Yule-Walker equations

Oliver Snellman

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Having observed data $y_1, \dots, y_T$, presumably created by the process

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t \tag{1}$$

we wish to estimate the autoregressive parameters $\phi = (\phi_1, \dots, \phi_p)'$.

Using the Yule-Walker equations, ϕ can be estimated on the basis of the empirical autocovariances γ or autocorrelations ρ, calculated from the observed data.

Autocovariances

Multiply both sides of the Equation 1 by y_{t-h} and take expectations to obtain the h^{th} autocovariance

$$\gamma_h = \mathbf{E}[y_t, y_{t-h}]$$

$$\iff$$

$$\mathbf{E}[y_t y_{t-h}] = \phi_1 \underbrace{\mathbf{E}[y_{t-1} y_{t-h}]}_{\gamma_{h-1}} + \cdots + \phi_p \underbrace{\mathbf{E}[y_{t-p} y_{t-h}]}_{\gamma_{h-p}} + \mathbf{E}[\varepsilon_t y_{t-h}]$$

$$\iff$$

$$\gamma_h = \begin{cases} \phi_1 \gamma_1 + \cdots + \phi_p \gamma_p + \sigma^2 & when \quad h = 0 \\ \phi_1 \gamma_{h-1} + \cdots + \phi_p \gamma_{h-p} & when \quad h > 0 \end{cases} \tag{2}$$

To construct the **Yule-Walker equations**, we first write explicitly open all of the autocovariances γ_h up to $h = p$. By doing so we get a system of equations with as many equations, p, as there are unknown parameters in ϕ. Here the indexes of γ terms look awkward, as the focus is in making their recursive logic clear. The simplified form can be found in Equation 4.

$$
\begin{aligned}
\gamma_1 &= \phi_1 \underbrace{\gamma_{1-1}}_{\gamma_0} + \phi_2 \gamma_{1-2} + \cdots + \phi_{p-1} \underbrace{\gamma_{1-(p-1)}}_{\gamma_{2-p}=\gamma_{p-2}} + \phi_p \gamma_{1-p} \\
\gamma_2 &= \phi_1 \gamma_{2-1} + \phi_2 \gamma_{2-2} + \cdots + \phi_{p-1} \gamma_{2-(p-1)} + \phi_p \gamma_{2-p} \\
&\vdots \\
\gamma_{p-1} &= \phi_1 \gamma_{(p-1)-1} + \phi_2 \gamma_{(p-1)-2} + \cdots + \phi_{p-1} \gamma_{(p-1)-(p-1)} + \phi_p \gamma_{(p-1)-p} \\
\gamma_p &= \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \cdots + \phi_{p-1} \gamma_{p-(p-1)} + \phi_p \gamma_{p-p}
\end{aligned}
\tag{3}
$$

Remind, that $\gamma_0 = \mathbf{E}[y_{t-h} y_{t-h}] \;\; \forall h$, and by symmetry

$$
\begin{gathered}
\gamma_h = \gamma_{-h} \\
\Longleftrightarrow \\
\mathbf{E}[y_t y_{t-h}] = \mathbf{E}[y_{t-h} y_t]
\end{gathered}
$$

The Equation 3 simplifies to

$$
\begin{aligned}
\gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 + \cdots + \phi_{p-1} \gamma_{p-2} + \phi_p \gamma_{p-1} \\
\gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 + \cdots + \phi_{p-1} \gamma_{p-3} + \phi_p \gamma_{p-2} \\
&\vdots \\
\gamma_{p-1} &= \phi_1 \gamma_{p-2} + \phi_2 \gamma_{p-3} + \cdots + +\phi_{p-1} \gamma_0 + \phi_p \gamma_1 \\
\gamma_p &= \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \cdots + \phi_{p-1} \gamma_1 + \phi_p \gamma_0
\end{aligned}
\tag{4}
$$

which in matrix form is given by

$$
\underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p-2} \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma}
=
\underbrace{\begin{bmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-3} & \gamma_{p-2} & \gamma_{p-1} \\
\gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-4} & \gamma_{p-3} & \gamma_{p-2} \\
\gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-5} & \gamma_{p-4} & \gamma_{p-3} \\
\vdots & & & \ddots & & & \vdots \\
\gamma_{p-3} & \gamma_{p-4} & \gamma_{p-5} & \cdots & \gamma_0 & \gamma_1 & \gamma_2 \\
\gamma_{p-2} & \gamma_{p-3} & \gamma_{p-4} & \cdots & \gamma_1 & \gamma_0 & \gamma_1 \\
\gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_2 & \gamma_1 & \gamma_0
\end{bmatrix}}_{\mathbf{\Gamma}}
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi}
\tag{5}
$$

Denote vectors $\gamma = (\gamma_1, \ldots, \gamma_p)'$, $\phi = (\phi_1, \ldots, \phi_p)'$, and the above matrix $\mathbf{\Gamma} = [\gamma_{i-j}]_{i,j=1,\ldots,p}$ to get

$$\gamma = \mathbf{\Gamma}\phi \qquad || \times \Gamma^{-1} \tag{6}$$
$$\Longleftrightarrow$$
$$\phi = \mathbf{\Gamma}^{-1}\gamma \tag{7}$$
$$\Longleftrightarrow$$
$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \\ \vdots \\ \\ \phi_{p-1} \\ \phi_p \end{bmatrix} = \begin{bmatrix} 1 & \gamma_1 & \dots & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & 1 & \dots & \gamma_{p-3} & \gamma_{p-2} \\ \\ \vdots & & \ddots & & \vdots \\ \\ \gamma_{p-2} & \gamma_{p-3} & \dots & 1 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \dots & \gamma_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \\ \vdots \\ \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix} \tag{8}$$

Equation 8 is the first important finding, allowing the unknown parameters ϕ to be expressed in terms of the empirical autocovariances γ.

Autocorrelations

Divide the Equation 2, when $h > 0$, by the variance $\gamma_0 = \mathbf{E}[y_t y_t]$ to get the h^{th} order autocorrelations (h^{th} order autocovariance divided by variance)

$$\rho_h = \frac{\gamma_h}{\gamma_0}$$
$$\Longleftrightarrow$$
$$\frac{\mathbf{E}[y_t y_{t-h}]}{\mathbf{E}[y_t y_t]} = \phi_1 \frac{\mathbf{E}[y_{t-1} y_{t-h}]}{\mathbf{E}[y_t y_t]} + \dots + \phi_p \frac{\mathbf{E}[y_{t-p} y_{t-h}]}{\mathbf{E}[y_t y_t]}$$
$$\Longleftrightarrow$$
$$\frac{\gamma_h}{\gamma_0} = \phi_1 \underbrace{\frac{\gamma_{h-1}}{\gamma_0}}_{\rho_{h-1}} + \dots + \phi_p \underbrace{\frac{\gamma_{h-p}}{\gamma_0}}_{\rho_{h-p}}$$
$$\Longleftrightarrow$$
$$\rho_h = \phi_1 \rho_{h-1} + \dots + \phi_p \rho_{h-p} \tag{9}$$

Note, that the 0^{th} order autocorrelation is $\rho_{h-h} = \rho_0 = \frac{\gamma_0}{\gamma_0} = 1, \ \ \forall h$.

Therefore we have

$$\rho = \frac{1}{\gamma_0}\gamma$$
$$\mathbf{P} = \frac{1}{\gamma_0}\mathbf{\Gamma}$$

where $\rho = (\rho_1, \dots, \rho_p)'$ and $\mathbf{P} = [\rho_{i-j}]_{i,j=1,\dots,p}$.

Using these to re-write Equation 6 as

$$
\begin{aligned}
\gamma &= \mathbf{\Gamma}\phi && || * \frac{1}{\gamma_0} \\
\frac{1}{\gamma_0}\gamma &= \frac{1}{\gamma_0}\mathbf{\Gamma}\phi \\
\rho &= \mathbf{P}\phi
\end{aligned} \tag{10}
$$

where the opened up Equation 10 is

$$
\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + \cdots + \phi_{p-1}\rho_{p-2} + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \cdots + \phi_{p-1}\rho_{p-3} + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_{p-1} &= \phi_1\rho_{p-2} + \phi_2\rho_{p-3} + \cdots + +\phi_{p-1} + \phi_p\rho_1 \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_{p-1}\rho_1 + \phi_p
\end{aligned} \tag{11}
$$

and the same in the matrix form

$$
\underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_{p-2} \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-3} & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-4} & \rho_{p-3} & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-5} & \rho_{p-4} & \rho_{p-3} \\ \vdots & & & \ddots & & & \vdots \\ \rho_{p-3} & \rho_{p-4} & \rho_{p-5} & \cdots & 1 & \rho_1 & \rho_2 \\ \rho_{p-2} & \rho_{p-3} & \rho p-4 & \cdots & \rho_1 & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & \rho_2 & \rho_1 & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{12}
$$

Equation 7 can then be re-written as

$$
\begin{aligned}
\rho &= \mathbf{P}\phi && || \times \mathbf{P^{-1}} \\
\phi &= \mathbf{P}^{-1}\rho \\
&\iff
\end{aligned}
$$

$$
\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-3} & \rho_{p-2} \\ \vdots & & \ddots & & \vdots \\ \rho_{p-2} & \rho_{p-3} & \cdots & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{p-1} \\ \rho_p \end{bmatrix} \tag{13}
$$

Equation 13 is the other central equation, which allows for the parameters ϕ to be recovered from the empirical autocorrelations ρ.