Overlapping Generations Model (OLG)

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Abstract

In these notes I compare the efficiency of two common pension schemes in the OLG model, pay as you go and fully funded, solve the model analytically for the fully funded case and end with a short conversation about dynamic inefficiency.

Introduction

The OLG model is built upon the problems of households and firms, like other common macroeconomic modeling frameworks, but it differs from them in two important ways, having **ageing and heterogenous households**.

Instead of having infinitely lived dynasty-like households, the OLG model has ageing households which live finitely many periods and face different economic circumstances through their life cycle. The households are also heterogeneous, which means that the OLG model doesn't have a representative household. Using a representative household requires assuming that the distribution of the households is such, that their whole mass can be represented using just the median household. In the OLG model the different types of households (by their age) coexist each period, and their incentives don't necessarily align.

In this document I will be concentrating on a specification of the OLG model with two types of households, young and old. The household could be defined to live for 80 periods (years) just as well.

An important feature of the model is the pension system, through which the old households get their livelihoods. Only young households work, and they either save for their own pensions or transfer a part of their income directly to currently old households, relying that the next generation is going to do the same for them. These are the two common pension schemes called fully funded and pay as you go systems, respectively. They both convey the same benefit to old households, but might inflict different sized costs to the young households.

Let's first compare the pension systems and then solve the model for the fully funded case.

Pension systems

Consider first the **Fully Funded (ff)** pension system, where every household saves for their own pension. Here the pension of currently old households was saved by themselves from the last period's salary.

$$\underbrace{(1+r)\tau_{t-1}^{\text{ff}} w_{t-1} N_{t-1}}_{\text{Value of savings on t}} = \underbrace{\bar{e} w_{t-1} N_{t-1}}_{\text{Size of the pension}} \qquad || \frac{1}{w_{t-1} N_{t-1}}$$

$$(1+r)\tau_{t-1}^{\text{ff}} = \bar{e}$$

$$\tau_{t-1}^{\text{ff}} = \tau^{\text{ff}} = \frac{\bar{e}}{1+r} \tag{1}$$

The second alternative is the **Pay As You Go (payg)** -pension system, where the currently working young households transfer part of their income to support the non-working old households.

The replacement rate $0 \leq \bar{e} \leq 1$ is a fraction of that income, which the old household received when working as a young household. The size of the pension of the currently old household is then given by $e_t = \bar{e} w_{t-1}$. The fraction of young households' total income, τ_t^{payg}, needed to finance this pensions scheme, can then be solved from

$$\tau_t^{\text{payg}} \underbrace{w_t N_t}_{\text{Current output}} = \bar{e} \underbrace{w_{t-1} N_{t-1}}_{\text{Previous output}} \qquad \text{Aggregate pensions}$$

$$\tau_t^{\text{payg}} \underbrace{(1+a) w_{t-1}}_{=w_t} \underbrace{(1+n) N_{t-1}}_{=N_t} = \bar{e} w_{t-1} N_{t-1} \qquad || \frac{1}{w_{t-1} N_{t-1}}$$

$$\tau_t^{\text{payg}} (1+a)(1+n) = \bar{e}$$

$$\tau_t^{\text{payg}} = \tau^{\text{payg}} = \frac{\bar{e}}{(1+a)(1+n)} \tag{2}$$

As both of the systems convey the benefit of identical size to old households, $\bar{e} w_{t-1} N_{t-1}$, which one is cheaper to implement? I will denote the unknown direction of inequality in the middle with a question mark.

$$
\begin{aligned}
\tau_t^{\text{payg}} \quad &? \quad \tau_t^{\text{ff}} \\
&\Longleftrightarrow \\
\underbrace{\frac{\bar{e}}{(1+a)(1+n)}}_{\tau_t^{\text{payg}}} \quad &? \quad \underbrace{\frac{\bar{e}}{1+r}}_{\tau_t^{\text{ff}}} \qquad\qquad\qquad\qquad || \bar{e}, a, n, r \geq 0 \\
&\Longleftrightarrow \\
1+r \quad &? \quad \underbrace{(1+a)(1+n)}_{1+a+n+\underbrace{an}_{\approx 0} \approx 1+a+n} \\
&\Longleftrightarrow \\
r \quad &? \quad \underbrace{a+n}_{=:g} \\
&\Longleftrightarrow \\
r \quad &? \quad g
\end{aligned}
$$

The question whether PAYG or FF can be funded with smaller costs boils down to whether the interest rate r or the growth rate of the economy g is larger. If the interest rate expands savings faster than the economy grows, then it's cheaper to save for your own pension. If the economy grows faster than savings, it doesn't make sense to finance the pensions by holding money through time.

$$
\begin{aligned}
&\text{If} && r > g \\
&\text{Then} && \tau_t^{\text{payg}} > \tau_t^{\text{ff}} \\
\\
&\text{If} && r < g \\
&\text{Then} && \tau_t^{\text{payg}} < \tau_t^{\text{ff}}
\end{aligned}
$$

Solving the model

Next we'll solve for the model under the Fully Funded pension system, i.e. the households fund their consumption at old age by saving. I will specify the functional form for the utility- (logarithmic) and production functions (Cobb-Douglas) straight away, so that we can analytically solve the model.

The procedure is as follows:

1. Solve the equilibrium savings rate s from the household's problem
2. Solve for the wage rate w from the Firm's problem
3. Based on s and w, solve for the capital accumulation $k_{t+1} = f(k_t)$
4. ... and the steady state level of capital k^*
5. Find the weighted consumption basket c=y-i of the households
6. Use k^* and c to find out the steady state consumption c^*
7. Find out the highest level of (golden rate) consumption attainable in the model c^{gr}, and the corresponding steady state level capital k^{gr}
8. See if the actual steady state consumption c^*, falls short of c^{gr}: Dynamic inefficiency

1. Solve for savings s from the problem of the household

$$\max_{c_{y,t},c_{o,t+1}} \quad U(c_{y,t}, c_{o,t+1}) = u(c_{y,t}) + \beta u(c_{o,t+1})$$

$$\Longleftrightarrow$$

$$\max_{c_{y,t},c_{o,t+1}} \quad log(c_{y,t}) + \beta log(c_{o,t+1})$$

s.t.

$$c_{y,t} + s_t = w_t \qquad \Longleftrightarrow \qquad c_{y,t} = w_t - s_t \tag{3}$$

$$c_{o,t+1} = \underbrace{(1 + R_{t+1})s_t}_{\text{Return to savings}} - \underbrace{\delta s_t}_{\text{Depreciation of capital}} \qquad \Longleftrightarrow \qquad c_{o,t+1} = (R_{t+1} + 1 - \delta)s_t \tag{4}$$

The young household receives wages and saves some of that for old age (for ff pension). The savings is used to acquire capital, which the household rents to firms during the next period.

When the household is old, it gets rental rate R_{t+1} for the whole capital stock they acquired with their savings s_t, and they also get to consume the leftover capital, that has not been depreciated $(1 - \delta)s_t$ during the production at period t+1.

More specifically, in equilibrium the old households sell their leftover capital $(1 - \delta)K_t$ to young households, and then use the money for consumption. The young households buy this leftover capital using their savings, and also acquire some more capital, by buying investment goods produced by the firms, to cover up the depreciated capital (so that $k_t = k_{t+1} = k^*$). Hence, in the OLG -model the savings typically does not equal investments, as savings cover the whole next period's capital stock K_{t+1}, but investments are defined as only the newly acquired capital, $I_t = \underbrace{K_{t+1}}_{S_t} - (1 - \delta)K_t$.

Plug the constraints, Eq 3 and 4, into the problem to get an unconstrained problem, with savings as the control variable

$$\max_{s} \quad log(\underbrace{w_t - s_t}_{c_{y,t}}) + \beta \log\Big(\underbrace{[R_{t+1} + 1 - \delta]s_t}_{c_{o,t+1}}\Big)$$

FOC:

$$\frac{\partial U}{\partial s_t} = -\frac{1}{w_t - s_t} + \cancel{[R_{t+1} + 1 - \delta]}\beta\frac{1}{\cancel{[R_{t+1} + 1 - \delta]}s_t} = 0$$

$$\frac{1}{w_t - s_t} = \frac{\beta}{s_t}$$

$$s_t = \beta w_t - \beta s_t$$

$$s_t = \frac{\beta}{1+\beta}w_t \qquad \text{Equilibrium savings} \quad (5)$$

We will later use this savings rate to pin down the capital accumulation, but for that we also need to find out the equilibrium wages by solving the Firm's problem. I also solve for the rental rate of capital, R_t, even though we don't need it here.

2. Solve for wages w from the problem of the Firm

$$\underset{K_t,N_t}{max} \quad \pi(K_t, N_t) = \underbrace{\overbrace{P}^{=1} * A * F(K_t, N_t)}_{\text{Value of output}} - \underbrace{R_t K_t - w_t N_t}_{\text{Costs of production}}$$

$$\iff$$

$$\underset{K_t,N_t}{max} \quad AK_t^{\alpha}N_t^{1-\alpha} - R_t K_t - w_t N_t$$

FOC:

$$\frac{\partial \pi}{\partial N_t} = (1-\alpha)AK_t^{\alpha}N_t^{1-\alpha-1} - w_t = 0$$

$$w_t = (1-\alpha)A\frac{K_t^{\alpha}}{N_t^{\alpha}}$$

$$w_t = (1-\alpha)A\Big(\underbrace{\frac{K_t}{N_t}}_{=:k_t}\Big)^{\alpha}$$

$$w_t = (1-\alpha)Ak_t^{\alpha} \qquad \text{Equilibrium wage} \quad (6)$$

$$\frac{\partial \pi}{\partial K_t} = \alpha AK_t^{\alpha-1}N_t^{1-\alpha} - R_t = 0$$

$$R_t = \alpha Ak_t^{\alpha-1}$$

Now that we know the savings- and the wage-rate, Eq 5 and 6 respectively, we can figure out how the capital accumulates through time.

3. Solve for capital accumulation

Capital is just the fraction saved by the young households out of their income. This saved money is used to acquire capital, which is then rented to firms by the same households at old age (next period), $K_t = N_{t-1}s_{t-1}$. The equation is identical for all periods, so let's roll the indices forward one period. Remember, that the uppercase letter K stands for aggregate capital stock.

$$K_{t+1} = N_t s_t \qquad || \frac{1}{N_t}$$

$$\frac{K_{t+1}}{N_t} = s_t$$

$$\frac{K_{t+1}}{N_t}\underbrace{\frac{N_{t+1}}{N_{t+1}}}_{=1} = s_t$$

$$\underbrace{\frac{K_{t+1}}{N_{t+1}}}_{=:k_{t+1}}\underbrace{\frac{N_{t+1}}{N_t}}_{(1+n)} = s_t \qquad ||N_{t+1} = (1+n)N_t$$

$$(1+n)k_{t+1} = \underbrace{\frac{\beta}{1+\beta}w_t}_{s_t}$$

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}\underbrace{(1-\alpha)Ak_t^{\alpha}}_{w_t}$$

$$k_{t+1} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}k_t^{\alpha} \qquad \text{Capital accumulation} \qquad (7)$$

This capital accumulation equation describes, how the next period's capital stock (per labor) k_{t+1} is determined by the current period's capital stock k_t.

4. Steady state level of capital

In the steady state the capital stock stays the same (or grows at the same rate) $k_t = k_{t+1} = k^*$. We can find this steady state value by plugging k^* into the Eq 7 for both k_t and k_{t+1}.

$$k^* = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}(k^*)^\alpha \qquad || \frac{1}{(k^*)^\alpha}$$

$$(k^*)^{1-\alpha} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} \qquad ||()^{\frac{1}{1-\alpha}}$$

$$k^* = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}} \qquad \text{Steady state capital} \tag{8}$$

5. Find the weighted consumption basket

To be able to find the steady state consumption, we first have to do some aggregation over young and old households, to find expressions for total income Y_t and investments I_t, and combine them to get an aggregate constraint for the whole economy, $C_t + I_t = Y_t$. Why is this necessary? Because $S_t \neq I_t$, as some of the savings of the young household is going to end up as the consumption of the old household (worth of $(1-\delta)K_t$). Hence, we need to subtract only the investments from the total income (instead of savings), to get the aggregate consumption, $C_t = Y_t - I_t$. Here C_t includes the total consumption of both household types at t.

By then dis-aggregating this constraint, we get a constraint for the working households, from which we can find out the consumption using $c_t = y_t - i_t$, including consumption of both household types. We don't need to care about population growth when working with the dis-aggregated constraint.

Aggregation. Start by multiplying the original constraints for young and old households, Eq 3 and 4, by the number of households N_t and N_{t-1}, respectively.

Aggregate young households at t

$$\begin{aligned} c_{y,t} + s_t &= w_t && || * N_t \\ \underbrace{N_t c_{y,t}}_{=:C_{y,t}} + \underbrace{N_t s_t}_{=S_t = K_{t+1}} &= N_t w_t \end{aligned}$$

Aggregate old households at t

$$\begin{aligned} c_{o,t} &= (R_t + 1 - \delta)s_{t-1} && || * N_{t-1} \\ \underbrace{N_{t-1} c_{o,t}}_{=:C_{o,t}} &= (R_t + 1 - \delta)\underbrace{N_{t-1} s_{t-1}}_{=K_t} \end{aligned}$$

Combine the above aggregate constraints for young and old households at time t by summing them up side by side (combined consumption equals combined income), to get an aggregate constraint for the whole economy.

$$\underbrace{C_{y,t} + K_{t+1}}_{young} + \underbrace{C_{o,t}}_{old} = \underbrace{N_t w_t}_{young} + \underbrace{(R_t + 1 - \delta)K_t}_{old}$$

$$\underbrace{C_{y,t} + C_{o,t}}_{=:C_t} + \underbrace{\overbrace{K_{t+1}}^{S_t} - (1-\delta)K_t}_{=:I_t} = \underbrace{N_t w_t + R_t K_t}_{=:Y_t}$$

$$C_t + I_t = Y_t$$

We have now constructed the aggregate recourse constraint for the whole economy, and also defined $I_t = K_{t+1} - (1-\delta)K_t$.

Dis-aggregation. By dividing the terms by N_t, we find the constraint for the working household. Why divide by N_t and not by $(N_t + N_{t-1})$? First one is more practical, because Y and I are something that are related only to the young households. The end results for everything relevant are the same anyway.

The dis-aggregated **consumption**, on the other hand, ends up being a consumption basket that takes into account the consumption of both ages, weighted by the relative number of young households with respect to the old ones:

$$\begin{aligned}
\frac{C_t}{N_t} &= \frac{C_{y,t} + C_{o,t}}{N_t} \\
&= \frac{N_t c_{y,t} + N_{t-1} c_{o,t}}{N_t} \\
&= c_{y,t} + \frac{c_{o,t}}{(1+n)} \\
&=: c_t
\end{aligned}$$

To disaggregate the **income** Y_t, remember that it equals the output, and that only the young households work, so we can define y_t using the production function

$$\begin{aligned}
\frac{Y_t}{N_t} &= AF(\frac{K_t}{N_t}, \frac{N_t}{N_t}) \\
&= AF(k_t, 1) \\
&= AF(k_t) \\
&= Ak_t^{\alpha} \\
&=: y_t
\end{aligned}$$

Doing the same for **investment** requires an extra trick. Remember also here, that only the young households are investing - buying capital that didn't exist before.

$$I_t = K_{t+1} - (1-\delta)K_t \qquad || * \frac{1}{N_t}$$

$$i_t = \frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} - (1-\delta)k_t$$

$$i_t = (1+n)k_{t+1} - (1-\delta)k_t \qquad || \ k_{t+1} = k_t = k^*$$

$$i^* = (1+n)k^* - (1-\delta)k^*$$

$$i^* = (\delta+n)k^*$$

We arrive at the desired result

$$\begin{aligned} c + i &= y \\ c &= y - i \qquad \textbf{Weighted consumption basket} \end{aligned} \tag{9}$$

6. Use k^* and c to find the steady state consumption c^*

Notice, that both y and i are functions of capital, so in the steady state they are solely defined by k^*.

To find out how much consumption there is in the steady state, c^*, just plug in the expression for k^*, Eq 8, for y and i in the consumption basket above, Eq 9.

$$c^* = y(k^*) - i(k^*) \tag{10}$$

$$c^* = A(k^*)^\alpha - (\delta+n)k^*$$

$$c^* = A\Bigg(\underbrace{\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}}_{k^*}\Bigg)^\alpha - (\delta+n)\underbrace{\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}}_{k^*}$$

$$c^* = A\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{\alpha}{1-\alpha}} - (\delta+n)\left[\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} \qquad \text{Steady state consumption} \tag{11}$$

This steady state consumption is not necessarily the best we can do, as we'll see next.

7.1 Find the capital k^{gr} providing maximal (golden rule) consumption

The steady state capital k^* does not necessarily allows for the highest possible steady state consumption, called the golden rate consumption c^{gr}. To find out how good or bad c^* is with regards to what it could be, we need to find out the golden rate capital stock k^{gr}, which gives rise to c^{gr}.

Recall from the basic Solow model, that to maximize the steady state consumption, we want to increase k^* as long as it increases c^*. Therefore we differentiate the Equation 10 describing the steady state consumption basket, with respect to the steady state capital k^*.

$$\frac{dc^*}{dk^*} = \alpha A(k^*)^{\alpha-1} - (\delta+n) = 0$$

$$(k^*)^{\alpha-1} = \frac{\delta+n}{\alpha A}$$

$$k^* = \left[\frac{\delta+n}{\alpha A}\right]^{\frac{1}{\alpha-1}}$$

$$k^{gr} = \left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}} \qquad \text{Consumption maximizing capital} \tag{12}$$

The amount of capital in steady state, which maximizes the consumption basket, is denoted by k^{gr}.

7.2 Find the golden rule consumption c^{gr}

To find out c^{gr}, we proceed similarly as when finding the steady state consumption in step 6. But instead of k^*, we plug in the golden rule capital k^{gr}, Eq 12, and define the consumption level it produces as c^{gr}.

$$c^{gr} = y(k^{gr}) - i(k^{gr})$$

$$c^{gr} = A(k^{gr})^{\alpha} - (\delta+n)k^{gr}$$

Golden rate consumption in terms of the model's parameters is then given by

$$c^{gr} = A\left(\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}\right)^{\alpha} - (\delta+n)\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}$$

$$c^{gr} = A\left[\frac{\alpha A}{\delta+n}\right]^{\frac{\alpha}{1-\alpha}} - (\delta+n)\left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}} \qquad \text{Golden rule consumption}$$

8. Dynamic inefficiency?

The economy is dynamically efficient, if the market equilibrium is Pareto optimal. Dynamic inefficiency stems from excess savings.

We now have two expressions for steady state capital, k^* from Eq 8 and the golden rule capital k^{gr} from Eq 12. If these are equal, then the savings rate is such that it produces the steady state capital $k^* = k^{gr}$, which then maximizes the consumption of the households.

$$k^* = k^{gr} \qquad\qquad \rightarrow c^* = c^{gr}$$

$$\left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}} = \left[\frac{\alpha A}{\delta+n}\right]^{\frac{1}{1-\alpha}}$$

$$\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} = \frac{\alpha}{\delta+n}$$

If this condition is not met, then the savings rate is either too low or too high, and the steady state capital stock ends up being such, that the highest level of consumption attainable is not reached, $c^* < c^{gr}$. The savings rate might be sub-optimal, because of the incentive structures of the economy.

If the savings rate is too high, fixing the situation is easy: just cut the rate down for a Pareto improvement by changing the incentives to save. Excess savings might be due to the fully funded pension system, which incentivizes people to save too much for their future pensions.

The situation is not as straight forward, when savings rate is too low. It might be beneficial to increase savings, which would accumulate capital and create a higher standard of living for the future generations. However, this might require a transition period of several generations, who would have to suffer reduced living standards. Therefore it's not always a Pareto improvement to move towards c^{gr}.

Another problem is, that there is a missing negotiations channel between currently living and yet unborn households. For example, moving from FF to PAYG- pension system might be beneficial in some situation, but the next future generation is not yet alive to consent to paying for the pensions of the current working population. During the transition between pension systems, the currently working households would have to pay transfers for currently old households, instead of saving for their own pension, relying that the future generation will do the same for them.