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On the complexity of higher order abstract Voronoi diagrams ☆

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ABSTRACT

Abstract Voronoi diagrams (AVDs) are based on bisecting curves enjoying simple combinatorial properties, rather than on the geometric notions of sites and circles. They serve as a unifying concept. Once the bisector system of any concrete type of Voronoi diagram is shown to fulfill the AVD properties, structural results and efficient algorithms become available without further effort.

In a concrete order-k Voronoi diagram, all points are placed into the same region that have the same k nearest neighbors among the given sites. This paper is the first to study *abstract Voronoi diagrams of arbitrary order k*. We prove that their complexity in the plane is upper bounded by $2k(n-k)$. So far, an $O(k(n-k))$ bound has been shown only for point sites in the Euclidean and L_p planes, and, recently, for line segments, in the L_p metric. These proofs made extensive use of the geometry of the sites.

Our result on AVDs implies a $2k(n-k)$ upper bound for a wide range of cases for which only trivial upper complexity bounds were previously known, and a slightly sharper bound for the known cases. Also, our proof shows that the reasons for this bound are combinatorial properties of certain permutation sequences.

1. Introduction

Voronoi diagrams are useful structures, known in many areas of science. The underlying idea goes back to Descartes [13]. There are sites p, q that exert influence on their surrounding space, M. Each point of M is assigned to that site p (resp. to those sites $p_1, \dots, p_k$) for which the influence is strongest. Points assigned to the same site(s) form *Voronoi regions*.

The nature of the sites, the measure of influence, and space M can vary. The *order*, k, can range from 1 to $n-1$ if n sites are given. For $k=1$ the standard *nearest* Voronoi diagram results, while for $k=n-1$ the *farthest* Voronoi diagram is obtained, where all points of M having the same farthest site belong in the same Voronoi region. In this paper we are interested in values of k between 1 and $n-1$; here an *order-k* Voronoi region contains all points that have the same k nearest sites. Observe that the order of the Voronoi diagram has nothing to do with the dimension of the space of its

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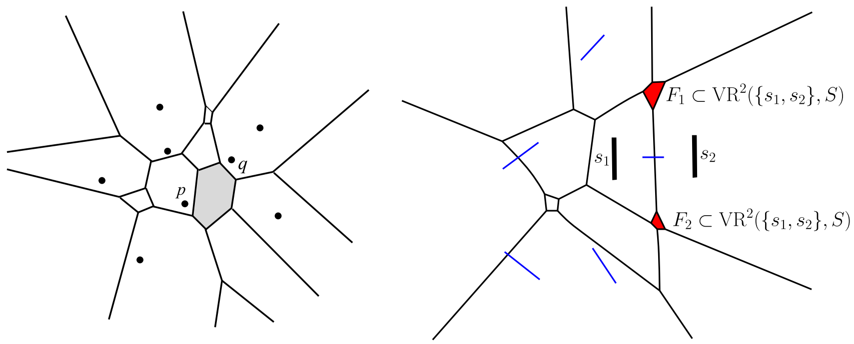

Fig. 1. Order-2 diagrams of points and line segments. The shadowed region in the left picture belongs to the sites p and q and is connected, in the right picture the shadowed faces belong to the region of s_1 and s_2 which is disconnected.

embedding. In this paper we consider diagrams in the 2-dimensional plane. For more Voronoi diagrams see the surveys and monographs [5–7,11,14,24].

A lot of attention has been given to nearest Voronoi diagrams in the plane. Many concrete cases have the following features in common. The locus of all points at identical distance to two sites p, q is an unbounded curve $J(p,q)$. It bisects the plane into two domains, $D(p,q)$ and $D(q,p)$; domain $D(p,q)$ consists of all points closer to p than to q. Intersecting all $D(p,q)$, where $q \neq p$ for a fixed p, results in the Voronoi region $\mathrm{VR}(p,S)$ of p with respect to site set S. It equals the set of all points with unique nearest neighbor p in S. If geodesics exist, Voronoi regions are pathwise connected, and the union of their closures covers the plane, since each point has at least one nearest neighbor in S.

In *abstract Voronoi diagrams* [17,18] (AVDs, for short) no sites or distance measures are given. Instead, one takes as primary objects unbounded curves $J(p,q) = J(q,p)$, homeomorphic to unbounded lines, together with the domains $D(p,q)$ and $D(q,p)$ they separate. *Nearest abstract Voronoi regions* are defined by

$$\mathrm{VR}(p,S) := \bigcap_{q \in S \setminus \{p\}} D(p,q),$$

and now one *requires* that the following properties hold true for each nonempty subset S' of S.

(A1) Each nearest Voronoi region $\mathrm{VR}(p,S')$ is pathwise connected.
(A2) Each point of the plane belongs to the closure of a nearest Voronoi region $\mathrm{VR}(p,S')$.

Two more, rather technical, assumptions on the curves $J(p,q)$ are stated in Definition 1 below. It has been shown that the resulting nearest AVDs—the plane minus all Voronoi regions—are planar graphs of complexity $O(n)$. They can be constructed, by randomized incremental construction, in $O(n \log n)$ many steps [18,19,22]. Moreover, properties (A1) and (A2) need only be checked for all subsets S' of size three [18]. This makes it easier to verify that a concrete Voronoi diagram is under the umbrella of the AVD concept. Examples of such applications can be found in [1–3,8,16,22]. *Farthest abstract Voronoi diagrams* consist of regions $\mathrm{VR}^*(p,S) := \bigcap_{q \in S \setminus \{p\}} D(q,p)$. They have been shown to be trees of complexity $O(n)$, computable in expected $O(n \log n)$ many steps [23].

In this paper we consider, for the first time, general *order-k abstract Voronoi regions*, defined by

$$\mathrm{VR}^k(P,S) := \bigcap_{p \in P,\ q \in S \setminus P} D(p,q),$$

for each subset P of S of size k. The order-k abstract Voronoi diagram $V^k(S)$ is defined to be the complement of all order-k Voronoi regions in the plane; it equals the collection of all edges that separate order-k Voronoi regions (Lemma 4). In addition to properties (A1) and (A2) we shall assume the following for each nonempty subset S' of S.

(A3) No nearest Voronoi region $\mathrm{VR}(p,S')$ is empty.

Observe that our axioms are required to hold only for the *nearest*-, i.e. for the order-1 Voronoi diagram. In Lemma 1 we prove that property (A3) need only be tested for all subsets S' of size four. Clearly, (A3) holds in all concrete cases where each nearest region contains its site.

Fig. 1 shows two concrete order-2 diagrams of points and line segments under the Euclidean metric. We observe that the order-2 Voronoi region of line segments s_1, s_2 is disconnected, whereas for points the higher-order regions are still connected. In general, a Voronoi region in $V^2(S)$ can have $n-1$ connected components [25]. Fig. 2 depicts a curve system fulfilling all the required properties, and the resulting abstract Voronoi diagrams of orders 1 to 4. An index p placed next to a curve indicates the side of the curve where $D(p,q)$ lies. The dashed curves represent arcs of the bisectors that do not appear as Voronoi edges in the diagram. The order-2 region of p_1 and p_2 consists of four connected components.

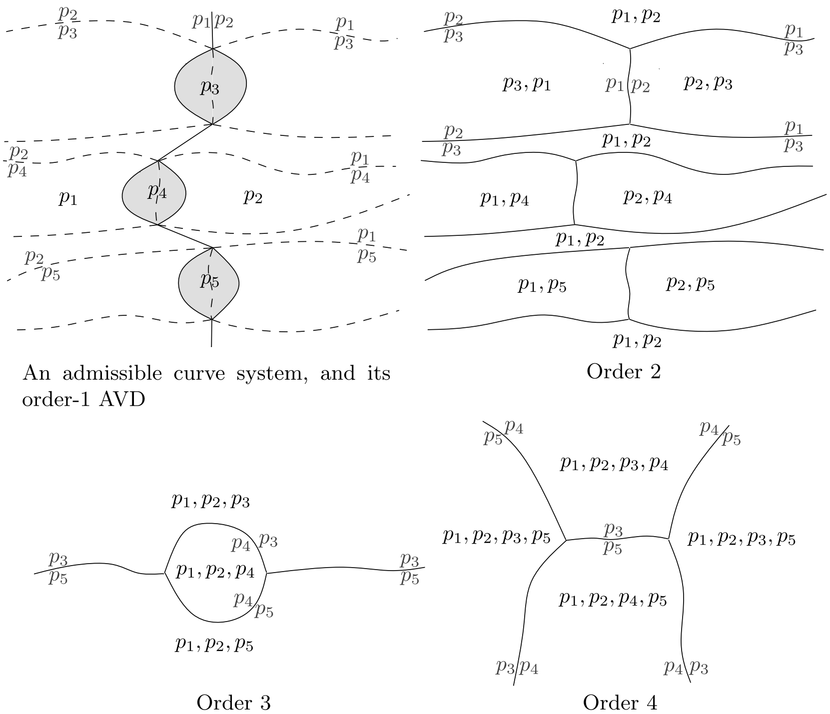

Fig. 2. AVD of 5 sites in all orders.

In this paper we are proving the following result on the number of 2-dimen-sional faces of order-k abstract Voronoi diagrams.

Theorem 1. *(Short version.) The abstract order-k Voronoi diagram* $V^k(S)$ *has at most* $2k(n-k)$ *many faces.*

So far, an $O(k(n-k))$ bound had been shown only for points [20,21], and recently for line-segments [25], in the L_p metric.[1] The structural properties of the diagram for line segments turned out surprisingly different from those for points, including disconnected Voronoi regions and a lack of symmetry between the number of unbounded regions in the order-k and order-$(n-k)$ diagrams. The differences clearly propagate in the abstract setting. The proofs of these results are based on geometric arguments, including results on k-sets,[2] point-line duality and $\leq k$-levels in arrangements. None of these arguments applies to abstract Voronoi diagrams.

However, the upper bound on k-sets established in [4] had a combinatorial proof; it was obtained by analyzing the cyclic permutation sequences that result when projecting n point sites onto a rotating line. In such a sequence, consecutive permutations differ by a switch of adjacent elements, and permutations at distance $\binom{n}{2}$ are inverse to each other.

In this paper we prove Theorem 1 using combinatorial arguments. We traverse the unbounded edges of higher order AVDs, and obtain a strictly larger class of cyclic permutation sequences, where consecutive permutations differ by switches and any two elements switch exactly twice. Our proof is based on a tight upper bound to the number of switches that can occur among the first $k+1$ elements; see Lemma 9. It is interesting to observe that in our class, each permutation sequence can be realized by an AVD (Lemma 10), while this is not the case for the sequences obtained by point projection [15].

It is tempting to think that one could easily use the techniques from Clarkson and Shor [12] to prove these bounds. But for abstract Voronoi diagrams there appear some special phenomena which complicate this task, as will be discussed in the conclusion, Section 6.

To avoid technical complications we are assuming, in the first 4 sections of this paper, that any two input curves $J(p,q)$ intersect in a finite number of points, these intersections are transversal, and that Voronoi vertices are of degree 3. How to get rid of the first assumption has been shown for the case of nearest AVDs in [18]. In Section 5 we show that also vertices of higher degree than 3 can be allowed.

Theorem 1 implies a $2k(n-k)$ upper complexity bound on a wide range of order-k Voronoi diagrams for which no good bounds were previously known. For example, sites may be disjoint convex objects of constant complexity in L_2 or under the

1 In the L_1 and L_∞ metrics, the bound is slightly tighter, i.e., $O((n-k)^2)$ for $k > n/2$ [25].

2 We call a subset of size k of n points a *k-set* if it can be separated by a line passing through two other points. Such k-sets correspond to unbounded order-$(k+1)$ Voronoi edges.

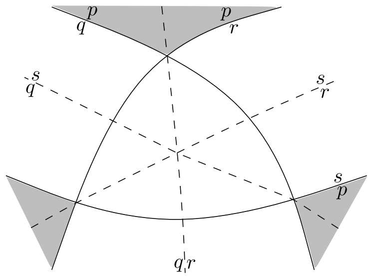

Fig. 3. An AVD of 4 sites p, q, r, s. In each diagram of three of these sites no Voronoi region is empty (the regions of p are shaded) but in the diagram of all 4 sites the region of p is empty.

Hausdorff metric. For point sites, distance can be measured by any metric d satisfying the following conditions: points in general position have unbounded bisector curves; d-circles are of constant algebraic complexity; each d-circle contains an L_2-circle and vice versa; for any two points $a \neq c$ there is a third point $b \neq a, c$ such that $d(a,c) = d(a,b) + d(b,c)$ holds. This includes all convex distance functions of constant complexity, but also the Karlsruhe metric where motions are constrained to radial or circular segments with respect to a fixed center point. A third example are point sites with additive weights a_p, a_q that satisfy $|a_p - a_q| < |p - q|$, for any two sites $p \neq q$; see [6] for a discussion of these examples.

Using Theorem 1, the first algorithm computing the abstract order-k Voronoi diagram has been presented in [10]. It runs in expected time $O(kn^{1+\epsilon})$.

The rest of this paper is organized as follows. In Section 2 we present some basic facts about AVDs. Then, in Section 3, permutation sequences will be studied, in order to establish an upper bound to the number of unbounded Voronoi edges of order at most k. This will lead, in Section 4, to a tight upper bound for the number of faces of order k. Finally, in Section 5 we show how to get rid of the general position assumption and maintain the same result for the number of order-k faces.

A preliminary version of this work with incomplete proofs and without generalizations appeared in [9].

2. Preliminaries

In this section we present some basic facts on abstract Voronoi diagrams of various orders.

Definition 1. A curve system $J := \{J(p,q) : p \neq q \in S\}$ is called *admissible* if it fulfills, axioms (A1), (A2), (A3) stated in the introduction, and in addition the following ones.

(A4) Each curve $J(p,q)$, where $p \neq q$, is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole.

(A5) Any two curves $J(p,q)$ and $J(r,t)$ have only finitely many intersection points, and these intersections are transversal.

Axiom (A4) implies that each bisecting curve $J(p,q)$ is homeomorphic to an unbounded line in the plane. As part of the Jordan curve theorem, $J(p,q)$ separates the plane into two domains, the two dominance regions $D(p,q)$ and $D(q,p)$. Further, for each point z on $J(p,q)$ there exists a sufficiently small neighborhood $U(z)$, such that $U(z) \setminus J(p,q)$ consists of exactly two connected components, see also [18, Theorem 4] for a more detailed discussion.

Each transversal intersection v between two related bisectors $J(p,q)$ and $J(p,r)$ is a Voronoi vertex in the order-1 diagram of $\{p,q,r\}$, and $J(q,r)$ runs also through v, compare [17]. Together with the *Euler formula* and our axioms we obtain that any two related bisectors $J(p,q)$ and $J(p,r)$ may intersect in at most two points.

Fortunately, verification of our axioms can be based on constant size examples. Pairs and quadruples of sites are clearly sufficient to verify axioms (A4) and (A5). Axioms (A1) and (A2) can be verified by checking all subsets of S of size 3, see [18, Section 4.3]. For axiom (A3) we need to check subsets of size 4 as shown in the following lemma; checking subsets of size 3 is not always enough, as shown in Fig. 3.

Lemma 1. *To verify axiom (A3), assuming that axioms (A1) and (A2) hold, it is sufficient to check all subsets S' of S of size 4.*

Proof. If all bisecting curves are straight lines, (A3) follows from Helly's theorem on convex sets, stating that for a finite collection of convex sets in the plane, if each intersection of 3 of these sets is nonempty, then the intersection of the whole collection is nonempty.

Our bisecting curves $J(p,q)$ do not necessarily define convex domains $D(p,q)$ from which the Voronoi region $\mathrm{VR}(p,S) = \bigcap_{q \in S \setminus \{p\}} D(p,q)$ is constructed. This is why a new proof is needed.

So, assume that $\mathrm{VR}(p,S')$ is nonempty for all subsets S' of size 4. Let $|S| > 4$ and for the sake of a contradiction assume that $\mathrm{VR}(p,S) = \emptyset$. Let $q_1, q_2, q_3 \in S \setminus \{p\}$ be pairwise different. By induction on the size of S, there exist points $x_i \in \mathrm{VR}(p, S \setminus \{q_i\})$, $i \in \{1,2,3\}$. Since no point lies in $\mathrm{VR}(p,S)$, we have $x_i \in D(q_i, p)$. By (A1), there are paths P_{ij} connecting

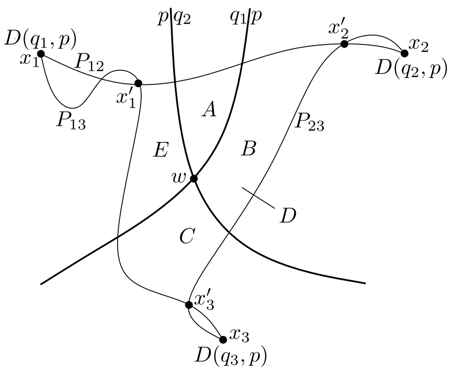

Fig. 4. In the proof of Lemma 1, curves $J(p,q_1)$ and $J(p,q_2)$ meet at point w.

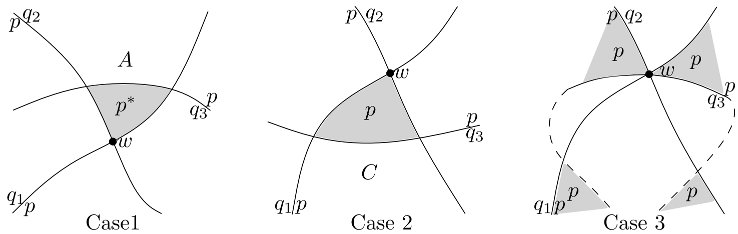

Fig. 5. Discussion of three cases.

x_i and x_j in $\mathrm{VR}(p, S \setminus \{q_i, q_j\}) \subseteq D(p, q_k)$, where $\{i, j, k\} = \{1, 2, 3\}$. Because $\mathrm{VR}(p, S)$ may not be empty, P_{ij} has to be contained in $D(q_i, p) \cup D(q_j, p)$, thus when traversing P_{ij} from x_i to x_j one must cross $J(p, q_j)$ first, before one crosses $J(p, q_i)$. W.l.o.g. we can assume that P_{ij} intersects $J(p, q_i)$ and $J(p, q_j)$ exactly once each.

Let x'_1 be the first point of P_{13} one meets, when traversing P_{12} from its intersection with $J(p, q_2)$ in the direction of x_1, let x'_2 be the first point on P_{23} one meets, when traversing P_{12} from its intersection with $J(p, q_1)$ in the direction of x_2, and let x'_3 be the first point on P_{23} one meets, when traversing P_{13} from its intersection with $J(p, q_1)$ in the direction of x_3. Because P_{23} cannot intersect P_{12} in $D(q_1, p)$, P_{13} cannot intersect P_{12} in $D(q_2, p)$, and P_{12} cannot intersect P_{23} in $D(q_3, p)$, the points x'_1, x'_2 and x'_2 together with the segments of P_{12}, P_{23}, and P_{13} in between form a simple closed curve which bounds a domain D.

We have x'_i in $D(q_i, p)$ and in $D(p, q)$ for all $q \in S \setminus \{q_i\}$ and $J(p, q_i)$ may not intersect P_{jk} for $\{i, j, k\} = \{1, 2, 3\}$. Further, because $P_{12} \subseteq D(q_1, p) \cup D(q_2, p)$ the two bisectors $J(p, q_1)$ and $J(p, q_2)$ must intersect transversally in a point w contained in the domain D; see Fig. 4. The two bisectors $J(p, q_1)$ and $J(p, q_2)$ divide the domain D into four subdomains around w, $A \subseteq D(q_1, p) \cap D(q_2, p)$, $B \subseteq D(p, q_1) \cap D(q_2, p)$, $C \subseteq D(p, q_2) \cap D(p, q_1)$, and $E \subseteq D(p, q_2) \cap D(q_1, p)$.

The bisector $J(p, q_3)$ enters D through P_{13} and E such that the segment of P_{13} in the direction of x'_1 is in $D(p, q_3)$ and the segment in the direction of x'_3 is in $D(q_3, p)$. $J(p, q_3)$ leaves D through B and P_{23} such that the segment of P_{23} in the direction of x'_2 is in $D(p, q_3)$ and the segment in the direction of x'_3 is in $D(q_3, p)$. Because $J(p, q_3)$ may intersect $J(p, q_1)$ and $J(p, q_2)$ in at most two points, if it traverses A then it cannot traverse C and vice versa, and it can traverse A and C in at most one segment.

Now there are three cases, see Fig. 5.

Case 1: $J(p, q_3)$ traverses A. Then it forms a bounded domain of the farthest region $\mathrm{VR}^*(p, \{p, q_1, q_2, q_3\})$ in A incident to w. This contradicts Lemma 7 (given later) whose proof is independent from this lemma.

Case 2: $J(p, q_3)$ traverses C. Then it bounds $\mathrm{VR}(p, \{p, q_1, q_2, q_3\})$ in C incident to w. Let $x \in \mathrm{VR}(p, \{p, q_1, q_2, q_3\})$. Because $\mathrm{VR}(p, S) = \emptyset$ there must be $q \in S \setminus \{p, q_1, q_2, q_3\}$ such that $x \in D(q, p)$. But the boundary of the domain D is contained in $D(p, q)$, implying that $J(p, q)$ is closed—a contradiction to (A4).

Case 3: $J(p, q_3)$ traverses neither A nor C and must hence run through w. Since $\mathrm{VR}(p, \{p, q_1, q_2, q_3\})$ must not be empty, by (A3), $J(p, q_3)$ has to traverse through $D(p, q_1) \cap D(p, q_2)$, but then it must intersect $J(p, q_1)$ or $J(p, q_2)$ in another point, resulting in a disconnection of $\mathrm{VR}(p, \{p, q_1, q_3\})$ or $\mathrm{VR}(p, \{p, q_2, q_3\})$ that contradicts (A1). □

The following fact will be very useful in the sequel. Its proof can be found in [18, Lemma 5].

Lemma 2. *For all p, q, r in S, $D(p, q) \cap D(q, r) \subseteq D(p, r)$ holds.*

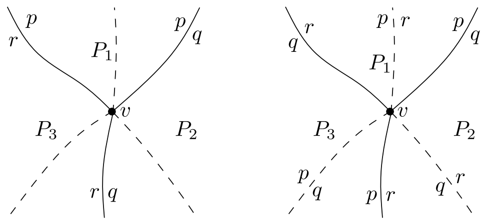

Fig. 6. A new vertex v to the left and an old vertex to the right. Solid curves indicate the Voronoi edges and dashed ones the prolongations of the bisectors.

Consequently, a total ordering of the set S is possible for $x \notin \bigcup_{p,q \in S} J(p,q)$, where

$$p <_x q :\Longleftrightarrow x \in D(p,q).$$

Informally, one can interpret $p <_x q$ as "x is closer to p than to q". We will write $p < q$ if it is clear which $x \in \mathbb{R}^2$ we are referring to.

As a direct consequence we show that property (A2) holds also for abstract order-k Voronoi regions.

Lemma 3. *Let $J = \{J(p,q) : p \neq q \in S\}$ be an admissible curve system. Then for each $k \in \{1,\dots,n-1\}$*

$$\mathbb{R}^2 = \bigcup_{P \subseteq S, |P|=k} \overline{VR^k(P,S)}.$$

Proof. Let $x \in \mathbb{R}^2$. If x is not contained in any bisecting curve $J(p,q)$ then it belongs to the order-k region $\mathrm{VR}^k(P,S)$, where $P = \{p_1,\dots,p_k\}$ are the k smallest elements of S with respect to the ordering $<_x$. Otherwise, x lies on the boundary of a domain $D \subset \mathbb{R}^2 \setminus \bigcup_{p \neq q \in S} J(p,q)$, and D fully belongs to an order-k region. $\square$

The proofs of the following Lemmata 4 and 5 are similar to the proof of Lemma 3. Lemma 5 indicates that two neighboring regions differ in exactly one site.

Lemma 4.

$$V^k(S) = \bigcup_{\substack{P \neq P' \subset S\\ |P|=|P'|=k}} \overline{VR^k(P,S)} \cap \overline{VR^k(P',S)}$$

Lemma 5. *If the intersection $E := \overline{VR^k(P,S)} \cap \overline{VR^k(P',S)}$ is not empty, there are sites $p \in P$ and $p' \in P'$ such that $P \setminus \{p\} = P' \setminus \{p'\}$, and $E \subseteq J(p,p')$ holds. For each point $x \in VR^k(P,S)$ near E, index p is the k-th with respect to $<_x$, while for points x' in $VR^k(P',S)$ index p' appears at position k.*

In particular, $D(p,p')$ is on the same side of $J(p,p')$ as $\mathrm{VR}^k(P,S)$.

If F, F' are connected components (*faces*) of $\mathrm{VR}^k(P,S)$ and $\mathrm{VR}^k(P',S)$, respectively, the intersection $\overline{F} \cap \overline{F'}$ can be empty, or otherwise be of dimension 0 (*Voronoi vertices*) or 1 (*Voronoi edges*).

For the next lemma we assume that all vertices are of degree 3. Recall that if two related bisectors $J(p,q)$ and $J(p,r)$ intersect in a point v, then $J(q,r)$ runs also through v, [17], and v is a Voronoi vertex of the order-1 Voronoi diagram of $\{p,q,r\}$. Thus, when assuming that all Voronoi vertices are of degree 3, no other bisector related to one of the sites p,q,r can run through v.

Let $P_1, P_2, P_3 \subset S$ be the sets defining the adjacent to v order-k Voronoi regions in clockwise order, see Fig. 6. As with the concrete order-k Voronoi diagrams, vertex v can be of two types, depending on the nature of sets P_1, P_2, P_3 [20]. There are two cases. In the first case, there exists a set $H \subset S$ of size $k-1$ and three more sites $p,q,r \in S$ satisfying

$$P_1 = H \cup \{p\}, \quad P_2 = H \cup \{q\}, \quad P_3 = H \cup \{r\}.$$

Then vertex v is called *new* in $V^k(S)$, or *of nearest type*. In the second case, there is a subset $K \subset S$ of size $k-2$ and three more sites $p,q,r \in S$ such that

$$P_1 = K \cup \{p,r\}, \quad P_2 = K \cup \{q,r\}, \quad P_3 = K \cup \{p,q\}.$$

Then vertex v is called *old* in $V^k(S)$, or *of farthest type*.

To see that these are indeed the only two cases, we walk around v in clockwise order along the boundary of an ϵ-neighborhood, see Fig. 6. Let the edge between P_1 and P_2 belong to the bisector $J(p,q)$, such that $P_1 \subseteq D(p,q)$ and $P_2 \subseteq D(q,p)$. If q is still last in the ordering of the sites when we reach the edge between P_2 and P_3, then this edge must

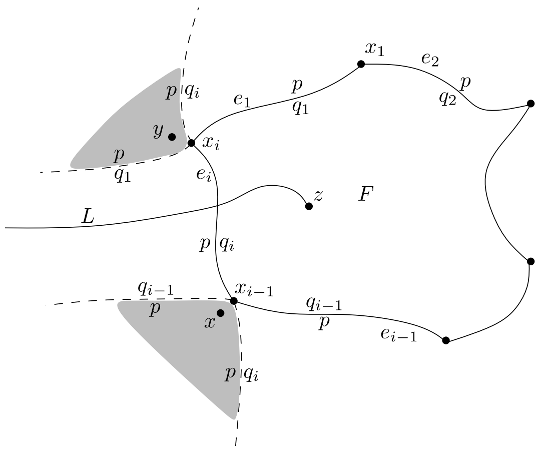

Fig. 7. Farthest AVDs cannot contain bounded regions.

belong to the bisector $J(q,r)$ for an $r \in S \setminus \{p,q\}$. Thus, bisector $J(r,p)$ must run through v, and because no other bisector related to r may contain v, a segment of $J(r,p)$ defines the edge between P_2 and P_3. Thus, we have the first case.

If q is no longer last in this ordering, then we must have crossed a bisector $J(q,r)$ related to q while traversing P_2. Again, $J(p,r)$ must contain v, and thus, no other bisector related to r may run through v. Thus, r is in last position when we reach the edge between P_2 and P_3 and the edge must belong to $J(r,p)$. Next, because all bisector intersections are transversal (A5), we cross the bisector $J(p,q)$, and p changes its position in the ordering with q, which is now in last position. The next bisector is $J(q,r)$, and because q is in last position, this defines the edge between P_2 and P_3. This implies the second case and no other configuration is possible.

The proof of the following lemma follows quite directly from these definitions.

Lemma 6. *Let v be a new vertex in $V^k(S)$. Then v is an old vertex of $V^{k+1}(S)$, and v lies in the interior of a face of $V^{k+2}(S)$, i.e., v is not a vertex of $V^{k+2}(S)$. Furthermore, every edge of $V^k(S)$ is enclosed by a face of $V^{k+1}(S)$.*

Already in [23] it has been shown that farthest abstract Voronoi diagrams are trees, under a slightly different definition of admissible curves. In this paper we give a short alternative proof of this fact based on our axioms (A1)–(A5).

Lemma 7. *The farthest abstract Voronoi diagram $V^*(S)$ is a tree.*

Proof. Let $|S| = n$. Suppose some farthest region $\mathrm{VR}^*(p,S)$ has a face F that is bounded. Let its boundary consist of edges $e_1, \dots, e_i$ in this order, such that each edge e_j is a segment of a bisector $J(p,q_j)$. The sites q_j need not be pairwise different, but consecutive edges belong to different bisecting curves. Let x_j be the intersection point between e_j and e_{j+1} on the boundary of F, and x_i the intersection point between e_i and e_1 on F, see Fig. 7.

If $n = 1$, then there exists no bounded Voronoi region. If $n = 2$, there can be at most one edge e_1 on the boundary of $V^*(p,S)$, implying that e_1 and $J(p,q_1)$ would be closed a contradiction to (A4).

Now let $n > 2$, and let $i \geq 2$. By induction on n, $\mathrm{VR}^*(p, S \setminus \{q_i\})$ is unbounded (it still contains F and is thus not empty). Let z be a point in F, then there exists an unbounded arc $L \subseteq V^*(p, S \setminus \{q_i\})$ emanating from z to infinity. L may not intersect any $J(p,q_1), \dots, J(p,q_i)$ except for $J(p,q_i)$, thus it must leave F through $J(p,q_i)$, w.l.o.g. let its last intersection with the boundary of F be on e_i.

Because of (A5), there is a point x in the ϵ-neighborhood of x_{i-1} contained in $D(p,q_{i-1}) \cap D(p,q_i)$ and a point y in the ϵ-neighborhood of x_i contained in $D(p,q_1) \cap D(p,q_i)$. We claim that any path $\pi_{x,y}$ from x to y must intersect $J(p,q_i)$ or L. This is immediate, because by construction x and y are on the same side of the unbounded curve $J(p,q_i)$ and on opposite sides of L.

Because of (A3), there exists a point $w \in \mathrm{VR}(p, \{p, q_1, q_{i-1}, q_i\})$ (q_1 may be equal to q_i). The point w is also contained in both $\mathrm{VR}(p, \{p, q_1, q_i\})$ and $\mathrm{VR}(p, \{p, q_{i-1}, q_i\})$. The point y is contained in $\mathrm{VR}(p, \{p, q_1, q_i\})$ and because of (A1) there exists a path π_{yw} from y to w in $\mathrm{VR}(p, \{p, q_1, q_i\})$ and thus intersecting neither L nor $J(p,q_i)$. The point x is contained in $\mathrm{VR}(p, \{p, q_{i-1}, q_i\})$ and thus there exists a path π_{xw} from x to w in $\mathrm{VR}(p, \{p, q_{i-1}, q_i\})$ also intersecting neither L nor $J(p,q_i)$. But then the concatenation of π_{xw} and π_{yw} is a path connecting x and y without intersecting L and $J(p,q_i)$, a contradiction.

It remains to show that $V^*(S)$ is connected. Suppose there is a curve C separating parts of $V^*(S)$. Then $C \subset \mathrm{VR}^*(p,S)$ for a $p \in S$, $C \cap D(p,q) = \emptyset$ for all $q \in S \setminus \{p\}$ and there are $q \neq r \in S$ such that $D(p,q)$ lies on one side of C and $D(p,r)$ on the other side. But then $\mathrm{VR}(p, \{p,q,r\})$ would be empty. $\square$

3. Bounding the number of unbounded edges of $V^{\leq k}(S)$

Let Γ be a closed Jordan curve in $\mathbb{R}^2$ large enough such that no pair of bisectors cross on or outside of Γ (axiom (A5)), each bisector crosses Γ exactly twice and these intersections are transversal (axiom (A4)). One could choose Γ as the boundary of a sufficiently small neighborhood of the north pole of the sphere, projected stereographically to the plane, such that for any bisector $J(p,q)$, the set $\Gamma \setminus J(p,q)$ consist of exactly two connected components. This is possible by Lemma 2.3.1 in [17].

If we traverse Γ around the Voronoi diagram, the ordering $<_x$ on S changes whenever we cross a bisector $J(p,q)$. Here indices p and q change their relation according to $<_x$.

Lemma 8. *When we cross a bisector $J(p,q)$, then p and q change their places in the ordering $<_x$ along Γ. Further, they are adjacent to each other just before and after the crossing.*

Proof. Let $p_i < p_j$, and assume that we cross $J(p_i, p_j)$, which means that $p_i < p_j$ changes to $p_j < p_i$. Let $p_k < p_i, p_j$ before we cross $J(p_i, p_j)$. Because of the construction of Γ no other bisector has been crossed at the same time, thus $p_k < p_i, p_j$ remains. The same happens for a $p_k > p_i, p_j$. Now let $p_i < p_k < p_j$ before we cross $J(p_i, p_j)$. Then we still have $p_i < p_k < p_j$ afterwards, but now $p_j < p_i$, a contradiction to the transitivity. Thus p_i and p_j must have been adjacent right before and after we cross $J(p_i, p_j)$ and hence they change their places in the ordering. □

We call such a change in the ordering of S a *switch* between the two sites p and q, which must be adjacent. There can be only one switch at a time and each pair of sites switches exactly two times while walking one round around Γ, resulting in $n(n-1)$ switches altogether.

Every time a switch among the first $k+1$ elements of the ordering occurs, there is an unbounded edge of a Voronoi diagram of order $\leq k$. This means that *the maximum number of unbounded edges of all diagrams of order $\leq k$ is equal to the maximum number of switches among the first $k+1$ elements in the ordering.*

Permutation sequences and estimates for the maximum number of switches among the first k elements have been used in [4] to bound the number of k-sets of n points in the plane. These sequences resulted from projecting n points in general position onto a rotating line. Hence, they were of length $2N$, where $N = \binom{n}{2}$, and they had the following properties. Adjacent permutations differ by a transposition of adjacent elements, and any two permutations a distance N apart are inverse to each other. It has been shown in [15] that not every permutation of this type can be realized by a point set.

In the following lemma we introduce a larger class of permutation sequences that fits the AVD framework. Recall that a switch can occur only between to adjacent elements in the ordering.

Lemma 9. *Let $P(S)$ be a cyclic sequence of permutations $P_0, \ldots, P_N = P_0$ such that*

(i) *P_{i+1} differs from P_i by a switch;*
(ii) *each pair of sites $p, q \in S$ switches exactly two times in $P(S)$.*

Then the number of switches occurring in $P(S)$ among the first $k+1$ sites is upper bounded by $k(2n-k-1)$. Furthermore, this bound is tight.

Proof. Call a switch *good* if it involves at least one of the k first sites of a permutation; otherwise call it *bad*. Let $S = \{p_1, \ldots, p_n\}$ and the initial ordering of the sites in the first permutation be $p_1 < \ldots < p_n$. For $i \in \{k+2, \ldots, n\}$, define B_i as the set of bad switches where p_i is switching with a site in $\{p_1, \ldots, p_{i-1}\}$. We remark that the sets B_i, for $i \in \{k+2, \ldots, n\}$, are pairwise disjoint. If p_i is not involved in a good switch, then all its $2i-2$ switches with sites in $\{p_1, \ldots, p_{i-1}\}$ are bad. Otherwise, for p_i to be involved in a good switch, it must first be involved in at least $i-k-1$ bad switches with sites in $\{p_1, \ldots, p_{i-1}\}$, in order to reach the first $k+1$ positions, and since $P_0 = P_N$, p_i has to be involved in as many bad switches in order to return to its original place in the ordering. In both cases, $|B_i| \geq 2(i-k-1)$. Because of (ii), the total number of switches is $N = 2\binom{n}{2}$. Therefore the number of good switches is at most

$$2\binom{n}{2} - \sum_{i=k+2}^{n} |B_i| \leq 2\binom{n}{2} - 2\sum_{j=1}^{n-k-1} j = k(2n-k-1),$$

where $j = i-k-1$.

To show that the bound is tight, let again the initial ordering of the first permutation be $p_1 < \ldots < p_n$. We switch each p_i with $p_{i-1}, \ldots, p_1$ in decreasing index order such that p_i now is the first element and then in inverse order back to its original position. Start with $i = 2$ and continue until $i = n$. Then the number of switches among the first $k+1$ sites is exactly $2\binom{n}{2} - 2\sum_{j=1}^{n-k-1} j$. □

In contradistinction to the result in [15], each such permutation sequence can be realized by an AVD. The following Lemma 10 will be used for proving that the upper bound shown in Lemma 11 is tight.

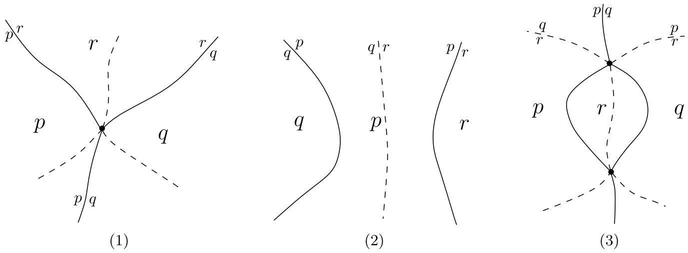

Fig. 8. Illustrations of cases (1) to (3) in the proof of Lemma 10.

Lemma 10. *Let $P(S)$ be a sequence of permutations as in Lemma 9. There exists an abstract Voronoi diagram where the ordering of the sites along Γ changes according to $P(S)$.*

Proof. We show that for $|S| = 3$ each $P(S)$ fulfilling the above properties can be realized by an AVD. If $|S| \geq 3$, we can consider $V(S)$ such that each triple p, q, r of sites changes its ordering on Γ according to $P(S)$. This is possible because if the curve system of each triple of sites is admissible then the curve system of S is admissible, too; see [18]. Now if there are two bisectors $J(p,q)$ and $J(r,t)$ having a different order on Γ than p, q, r, t have in $P(S)$, then p, q, r, t are pairwise different, and neither of the bisectors $J(p,r)$, $J(p,t)$, $J(q,r)$ or $J(q,t)$ can occur between the two bisectors $J(p,q)$ and $J(r,t)$ on Γ. Otherwise, suppose w.l.o.g. that $J(p,r)$ occurs between $J(p,q)$ and $J(r,t)$; then $\{p,q,r\}$ would not have the same ordering on Γ as in $P(S)$, a contradiction to our assumption. Thus the bisectors $J(p,q)$ and $J(r,t)$ can be deformed such that their ordering on Γ changes, but the structure of $V(S)$ remains the same.

Now let $S = \{p,q,r\}$. Then there are three different cases:

(1) Each site switches into the first position exactly once.
(2) One site switches into the first position exactly twice; it cannot do so more often because then it would have to switch with one of the other sites more often than twice. Further, it implies that all other sites must switch themselves into first position exactly once.
(3) One site never moves to first position. This implies that both the other sites switch to first position exactly once; otherwise, either one site would remain in first position during the whole permutation, but then it would never switch with any other site, or the two other sites would have to switch more than twice.

Let $P_0 = (p,q,r)$. Then there are two possibilities for P_1 in case (1): Either $P_1 = (q,p,r)$, which leads to the sequence
$P_0 = (p,q,r)$
$P_1 = (q,p,r)$
$P_2 = (q,r,p)$, otherwise r never switches into first position or p switches into first position twice
$P_3 = (r,q,p)$, otherwise r never switches into first position
$P_4 = (r,p,q)$, otherwise q switches into first position a second time
$P_5 = (p,r,q)$, otherwise p and q switch more than twice
$P_6 = (p,q,r)$, otherwise $P_0 \neq P_6$;
or $P_1 = (p,r,q)$, which leads to the same permutation sequence in inverse order.

Assume that p is the site that switches into first position twice in case (2). Then we get the following permutation sequence:
$P_0 = (p,q,r)$
$P_1 = (q,p,r)$, then
$P_2 = (p,q,r)$, otherwise $P_2 = (q,r,p)$ which leads to the permutation sequence as in case (1)
$P_3 = (p,r,q)$, otherwise p and q switch more than twice
$P_4 = (r,p,q)$, otherwise r never switches into first position
$P_5 = (p,r,q)$, otherwise p and q switch more than twice
$P_6 = (p,q,r)$, otherwise $P_0 \neq P_6$;
or in inverse order.

Assume that r is the site that never switches into first position in case (3). Then we get the following permutation sequence:
$P_0 = (p,q,r)$
$P_1 = (q,p,r)$, then
$P_2 = (q,r,p)$, otherwise p switches into first position twice
$P_3 = (q,p,r)$, otherwise r switches into first position
$P_4 = (p,q,r)$, otherwise p and r switch more than twice
$P_5 = (p,r,q)$, otherwise p and q switch more than twice
$P_6 = (p,q,r)$, otherwise $P_0 \neq P_6$;
or in inverse order.

These permutation sequences can be realized by the AVDs depicted in Fig. 8. □

Let S_i be the number of unbounded edges in $V^i(S)$. If an edge e has got two unbounded endpieces, i.e. edge e bounding a p- and q-region is the whole bisector $J(p,q)$, then e is counted twice as an unbounded edge.

Lemma 11. *Let $k \in \{1,\dots,n-1\}$. Then,*

$$k(k+1) \le \sum_{i=1}^{k} S_i \le k(2n-k-1).$$

Both bounds can be attained.

Proof. The second bound follows directly from Lemma 9. The first bound follows from the fact that the minimum number of switches among the first $(k+1)$ sites is greater or equal to the total number of switches, $n(n-1)$, minus the maximum number of switches among the last $(n-k)$ sites, which again is equal to the maximum number of switches among the first $(n-k)$ sites. Using Lemma 9 this implies

$$\sum_{i=1}^{k} S_i \ge n(n-1) - (n-k-1)(2n-(n-k-1)-1) = k(k+1).$$

The tightness of the bounds follows from Lemma 10. □

4. Bounding the number of faces of $V^k(S)$

In the following, we assume that each Voronoi vertex is of degree 3. The following two lemmata give combinatorial proofs for facts that were previously shown by geometric arguments [20,25].

Lemma 12. *Let H be a subset of S of size $k+1$ and F a face of $\mathrm{VR}^{k+1}(H,S)$. The portion of $V^k(S)$ enclosed in F is exactly the farthest Voronoi diagram $V^*(H)$ intersected with F.*

Proof. "$\Rightarrow$": Let $x \in F$ and suppose $x \in \mathrm{VR}^k(H',S)$ for $H' \subset S$ of size k. Since $F \subseteq \mathrm{VR}^{k+1}(H,S)$ it follows that $x \in D(p,q)$ for all $p \in H$ and $q \in S \setminus H$, implying $H' \subset H$. Let $H \setminus H' = \{r\}$, then $x \in D(p,r)$ for all $p \in H'$ and hence $x \in \mathrm{VR}^*(r,H)$.

"$\Leftarrow$": Let $x \in F$ and $x \in \mathrm{VR}^*(r,H)$. Then $x \in D(p,q)$ for all $p \in H$ and $q \in S \setminus H$ and $x \in D(p,r)$ for all $p \in H \setminus \{r\}$. This implies $x \in \mathrm{VR}^k(H \setminus \{r\}, S)$. □

Lemma 13. *Let F be a face of $\mathrm{VR}^k(H,S)$, $H \subseteq S$, $|H| = k \ge 2$. Then $V^*(H) \cap F$ is a nonempty tree.*

Proof. First we show that $V^*(H) \cap F$ is not empty by assuming the opposite. Then there is a $p \in H$ such that $F \subseteq \mathrm{VR}^*(p,H)$. Let $F' \subseteq \mathrm{VR}^k(H',S)$ be a face of $V^k(S)$ adjacent to F along an edge e. By Lemma 5, we have $H = U \cup \{q\}$ and $H' = U \cup \{q'\}$, where q, q' are different and not contained in U. Also, $e \subseteq J(q,q')$ holds. If p were in U, we would obtain $F' \subseteq D(p,q)$ and $F \subseteq V^*(p,H) \subseteq D(q,p)$, hence $e \subseteq J(p,q)$—a contradiction to axiom (A5). Thus, $p \notin U$, which means $p = q$. Now Lemma 5 implies that each edge on the boundary of F has to be a segment of a curve $J(p,q_j)$ such that $D(p,q_j)$ lies on the F-side. Let $q_1,\dots,q_i$ be the sites for which there is such an edge e on the boundary of F. Then $\mathrm{VR}^1(p,\{p,q_1,\dots,q_i\}) = F$, because nearest Voronoi regions are connected thanks to axiom (A1). But from $F \subseteq V^*(p,H)$ it follows that $\mathrm{VR}^1(p,H) \subseteq \mathbb{R}^2 \setminus F$, and hence, $\mathrm{VR}^1(p,S) \subseteq F \cap \mathbb{R}^2 \setminus F = \emptyset$, a contradiction to axiom (A3).

Next we show that $V^*(H) \cap F$ is a tree. Because of Lemma 7 it is clear that it is a forest. So it remains to prove that it is connected. Otherwise, there would be a domain $D \subset F$, bounded by two paths $P_1, P_2 \subset F$ of $V^*(H)$ and two disconnected segments e_1 and e_2 of the boundary of F. There is an index $p \in H$ such that $D \subseteq \mathrm{VR}^*(p,H)$. Since $V^*(H)$ is a tree, by Lemma 7, the upper (or: the lower) two endpoints of P_1 and P_2 must be connected by a path P in $V^*(H)$ that belongs to the boundary of $\mathrm{VR}^*(p,H)$; see Fig. 9. Here path P connects the endpoints of e_1; both curves together encircle a domain D', which is part of $\mathrm{VR}^*(p,H)$. By definition of the farthest Voronoi diagram and because e_1 is on the boundary of F and contained in $\mathrm{VR}^*(p,H)$, there are $q_1,\dots,q_i$, such that $e_1 \cup P$ consists of segments of $J(p,q_1),\dots,J(p,q_i)$, and all $D(p,q_j)$ are situated outside of D'; compare Lemma 5. But then $\mathrm{VR}^*(p,\{p,q_1,\dots,q_i\})$ would be bounded, a contradiction to Lemma 7. □

Lemma 14. *Let F be a face of $\mathrm{VR}^{k+1}(H,S)$ and m the number of Voronoi vertices of $V^k(S)$ enclosed in its interior. Then F encloses $2m+1$ Voronoi edges of $V^k(S)$.*

Proof. See Lemmata 12 and 13. □

The formulae in the next two lemmata originate in [20]. Below we include their proofs from [25] for completeness.

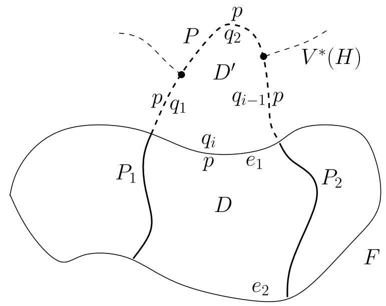

Fig. 9. The intersection of an order-k face F and the farthest Voronoi diagram of its defining sites must be a tree.

Lemma 15. *Let F_k, E_k, V_k and S_k denote, respectively, the number of faces, edges, vertices, and unbounded edges in $V^k(S)$. Then,*

$$E_k = 3(F_k - 1) - S_k \tag{1}$$

$$V_k = 2(F_k - 1) - S_k. \tag{2}$$

Proof. Consider $V^k(S) \cup \Gamma$, cut off all edges outside of Γ, and let G be the resulting graph. Then G is a connected planar graph and for its number of faces, f, of vertices, v, and edges, e, we have $f = F_k + 1$, $v = V_k + S_k$, $e = E_k + S_k$. Because of the general position assumption each vertex is of degree 3 and hence $2e = 3v$. Now the Euler formula $v - e + f = c + 1$ implies the lemma. □

Lemma 16. *The number of faces in an AVD of order k is*

$$F_k = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i.$$

Proof. Let V_k, V'_k and V''_k be the number of Voronoi vertices, *new* Voronoi vertices and *old* Voronoi vertices in $V^k(S)$, respectively. Then because of Lemma 6 we have $V_k = V'_k + V''_k = V'_k + V'_{k-1}$.

Claim 1: $F_{k+2} = E_{k+1} - 2V'_k$.

Because of Lemma 6, every *old* vertex of $V^{k+1}(S)$ lies in the interior of a face of $V^{k+2}(S)$. Consider a face F_i of $V^{k+2}(S)$. Let m_i be the number of *old* vertices of $V^{k+1}(S)$ enclosed in its interior. Then F_i encloses $e_i = 2m_i + 1$ edges of $V^{k+1}(S)$; see Lemma 14. If we sum up through all the faces in $V^{k+2}(S)$, we obtain

$$\sum_{i=1}^{F_{k+2}} e_i = 2\sum_{i=1}^{F_{k+2}} m_i + F_{k+2}.$$

Note that $\sum_{j=1}^{F_{k+2}} m_j = V''_{k+1} = V'_k$ and $\sum_{j=1}^{F_{k+2}} e_j = E_{k+1}$, hence $F_{k+2} = E_{k+1} - 2V'_k$.

Claim 2: The number of faces in $V^1(S)$ is $F_1 = n$ and the number of faces in $V^2(S)$ is $F_2 = 3(n-1) - S_1$.

The first part follows from axioms (A1) and (A3). To prove the second part, consider a face of $V^2(S)$. There are no *old* vertices in $V^1(S)$, therefore the face encloses exactly one edge of $V^1(S)$ and hence $F_2 = E_1$. Eq. (1) implies $F_2 = 3(n-1) - S_1$.

Now we sum up F_{k+2} and F_{k+3} to obtain

$$F_{k+3} = E_{k+2} + E_{k+1} - F_{k+2} - 2V'_{k+1} - 2V'_k = E_{k+2} + E_{k+1} - F_{k+2} - 2V_{k+1};$$

(see Claim 1). Substituting (1) and (2) of Lemma 15 into it results in

$$F_{k+3} = 2F_{k+2} - F_{k+1} - 2 - S_{k+2} + S_{k+1}.$$

Using the iterative formula, the base cases $F_1 = n$ and $F_2 = 3(n-1) - S_1$, we derive the lemma by strong induction. □

Theorem 1. *The number of faces F_k in an AVD of order k is bounded as follows*

$$\begin{aligned} n - k + 1 \le F_k &\le 2k(n-k) + k + 1 - n \\ &\le 2k(n-k) \\ &\in O(k(n-k)). \end{aligned}$$

Both the upper bound $2k(n-k) + k + 1 - n$ and the lower bound $n - k + 1$ can be attained.

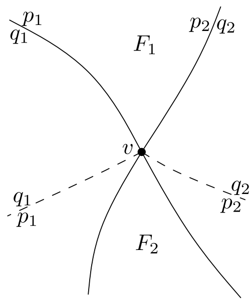

Fig. 10. Proof of Lemma 17.

Proof. Lemma 11 implies tight bounds $k(k-1) \le \sum_{i=1}^{k-1} S_i \le (k-1)(2n-k)$. Together with Lemma 16 this proves the theorem. □

5. Generalizations

In [18] it has been shown, with some technical effort, that it is not necessary to require only finitely many intersection points between each pair of bisecting curves $J(p,q)$ and $J(r,t)$. The abstract Voronoi diagram is still a finite planar graph with finitely many vertices even if bisectors intersect in infinitely many points. With the same proofs one can show that this is also true for higher order AVDs. Observe that bisector intersections have to be considered only when resulting in a Voronoi vertex of some order. Thus in axiom (A5) it would suffice to require only transversal intersections.

Another natural question is, whether the general position assumption, where each Voronoi vertex is of degree 3, is really necessary. We can show that it is not and the number of faces of an order k AVD can only decrease if vertices have higher degree than 3 and all bisectors intersect transversally. First we prove that all "pieces of pie" around an order k vertex are from different order k Voronoi regions.

Lemma 17. *Let v be a vertex of $V^k(S)$. All faces adjacent to v are from pairwise different Voronoi regions and if we walk around v, no face appears more than once.*

Proof. The proof that no face appears more than once is analogous to the proof of the simple connectivity of order-1 Voronoi regions, see [18].

Suppose there are two different faces F_1 and F_2 incident to v, both belonging to the same order k Voronoi region of $H \subseteq S$. For $k=1$ the regions are connected and we have a contradiction.

So let $k \ge 2$. Let the left border-edge of F_1 incident to v belong to the bisector $J(p_1,q_1)$ and the right one to $J(p_2,q_2)$ such that $p_1, p_2 \in H$ and $q_1, q_2 \notin H$, in other words $\mathrm{VR}^k(H,S) \subseteq D(p_1,q_1)$ and $\subseteq D(p_2,q_2)$, see Fig. 10. Because F_2 is also a face of the region of H, the bisector $J(p_1,q_1)$ has to pass on the left side between F_1 and F_2 and $J(p_2,q_2)$ has to pass on the right side between F_1 and F_2. But then $\{p_1,q_1\} \ne \{p_2,q_2\}$ and the two bisectors $J(p_1,q_1)$ and $J(p_2,q_2)$ intersect non-transversally in v contradicting axiom (A5). □

Now we can prove a more relaxed version of Theorem 1.

Theorem 2. *Let $V^k(S)$ be an AVD of order k, satisfying axioms (A1) to (A5), with vertex degree ≥ 3 for all Voronoi vertices. Then the number of faces of $V^k(S)$ is bounded from above by $2k(n-k)$.*

Proof. For each vertex v of $V^k(S)$ with degree ≥ 4 consider a sufficiently small ε-neighborhood $U_\varepsilon(v)$ around v, such that no other intersection between bisectors lies in $U_\varepsilon(v)$. Use the perturbation technique of [17] to obtain degree 3 vertices in $U_\varepsilon(v)$. Because of Lemma 17 the modified Voronoi diagram $\overline{V^k}(S)$ has got at least as many faces as $V^k(S)$. Now Theorem 1 implies that $V^k(S)$ has got at most $2k(n-k)$ many faces. □

6. Concluding remarks

One may wonder if the Clarkson–Shor technique [12] can be applied to get the same complexity bound. The Clarkson–Shor technique proves that for point sites in the Euclidean metric $V_k(S)$ has at most $2k(n-k-1)$ new Voronoi vertices, leading to at most $4kn - 4k^2 - 2n$ Voronoi vertices. However, there are three problems in applying this technique to the abstract setting. Let $V_{k,S}$, $V'_{k,S}$, and $V''_{k,S}$ be the numbers of Voronoi vertices, new Voronoi vertices, and old Voronoi vertices of $V_k(S)$, respectively, and let $F_{k,S}$ and $S_{k,S}$ be the numbers of faces and unbounded faces of $V_k(S)$, respectively.

First, it is shown that $V'_{k,S} + V'_{n-k,S}$ is $2k(n-k-1)$. This proof depends on the fact that in the Euclidean metric, $V'_{1,R} + V'_{r-1,R}$ is $2r-4$ for any r-element subset R of S. However, the equation $V'_{1,R} + V'_{r-1,R} = 2r-4$ does not hold in the abstract version since $S_{1,R}$ is not necessarily $S_{r-1,R}$. Actually, $r-2 \le V'_{1,R} + V'_{r-1,R} \le 2r-4$. It is not clear how to make use of this inequality and their technique to derive an upper bound. Second, the Clarkson–Shor technique mainly focuses on the upper bound. No bound for the minimum number of faces in $V_k(S)$ is derived. Last but not least, if we assume that $V'_{1,R} + V'_{r-1,R} = 2r-4$ holds, by the Euler formula, we have an upper bound of $2k(n-k) + S_{k,S}/2 + 1 - n$, but our derived upper bound is $2k(n-k) + k + 1 - n$. In the abstract version, it is trivial to find a case in which $S_{k,S}/2 > k$. Moreover, since we prove the existence of an instance satisfying the bound $2k(n-k) + k + 1 - n$, our bound is tight.

Another natural question is whether axioms weaker than (A1)–(A5) can still imply Theorem 1. In Section 5, we showed that axiom (A5) can be simplified, requiring only transversal intersections, and the general position assumption can be omitted. However, what about disconnected and empty Voronoi regions?

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