

Parabola equation from directrix and focus

Parabola equation

The equation of the parabola ϕ has the form:

$$\phi =: \{ (x, y) \mid rx^2 + sy^2 + txy + ux + vy + w = 0 \}$$

Line equation and point coordinates

The equation of a line l is given as:

$$l =: \{ (x, y) \mid ax + by + c = 0 \}$$

the point coordinates for the focus are:

$$f(f_x, f_y)$$

Definition of a parabola

A parabola is defined as the set of all points (x, y) that are equidistant from a line (the directrix l) and a point (the focus f):

$$\phi := \{ (x, y) \mid d((x, y), l) = d((x, y), f) \}$$

therefore we can find the equation of a parabola in function of its directrix and focus.

$$\begin{aligned} d((x, y), l) &= d((x, y), f) \\ [d((x, y), l)]^2 &= [d((x, y), f)]^2 \\ \left(\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} \right)^2 &= \left(\sqrt{|x - f_x|^2 + |y - f_y|^2} \right)^2 \\ a^2x^2 + b^2y^2 + 2abxy + 2acx + 2bcy + c^2 &= (x^2 - 2f_x x + f_x^2 + y^2 - 2f_y y + f_y^2) \cdot (a^2 + b^2) \\ 2abxy + 2acx + 2bcy + c^2 &= b^2x^2 + a^2y^2 + (-2a^2f_x - 2b^2f_x)x + (-2a^2f_y - 2b^2f_y)y + a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2 \\ 0 &= b^2x^2 + a^2y^2 + (-2ab)xy + (-2ac - 2a^2f_x - 2b^2f_x)x + (-2bc - 2a^2f_y - 2b^2f_y)y + a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2 - c^2 \end{aligned}$$

We get an equation for the parabola $\phi : rx^2 + sy^2 + txy + ux + vy + w = 0$ with coefficients:

- ▷ $r = b^2$
- ▷ $s = a^2$
- ▷ $t = -2ab$
- ▷ $u = -2ac - 2a^2f_x - 2b^2f_x$
- ▷ $v = -2bc - 2a^2f_y - 2b^2f_y$
- ▷ $w = a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2 - c^2$