50003 - Models of Computation - Lecture 3

Oliver Killane

24/10/21

Lecture Recording

Lecture recording is available here

Syntax of While

We can define a simple **While** language (if, else, while loops) to build programs from & to analyse.

$$\begin{array}{rcl} B \in Bool & ::= & true|false|E = E|E < E|B\&B|\neg B \dots \\ E \in Exp & ::= & x|n|E + E|E \times E| \dots \\ C \in Com & ::= & x := E|if\ B\ then\ C\ else\ C|C; C|skip|while\ B\ do\ C \end{array}$$

Where $x \in Var$ ranges over variable identifiers, and $n \in \mathbb{N}$ ranges over natural numbers.

States

A **state** is a partial function from variables to numbers. For state s, and variable x, $s(x)$ is defined, e.g:

$$s = (x \mapsto 2, y \mapsto 200, z \mapsto 20)$$

(In the current state, $x = 2, y = 200, z = 20$).

Partial Functions

A partial function is a mapping of every member of its domain, to at most one member of its codomain.

A state is a partial function as it is only defined for some variables.

For example:

$$\begin{array}{rcll} s[x \mapsto 7](u) & = & 7 & \text{if } u = x \\ & = & s(u) & \text{otherwise} \end{array}$$

The small step semantics of **While** are defined using **configurations** of form:

$$\langle E, s\rangle, \langle B, s\rangle, \langle C, s\rangle$$

(Evaluating E, B, or C with respect to state s)

We can create a new state, where variable x equals value a, from an existing state s:

$$s'(u) \triangleq \alpha(x) = \begin{cases} a & u = x \\ s(u) & otherwise \end{cases}$$

$s' = s[x \mapsto u]$ is equivalent to $dom(s') = dom(s) \wedge \forall y.[y \neq x \rightarrow s(y) = s'(y) \wedge s'(x) = a]$

(s' equals s where x maps to a)

Expressions

$$\text{(W-EXP.LEFT)}\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 + E_2, s\rangle \rightarrow_e \langle E_1' + E_2, s'\rangle}$$

$$\text{(W-EXP.RIGHT)}\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n + E, s\rangle \rightarrow_e \langle n + E', s'\rangle}$$

$$\text{(W-EXP.VAR)}\frac{}{\langle x, s\rangle \rightarrow_e \langle n, s\rangle}\ s(x) = n$$

$$\text{(W-EXP.ADD)}\frac{}{\langle n_1 + n_2, s\rangle}\ \langle n_3, s\rangle n_3 = n_1 + n_2$$

These rules allow for side effects, despite the While language being side effect free in expression evaluation. We show this by changing state $s \rightarrow_e s'$.

We can show inductively (from the base cases W-EXP.VAR and W-EXP.ADD) that expression evaluation is side effect free.

Booleans

(Based on expressions, one can create the same for booleans) ($b \in \{true, false\}$)

AND

$$\text{(W-BOOL.AND.LEFT)}\frac{\langle B_1, s\rangle \rightarrow_b \langle B_1', s'\rangle}{\langle B_1 \& B_2, s\rangle \rightarrow_b \langle B_1' \& B_2, s'\rangle}$$

$$\text{(W-BOOL.AND.RIGHT)}\frac{\langle B, s\rangle \rightarrow_b \langle B', s'\rangle}{\langle b \& B_2, s\rangle \rightarrow_b \langle b \& B', s'\rangle}$$

$$\text{(W-BOOL.AND.TRUE)}\frac{}{\langle true \& true, s\rangle \rightarrow_b \langle true, s\rangle}$$

$$\text{(W-BOOL.AND.FALSE)}\frac{}{\langle false \& b, s\rangle \rightarrow_b \langle true, s\rangle}$$

(Notice we do not short circuit, as the right arm may change the state. In a side effect free language, we could.)

EQUAL

$$\text{(W-BOOL.EQUAL.LEFT)}\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 = E_2, s\rangle \rightarrow_b \langle E_1' = E_2, s'\rangle}$$

$$\text{(W-BOOL.EQUAL.RIGHT)}\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n = E, s\rangle \rightarrow_b \langle n = E, s'\rangle}$$

$$\text{(W-BOOL.EQUAL.TRUE)}\frac{}{\langle n_1 = n_2, s\rangle \rightarrow_b \langle true, s\rangle}\ n_1 = n_2$$

$$\text{(W-BOOL.EQUAL.FALSE)}\frac{}{\langle n_1 = n_2, s\rangle \rightarrow_b \langle false, s\rangle}\ n_1 \neq n_2$$

LESS

$$\text{(W-BOOL.LESS.LEFT)}\frac{\langle E_1, s\rangle \rightarrow_e \langle E_1', s'\rangle}{\langle E_1 < E_2, s\rangle \rightarrow_b \langle E_1' < E_2, s'\rangle}$$

$$\text{(W-BOOL.LESS.RIGHT)}\frac{\langle E, s\rangle \rightarrow_e \langle E', s'\rangle}{\langle n < E, s\rangle \rightarrow_b \langle n < E, s'\rangle}$$

$$\text{(W-BOOL.LESS.TRUE)}\frac{}{\langle n_1 < n_2, s\rangle \rightarrow_b \langle true, s\rangle}\ n_1 < n_2$$

$$\text{(W-BOOL.EQUAL.FALSE)}\frac{}{\langle n_1 < n_2, s\rangle \rightarrow_b \langle false, s\rangle}\ n_1 \geq n_2$$

NOT

$$\text{(W-BOOL.NOT)}\frac{}{\langle \neg true, s\rangle \rightarrow_b \langle false, s\rangle}$$

$$\text{(W-BOOL.NOT)}\frac{}{\langle \neg false, s\rangle \rightarrow_b \langle true, s\rangle}$$