

Yule-Walker equations

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Having observed data y_1, \dots, y_T , presumably created by the process

$$y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t \quad (1)$$

we wish to estimate the autoregressive parameters $\phi = (\phi_1, \dots, \phi_p)'$.

Using the Yule-Walker equations, ϕ can be estimated on the basis of the empirical autocovariances γ and or autocorrelations ρ , calculated from the observed data.

$$\phi = \mathbf{P}^{-1} \rho = \mathbf{\Gamma}^{-1} \gamma \quad (2)$$

Autocovariances

Multiply both sides of the Equation 1 by y_{t-h} and take expectations to obtain the h^{th} autocovariance

$$\begin{aligned} \gamma_h &= \mathbf{E}[y_t y_{t-h}] \\ &\iff \\ \mathbf{E}[y_t y_{t-h}] &= \phi_1 \underbrace{\mathbf{E}[y_{t-1} y_{t-h}]}_{\gamma_{h-1}} + \dots + \phi_p \underbrace{\mathbf{E}[y_{t-p} y_{t-h}]}_{\gamma_{h-p}} + \mathbf{E}[\varepsilon_t y_{t-h}] \\ &\iff \\ \gamma_h &= \begin{cases} \phi_1 \gamma_1 + \dots + \phi_p \gamma_p + \sigma^2 & \text{when } h = 0 \\ \phi_1 \gamma_{h-1} + \dots + \phi_p \gamma_{h-p} & \text{when } h > 0 \end{cases} \quad (3) \end{aligned}$$

To construct the **Yule-Walker equations** (Equations 6, 13), we write explicitly open all of the autocovariances γ_h up to $h = p$. By doing so we get a system of equations with as many equations, p , as there are unknown parameters in ϕ . Here the indexes of γ terms look awkward, as the focus is in making their recursive logic clear. The simplified form can be found in Equation 5.

$$\begin{aligned}
\gamma_1 &= \phi_1 \underbrace{\gamma_{1-1}}_{\gamma_0} + \phi_2 \gamma_{1-2} + \cdots + \phi_{p-1} \underbrace{\gamma_{1-(p-1)}}_{\gamma_{2-p}=\gamma_{p-2}} + \phi_p \gamma_{1-p} \\
\gamma_2 &= \phi_1 \gamma_{2-1} + \phi_2 \gamma_{2-2} + \cdots + \phi_{p-1} \gamma_{2-(p-1)} + \phi_p \gamma_{2-p} \\
&\vdots \\
\gamma_{p-1} &= \phi_1 \gamma_{(p-1)-1} + \phi_2 \gamma_{(p-1)-2} + \cdots + \phi_{p-1} \gamma_{(p-1)-(p-1)} + \phi_p \gamma_{(p-1)-p} \\
\gamma_p &= \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \cdots + \phi_{p-1} \gamma_{p-(p-1)} + \phi_p \gamma_{p-p}
\end{aligned} \tag{4}$$

Remind, that $\gamma_0 = \mathbf{E}[y_{t-h}y_{t-h}] \ \forall h$, and by symmetry

$$\begin{aligned}
\gamma_h &= \gamma_{-h} \\
&\iff \\
\mathbf{E}[y_t y_{t-h}] &= \mathbf{E}[y_{t-h} y_t]
\end{aligned}$$

The Equation 4 simplifies to

$$\begin{aligned}
\gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 + \cdots + \phi_{p-1} \gamma_{p-2} + \phi_p \gamma_{p-1} \\
\gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 + \cdots + \phi_{p-1} \gamma_{p-3} + \phi_p \gamma_{p-2} \\
&\vdots \\
\gamma_{p-1} &= \phi_1 \gamma_{p-2} + \phi_2 \gamma_{p-3} + \cdots + \phi_{p-1} \gamma_0 + \phi_p \gamma_1 \\
\gamma_p &= \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \cdots + \phi_{p-1} \gamma_1 + \phi_p \gamma_0
\end{aligned} \tag{5}$$

which in matrix form is given by

$$\underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p-2} \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix}}_{\gamma} = \underbrace{\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-3} & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-4} & \gamma_{p-3} & \gamma_{p-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-5} & \gamma_{p-4} & \gamma_{p-3} \\ \vdots & & & \ddots & & & \vdots \\ \gamma_{p-3} & \gamma_{p-4} & \gamma_{p-5} & \cdots & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_{p-2} & \gamma_{p-3} & \gamma_{p-4} & \cdots & \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix}}_{\mathbf{\Gamma}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{6}$$

Denote vectors $\gamma = (\gamma_1, \dots, \gamma_p)'$, $\phi = (\phi_1, \dots, \phi_p)'$, and the above matrix $\mathbf{\Gamma} = [\gamma_{i-j}]_{i,j=1,\dots,p}$ to get

$$\gamma = \mathbf{\Gamma} \phi \quad || \times \mathbf{\Gamma}^{-1} \quad (7)$$

$$\begin{aligned} &\Longleftrightarrow \\ \phi &= \mathbf{\Gamma}^{-1} \gamma \end{aligned} \quad (8)$$

$$\begin{aligned} &\Longleftrightarrow \\ \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix} &= \begin{bmatrix} 1 & \gamma_1 & \dots & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & 1 & \dots & \gamma_{p-3} & \gamma_{p-2} \\ & & \ddots & & \\ \gamma_{p-2} & \gamma_{p-3} & \dots & 1 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \dots & \gamma_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix} \end{aligned} \quad (9)$$

Equation 9 is the first important finding, allowing the unknown parameters ϕ to be expressed in terms of the empirical autocovariances γ .

Autocorrelations

Divide the Equation 3, when $h > 0$, by the variance $\gamma_0 = \mathbf{E}[y_t y_t]$ to get the h^{th} order autocorrelations (h^{th} order autocovariance divided by variance)

$$\begin{aligned} \rho_h &= \frac{\gamma_h}{\gamma_0} \\ &\Longleftrightarrow \\ \frac{\mathbf{E}[y_t y_{t-h}]}{\mathbf{E}[y_t y_t]} &= \phi_1 \frac{\mathbf{E}[y_{t-1} y_{t-h}]}{\mathbf{E}[y_t y_t]} + \dots + \phi_p \frac{\mathbf{E}[y_{t-p} y_{t-h}]}{\mathbf{E}[y_t y_t]} \\ &\Longleftrightarrow \\ \frac{\gamma_h}{\gamma_0} &= \phi_1 \underbrace{\frac{\gamma_{h-1}}{\gamma_0}}_{\rho_{h-1}} + \dots + \phi_p \underbrace{\frac{\gamma_{h-p}}{\gamma_0}}_{\rho_{h-p}} \\ &\Longleftrightarrow \\ \rho_h &= \phi_1 \rho_{h-1} + \dots + \phi_p \rho_{h-p} \end{aligned} \quad (10)$$

Note, that the 0^{th} order autocorrelation is $\rho_{h-h} = \rho_0 = \frac{\gamma_0}{\gamma_0} = 1, \forall h$.

Therefore we have

$$\begin{aligned} \rho &= \frac{1}{\gamma_0} \gamma \\ \mathbf{P} &= \frac{1}{\gamma_0} \mathbf{\Gamma} \end{aligned}$$

where $\rho = (\rho_1, \dots, \rho_p)'$ and $\mathbf{P} = [\rho_{i-j}]_{i,j=1,\dots,p}$.

Using these to re-write Equation 7 as

$$\begin{aligned}
\gamma &= \mathbf{\Gamma} \phi & || * \frac{1}{\gamma_0} \\
\frac{1}{\gamma_0} \gamma &= \frac{1}{\gamma_0} \mathbf{\Gamma} \phi \\
\rho &= \mathbf{P} \phi
\end{aligned} \tag{11}$$

where the opened up Equation 11 is

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2 \rho_1 + \cdots + \phi_{p-1} \rho_{p-2} + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 + \cdots + \phi_{p-1} \rho_{p-3} + \phi_p \rho_{p-2} \\
&\vdots \\
\rho_{p-1} &= \phi_1 \rho_{p-2} + \phi_2 \rho_{p-3} + \cdots + \phi_{p-1} + \phi_p \rho_1 \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \cdots + \phi_{p-1} \rho_1 + \phi_p
\end{aligned} \tag{12}$$

and the same in the matrix form

$$\underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_{p-2} \\ \rho_{p-1} \\ \rho_p \end{bmatrix}}_{\rho} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-3} & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-4} & \rho_{p-3} & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-5} & \rho_{p-4} & \rho_{p-3} \\ \vdots & & & \ddots & & & \vdots \\ \rho_{p-3} & \rho_{p-4} & \rho_{p-5} & \cdots & 1 & \rho_1 & \rho_2 \\ \rho_{p-2} & \rho_{p-3} & \rho_{p-4} & \cdots & \rho_1 & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & \rho_2 & \rho_1 & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-2} \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi} \tag{13}$$

Equation 8 can then be re-written as

$$\begin{aligned}
\rho &= \mathbf{P} \phi & || \times \mathbf{P}^{-1} \\
\phi &= \mathbf{P}^{-1} \rho \\
&\iff \\
\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix} &= \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{p-2} & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-3} & \rho_{p-2} \\ \vdots & & \ddots & & \vdots \\ \rho_{p-2} & \rho_{p-3} & \cdots & 1 & \rho_1 \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{p-1} \\ \rho_p \end{bmatrix}
\end{aligned} \tag{14}$$

Equation 14 is the other central equation, which allows for the parameters ϕ to be recovered from the empirical autocorrelations ρ .