50001 - Algorithm Analysis and Design - Lecture 3

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Lecture Recording

Lecture recording is available here

Asymptotics

L-Function

A **Logarithmico-exponential** function f is:

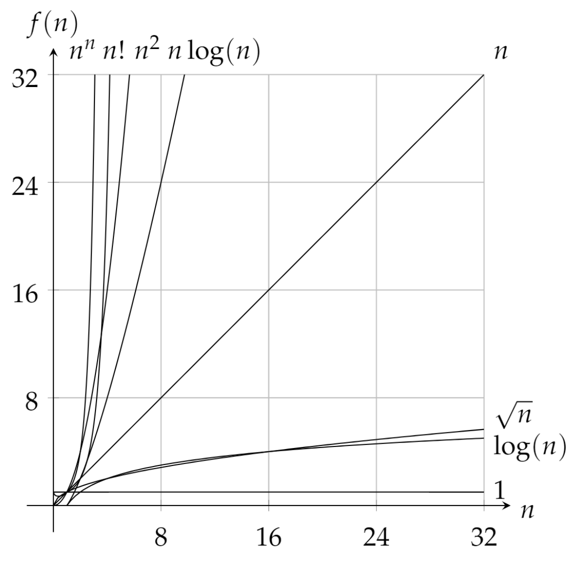
- real: $f \in X \to Y$ where $X, Y \subset \mathbb{R}$
- positive: $\forall x \in X.[f(x) \leq 0]$
- monotonic: $\forall x_1, x_2 \in X.[x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)]$ (positive monotonic) or $\forall x_1, x_2 \in X.[x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)]$ (negative monotonic)
- one valued: $\forall x \in X, y_1, y_2 \in Y.[f(x) = y_1 \land f(x) = y_2 \Rightarrow y_1 = y_2]$
- on a real variable defined for all values greater than some definite value: $X \equiv \{x | x > \text{definite limit} \land x \in \mathbb{R}\}$

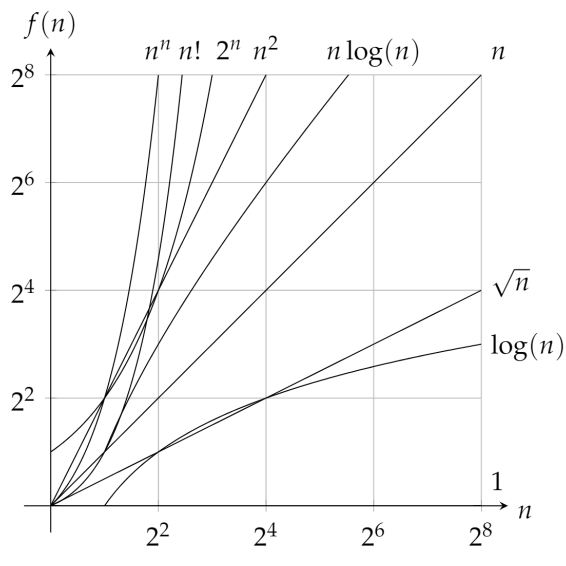
L-Functions are continuous, of constant sign and as $n \to \infty$ the value $f(n)$ tends to $0, \infty$ or some other positive definite limit.

Functions that aren't **L-Functions** are called **Wild Functions**.

In asymptotics we use **L-Functions** to describe the growth of time used by algorithms as the size of the input to an algorithm grows.

Common functions are shown below:





Du Bois-Reymond Theorem

Defines inequalities for the rate of increase of functions.

Where $lim = \lim_{n\to\infty} \dfrac{f(n)}{g(n)}$

$(<)$	$f \prec g \Leftrightarrow lim = 0$	g grows much faster than f
$(\leq)$	$f \preccurlyeq g \Leftrightarrow lim < \infty$	g grows much faster than f or some multiple of f
$(=)$	$f \asymp g \Leftrightarrow 0 < lim < \infty$	$g(n)$ grows towards some constant times $f(n)$
$(\geq)$	$f \succcurlyeq g \Leftrightarrow lim > 0$	f grows much faster than g or some multiple of g
$(>)$	$f \succ g \Leftrightarrow lim = \infty$	f grows much faster than g

These operators form a trichotomy such that one of the below will always hold:

$$f \prec g \quad f \asymp g \quad f \succ g$$

Further the operators $\succ$ and $\prec$ are converse:

$$f \succ g \Leftrightarrow g \prec f$$

And transitive:

$$f \prec g \wedge g \prec h \Rightarrow f \prec h$$
$$f \preccurlyeq g \wedge g \preccurlyeq h \Rightarrow f \preccurlyeq h$$

We can place the common **L-Functions** in order:

$$1 \prec \log n \prec \sqrt{n} \prec n \prec n \log n \prec n^2 \prec n^3 \prec n! \prec n^n$$

Bachman-Landau Notation

Comparison with Bois-Reymond	Set definition
$f \in o(g(n)) \Leftrightarrow f \prec g$	$o(g(n)) = \{f \mid \forall\delta > 0.\exists n_0 > 0.\forall n > n_0[f(n) < \delta g(n)]\}$
$f \in O(g(n)) \Leftrightarrow f \preccurlyeq g$	$O(g(n)) = \{f \mid \exists\delta > 0.\exists n_0 > 0.\forall n > n_0[f(n) \leq \delta g(n)]\}$
$f \in \Theta(g(n)) \Leftrightarrow f \asymp g$	$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
$f \in \Omega(g(n)) \Leftrightarrow f \succcurlyeq g$	$\Omega(g(n)) = \{f \mid \exists\delta > 0.\exists n_0 > 0.\forall n > n_0[f(n) \geq \delta g(n)]\}$
$f \in \omega(g(n)) \Leftrightarrow f \succ g$	$\omega(g(n)) = \{f \mid \forall\delta > 0.\exists n_0 > 0.\forall n > n_0[f(n) > \delta g(n)]\}$