50003 - Models Of Computation - (Prof Wiklicky) Lecture 3

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Lecture Recording

Lecture recording is available here

Halting Problem for Register Machines

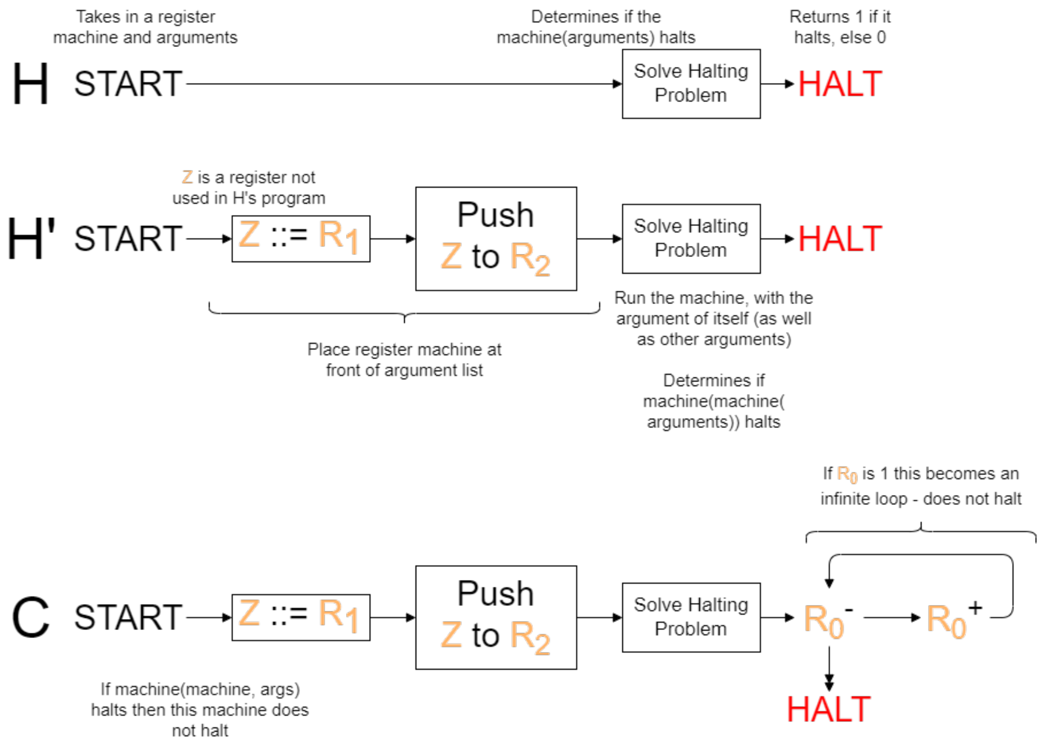
A register machine H decides the halting problem if for all $e, a_1, \dots, a_n \in \mathbb{N}$:

$$R_0 = 0 \quad R_1 = e \quad R_2 = \ulcorner [a_1, \dots, a_n] \urcorner \quad R_{3..} = 0$$

And where H halt with the state as follows:

$$R_0 = \begin{cases} 1 & \text{Register machine encoded as } e \text{ halts when started with } R_0 = 0, R_1 = a_1, \dots, R_n = a_n \\ 0 & otherwise \end{cases}$$

We can prove that there is no such machine H through a contradiction.



Hence when we run C with the argument C we get a contradiction.

- $C(C)$ **Halts** Then C with $R_1 = \ulcorner C \urcorner$ as an argument does not halt, which is a contradiction
- $C(C)$ **Does not Halt** Then C with $R_1 = \ulcorner C \urcorner$ as an argument halts, which is a contradiction

Computable Functions

Enumerating the Computable Functions

Definition: Onto (Surjective)

Each element in the codomain is mapped to by at least one element in the domain.

$$\forall y \in Y.\ \exists x \in X.\ [f(x) = y] \Rightarrow f \text{ is onto}$$

For each $e \in \mathbb{N}$, $\varphi_e \in \mathbb{N} \rightharpoonup \mathbb{N}$ (partial function computed by $program(e)$):

$$\varphi_e(x) = y \Leftrightarrow program(e) \text{ with } R_0 = 0 \land R_1 = x \text{ halts with } R_0 = y$$

Hence for a given program $\in \mathbb{N}$ we can get the computable partial function of the program.

$$e \mapsto \varphi_e$$

Therefore the above mapping represents an **onto/surjective** function from $\mathbb{N}$ to all computable partial functions from $\mathbb{N} \rightharpoonup \mathbb{N}$.

Uncomputable Functions

$\uparrow$ and $\downarrow$

As in Prof Wicklicky's first lecture, for $f : X \rightharpoonup Y$ (partial function from X to Y):

$$f(x)\uparrow \triangleq \neg\exists y \in Y.\ [f(x) = y]$$
$$f(x)\downarrow \triangleq \exists y \in Y.\ [f(x) = y]$$

Hence we can attempt to define a function to determine if a function halts.

$$f \in \mathbb{N} \rightharpoonup \mathbb{N} \triangleq \{(x, 0) | \varphi_x(x)\uparrow\} \triangleq f(x) = \begin{cases} 0 & \varphi_x(x)\uparrow \\ undefined & \varphi_x(x)\downarrow \end{cases}$$

However we run into the halting problem:

Assume f is computable, then $f = \varphi_e$ for some $e \in \mathbb{N}$.

- **if** $\varphi_e(e)\uparrow$ by definition of f, $\varphi_e(e) = 0$ so $\varphi_e(e)\downarrow$ which is a contradiction
- **if** $\varphi_e(e)\downarrow$ by definition of f, $f(e)\uparrow$, and hence as $f = \varphi_e$, $\varphi_e\uparrow$ which is a contradiction

Here we have ended up with the halting problem being uncomputable.

Undecidable Set of Numbers

Given a set $S \subseteq \mathbb{N}$, its characteristic function is:

$$\chi_S \in \mathbb{N} \to \mathbb{N} \quad \chi_S(x) \triangleq \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

S is **register machine decidable** if its characteristic function is a register machine computable function.

S is decidable iff there is a register machine M such that for all $x \in \mathbb{N}$ when run with $R_0 = 0, R_1 = x$ and $R_{2..} = 0$ it eventually halts with:

- $R_0 = 1 \quad$ if and only if $x \in S$
- $R_0 = 1 \quad$ if and only if $x \notin S$

Hence we are effectively asking if a register machine exists that can determine if any number is in some set S.

We can then define subsets of $\mathbb{N}$ that are decidable/undecidable.

The set of functions mapping 0 is undecidable

Given a set:

$$S_0 \triangleq \{e|\varphi_e(0) \downarrow\}$$

Hence we are finding the set of the indexes (numbers representing register machines) that halt on input 0.

If such a machine exists, we can use it to create a register machine to solve the halting problem. Hence this is a contradiction, so the set is undecidable.

The set of total functions is undecidable

Take set $S_1 \subseteq \mathbb{N}$:

$$S_1 \triangleq \{e|\varphi_e \text{total function}\}$$

If such a register machine exists to compute χ_{S_1}, we can create another register machine, simply checking 0. Hence as from the previous example, there is no register machine to determine S_0, there is none to determine S_1.