50001 - Algorithm Analysis and Design - Lecture 6

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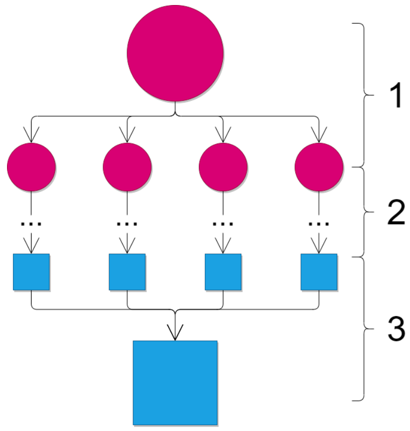
Lecture Recording

Lecture recording is available here

Divide & Conquer

1. Divide a problem into subproblems
2. Divide and conquer subproblems into subsolutions
3. Conquer subsolutions into a solution



Merge Sort

```haskell
msort :: Ord a => [a] -> [a]
msort []   = []
msort [x]  = [x]
msort xs   = merge (msort us) (msort vs)
  where (us,vs) = splitAt (length xs `div` 2) xs

merge :: Ord a => [a] -> [a] -> [a]
merge [] ys = ys
merge xs [] = xs
merge xss@(x:xs) yss@(y:ys)
  | x <= y     = x : merge xs yss
  | otherwise = y : merge xss ys
```

SplitAt divides, and **merge** Conquers. We can calculate the time complexity for the recurrence relations below (based on recursive structure of **msort**):

$$T_{msort}(0) = 1$$
$$T_{msort}(1) = 1$$
$$T_{msort}(n) = T_{length}(n) + T_{splitAt}(\frac{n}{2}) + T_{merge}(\frac{n}{2}) + 2 \times T_{msort}(\frac{n}{2})$$

We can simplify the complexity of **msort**

$$
\begin{aligned}
T_{msort}(n) &= T_{length}(n) + T_{splitAt}(\frac{n}{2}) + T_{merge}(\frac{n}{2}) + 2 \times T_{msort}(\frac{n}{2}) \\
&= n + \frac{n}{2} + \frac{n}{2} + \frac{n}{2} + 2 \times T_{msort}(\frac{n}{2}) \\
&= \frac{5}{2} \times n + 2 \times T_{msort}(\frac{n}{2})
\end{aligned}
$$

Master Theorem

For an algorithm *algo* such that:

$$T_{algo}(n) = a \times T_{algo}(\frac{n}{b}) + f(n) + \text{base cases}$$

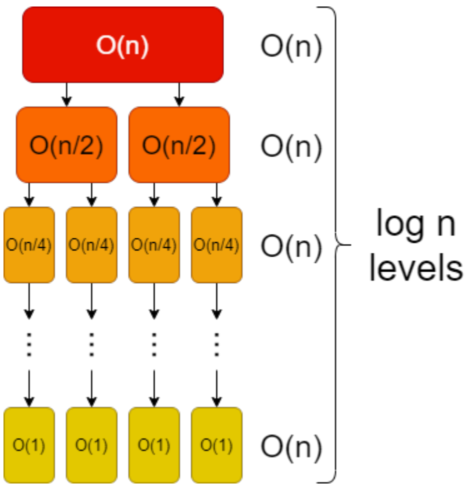
The work at recursion level $\log_b n$ is $\Theta(a^{\log_b n})$ To calculate the order of the time complexity:

1. Get the recurrence relation in the form above.
2. Get the critical exponent E by formula: $E = \log_b a = \dfrac{\log a}{\log b}$.
3. Given $f(n) = n^c$ we can express the work as a geometric sum $\sum_{i=0}^{\log n} ar^i$ where $r = \dfrac{a}{b^c}$.

$$r > 1 \Leftrightarrow a > b^c \Leftrightarrow \log_b a > c \Leftrightarrow E > c$$

$$
\begin{array}{lc}
E < c & T_{algo}(n) \in \Theta(f(n)) \\
E = c & T_{algo}(n) \in \Theta(f(n) \log_b n) = \Theta(f(n) \log n) \\
E > c & T_{algo}(n) \in \Theta(a^{\log_b n}) = \Theta(n^{\log_b a}) = \Theta(n^E)
\end{array}
$$

By master theorem we can easily see $T_{msort}(n) \in \Theta(n \log n)$. We can also calculate it using a graph:



Quicksort

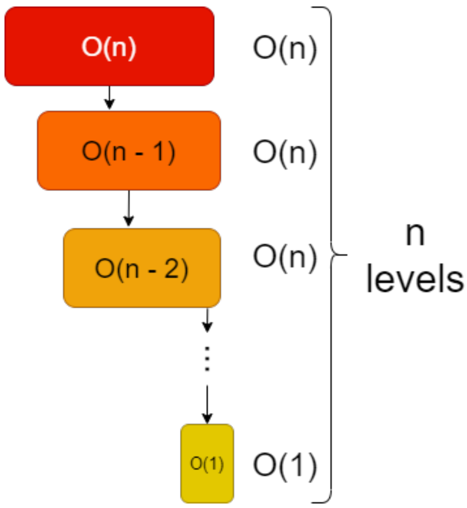
```haskell
qsort :: Ord a => [a] -> [a]
qsort [] = []
qsort [x] = [x]
qsort (x:xs) = qsort us ++ x:qsort vs
  where (us, vs) = partition (<x) xs

partition :: (a -> Bool) -> [a] -> ([a],[a])
partition p xs = (filter p xs, filter (not . p) xs)
```

Note for simplicity, we assume the lists are split into equal parts.

$$\begin{aligned} T_{qsort}(0) &= 1 \\ T_{qsort}(1) &= 1 \\ T_{qsort}(n) &= T_{partition}(n-1) + T_{++}(\frac{n}{2}) + 2 \times T_{qsort}(\frac{n}{2}) \end{aligned}$$

In the worst case, the partition splits xs into $(xs, [])$, we have complexity:

$$\begin{aligned} T_{qsort}(n) &= T_{partition}(n-1) + T_{++}(n-1) + T_{qsort}(0) + T_{qsort}(n-1) \\ &= 2(n-1) + n + 1 + T_{qsort}(n-1) \end{aligned}$$



We can once again use master theorem, or a diagram such as below to see the complexity: Hence in the worst case $T_{qsort}(n) \in O(n^2)$ (same as insertion sort).