

# Parabola construction and intersection

## Parabola equation

The equation of the parabola  $\phi$  has the form:

$$\phi =: \{ (x, y) \mid rx^2 + sy^2 + txy + ux + vy + w = 0 \}$$

## Line equation and point coordinates

The equation of a line  $l$  is given as:

$$l =: \{ (x, y) \mid ax + by + c = 0 \}$$

the point coordinates for the focus are:

$$f(f_x, f_y)$$

## Definition of a parabola

A parabola is defined as the set of all points  $(x, y)$  that are equidistant from a line (the directrix  $l$ ) and a point (the focus  $f$ ):

$$\phi := \{ (x, y) \mid d((x, y), l) = d((x, y), f) \}$$

therefore we can find the equation of a parabola in function of its directrix and focus.

$$\begin{aligned} d((x, y), l) &= d((x, y), f) \\ [d((x, y), l)]^2 &= [d((x, y), f)]^2 \\ \left( \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} \right)^2 &= \left( \sqrt{|x - f_x|^2 + |y - f_y|^2} \right)^2 \\ a^2x^2 + b^2y^2 + 2abxy + 2acx + 2bcy + c^2 &= (x^2 - 2f_x x + f_x^2 + y^2 - 2f_y y + f_y^2) \cdot (a^2 + b^2) \\ 2abxy + 2acx + 2bcy + c^2 &= b^2x^2 + a^2y^2 + (-2a^2f_x - 2b^2f_x)x + (-2a^2f_y - 2b^2f_y)y + a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2 \\ -b^2x^2 - a^2y^2 + (2ab)xy + (2ac + 2a^2f_x + 2b^2f_x)x + (2bc + 2a^2f_y + 2b^2f_y)y + c^2 - a^2f_x^2 - a^2f_y^2 - b^2f_x^2 - b^2f_y^2 &= 0 \end{aligned}$$

We get an equation for the parabola  $\phi : rx^2 + sy^2 + txy + ux + vy + w = 0$  with coefficients:

- ▷  $r = -b^2$
- ▷  $s = -a^2$
- ▷  $t = 2ab$
- ▷  $u = 2ac + 2a^2f_x + 2b^2f_x$
- ▷  $v = 2bc + 2a^2f_y + 2b^2f_y$
- ▷  $w = c^2 - a^2f_x^2 - a^2f_y^2 - b^2f_x^2 - b^2f_y^2$

## Parabola–line intersection

The intersections of a line and a parabola can be found by solving the equation of the line for  $y$ , then inserting it into the equation of the parabola to get a quadratic equation in  $x$ . If the coefficient  $b$  of the line is equal to 0, instead we can substitute  $x$  in the parabola equation and get a quadratic equation in  $y$ .

The resulting  $x(s)$  (or  $y(s)$ ) of the equation can be input in the line equation to get the corresponding value(s) of  $y$  (or  $x$ ), each indicating a point of intersection.

There can be 0, 1, or 2 intersections.

If  $b \neq 0$ :

$$\begin{aligned}
 \phi \cap l &:= \\
 \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ 0 &= ax + by + c \end{cases} &\iff \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ y &= \frac{-c}{b} + \frac{-ax}{b} \end{cases} \iff \\
 rx^2 + s \left( \frac{-c}{b} + \frac{-ax}{b} \right)^2 + tx \left( \frac{-c}{b} + \frac{-ax}{b} \right) + ux + v \left( \frac{-c}{b} + \frac{-ax}{b} \right) + w &= 0 \\
 rx^2 + s \left( \frac{(-c) + (-ax)}{b} \right)^2 + tx \left( \frac{(-c) + (-ax)}{b} \right) + ux + v \left( \frac{(-c) + (-ax)}{b} \right) + w &= 0 \\
 rx^2 + s \frac{((-c) + (-ax))^2}{b^2} + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
 rx^2 + s \frac{c^2 + 2acx + a^2x^2}{b^2} + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
 rx^2 + \frac{sc^2}{b^2} + \frac{2acs}{b^2}x + \frac{sa^2}{b^2}x^2 + \frac{-ct}{b}x + \frac{-at}{b}x^2 + ux + \frac{-cv}{b} + \frac{-av}{b}x + w &= 0 \\
 rx^2 + \frac{sa^2}{b^2}x^2 + \frac{-at}{b}x^2 + \frac{2acs}{b^2}x + \frac{-ct}{b}x + \frac{-av}{b}x + ux + \frac{sc^2}{b^2} + \frac{-cv}{b} + w &= 0 \\
 \left( r + \frac{sa^2}{b^2} + \frac{-at}{b} \right) x^2 + \left( \frac{2acs}{b^2} + \frac{-ct}{b} + \frac{-av}{b} + u \right) x + \left( \frac{sc^2}{b^2} + \frac{-cv}{b} + w \right) &= 0
 \end{aligned}$$

Instead if  $b = 0$ :

$$\begin{aligned}
 \phi \cap l &:= \\
 \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ 0 &= ax + c \end{cases} &\iff \begin{cases} 0 &= rx^2 + sy^2 + txy + ux + vy + w \\ x &= \frac{-c}{a} \end{cases} \iff \\
 r \left( \frac{-c}{a} \right)^2 + sy^2 + t \left( \frac{-c}{a} \right) y + u \left( \frac{-c}{a} \right) + vy + w &= 0 \\
 \frac{rc^2}{a^2} + sy^2 + \frac{-ct}{a}y + \frac{-cu}{a} + vy + w &= 0 \\
 (s)y^2 + \left( v + \frac{-ct}{a} \right) y + \left( \frac{rc^2}{a^2} + \frac{-cu}{a} + w \right) &= 0
 \end{aligned}$$

## Parabola tangent at a point