Overlapping Generations Model (OLG)

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Abstract

In these notes I solve the OLG model analytically step by step for the fully funded case, compare the efficiency of two common pension schemes, pay as you go and fully funded, and end with a short conversation about dynamic inefficiency.

Introduction

The OLG model is built upon the problems of households and firms, like other common macroeconomic modeling frameworks. But an important difference is, that the OLG model **doesn't utilize the assumption of Representative household**. Using a representative household requires an assumption, that the distribution of the households is such, that the whole mass of households can be represented using just the median household.

Instead of having infinitely lived dynasty-like (representative) households, in the OLG model there are young and old (heterogenous) households each period, whose incentives don't necessarily align. We could equally well specify 80 types of households, one for each year of an average life expectancy, but we'll stick with two for now, for simplicity.

Another important feature of the model is the **pension system**, through which the aging households get their livelihoods when they become unable to work. There are two common pension schemes, **pay as you go (PAYG)** and the **fully funded (FF)** system. They both convey the same benefit to old households, but might inflict different sized costs to the young households.

Let's first solve the model for the fully funded case (PAYG goes quite similarly). Then we consider the possibility of **dynamic inefficiency** (page 8), where the actual steady state consumption c^* falls short of the golden rule consumption c^{gr}. Finally, we compare the two pensions schemes (page 9) and find out that PAYG is cheaper when $r < g$ and FF is cheaper otherwise.

Solving the model

In the Fully Funded pension system the households fund their consumption at old age by saving some of their salary when they are young. I will specify the functional form for the utility- (logarithmic) and production functions (Cobb-Douglas) straight away, so that we can analytically solve the model.

The procedure is as follows:

1. Solve the equilibrium savings rate s from the HH's problem
2. Solve for the wage rate w from the Firm's problem
3. Based on s and w, solve for the capital accumulation $k_{t+1} = f(k_t)$
4. ... and the steady state level of capital k^*
5. Find the weighted consumption basket c=y-i of the households
6. Use k^* and c to find out the steady state consumption c^*
7. Find out the highest level of (golden rate) consumption attainable in the model c^{gr}, and the corresponding steady state level capital k^{gr}

1. Solve for savings s from the problem of the HH

$$\max_{c_{y,t},c_{o,t+1}} \quad U(c_{y,t}, c_{o,t+1}) = u(c_{y,t}) + \beta u(c_{o,t+1})$$

$$\iff$$

$$\max_{c_{y,t},c_{o,t+1}} \quad log(c_{y,t}) + \beta log(c_{o,t+1})$$

s.t.

$$c_{y,t} + s_t = w_t \qquad \iff \qquad c_{y,t} = w_t - s_t \tag{1}$$

$$c_{o,t+1} = \underbrace{(R_{t+1})s_t}_{\text{Return to savings}} + \underbrace{(1-\delta)s_t}_{\text{Leftover capital}} \qquad \iff \qquad c_{o,t+1} = (R_{t+1} + 1 - \delta)s_t \tag{2}$$

The young household receives wages and saves some of that for old age (for ff pension). The savings is used to acquire capital, which the household rents to firms during the next period.

When the household is old, it gets rental rate R_{t+1} for the whole capital stock they acquired with their savings s_t, and they also get to consume the leftover capital, that has not been depreciated $(1-\delta)s_t$ during the production at period t+1.

More specifically, in equilibrium the old households sell their leftover capital $(1-\delta)K_t$ to young households, and then use the money for consumption. The young households buy this leftover capital using their savings, and also acquire some more capital, by buying investment goods produced by the firms, to cover up the depreciated capital (so that $k_t = k_{t+1} = k^*$). Hence, in the OLG -model the savings typically does not equal investments, as savings cover the whole next period's capital stock K_{t+1}, but investments are defined as only the newly acquired capital, $I_t = \underbrace{K_{t+1}}_{S_t} - (1-\delta)K_t$.

Plug the constraints into the problem to get an unconstrained problem, with savings as the control variable

$$\max_{s} \quad log(\underbrace{w_t - s_t}_{c_{y,t}}) + \beta \, \log\Big(\underbrace{[R_{t+1} + 1 - \delta]s_t}_{c_{o,t+1}}\Big)$$

FOC:

$$\frac{\partial U}{\partial s_t} = -\frac{1}{w_t - s_t} + \cancel{[R_{t+1} + 1 - \delta]}\beta \frac{1}{\cancel{[R_{t+1} + 1 - \delta]}s_t} = 0$$

$$\frac{1}{w_t - s_t} = \frac{\beta}{s_t}$$

$$s_t = \beta w_t - \beta s_t$$

$$s_t = \frac{\beta}{1+\beta} w_t \qquad \text{Equilibrium savings} \quad (3)$$

We will later use this savings rate to pin down the capital accumulation, but for that we also need to find out the equilibrium wages by solving the Firm's problem. I also solve for the rental rate of capital, R_t, even though we don't need it here.

2. Solve for wages w from the problem of the Firm

$$\max_{K_t, N_t} \; \pi(K_t, N_t) = \underbrace{\overbrace{P}^{=1} * A * F(K_t, N_t)}_{\text{Value of output}} - \underbrace{R_t K_t - w_t N_t}_{\text{Costs of production}}$$

$$\iff$$

$$\max_{K_t, N_t} \; A K_t^{\alpha} N_t^{1-\alpha} - R_t K_t - w_t N_t$$

FOC:

$$\frac{\partial \pi}{\partial N_t} = (1-\alpha) A K_t^{\alpha} N_t^{1-\alpha-1} - w_t = 0$$

$$w_t = (1-\alpha) A \frac{K_t^{\alpha}}{N_t^{\alpha}}$$

$$w_t = (1-\alpha) A \Big(\underbrace{\frac{K_t}{N_t}}_{=:k_t} \Big)^{\alpha}$$

$$w_t = (1-\alpha) A k_t^{\alpha} \qquad \text{Equilibrium wage} \quad (4)$$

$$\frac{\partial \pi}{\partial K_t} = \alpha A K_t^{\alpha-1} N_t^{1-\alpha} - R_t = 0$$

$$R_t = \alpha A k_t^{\alpha-1}$$

Now that we know the savings- and the wage-rate, Eq 3 and 4 respectively, we can figure out how the capital accumulates through time.

3. Solve for capital accumulation

Capital is just the fraction saved by the young households out of their income. This saved money is used to acquire capital, which is then rented to firms by the same households at old age (next period), $K_t = N_{t-1}s_{t-1}$. The equation is identical for all periods, so let's roll the indices forward one period. Remember, that the uppercase letter K stands for aggregate capital stock.

$$K_{t+1} = N_t s_t \qquad\qquad || \frac{1}{N_t}$$

$$\frac{K_{t+1}}{N_t} = s_t$$

$$\frac{K_{t+1}}{N_t} \underbrace{\frac{N_{t+1}}{N_{t+1}}}_{=1} = s_t$$

$$\underbrace{\frac{K_{t+1}}{N_{t+1}}}_{=:k_{t+1}} \underbrace{\frac{N_{t+1}}{N_t}}_{(1+n)} = s_t \qquad\qquad || N_{t+1} = (1+n)N_t$$

$$(1+n)k_{t+1} = \underbrace{\frac{\beta}{1+\beta} w_t}_{s_t}$$

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta} \underbrace{(1-\alpha)Ak_t^{\alpha}}_{w_t}$$

$$k_{t+1} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} k_t^{\alpha} \qquad \text{Capital accumulation} \tag{5}$$

This capital accumulation equation describes, how the next period's capital stock (per labor) k_{t+1} is determined by the current period's capital stock k_t.

4. Steady state level of capital

In the steady state the capital stock stays the same (or grows at the same rate) $k_t = k_{t+1} = k^*$. We can find this steady state value by plugging k^* into the Eq 5 for both k_t and k_{t+1}.

$$k^* = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}(k^*)^\alpha \qquad || \frac{1}{(k^*)^\alpha}$$

$$(k^*)^{1-\alpha} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} \qquad ||()^{\frac{1}{1-\alpha}}$$

$$k^* = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}} \qquad \text{Steady state capital} \tag{6}$$

5. Find the weighted consumption basket

To be able to find the steady state consumption, we first have to do some aggregation over young and old households, to find expressions for total income Y_t and investments I_t, and combine them to get an aggregate constraint for the whole economy, $C_t + I_t = Y_t$. Why is this necessary? Because $S_t \neq I_t$, as some of the savings of the young household is going to end up as the consumption of the old household (worth of $(1-\delta)K_t$). Hence, we need to subtract only the investments from the total income (instead of savings), to get the aggregate consumption, $C_t = Y_t - I_t$. Here C_t includes the total consumption of both household types at t.

By then dis-aggregating this constraint, we get a constraint for the working households, from which we can find out the consumption using $c_t = y_t - i_t$, including consumption of both household types. We don't need to care about population growth when working with the dis-aggregated constraint.

Aggregation. Start by multiplying the original constraints for young and old households, Eq 1 and 2, by the number of households N_t and N_{t-1}, respectively.

Aggregate young households at t

$$\begin{aligned} c_{y,t} + s_t &= w_t && || * N_t \\ \underbrace{N_t c_{y,t}}_{=:C_{y,t}} + \underbrace{N_t s_t}_{=S_t = K_{t+1}} &= N_t w_t \end{aligned}$$

Aggregate old households at t

$$\begin{aligned} c_{o,t} &= (R_t + 1 - \delta)s_{t-1} && || * N_{t-1} \\ \underbrace{N_{t-1} c_{o,t}}_{=:C_{o,t}} &= (R_t + 1 - \delta)\underbrace{N_{t-1} s_{t-1}}_{=K_t} \end{aligned}$$

Combine the above aggregate constraints for young and old households at time t by summing them up side by side (combined consumption equals combined income), to get an aggregate constraint for the whole economy.

$$\underbrace{C_{y,t} + K_{t+1}}_{young} + \underbrace{C_{o,t}}_{old} = \underbrace{N_t w_t}_{young} + \underbrace{(R_t + 1 - \delta)K_t}_{old}$$

$$\underbrace{C_{y,t} + C_{o,t}}_{=:C_t} + \underbrace{\overbrace{K_{t+1}}^{S_t} - (1-\delta)K_t}_{=:I_t} = \underbrace{N_t w_t + R_t K_t}_{=:Y_t}$$

$$C_t + I_t = Y_t$$

We have now constructed the aggregate recourse constraint for the whole economy, and also defined $I_t = K_{t+1} - (1-\delta)K_t$.

Dis-aggregation. By dividing the terms by N_t, we find the constraint for the working household. Why divide by N_t and not by $(N_t + N_{t-1})$? First one is more practical, because Y and I are something that are related only to the young households. The end results for everything relevant are the same anyway.

The dis-aggregated **consumption**, on the other hand, ends up being a consumption basket that takes into account the consumption of both ages, weighted by the relative number of young households with respect to the old ones:

$$\begin{aligned}\frac{C_t}{N_t} &= \frac{C_{y,t} + C_{o,t}}{N_t} \\ &= \frac{N_t c_{y,t} + N_{t-1} c_{o,t}}{N_t} \\ &= c_{y,t} + \frac{c_{o,t}}{(1+n)} \\ &=: c_t\end{aligned}$$

To disaggregate the **income** Y_t, remember that it equals the output, and that only the young households work, so we can define y_t using the production function

$$\begin{aligned}\frac{Y_t}{N_t} &= AF(\frac{K_t}{N_t}, \frac{N_t}{N_t}) \\ &= AF(k_t, 1) \\ &= AF(k_t) \\ &= Ak_t^\alpha \\ &=: y_t\end{aligned}$$

Doing the same for **investment** requires an extra trick. Remember also here, that only the young households are investing - buying capital that didn't exist before.

$$I_t = K_{t+1} - (1-\delta)K_t \qquad || * \frac{1}{N_t}$$

$$i_t = \frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} - (1-\delta)k_t$$

$$i_t = (1+n)k_{t+1} - (1-\delta)k_t \qquad || \; k_{t+1} = k_t = k^*$$

$$i^* = (1+n)k^* - (1-\delta)k^*$$

$$i^* = (\delta+n)k^*$$

We arrive at the desired result

$$\begin{aligned} c + i &= y \\ c &= y - i \qquad \textbf{Weighted consumption basket} \end{aligned}$$

6. Use k^* and c to find the steady state consumption c^*

Notice, that both y and i are functions of capital. In the steady state they are solely defined by k^*.

To find out how much consumption there is in the steady state, c^*, just plug in the expression for k^*, Eq 6, for y and i in the consumption basket above.

$$c^* = y(k^*) - i(k^*)$$

$$c^* = A(k^*)^\alpha - (\delta+n)k^*$$

$$c^* = A\left(\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}\right)^\alpha - (\delta+n)\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

$$c^* = A\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{\alpha}{1-\alpha}} - (\delta+n)\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} \qquad \text{Steady state consumption}$$

This steady state consumption is not necessarily the best we can do, as we'll see next.

7.1 Find the capital k^{gr} providing maximal (golden rule) consumption

The steady state capital k^* does not necessarily allows for the highest possible steady state consumption, called the golden rate consumption c^{gr}. To find out how good or bad c^* is with regards to what it could be, we need to find out the golden rate capital stock k^{gr}, which gives rise to c^{gr}.

Recall from the basic Solow model, that to maximize the steady state consumption, we want to increase k^* as long as it increases c^*.

$$
\begin{aligned}
\frac{dc^*}{dk^*} = \alpha A(k^*)^{\alpha-1} - (\delta+n) = 0 & \\
(k^*)^{\alpha-1} &= \frac{\delta+n}{\alpha A} \\
k^* &= \left[\frac{\delta+n}{\alpha A}\right]^{\frac{1}{\alpha-1}} \\
k^{gr} &= \left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}} \qquad \text{Consumption maximizing capital} \qquad (7)
\end{aligned}
$$

The amount of capital in steady state, which maximizes the consumption basket, is denoted by k^{gr}.

7.2 Find the golden rule consumption c^{gr}

To find out c^{gr}, we proceed similarly as when finding the steady state consumption in step 6. But instead of k^*, we plug in the golden rule capital k^{gr}, Eq 7, and define the consumption level it produces as c^{gr}.

$$c^{gr} = y(k^{gr}) - i(k^{gr})$$

$$c^{gr} = A(k^{gr})^{\alpha} - (\delta+n)k^{gr}$$

Golden rate consumption in terms of the model's parameters is then given by

$$c^{gr} = A\left(\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}\right)^{\alpha} - (\delta+n)\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}$$

$$c^{gr} = A\left[\frac{\alpha A}{\delta+n}\right]^{\frac{\alpha}{1-\alpha}} - (\delta+n)\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}} \qquad \text{Golden rule consumption}$$

Dynamic inefficiency: Too much or little savings?

We now have two expressions for steady state capital, k^* from Eq 6 and the golden ratio capital k^{gr} from Eq 7. If these are equal, then the savings rate is such that it produces the steady state capital, which then maximizes the consumption of the households.

$$k^* = k^{gr} \qquad\qquad \to c^* = c^{gr}$$

$$\left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}} = \left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}$$

$$\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} = \frac{\alpha}{\delta+n}$$

If this condition is not met, then the savings rate is either too low or too high, and the steady state capital stock ends up being such, that the highest level of consumption attainable is not reached, $c^* < c^{gr}$. The savings rate might be sub-optimal, because of the incentive structures of the economy.

If savings rate is too high, it means that the capital stock is kept at larger than optimal level. The excess savings might be due to the fully funded pension system, which incentivizes people to save too much for their future pensions. The savings rate should be lowered with some policy choice, for example by changing to PAYG-pension system for Pareto improvement.

When savings rate is too low, it might be beneficial to incentivize increased savings, which would accumulate capital and create a higher standard of living for the future generations. However, this might require a transition period of several generations, who would have to suffer reduced living standards. Therefore it's not necessarily a Pareto improvement to move towards c^{gr}.

Comparing pension systems

How much will the old household receive as pension payments? The size of the pension is given by the replacement rate $0 \leq \bar{e} \leq 1$, which is a fraction of that income, which the old HH received when working as a young HH. The pension of one old HH is then $e_t = \bar{e}w_{t-1}$, and the aggregate pension payments are $N_{t-1}e_t$. The size of the pension is the same regardless of whether payg of ff is implemented. Therefore that pension scheme is better, which can be financed with smaller reduction in consumption of the young households.

Consider first the **Pay As You Go (payg)** -pension system, where the currently working young households transfer part of their income to support the non-working old households. The fraction of young HHs' total income, τ_t^{payg}, needed to finance this pensions scheme, can then be solved from

$$\tau_t^{\text{payg}} \underbrace{w_t N_t}_{\text{Current output}} = \bar{e} \underbrace{w_{t-1} N_{t-1}}_{\text{Previous output}} \qquad \text{Aggregate pensions}$$

$$\tau_t^{\text{payg}} \underbrace{(1+a)w_{t-1}}_{=w_t} \underbrace{(1+n)N_{t-1}}_{=N_t} = \bar{e} w_{t-1} N_{t-1} \qquad || \frac{1}{w_{t-1}N_{t-1}}$$

$$\tau_t^{\text{payg}}(1+a)(1+n) = \bar{e}$$

$$\tau_t^{\text{payg}} = \tau^{\text{payg}} = \frac{\bar{e}}{(1+a)(1+n)} \tag{8}$$

In the last line I dropped the time index t from τ^{payg}, as there are only time invariant parameters left on the RHS.

The second alternative is the **Fully Funded (ff)** pension system, where every household saves for their own pension. Here the pension of currently old households was saved by themselves from the last period's salary.

$$\underbrace{(1+r)\tau_{t-1}^{\text{ff}} w_{t-1} N_{t-1}}_{\text{Value of savings on t}} = \underbrace{\bar{e} w_{t-1} N_{t-1}}_{\text{Size of the pension}} \qquad || \frac{1}{w_{t-1}N_{t-1}}$$

$$(1+r)\tau_{t-1}^{\text{ff}} = \bar{e}$$

$$\tau_{t-1}^{\text{ff}} = \tau^{\text{ff}} = \frac{\bar{e}}{1+r} \tag{9}$$

As both of the systems convey the benefit of identical size to old households, $\bar{e} w_{t-1} N_{t-1}$, the important question is: which one is cheaper to implement? In other words, what is the situation where FF is cheaper to fund than PAYG?

$$\begin{aligned}
\tau_t^{\text{payg}} \quad &> \quad \tau_t^{\text{ff}} \\
&\Longleftrightarrow \\
\underbrace{\frac{\bar{e}}{(1+a)(1+n)}}_{\tau_t^{\text{payg}}} \quad &> \quad \underbrace{\frac{\bar{e}}{1+r}}_{\tau_t^{\text{ff}}} \qquad\qquad || \bar{e}, a, n, r \geq 0 \\
&\Longleftrightarrow \\
1+r \quad &> \quad \underbrace{(1+a)(1+n)}_{1+a+n+\underbrace{an}_{\approx 0} \approx 1+a+n} \\
&\Longleftrightarrow \\
r \quad &> \quad \underbrace{a+n}_{=:g} \\
&\Longleftrightarrow \\
r \quad &> \quad g
\end{aligned}$$

The question whether PAYG or FF can be funded with smaller costs boils down to whether the interest rate r or the growth rate of the economy g is larger. The term g takes into account both population growth and technological progress.

If	$r > g$	
Then	$\tau_t^{\text{payg}} > \tau_t^{\text{ff}}$	**FF is cheaper**
If	$r < g$	
Then	$\tau_t^{\text{payg}} < \tau_t^{\text{ff}}$	**PAYG is cheaper**

However, it is not always straight forward to change the pension system when the circumstances make the current system unfavorable. When planning to move from FF to PAYG, there is a missing negotiations channel, as the next future generation is not yet alive to consent to paying for the pensions of the current working population at their old age.

It is, in principe, even harder to go from PAYG to FF system. It produces a temporary Pareto-diminishment, as the young households have to pay for the pensions of the old people (who contributed to the PAYG system), additional to saving for their own pensions. From that point onwards, the FF system would be an improvement for the rest of the generations to come.