50003 - Models of Computation - Lecture 4

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Factorial Program

$$C = y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$$

We can attempt to evaluate this for a given input, for example:

$$s = [x \mapsto 3, y \mapsto 17, z \mapsto 42]$$

The evaluation path is as follows:

Start

$$\langle y := x; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), [x \mapsto 3, y \mapsto 17, z \mapsto 42] \rangle$$

Get x variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)} \frac{\text{(W-ASS.EXP)} \dfrac{\text{(W-EXP.VAR)} \dfrac{}{\langle x, s \rangle \to_e \langle 3, s \rangle}}{\langle y := x, s \rangle \to_c \langle y := 3, s \rangle}}{\langle y := x; C, s \rangle \to_c \langle y := 3; C, s \rangle}$$

Result:

$$\langle y := 3; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 17, z \mapsto 42) \rangle$$

Assign to y variable

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 17, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)} \frac{\text{(W-ASS.NUM)} \dfrac{}{\langle y := 3, s \rangle \to_c \langle \mathit{skip}, s[y \mapsto 3] \rangle}}{\langle y := 3; C, s \rangle \to_c \langle \mathit{skip}; C, s[y \mapsto 3] \rangle}$$

Result:

$$\langle \mathit{skip}; a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42) \rangle$$

Eliminate skip

where $C = a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$\text{(W-SEQ.SKIP)} \frac{}{\langle \mathit{skip}; C, s \rangle \to_c \langle C, s \rangle}$$

Result:

$$\langle a := 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42) \rangle$$

Assign a

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42)$:

$$\text{(W-SEQ.LEFT)} \frac{\text{(W-ASS.NUM)} \dfrac{}{\langle a := 1, s \rangle \to_c \langle \mathit{skip}, s[a \mapsto 1] \rangle}}{\langle a := 1; C, s \rangle \to_c \langle \mathit{skip}; C, s[a \mapsto 1] \rangle}$$

Result:

$$\langle \mathit{skip}; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1) \rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$

$$\text{(W-SEQ.SKIP)}\frac{}{\langle skip; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Expand while

where $C = (a := a \times y; y := y - 1)$, $B = 0 < y$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-WHILE)}\frac{}{\langle \text{while } B \text{ do } C, s\rangle \to_c \langle \text{if } B \text{ then } (C; \text{while } B \text{ do } C) \text{ else } skip, s\rangle}$$

Result:

$$\langle \text{if } 0 < y \text{ then } (a := a{\times}y; y := y{-}1; \text{while } 0 < y \text{ do } a := a{\times}y; y := y{-}1) \text{ else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Get y variable

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.BEXP)}\frac{\text{(W-BOOL.LESS.RIGHT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \to \langle 3, s\rangle}}{\langle 0 < y, s\rangle \to_b \langle 0 < 3, s\rangle}}{\langle \text{if } 0 < y \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \to_c \langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip,}$$

Result:

$$\langle \text{if } 0 < 3 \text{ then } (a := a{\times}y; y := y{-}1; \text{while } 0 < y \text{ do } a := a{\times}y; y := y{-}1);\ \text{else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Complete if boolean

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.EXP)}\frac{\text{(W-BOOl.LESS.TRUE)}\dfrac{}{\langle 0 < 3, s\rangle \to_b \langle true, s\rangle}}{\langle \text{if } 0 < 3 \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \to_c \langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle}$$

Result:

$$\langle \text{if } true \text{ then } (a := a{\times}y; y := y{-}1; \text{while } 0 < y \text{ do } a := a{\times}y; y := y{-}1);\ \text{else } skip, (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate if

where $C = (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-COND.TRUE)}\frac{}{\langle \text{if } true \text{ then } (C; \text{while } 0 < y \text{ do } C) \text{ else } skip, s\rangle \to_c \langle C; \text{while } 0 < y \text{ do } C, s\rangle}$$

Result:

$$\langle a := a \times y; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression a

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle a, s\rangle \to \langle 1, s\rangle}}{\langle a \times y, s\rangle \to_e \langle 1 \times y, s\rangle}}{\langle a := a \times y, s\rangle \to_c \langle a := 1 \times y, s\rangle}}{\langle a := a \times y; C, s\rangle \to_c \langle a := 1 \times y; C, s\rangle}$$

Result:

$$\langle a := 1 \times y; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Expression y

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL.RIGHT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \to_e \langle 3, s\rangle}}{\langle 1 \times y, s\rangle \to_e \langle 1 \times 3, s\rangle}}{\langle a := 1 \times y, s\rangle \to_c \langle a := 1 \times 3, s\rangle}}{\langle a := 1 \times y; C, s\rangle \to \langle a := 1 \times 3; C, s\rangle}$$

Result:

$$\langle a := 1 \times 3; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Evaluate Multiply

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.MUL)}\dfrac{}{\langle 1 \times 3, s\rangle \to_e \langle 3, s\rangle}}{\langle a := 1 \times 3, s\rangle \to_c \langle a := 3, s\rangle}}{\langle a := 1 \times 3; C, s\rangle \to_c \langle a := 3; C, s\rangle}$$

Result:

$$\langle a := 3; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)\rangle$$

Assign 3 to a

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 1)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle a := 3, s\rangle \to_c \langle \mathit{skip}, s[a \mapsto 3]\rangle}}{\langle a := 3; C, s\rangle \to_c \langle \mathit{skip}; C, s[a \mapsto 3]\rangle}$$

Result:

$$\langle \mathit{skip}; y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate Skip

where $C = y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle \mathit{skip}; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle y := y - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 3 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB.LEFT)}\dfrac{\text{(W-EXP.VAR)}\dfrac{}{\langle y, s\rangle \to \langle 3, s\rangle}}{\langle y - 1, s\rangle \to_e \langle 3 - 1, s\rangle}}{\langle y := y - 1, s\rangle \to_c \langle y := 3 - 1, s\rangle}}{\langle y := y - 1; C, s\rangle \to_c \langle y := 3 - 1, s\rangle}$$

Result:

$$\langle y := 3 - 1; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Evaluate Subtraction

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.EXP)}\dfrac{\text{(W-EXP.SUB)}\dfrac{}{\langle 3 - 1, s\rangle \to_e \langle 2, s\rangle}}{\langle y := 3 - 1, s\rangle \to_c \langle y := 2, s\rangle}}{\langle y := 3 - 1; C, s\rangle \to_c \langle y := 2; C, s\rangle}$$

Result:

$$\langle y := 2; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)\rangle$$

Assign 2 to y

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 3, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.LEFT)}\frac{\text{(W-ASS.NUM)}\dfrac{}{\langle y := 2, s\rangle \to_c \langle \mathit{skip}, s[y \mapsto 2]\rangle}}{\langle y := 2; C, s\rangle \to_c \langle \mathit{skip}; C, s[y \mapsto 2]\rangle}$$

Result:

$$\langle \mathit{skip}; \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$

Eliminate skip

where $C = \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1)$ and $s = (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)$:

$$\text{(W-SEQ.SKIP)}\frac{}{\langle \mathit{skip}; C, s\rangle \to_c \langle C, s\rangle}$$

Result:

$$\langle \text{while } 0 < y \text{ do } (a := a \times y; y := y - 1), (x \mapsto 3, y \mapsto 2, z \mapsto 42, a \mapsto 3)\rangle$$