

# 50008 - Probability and Statistics - Lecture 3

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24/01/22

# Continuous Random Variables

## Lecture Recording

Lecture recording is available here

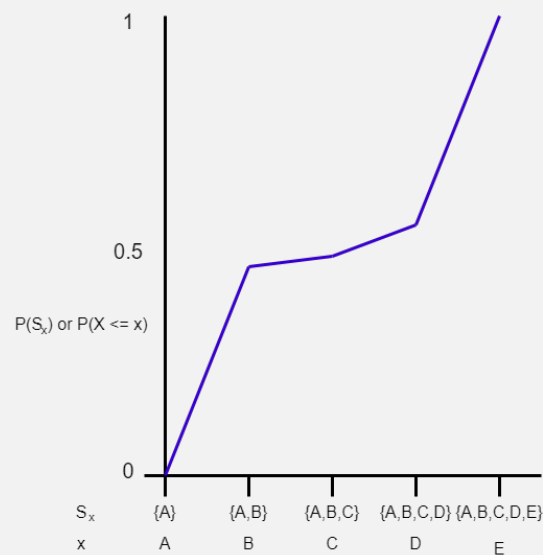
For continuous random variables we want to track quantities in  $\mathbb{R}$  (e.g temperature, volume, other probabilities).

## Induced Probability Terms

$$S_x = \{s \in S | X(s) \leq x\}$$

$$P_X(X \leq x) = P(S_x) = F_X(x)$$

$S_x$  is the elements of the sample space up to and including  $x$ . Hence the probability of getting  $S_x$  is the cumulative probability.



### Definition: Probability Density Function

For a random variable  $X : S \rightarrow \mathbb{R}$  the induced probability is defined as:

$$P_X((-\infty, x]) = P(S_X) = F_X(x)$$

A variable  $X$  is **absolutely continuous** if  $\exists f_X : \mathbb{R} \rightarrow \mathbb{R}$  such that:

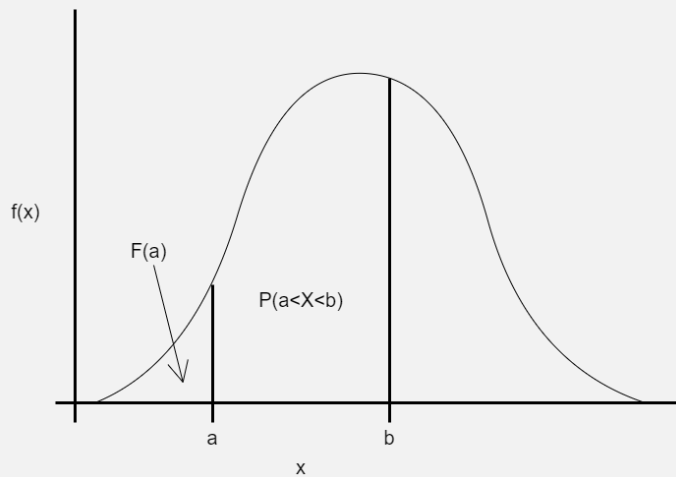
$$F_X(x) = \int_{u=-\infty}^x f_X(u) du$$

$$f_X(x) = F'_X(x) = \frac{d}{dx} F_X(x)$$

Where  $f_X$  is the **probability density function (pdf)**.

To find probability that  $X \in (a, b]$ :

$$P_X(a < X \leq b) = P_X(X \leq b) - P_X(X \leq a) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx$$



- We can use  $<$  and  $\leq$  interchangeably as  $P(X = x) = 0 \Leftrightarrow P(X \leq x) \equiv P(X < x)$ .
- Probability of any event is zero:  $P_X(X = y) = 0$ , any elementary event  $\{x\}$  where  $x \in \mathbb{R}$  has zero probability.
- However the sum of a range of events probabilities is not zero.
- Hence the range of a continuous random variable is uncountable (i.e as  $\mathbb{R}$  is also).

$$\forall x \in \mathbb{R}. f_X(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f_X(x) dx = 1$$

### Example: Defining a continuous random variable

Given some continuous random variable  $x$  with a probability density function given as:

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

For some unknown constant  $c$

To find the value of  $c$  we use the requirement that the cumulative distribution must sum to 1:

$$\int_0^3 cx^2 = 1 \rightsquigarrow \left[\frac{cx^3}{3}\right]_0^3 = 1 \rightsquigarrow (9c) - 0 = 1 \rightsquigarrow c = 1/9$$

Hence:

$$f(x) = \begin{cases} \frac{x^2}{9} & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

Hence we can specify the cumulative probability distribution as:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^3}{27} & 0 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

We can then calculate probabilities using the cumulative distribution:

$$P(1 < X < 2) = F(2) - F(1) = \frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27} \approx 0.259$$

## Mean, Variance and Quantiles

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Definition: Expected (Continuous)

The **mean** or **expected** of a continuous random variable  $X$ :

$$\mu_X = E_X(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

For a function of interest that is applied to the random variable  $g : \mathbb{R} \rightarrow \mathbb{R}$ :

$$E_X(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- $E(aX + b) = aE(X) + b$
- $E(g(X) + h(X)) = E(g(X)) + E(h(X))$

Definition: Variance (Continuous)

The variance of a continuous random variable  $X$ :

$$\sigma_X^2 = Var_X(X) = E((X - \mu_X)^2) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

We can show this as:

$$\begin{aligned} Var_X(X) &= \int_{-\infty}^{\infty} x^2 f_X(x) dx - \mu_X^2 \\ &= E(X^2) - (E(X))^2 \end{aligned}$$

For a linear transformation:

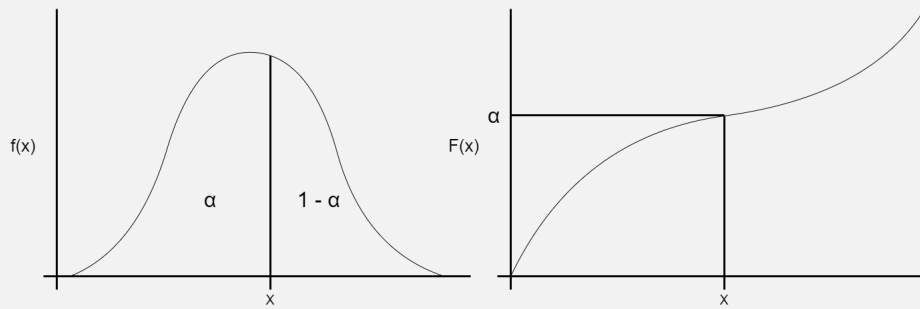
$$Var(aX + b) = a^2 Var(X)$$

### Definition: Quartiles

The lower, upper quartiles and median are points

For a continuous random variable  $X$ , we define the  $\alpha$ -**Quantile**  $Q_X(\alpha)$  where  $0 \leq \alpha \leq 1$  as the lowest  $X$  such that:

$$P(X \leq Q_X(\alpha)) = \alpha \quad \text{or in other words} \quad Q_X(\alpha) = F_X^{-1}(\alpha)$$



Using  $Q_X$  we can define some standard quantiles:

- **Quartiles** Lower Quartile ( $\alpha = 1/4$ ), Median ( $\alpha = 1/2$ ) and Upper Quartile ( $\alpha = 3/4$ )
- **Percentiles** The  $n$ th percentile:  $\alpha = \frac{n}{100}$

### Example: Basic continuous random variable

Given continuous random variable  $X$ :

$$f(x) = \begin{cases} \frac{x^2}{9} & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the expected:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x f(x) dx + \int_0^3 x f(x) dx + \int_3^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x \times 0 dx + \int_0^3 x f(x) dx + \int_3^{\infty} x \times 0 dx \\ &= \int_0^3 x f(x) dx = \int_0^3 \frac{x^3}{9} dx = \left[ \frac{x^4}{36} \right]_0^3 \\ &= \frac{9}{4} = 2.25 \end{aligned}$$

We can calculate the variance:

$$\begin{aligned} Var(X) &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu_X^2 \\ &= \int_{-\infty}^0 x^2 f(x) dx + \int_0^3 x^2 f(x) dx + \int_3^{\infty} x^2 f(x) dx - \mu_X^2 \\ &= \int_0^3 x^2 f(x) dx - \mu_X^2 = \int_0^3 \frac{x^5}{9} dx - \mu_X^2 \\ &= 27 - \mu_X^2 = 27 - 2.25 = 24.75 \end{aligned}$$

we can calculate the median, we ignore the range  $x > 3$  as the median must be below this.

$$\begin{aligned} 0.5 &= \int_{-\infty}^x f(y) dy = \int_{-\infty}^0 f(y) dy + \int_0^x f(y) dy = \int_0^x f(y) dy \\ 0.5 &= \int_0^x \frac{y^2}{9} dy = \left[ \frac{y^3}{27} \right]_0^x = \frac{x^3}{27} \\ x &= \sqrt[3]{0.5 \times 27} \approx 2.38 \end{aligned}$$

## Notable Continuous Distributions

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### Definition: Continuous Uniform Distribution

A continuous random variable with equal probability of being any value within a range:

For  $X \sim U(a, b)$ :

PDF	CDF	Expected	Variance
$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$	$\mu = \frac{a+b}{2}$	$\sigma^2 = \frac{(b-a)^2}{12}$

The standard uniform distribution is defined as  $X \sim U(0, 1)$ :

PDF	CDF	Expected	Variance
$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$	$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$	$\mu = 1/2$	$\sigma^2 = 1/12$

Other uniform distributions can be mapped linearly to the standard uniform.

#### Example: Mapping to Standard Uniform

Given  $X \sim U(2, 5)$  find the expected, variance and median.

Take  $Y \sim U(0, 1)$ ,  $X = 3 \times Y + 2$ .

Distribution	Expected	Variance	Median
$Y$	0.5	$1/12$	0.5
$X$	3.5	$3/4$	3.5



### Definition: Exponential Distribution

Given a rate of events  $\lambda$ , what is the probability of waiting  $X$  time for the event to occur.

For  $X \sim \text{Exponential}(\lambda)$  or  $X \sim \text{Exp}(\lambda)$  where  $\lambda > 0$ :

PDF	CDF	Expected	Variance
$f_X(x) = \lambda e^{-\lambda x}$ where $x \geq 0$	$F_X(x) = 1 - e^{-\lambda x}$ where $x \geq 0$	$\mu_X = \frac{1}{\lambda}$	$\sigma^2 = \frac{1}{\lambda^2}$

The distribution has the **Lack of memory property**, namely the time waited already does not affect the next part of the distribution (same shape).

$$P(X > x+t | X > t) = \frac{P(X > x+t \cap X > t)}{P(X > t)} = \frac{P(X > x+t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x} = P(X > x)$$

$$P(X > x+t | X > t) = P(X > x)$$

This distribution can be combined with Poisson. Given  $X \sim \text{Poisson}(\lambda)$  (events occurring in a given time frame), the time between events is modelled by  $X \sim \text{Exponential}(\lambda)$  (interval time for one event).

There is a variant with  $\theta$  as the parameter for the distribution where  $\theta = \frac{1}{\lambda}$ .

### Definition: Normal Distribution

Given a mean value ( $\mu$ ) and a variance ( $\sigma^2$ ) from the mean the symmetrical distribution is a **Normal Distribution**.

For  $X \sim \text{Normal}(\mu, \sigma^2)$  or  $X \sim N(\mu, \sigma^2)$  where  $\sigma > 0$ :

PDF	CDF
$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$	$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\} dt$

The **Standard/Unit Normal Distribution** is  $X \sim N(0, 1)$ :

PDF	CDF
$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}$	$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$

We can apply linear functions:

$$X \sim N(\mu, \sigma^2) \rightarrow \text{and } aX + b \sim N(a\mu + b, a^2\sigma^2)$$

Hence we can use the **Standard Normal Distribution**:

$$X \sim N(\mu, \sigma^2) \Rightarrow \frac{X - \mu}{\sigma} \sim N(0, 1) \text{ and hence } P(X \leq x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

#### Definition: Lognormal Distribution

Given  $X \sim N(\mu, \sigma^2)$  and  $Y = e^X$  we can compute the **PDF** of  $Y$ :

$$f_Y(y) = \frac{1}{\sigma y \sqrt{2\pi}} \exp \left[ -\frac{(\log y - \mu)^2}{2\sigma^2} \right]$$

## Central Limit Theorem

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#### Definition: Moment Generating Function

The moment generating function  $M_X$  for a continuous random variable  $X$  is:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

Assuming the calculus within the  $E(\dots)$  is valid, the  $n$ th moment is given by:

$$E[X^n] = \left. \frac{d^n M_x(t)}{dt^n} \right|_{t=0}$$

If the integral does not exist, the **characteristic function**  $\phi_X(t) = M_X(it)$  can be used ( $i$  is imaginary unit).

### Example: Expected and Variance

$$\begin{aligned} E[X] &= \left. \frac{dM_x(t)}{dt} \right|_{t=0} \\ &= \left. \frac{dE[e^{tX}]}{dt} \right|_{t=0} \\ &= \left. \frac{d \int_{-\infty}^{\infty} e^{tx} f_X(x) dx}{dt} \right|_{t=0} \\ &= \left. \int_{-\infty}^{\infty} x e^{tx} f_X(x) dx \right|_{t=0} \\ &= \int_{-\infty}^{\infty} x e^{0x} f_X(x) dx \\ &= \int_{-\infty}^{\infty} x f_X(x) dx \end{aligned}$$

$$\begin{aligned} E[X^2] &= \left. \frac{d^2 M_x(t)}{dt^2} \right|_{t=0} \\ &= \left. \frac{d^2 E(e^{tX})}{dt^2} \right|_{t=0} \\ &= \left. \frac{d^2 \int_{-\infty}^{\infty} e^{tx} f_X(x) dx}{dt^2} \right|_{t=0} \\ &= \left. \frac{d \int_{-\infty}^{\infty} x e^{tx} f_X(x) dx}{dt} \right|_{t=0} \\ &= \left. \int_{-\infty}^{\infty} x^2 e^{tx} f_X(x) dx \right|_{t=0} \\ &= \int_{-\infty}^{\infty} x^2 e^{0x} f_X(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \end{aligned}$$

$$Var[X] = E[X^2] - (E[X])^2$$

### Product of Random Variables

Given independent random variables  $Z_1, Z_2, \dots, Z_n$ :

$$E\left[\prod_{i=1}^n Z_i\right] = \prod_{i=1}^n E[Z_i]$$

The sum of the random variables is the products of their **Moment Generating Functions**.

$$M_{Z_1+Z_2}(t) = E[e^{t(Z_1+Z_2)}] = E[e^{tZ_1}e^{tZ_2}] = E[e^{tZ_1}]E[e^{tZ_2}] = M_{Z_1}(t)M_{Z_2}(t)$$

$$S_n = \sum_{i=1}^n Z_i \Rightarrow M_{S_n}(t) = \prod_{j=1}^n M_{Z_j}(t)$$

## Central Limit Theorem

### Definition: Central Limit Theorem

Given  $X_1, X_2, \dots, X_n$  are independent and identically distributed random variables from any distribution with mean  $\mu$  and finite variance  $\sigma^2$ .

$$S_n = \sum_{i=1}^n X_i$$

Hence we have a distribution with a known expected and variance, so can form a **Normal Distribution**.

$$\begin{aligned} Y &= S_n & E(Y) &= n\mu & \text{Var}(Y) &= n\sigma^2 \\ Y &= S_n - n\mu & E(Y) &= 0 & \text{Var}(Y) &= n\sigma^2 \\ Y &= \frac{S_n - n\mu}{\sqrt{n}\sigma} & E(Y) &= 0 & \text{Var}(Y) &= 1 \end{aligned}$$

$Y$  can now be used to approximate a **Standard Normal Distribution**.

$$\lim_{n \rightarrow \infty} \frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0, 1)$$

This implies that for large (but finite  $n$ ):

$$\bar{X} \approx N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{and} \quad \sum_{i=1}^n X_i \approx N(n\mu, n\sigma^2)$$

Where  $\bar{X}$  is the average value of the random variables  $\frac{\sum_{i=1}^n X_i}{n}$ .

The approximation holds for all distributions (including discrete), and is exact when the random variables are from the same **normal distribution**.

### An attempt at CLT proof

Given the random variables  $X_1, X_2, \dots, X_n$  we can standardize and get their sum:

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^n (X_i - \mu)}{\sqrt{n}\sigma} = \sum_{i=1}^n \frac{Y_i}{\sqrt{n}\sigma} \quad \text{where } Y_i = X_i - \mu$$

The moment generating function of  $Z_n$  is the product of the **moment generating functions** of the  $Y$  (all identically distributed, so identical **MGFs**).

$$M_{Z_n}(t) = \left( M_Y \left( \frac{t}{\sqrt{n}\sigma} \right) \right)^n \quad \text{where } M_Y \text{ is the moment generating function for all } Y_i$$

We can then expand the  $M_Y$  around 0 using Taylor's Theorem:

$$M_Y(t) = M_Y(0) + M_Y'(0)t + \frac{1}{2}M_Y''(0)t^2 + O(t^3)$$

$O(t^3)$  is the error term of our approximation, as this is for higher powers, it has a small effect so can be ignored

The derivatives of the **MFG** are:

$$M_Y'(0) = E(Y_i) = 0 \text{ due to shift performed earlier and } M_Y''(0) = E(Y_i^2) = \sigma^2 + E(Y_i)^2 = \sigma^2 + 0 = \sigma^2$$

Hence we can derive:

$$M_Y(t) = 1 + \frac{\sigma^2 t^2}{2} + O(t^3)$$

Hence we can scale  $t$ , and ignore the error term for simplicity:

$$M_Y \left( \frac{t}{\sqrt{n}\sigma} \right) = 1 + \frac{t^2}{2n}$$

As the error term gets very small, we can use limits to get an approximation for  $M_{Z_n}(t)$ .

$$\lim_{n \rightarrow \infty} M_{Z_n}(t) = \lim_{n \rightarrow \infty} \left( 1 + \frac{t^2}{2n} + O(n^{-3/2}) \right)^n = e^{t^2/2}$$

Note that  $\lim_{m \rightarrow \infty} \left( 1 + \frac{x}{m} \right)^m = e^x$ .

#### Example: Coin Tossing

Consider a set of count tosses, each are Bernoulli discrete random variables (take values 0 or 1).

$$X_1, X_2, X_3, \dots, X_n \text{ where } \mu = p \text{ and } \sigma^2 = p(1-p)$$

The total score of coin tosses can be modelled as a binomial distribution:

$$\sum_{i=1}^n X_i \text{ is } X \sim \text{Binomial}(n, p) \text{ with } E(X) = np \text{ and } \text{Var}(X) = np(1-p)$$

For large  $n$  can also model it as a normal distribution:

$$\sum_{i=1}^n X_i \text{ is } X \sim N(n\mu, n\sigma^2) \equiv N(np, np(1-p))$$

As the number of events (coin tosses) tends to infinity, the distributions tend to look identical.