50003 - Models of Computation - Lecture 6

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Lecture Recording

Lecture recording is available here

Definition by Induction for SimpleExp

To define a function on all expressions in **SimpleExp**:

- define $f(n)$ directly, for each number n.
- define $f(E_1 + E_2)$ in terms of $f(E_1)$ and $f(E_2)$.
- define $f(E_1 \times E_2)$ in terms of $f(E_1)$ and $f(E_2)$.

For example, we can do this with *den*:

$$den(E) = n \leftrightarrow E \Downarrow n$$

Evaluation

Many Steps of Evaluation

Given $\rightarrow$ we can define a new relation $\rightarrow^*$ as:

$$E \rightarrow^* E' \leftrightarrow (E = E' \vee E \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_k \rightarrow E')$$

For expressions, the final answer is n if $E \rightarrow^* n$.

Multi-Step Reductions

The relation $E \rightarrow^n E'$ is defined using mathematics induction by:

- **Base Case**
  $E \rightarrow^0 E$ for all $E \in \mathit{SimpleExp}$
- **Inductive Case**
  For every $E, E' \in \mathit{SimpleExp}$, $E \rightarrow^{k+1} E'$ if and only if there is some E'' such that:

  $$E \rightarrow^k E'' \wedge E'' \rightarrow E'$$
- **Definition**
  $\rightarrow^*$ - there are some number of steps to evaluate to E'.

  $$E \rightarrow^* E' \Leftrightarrow \exists n.[E \rightarrow^n E']$$

Properties of $\rightarrow$

- **Determinacy** If $E \rightarrow E_1$ and $E \rightarrow E_2$ then $E_1 = E_2$.
- **Confluence** If $E \rightarrow^* E_1$ and $E \rightarrow^* E_2$ then there exists E' such that $E_1 \rightarrow^* E'$ and $E_2 \rightarrow^* E'$.

- **Unique answer** If $E \to^* n_1$ and $E \to^* n_2$ then $n_1 = n_2$.
- **Normal Forms** Normal form is numbers ($\mathbb{N}$) for any E, $E = n$ or $E \to E'$ for some E'.
- **Normalisation** No infinite sequences of expressions $E_1, E_2, E_3, \dots$ such that for all $i \in \mathbb{N}$ $E_1 \to E_{i+1}$ (Every path goes to a normal form).

Confluence of Small Step

We can prove a lemma expressing confluence:

$L_1 : \forall n \in \mathbb{N}.\forall E, E_1, E_2 \in \mathit{SimpleExp}.[E \to^n E_1 \wedge E \to^* E_2 \Rightarrow \exists E' \in \mathit{SimpleExp}.[E_1 \to^* E' \wedge E_2 \to^* E']]$

Lemma $\Rightarrow$ Confluence

Confluence is: $\forall E, E_1, E_2 \in \mathit{SimpleExp}.[E \to^* E_1 \wedge E \to^* E_2 \Rightarrow \exists E' \in \mathit{SimpleExp}.[E_1 \to^* E' \wedge E_2 \to^* E']]$ From lemma L_1

(1)	Take some arbitrary $E, E_1, E_2 \in \mathit{SimpleExp}$, assume confluence holds.	(Initial Setup)
(2)	$E \to^* E_1$	(By Confluence)
(3)	$\exists n \in \mathbb{N}.[E \to^n E_1]$	(By 2 & definition of $\to^*$)
(4)	Hence L_1	(By 3)

Determinacy of Small Step

We create a property P:

$$P(E) \overset{def}{=} \forall E_1, E_2 \in \mathit{SimpleExp}.[E \to E_1 \wedge E \to E_2 \Rightarrow E_1 = E_2]$$

There are 3 rules that apply:

$$(\text{A})\frac{}{n_1 + n_2 \to n}\ n = n_1 + n_2 \quad (\text{B})\frac{E \to E'}{n + E \to n + E'} \quad (\text{C})\frac{E_1 \to E_1'}{E_1 + E_2 \to E_1' + E_2}$$

Base Case

Take arbitrary $n \in \mathbb{N}$ and $E_1, E_2 \in \mathit{SimpleExp}$ such that $n \to E_1 \wedge n \to E_2$ to show $E_1 = E_2$.

(1)	$n \not\to$	(By inversion on A,B & C)
(2)	$\neg(n \to E_1)$	(By 1)
(3)	$\neg(n \to E_1 \wedge n \to E_2)$	(By 2)
(4)	$n \to E_1 \wedge n \to E_2 \Rightarrow E_1 = E_2$	(By 3)
(5)	$E \to E_1 \wedge E \to E_2 \Rightarrow E_1 = E_2$	(By 4)

Hence $P(n)$

Inductive Step

Take arbitrary E, E_1, E_2 such that $E = E_1 + E_2$
Inductive Hypothesis:

$$IH_1 = P(E_1)$$

$$IH_2 = P(E_2)$$

Assume there exists $E_3, E_4 \in \mathit{SimpleExp}$ such that $E_1 + E_2 \to E_3$ and $E_1 + E_2 \to E_4$.
To show $E_2 = E_4$.

From inversion on A, B & C there are 3 cases to consider:
For A:

(1)	There exists $n_1, n_2 \in \mathbb{N}$ such that $E_1 = n_1$ and $E_2 = n_2$	(By case A)
(3)	$E_3 = n_1 + n_2$	(By 1, A)
(4)	$E_4 = n_1 + n_2$	(By 1, A)
(5)	$E_3 = E_4$	(By 3 & 4)

For B:

(1)	There exists $n \in \mathbb{N}$ such that $E_1 = n$	(By case B)
(2)	There exists $E' \in \mathit{SimpleExp}$ such that $E_2 \to E'$	(By case B)
(3)	$E_3 = n + E'$	(By case B)
(4)	There exists $E'' \in \mathit{SimpleExp}$ such that $E_2 \to E''$	(By case B)
(5)	$E_4 = n + E''$	(By case B)
(6)	$E' = E''$	(By IH_2)
(7)	$E_3 = E_4$	(By 3,5 & 6)

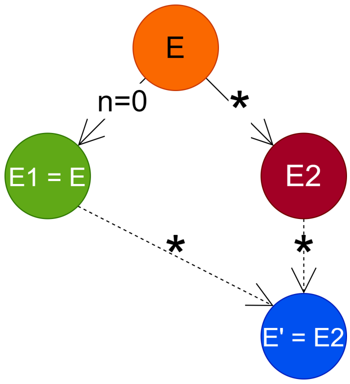
For C:

(1)	There exists $E' \in \mathit{SimpleExp}$ such that $E_1 \to E'$	(By case C)
(2)	There exists $E'' \in \mathit{SimpleExp}$ such that $E_1 \to E''$	(By case C)
(3)	$E_3 = E' + E_2$	(By case C)
(4)	$E_4 = E'' + E_2$	(By case C)
(5)	$E' = E''$	(By IH_1)
(6)	$E_3 = E_4$	(By 3,4 & 5)

(If E reduces to E_1 in n steps, and to E_2 in some number of steps, then there must be some E' that E_1 and E_2 reduce to.)

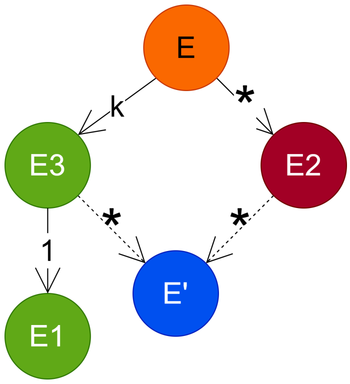
Base Case

The base cases has $n = 0$. Hence $E = E_1$, and hence $E_1 \to^* E_2$ and $E_1 \to^* E'$

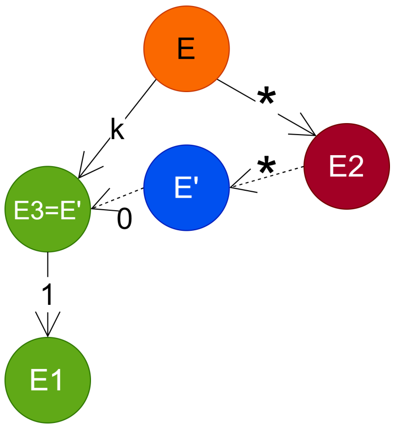


Inductive Case

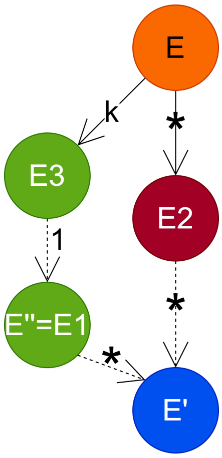
Next we assume confluence for up to k steps, and attempt to prove for $k + 1$ steps.



We have two cases:
Case 1: $E_3 = E'$, this is easy as $E_2 \rightarrow^* E' \rightarrow^0 E3 \rightarrow^1 E1$.



Case 2: $E_3 \rightarrow^1 E'' \rightarrow^* E'$, in this case as $E_3 \rightarrow^1 E1$ we know by determinacy that $E'' = E_1$ and hence $E_1 \rightarrow^* E'$.


E
k
*
E3
E2
1
*
E"=E1
*
E'