1 Bounded Decision Tree Learner for Horn Samples

Theorem 1.1. *If the input set of points X is separable and the input Horn Constraints are satisfiable, then the Learner always terminates with a decision tree consistent with the Horn sample (X,C).[1]*

Definition 1.2. *Given an Horn sample $S = (X, C)$. Define the equivalence class $\equiv_m$ m on X:*

$$s \equiv_m s' \iff \text{there is no predicate of the form } a_i \leq c \text{ with } |c| \leq m \\ \text{that separates } s \text{ and } s'.$$

Definition 1.3. *We can augment a sample $S = (X, C)$ to obtain a* m-augmented Horn Sample of S, *denoted by $S \oplus m$. To obtain $S \oplus m$, we first construct the set $E = \{(s, s') \mid s \equiv_m s'\}$ and add it to a our Horn Constraints, $(X, C \cup E)$.*

Theorem 1.4. *The following holds:*

1. *If $S \oplus m$ is not valid, then there is* no Boolean formula with absolute maximum threshold m *that is consistent with (X, C). In the overall learning loop, we would increment m in this case and restart learning from (X, C).*
2. *If $S \oplus m$ is valid, then calling our Decision Tree Learner for Horn Samples on $S \oplus m$ while restricting it to predicates that use thresholds with absolute values at most m is* guaranteed to terminate and return a tree that is consistent with S.

Proof.

1. Assume $S \oplus m$ is not valid. This means there exists a horn clause$(s_1 \wedge s_2 \wedge \cdots \wedge s_n \rightarrow s) \in C$ where $s_i \in X$ is forced to a positive classification and $s \in X$ is forced to a negative classification for every valuation of X. Now for the sake of contradiction assume that there is a Boolean formula f with absolute maximum threshold m that is consistent with S. Thus, f satisfies all the new horn constraints added and consequently $S \oplus m$. Therefore, f must classify the s_i as positive and s as negative, which doesn't satisfy $s_1 \wedge s_2 \wedge \cdots \wedge s_n \rightarrow s$.
2. Assume now that $S \oplus m$ is valid. We need to show that the input set of points is separable. For this assume that our bounded Decision Tree Learner for Horn Samples processes a node with the sample of $S \oplus m$ and the nodes contains a point p that is forced to positive and a point n that is forced to negative. For the sake of contradiction assume that the learner can't find an split within a threshold $|c| \leq m$. Thus, p and n are in the same m-equivalence class $p \equiv_m n$. This means $(p, n) \in C$. This is a contradiction with S' being consistent since p is forced to positive and n is forced to negative.

   Now, we have that the input set of points is separable and that $S \oplus m$ is valid and can use Theorem 1.1 to get that the Learner always terminates with a decision tree consistent with the Horn sample (X, C).

□

2 Questions

1. is forced to positive/negative the right thing? (for ICE it was just positive or negative)
2. Bei 2. show is separable for pos/neg? what about unclassified points?

References

[1] Deepak D'Souza, P Ezudheen, Pranav Garg, P Madhusudan, and Daniel Neider. Horn-ice learning for synthesizing invariants and contracts. *arXiv preprint arXiv:1712.09418*, 2017.