Three equation model

Oliver Snellman
oliver.snellman@helsinki.fi

January 31, 2021

Abstract

In this document I derive the three equation model for macro economy, which focuses on the role of the Central Bank (CB) and monetary policy on stabilizing the economy over the short run business cycles. As the name suggests, the model consists of three equations: IS curve describing the demand side, Phillips curve for the supply side and the MR curve for the policymaker (CB). Households and wage setters have adaptive expectations, whereas the CB has rational expectations. The frictions in the model stem from staggered wage setting.

$y_t = A - ar_{t-1}$	**IS curve**	**Demand**	(1)
$\pi_t = \pi_{t-1} + \alpha(y_t - y^e)$	**Phillips curve**	**Supply**	(2)
$\pi_t = \pi^T - \dfrac{1}{\alpha\beta}(y_t - y^e)$	**MR curve**	**Central Bank**	(3)

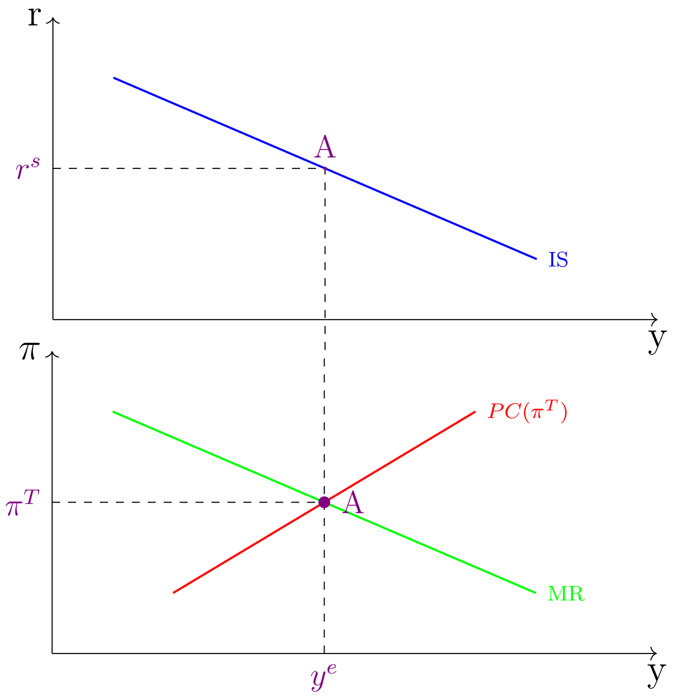

Figure 1: The three equation model

The three equation model provides a rather simple way to study the impact of shocks and policies on the macro economy in the short to medium run (simple, compared to DSGE-models, for example). The first equation (IS curve) describes the demand side behavior, the second equation (Phillips curve) describes the supply side and the third equation (MR curve) gives the behavior of the policy maker, the Central Bank (CB).[1]

The model economy in Figure 1 is in an equilibrium point. At this point nothing happens, unless an exogenous shock (coming from outside of the model) pushes the economy in some direction. After such an event the economy typically converges back to equilibrium, and we are interested in the particular steps and path by which this happens, especially on what is the role of the CB in the process. A handy rule of thumb is, that **the economy is always on both the IS and the Phillips curves, but a shock might push the economy temporarily off from the MR curve**.

More specifically: The CB wants to keep the economy close to an equilibrium point, where the inflation is at its target rate $\pi_t = \pi^T$ and output is at it's natural level $y_t = y^e$. If the economy is pushed off from this equilibrium, the CB uses monetary policy (MR curve) to influence the aggregate demand (IS-curve), while taking into account the impact of the monetary policy on the supply side (Phillips curve). In other words, the CB chooses each period optimally one point compatible with the IS and the Phillips curves, until the economy is back in equilibrium.

Let's start off by deriving the IS curve. Derivation of the Phillips curve starts on page 5 and the MR curve on page 15.

IS curve

The IS-curve gives the combinations of output and the real interest rate, which are compatible with the equilibrium in the goods market. From the point of view of the CB, when setting the real interest rate, the IS curve gives the corresponding aggregate demand for output Y. Here the IS-curve is used to represent the demand side of the economy.[2] To derive the IS-curve, we start from the national accounting identity, then postulate simple functional forms for consumption C and investments I, and re-package the equation in such a way, that the impact of the real interest rate r on the output y is the main focus.

[1] The word "curve" is used in this context as a synonym for "equation" or "relation".

[2] Why not use the AD-curve to represent the demand side, which consists of the intersections of IS and LM/TR -curves? The three equation model takes the actions of the CB into account with the MR-curve, which serves the same function as the LM/TR-curve in the AD-AS model. But unlike as in the AD-AS model, we don't want to "hide" the CB's monetary policy inside the AD-curve, and instead we leave it more explicitly in the main diagram as the MR curve, along with the Phillips curve. This makes sense, because we are specifically interested in the macro policy.

$$
\begin{aligned}
y &\equiv C + I + G \\
y &= \underbrace{c_0 + c_1(1-t)y}_{C} + \underbrace{a_0 - a_1 r}_{I} + G \\
y - c_1(1-t)y &= c_0 + a_0 - a_1 r + G \\
[1 - c_1(1-t)]y &= c_0 + a_0 + G + a_1 r \\
y &= \underbrace{\frac{1}{1 - c_1(1-t)}}_{=:k}[c_0 + a_0 + G - a_1 r] \\
y &= \underbrace{k[c_0 + a_0 + G]}_{=:A} - \underbrace{k a_1}_{=:a} r \\
y &= A - ar \qquad\qquad\qquad\qquad \textbf{IS curve}
\end{aligned}
$$

Let's add time indices to the variables. Next we make an important assumption regarding the policy lag of the CB: The aggregate demand responds negatively to changes in the real interest rate (set by the CB) with a one period lag. This is called the dynamic IS curve and will be used from now on.

$$
y_t = A - ar_{t-1} \qquad\qquad \textbf{dynamic IS curve} \tag{4}
$$

I will refer to the dynamic IS curve as just the IS curve. The IS curve is often derived using the Keynes cross model. For some of you that might be more familiar.

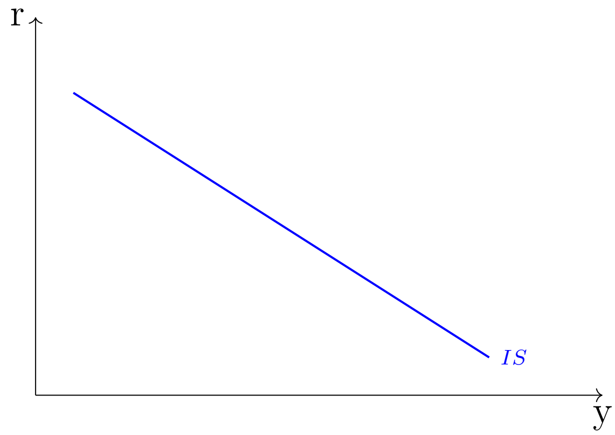

Figure 2: IS curve

What shifts the IS curve? The IS curve depicts the relation between the real interest rate and aggregate demand for output. For that reason the IS curve can't shift in response to changes in real interest rate. Make sure that feels intuitive.

$$y = \underbrace{\overbrace{\frac{1}{1-c_1(1-t)}}^{=:k}[c_0+a_0+G]}_{=:A} - \underbrace{\overbrace{\frac{1}{1-c_1(1-t)}}^{=:k} a_1}_{=:a} r \qquad \textbf{IS curve}$$

The IS curve shifts up, if government spending G increases, or the autonomous spending or investment components c_0, a_0 increase.

G ↑	IS ↑
c_0 ↑	IS ↑
a_0 ↑	IS ↑

Table 1: Shifting the IS-curve

The slope of the IS curve depends on the term a multiplying r in the equation. Hence, the slope of the IS curve flattens, when the marginal propensity to consume c_1 increases, the tax rate t lowers, or the sensitivity of investments to interest rate a_1 increases.[3]

c_1 ↑	IS flattens
t ↓	IS flattens
a_1 ↑	IS flattens

Table 2: Changing the slope of the IS-curve (rotating counter clockwise around the current location)

Why doesn't the IS curve shift in response to changes in inflation? The IS curve only concerns the goods market equilibrium at different interest rates. There are no terms including inflation or expected inflation in the IS equation we use in the first exercises. In this model it's the CB that responds to changes in inflation by setting the nominal interest rate (while taking the Fisher's equation into account: $i = \pi^E + r$), and thereby choosing the real interest rate. The impact of inflation on the demand side, the change in output demanded, can then be seen as a movement along the IS curve.

If you're more familiar with the IS-LM model, it has the same idea. Changes in inflation and inflation expectations shift the LM curve (or the TR curve), meaning that we move along the IS curve. Here the MR curve performs the same function.

[3] You might notice, that c_1 and t are also present in the intercept term A, so does the curve also shift with them? No, but instead the IS curve rotates counter clockwise using the current location as a pivoting point.

In more modern models, like the DSGE framework, everything tends to be more interlinked, and this kind of compartmentalization is not as rigid. There are also modifications that could be made to the three equation model, which result in the IS curve shifting in response to inflation.

Phillips curve

The Phillips curve (PC), Eq 1, describes the supply side relationship between real and nominal quantities, the prices and the output. More specifically, the PC relates the deviation of inflation from its expected level to deviation of output from its natural level.

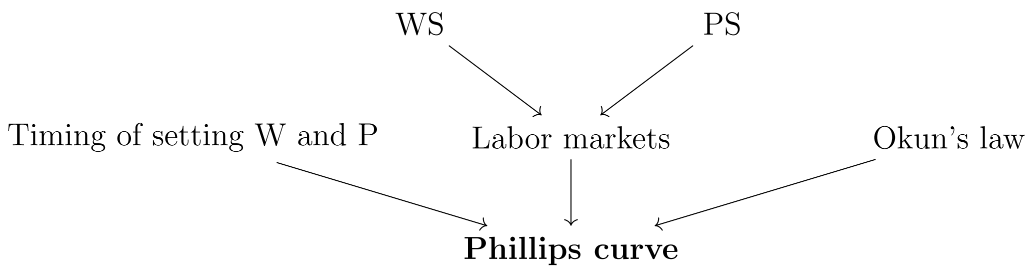

Figure 3: Construction of the Phillips curve

To derive the Phillips curve, we first take a look at the labor market dynamics, which describes the wage formation and unemployment.

Timing and order of events

In the beginning of a new period, the agents form the expectation regarding the key variables with uncertain values, namely the inflation π_t^E. Then the nominal wage is negotiated based on this expected inflation. Then the exogenous shocks impact the economy. Finally, the firms set their prices P_t optimally, with the information about the nominal wages and shocks. Setting the prices will also fix the actual inflation π_t.

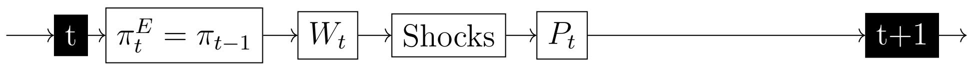

Figure 4: Timing of setting W and P

Nominal frictions. If we want to model short run business cycles and the impact of shocks and policies, we need to introduce frictions. Without frictions, the model economy's nominal variables adjust immediately to changes for example in aggregate demand, leaving the real quantities untouched. Typically frictions are modelled by preventing either nominal prices or wages from being immediately updated after changes occur.

The order of events presented above implies, that there are **frictions in wage setting**: The wages can only be re-negotiated with a delay of one period. Prices, on the other hand, are fully flexible in this model, as they are set with all of the relevant information being available.

Inflation expectations. The workers are not interested in the nominal wages W, because what actually matters for their purchasing power are the real wages $w = \frac{W}{P}$. The problem is that the nominal wage W_t has to be negotiated before the prices P_t are known. The expectations regarding the prices, P_t^E are therefore crucial in the process. How do the agents negotiating for wages form these expectations? We assume **adaptive expectations**[4], meaning that the expectations are updated each period to match the latest actual price level $P_t^E = P_{t-1}$. Prices and inflation have a direct link, $\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$, and it is more typical to speak about expectations regarding inflation, $\pi_t^E = \pi_{t-1}$.

The Wage Setting curve (WS)

The WS curve characterizes the **labor supply** and gives the real wage w that has to be offered for each unemployment level u, to incentivize the employed workers to exert the desired effort in their work. There are different ways the labor supply can be modelled, but in the general form the idea is to obtain a positive relation between the real wages w and the employment level u.

$$\begin{aligned} w &= F(u) \qquad \textbf{general WS curve} \qquad (5) \\ F_u' &< 0 \end{aligned}$$

We can also add an iid shock process z_w to the function, which then would influence the wages exogenously.

Example: Here is a particular shirking model, with which the WS curve could be obtained. For simplicity, we assume here that the workers have a binary choice between exerting effort (costing them a) in their work (producing something of value for the firm) and not exerting effort (doing nothing of value). Not exerting effort is called shirking. The firms want to incentivize the workers to exert effort. They try to catch the low effort workers with a monitoring system

[4]More general specification for adaptive expectations is $\pi_{t+1}^E = (1-a)\pi_t^E + a\pi_t$, where the speed by which the expectation adapt to changes in realized inflation is given by the parameter $a \in [0, 1]$. There are also other ways we can use to model the formation of expectations. Another typical assumption is, that the agents have **rational expectations**, meaning that their expectations adjust immediately to changes in inflation, $\pi_t^E = \pi_t$, and they also have a full understanding of the functioning of the model they are inside of.

that works q % of the time and by offering efficiency wages, to make exerting effort more profitable for the worker than taking the chances with shirking. The unemployment benefit b also impacts the value of shirking option.

$$\overbrace{w-a}^{\text{Outcome with high effort}} \geq \overbrace{\underbrace{(1-q)w}_{\text{Not caught}} + q[\underbrace{(1-u)w}_{\text{Finds a new job}} + \underbrace{ub}_{\text{Unemployed}}]}^{\text{Expected outcome with low effort}}$$

$$\begin{aligned}
w-a &\geq w - \cancel{qw} + \cancel{qw} - quw + qub && \| -w \\
-a &\geq -quw + qub && \| \frac{1}{qu} \\
-\frac{a}{qu} &\geq -w + b \\
w &\geq b + \frac{a}{qu} \\
&\rightarrow \\
w &= b + \frac{a}{qu} && \textbf{WS curve} \quad (6)
\end{aligned}$$

Eq 6 gives one possible functional form for Eq 5, $w = F(u) = b + \frac{a}{qu}$.

This WS curve defines the wage w, which makes the workers want to exert effort instead of shirking. This wage depends on the prevailing unemployment level u, because that in turn impacts the probability of finding a new job after getting caught shirking, and thereby the expected value of shirking.

Eq 6 implies also how the nominal wage is set using the inflation expectations.

$$\begin{aligned}
\mathbb{E}\left[\frac{W_t}{P_t^E}\right] &= b + \frac{a}{qu} \\
W_t &= \left[b + \frac{a}{qu}\right]\mathbb{E}[P_t^E] \\
W_t &= \left[b + \frac{a}{qu}\right]P_{t-1}
\end{aligned}$$

Price Setting curve (PS)

The PS curve describes the optimal behavior of the **labor demand** side, the firms. Next we make several simplifying assumptions.

- The firms have a production function with only labor inputs, and the labor has constant productivity $Y = F(N) = \lambda N$ (each additional worker produces as much as the previous one).[5]

- The firms operate on imperfectly competitive markets, so they do have some market power (charging more reduces the number of units sold). More specifically, the firms face the following demand equation, $Q(P) = AP^{-\varepsilon}$.[6]
- We don't permit any strategic behavior in the pricing decision (as in the oligopoly models, for example).

We can now formulate the firm's problem and solve for the optimal prices from there. It is a constrained optimization problem where the firm chooses the price of its output, P, to maximize the profits. This can be turned into an unconstrained problem by plugging the constraints into the objective function.

$$\max_P \quad \underbrace{PQ(P)}_{\text{Revenue}} - \underbrace{C(W)}_{\text{Unit cost}} Q(P)$$

s.t.

$$Q(P) = AP^{-\varepsilon} \qquad \text{Demand}$$

$$C(W) = \frac{WN}{Y} = \frac{W\cancel{N}}{\lambda \cancel{N}} = \frac{W}{\lambda} \qquad \text{Unit costs}$$

$$\Longleftrightarrow$$

$$\max_P \quad AP^{1-\varepsilon} - \frac{W}{\lambda} AP^{-\varepsilon} \qquad \text{Firm's problem}$$

FOC:

[5]Notice, that we could as well assume diminishing marginal product for the labor. This would make the PS curve downwards sloping and would also have an impact on the slope of the Phillips curve. However, the cost of adding complexity to the model is not necessarily worth wile here, as most of the results would be qualitatively similar.

[6]Notice, that this demand experienced by an individual firm (maybe on one sector out of many) is not the same as the aggregate demand, described by the IS curve. We are using this demand function here just to build the supply side dynamic. It would require more steps and rigour to connect the individual demands and the aggregate demand.

$$
\begin{aligned}
\frac{\partial \pi}{\partial P} = (1-\varepsilon)AP^{-\varepsilon} + \frac{\varepsilon WAP^{-\varepsilon-1}}{\lambda} &= 0 && || \times P^{\varepsilon} \\
(1-\varepsilon)A &= -\frac{\varepsilon WAP^{-1}}{\lambda} && ||\frac{1}{A} \\
(1-\varepsilon) &= -\frac{\varepsilon WP^{-1}}{\lambda} && || \times P \\
(1-\varepsilon)P &= -\frac{\varepsilon W}{\lambda} && ||\frac{1}{1-\varepsilon} \\
P &= -\frac{\varepsilon}{1-\varepsilon}\frac{W}{\lambda} && ||(1-\varepsilon) = -(\varepsilon-1) \\
P &= \frac{\varepsilon}{\varepsilon-1}\frac{W}{\lambda} && \textbf{Optimal price}
\end{aligned}
$$

This is the optimal price set by the firms. Given that the nominal wages and prices has now been set, the real wages have thereby also been implicitly formed in the process.

$$
\begin{aligned}
w &= \frac{W}{P} && ||P = \frac{\varepsilon}{\varepsilon-1}\frac{W}{\lambda} \\
w &= \frac{W}{\frac{\varepsilon}{\varepsilon-1}\frac{W}{\lambda}} && ||\frac{A}{\frac{B}{C}} = \frac{AC}{B}
\end{aligned}
$$

$$
\begin{aligned}
w &= \frac{\cancel{W}(\varepsilon-1)\lambda}{\varepsilon\cancel{W}} \\
w &= \frac{\varepsilon\lambda-\lambda}{\varepsilon}
\end{aligned}
$$

Then some boring algebra for the RHS

$$
\begin{aligned}
\frac{\varepsilon\lambda-\lambda}{\varepsilon} &= \lambda - \frac{\lambda}{\varepsilon} = \lambda\Big(1-\frac{1}{\varepsilon}\Big) = \lambda\Big(\frac{\varepsilon}{\varepsilon}-\frac{1}{\varepsilon}\Big) = \lambda\Big(\frac{\varepsilon-1}{\varepsilon}\Big) \\
&= \frac{\lambda}{\frac{\varepsilon}{\varepsilon-1}} = \frac{\lambda}{\frac{\varepsilon-1+1}{\varepsilon-1}} = \frac{\lambda}{\frac{\varepsilon-1}{\varepsilon-1}+\frac{1}{\varepsilon-1}} = \frac{\lambda}{1+\frac{1}{\varepsilon-1}} \\
&= \frac{\lambda}{1+\mu} && ||\frac{1}{\varepsilon-1} =: \mu
\end{aligned}
$$

Hence,

$$
w = \frac{\lambda}{1+\mu} \qquad \textbf{PS curve} \tag{7}
$$

This is the real wage "offered" by the markets, also called the PS curve. Because we assume that the prices can adjust immediately to changes in costs of production (P is set after W and shocks), the real wage is always on the PS curve. If W increases, so does P in same proportion, to maintain the constant $w = \frac{\lambda}{1+\mu}$.[7]

A bit later a variant of the PS curve will become useful, where the nominal wage is isolated to the LHS.

$$
\begin{aligned}
\frac{W}{P} &= \frac{\lambda}{1+\mu} \\
W &= \frac{\lambda}{1+\mu} P
\end{aligned}
\tag{8}
$$

Combining the labor supply and demand. Labor markets are in equilibrium, when the labor supply and labor demand are equal, WS=PS, and there are no internal dynamics at play pushing the unemployment or real wages in any direction (in absence of shocks). At this point the real wage is enough to motivate the workers to exert effort, and it is also in accordance with the firm's optimal pricing decisions. This equilibrium can always be achieved by altering the employment level (there is a unique unemployment level which fulfils this equilibrium).

Let's define the **natural level of employment** N^e (and by the same token natural unemployment u^e) such that it fulfils WS=PS. This natural unemployment is unique and is determined by the supply side dynamic in the medium term, and is sometimes also called the non-accelerating inflation rate of unemployment (NAIRU).

What happens if $WS \neq PS$? The model doesn't explicitly describe the dynamic, by which the labor market is driven towards the equilibrium. All the points on the Phillips curve will be compatible with the labor market equilibrium, hence, from now on we will not face a situation, where $WS \neq PS$.

Nonetheless, I'll try to provide you an intuition, even though it is a bit beyond the scope of the model: In order to produce a certain level of output y, the firms employ the needed number of workers with the real wage determined by the PS curve, leaving the rest of the labor force unemployed. If this real wage is below the real wage given by the WS curve (for the current unemployment level), shirking will increase, leading to firms laying caught workers off, increasing the unemployment. Increased unemployment reduces the expected utility from

[7]Because of the simplifying assumptions and skipped mathematical steps (aggregation of demand for example), we don't explicitly see the impact of demand shocks in the PS equation, Eq 7. Nonetheless, the shocks to aggregate demand also impact the demand curve faced by an imperfectly competitive firm. Therefore a demand shock also shift the PS curve, leading to a new temporary intersection point with the WS curve. This enables the fluctuation in real wages in the aftermath of demand side shocks, despite the PS curve being flat.

shirking, meaning that also the real wage required by the WS curve gets smaller. This means, that more workers are being incentivized to exert effort with the actual real wage given by the PS curve, because the cost of getting fired has increased. Unemployment keeps rising, until the WS curve issues the same real wage as the PS curve. Similarly, if the real wages offered by the PS curve are above the WS curve, shirking and unemployment will decrease until the WS=PS.

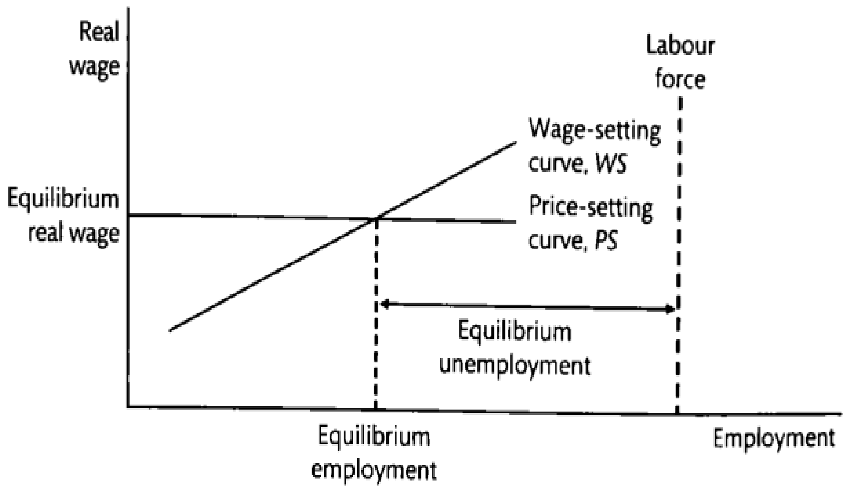

Figure 5: In the intersection of WS and PS curves the labor supply and demand meet and the labor markets are in equilibrium.

Okun's law.

We now have established the labor market dynamic: The relation between real wages and unemployment. The next step towards the Phillips curve is to establish a link between the unemployment and aggregate output, called Okun's law.

We start by assuming that the aggregate output is a function of (only) labor inputs, $Y = F(N)$. The full labor force is denoted by N^* and unemployment and employment have a direct link $(1 - u)N^* = N$. Other way of expressing the natural employment, defined in the previous paragraph, is $(1 - u^e)N^* = N^e$. **The natural level of output** is the one reached with the natural employment level, $Y^e = F(N^e)$.

Next we derive an expression for percentage deviation of output Y around the natural equilibrium output Y^e. The first row below expresses the output $Y = F(N)$ using the first order Taylor approximation (I don't write the approximations $\approx$ to be consistent with the slides). This approximation is quite accurate for small deviations of Y around Y^e. On the LHS the percentage deviation is approximated using logarithmic differences.

$$
\begin{aligned}
Y &= F(N^e) + F'(N^e)(N - N^e) && \|- Y^e \\
Y - Y^e &= \cancel{F(N^e)} + F'(N^e)(N - N^e) - \underbrace{\cancel{F(N^e)}}_{=Y^e} \\
Y - Y^e &= F'(N^e)(N - N^e) \\
Y - Y^e &= F'(N^e)((1-u)N^* - (1-u^e)N^*) \\
Y - Y^e &= F'(N^e)[1 - u - 1 + u^e]N^* \\
Y - Y^e &= F'(N^e)[u^e - u]N^* && \|\frac{1}{Y^e} \\
\frac{Y - Y^e}{Y^e} &= \frac{F'(N^e)[u^e - u]N^*}{F(N^e)} \\
\frac{Y - Y^e}{Y^e} &= \underbrace{\frac{F'(N^e)N^*}{F(N^e)}}_{=:c}(u^e - u) \\
\frac{Y - Y^e}{Y^e} &= c(u^e - u) && \|\frac{Y - Y^e}{Y^e} \approx \ln(Y) - \ln(Y^e) \\
\ln(Y) - \ln(Y^e) &= c(u^e - u) && \|\ln(Y) =: y
\end{aligned}
$$

$$
\begin{aligned}
y - y^e &= c(u^e - u) \\
u^e - u &= \frac{y - y^e}{c} && \textbf{Okun's law} \qquad (9)
\end{aligned}
$$

Okun's law states, that the output gap and the unemployment gap are negatively correlated: Increasing the output is accompanied by diminishing unemployment.

Next we transform the general WS curve (without specifying the particular model of wage setting), $w = F(u)$, Eq 5, into a similar log linear form, where the LHS is approximated using logarithmic differences and the RHS using a first order Taylor approximation around the natural real wage (resulting in WS=PS).

$$
\begin{aligned}
w &= F(u^e) + F'(u^e)(u - u^e) && \|- w^e \\
w - w^e &= \cancel{F(u^e)} + F'(u^e)(u - u^e) - \cancel{F(u^e)} && \|\frac{1}{w^e} \\
\ln(w) - \ln(w^e) &= -\underbrace{\frac{F'(u^e)}{F(u^e)}}_{=:h}(u - u^e) \\
\ln(w) - \ln(w^e) &= h(u^e - u) && (10)
\end{aligned}
$$

The scalar h includes everything that ends up multiplying the unemployment gap in the Taylor expansion.

Because the prices are flexible, the real wages w become the same as natural real wages w^e after the price setting round, $w_{t-1} = w^e$. The reason for this is, that the economy is always on the PS curve; if the nominal wages are increased by the wage setters to increase the real wage, the firms increase the prices similarly to keep the real wage at the level dictated by the PS curve. We can then re-write Eq 10 as

$$\ln(w_t) - \ln(w_{t-1}) = h(u^e - u_t) \tag{11}$$

Next we deal with the RHS and the LHS of the Eq 11 separately and pull them back together in the end, yielding the Phillips curve.

To the RHS we apply the Okun's law, Eq 9

$$\begin{aligned} h(u^e - u) &= h(\frac{y - y^e}{c}) \\ &= \underbrace{\frac{h}{c}}_{=:\alpha} (y - y^e) \\ &= \alpha(y - y^e) \end{aligned} \tag{12}$$

The LHS of Eq 11 requires a bit more wrangling. Let's start by expressing the change in real wages as changes in nominal wages and inflation.

$$\begin{aligned} \ln(w_t) - \ln(w_{t-1}) &= ln\Big(\frac{W_t}{P_t^E}\Big) - ln\Big(\frac{W_{t-1}}{P_{t-1}}\Big) \\ &= \ln(W_t) - \ln(P_t^E) - \ln(W_{t-1}) + \ln(P_{t-1}) \\ &= \ln(W_t) - \ln(W_{t-1}) - \underbrace{[\ln(P_t^E) - \ln(P_{t-1})]}_{=\pi_t^E} \\ &= \ln(W_t) - \ln(W_{t-1}) - \pi_t^E \end{aligned}$$

Recall the expression for nominal wages from PS curve, Eq 8: $W_t = \frac{\lambda}{1+\mu} P_t$. Use it to include the firms' price setting rule in the end result (Phillips curve, which we will arrive shortly).

$$\begin{aligned}
\ln(w_t) - \ln(w_{t-1}) &= \ln(\underbrace{\frac{\lambda}{1+\mu}P_t}_{=W_t}) - \ln(\underbrace{\frac{\lambda}{1+\mu}P_{t-1}}_{=W_{t-1}}) - \pi_t^E \\
&= \ln(\cancel{\frac{\lambda}{1+\mu}}) + \ln(P_t) - \ln(\cancel{\frac{\lambda}{1+\mu}}) - \ln(P_{t-1}) - \pi_t^E \\
&= \underbrace{\ln(P_t) - \ln(P_{t-1})}_{=\pi_t} - \pi_t^E \\
&= \pi_t - \pi_t^E
\end{aligned} \tag{13}$$

Now we can finally go back to Eq 11 and combine the re-written RHS (Eq 12) and LHS (Eq 13) to arrive at the Phillips curve

$$\ln(w_t) - \ln(w_{t-1}) = h(u^e - u)$$

$$\Longleftrightarrow$$

$$\pi_t - \pi_t^E = \alpha(y - y^e)$$

$$\pi_t = \pi_t^E + \alpha(y_t - y^e) \qquad \textbf{Phillips curve}$$

Notice, that we combined the WS and PS curves and the Okun's law to get this equation, so the Phillips curve fulfils all of them at the same time.[8] The slope of the Phillips curve is given by the coefficient α, and the location of the curve is determined by the inflation expectations π^E (the inflation at $y = y^e$ is π^E on the curve).

As was mentioned in the beginning of this section, the agents on the supply side of the model form their expectations adaptively, meaning that they base their expectations on what has happened previously. Therefore $\pi_t^e = \pi_{t-1}$.

$$\pi_t = \pi_{t-1} + \alpha(y_t - y^e) \qquad \textbf{Adaptive expectations Phillips curve}$$

I will be referring to this Adaptive expectations Phillips curve just as the Phillips curve (or PC) from now on.

[8]We use the Phillips curve to represent the relation between inflation and output gaps, but it is also sometimes specified as the relation between inflation and unemployment gaps (using the Okun's law).

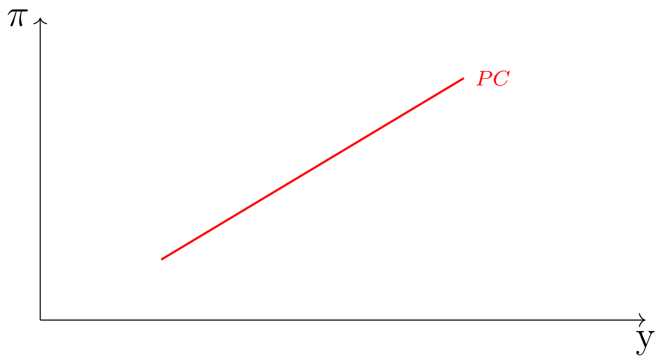

Figure 6: The Phillips curve

How exactly does the Phillips curve shift when inflation expectations change? The slope of the Phillips curve is given by α and the location (intercept) by the inflation expectations π^E. When the inflation expectations change, in our case adapt to the previous period's inflation, the Phillips curve shifts to a new location. But where exactly? The height of the Phillips curve at the natural output level is always π^E, so the new Phillips curve will cut through the point (y^e, π_{t-1}). Why? Look at the equation for Phillips curve and see what the inflation will be if the output gap is zero, i.e. $y = y^e$: Only $\pi_t = \pi_{t-1}$ is left at that point.

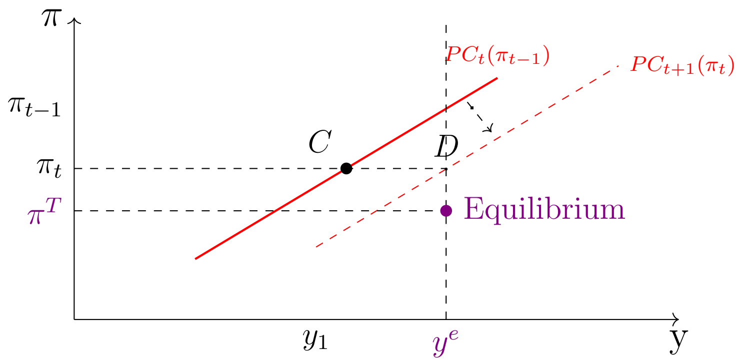

Figure 7: Shifting of the Phillips curve. At time t we find ourselves in point C. We can deduce from the current Phillips curve $PC_t(\pi_{t-1})$, that the inflation on the last period must have been $\pi_{t-1} > \pi_t$, but we don't know where the previous Phillips curve was located at. The Next Phillips curve $PC_{t+1}(\pi_t)$ will be such, that its height at y^e is π_t, point D. Notice, however, that we don't know the point on which the economy will be located at on this next Phillips curve, without specifying the MR and IS curves.

MR curve

The Monetary policy Rule (MR) gives the CB's preferred output-inflation combinations for any Phillips curve. In other words, MR curve tells the CB what output gap to choose, given that inflation deviates from it's target. Still in other words, the MR curve gives the CB's Best Response to a shock; what output-inflation pair to aim for given the situation.

The economy is always on the Phillips curve, from which the CB can choose one point, the intersecting point with the MR curve. In case the economy is off-equilibrium, the MR curve forms a path, along which the CB seeks to guide the economy back towards the point of target inflation and equilibrium output (π^T, y^e).

How is the monetary policy actually conducted? The CB sets the nominal interest rate[9], which influences the real interest rate via Fisher equation ($i = \pi^E + r$), which in turn impacts the aggregate demand, captured by the IS curve. The IS curve gives the output demanded at the new interest rate. This level of output then influences the employment, wages and prices through the labor markets, and the Phillips curve gives the new inflation rate.

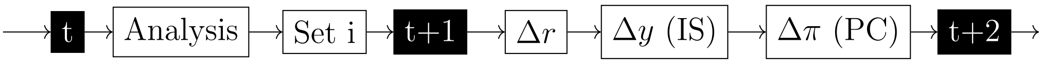

Figure 8: Executing the monetary policy

The location of the Phillips curve, which the CB has to take into account, depends on the inflation expectations (last period's inflation). Also in turn the current inflation becomes the next period's expected inflation, fixing the location of the next Phillips curve.

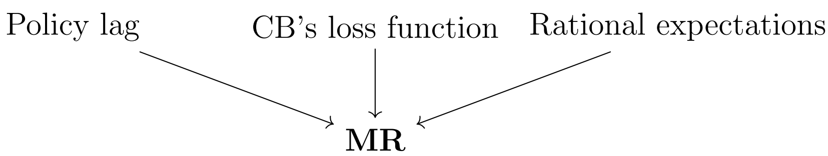

Figure 9: Constructing the MR curve

The CB's optimal policy is the solution to the following constrained optimization problem, involving both inflation and output gaps.[10]

[9]Nowadays the CB has many tools to conduct the monetary policy, additional to setting the nominal interest rate.

[10]In the textbook the CB chooses y, and in the slides it chooses π. Both ways lead to the same conclusion. Actually the CB influences directly the interest rate, and only indirectly first the output and then inflation as a result. But the CB is interested in inflation and output, not the interest rate in itself. Hence, we can cut the corners and say that the CB is "choosing" inflation or output.

$$\min_{\pi_t} L = \beta(\pi_t - \pi^T)^2 + (y_t - y^e)^2 \qquad \text{CB's loss function} \quad (14)$$

s.t.

$$\pi_t = \pi_{t-1} + \alpha(y_t - y^e) \iff y_t - y^e = \frac{\pi_t - \pi_{t-1}}{\alpha} \quad \text{Phillips curve}$$

$$\iff$$

$$\min_{\pi_t} L = \beta(\pi_t - \pi^T)^2 + \Big[\underbrace{\frac{\pi_t - \pi_{t-1}}{\alpha}}_{y_t - y^e}\Big]^2$$

The Phillips curve is the supply side constraint, which the CB has to take into account. This means, that the economy is always on the Phillips curve, but the CB can choose the particular point from the Phillips curve (π, y) for the economy.

By substituting the constraint into the problem, we get an unconstrained optimization problem. The solution can be found by taking the FOC

$$\begin{aligned}
\frac{\partial L}{\partial \pi_t} = 2\beta(\pi_t - \pi^T) + 2\Big[\frac{\pi_t - \pi_{t-1}}{\alpha^2}\Big] = 0 & \\
\beta(\pi_t - \pi^T) &= -\Big[\frac{\pi_t - \pi_{t-1}}{\alpha^2}\Big] \\
\pi_t - \pi^T &= -\frac{1}{\alpha^2\beta}[\pi_t - \pi_{t-1}] \qquad || \pi_t - \pi_{t-1} = \alpha(y_t - y^e) \\
\pi_t - \pi^T &= -\frac{1}{\alpha^2\beta}[\alpha(y_t - y^e)] \\
\pi_t - \pi^T &= -\frac{1}{\alpha\beta}(y_t - y^e) \\
\pi_t &= \pi^T - \frac{1}{\alpha\beta}(y_t - y^e) \quad \textbf{MR curve}
\end{aligned}$$

On the third row we used the Phillips curve to get the output gap in the MR equation.[11] Also notice, that the last two rows can both be presented as the MR curve, as they express exactly the same relation. The textbook formulates the problem in such a way that the CB chooses y instead of π, but this leads to the same end result, because the CB in essence chooses both of them (y directly and

[11] Why was it necessary to first plug in the Phillips equation into the problem before taking the FOC, if we then do the same in reverse afterwards? Because when the CB sets the inflation, it also has an impact on the supply side, characterized by the Phillips curve. This impact wouldn't have been taken into account, if the FOC was taken without substituting in the Phillips curve, yielding a different (wrong) optimal solution.

π idirectly).

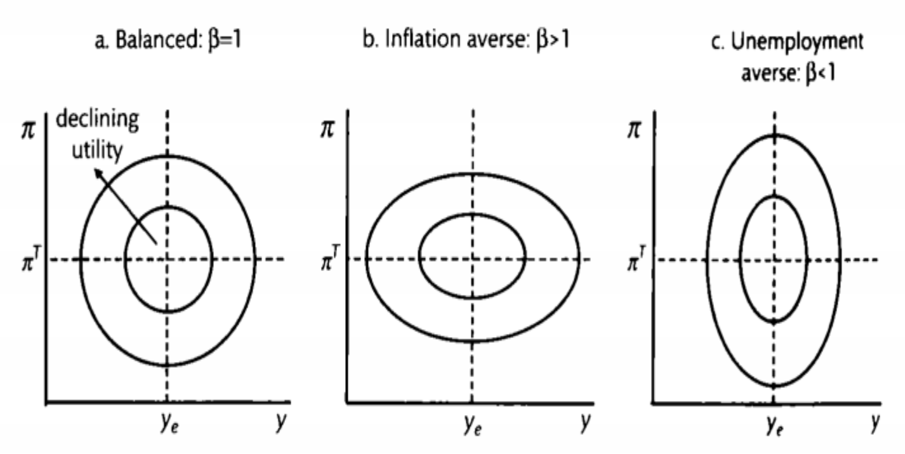

Figure 10: CB's indifference ellipses; All points on an ellipse yield the same loss for the CB, Eq 14. The relative weight between minimizing the inflation and output gaps is given by the parameter β in the loss function.

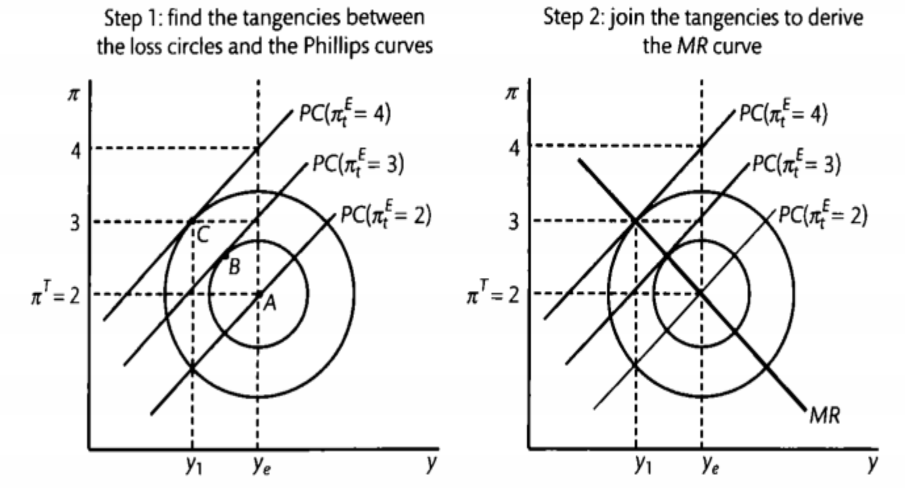

Figure 11: Deriving the MR curve graphically

Policy lag. The impact of a policy typically has a lag, which the CB has to take into account when deciding the policy to be implemented. This policy lag was already built into the dynamic IS curve, Eq 4: Aggregate demand responds to changes in the real interest rate with a one period lag.

CB has rational expectations. The rational expectations hypothesis means, that the CB understands how the model works, in which it is operating.

Another name for the hypothesis is model consistent expectations. Therefore the CB knows whether a shock is transitory or permanent. The CB can also anticipate what the situation will be on the next period, when the monetary policy will kick in.