

50003 - Models of Computation - Lecture 6

Oliver Killane

29/10/21

Definition by Induction for SimpleExp

To define a function on all expressions in **SimpleExp**:

- define $f(n)$ directly, for each number n .
- define $f(E_1 + E_2)$ in terms of $f(E_1)$ and $f(E_2)$.
- define $f(E_1 \times E_2)$ in terms of $f(E_1)$ and $f(E_2)$.

For example, we can do this with *den*:

$$\text{den}(E = n \leftrightarrow E \Downarrow n$$

Evaluation

Many Steps of Evaluation

Given \rightarrow we can define a new relation \rightarrow^* as:

$$E \rightarrow^* E' \leftrightarrow (E = E' \vee E \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_k \rightarrow E')$$

For expressions, the final answer is n if $E \rightarrow^* n$.

Multi-Step Reductions

The relation $E \rightarrow^n E'$ is defined using mathematics induction by:

- **Base Case**

$$E \rightarrow^0 E \text{ for all } E \in \text{SimpleExp}$$

- **Inductive Case**

For every $E, E' \in \text{SimpleExp}$, $E \rightarrow^{k+1} E'$ if and only if there is some E'' such that:

$$E \rightarrow^k E'' \wedge E'' \rightarrow E'$$

- **Definition**

\rightarrow^* - there are some number of steps to evaluate to E' .

$$E \rightarrow^* E' \Leftrightarrow \exists n. [E \rightarrow^n E']$$

Properties of \rightarrow

- **Determinacy** If $E \rightarrow E_1$ and $E \rightarrow E_2$ then $E_1 = E_2$.
- **Confluence** If $E \rightarrow^* E_1$ and $E \rightarrow^* E_2$ then there exists E' such that $E_1 \rightarrow^* E'$ and $E_2 \rightarrow^* E'$.

- **Unique answer** If $E \rightarrow^* n_1$ and $E \rightarrow^* n_2$ then $N_1 = n_2$.
- **Normal Forms** Normal form is numbers (\mathbb{N}) for any E , $E = n$ or $E \rightarrow E'$ for some E' .
- **Normalisation** No infinite sequences of expressions E_1, E_2, E_3, \dots such that for all $i \in \mathbb{N}$ $E_i \rightarrow E_{i+1}$ (Every path goes to a normal form).

Confluence of Small Step

We can prove a lemma expressing confluence:

$$L_1 : \forall n \in \mathbb{N}. \forall E, E_1, E_2 \in SimpleExp. [E \rightarrow^n E_1 \wedge E \rightarrow^* E_2 \Rightarrow \exists E' \in SimpleExp. [E_1 \rightarrow^* E' \wedge E_2 \rightarrow^* E']]$$

Lemma \Rightarrow Confluence

Confluence is: $\forall E, E_1, E_2 \in SimpleExp. [E \rightarrow^* E_1 \wedge E \rightarrow^* E_2 \Rightarrow \exists E' \in SimpleExp. [E_1 \rightarrow^* E' \wedge E_2 \rightarrow^* E']]$ From lemma L_1

- (1) Take some arbitrary $E, E_1, E_2 \in SimpleExp$, assume confluence holds. (Initial Setup)
- (2) $E \rightarrow^* E_1$ (By Confluence)
- (3) $\exists n \in \mathbb{N}. [E \rightarrow^n E_1]$ (By 2 & definition of \rightarrow^*)
- (4) Hence L_1 (By 3)

Determinacy of Small Step

We create a property P :

$$P(E) \stackrel{def}{=} \forall E_1, E_2 \in SimpleExp. [E \rightarrow E_1 \wedge E \rightarrow E_2 \Rightarrow E_1 = E_2]$$

There are 3 rules that apply:

$$(A) \frac{}{n_1 + n_2 \rightarrow n} \quad n = n_1 + n_2 \quad (B) \frac{E \rightarrow E'}{n + E \rightarrow n + E'} \quad (C) \frac{E_1 \rightarrow E'_1}{E_1 + E_2 \rightarrow E'_1 + E_2}$$

Base Case

Take arbitrary $n \in \mathbb{N}$ and $E_1, E_2 \in SimpleExp$ such that $n \rightarrow E_1 \wedge n \rightarrow E_2$ to show $E_1 = E_2$.

- (1) $n \not\rightarrow$ (By inversion on A,B & C)
- (2) $\neg(n \rightarrow E_1)$ (By 1)
- (3) $\neg(n \rightarrow E_1 \wedge n \rightarrow E_2)$ (By 2)
- (4) $E \rightarrow E_1 \wedge E \rightarrow E_2 \Rightarrow E_1 = E_2$ (By 3)

Hence $P(n)$

Inductive Step

Take arbitrary E, E_1, E_2 such that $E = E_1 + E_2$

Inductive Hypothesis:

$$IH_1 = P(E_1)$$

$$IH_2 = P(E_2)$$

Assume there exists $E_3, E_4 \in SimpleExp$ such that $E_1 + E_2 \rightarrow E_3$ and $E_1 + E_2 \rightarrow E_4$.
To show $E_2 = E_4$.

From inversion on A, B & C there are 3 cases to consider:

For A:

- (1) There exists $n_1, n_2 \in \mathbb{N}$ such that $E_1 = n_1$ and $E_2 = n_2$ (By case A)
- (3) $E_3 = n_1 + n_2$ (By 1, A)
- (4) $E_4 = n_1 + n_2$ (By 1, A)
- (5) $E_3 = E_4$ (By 3 & 4)

For B:

- (1) There exists $n \in \mathbb{N}$ such that $E_1 = n$ (By case B)
- (2) There exists $E' \in SimpleExp$ such that $E_2 \rightarrow E'$ (By case B)
- (3) $E_3 = n + E'$ (By case B)
- (4) There exists $E'' \in SimpleExp$ such that $E_2 \rightarrow E''$ (By case B)
- (5) $E_4 = n + E''$ (By case B)
- (6) $E' = E''$ (By IH_2)
- (7) $E_3 = E_4$ (By 3,5 & 6)

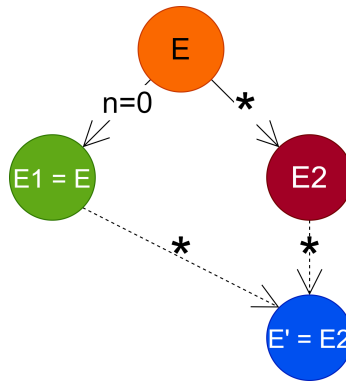
For C:

- (1) There exists $E' \in SimpleExp$ such that $E_1 \rightarrow E'$ (By case C)
- (2) There exists $E'' \in SimpleExp$ such that $E_1 \rightarrow E''$ (By case C)
- (3) $E_3 = E' + E_2$ (By case C)
- (4) $E_4 = E'' + E_2$ (By case C)
- (5) $E' = E''$ (By IH_1)
- (6) $E_3 = E_4$ (By 3,4 & 5)

(If E reduces to E_1 in n steps, and to E_2 in some number of steps, then there must be some E' that E_1 and E_2 reduce to.)

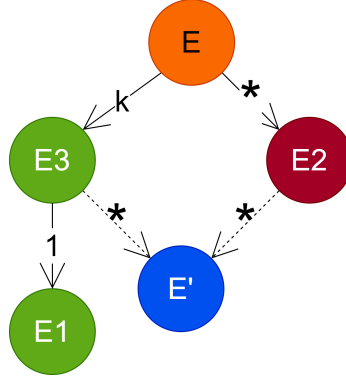
Base Case

The base cases has $n = 0$. Hence $E = E_1$, and hence $E_1 \rightarrow^* E_2$ and $E_1 \rightarrow^* E'$



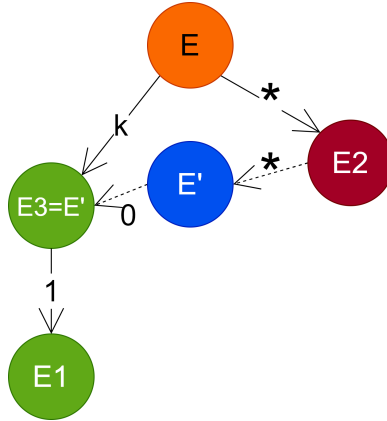
Inductive Case

Next we assume confluence for up to k steps, and attempt to prove for $k + 1$ steps.



We have two cases:

Case 1: $E_3 = E'$, this is easy as $E_2 \rightarrow^* E' \rightarrow^0 E_3 \rightarrow^1 E_1$.



Case 2: $E_3 \rightarrow^1 E'' \rightarrow^* E'$, in this case as $E_3 \rightarrow^1 E_1$ we know by determinacy that $E'' = E_1$ and hence $E_1 \rightarrow^* E'$.

