Parabola equation from directrix and focus

Parabola equation

The equation of the parabola ϕ has the form:

$$\phi =: \{ \ (x, y) \mid rx^2 + sy^2 + txy + ux + vy + w = 0 \ \}$$

Line equation and point coordinates

The equation of a line l is given as:

$$l =: \{ \ (x, y) \mid ax + by + c = 0 \ \}$$

the point coordinates for the focus are:

$$f(f_x, f_y)$$

Definition of a parabola

A parabola is defined as the set of all points (x, y) that are equidistant from a line (the directrix l) and a point (the focus f):

$$\phi := \{ \ (x, y) \mid d\left((x, y), l\right) = d\left((x, y), f\right) \ \}$$

therefore we can find the equation of a parabola in function of its directrix and focus.

$$d\left((x, y), l\right) = d\left((x, y), f\right)$$
$$\left[d\left((x, y), l\right)\right]^2 = \left[d\left((x, y), f\right)\right]^2$$
$$\left(\frac{|ax + by + c|}{\sqrt{a^2 + b^2}}\right)^2 = \left(\sqrt{|x - f_x|^2 + |y - f_y|^2}\right)^2$$
$$a^2x^2 + b^2y^2 + 2abxy + 2acx + 2bcy + c^2 = \left(x^2 - 2f_x x + f_x^2 + y^2 - 2f_y y + f_y^2\right) \cdot \left(a^2 + b^2\right)$$

$$2abxy+2acx+2bcy+c^2 = b^2x^2+a^2y^2+(-2a^2f_x-2b^2f_x)x+(-2a^2f_y-2b^2f_y)y+a^2f_x^2+a^2f_y^2+b^2f_x^2+b^2f_y^2$$

$$0 = b^2x^2+a^2y^2+(-2ab)xy+(-2ac-2a^2f_x-2b^2f_x)x+(-2bc-2a^2f_y-2b^2f_y)y+a^2f_x^2+a^2f_y^2+b^2f_x^2+b^2f_y^2-c^2$$

We get an equation for the parabola $\phi : rx^2 + sy^2 + txy + ux + vy + w = 0$ with coefficients:

- ▷ $r = b^2$
- ▷ $s = a^2$
- ▷ $t = -2ab$
- ▷ $u = -2ac - 2a^2f_x - 2b^2f_x$
- ▷ $v = -2bc - 2a^2f_y - 2b^2f_y$
- ▷ $w = a^2f_x^2 + a^2f_y^2 + b^2f_x^2 + b^2f_y^2 - c^2$

Parabola–line intersection

The intersections of a line and a parabola can be found by solving the equation of the line for y, then inserting it into the equation of the parabola to get a quadratic equation in x.
The resulting x(s) of the equation can be input in the line equation to get the corresponding value(s) of y, each indicating a point of intersection.
There can be 0, 1, or 2 intersections.