



## 连续系统动力学离散化:

$$\dot{\mathbf{p}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = -a_T \mathbf{R} \mathbf{e}_3 - \frac{1}{m} \mathbf{R} \mathbf{D} \mathbf{R}^T \mathbf{v} + \mathbf{g}$$

$$\dot{\mathbf{R}} = \mathbf{R} [\boldsymbol{\omega}]$$

$$(\mathbf{e}_3 = [0, 0, -1]^T)$$

$$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\boxplus: \mathcal{M} \times \mathbb{R}^n \rightarrow \mathcal{M}; \quad \boxminus: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^n$$

$$\mathcal{M} = SO(3): \mathbf{R} \boxplus \mathbf{r} = \mathbf{R} \text{Exp}(\mathbf{r});$$

$$\mathbf{R}_1 \boxminus \mathbf{R}_2 = \text{Log}(\mathbf{R}_2^T \mathbf{R}_1)$$

$$\mathcal{M} = \mathbb{R}^n: \mathbf{a} \boxplus \mathbf{b} = \mathbf{a} + \mathbf{b};$$

$$\mathbf{a} \boxminus \mathbf{b} = \mathbf{a} - \mathbf{b}$$



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$$\mathbf{x}_{k+1} = \mathbf{x}_k \boxplus (\Delta t \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k))$$

其中

$$\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3), \quad \dim(\mathcal{M}) = 9$$

$$\mathbf{x} \doteq [\mathbf{p} \quad \mathbf{v} \quad \mathbf{R}] \in \mathcal{M}$$

$$\mathbf{u} \doteq [a_T \quad \boldsymbol{\omega}]^T \in \mathbb{R}^4$$

$$\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) = \begin{bmatrix} \mathbf{v}_k \\ \mathbf{g} - a_{Tk} \mathbf{R}_k \mathbf{e}_3 - \frac{1}{m} \mathbf{R}_k \mathbf{D} (\mathbf{R}_k^d)^T \mathbf{v}_k \\ \boldsymbol{\omega}_k \end{bmatrix} \in \mathbb{R}^9$$

[1] G. Lu, W. Xu and F. Zhang, "On-Manifold Model Predictive Control for Trajectory Tracking on Robotic Systems," in IEEE Transactions on Industrial Electronics, vol. 70, no. 9, pp. 9192-9202, Sept. 2023, doi: 10.1109/TIE.2022.3212397.

[2] Yunfan Ren et al., Safety-assured high-speed navigation for MAVs. Sci. Robot. 10, eado6187(2025). <https://github.com/hku-mars/SUPER>



## 定义误差状态:

$$\delta \mathbf{x} \triangleq \mathbf{x}_d \boxminus \mathbf{x} = \begin{bmatrix} \delta \mathbf{p}^T & \delta \mathbf{v}^T & \delta \mathbf{R}^T \end{bmatrix}^T \in \mathbb{R}^9,$$

$$\delta \mathbf{p} \triangleq \mathbf{p}_d \boxminus \mathbf{p} = \mathbf{p}_d - \mathbf{p} \in \mathbb{R}^3,$$

$$\delta \mathbf{v} \triangleq \mathbf{v}_d \boxminus \mathbf{v} = \mathbf{v}_d - \mathbf{v} \in \mathbb{R}^3,$$

$$\delta \mathbf{R} \triangleq \mathbf{R}_d \boxminus \mathbf{R} = \text{Log}(\mathbf{R}^T \mathbf{R}_d) \in \mathbb{R}^3,$$

$$\delta \mathbf{u} \triangleq \mathbf{u}_d - \mathbf{u} = \begin{bmatrix} \delta a_T & \delta \boldsymbol{\omega}^T \end{bmatrix}^T \in \mathbb{R}^4,$$

$$\delta a_T \triangleq a_{T_d} - a_T \in \mathbb{R},$$

$$\delta \boldsymbol{\omega} \triangleq \boldsymbol{\omega}_d - \boldsymbol{\omega} \in \mathbb{R}^3,$$

## 误差传播:

$$\begin{aligned} \delta \mathbf{x}_{k+1} &= \mathbf{x}_{k+1}^d \boxminus \mathbf{x}_{k+1} \\ &= (\mathbf{x}_k^d \boxplus \Delta \text{tf}(\mathbf{x}_k^d, \mathbf{u}_k^d)) \boxminus (\mathbf{x}_k \boxplus \Delta \text{tf}(\mathbf{x}_k, \mathbf{u}_k)) \\ &= (\mathbf{x}_k^d \boxplus \Delta \text{tf}(\mathbf{x}_k^d, \mathbf{u}_k^d)) \\ &\quad \boxminus ((\mathbf{x}_k^d \boxminus (-\delta \mathbf{x}_k)) \boxplus \Delta \text{tf}(\mathbf{x}_k^d \boxminus (-\delta \mathbf{x}_k), \mathbf{u}_k^d - \delta \mathbf{u}_k)) \end{aligned}$$

**定义**  $g(\delta \mathbf{x}_k, \delta \mathbf{u}_k) = \Delta \text{tf}(\mathbf{x}_k^d \boxminus (-\delta \mathbf{x}_k), \mathbf{u}_k^d - \delta \mathbf{u}_k)$

**则有**

$$\delta \mathbf{x}_{k+1} = \underbrace{(\mathbf{x}_k^d \boxplus g(\mathbf{0}, \mathbf{0})) \boxminus ((\mathbf{x}_k^d \boxminus (-\delta \mathbf{x}_k)) \boxplus g(\delta \mathbf{x}_k, \delta \mathbf{u}_k))}_{G(\delta \mathbf{x}_k, g(\delta \mathbf{x}_k, \delta \mathbf{u}_k))}$$



线性化:

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$$

线性化于  $\delta \mathbf{x}_k = \mathbf{0}, \delta \mathbf{u}_k = \mathbf{0}$

$$\text{其中 } \mathbf{F}_x = \left( \frac{\partial G(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial \delta \mathbf{x}_k} + \frac{\partial G(\mathbf{0}, \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0}))}{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})} \frac{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})}{\partial \delta \mathbf{x}_k} \right) \bigg|_{\delta \mathbf{x}_k = \mathbf{0}} \quad \mathbf{F}_u = \left( \frac{\partial G(\mathbf{0}, \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k))}{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)} \frac{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)}{\partial \delta \mathbf{u}_k} \right) \bigg|_{\delta \mathbf{u}_k = \mathbf{0}}$$

$$\begin{aligned} G(\delta \mathbf{x}_k, \mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k)) &= (\mathbf{x}_k^d \boxplus \mathbf{g}(\mathbf{0}, \mathbf{0})) \boxminus ((\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k)) \boxplus \mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k)) \\ &:= \mathbf{d} \boxminus ((\mathbf{a} \boxplus \mathbf{b}) \boxplus \mathbf{c}), \quad \mathbf{G}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^9, \mathbf{a}, \mathbf{d} \in \mathbb{R}^6 \times SO(3) \end{aligned}$$

情况 1:  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n \quad \frac{\partial \mathbf{G}}{\partial \mathbf{b}} = \frac{\partial \mathbf{G}}{\partial \mathbf{c}} = -\mathbf{I}_n$

情况 2:  $\mathbf{G}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3, \mathbf{a}, \mathbf{d} \in SO(3) \quad \frac{\partial \mathbf{G}}{\partial \mathbf{b}} = -\mathbf{A}(\mathbf{G})^{-1} \text{Exp}(-\mathbf{c}) \mathbf{A}(\mathbf{b})^T, \quad \frac{\partial \mathbf{G}}{\partial \mathbf{c}} = -\mathbf{A}(\mathbf{G})^{-1} \mathbf{A}(\mathbf{c})^T$

其中  $\mathbf{A}(\mathbf{u}) = \mathbf{I} + \frac{1 - \cos \|\mathbf{u}\|}{\|\mathbf{u}\|} \frac{\lfloor \mathbf{u} \rfloor}{\|\mathbf{u}\|} + \left( 1 - \frac{\sin \|\mathbf{u}\|}{\|\mathbf{u}\|} \right) \frac{\lfloor \mathbf{u} \rfloor^2}{\|\mathbf{u}\|^2}$  (左雅可比矩阵, 源于 Baker-Campbell-Hausdorff (BCH) 公式)



线性化:

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta u_k$$

情况 2 推导:

$\frac{\partial G}{\partial b}$ :  $G = \text{Log}((a \otimes b) \otimes c)^T d$      $\text{Exp}(G) = ((a \otimes b) \otimes c)^T d = \text{Exp}(-c) \text{Exp}(-b) a^{-1} d$

对  $G$  施加扰动:  $\text{Exp}(G + \Delta G) = \text{Exp}(-c) \text{Exp}[-(b + \Delta b)] a^{-1} d$

$\text{Exp}(A(G) \Delta G) \text{Exp}(G) = \text{Exp}(-c) \left( \text{Exp}(b) \text{Exp}(A^T(b) \Delta b) \right)^T a^{-1} d$      $\leftarrow$  Lie 群上的 BCH 公式

$\text{Exp}(A(G) \Delta G) = \text{Exp}(-c) (\dots)^T a^{-1} d d a \text{Exp}(b) \text{Exp}(c)$      $A$  为左雅可比阵

$= \text{Exp}(-c) \text{Exp}^T(A^T(b) \Delta b) \text{Exp}(-b) a^{-1} d d a \text{Exp}(b) \text{Exp}(c)$

$= \text{Exp}(-c) \text{Exp}^T(A^T(b) \Delta b) \text{Exp}(c)$      $A(\theta) = I_3 + \left( \frac{1 - \cos \|\theta\|}{\|\theta\|} \right) \frac{L\theta}{\|\theta\|} + \left( 1 - \frac{\sin \|\theta\|}{\|\theta\|} \right) \frac{L\theta^2}{\|\theta\|^2}$

$R [P] R^T = [RP]$     更常见的形式是

$\text{Exp}(-c) = \text{Exp}^T(c)$      $A(\vec{\theta}) = \frac{\sin \theta}{\theta} I_3 + \left( 1 - \frac{\sin \theta}{\theta} \right) a a^T + \left( \frac{1 - \cos \theta}{\theta} \right) [a]$

$\text{Exp}(A(G) \Delta G) = \text{Exp} \left( \text{Exp}(-c) (-A^T(b) \Delta b) \right)$

$A(G) \Delta G = -\text{Exp}(-c) A^T(b) \Delta b$

$\frac{\partial G}{\partial b} = -A(G)^{-1} \text{Exp}(-c) A^T(b)$

$\frac{\partial G}{\partial c}$ :  $G = \text{Log}((a \otimes b) \otimes c)^T d$      $\text{Exp}(G) = ((a \otimes b) \otimes c)^T d = \text{Exp}(-c) \text{Exp}(-b) a^{-1} d$

$\text{Exp}(G + \Delta G) = \text{Exp}(-(c + \Delta c)) \text{Exp}(-b) a^{-1} d$

$\text{Exp}(A(G) \Delta G) \text{Exp}(G) = \left( \text{Exp}(c) \text{Exp}(A^T(c) \Delta c) \right)^T \text{Exp}(-b) a^{-1} d$      $\leftarrow$  Lie 群上的 BCH 公式

$\text{Exp}(A(G) \Delta G) = \text{Exp}^T(A^T(c) \Delta c) \text{Exp}(c) \text{Exp}(-b) a^{-1} d d a \text{Exp}(b) \text{Exp}(c)$

$\text{Exp}(A(G) \Delta G) = \text{Exp}(-A^T(c) \Delta c)$

$\frac{\Delta G}{\Delta c} = -A(G)^{-1} A^T(c)$



线性化:

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$$

$$\left. \frac{\partial G(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial \delta \mathbf{x}_k} \right|_{\delta \mathbf{x}_k = \mathbf{0}} = (-1) \left. \frac{\partial G(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial (-\delta \mathbf{x}_k)} \right|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} \mathbf{I}_6 & \\ & \mathbf{A}(\mathbf{0})^{-1} \text{Exp}(-\mathbf{g}(\mathbf{0}, \mathbf{0})) \mathbf{A}(\mathbf{0})^T \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 & \\ & \text{Exp}(-\boldsymbol{\omega}_k^d \Delta t) \end{bmatrix}$$

$$\left. \frac{\partial G(\mathbf{0}, \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0}))}{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})} \right|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} -\mathbf{I}_6 & \\ & -\mathbf{A}(\mathbf{0})^{-1} \mathbf{A}(\mathbf{g}(\mathbf{0}, \mathbf{0}))^T \end{bmatrix} = \begin{bmatrix} -\mathbf{I}_6 & \\ & -\mathbf{A}(\boldsymbol{\omega}_k^d \Delta t)^T \end{bmatrix}, \quad \left. \frac{\partial G(\mathbf{0}, \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k))}{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)} \right|_{\delta \mathbf{u}_k = \mathbf{0}} = -\mathbf{I}_4$$

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$$\mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k) = \Delta t \begin{bmatrix} \mathbf{v}_k^d - \delta \mathbf{v}_k \\ \mathbf{g} - \left( a_{Tk}^d - \delta a_{Tk} \right) \mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k) \mathbf{e}_3 - \frac{1}{m} \mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k) \mathbf{D} \left( \mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k) \right)^T \left( \mathbf{v}_k^d - \delta \mathbf{v}_k \right) \\ \boldsymbol{\omega}_k^d - \delta \boldsymbol{\omega}_k \end{bmatrix}$$

$$\left. \frac{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})}{\partial \delta \mathbf{x}_k} \right|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} \delta p & \delta \mathbf{v} & \delta \mathbf{R} \\ \mathbf{0} & -\mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \frac{\Delta t}{m} \mathbf{R}_k^d \mathbf{D} \left( \mathbf{R}_k^d \right)^T & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \left. \frac{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)}{\partial \delta \mathbf{u}_k} \right|_{\delta \mathbf{u}_k = \mathbf{0}} = \begin{bmatrix} \delta a & \delta \boldsymbol{\omega} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{R}_k^d \mathbf{e}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 \Delta t \end{bmatrix}$$



## 线性化:

★ 的推导:

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$$

\* 的推导: 为简便, 略去重力项、 $\Delta t$  和下标, 记  $G = -a R e_3 - \frac{1}{m} R D R^T v$

对  $G$  施加扰动有  $R' = R \exp(-[\delta]) \approx R(I - L\delta)$

$$\text{则 } G(R') = -a R' e_3 - \frac{1}{m} R' D R'^T v$$

展开至一阶有  $R' e_3 = R e_3 - R L \delta e_3$

$$R' D R'^T v = R(I - L\delta) D(I - L\delta) R^T v$$

$$\approx R D R^T v + R(-L\delta D + D L\delta) R^T v \quad (\text{忽略高阶项})$$

$$\text{因此 } G(R') = -a(R e_3 - R L \delta e_3) - \frac{1}{m} [R D R^T v + R(-L\delta D + D L\delta) R^T v]$$

$$\Rightarrow \Delta G = G(R') - G(R) = a R L \delta e_3 - \frac{1}{m} [R(-L\delta D + D L\delta) R^T v]$$

$$\text{利用性质 } L\delta u = \delta \times u = -u \times \delta = -[u] \delta$$

$$\Delta G = -a R L e_3 \delta + \frac{1}{m} R L \delta D R^T v - \frac{1}{m} R D L \delta R^T v$$

$$= -a R L e_3 \delta - \frac{1}{m} R [D R^T v] \delta + \frac{1}{m} R D [R^T v] \delta$$

$$\Rightarrow \frac{\Delta G}{\delta} = -a R L e_3 - \frac{1}{m} R [D R^T v] + \frac{1}{m} R D [R^T v]$$

$$\mathbf{F}_x = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 - \frac{\Delta t}{m} \mathbf{R}_k^d \mathbf{D} (\mathbf{R}_k^d)^T & \frac{\Delta t}{m} \mathbf{R}_k^d \left( a_{Tk}^d m [e_3] + \left[ \mathbf{D} (\mathbf{R}_k^d)^T \mathbf{v}_k^d \right] - \mathbf{D} \left[ (\mathbf{R}_k^d)^T \mathbf{v}_k^d \right] \right) \\ \mathbf{0} & \mathbf{0} & \text{Exp}(-\omega_k^d \Delta t) \end{bmatrix}, \quad \mathbf{F}_u = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{R}_k^d e_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(\omega_k^d \Delta t)^T \end{bmatrix} \Delta t$$

$$\xrightarrow{D=0} \mathbf{F}_x = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \Delta t \mathbf{R}_k^d a_{Tk}^d [e_3] \\ \mathbf{0} & \mathbf{0} & \text{Exp}(-\omega_k^d \Delta t) \end{bmatrix}, \quad \text{与质量无关}$$



## 优化问题构建:

$$\begin{aligned} \min_{\delta \mathbf{u}_k} \quad & \sum_{k=0}^{N-1} (\|\delta \mathbf{x}_k\|_{\mathbf{Q}}^2 + \|\delta \mathbf{u}_k\|_{\mathbf{R}}^2) + \|\delta \mathbf{x}_N\|_{\mathbf{P}}^2, \\ \text{s.t.} \quad & \delta \mathbf{x}_{k+1} = \mathbf{F}_{\mathbf{x}_k} \delta \mathbf{x}_k + \mathbf{F}_{\mathbf{u}_k} \delta \mathbf{u}_k, \\ & \delta \mathbf{u}_k \in \delta \mathbf{U}_k, \quad k = 0, \dots, N-1, \end{aligned}$$

与 FAST-LIO 的比较:

线性化误差动力学

- LIO
- ommpc

真值(G.T.)估计  $\mathbf{x}_R \triangleq \hat{\mathbf{x}}_R$

扰动  $\uparrow$

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{F}}_{\tilde{\mathbf{x}}} \tilde{\mathbf{x}}_k + \tilde{\mathbf{F}}_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}_k$$

在(0,0)处线性化

$$\delta \mathbf{x}_{k+1} = \mathbf{F}_{\mathbf{x}} \delta \mathbf{x}_k + \mathbf{F}_{\mathbf{u}} \delta \mathbf{u}_k$$

$\mathbf{x}_k^d \triangleq \mathbf{x}_k$  与期望状态的偏差

$\mathbf{u}_k^d \triangleq \mathbf{u}_k$  与期望控制输入的偏差