Parabola construction and intersection

Parabola equation

The equation of the parabola ϕ has the form:

$$\phi =: \{\ (x, y) \mid rx^2 + sy^2 + txy + ux + vy + w = 0\ \}$$

Line equation and point coordinates

The equation of a line l is given as:

$$l =: \{\ (x, y) \mid ax + by + c = 0\ \}$$

the point coordinates for the focus are:

$$f(f_x, f_y)$$

Definition of a parabola

A parabola is defined as the set of all points (x, y) that are equidistant from a line (the directrix l) and a point (the focus f):

$$\phi := \{\ (x, y) \mid d\left((x, y), l\right) = d\left((x, y), f\right)\ \}$$

therefore we can find the equation of a parabola in function of its directrix and focus.

$$\begin{gathered}
d\left((x, y), l\right) = d\left((x, y), f\right) \\
\left[d\left((x, y), l\right)\right]^2 = \left[d\left((x, y), f\right)\right]^2 \\
\left(\frac{|ax + by + c|}{\sqrt{a^2 + b^2}}\right)^2 = \left(\sqrt{|x - f_x|^2 + |y - f_y|^2}\right)^2 \\
a^2x^2 + b^2y^2 + 2abxy + 2acx + 2bcy + c^2 = \left(x^2 - 2f_x x + f_x^2 + y^2 - 2f_y y + f_y^2\right) \cdot \left(a^2 + b^2\right)
\end{gathered}$$

$$2abxy + 2acx + 2bcy + c^2 = b^2x^2 + a^2y^2 + (-2a^2f_x - 2b^2f_x)x + (-2a^2f_y - 2b^2f_y)y + a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2$$

$$-b^2x^2 + -a^2y^2 + (2ab)xy + (2ac + 2a^2f_x + 2b^2f_x)x + (2bc + 2a^2f_y + 2b^2f_y)y + c^2 - a^2f_x^2 - a^2f_y^2 - b^2f_x^2 - b^2f_y^2 = 0$$

We get an equation for the parabola $\phi : rx^2 + sy^2 + txy + ux + vy + w = 0$ with coefficients:

- $r = -b^2$
- $s = -a^2$
- $t = 2ab$
- $u = 2ac + 2a^2f_x + 2b^2f_x$
- $v = 2bc + 2a^2f_y + 2b^2f_y$
- $w = c^2 - a^2f_x^2 - a^2f_y^2 - b^2f_x^2 - b^2f_y^2$

Parabola–line intersection

The intersections of a line and a parabola can be found by solving the equation of the line for y, then inserting it into the equation of the parabola to get a quadratic equation in x. If the coefficient b of the line is equal to 0, instead we can substitute x in the parabola equation and get a quadratic equation in y.
The resulting x(s) (or y(s)) of the equation can be input in the line equation to get the corresponding value(s) of y (or x), each indicating a point of intersection.
There can be 0, 1, or 2 intersections.

If $b \neq 0$:

$$\phi \cap l :=$$

$$\begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ 0 &= ax + by + c \end{cases} \iff \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ y &= \dfrac{-c}{b} + \dfrac{-ax}{b} \end{cases} \iff$$

$$\begin{aligned}
rx^2 + s\left(\frac{-c}{b} + \frac{-ax}{b}\right)^2 + tx\left(\frac{-c}{b} + \frac{-ax}{b}\right) + ux + v\left(\frac{-c}{b} + \frac{-ax}{b}\right) + w &= 0 \\
rx^2 + s\left(\frac{(-c) + (-ax)}{b}\right)^2 + tx\left(\frac{(-c) + (-ax)}{b}\right) + ux + v\left(\frac{(-c) + (-ax)}{b}\right) + w &= 0 \\
rx^2 + s\frac{\big((-c) + (-ax)\big)^2}{b^2} + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
rx^2 + s\frac{c^2 + 2acx + a^2x^2}{b^2} + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
rx^2 + \frac{sc^2}{b^2} + \frac{2acs}{b^2}x + \frac{sa^2}{b^2}x^2 + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
rx^2 + \frac{sa^2}{b^2}x^2 + \frac{-at}{b}x^2 + \frac{2acs}{b^2}x + \frac{-ct}{b}x + \frac{-av}{b}x + ux + \frac{sc^2}{b^2} + \frac{-cv}{b} + w &= 0 \\
\left(r + \frac{sa^2}{b^2} + \frac{-at}{b}\right)x^2 + \left(\frac{2acs}{b^2} + \frac{-ct}{b} + \frac{-av}{b} + u\right)x + \left(\frac{sc^2}{b^2} + \frac{-cv}{b} + w\right) &= 0
\end{aligned}$$

Instead if $b = 0$:

$$\phi \cap l :=$$

$$\begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ 0 &= ax + c \end{cases} \iff \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ x &= \dfrac{-c}{a} \end{cases} \iff$$

$$\begin{aligned}
r\left(\frac{-c}{a}\right)^2 + sy^2 + t\left(\frac{-c}{a}\right)y + u\left(\frac{-c}{a}\right) + vy + w &= 0 \\
\frac{rc^2}{a^2} + sy^2 + \frac{-ct}{a}y + \frac{-cu}{a} + vy + w &= 0 \\
(s)\,y^2 + \left(v + \frac{-ct}{a}\right)y + \left(\frac{rc^2}{a^2} + \frac{-cu}{a} + w\right) &= 0
\end{aligned}$$