Yule-Walker equations

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Having observed time series data $y_1, \dots, y_T$, presumably created by the AR(p) process

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma^2) \tag{1}$$

we wish to estimate the autoregressive parameters $\phi = (\phi_1, \dots, \phi_p)'$ and the error variance σ^2.

Using the Yule-Walker equations, ϕ and σ^2 can be estimated on the basis of the empirical estimates for autocovariances γ and autocorrelations ρ, calculated from the observed data.

$$\phi = \mathbf{\Gamma^{-1}}\gamma = \mathbf{P^{-1}}\rho \tag{2}$$

Autocovariances

Empirical estimate c_h for the h^{th} order autocovariance function can be easily calculated by

$$c_h = \frac{1}{T-h} \sum_{t=1}^{T-h} (y_t - \bar{y})(y_{t+h} - \bar{y})$$

To get the corresponding theoretical h^{th} order autocovariance function $\gamma_h = \mathbf{E}[y_t y_{t-h}]$, multiply both sides of the Equation 1 by y_{t-h} and take expectations to obtain

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t \qquad || * y_{t-h}$$

$$\underbrace{\mathbf{E}[y_t y_{t-h}]}_{\gamma_h} = \phi_1 \underbrace{\mathbf{E}[y_{t-1} y_{t-h}]}_{\gamma_{h-1}} + \cdots + \phi_p \underbrace{\mathbf{E}[y_{t-p} y_{t-h}]}_{\gamma_{h-p}} + \mathbf{E}[\varepsilon_t y_{t-h}]$$

$$\iff$$

$$\gamma_h = \begin{cases} \phi_1\gamma_1 + \cdots + \phi_p\gamma_p + \sigma^2 & when \quad h = 0 \\ \phi_1\gamma_{h-1} + \cdots + \phi_p\gamma_{h-p} & when \quad h > 0 \end{cases} \tag{3}$$

To construct the **Yule-Walker equations** (Equations 6 and 14), we write explicitly open all of the autocovariances γ_h up to $h = p$. By doing so we get a system of equations with as many equations, p, as there are unknown parameters in ϕ. Here the indexes of γ terms look awkward, as the focus is in making their recursive logic clear. The simplified form can be found in Equation 5.

$$
\begin{aligned}
\gamma_1 &= \phi_1 \underbrace{\gamma_{1-1}}_{\gamma_0} + \phi_2\gamma_{1-2} + \cdots + \phi_{p-1} \underbrace{\gamma_{1-(p-1)}}_{\gamma_{2-p}=\gamma_{p-2}} + \phi_p \underbrace{\gamma_{1-p}}_{\gamma_{p-1}} \\
\gamma_2 &= \phi_1\gamma_{2-1} + \phi_2\gamma_{2-2} + \cdots + \phi_{p-1}\gamma_{2-(p-1)} + \phi_p\gamma_{2-p} \\
&\vdots \\
\gamma_{p-1} &= \phi_1\gamma_{(p-1)-1} + \phi_2\gamma_{(p-1)-2} + \cdots + \phi_{p-1}\gamma_{(p-1)-(p-1)} + \phi_p\gamma_{(p-1)-p} \\
\gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_{p-1}\gamma_{p-(p-1)} + \phi_p\gamma_{p-p}
\end{aligned} \tag{4}
$$

Remind, that $\mathbf{E}[y_{t-h}y_{t-h}] = \gamma_0 \ \forall h$, and by symmetry $\gamma_h = \gamma_{-h}$, because $\mathbf{E}[y_t y_{t-h}] = \mathbf{E}[y_{t-h}y_t]$.

The Equation 4 simplifies to

$$
\begin{aligned}
\gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 + \cdots + \phi_{p-1}\gamma_{p-2} + \phi_p\gamma_{p-1} \\
\gamma_2 &= \phi_1\gamma_1 + \phi_2\gamma_0 + \cdots + \phi_{p-1}\gamma_{p-3} + \phi_p\gamma_{p-2} \\
&\vdots \\
\gamma_{p-1} &= \phi_1\gamma_{p-2} + \phi_2\gamma_{p-3} + \cdots + +\phi_{p-1}\gamma_0 + \phi_p\gamma_1 \\
\gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_{p-1}\gamma_1 + \phi_p\gamma_0
\end{aligned} \tag{5}
$$

which in matrix form is given by

$$
\underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p-2} \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma}
=
\underbrace{\begin{bmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-3} & \gamma_{p-2} & \gamma_{p-1} \\
\gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-4} & \gamma_{p-3} & \gamma_{p-2} \\
\gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-5} & \gamma_{p-4} & \gamma_{p-3} \\
\vdots & & & \ddots & & & \vdots \\
\gamma_{p-3} & \gamma_{p-4} & \gamma_{p-5} & \cdots & \gamma_0 & \gamma_1 & \gamma_2 \\
\gamma_{p-2} & \gamma_{p-3} & \gamma_{p-4} & \cdots & \gamma_1 & \gamma_0 & \gamma_1 \\
\gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_2 & \gamma_1 & \gamma_0
\end{bmatrix}}_{\mathbf{\Gamma}}
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{6}
$$

Denote vectors $\gamma = (\gamma_1, \ldots, \gamma_p)'$, $\phi = (\phi_1, \ldots, \phi_p)'$, and the above matrix $\mathbf{\Gamma} = [\gamma_{i-j}]_{i,j=1,\ldots,p}$ to get

$$
\gamma = \mathbf{\Gamma}\phi \qquad || \times \Gamma^{-1} \tag{7}
$$

$$
\iff
$$

$$
\phi = \mathbf{\Gamma}^{-1}\gamma \tag{8}
$$

$$\iff$$

$$\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \\ \vdots \\ \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-3} & \gamma_{p-2} \\ \\ \vdots & & \ddots & & \vdots \\ \\ \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}^{-1}}_{\Gamma^{-1}} \underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \\ \vdots \\ \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma} \tag{9}$$

σ^2 can be expressed in terms of γ by using the Equation 3 with $h = 0$

$$\begin{aligned} \gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_0 + \cdots + \phi_{p-1}\gamma_{p-3} + \phi_p\gamma_{p-2} + \sigma^2 \\ &\iff \\ \sigma^2 &= \gamma_0 - \phi_1\gamma_1 - \phi_2\gamma_0 - \cdots - \phi_{p-1}\gamma_{p-3} - \phi_p\gamma_{p-2} \end{aligned} \tag{10}$$

Equations 9 and 10 are the first important findings, allowing the unknown parameters ϕ and σ^2 to be expressed in terms of the autocovariances γ. The Yule-Walker estimators $\tilde{\phi}_{YW}$ and $\tilde{\sigma}^2_{YW}$ can then be obtained by inserting c_h for γ_h in the formulas.

Autocorrelations

Empirical autocorrelations are given by $r_h = c_0^{-1}c_h$. To get their theoretical counterparts, divide the Equation 3, when $h > 0$, by the variance $\gamma_0 = \mathbf{E}[y_t y_t]$ to get the h^{th} order autocorrelations (h^{th} order autocovariance divided by variance)

$$\begin{aligned} \rho_h &= \frac{\gamma_h}{\gamma_0} \\ &\iff \\ \frac{\mathbf{E}[y_t y_{t-h}]}{\mathbf{E}[y_t y_t]} &= \phi_1 \frac{\mathbf{E}[y_{t-1} y_{t-h}]}{\mathbf{E}[y_t y_t]} + \cdots + \phi_p \frac{\mathbf{E}[y_{t-p} y_{t-h}]}{\mathbf{E}[y_t y_t]} \\ &\iff \\ \underbrace{\frac{\gamma_h}{\gamma_0}}_{\rho_h} &= \phi_1 \underbrace{\frac{\gamma_{h-1}}{\gamma_0}}_{\rho_{h-1}} + \cdots + \phi_p \underbrace{\frac{\gamma_{h-p}}{\gamma_0}}_{\rho_{h-p}} \\ &\iff \\ \rho_h &= \phi_1\rho_{h-1} + \cdots + \phi_p\rho_{h-p} \end{aligned} \tag{11}$$

Therefore we have $\rho = \frac{1}{\gamma_0}\gamma$ and $\mathbf{P} = \frac{1}{\gamma_0}\mathbf{\Gamma}$, where $\rho = (\rho_1, \ldots, \rho_p)'$ and $\mathbf{P} = [\rho_{i-j}]_{i,j=1,\ldots,p}$. Note, that the 0^{th} order autocorrelation is $\rho_{h-h} = \rho_0 = \frac{\gamma_0}{\gamma_0} = 1, \ \forall h$.

Using these to re-write the Equation 7 as

$$\begin{aligned} \gamma &= \mathbf{\Gamma}\phi && || * \frac{1}{\gamma_0} \\ \frac{1}{\gamma_0}\gamma &= \frac{1}{\gamma_0}\mathbf{\Gamma}\phi \\ \rho &= \mathbf{P}\phi \end{aligned} \tag{12}$$

where the opened up Equation 12 is

$$
\begin{aligned}
\underbrace{\rho_1}_{\frac{\gamma_1}{\gamma_0}} &= \underbrace{\phi_1}_{\phi_1 \frac{\gamma_0}{\gamma_0}} + \phi_2 \underbrace{\rho_1}_{\frac{\gamma_1}{\gamma_0}} + \cdots + \phi_{p-1} \underbrace{\rho_{p-2}}_{\frac{\gamma_{p-2}}{\gamma_0}} + \phi_p \underbrace{\rho_{p-1}}_{\frac{\gamma_{p-1}}{\gamma_0}} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \cdots + \phi_{p-1}\rho_{p-3} + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_{p-1} &= \phi_1\rho_{p-2} + \phi_2\rho_{p-3} + \cdots + +\phi_{p-1} + \phi_p\rho_1 \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_{p-1}\rho_1 + \phi_p
\end{aligned} \tag{13}
$$

and the same in the matrix form

$$
\underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_{p-2} \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-3} & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-4} & \rho_{p-3} & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-5} & \rho_{p-4} & \rho_{p-3} \\ \vdots & & & \ddots & & & \vdots \\ \rho_{p-3} & \rho_{p-4} & \rho_{p-5} & \cdots & 1 & \rho_1 & \rho_2 \\ \rho_{p-2} & \rho_{p-3} & \rho p-4 & \cdots & \rho_1 & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & \rho_2 & \rho_1 & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{14}
$$

Equation 8 can then be re-written as

$$
\begin{aligned}
\rho &= \mathbf{P}\phi && || \times \mathbf{P^{-1}} \\
\phi &= \mathbf{P}^{-1}\rho \\
&\iff
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-3} & \rho_{p-2} \\ \vdots & & \ddots & & \vdots \\ \rho_{p-2} & \rho_{p-3} & \cdots & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_1 & 1 \end{bmatrix}^{-1}}_{\mathbf{P^{-1}}} \underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} \tag{15}
$$

Similarly as in Equation 10, an estimate for σ^2 can be obtained by using the

$$
\begin{aligned}
1 &= \phi_1\rho_1 + \phi_2 + \cdots + \phi_{p-1}\rho_{p-3} + \phi_p\rho_{p-2} + \sigma^2 \\
&\iff \\
\sigma^2 &= 1 - \phi_1\rho_1 - \phi_2 - \cdots - \phi_{p-1}\rho_{p-3} - \phi_p\rho_{p-2}
\end{aligned} \tag{16}
$$

Equations 15 and 16 are the other central equations, which allow for the parameters ϕ and σ^2 to be recovered from the empirical autocorrelations ρ. Likewise with autocorrelations, the estimators $\tilde{\phi}_{YW}$ and $\tilde{\sigma}^2_{YW}$ can also be obtained using r_h for ρ_h in the above equations.