

# On-Manifold MPC



## 连续系统动力学离散化:

$$\dot{p} = v$$

$$\dot{v} = -a_T \mathbf{R} e_3 - \frac{1}{m} \mathbf{R} D \mathbf{R}^T v + g$$

$$\dot{\mathbf{R}} = \mathbf{R} \lfloor \boldsymbol{\omega} \rfloor$$

$$(\mathbf{e}_3 = [0, 0, -1]^T)$$

$$\lfloor \boldsymbol{\omega} \rfloor = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\boxplus: \mathcal{M} \times \mathbb{R}^n \rightarrow \mathcal{M}; \quad \boxminus: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^n$$

$$\mathcal{M} = SO(3): \mathbf{R} \boxplus \mathbf{r} = \mathbf{R} \text{Exp}(\mathbf{r});$$

$$\mathcal{M} = \mathbb{R}^n: \quad \mathbf{a} \boxplus \mathbf{b} = \mathbf{a} + \mathbf{b};$$

$$\mathbf{a} \boxminus \mathbf{b} = \mathbf{a} - \mathbf{b}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k \boxplus (\Delta t f(\mathbf{x}_k, \mathbf{u}_k))$$

其中

$$\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3), \quad \dim(\mathcal{M}) = 9$$

$$\mathbf{x} \doteq [p \quad v \quad \mathbf{R}] \in \mathcal{M}$$

$$\mathbf{u} \doteq [a_T \quad \boldsymbol{\omega}]^T \in \mathbb{R}^4$$

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$$f(\mathbf{x}_k, \mathbf{u}_k) = \begin{bmatrix} v_k \\ \mathbf{g} - a_{Tk} \mathbf{R}_k \mathbf{e}_3 - \frac{1}{m} \mathbf{R}_k \mathbf{D} (\mathbf{R}_k^d)^T v_k \\ \boldsymbol{\omega}_k \end{bmatrix} \in \mathbb{R}^9$$

[1] G. Lu, W. Xu and F. Zhang, "On-Manifold Model Predictive Control for Trajectory Tracking on Robotic Systems," in IEEE Transactions on Industrial Electronics, vol. 70, no. 9, pp. 9192-9202, Sept. 2023, doi: 10.1109/TIE.2022.3212397.

[2] Yunfan Ren et al., Safety-assured high-speed navigation for MAVs. Sci. Robot. 10, eado6187 (2025). <https://github.com/hku-mars/SUPER>



定义误差状态：

$$\delta \mathbf{x} \triangleq \mathbf{x}_d \boxminus \mathbf{x} = [\delta \mathbf{p}^T \quad \delta \mathbf{v}^T \quad \delta \mathbf{R}^T]^T \in \mathbb{R}^9,$$

$$\delta \mathbf{p} \triangleq \mathbf{p}_d \boxminus \mathbf{p} = \mathbf{p}_d - \mathbf{p} \in \mathbb{R}^3,$$

$$\delta \mathbf{v} \triangleq \mathbf{v}_d \boxminus \mathbf{v} = \mathbf{v}_d - \mathbf{v} \in \mathbb{R}^3,$$

$$\delta \mathbf{R} \triangleq \mathbf{R}_d \boxminus \mathbf{R} = \text{Log}(\mathbf{R}^T \mathbf{R}_d) \in \mathbb{R}^3,$$

$$\delta \mathbf{u} \triangleq \mathbf{u}_d - \mathbf{u} = [\delta a_T \quad \delta \boldsymbol{\omega}^T]^T \in \mathbb{R}^4,$$

$$\delta a_T \triangleq a_{T_d} - a_T \in \mathbb{R},$$

$$\delta \boldsymbol{\omega} \triangleq \boldsymbol{\omega}_d - \boldsymbol{\omega} \in \mathbb{R}^3,$$

误差传播：

$$\begin{aligned} \delta \mathbf{x}_{k+1} &= \mathbf{x}_{k+1}^d \boxminus \mathbf{x}_{k+1} \\ &= (\mathbf{x}_k^d \boxplus \Delta tf(\mathbf{x}_k^d, \mathbf{u}_k^d)) \boxminus (\mathbf{x}_k \boxplus \Delta tf(\mathbf{x}_k, \mathbf{u}_k)) \\ &= (\mathbf{x}_k^d \boxplus \Delta tf(\mathbf{x}_k^d, \mathbf{u}_k^d)) \\ &\quad \boxminus ((\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k)) \boxplus \Delta tf(\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k), \mathbf{u}_k^d - \delta \mathbf{u}_k)) \end{aligned}$$

定义  $g(\delta \mathbf{x}_k, \delta \mathbf{u}_k) = \Delta tf(\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k), \mathbf{u}_k^d - \delta \mathbf{u}_k)$

则有

$$\delta \mathbf{x}_{k+1} = \underbrace{(\mathbf{x}_k^d \boxplus g(\mathbf{0}, \mathbf{0}))}_{G(\delta \mathbf{x}_k, g(\delta \mathbf{x}_k, \delta \mathbf{u}_k))} \boxminus ((\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k)) \boxplus g(\delta \mathbf{x}_k, \delta \mathbf{u}_k))$$

# On-Manifold MPC



**线性化:**

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$$

线性化于  $\delta \mathbf{x}_k = \mathbf{0}, \delta \mathbf{u}_k = \mathbf{0}$

$$\text{其中 } \mathbf{F}_x = \left( \frac{\partial \mathbf{G}(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial \delta \mathbf{x}_k} + \frac{\partial \mathbf{G}(\mathbf{0}, \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0}))}{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})} \frac{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})}{\partial \delta \mathbf{x}_k} \right) \Big|_{\delta \mathbf{x}_k = \mathbf{0}} \quad \mathbf{F}_u = \left( \frac{\partial \mathbf{G}(\mathbf{0}, \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k))}{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)} \frac{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)}{\partial \delta \mathbf{u}_k} \right) \Big|_{\delta \mathbf{u}_k = \mathbf{0}}$$

$$\begin{aligned} \mathbf{G}(\delta \mathbf{x}_k, \mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k)) &= (\mathbf{x}_k^d \boxplus \mathbf{g}(\mathbf{0}, \mathbf{0})) \boxminus ((\mathbf{x}_k^d \boxplus (-\delta \mathbf{x}_k)) \boxplus \mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k)) \\ &:= \mathbf{d} \boxminus ((\mathbf{a} \boxplus \mathbf{b}) \boxplus \mathbf{c}), \quad \mathbf{G}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^9, \mathbf{a}, \mathbf{d} \in \mathbb{R}^6 \times SO(3) \end{aligned}$$

**情况 1:**

$$a, b, c, d \in \mathbb{R}^n \quad \frac{\partial \mathbf{G}}{\partial \mathbf{b}} = \frac{\partial \mathbf{G}}{\partial \mathbf{c}} = -\mathbf{I}_n$$

**情况 2:**

$$\mathbf{G}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3, \mathbf{a}, \mathbf{d} \in SO(3) \quad \frac{\partial \mathbf{G}}{\partial \mathbf{b}} = -A(\mathbf{G})^{-1} \text{Exp}(-\mathbf{c}) A(\mathbf{b})^T, \quad \frac{\partial \mathbf{G}}{\partial \mathbf{c}} = -A(\mathbf{G})^{-1} A(\mathbf{c})^T$$

**其中**

$$A(\mathbf{u}) = \mathbf{I} + \frac{1 - \cos \|\mathbf{u}\|}{\|\mathbf{u}\|} \frac{\lfloor \mathbf{u} \rfloor}{\|\mathbf{u}\|} + \left(1 - \frac{\sin \|\mathbf{u}\|}{\|\mathbf{u}\|}\right) \frac{\lfloor \mathbf{u} \rfloor^2}{\|\mathbf{u}\|^2}$$

(左雅可比矩阵, 源于 Baker-Campbell-Hausdorff (BCH) 公式)

# On-Manifold MPC



**线性化:**  $\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$

**情况 2 推导:**

$$\frac{\partial G}{\partial b} : G = \log((a \otimes b) \otimes c)^T d \quad \text{Exp}(G) = ((a \otimes b) \otimes c)^T d = \text{Exp}(-c) \text{Exp}(-b) a^{-1} d$$

$$\text{对 } G \text{ 和 } b \text{ 分别求偏导: } \text{Exp}(G + \Delta G) = \text{Exp}(-c) \text{Exp}\left[-(b + \Delta b)\right] a^{-1} d$$

$$\text{Exp}(A(G) \Delta G) \text{Exp}(G) = \text{Exp}(-c) \left( \text{Exp}(b) \text{Exp}(A^T(b) \Delta b) \right)^T a^{-1} d$$

$$\text{Exp}(A(G) \Delta G) = \text{Exp}(-c) (\dots)^T a^{-1} d d \otimes \text{Exp}(b) \text{Exp}(c)$$

← Lie 群上的 BCH 公式

A 为左乘变换

$$A(\theta) = I_3 + \left( \frac{1 - \cos\|\theta\|}{\|\theta\|} \right) \frac{\theta}{\|\theta\|} + \left( 1 - \frac{\sin\|\theta\|}{\|\theta\|} \right) \frac{\theta\theta^T}{\|\theta\|^2}$$

$$= \text{Exp}(-c) \text{Exp}^T(A^T(b) \Delta b) \text{Exp}(-b) a^{-1} d d \otimes \text{Exp}(b) \text{Exp}(c)$$

$$= \text{Exp}(-c) \text{Exp}^T(A^T(b) \Delta b) \text{Exp}(c)$$

$$R$$

$$[P]$$

$$R^T P^T R = [RP]$$

$$R^T$$

$$\text{Exp}(-c) = \text{Exp}^T(c)$$

$$\text{更靠近的形式是}$$

$$A(\vec{\theta}) = \frac{\sin\theta}{\theta} I_3 + \left( 1 - \frac{\sin\theta}{\theta} \right) \theta\theta^T + \left( \frac{1 - \cos\theta}{\theta} \right) \theta\theta^T$$

$$\text{Exp}(A(G) \Delta G) = \text{Exp} \left( \text{Exp}(-c) (-A^T(b) \Delta b) \right)$$

$$A(G) \Delta G = -\text{Exp}(-c) A^T(b) \Delta b$$

$$\frac{\partial G}{\partial b} = -A(G)^{-1} \text{Exp}(-c) A^T(b)$$

$$\frac{\partial G}{\partial c} : G = \log((a \otimes b) \otimes c)^T d \quad \text{Exp}(G) = ((a \otimes b) \otimes c)^T d = \text{Exp}(-c) \text{Exp}(-b) a^{-1} d$$

$$\text{Exp}(G + \Delta G) = \text{Exp}\left(-(c + \Delta c)\right) \text{Exp}(-b) a^{-1} d$$

$$\text{Exp}(A(G) \Delta G) \text{Exp}(G) = \left( \text{Exp}(c) \text{Exp}(A^T(c) \Delta c) \right)^T \text{Exp}(-b) a^{-1} d \quad \leftarrow \text{Lie 群上的 BCH 公式}$$

$$\text{Exp}(A(G) \Delta G) = \text{Exp}^T(A^T(c) \Delta c) \text{Exp}^T(-c) \text{Exp}(-b) a^{-1} d \times \Delta c \text{Exp}(b) \text{Exp}(c)$$

$$\text{Exp}(A(G) \Delta G) = \text{Exp}(-A^T(c) \Delta c)$$

$$\frac{\Delta G}{\Delta c} = -A(G)^{-1} A^T(c)$$

# On-Manifold MPC



线性化:  $\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$

$$\frac{\partial G(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial \delta \mathbf{x}_k} \Big|_{\delta \mathbf{x}_k = \mathbf{0}} = (-1) \frac{\partial G(\delta \mathbf{x}_k, \mathbf{g}(\mathbf{0}, \mathbf{0}))}{\partial (-\delta \mathbf{x}_k)} \Big|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} \mathbf{I}_6 \\ & \mathbf{A}(\mathbf{0})^{-1} \text{Exp}(-\mathbf{g}(\mathbf{0}, \mathbf{0})) \mathbf{A}(\mathbf{0})^T \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 \\ & \text{Exp}(-\boldsymbol{\omega}_k^d \Delta t) \end{bmatrix}$$

$$\frac{\partial \mathbf{g}(\mathbf{0}, \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0}))}{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})} \Big|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} -\mathbf{I}_6 \\ & -\mathbf{A}(\mathbf{0})^{-1} \mathbf{A}(\mathbf{g}(\mathbf{0}, \mathbf{0}))^T \end{bmatrix} = \begin{bmatrix} -\mathbf{I}_6 \\ & -\mathbf{A}(\boldsymbol{\omega}_k^d \Delta t)^T \end{bmatrix}, \quad \frac{\partial \mathbf{g}(\mathbf{0}, \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k))}{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)} \Big|_{\delta \mathbf{u}_k = \mathbf{0}} = -\mathbf{I}_4$$

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$$\mathbf{g}(\delta \mathbf{x}_k, \delta \mathbf{u}_k) = \Delta t \begin{bmatrix} \mathbf{v}_k^d - \delta \mathbf{v}_k \\ \mathbf{g} - (\mathbf{a}_{Tk}^d - \delta \mathbf{a}_{Tk}) \mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k) \mathbf{e}_3 - \frac{1}{m} \mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k) \mathbf{D} (\mathbf{R}_k^d \text{Exp}(-\delta \mathbf{R}_k))^T (\mathbf{v}_k^d - \delta \mathbf{v}_k) \\ \boldsymbol{\omega}_k^d - \delta \boldsymbol{\omega}_k \end{bmatrix}$$

$$\frac{\partial \mathbf{g}(\delta \mathbf{x}_k, \mathbf{0})}{\partial \delta \mathbf{x}_k} \Big|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} \delta p & \delta \mathbf{v} & \delta \mathbf{R} \\ \mathbf{0} & -\mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \frac{\Delta t}{m} \mathbf{R}_k^d \mathbf{D} (\mathbf{R}_k^d)^T & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \frac{\partial \mathbf{g}(\mathbf{0}, \delta \mathbf{u}_k)}{\partial \delta \mathbf{u}_k} \Big|_{\delta \mathbf{u}_k = \mathbf{0}} = \begin{bmatrix} \delta a & \delta \boldsymbol{\omega} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{R}_k^d \mathbf{e}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 \Delta t \end{bmatrix}$$

# On-Manifold MPC



线性化:

$$\delta \mathbf{x}_{k+1} \simeq \mathbf{F}_x \delta \mathbf{x}_k + \mathbf{F}_u \delta \mathbf{u}_k$$

★ 的推导:

\*<sup>的</sup>选取: 为求简便, 忽略重力项、 $\Delta t$  和下标, 则  $G = -\alpha R e_3 - \frac{1}{m} R D R^T v$

对  $G$  施加扰动有  $R' = R \exp(-\delta J) \approx R(I - L \delta J)$

$$\text{则 } G(R') = -\alpha R' e_3 - \frac{1}{m} R' D R'^T v$$

$$\text{展开至一阶有 } R' e_3 = R e_3 - R[\delta J] e_3$$

$$R' D R'^T v = R(I - L \delta J) D(I - L \delta J) R^T v$$

$$\approx R D R^T v + R(-L \delta J D + D L \delta J) R^T v \quad (\text{忽略高阶项})$$

因此

$$G(R') = -\alpha(R e_3 - R[\delta J] e_3) - \frac{1}{m} [R D R^T v + R(-L \delta J D + D L \delta J) R^T v]$$

$$\Rightarrow \Delta G = G(R') - G(R) = \alpha R[\delta J] e_3 - \frac{1}{m} [R(-L \delta J D + D L \delta J) R^T v]$$

利用性质  $[L \delta J] u = \delta \times u = -u \times \delta = -[u] \delta$

$$\Delta G = -\alpha R[e_3] \delta + \frac{1}{m} R[\delta J] D R^T v - \frac{1}{m} R D[\delta J] R^T v$$

$$= -\alpha R[e_3] \delta - \frac{1}{m} R[D R^T v] \delta + \frac{1}{m} R D[R^T v] \delta$$

$$\Rightarrow \frac{\Delta G}{\delta} = -\alpha R[e_3] - \frac{1}{m} R[D R^T v] + \frac{1}{m} R D[R^T v]$$

$$\mathbf{F}_x = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 - \frac{\Delta t}{m} \mathbf{R}_k^d \mathbf{D} (\mathbf{R}_k^d)^T & \frac{\Delta t}{m} \mathbf{R}_k^d \left( a_{Tk}^d m \lfloor \mathbf{e}_3 \rfloor + \left[ \mathbf{D} (\mathbf{R}_k^d)^T \mathbf{v}_k^d \right] - \mathbf{D} \left[ (\mathbf{R}_k^d)^T \mathbf{v}_k^d \right] \right) \\ \mathbf{0} & \mathbf{0} & \text{Exp}(-\omega_k^d \Delta t) \end{bmatrix}, \quad \mathbf{F}_u = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{R}_k^d \mathbf{e}_3 & \mathbf{0} \\ \mathbf{0} & A(\omega_k^d \Delta t)^T \end{bmatrix} \Delta t$$

$$\xrightarrow{\mathbf{D} = \mathbf{0}} \mathbf{F}_x = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \Delta t \mathbf{R}_k^d a_{Tk}^d \lfloor \mathbf{e}_3 \rfloor \\ \mathbf{0} & \mathbf{0} & \text{Exp}(-\omega_k^d \Delta t) \end{bmatrix}, \quad \text{与质量无关}$$

# On-Manifold MPC



优化问题构建:

$$\begin{aligned} & \min_{\delta \mathbf{u}_k} \sum_{k=0}^{N-1} (\|\delta \mathbf{x}_k\|_{\mathbf{Q}}^2 + \|\delta \mathbf{u}_k\|_{\mathbf{R}}^2) + \|\delta \mathbf{x}_N\|_{\mathbf{P}}^2, \\ \text{s.t. } & \delta \mathbf{x}_{k+1} = \mathbf{F}_{\mathbf{x}_k} \delta \mathbf{x}_k + \mathbf{F}_{\mathbf{u}_k} \delta \mathbf{u}_k, \\ & \delta \mathbf{u}_k \in \delta \mathbb{U}_k, \quad k = 0, \dots, N-1, \end{aligned}$$

与 FAST-LIO 比较:

线性化误差动力学  
OMMPC

直值(G.T.)  
 $\tilde{x}_k$  由  $\tilde{x}_k$  估计  
 $\sim \tilde{x}_{k+1} = F_{\tilde{x}} \tilde{x}_k + F_w w_k$   
扰动  
在(0,0)处线性化  
 $\delta \tilde{x}_{k+1} = F_x \delta \tilde{x}_k + F_u u_k$   
 $\delta \tilde{x}_k$  由  $\tilde{x}_k$  估计  
与期望状态  
的偏差  
 $u_k^d$  由  $u_k$  估计  
与期望控制输入  
的偏差