $E = n$ or $E \rightarrow E'$ for some E'.
- **Normalisation** No infinite sequences of expressions $E_1, E_2, E_3, \ldots$ such that for all $i \in \mathbb{N}$ $E_1 \rightarrow E_{i+1}$ (Every path goes to a normal form).

Confluence of Small Step

We can prove a lemma expressing confluence:

$L_1 : \forall n \in \mathbb{N}.\forall E, E_1, E_2 \in \mathit{SimpleExp}.[E \rightarrow^n E_1 \wedge E \rightarrow^* E_2 \Rightarrow \exists E' \in \mathit{SimpleExp}.[E_1 \rightarrow^* E' \wedge E_2 \rightarrow^* E']]$

Lemma $\Rightarrow$ Confluence

Confluence is: $\forall E, E_1, E_2 \in \mathit{SimpleExp}.[E \rightarrow^* E_1 \wedge E \rightarrow^* E_2 \Rightarrow \exists E' \in \mathit{SimpleExp}.[E_1 \rightarrow^* E' \wedge E_2 \rightarrow^* E']]$ From lemma L_1

(1)	Take some arbitrary $E, E_1, E_2 \in \mathit{SimpleExp}$, assume confluence holds.	(Initial Setup)
(2)	$E \rightarrow^* E_1$	(By Confluence)
(3)	$\exists n \in \mathbb{N}.[E \rightarrow^n E_1]$	(By 2 & definition of $\rightarrow^*$)
(4)	Hence L_1	(By 3)

Determinacy of Small Step

We create a property P:

$$P(E) \stackrel{def}{=} \forall E_1, E_2 \in \mathit{SimpleExp}.[E \rightarrow E_1 \wedge E \rightarrow E_2 \Rightarrow E_1 = E_2]$$

There are 3 rules that apply:

$$(\text{A})\frac{}{n_1 + n_2 \rightarrow n}\ n = n_1 + n_2 \quad (\text{B})\frac{E \rightarrow E'}{n + E \rightarrow n + E'} \quad (\text{C})\frac{E_1 \rightarrow E_1'}{E_1 + E_2 \rightarrow E_1' + E_2}$$

Base Case

Take arbitrary $n \in \mathbb{N}$ and $E_1, E_2 \in \mathit{SimpleExp}$ such that $n \rightarrow E_1 \wedge n \rightarrow E_2$ to show $E_1 = E_2$.

(1)	$n \not\rightarrow$	(By inversion on A,B & C)
(2)	$\neg(n \rightarrow E_1)$	(By 1)
(3)	$\neg(n \rightarrow E_1 \wedge n \rightarrow E_2)$	(By 2)
(4)	$n \rightarrow E_1 \wedge n \rightarrow E_2 \Rightarrow E_1 = E_2$	(By 3)
(5)	$E \rightarrow E_1 \wedge E \rightarrow E_2 \Rightarrow E_1 = E_2$	(By 4)

Hence $P(n)$

Inductive Step

Take arbitrary E, E_1, E_2 such that $E = E_1 + E_2$
Inductive Hypothesis:

$$IH_1 = P(E_1)$$

$$IH_2 = P(E_2)$$

Assume there exists $E_3, E_4 \in \mathit{SimpleExp}$ such that $E_1 + E_2 \to E_3$ and $E_1 + E_2 \to E_4$.
To show $E_3 = E_4$.

From inversion on A, B & C there are 3 cases to consider:
For A:

(1)	There exists $n_1, n_2 \in \mathbb{N}$ such that $E_1 = n_1$ and $E_2 = n_2$	(By case A)
(3)	$E_3 = n_1 + n_2$	(By 1, A)
(4)	$E_4 = n_1 + n_2$	(By 1, A)
(5)	$E_3 = E_4$	(By 3 & 4)

For B:

(1)	There exists $n \in \mathbb{N}$ such that $E_1 = n$	(By case B)
(2)	There exists $E' \in \mathit{SimpleExp}$ such that $E_2 \to E'$	(By case B)
(3)	$E_3 = n + E'$	(By case B)
(4)	There exists $E'' \in \mathit{SimpleExp}$ such that $E_2 \to E''$	(By case B)
(5)	$E_4 = n + E''$	(By case B)
(6)	$E' = E''$	(By IH_2)
(7)	$E_3 = E_4$	(By 3,5 & 6)

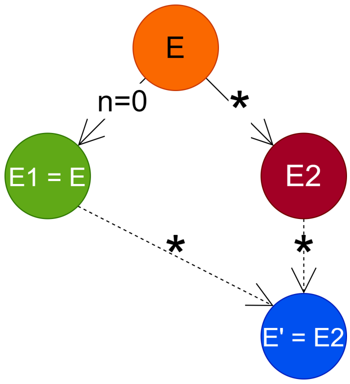
For C:

(1)	There exists $E' \in \mathit{SimpleExp}$ such that $E_1 \to E'$	(By case C)
(2)	There exists $E'' \in \mathit{SimpleExp}$ such that $E_1 \to E''$	(By case C)
(3)	$E_3 = E' + E_2$	(By case C)
(4)	$E_4 = E'' + E_2$	(By case C)
(5)	$E' = E''$	(By IH_1)
(6)	$E_3 = E_4$	(By 3,4 & 5)

(If E reduces to E_1 in n steps, and to E_2 in some number of steps, then there must be some E' that E_1 and E_2 reduce to.)

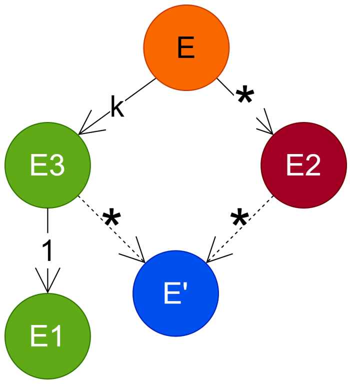
Base Case

The base cases has $n = 0$. Hence $E = E_1$, and hence $E_1 \to^* E_2$ and $E_1 \to^* E'$

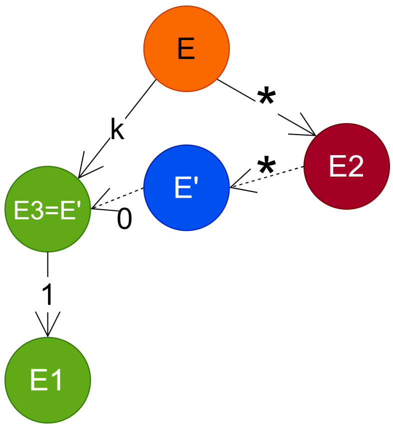


Inductive Case

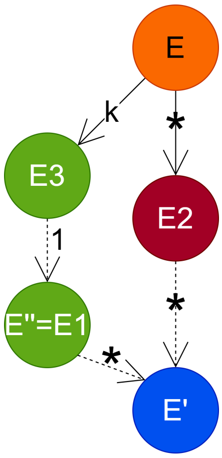
Next we assume confluence for up to k steps, and attempt to prove for $k+1$ steps.



We have two cases:
Case 1: $E_3 = E'$, this is easy as $E_2 \to^* E' \to^0 E3 \to^1 E1$.



Case 2: $E_3 \to^1 E'' \to^* E'$, in this case as $E_3 \to^1 E1$ we know by determinacy that $E'' = E_1$ and hence $E_1 \to^* E'$.


E
k
*
E3
E2
1
*
E"=E1
*
E'

50003 - Models of Computation - (Dr Raad) Lecture 7

Oliver Killane

02/11/21

Lecture Recording

Lecture recording is available here

Note for reader

We will reference to state by set $State \triangleq (Var \rightharpoonup \mathbb{N})$.

Lemmas

Lemma

A small proven proposition that can be used in a proof. Used to make the proof smaller.

Also know as an "auxiliary theorem" or "helper theorem".

Corollary

A theorem connected by a short proof to another existing theorem.

If B is can be easily deduced from A (or is evident in A's proof) then B is a corollary of A.

Lemmas

1. $\forall r \in \mathbb{N}.\forall E_1, E_1', E_2 \in \mathit{SimpleExp}.[E_1 \rightarrow^r E_1' \Rightarrow (E_1 + E_2) \rightarrow^r (E_1' + E_2)]$
2. $\forall r, n \in \mathbb{N}.\forall E_2, E_2' \in \mathit{SimpleExp}.[E_2 \rightarrow^r E_2' \Rightarrow (n + E_2) \rightarrow^r (n + E_2')]$

Corollaries

1. $\forall n_1 \in \mathbb{N}.\forall E_1, E_2 \in \mathit{SimpleExp}.[E_1 \rightarrow^* n_1 \Rightarrow (E_1 + E_2) \rightarrow^* (n_1 + E_2)]$
2. $\forall n_1, n_2 \in \mathbb{N}.\forall E_2 \in \mathit{SimpleExp}.[E_2 \rightarrow^* n_2 \Rightarrow (n_1 + E_2) \rightarrow^* (n_1 + n_2)]$
3. $\forall n, n_1, n_2, \in \mathbb{N}.\forall E_1, E_2 \in \mathit{SimpleExp}.[E_1 \rightarrow^* n_1 \land E_2 \rightarrow^* n_2 \land n = n_1 + n_2 \Rightarrow (E_1 + E_2) \rightarrow^* n]$

Connecting $\Downarrow$ and $\rightarrow^*$ for SimpleExp

$$\forall E \in \mathit{SimpleExp}, n \in \mathbb{N}.[E \Downarrow n \Leftrightarrow E \rightarrow^* n]$$

We prove each direction of implication separately. First we prove by induction over E using the property P:

$$P(E) =^{def} \forall n \in \mathbb{N}.[E \Downarrow n \Rightarrow E \rightarrow^* n]$$

Base Case

Take arbitrary $m \in \mathbb{N}$ to show $P(m) = m \Downarrow n \Rightarrow m \to^* n$.

(1)	Assume $m \Downarrow n$	
(2)	$m = n$	(From Inversion of $\Downarrow$)
(3)	$m \to^* n$	(By 2 and definition of $\to^*$)

Inductive Step

Take some arbitrary E, E_1, E_2 such that $E = E_1 + E_2$.
Inductive Hypothesis

$$\forall n_1 \in \mathbb{N}.[E_1 \Downarrow n_1 \Rightarrow E_1 \to^* n_1]$$

$$\forall n_2 \in \mathbb{N}.[E_2 \Downarrow n_2 \Rightarrow E_2 \to^* n_2]$$

To show $P(E)$: $\forall n \in \mathbb{N}.[(E_1 + E_2) \Downarrow n \Rightarrow (E_1 + E_2) \to^* n]$.

(1)	Assume $(E_1 + E_2) \Downarrow n$	
(2)	$\exists n_1, n_2 \in \mathbb{N}.[E_1 \Downarrow n_1 \wedge E_2 \Downarrow n_2]$	(By 1 & definition of B-ADD)
(3)	$E_1 \to^* n_1$	(By 2 & IH)
(4)	$E_2 \to^* n_2$	(By 2 & IH)
(5)	Chose some $n \in \mathbb{N}$ such that $n = n_1 + n_2$	
(6)	$(E_1 + E_2) \to^* n$	(By 3,4,5 Corollary 3)
(7)	$E \to^* n$	(By 6, definition of E)

Hence assuming $E \Downarrow n$ implies $E \to^* n$, so $P(E)$.
Next we work the other way, to show:

$$\forall E \in \mathit{SimpleExp}.\forall n \in \mathbb{N}.[E \to^* n \Rightarrow E \Downarrow n]$$

(1)	Take arbitrary $E \in \mathit{SimplExp}$ such that $E \to^* n$	(Initial setup)
(2)	Take some $m \in \mathbb{N}$ such that $E \Downarrow m$	(By totality of $\Downarrow$)
(3)	$n = m$	(By 1,2 & uniqueness of result for $\to$)
(4)	$E \Downarrow n$	(By 3)

It is also possible to prove this without using normalisation and determinacy, by induction on E.

Multi-Step Reductions

Lemma:

$$\forall r \in \mathbb{N}.\forall E_1, E_1', E_2.[E_1 \to^r E_1' \Rightarrow (E_1 + E_2) \to^r (E_1' + E_2)]$$

To prove $\forall r \in \mathbb{N}.[P(r)]$ by induction on r:

Base Case

- Base case is $r = 0$.
- Prove that $P(0)$ holds.

Inductive Step

- Inductive Case is $r = k + 1$ for arbitrary $k \in \mathbb{N}$.
- Inductive hypothesis is $P(k)$.
- Prove $P(k + 1)$ using inductive hypothesis.

Proof of the Lemma

By induction on r: **Base Case:** Take some arbitrary $E_1, E_1', E_2 \in \mathit{SimpleExp}$ such that $E_1 \rightarrow^0 E_1'$.

$$\begin{array}{llc}
(1) & E_1 = E_1' & \text{(By definition of } \rightarrow^0\text{)} \\
(2) & (E_1 + E_2) = (E_1' + E_2) & \text{(By 1)} \\
(3) & (E_1 + E_2) \rightarrow^0 (E_1' + E_2) & \text{(By definition of } \rightarrow^0\text{)}
\end{array}$$

Inductive Step: Take arbitrary $k \in \mathbb{N}$ such that $P(k)$

$$\begin{array}{lcl}
(1) & \text{Take arbitrary } E_1, E_1', E_2 \text{ such that } E_1 \rightarrow E_1' & \text{(Initial setup)} \\
(2) & \text{Take arbitrary } E_1'' \text{ such that } E_1'' \rightarrow E_1' & \\
(3) & (E_1 + E_2) \rightarrow^k (E_1'' + E_2) & \text{(By 2 \& IH)} \\
(4) & (E_1'' + E_2) \rightarrow (E_1' + E_2) & \text{(By 2 \& rule S-LEFT)} \\
(5) & (E_1 + E_2) \rightarrow^{k+1} (E_1' + E_2) & \text{(3,4, definition of } \rightarrow^{k+1}\text{)}
\end{array}$$

Determinacy of $\rightarrow$ for Exp

We extend simple expressions configurations of the form $\langle E, s \rangle$.

$$E \in \mathit{Exp} ::= n | x | E + E | \ldots$$

Determinacy:

$$\forall E, E_1, E_2 \in \mathit{Exp}. \forall s, s_1, s_2 \in \mathit{State}. [\langle E, s \rangle \rightarrow \langle E_1, s_1 \rangle \wedge \langle E, s \rangle \rightarrow \langle E_2, s_2 \rangle \Rightarrow \langle E_1, s_1 \rangle = \langle E_2, s_2 \rangle]$$

We prove this using property P:

$$P(E, s) \triangleq \forall E_1, E_2 \in \mathit{Exp}. \forall s_1, s_2 \in \mathit{State}. [\langle E, s \rangle \rightarrow \langle E_1, s_1 \rangle \wedge \langle E, s \rangle \rightarrow \langle E_2, s_2 \rangle \Rightarrow \langle E_1, s_1 \rangle = \langle E_2, s_2 \rangle]$$

Base Case: $E = x$

Take arbitrary $n \in \mathbb{N}$ and $s \in \mathit{State}$ to show $P(n, s)$

$$\begin{array}{lcl}
(1) & \text{take } E_1 \in \mathit{Exp},\ s_1 \in \mathit{State} \text{ such that } \langle n, s \rangle \rightarrow \langle E_1, s_1 \rangle & \text{(Initial setup)} \\
(2) & \text{take } E_2 \in \mathit{Exp},\ s_2 \in \mathit{State} \text{ such that } \langle n, s \rangle \rightarrow \langle E_2, s_2 \rangle & \text{(Initial setup)} \\
(3) & n = E_1 \wedge s = s_1 & \text{(By 1 \& inversion on definition of E.NUM)} \\
(4) & n = E_2 \wedge s = s_2 & \text{(By 2 \& inversion on definition of E.NUM)} \\
(5) & E_1 = E_2 \wedge s_1 = s_2 & \text{(By 3 \& 4)} \\
(6) & \langle E_1, s_1 \rangle = \langle E_2, s_2 \rangle & \text{(By 5 \& definition of configurations)}
\end{array}$$

Base Case: $E = x$

Take arbitrary $x \in Var$ and $s \in State$ to show $P(n, s)$

(1)	take $E_1 \in \mathbb{N}$, $s_1 \in State$ such that $\langle x, s\rangle \to \langle E_1, s_1\rangle$	(Initial setup)
(2)	take $E_2 \in \mathbb{N}$, $s_2 \in State$ such that $\langle x, s\rangle \to \langle E_2, s_2\rangle$	(Initial setup)
(3)	$E_1 = s(x) \land s_1 = s$	(By 1 & inversion on definition of E.VAR)
(3)	$E_2 = s(x) \land s_2 = s$	(By 2 & inversion on definition of E.VAR)
(5)	$E_1 = E_2 \land s_1 = s_2$	(By 3 & 4)
(6)	$\langle E_1, s_1\rangle = \langle E_2, s_2\rangle$	(By 5 & definition of configurations)

...Inductive Step ...

Syntax of Commands

$$C \in Com ::= x := E | \text{if } B \text{ then } C \text{ else } C | C; C | skip | \text{while } B \text{ do } C$$

- **Determinacy**

  $\forall C, C_1, C_2 \in Com. \forall s, s_1, s_2 \in State. [\langle C, s\rangle \to_c \langle C_1, s_1\rangle \land \langle C, s\rangle \to_c \langle C_2, s_2\rangle \Rightarrow \langle C_1, s_1\rangle = \langle C_2, s_2\rangle]$

- **Confluence**

  $\forall C, C_1, C_2 \in Com. \forall s, s_1, s_2 \in State. [\langle C, s\rangle \to_c^* \langle C_1, s_1\rangle \land \langle C, s\rangle \to_c^* \langle C_2, s_2\rangle \Rightarrow \exists C' \in Com. \exists s' \in State. [\langle C_1, s_1\rangle \to$

- **Unique Answer**

  $$\forall C \in Com. s_1 s_2 \in State. [\langle C, s\rangle \to_c^* \langle skip, s_1\rangle \land \langle C, s\rangle \to_c^* \langle skip, s_2\rangle \Rightarrow s_1 = s_2]$$

- **No Normalisation**
  There exist derivations of infinite length for while.

Connecting $\Downarrow$ and $\to^*$ for While

1. $\forall E, n \in Exp. \forall s, s' \in State. [\langle E, s\rangle \Downarrow_e \langle n, s'\rangle \Leftrightarrow \langle E, s\rangle \to_e^* \langle n, s'\rangle]$
2. $\forall B, b \in Bool. \forall s, s' \in State. [\langle B, s\rangle \Downarrow_b \langle b, s'\rangle \Leftrightarrow \langle B, s\rangle \to_b^* \langle b, s'\rangle]$
3. $\forall C \in Com. \forall s, s' \in State. [\langle C, s\rangle \Downarrow_c \langle s'\rangle \Leftrightarrow \langle C, s\rangle \to_c^* \langle skip, s'\rangle]$

For Exp and $Bool$ we have proofs by induction on the structure of expressions/booleans.

For $\Downarrow_c$ it is more complex as the $\Downarrow_c \Leftarrow \to_c^*$ cannot be proven using totality. Instead **complete/strong induction** on length of $\to_c^*$ is used.

50003 - Models Of Computation - (Prof Wiklicky) Lecture 1

Oliver Killane

28/03/22

Algorithms

Lecture Recording

Lecture recording is available here

Hilbert's Entscheidungsproblem (Decision Problem)

A problem proposed by David Hilbert and Wilhem Ackermann in 1928. Considering if there is an algorithm to determine if any statement is universally valid (valid in every structure satisfying the axioms - facts within the logic system assumed to be true (e.g in maths $1+0 = 1$)).

This can be also be expressed as an algorithm that can determine if any first-order logic statement is provable given some axioms.

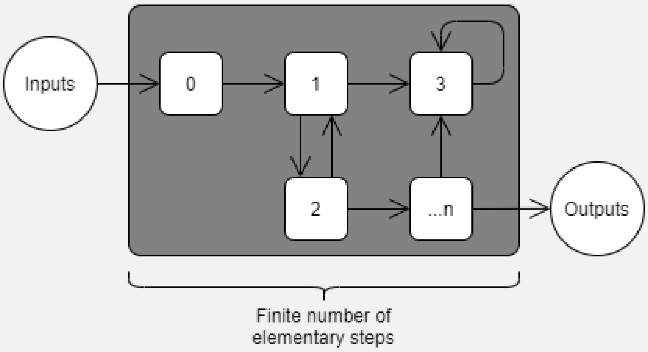
It was proven that no such algorithm exists by Alonzo Church and Alan Turing using their notions of Computing which show it is not computable.

Definition: Algorithms Informally

One definition is: *A finite, ordered series of steps to solve a problem.*

Common features of the many definitions of algorithms are:

- **Finite** Finite number of elementary (cannot be broken down further) operations.
- **Deterministic** Next step uniquely defined by the current.
- **Terminating?** May not terminate, but we can see when it does & what the result is.



Register Machines

Lecture Recording

Lecture recording is available here

Definition: Register Machine

A turing-equivalent (same computational power as a turing machine) abstract machine that models what is computable.

- Infinitely many registers, each storing a natural number ($\mathbb{N} \triangleq \{0, 1, 2, \dots\}$)
- Each instruction has a label associated with it.
- **3 Instructions**

$R_i^+ \rightarrow L_m$	Add 1 to register R_i and then jump to the instruction at L_m
$R_i^- \rightarrow L_n, L_m$	If $R_i > 0$ then decrement it and jump to L_n, else jump to L_m
HALT	Halt the program.

At each point in a program the registers are in a configuration $c = (l, r_0, \dots, r_n)$ (where r_i is the value of R_i and l is the instruction label L_l that is about to be run).

- c_0 is the initial configuration, next configurations are $c_1, c_2, \dots$.
- In a finite computation, the final configuration is the **halting configuration**.
- In a **proper halt** the program ends on a **HALT**.
- In an **erroneous halt** the program jumps to a non-existent instruction, the **halting configuration** is for the instruction immediately before this jump.

Example: Sum of three numbers

The following register machine computes:

$$R_0 = R_0 + R_1 + R_2 \quad R_1 = 0 \quad R_2 = 0$$

Or as a partial function:

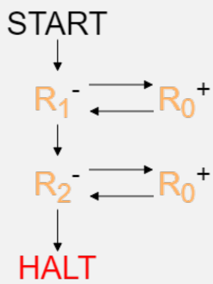
$$f(x, y, z) = x + y + z$$

Registers

$R_0 \quad R_1 \quad R_2$

Program

$L_0 :\ {R_1}^- \to L_1, L_2$
$L_1 :\ {R_0}^+ \to L_0$
$L_2 :\ {R_2}^- \to L_3, L_4$
$L_3 :\ {R_0}^+ \to L_2$
$L_4 :$ **HALT**



Example Configuration

L_i	R_0	R_1	R_2
0	1	2	3
1	1	1	3
0	2	1	3
1	2	0	3
0	3	0	3
2	3	0	3
3	3	0	2
2	4	0	2
3	4	0	1
2	5	0	1
3	5	0	0
2	6	0	0
4	6	0	0

Partial Functions

Definition: Partial Function

A partial function maps some members of the domain X, with each mapped member going to at most one member of the codomain Y.

$$f \subseteq X \times Y \quad \text{and} \quad (x, y_1) \in f \land (x, y_2) \in f \Rightarrow y_1 = y_2$$

$f(x) = y$	$(x, y) \in f$
$f(x) \downarrow$	$\exists y \in Y.[f(x) = y]$
$f(x) \uparrow$	$\neg \exists y \in Y.[f(x) = y]$
$X \rightharpoonup Y$	Set of all **partial functions** from X to Y.
$X \to Y$	Set of all **total functions** from X to Y.

A partial function from X to Y is total if it satisfies $f(x) \downarrow$.

Register machines can be considered as partial functions as for a given input/initial configuration, they produce at most one halting configuration (as they are deterministic, for non-finite computations/non-halting there is no halting configuration).

We can consider a register machine as a partial function of the input configuration, to the value of the first register in the halting configuration.

$$f \in \mathbb{N}^n \rightharpoonup \mathbb{N} \text{ and } (r_0, \ldots, r_n) \in \mathbb{N}^n, r_0 \in \mathbb{N}$$

Note that many different register machines may compute the same partial function.

Computable Functions

The following arithmetic functions are computable. Using them we can derive larger register machines for more complex arithmetic (e.g logarithms making use of repeated division).

Projection

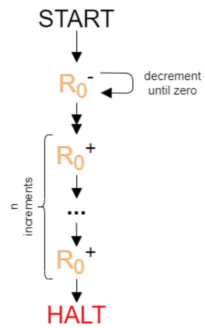
$$p(x, y) \triangleq x$$
$$(r_0, r_1) \to r_0$$

HALT

Constant

$$c(x) \triangleq n$$
$$(r_0) \to n$$

$$\begin{array}{ll} L_0: & {R_0}^- \to L_0, L_1 \\ L_1: & {R_0}^+ \to L_2 \\ \vdots & \vdots \\ L_n: & {R_0}^+ \to L_{n+1} \\ L_{n+1}: & \textbf{HALT} \end{array}$$

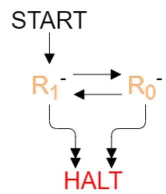


Truncated Subtraction

$$x - y \triangleq \begin{cases} x - y & y \le x \\ 0 & y > x \end{cases}$$

$$(r_0, r_1) \to r_0 - r_1$$

$$\begin{array}{ll} L_0: & {R_1}^- \to L_1, L_2 \\ L_1: & {R_0}^- \to L_0, L_2 \\ L_2: & \textbf{HALT} \end{array}$$

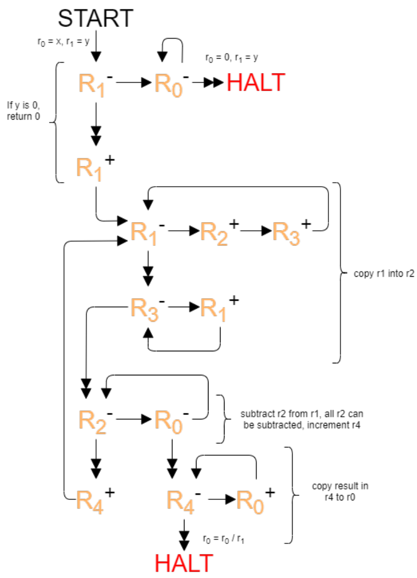


Integer Division

Note that this is an inefficient implementation (to make it easy to follow) we could combine the halts and shortcut the initial zero check (so we don't increment, then re-decrement).

$$x \; div \; y \triangleq \begin{cases} \left\lfloor \dfrac{x}{y} \right\rfloor & y > 0 \\ 0 & y = 0 \end{cases}$$

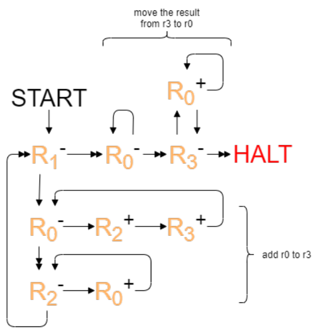
$$\begin{array}{ll}
L_0: & R_1^- \to L_3, L_2 \\
L_1: & R_0^- \to L_1, L_2 \\
L_2: & \textbf{HALT} \\
L_3: & R_1^+ \to L_4 \\
L_4: & R_1^- \to L_5, L_7 \\
L_5: & R_2^+ \to L_6 \\
L_6: & R_3^+ \to L_4 \\
L_7: & R_3^- \to L_8, L_9 \\
L_8: & R_1^+ \to L_9 \\
L_9: & R_2^- \to L_{10}, L_4 \\
L_{10}: & R_0^- \to L_9, L_{11} \\
L_{11}: & R_4^- \to L_{12}, L_{13} \\
L_{12}: & R_0^+ \to L_{11} \\
L_{13}: & \textbf{HALT}
\end{array}$$



Multiplication

$$x \times y$$

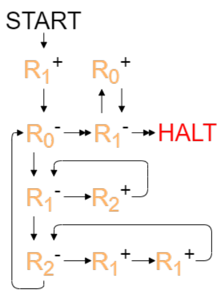
$$\begin{array}{ll}
L_0: & R_1^- \to L_5, L_1 \\
L_1: & R_0^- \to L_1, L_2 \\
L_2: & R_3^- \to L_3, L_4 \\
L_3: & R_0^+ \to L_2 \\
L_4: & \textbf{HALT} \\
L_5: & R_0^- \to L_6, L_8 \\
L_6: & R_2^+ \to L_7 \\
L_7: & R_3^+ \to L_5 \\
L_8: & R_2^- \to L_9, L_0 \\
L_9: & R_0^+ \to L_8
\end{array}$$



Exponent of base 2

$$e(x) \triangleq 2^x$$

$$\begin{array}{ll}
L_0: & R_1^+ \to L_1 \\
L_1: & R_0^- \to L_5, L_2 \\
L_2: & R_1^- \to L_3, L_4 \\
L_3: & R_0^+ \to L_2 \\
L_4: & \textbf{HALT} \\
L_5: & R_1^- \to L_6, L_7 \\
L_6: & R_2^+ \to L_5 \\
L_7: & R_2^- \to L_8, L_1 \\
L_8: & R_1^+ \to L_9 \\
L_9: & R_1^+ \to L_7
\end{array}$$



Encoding Programs as Numbers

Lecture Recording

Lecture recording is available here

Definition: Halting Problem

Given a set S of pairs (A, D) where A is an algorithm and D is some input data A operates on $(A(D))$.

We want to create some algorithm H such that:

$$H(A, D) \triangleq \begin{cases} 1 & A(D) \downarrow \\ 0 & otherwise \end{cases}$$

Hence if $A(D) \downarrow$ then $A(D)$ eventually halts with some result.

We can use proof by contradiction to show no such algorithm H can exist.

Assume an algorithm H exists:

$$B(p) \triangleq \begin{cases} halts & H(p(p)) = 0 \quad (p(p) \text{ does not halt}) \\ forever & H(p(p)) = 1 \quad (p(p) \text{ halts}) \end{cases}$$

Hence using H on any $B(p)$ we can determine if $p(p)$ halts $(H(B(p)) \Rightarrow \neg H(p(p)))$.

Now we consider the case when $p = B$.

- $B(B)$ **halts** Hence $B(B)$ does not halt. Contradiction!
- $B(B)$ **does not halt** Hence $B(B)$ halts. Contradiction!

Hence by contradiction there is not such algorithm H.

In order to reason about programs consuming/running programs (as in the halting problem), we need a way to encode programs as data. Register machines use natural numbers as values for input, and hence we need a way to encode any register machine as a natural number.

Pairs

$\langle\langle x, y\rangle\rangle$	$= 2^x(2y+1)$	$y\ 1\ 0_1 \dots 0_x$	Bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}^+ = \{n \in \mathbb{N} \mid n \neq 0\}$
$\langle x, y\rangle$	$= 2^x(2y+1) - 1$	$y\ 0\ 1_1 \dots 1_x$	Bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$

Lists

We can express lists and right-nested pairs.

$$[x_1, x_2, \dots, x_n] = x_1 : x_2 : \dots : x_n = (x_1, (x_2, (\dots, x_n) \dots))$$

We use zero to define the empty list, so must use a bijection that does not map to zero, hence we use the pair mapping $\langle\langle x, y\rangle\rangle$.

$$l : \begin{cases} \ulcorner [] \urcorner \triangleq 0 \\ \ulcorner x_1 :: l_{inner} \urcorner \triangleq \langle\langle x, \ulcorner l_{inner} \urcorner \rangle\rangle \end{cases}$$

Hence:

$$\ulcorner x_1, \dots, x_n \urcorner = \langle\langle x_1, \langle\langle \dots, x_n \rangle\rangle \dots \rangle\rangle$$

Instructions

$$\ulcorner {R_i}^+ \to L_n \urcorner = \langle\langle 2i, n \rangle\rangle$$
$$\ulcorner {R_i}^- \to L_n, L_m \urcorner = \langle\langle 2i + 1, \langle n, m \rangle \ \rangle\rangle$$
$$\ulcorner \textbf{HALT} \urcorner = 0$$

programs

Given some program:

$$\ulcorner \begin{pmatrix} L_0 : & instruction_0 \\ \vdots & \vdots \\ L_n : & instruction_n \end{pmatrix} \urcorner = \ulcorner [\ulcorner instruction_0 \urcorner, \dots, \ulcorner instruction_n \urcorner] \urcorner$$

Tools

In order to simplify checking workings, I have created a basic python script for running, encoding and decoding register machines.

It is designed to be used in the python shell, to allow for easy manipulation, storing, etc of register machines, encoding/decoding results.

It also produces latex to show step-by-step workings for calculations.

```
from typing import List, Tuple
from collections import namedtuple

# Register Instructions
Inc = namedtuple('Inc', 'reg label')
Dec = namedtuple('Dec', 'reg label1 label2')
Halt = namedtuple('Halt', '')

'''
 --           -------- -      ----- ----  --  --  ------
 \ \         / /  ----| |     / ----/ -- \|  \/  |   ----|
  \ \  /\  / /| |--  | |    | |   | |  | | \  / | |--
   \ \/  \/ / |  --| | |    | |   | |  | | |\/| |  --|
    \  /\  /  | |----| |---| |---| |--| | |  | | |----
     \/  \/   |------|------\-----\----/|-|  |-|------|

This file can be used to quickly create, run, encode & decode register machine
programs. Furthermore it prints out the workings as formatted latex for easy
use in documents.
```

```

Here making use of python's ints as they are arbitrary size (Rust's bigInts
are 3rd party and awful by comparison).

To create register Instructions simply use:
Dec(reg, label 1, label 2)
Inc(reg, label)
Halt()

To ensure your latex will compile, make sure you have commands for, these are
available on my github (Oliver Killane) (Imperial-Computing-Year-2-Notes):

% register machine helper commands:
\newcommand{\instrlabel}[1]{\text{\textcolor{teal}{$L_{#1}$}}}
\newcommand{\reglabel}[1]{\text{\textcolor{orange}{$R_{#1}$}}}
\newcommand{\instr}[2]{\instrlabel{#1} : & $#2$ \\}
\newcommand{\dec}[3]{\reglabel{#1}^- \to \instrlabel{#2}, \instrlabel{#3}}
\newcommand{\inc}[2]{\reglabel{#1}^+ \to \instrlabel{#2}}
\newcommand{\halt}{\text{\textcolor{red}{\textbf{HALT}}}}

To see examples, go to the end of this file.
'''

# for encoding numbers as <a,b>
def encode_large(x: int, y: int) -> int:
    return (2 ** x) * (2 * y+1)

# for decoding n -> <a,b>
def decode_large(n: int) -> Tuple[int, int]:
    x = 0;

    # get zeros from LSB
    while (n % 2 == 0 and n != 0):
        x += 1
        n /= 2
    y = int((n - 1) // 2)
    return (x,y)

# for encoding <<a,b>> -> n
def encode_small(a: int, b: int) -> int:
    return encode_large(a,b) - 1

# for decoding n -> <<a,b>>
def decode_small(n: int) -> Tuple[int, int]:
    return decode_large(n+1)

# for encoding [a0,a1,a2,...,an] -> <<a0, <<a1, <<a2, <<... <<an, 0 >>...>> >> >> >> -> n
def encode_large_list(lst: List[int]) -> int:
    return encode_large_list_helper(lst, 0)[0]

def encode_large_list_helper(lst: List[int], step: int) -> Tuple[int, int]:
    buffer = r"\to" * step
    if (step == 0):
        print(r"\begin{center}\begin{tabular}{r l l}")
    if len(lst) == 0:
        print(f"{step} &" + rf"$ {buffer} 0$ & (No more numbers in the list, can unwrap recursion) \\")
        return (0, step)
    else:
```

```python

        print(rf"{step} & $ {buffer} \langle \langle {lst[0]}, \ulcorner {lst[1:]} \urcorner \rangle \rangle $ & (Take next element {lst[0]}, and encode the rest {lst[1:]}) \\")

        (b, step2) = encode_large_list_helper(lst[1:], step + 1)
        c = encode_large(lst[0], b)

        step2 += 1

        print(f"{step2} & $ {buffer} \langle \langle {lst[0]}, {b} \\rangle \\rangle = {c} $ & (Can now encode) \\\\")

        if (step == 0):
            print(r"\end{tabular}\end{center}")
        return (encode_large(lst[0], b), step2)

# decode a list from an integer
def decode_large_list(n : int) -> List[int]:
    return decode_large_list_helper(n, [], 0)

def decode_large_list_helper(n : int, prev : List[int], step : int = 0) -> List[int]:
    if (step == 0):
        print(r"\begin{center}\begin{tabular}{r l l l}")
    if n == 0:
        print(rf"{step} & $0$ & ${prev}$ & (At the list end) \\")
        return prev
    else:
        (a,b) = decode_large(n)
        prev.append(a)
        print(rf"{step} & ${n} = \langle \langle {a}, {b} \rangle \rangle \ \ $&$ {prev}$ & (Decode into two integers) \\ ")

        next = decode_large_list_helper(b, prev, step + 1)

        if (step == 0):
            print(r"\end{tabular}\end{center}")

        return next

# For encoding register machine instructions
# R+(i) -> L(j)
def encode_inc(instr: Inc) -> int:
    encode = encode_large(2 * instr.reg, instr.label)
    print(rf"$\ulcorner \inc{{{instr.reg}}}{{{instr.label}}} \urcorner = \langle \langle 2 \times {instr.reg}, {instr.label} \rangle \rangle = {encode}$")
    return encode

# R-(i) -> L(j), L(k)
def encode_dec(instr: Dec) -> int:
    encode: int =  encode_large(2 * instr.reg + 1, encode_small(instr.label1 ,instr.label2))
    print(rf"$\ulcorner \dec {{{instr.reg}}}{{{instr.label1}}}{{{instr.label2}}} \urcorner = \langle \langle 2 \times {instr.reg} + 1, \langle {instr.label1}, {instr.label2} \rangle \rangle \rangle = {encode}$")
    return encode

# Halt
def encode_halt() -> int:
    print(rf"$\ulcorner \halt \urcorner = 0 $")
```

```python
    return 0

# encode an instruction
def encode_instr(instr) -> int:
    if type(instr) == Inc:
        return encode_inc(instr)
    elif type(instr) == Dec:
        return encode_dec(instr)
    else:
        return encode_halt()

# display register machine instruction in latex format
def instr_to_str(instr) -> str:
    if type(instr) == Inc:
        return rf"\inc{{{instr.reg}}}{{{instr.label}}}"
    elif type(instr) == Dec:
        return rf"\dec{{{instr.reg}}}{{{instr.label1}}}{{{instr.label2}}}"
    else:
        return r"\halt"

# decode an instruction
def decode_instr(x: int) -> int:
    if x == 0:
        return Halt()
    else:
        assert(x > 0)
        (y,z) = decode_large(x)
        if (y % 2 == 0):
            return Inc(int(y / 2), z)
        else:
            (j,k) = decode_small(z)
            return Dec(y // 2, j, k)

# encode a program to a number by encoding instructions, then list
def encode_program_to_list(prog : List) -> List[int]:
    encoded = []
    print(r"\begin{center}\begin{tabular}{r l l}")
    for (step, instr) in enumerate(prog):
        print(f"{step} & ")
        encoded.append(encode_instr(instr))
        print(r"& \\")
    print(r"\end{tabular}\end{center}")
    print(f"\[{encoded}\]")
    return encoded

# encode a program as an integer
def encode_program_to_int(prog: List) -> int:
    return encode_large_list(encode_program_to_list(prog))

# decode a program by decoding to a list, then decoding each instruction
def decode_program(n : int):
    decoded = decode_large_list(n)
    prog = []
    prog_str = []
    for num in decoded:
        instr = decode_instr(num)
        prog_str.append(instr_to_str(instr))
        prog.append(instr)
    print(f"\[ [ {', '.join(prog_str)} ] \]")
    return prog
```

```python

# print program in latex form
def program_str(prog) -> str:
    prog_str = []
    for (num, instr) in enumerate(prog):
        prog_str.append(rf"\instr{{{num}}}{{{instr_to_str(instr)}}}")
    print(r"\begin{center}\begin{tabular}{l l}")
    print("\n".join(prog_str))
    print(r"\end{tabular}\end{center}")

# run a register machine with an input:
def program_run(prog, instr_no : int, registers : List[int])-> Tuple[int, List[int]]:
    # step instruction label R0 R1 R2 ... (info)
    print(rf"\begin{{center}}\begin{{tabular}}{{l l l c" + " c" * len(registers) + " }")
    print(r"\textbf{Step} & \textbf{Instruction} & \instrlabel{{i}} &" + " & ".join([rf"$\reglabel{{{n}}}$" for n in range(0, len(registers))]) + r" & \textbf{Description}\\")
    print(r"\hline")
    step = 0
    while True:
        step_str = rf"{step} & ${instr_to_str(prog[instr_no])}$ & ${instr_no}$ & " + "&".join([f"${n}$" for n in registers]) + "&"
        instr = prog[instr_no]
        if type(instr) == Inc:
            if (instr.reg >= len(registers)):
                print(step_str + rf"(register {instr.reg} is does not exist)\\")
                break
            elif instr.label >= len(prog):
                print(step_str + rf"(label {instr.label} is does not exist)\\")
                break
            else:
                registers[instr.reg] += 1
                instr_no = instr.label
                print(step_str + rf"(Add 1 to register {instr.reg} and jump to instruction {instr.label})\\")
        elif type(instr) == Dec:
            if (instr.reg >= len(registers)):
                print(step_str + rf"(register {instr.reg} is does not exist)\\")
                break
            elif registers[instr.reg] > 0:
                if instr.label1 >= len(prog):
                    print(step_str + rf"(label {instr.label1} is does not exist)\\")
                    break
                else:
                    registers[instr.reg] -= 1
                    instr_no = instr.label1
                    print(step_str + rf"(Subtract 1 from register {instr.reg} and jump to instruction {instr.label1})\\")
            else:
                if instr.label2 >= len(prog):
                    print(step_str + rf"(label {instr.label2} is does not exist)\\")
                    break
                else:
                    instr_no = instr.label2
                    print(step_str + rf"(Register {instr.reg} is zero, jump to instruction {instr.label2})\\")
        else:
            print(step_str + rf"(Halt!)\\")
            break
```

```
        step += 1
    print(r"\end{tabular}\end{center}")
    print("\[(" + ", ".join([str(instr_no)] + list(map(str, registers))) + ")\]")
    return (instr_no, registers)

# Basic tests for program decode and encode
def test():
    prog_a = [
        Dec(1,2,1),
        Halt(),
        Dec(1,3,4),
        Dec(1,5,4),
        Halt(),
        Inc(0,0)]

    prog_b = [
        Dec(1,1,1),
        Halt()
    ]

    # set R0 to 2n for n+3 instructions
    prog_c = [
        Inc(1,1),
        Inc(0,2),
        Inc(0,3),
        Inc(0,4),
        Inc(0,5),
        Inc(0,6),
        Inc(0,7),
        Dec(1, 0, 9),
        Halt()
    ]

    assert decode_program(encode_program_to_int(prog_a)) == prog_a
    assert decode_program(encode_program_to_int(prog_b)) == prog_b
    assert decode_program(encode_program_to_int(prog_c)) == prog_c

# Examples usage
def examples():
    program_run([
        Dec(1,2,1),
        Halt(),
        Dec(1,3,4),
        Dec(1,5,4),
        Halt(),
        Inc(0,0)
    ], 0, [0,7])

    encode_program_to_list([
        Inc(1,1),
        Inc(0,2),
        Inc(0,3),
        Inc(0,4),
    ])

    encode_program_to_int([
        Dec(1,2,1),
        Halt(),
        Dec(1,3,4),
        Dec(1,5,4),
```

```
            Halt(),
            Inc(0,0)
        ])

        decode_program((2 ** 46) * 20483)

examples()
```

50003 - Models Of Computation - (Prof Wiklicky) Lecture 2

Oliver Killane

31/03/22

Lecture Recording

Lecture recording is available here

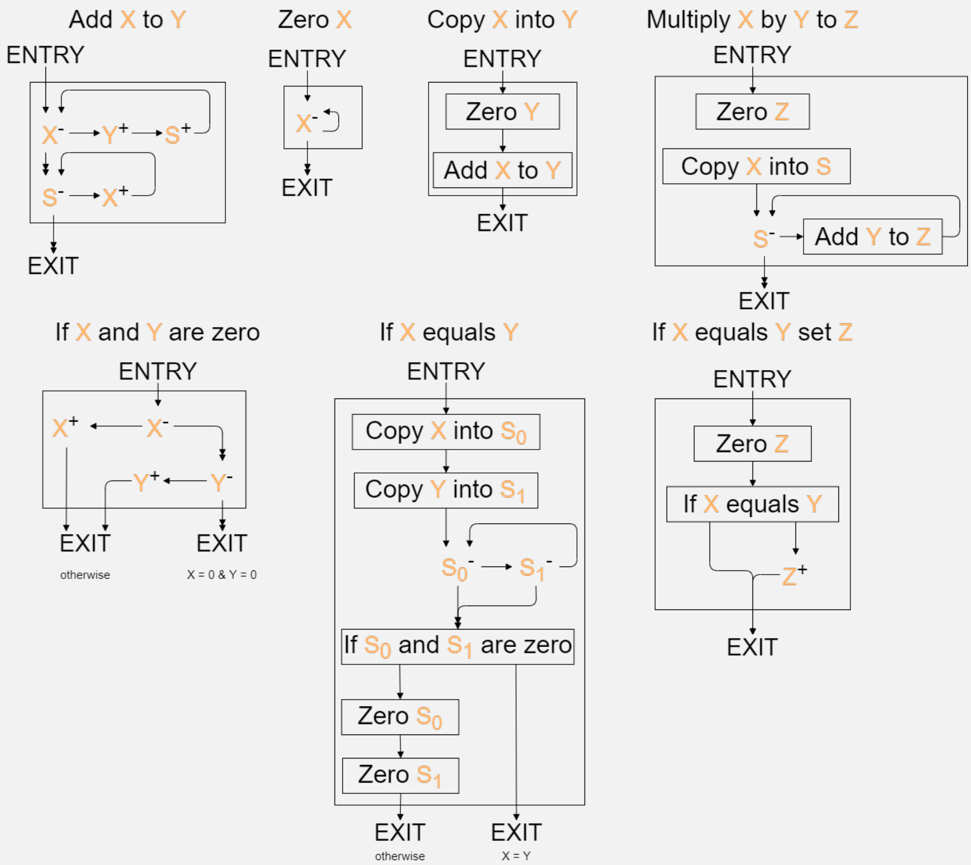
Gadgets

Definition: Register Machine Gadget

A gadget is a partial register machine graph, used as components in more complex programs, that can be composed into larger register machines or gadgets.

- Has a single $ENTRY$ (much like $START$).
- Can have many $EXIT$ (much like **HALT**).
- Operates on registers specified in the name of the gadget (e.g *"Add R_1 to R_2"*).
- Can use scratch registers (assumed to be zero prior to gadget and set to zero by the gadget before it exits - allows usage in loops)
- We can rename the registers used in gadgets (simply change the registers used in the name (*push R_0 to R_1* $\to$ *push X to Y*), and have all scratch registers renamed to registers unused by other parts of the program)

For example we can create several gadgets in terms of registers that we can rename.



And then can use these to create larger programs.

Analysing Register Machines

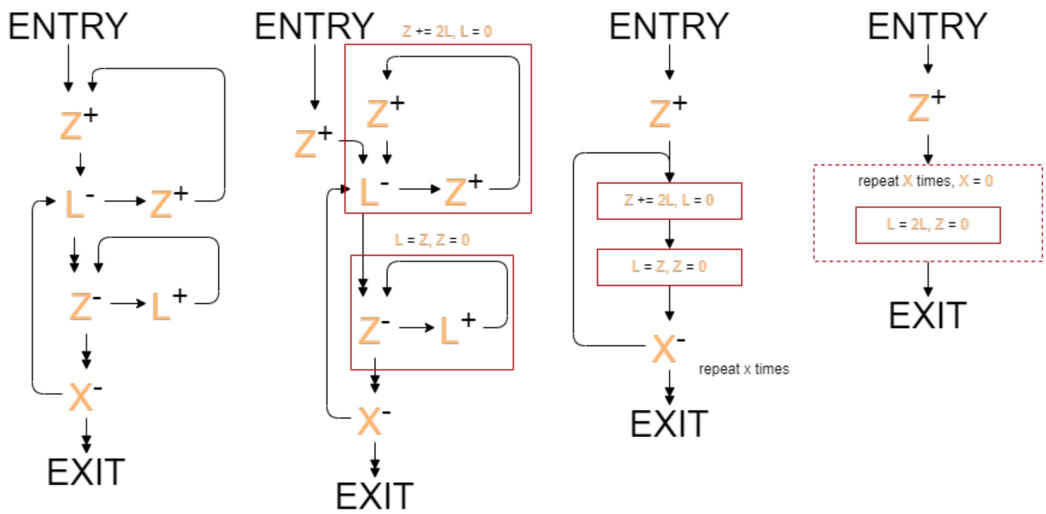
There is no general algorithm for determining the operations of a register machine (i.e halting problem)

However there are several useful strategies one can use:

- **Experimentation**
  Can create a table of input values against outputs to attempt to fetermine the relation - however the values could match many different relations.

- **Creating Gadgets**
  We can group instructions together into gadgets to identify simple behaviours, and continue to merge to develop an understanding of the entire machine.

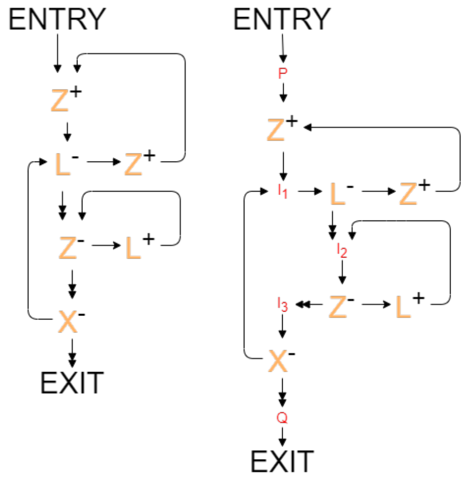
  For example below, we can deduce the result as $L = 2^X(2L + 1)$



- **Invariants**
  We can use logical assertions on the register machine state at certain instructions, both to get the result of the register machine, and to prove the result.

  If correct, every execution path to a given instruction's invariant, establishes that invariant.

  We could attach invariants to every instruction, however it is usually only necessary to use them at the start, end and for loops (preconditions/postconditions).



Our first invariant (P) can be defined as:

$$P \equiv (X = x \wedge L = l \wedge Z = 0)$$

Next we can use the instructions between invariant to find the states under which the invariants must hold.

1.	$P[Z-1/Z] \Rightarrow I_1$	After incrementing Z needs to go to the start of the first loop.
2.	$I_1[L+1/L, Z-2/Z] \Rightarrow I_1$	The loop decrements L and increases Z by two. After each loop iteration, I_1 must still hold.
3.	$I_1 \wedge L = 0 \Rightarrow I_2$	If $L = 0$ the loop is escaped, and we move to I_2.
4.	$I_2[Z+1/Z, L-1/L] \Rightarrow I_2$	Loop increments L and decrements Z on each iteration, after this, I_2 must still hold.
5.	$I_2 \wedge Z = 0 \Rightarrow I_3$	Loop ends when $Z = 0$, moves to I_3.
6.	$I_3[X+1/X] \Rightarrow I_1$	Large loop decrements X on each iteration, invariant must hold with the new/decremented X.
7.	$I_3 \wedge X = 0 \Rightarrow Q$	When the main X-decrementing loop is escaped, we move to exit, so Q must hold.

We can now use these constraints (also called **verification conditions**) to determine an invariant.

For each constraint we do:

1. Get the basic for (potentially one already derived) for the invariant in question.
2. If there is iteration, iterate to build up a disjunction.
3. Find the pattern, and then re-form the invariant based on it.

Constraint 1.

Hence we can deduce I_1 as:

$$I_1 = (X = x \wedge L = l \wedge Z = 1)$$

(Take P and increment Z)

Constraint 2.

We can iterate to get the disjunction:

$I_1 \equiv (X = x \wedge L = l \wedge Z = 1) \vee (X = x \wedge L+1 = l \wedge Z-2 = 1) \vee (X = x \wedge L+2 = l \wedge Z-4 = 1) \vee \ldots$

Hence we can determine the pattern for each disjunct as:

$$Z + 2L = 2l + 1$$

Hence we create our invariant:

$$I_1 = (X = x \wedge Z + 2L = 2l + 1)$$

Constraint 3.

Hence as $L = 0$ we can determine that $Z = 2l + 1$.

$$I_2 = (X = x \wedge Z = 2l + 1 \wedge L = 0)$$

Constraint 4.

We iterate to get the disjunction:

$I_2 = (X = x \wedge Z = 2l+1 \wedge L = 0) \vee (X = x \wedge Z = 2l+0 \wedge L = 1) \vee (X = x \wedge Z = 2l-1 \wedge L = 2) \vee \ldots$

Hence we notice the pattern:

$$Z + L = 2l + 1$$

So can deduce the invariant:

$$I_2 = (X = x \wedge Z + L = 2l + 1)$$

Constraint 5.

We can derive an invariant I_3 using $Z = 0$.

$$I_3 = (X = x \wedge L = 2l + 1 \wedge Z = 0)$$

Constraint 6.

We can use the constraint, and the currently derived I_1 to get a disjunction:

$$I_1 = (X = x - 1 \wedge L = 2l + 1 \wedge Z = 0) \vee (X = x \wedge Z + 2L = 2l + 1)$$

We can apply constraint **2.** on the first part of this disjunction, iterating to get the disjunction:

$$I_1 = (X = x \wedge Z + 2L = 2l + 1) \vee \begin{pmatrix} (X = x - 1 \wedge L = 2l + 1 \wedge Z = 0)\vee \\ (X = x - 1 \wedge L = 2l + 0 \wedge Z = 2)\vee \\ (X = x - 1 \wedge L = 2l - 1 \wedge Z = 4)\vee \\ (X = x - 1 \wedge L = 2l - 2 \wedge Z = 8) \vee \ldots \end{pmatrix}$$

Hence for the second group of disjuncts we have the relation:

$$Z + 2L = 2(2l + 1)$$

Hence we have:

$$I_1 = (X = x \wedge Z + 2L = 2l + 1) \vee (X = x - 1 \wedge Z + 2L = 2(2l + 1))$$

Hence when we repeat on the larger loop, we will double again, iterating we get:

$$I_1 = (X = x \wedge Z + 2L = 2l + 1) \vee (X = x - 1 \wedge Z + 2L = 2(2l + 1)) \vee (X = x - 2 \wedge Z + 2L = 4(2l + 1)) \vee \ldots$$

Hence we have the relation:

$$I_1 = (Z + 2L = 2^{X-x}(2l + 1))$$

We can apply this doubling to L_2 also as it forms part of the larger loop:

$$I_2 = (Z + L = 2^{X-x}(2l + 1))$$

And to I_3:

$$I_3 = (L = 2^{X-x}(2l + 1) \wedge Z = 0)$$

Constraint 7.

Hence we can now derive Q as:

$$Q = (L = 2^{x}(2l + 1) \wedge Z = 0)$$

Termination

We also need to show that each of our loops eventually terminate, we can do this by showing that sme variant (e.g register, or combination of) decreases every time the invariant is reached/visited.

For I_1 we can use the lexicographical ordering (X, L) as in each inner loop L decreases, but for the larger loop while L is reset/does not increase, X does.

For I_2 we can similarly use the lexicographical ordering (X, Z)

For I_3 we can just use X.

Universal Register Machine

A register machine that simulates a register machine.

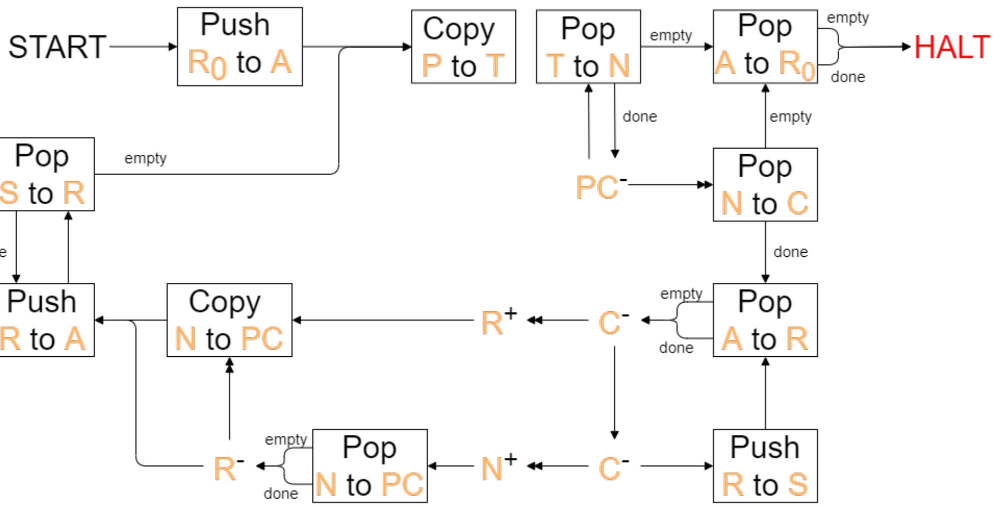
It takes the arguments:

- R_0 $= 0$
- R_1 $=$ the program encoded as a number
- R_2 $=$ the argument list encoded as a number
- **All other registers zeroed**

The registers used are:

R_1	P	Program code of the register machine being simulated/emulated.
R_2	A	Arguments provided to the simulated register machine.
R_3	PC	Program Counter - The current register machine instruction.
R_4	N	Next label num,ber/next instruction to go to. Is also used to store the current instruction
R_5	C	The current instruction.
R_6	R	The value of the register used by the current instruction.
R_7	S	Auxiliary Register
R_8	T	Auxiliary Register
$R_9 \ldots$		Scratch Registers

```
while true:
    if PC >=  length P:
        HALT!

    N = P[PC]

    if N == 0:
        HALT!

    (curr, next) = decode(N)
    C = curr
    N = next

    # either C = 2i (R+) or C = 2i + 1 (R-)
    R = A[C // 2]

    # Execute C on data R, get next label and write back to registers
    (PC, R_new) = Execute(C, R)
    A[C//2] = R_new
```


START
Push
R0 to A
Copy
P to T
Pop
T to N
empty
Pop
A to R0
empty
HALT
done
done
empty
PC-
Pop
N to C
Pop
S to R
empty
done
done
Push
R to A
Copy
N to PC
R+
C-
empty
done
Pop
A to R
empty
Pop
N to PC
R-
done
N+
C-
Push
R to S

50003 - Models Of Computation - (Prof Wiklicky) Lecture 3

Oliver Killane

31/03/22

Lecture Recording
Lecture recording is available here

Halting Problem for Register Machines

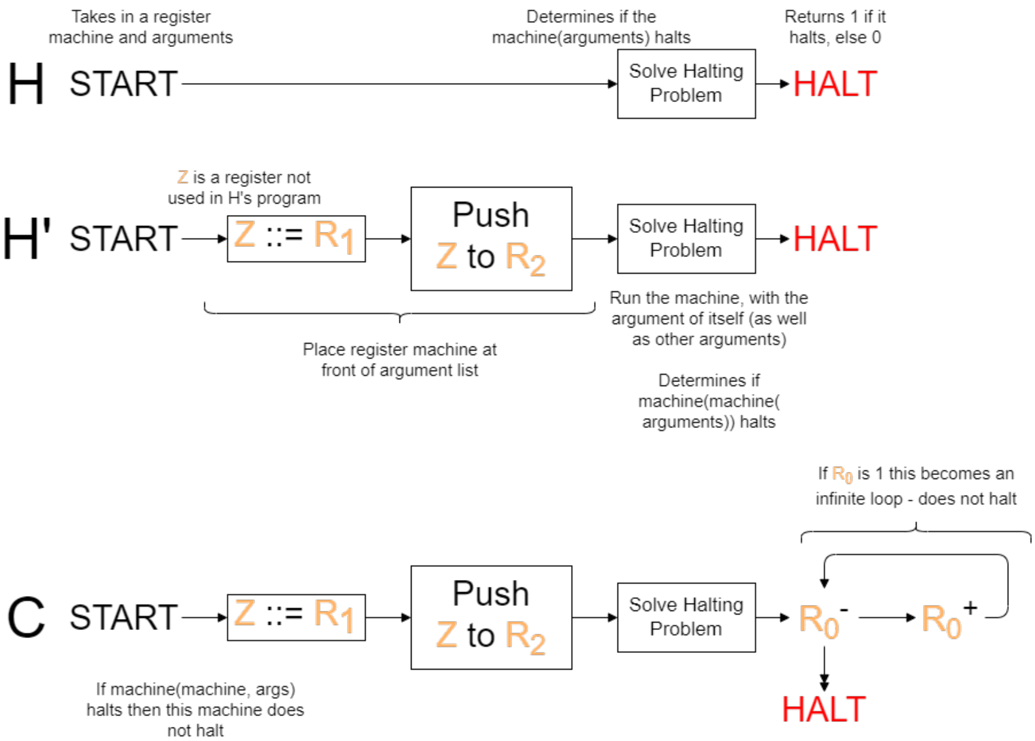
A register machine H decides the halting problem if for all $e, a_1, \ldots, a_n \in \mathbb{N}$:

$$R_0 = 0 \quad R_1 = e \quad R_2 = \ulcorner [a_1, \ldots, a_n] \urcorner \quad R_{3..} = 0$$

And where H halt with the state as follows:

$$R_0 = \begin{cases} 1 & \text{Register machine encoded as } e \text{ halts when started with } R_0 = 0, R_1 = a_1, \ldots, R_n = a_n \\ 0 & otherwise \end{cases}$$

We can prove that there is no such machine H through a contradiction.



Hence when we run C with the argument C we get a contradiction.

- $C(C)$ **Halts** Then C with $R_1 = \ulcorner C \urcorner$ as an argument does not halt, which is a contradiction
- $C(C)$ **Does not Halt** Then C with $R_1 = \ulcorner C \urcorner$ as an argument halts, which is a contradiction

Computable Functions

Enumerating the Computable Functions

Definition: Onto (Surjective)

Each element in the codomain is mapped to by at least one element in the domain.

$$\forall y \in Y.\ \exists x \in X.\ [f(x) = y] \Rightarrow f \text{ is onto}$$

For each $e \in \mathbb{N}$, $\varphi_e \in \mathbb{N} \rightharpoonup \mathbb{N}$ (partial function computed by $program(e)$):

$$\varphi_e(x) = y \Leftrightarrow program(e) \text{ with } R_0 = 0 \wedge R_1 = x \text{ halts with } R_0 = y$$

Hence for a given program $\in \mathbb{N}$ we can get the computable partial function of the program.

$$e \mapsto \varphi_e$$

Therefore the above mapping represents an **onto/surjective** function from $\mathbb{N}$ to all computable partial functions from $\mathbb{N} \rightharpoonup \mathbb{N}$.

Uncomputable Functions

$\uparrow$ and $\downarrow$

As in Prof Wicklicky's first lecture, for $f : X \rightharpoonup Y$ (partial function from X to Y):

$$f(x)\uparrow \triangleq \neg\exists y \in Y.\ [f(x) = y]$$
$$f(x)\downarrow \triangleq \exists y \in Y.\ [f(x) = y]$$

Hence we can attempt to define a function to determine if a function halts.

$$f \in \mathbb{N} \rightharpoonup \mathbb{N} \triangleq \{(x, 0) | \varphi_x(x)\uparrow\} \triangleq f(x) = \begin{cases} 0 & \varphi_x(x)\uparrow \\ undefined & \varphi_x(x)\downarrow \end{cases}$$

However we run into the halting problem:

Assume f is computable, then $f = \varphi_e$ for some $e \in \mathbb{N}$.

- **if** $\varphi_e(e)\uparrow$ by definition of f, $\varphi_e(e) = 0$ so $\varphi_e(e)\downarrow$ which is a contradiction
- **if** $\varphi_e(e)\downarrow$ by definition of f, $f(e)\uparrow$, and hence as $f = \varphi_e$, $\varphi_e\uparrow$ which is a contradiction

Here we have ended up with the halting problem being uncomputable.

Undecidable Set of Numbers

Given a set $S \subseteq \mathbb{N}$, its characteristic function is:

$$\chi_S \in \mathbb{N} \to \mathbb{N} \quad \chi_S(x) \triangleq \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

S is **register machine decidable** if its characteristic function is a register machine computable function.

S is decidable iff there is a register machine M such that for all $x \in \mathbb{N}$ when run with $R_0 = 0, R_1 = x$ and $R_{2..} = 0$ it eventually halts with:

- $R_0 = 1$ if and only if $x \in S$
- $R_0 = 1$ if and only if $x \notin S$

Hence we are effectively asking if a register machine exists that can determine if any number is in some set S.

We can then define subsets of $\mathbb{N}$ that are decidable/undecidable.

The set of functions mapping 0 is undecidable

Given a set:

$$S_0 \triangleq \{e | \varphi_e(0) \downarrow\}$$

Hence we are finding the set of the indexes (numbers representing register machines) that halt on input 0.

If such a machine exists, we can use it to create a register machine to solve the halting problem. Hence this is a contradiction, so the set is undecidable.

The set of total functions is undecidable

Take set $S_1 \subseteq \mathbb{N}$:

$$S_1 \triangleq \{e | \varphi_e \text{total function}\}$$

If such a register machine exists to compute χ_{S_1}, we can create another register machine, simply checking 0. Hence as from the previous example, there is no register machine to determine S_0, there is none to determine S_1.

50003 - Models Of Computation - (Prof Wiklicky) Lecture 4

Oliver Killane

01/04/22

Lecture Recording

Lecture recording is available here

Turing Machines

Definition: Turing Machine

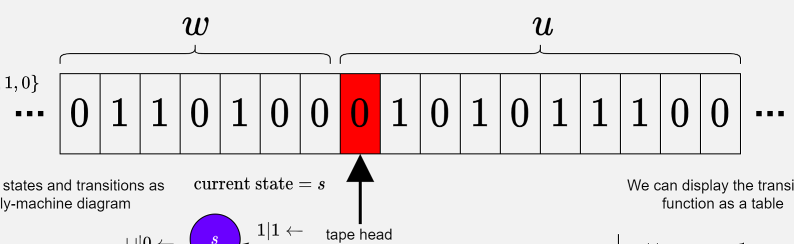
Turing Machine Definition → $M = (Q, \Sigma, s, \delta)$

- Q: Finite set of machine states
- Σ: Finite set of tape symbols
- s: Initial state, $s \in Q$
- δ: transition function

$\delta \in (Q \times \sum) \rightharpoonup (Q \times \sum \times \{L, R\})$

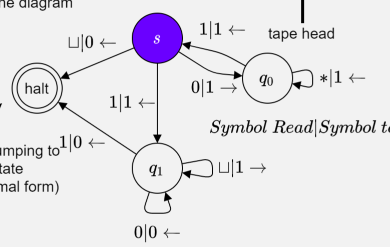
- $(Q \times \sum)$: symbol at head (on tape) & current state
- $(Q \times \sum \times \{L, R\})$: symbol to write, new state, and move tape left or right
- $\rightharpoonup$: partial function (some symbols & states have no mapping)

$\sum \triangleq \{\sqcup, 1, 0\}$



Display the states and transitions as a mealy-machine diagram

$\text{current state} = s$



Halt is encoded as jumping to an undefined state (configuration is normal form)

$\textit{Symbol Read}|\textit{Symbol to write}\{\leftarrow, \rightarrow\}$

We can display the transition function as a table

	$\sqcup$	1	0
s		$(q_1, 1, L)$	$(q_0, 1, R)$
q_0	$(q_0, 1, L)$	$(q_0, 1, L)$	$(q_0, 1, L)$
q_1	$(q_1, 1, R)$		$(q_1, 0, L)$

No entry mean undefined, which results in a halt

Configuration:

$(q, \underbrace{w, u})$

- q: Current State, $q \in Q$
- w: left of tape head
- u: right of tape head
- finite string of symbols (possibly empty): $w, u \in \sum *$

Start State: With the start state, no symbols to the left and the input symbols to the right

(s, ϵ, u) where $\epsilon = \text{empty}$

Normal form: when there is no transition for the current state and symbol at the tape head

(q, w, u) where $\delta(q, \textit{first } u) \uparrow$

Transition:

Note: When matching on an empty list, we get a new, empty element

$\sqcup : [\,] = [\,]$

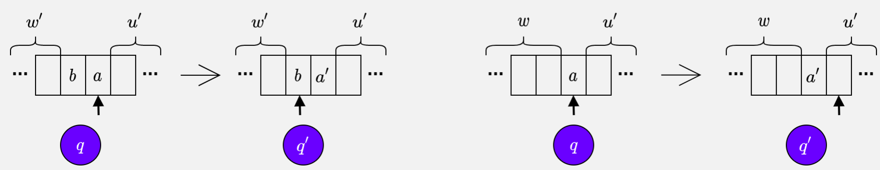
$$(q, w, u) \rightarrow_M (q', w', u')$$

Transition moves to the left

$$\frac{a : u' = u,\ b : w' = w,\ \delta(q, a) = (q', a', L)}{(q, w, u) \rightarrow_M (q', w', b : a' : u')}$$

Transition moves to the right

$$\frac{a : u' = u,\ \delta(q, a) = (q', a', R)}{(q, w, u) \rightarrow_M (q', w : a', u')}$$



$$c_0 \rightarrow_M c_1 \rightarrow_M c_2 \rightarrow_M \cdots \rightarrow_M \begin{cases} c_n & \text{finite sequence - the turing machine halts} \\ \ldots & \text{infinite sequence - turing machine does not halt} \end{cases}$$

Register machines abstract away the representation of numbers and operations on numbers (just uses $\mathbb{N}$ with increment, decrement operations), **Turing machines** are a more concrete representation of computing.

Turing $\rightarrow$ Register Machine

We can show that any computation by a **Turing Machine** can be implemented by a **Register Machine**. Given a **Turing Machine** M:

1. Create a numerical encoding of M's finite number of states, tape symbols, and initial tape contents.
2. Implement the transition table as a register machine.
3. Implement a register machine program to repeatedly carry out $\rightarrow_M$

Hence **Turing Machine Computable** $\Rightarrow$ **Register Machine Computable**.

Turing Machine Number Lists

In order to take arguments, and return value we need to encode lists on number on the tape of a turing machine. This is done as strings of unary values.

$$\sum = \{\sqcup, 0, 1\} \quad \overbrace{\underbrace{\dots}_{\text{all }\sqcup} \; 0 \; \underbrace{1\dots1}_{n_1} \sqcup \underbrace{1\dots1}_{n_2} \sqcup \dots \sqcup \underbrace{1\dots1}_{n_k} \; 0 \; \underbrace{\dots}_{\text{all }\sqcup}}^{\text{Turing Machine Tape}}$$

Definition: Turing Computable

If $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ is **Turing Computable** iff there is a turing machine M such that:

From initial state $(s, \epsilon, [x_1, \dots, x_n])$ (tape head at the leftmost 0), M halts if and only if $f(x_1, \dots, x_n) \downarrow$, and halts with the tape containing a list, the first element of which is y such that $f(x_1, \dots, x_n) = y$.

More formally, given $M = (Q, \sum, s, \delta)$ to compute f:

$$f(x_1, \dots, x_n) \downarrow \wedge f(x_1, \dots, x_n) = y \Leftrightarrow (s, \epsilon, [x_1, \dots, x_n]) \rightarrow_M^* (*, \epsilon, [y, \dots])$$

Register $\rightarrow$ Turing Machine

It is also possible to simulate any register machine on a turing machine. As we can encode lists of numbers on the tape, we can simply implement the register machine operations as operations on integers on the tape.

Hence **Register Machine Computable** $\Rightarrow$ **Turing Machine Computable**.

Notions of Computability

Every computable algorithm can be expressed as a turing machine (**Church-Turing Thesis**). In fact **Turing Machines**, **Register Machines** and the **Lambda Calculus** are all equivalent (all determine what is computable).

- **Partial Recursive Functions** Godel and Kleene (1936)
- **λ-Calculus** Church (1936)
- **canonical systems for generating the theorems of a formal system** Post (1943) and Markov (1951)
- **Register Machines** Lambek and Minsky (1961)
- **And many more** (multi-tape turing machines, parallel computation, turing machines embeded in cellular automata etc))

50003 - Models Of Computation - (Prof Wiklicky) Lecture 5

Oliver Killane

04/04/22

Lambda Calculus

Lecture Recording

Lecture recording is available here

$$\overset{}{M ::=} \quad \overset{\text{Variable}}{x} \quad \overset{\text{Abstraction}}{\lambda x.\, M} \quad \underset{\text{Left associative}}{\overset{\text{Application}}{M\ M}}$$

$$((M)\ M)\ M$$

Syntax

- **Bound Variables** x is bound inside $\lambda x\ .\ M$ (it is bound within the scope of M)
- **Free Variables** y is free inside $\lambda x\ .\ M$ (it is not bound)
- **Closed Term** A λ-term with no free variables, e.g $\lambda x\ y\ z\ .\ x\ y$
- **Binding Occurences** The λ-term's parameters $\lambda x\ y\ z\ .\ (\dots)$, here the x, y and z before the dot.
- **Left Associativity** Lambda Terms are left associative, hence $A\ B\ C\ D \equiv (((A)\ (B))\ (C))\ (D)$

Bound and Free Formally

$$\begin{aligned} FreeVariables &\quad (x) &&= \{x\} \\ FreeVariables &\quad (\lambda x\ .\ M) &&= FreeVariables(M) \setminus \{x\} \\ FreeVariables &\quad (M\ N) &&= FreeVariables(M) \cup FreeVariables(N) \end{aligned}$$

Definition: α-equivalence

$M =_\alpha N$ if and only if N can be obtained from M by renaming bound variables (or vice-versa)

Hence the free variable set must be the same (not renamed).

Substitution

$$M[new/old] \text{ means replace free variable } old \text{ with } new \text{ in } M$$

Only free variables can be substituted. Formally we can describe this as:

$$x[M/y] = \begin{cases} M & x = y \\ x & x \neq y \end{cases}$$

$$(\lambda x\ .\ N)[M/y] = \begin{cases} \lambda x\ .\ N & x = y \text{ } (x \text{ will be bound inside, so cannot go further}) \\ \lambda z\ .\ N[z/x][M/y] & x \neq y \text{ (To avoid name conflicts with } M\text{, } z \notin ((FV(N) \setminus \{x\}) \cup FV(M) \cup \{y\})) \end{cases}$$

$$(A\ B)[M/y] = (A[M/y])\ (B[M/y])$$

- For variables, simply check if equal.

- For lambda abstractions, if the old term is bound, cannot go further, else, switch the bound term for some term not free inside, in the substitution, and not the new value replacing.
- For applications, simply substitute into both λ-terms.

Example: Basic Substitution

$$x[y/x] = y$$
$$y[y/x] = y$$
$$(x\ y)[y/x] = y\ y$$
$$\lambda x\ .\ x\ y[y/x] = \lambda x\ .\ x\ y$$

Semantics

Lecture Recording

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$$\frac{}{(\lambda x\ .\ M)\ N \to_\beta M[N/x]} \quad \frac{M \to_\beta M'}{\lambda x\ .\ M \to_\beta \lambda x\ .\ M'} \quad \frac{M \to_\beta M'}{M\ N \to_\beta M'\ N} \quad \frac{N \to_\beta N'}{M\ N \to_\beta M\ N'}$$

$$\frac{M =_\alpha M' \quad M' \to_\beta N' \quad N' =_\alpha N}{M \to_\beta N}$$

- A term of the form $(\lambda x\ .\ M)$ is called a **redex**.
- A λ-term may have several different reductions. These different reductions for a **derivation tree**.

Multi-Step Reductions

Steps can be combined using the transitive closure of $\to_\beta$ under α-conversion.

$$\frac{M =_\alpha M'}{M \to_\beta^* M'} \qquad \text{(Reflexivity of } \alpha\text{-conversion)}$$

$$\frac{M \to_\beta M' \quad M' \to_\beta^* M''}{M \to_\beta^* M''} \qquad \text{(Transitivity)}$$

Definition: Confluence

All derivation paths in the derivation tree that reach some normal form, reach the same normal form.

$$\forall M, M_1, M_2.\ [M \to_\beta^* M_1 \land M \to_\beta^* M_2 \Rightarrow \exists M'.[M_1 \to_\beta^* M' \land M_2 \to_\beta^* M']]$$

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr] & \\
M_1 \arrow[dr] & & M_2 \arrow[dl] \\
 & M' &
\end{tikzcd}$$

Definition: β Normal Forms

A λ-term is in β-normal form if it contains no **redexes**, and hence cannot be further reduced.

$$\text{is in normal form}(M) \triangleq \forall N.\ M \not\to_\beta N$$
$$\text{has a normal form}(M) \triangleq \exists M'.\ M \to_\beta^* M' \land \text{is in normal form}(M)$$

If a normal form exists, it is unique.

$$\forall M, N_1, N_2, .\ [M \to_\beta^* N_1 \land M \to_\beta^* N_2 \land \text{is-norm-form}(N_1) \land \text{is-norm-form}(N_2) \Rightarrow N_1 =_\alpha N_2]$$

Definition: β-equivalence

An equivalence relation for $\to_\beta$.

$$M =_\beta N \Leftrightarrow \exists T.\ [M \to_\beta^* T \land N \to_\beta^* T]$$

Reduction Order

For a **redex** $E = (\lambda x\ .\ M)\ N$:

- Any **redex** in M or N is inside of E
- E is outside of any **redex** in M or N

Definition: Innermost Redex

A **Redex** with no **redexes** inside of it.

Definition: Outermost Redex

A **Redex** with no **redexes** outside of it.

We can choose several different orders by which to reduce.

- **Normal Order**
  - Reduce the **leftmost outermost redex** first.
  - This always reduces a λ-term to its normal form if one exists.
  - Can perform computations on unevaluated function bodies.
  - Not used in any programming languages.
- **Call By Name**
  - Reduce the **leftmost outermost** first.
  - Does not reduce the inside of λ-abstractions.
  - Does not always reduce a λ-term to its normal form.
  - Passes unevaluated function parameters into function body. Only evaluating a parameter when it is used.
  - Used with some variation by haskell, R, and LaTeX.
- **Call By Values**
  - Reduce the **leftmost innermost redex** first.
  - Does not reduce the inside of λ-abstractions.
  - Does not always reduce a λ-term to its normal form.
  - Evaluate parameters before passing them to function body.
  - Terminates less often than **call by name** (e.g if a parameter cannot be normalised, but is never used), but evaluated the parameters only once.
  - Used by C, Rust, Java, etc.

Definition: η-equivalence

Captures equality better than $=_\beta$.

$$\frac{x \notin FV(M)}{\lambda x\ .\ M\ x =_\eta M} \qquad \frac{\forall N.\ M\ N =_{\eta^+} M'\ N}{M =_{\eta^+} M'}$$

Namely if the application of M to another λ-term is equivalent to M' applied to the same λ-terms then M and M' are equivalent.

For example with the basic application of f:

$$\lambda x\ .\ f\ x \neq_\beta f \quad \text{however} \quad (\lambda x\ .\ f\ x)\ M =_\beta f\ M \quad \text{and} \quad \lambda x\ .\ f\ x \neq_\eta f$$

Definability

Lecture Recording

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Definition: λ-definable

Partial function $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ is **λ-definable** if and only if there is a **closed** λ-term M where:

$$f(x_1, \dots, x_n) = y \Leftrightarrow M \ \underline{x_1} \ \dots \ \underline{x_n} =_\beta y$$

And

$$f(x_1, \dots, x_n) \uparrow \Leftrightarrow M \ \underline{x_1} \ \dots \ \underline{x_n} \text{ has no } \textbf{normal form}$$

λ-definable specifies what can be computed by the lambda calculus, and is equivalent to **Register Machine Computable** or **Turing Machine Computable**.

Encoding Mathematics

Encoding Numbers

We represent natural numbers as **Church Numerals**. These are n repeated applications of some function f.

$$\underline{n} \triangleq \lambda f \ . \ \lambda x \ . \ \underbrace{f(\dots(f \ x)}_{n \text{ times}}\dots) \text{ with } n \text{ applications of } f$$

$$\begin{aligned}
\underline{0} &\triangleq \lambda f \ . \ \lambda x \ . \ x \\
\underline{1} &\triangleq \lambda f \ . \ \lambda x \ . \ f \ x \\
\underline{2} &\triangleq \lambda f \ . \ \lambda x \ . \ f \ f \ x \\
\underline{3} &\triangleq \lambda f \ . \ \lambda x \ . \ f \ f \ f \ x \\
\underline{4} &\triangleq \lambda f \ . \ \lambda x \ . \ f \ f \ f \ f \ x \\
\underline{5} &\triangleq \lambda f \ . \ \lambda x \ . \ f \ f \ f \ f \ f \ x \\
&\vdots
\end{aligned}$$

Encoding Addition

Addition is represented as a function application:

$$\underline{m} = \lambda f \ . \ \lambda x \ . \ \underbrace{f(\dots(f \ x)}_{m \text{ times}}\dots) \quad \underline{n} = \lambda f \ . \ \lambda x \ . \ \underbrace{f(\dots(f \ x)}_{n \text{ times}}\dots)$$

$$\underline{m+n} \triangleq \underbrace{(\lambda m \ . \ \lambda n \ . \ \lambda f \ . \ \lambda x \ . \ m \ f \ (n \ f \ x))}_{+} \ \underline{m} \ \underline{n}$$

By applying the functions, we have f applied $m + n$ times, representing the **Church Numeral** $\underline{m+n}$.

Encoding Multiplication

$$\underline{m} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{m \text{ times}}\ldots) \quad \underline{n} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{n \text{ times}}\ldots)$$

$$\underline{m \times n} \triangleq \underbrace{(\lambda m\ .\ \lambda n\ .\ \lambda f\ .\ m\ (n\ f))}_{\times}\ \underline{m}\ \underline{n}$$

Each application of the f inside m is substituted for n applications of f, using the above λ-abstraction we get $m \times n$ applications of f.

Exponentiation

$$\underline{m} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{m \text{ times}}\ldots) \quad \underline{n} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{n \text{ times}}\ldots)$$

$$\underline{m^n} \triangleq \underbrace{(\lambda m\ .\ \lambda n\ .\ n\ m)}_{\text{exponential}}\ \underline{m}\ \underline{n}$$

Conditional

$$\underline{m} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{m \text{ times}}\ldots)$$

$$\text{if } m = 0 \text{ then } x_1 \text{ else } x_2 \triangleq \underbrace{(\lambda m\ .\ \lambda x_1\ .\ \lambda x_2\ .\ m\ (\lambda z\ .\ x_2)\ x_1)}_{\text{if zero}}\ \underline{m}$$

If $\underline{m} = \underline{0} = \lambda f\ .\ \lambda x\ .\ x$ then x is returned, which will be x_1.

If not zero, then the f applied returns x_2, so any number of applications of f, results in x_2.

Successor

$$\underline{m} = \lambda f\ .\ \lambda x\ .\ \underbrace{f(\ldots(f\ x)}_{m \text{ times}}\ldots)$$

We simply take $\underline{m}$ and apply f one more time

$$\underline{m+1} \triangleq \underbrace{(\lambda m\ .\ \lambda f\ .\ \lambda x\ .\ f\ (m\ f\ x))}_{\text{succ}}\ \underline{m}$$

Pairs

We can encode pairs as a function, with a selector s function. Hence by supplying $first$ or $second$ as the selector, we can use the pair.

$$newpair(a,b) \triangleq \underbrace{(\lambda a\ .\ \lambda b\ .\ \lambda s\ .\ s\ a\ b)}_{\text{newpair}}\ a\ b \equiv \underbrace{(\lambda a\ b\ s\ .\ s\ a\ b)}_{\text{newpair}}\ a\ b$$

$$first(p) \triangleq p\ \underbrace{(\lambda x\ .\ \lambda y\ .\ x)}_{\text{first}} \equiv p\ \underbrace{(\lambda x\ y\ .\ x)}_{\text{first}}$$

$$second(p) \triangleq p\ \underbrace{(\lambda x\ .\ \lambda y\ .\ y)}_{\text{second}} \equiv p\ \underbrace{(\lambda x\ y\ .\ y)}_{\text{second}}$$

Predecessor

$$\underline{m} = \lambda f \; . \; \lambda x \; . \; \underbrace{f(\dots(f\ x)}_{m \text{ times}}\dots)$$

We cannot remove applications of f, however we can use a pair to count up until the successor is $\underline{m}$.

Hence we first need a function to get the next pair from the current:

$$\mathit{transition}\ p \triangleq \underbrace{(\lambda n \; . \; \mathit{newpair}\ (\mathit{second}\ n)\ ((\mathit{second}\ n) + 1))}_{\text{transition function}}\ p$$

We can then simply run the transition n times on a pair starting by using $f = \mathit{transition}$ and $x = \mathit{newpair}\ \underline{0}\ \underline{0}$.

$$\mathit{pred}(n) \triangleq \begin{cases} 0 & n = 0 \\ n - 1 & \mathit{otherwise} \end{cases}$$

$$\mathit{pred}(n) \triangleq \underbrace{(\lambda n \; . \; n\ \mathit{transition}\ (\mathit{newpair}\ \underline{0}\ \underline{0})\ \mathit{first})}_{\text{predecessor}}\ \underline{n}$$

A simpler definition of predecessor is:

$$\mathit{pred}(n) \triangleq \underbrace{(\lambda n \; . \; \lambda f \; . \; \lambda x \; . \; n\ (\lambda g \; . \; \lambda h \; . \; h\ (g\ f))\ (\lambda u \; . \; x)\ (\lambda u \; . \; u))}_{\text{predecessor}}\ \underline{n}$$

Subtraction

We can use the predecessor function for subtraction. By applying the predecessor function $\underline{n}$ times on some number $\underline{m}$ we get $\underline{m - n}$.

$$\underline{m - n} \triangleq \underbrace{(\lambda m \; . \; \lambda n \; . \; m\ \mathit{pred}\ n)}_{\text{subtract}}\ \underline{m}\ \underline{n}$$

Combinators

Definition: Combinator

A **closed** λ-term (no free variables), usually denoted by capital letters that describe

$$\begin{array}{ccl}
I & \triangleq & \lambda x \; . \; x \\
K & \triangleq & \lambda x\ y \; . \; x \\
S & \triangleq & \lambda x\ y\ z \; . \; x\ z\ (y\ z) \\
T & \triangleq & \lambda x\ y \; . \; y\ x \\
C & \triangleq & \lambda x\ y\ z \; . \; x\ z\ y \\
V & \triangleq & \lambda x\ y\ z \; . \; z\ x\ y \\
B & \triangleq & \lambda x\ y\ z \; . \; x\ (y\ z) \\
B' & \triangleq & \lambda x\ y\ z \; . \; y\ (x\ z) \\
W & \triangleq & \lambda x\ y \; . \; x\ y\ y \\
Y & \triangleq & \lambda g \; . \; (\lambda x \; . \; g\ (x\ x))\ (\lambda x \; . \; g\ (x\ x))
\end{array}
\qquad
\begin{array}{rcl}
I(x) & \triangleq & x \\
K(x, y) & \triangleq & x \\
S(x, y, z) & \triangleq & x(z)(y(z)) \\
T(x, y) & \triangleq & y(x) \\
C(x, y, z) & \triangleq & x(z)(y) \\
V(x, y, z) & \triangleq & z(x)(y) \\
B(x, y, z) & \triangleq & x(y(z)) \\
B'(x, y, z) & \triangleq & y(x(z)) \\
W(x, y) & \triangleq & x(y)(y) \\
Y(f) & \triangleq & (\lambda x \to f(x(x)))(\lambda x \to f(x(x)))
\end{array}$$

Only SKI are required to define any **computable function** (can remove even λ-abstraction, this is called SKI**-Combinator Calculus**).

The Y-Combinator is used for recursion. In one step of β-reduction:

$$Y\ f \to_\beta f\ (Y\ f)$$

We cannot define λ-terms in terms of themselves, as the λ-term is not yet defined, and infinitely large λ-terms are not allowed.

We can use the $Y - Combinator$ to create recursion in the absence of recursive λ-term definitions.

Definition: Fixed-Point Combinator

A higher order function (e.g fix) that returns some function of itself:

$$\begin{aligned} &fix\ f = f(fix\ f) \\ &fix\ f = f(f(\dots f(fix\ f)\dots))\ \text{(after repeated application)} \end{aligned}$$

Example: Factorial

$$fact(n) = \begin{cases} 1 & n = 0 \\ n \times fact(n-1) & otherwise \end{cases}$$

If recursive definitions for λ-terms were allows, we could express this as:

$$\begin{aligned} fact &\triangleq \lambda n\ .\ \text{if zero}\ n\ \underline{1}\ (multiply\ n\ (fact\ (pred\ n))) \\ &\triangleq (\lambda f\ .\ \lambda n\ .\ \text{if zero}\ n\ \underline{1}\ (multiply\ n\ (f\ (pred\ n))))\ fact \end{aligned}$$

Since we can use the above form (higher order function applied to itself) with the Y combinator.

$$fact \triangleq Y(\lambda f\ .\ \lambda n\ .\ \text{if zero}\ n\ \underline{1}\ (multiply\ n\ (f\ (pred\ n))))$$