Yule-Walker equations

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Having observed data $y_1, \dots, y_T$, presumably created by the process

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t \tag{1}$$

we wish to estimate the autoregressive parameters $\phi = (\phi_1, \dots, \phi_p)'$.

Using the Yule-Walker equations, ϕ can be estimated on the basis of the empirical autocovariances γ and or autocorrelations ρ, calculated from the observed data.

$$\phi = \mathbf{\Gamma^{-1}}\gamma = \mathbf{P^{-1}}\rho \tag{2}$$

Autocovariances

Multiply both sides of the Equation 1 by y_{t-h} and take expectations to obtain the h^{th} autocovariance

$$\begin{aligned} \gamma_h &= \mathbf{E}[y_t, y_{t-h}] \\ &\iff \\ \mathbf{E}[y_t y_{t-h}] &= \phi_1 \underbrace{\mathbf{E}[y_{t-1} y_{t-h}]}_{\gamma_{h-1}} + \cdots + \phi_p \underbrace{\mathbf{E}[y_{t-p} y_{t-h}]}_{\gamma_{h-p}} + \mathbf{E}[\varepsilon_t y_{t-h}] \\ &\iff \end{aligned}$$

$$\gamma_h = \begin{cases} \phi_1\gamma_1 + \cdots + \phi_p\gamma_p + \sigma^2 & \textit{when} \quad h = 0 \\ \phi_1\gamma_{h-1} + \cdots + \phi_p\gamma_{h-p} & \textit{when} \quad h > 0 \end{cases} \tag{3}$$

To construct the **Yule-Walker equations** (Equations 6 and 13), we write explicitly open all of the autocovariances γ_h up to $h = p$. By doing so we get a system of equations with as many equations, p, as there are unknown parameters in ϕ. Here the indexes of γ terms look awkward, as the focus is in making their recursive logic clear. The simplified form can be found in Equation 5.

$$
\begin{aligned}
\gamma_1 &= \phi_1 \underbrace{\gamma_{1-1}}_{\gamma_0} + \phi_2\gamma_{1-2} + \cdots + \phi_{p-1} \underbrace{\gamma_{1-(p-1)}}_{\gamma_{2-p}=\gamma_{p-2}} + \phi_p \underbrace{\gamma_{1-p}}_{\gamma_{p-1}} \\
\gamma_2 &= \phi_1\gamma_{2-1} + \phi_2\gamma_{2-2} + \cdots + \phi_{p-1}\gamma_{2-(p-1)} + \phi_p\gamma_{2-p} \\
&\vdots \\
\gamma_{p-1} &= \phi_1\gamma_{(p-1)-1} + \phi_2\gamma_{(p-1)-2} + \cdots + \phi_{p-1}\gamma_{(p-1)-(p-1)} + \phi_p\gamma_{(p-1)-p} \\
\gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_{p-1}\gamma_{p-(p-1)} + \phi_p\gamma_{p-p}
\end{aligned} \tag{4}
$$

Remind, that $\gamma_0 = \mathbf{E}[y_{t-h}y_{t-h}] \;\; \forall h$, and by symmetry

$$
\begin{gathered}
\gamma_h = \gamma_{-h} \\
\Longleftrightarrow \\
\mathbf{E}[y_t y_{t-h}] = \mathbf{E}[y_{t-h} y_t]
\end{gathered}
$$

The Equation 4 simplifies to

$$
\begin{aligned}
\gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 + \cdots + \phi_{p-1}\gamma_{p-2} + \phi_p\gamma_{p-1} \\
\gamma_2 &= \phi_1\gamma_1 + \phi_2\gamma_0 + \cdots + \phi_{p-1}\gamma_{p-3} + \phi_p\gamma_{p-2} \\
&\vdots \\
\gamma_{p-1} &= \phi_1\gamma_{p-2} + \phi_2\gamma_{p-3} + \cdots + +\phi_{p-1}\gamma_0 + \phi_p\gamma_1 \\
\gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_{p-1}\gamma_1 + \phi_p\gamma_0
\end{aligned} \tag{5}
$$

which in matrix form is given by

$$
\underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p-2} \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma}
=
\underbrace{\begin{bmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-3} & \gamma_{p-2} & \gamma_{p-1} \\
\gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-4} & \gamma_{p-3} & \gamma_{p-2} \\
\gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-5} & \gamma_{p-4} & \gamma_{p-3} \\
\vdots & & & \ddots & & & \vdots \\
\gamma_{p-3} & \gamma_{p-4} & \gamma_{p-5} & \cdots & \gamma_0 & \gamma_1 & \gamma_2 \\
\gamma_{p-2} & \gamma_{p-3} & \gamma_{p-4} & \cdots & \gamma_1 & \gamma_0 & \gamma_1 \\
\gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_2 & \gamma_1 & \gamma_0
\end{bmatrix}}_{\mathbf{\Gamma}}
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi}
\tag{6}
$$

Denote vectors $\gamma = (\gamma_1, \ldots, \gamma_p)'$, $\phi = (\phi_1, \ldots, \phi_p)'$, and the above matrix $\mathbf{\Gamma} = [\gamma_{i-j}]_{i,j=1,\ldots,p}$ to get

$$
\begin{aligned}
\gamma &= \mathbf{\Gamma}\phi && || \times \Gamma^{-1} \quad (7)\\
&\Longleftrightarrow \\
\phi &= \mathbf{\Gamma}^{-1}\gamma && (8)\\
&\Longleftrightarrow
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \\ \vdots \\ \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-3} & \gamma_{p-2} \\ & & & & \\ \vdots & & \ddots & & \vdots \\ & & & & \\ \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}^{-1}}_{\Gamma^{-1}} \underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \\ \vdots \\ \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma} \tag{9}
$$

Equation 9 is the first important finding, allowing the unknown parameters ϕ to be expressed in terms of the empirical autocovariances γ.

Autocorrelations

Divide the Equation 3, when $h > 0$, by the variance $\gamma_0 = \mathbf{E}[y_t y_t]$ to get the h^{th} order autocorrelations (h^{th} order autocovariance divided by variance)

$$
\begin{aligned}
\rho_h &= \frac{\gamma_h}{\gamma_0} \\
&\Longleftrightarrow \\
\frac{\mathbf{E}[y_t y_{t-h}]}{\mathbf{E}[y_t y_t]} &= \phi_1 \frac{\mathbf{E}[y_{t-1} y_{t-h}]}{\mathbf{E}[y_t y_t]} + \cdots + \phi_p \frac{\mathbf{E}[y_{t-p} y_{t-h}]}{\mathbf{E}[y_t y_t]} \\
&\Longleftrightarrow \\
\frac{\gamma_h}{\gamma_0} &= \phi_1 \underbrace{\frac{\gamma_{h-1}}{\gamma_0}}_{\rho_{h-1}} + \cdots + \phi_p \underbrace{\frac{\gamma_{h-p}}{\gamma_0}}_{\rho_{h-p}} \\
&\Longleftrightarrow \\
\rho_h &= \phi_1 \rho_{h-1} + \cdots + \phi_p \rho_{h-p} && (10)
\end{aligned}
$$

Note, that the 0^{th} order autocorrelation is $\rho_{h-h} = \rho_0 = \frac{\gamma_0}{\gamma_0} = 1, \ \forall h$.

Therefore we have

$$
\begin{aligned}
\rho &= \frac{1}{\gamma_0}\gamma \\
\mathbf{P} &= \frac{1}{\gamma_0}\mathbf{\Gamma}
\end{aligned}
$$

where $\rho = (\rho_1, \ldots, \rho_p)'$ and $\mathbf{P} = [\rho_{i-j}]_{i,j=1,\ldots,p}$.

Using these to re-write Equation 7 as

$$
\begin{aligned}
\gamma &= \boldsymbol{\Gamma}\phi && \quad || * \frac{1}{\gamma_0} \\
\frac{1}{\gamma_0}\gamma &= \frac{1}{\gamma_0}\boldsymbol{\Gamma}\phi \\
\rho &= \mathbf{P}\phi
\end{aligned} \tag{11}
$$

where the opened up Equation 11 is

$$
\begin{aligned}
\rho_1 &= \underbrace{\phi_1}_{\phi_1 \frac{\gamma_0}{\gamma_0}} + \phi_2 \underbrace{\rho_1}_{\frac{\gamma_1}{\gamma_0}} + \cdots + \phi_{p-1} \underbrace{\rho_{p-2}}_{\frac{\gamma_{p-2}}{\gamma_0}} + \phi_p \underbrace{\rho_{p-1}}_{\frac{\gamma_{p-1}}{\gamma_0}} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \cdots + \phi_{p-1}\rho_{p-3} + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_{p-1} &= \phi_1\rho_{p-2} + \phi_2\rho_{p-3} + \cdots + +\phi_{p-1} + \phi_p\rho_1 \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_{p-1}\rho_1 + \phi_p
\end{aligned} \tag{12}
$$

and the same in the matrix form

$$
\underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \\ \vdots \\ \\ \rho_{p-2} \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-3} & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-4} & \rho_{p-3} & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-5} & \rho_{p-4} & \rho_{p-3} \\ & & & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & & & \\ \rho_{p-3} & \rho_{p-4} & \rho_{p-5} & \cdots & 1 & \rho_1 & \rho_2 \\ \rho_{p-2} & \rho_{p-3} & \rho p-4 & \cdots & \rho_1 & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & \rho_2 & \rho_1 & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \\ \vdots \\ \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{13}
$$

Equation 8 can then be re-written as

$$
\begin{aligned}
\rho &= \mathbf{P}\phi && \quad || \times \mathbf{P^{-1}} \\
\phi &= \mathbf{P}^{-1}\rho \\
&\iff
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \\ \vdots \\ \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-3} & \rho_{p-2} \\ & & & & \\ \vdots & & \ddots & & \vdots \\ & & & & \\ \rho_{p-2} & \rho_{p-3} & \cdots & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_1 & 1 \end{bmatrix}^{-1}}_{\mathbf{P^{-1}}} \underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \\ \vdots \\ \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} \tag{14}
$$

Equation 14 is the other central equation, which allows for the parameters ϕ to be recovered from the empirical autocorrelations ρ.