CS 726: Homework #1

Posted: 01/28/2020, due: 02/10/2020 in class

Please typeset or write your solutions neatly! If we cannot read it, we cannot grade it.

Q 1. Prove that all isolated minima are strict.
Hint: One way to prove this is by contrapositive. [7pts]

Q 2. A saddle point is a stationary point (a point $\mathbf{x}$ with $\nabla f(\mathbf{x}) = 0$) that is neither a local minimum nor a local maximum. List all stationary points of the following functions, where $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, and for each stationary point indicate whether it is a local minimum/maximum, a global minimum/maximum, or a saddle point.

(i) $f(\mathbf{x}) = \frac{{x_1}^4}{20} - 10{x_1}^2 + 10{x_2}^2$; [5pts]

(ii) $f(\mathbf{x}) = {x_1}^2 + {x_2}^2 + \frac{1}{\pi}\big(\sin(\pi \cdot {x_1}^2) + \cos(\pi \cdot {x_2}^2)\big)$; [5pts]

(iii) $f(\mathbf{x}) = \frac{1}{2}{x_1}^2 + x_1 \cos(x_2)$. [5pts]

Q 3. Let $\mathbf{A}$ be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Prove that, $\forall \mathbf{x} \in \mathbb{R}^n$:

(i) $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq \lambda_1 \|\mathbf{x}\|_2^2$; [5pts]

(ii) $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq \lambda_n \|\mathbf{x}\|_2^2$. [5pts]

Q 4. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real valued *convex* function. Assume that f is such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $f(\mathbf{y}) \leq \underline{\lim}_{\alpha \downarrow 0} f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})$. Prove that:

(i) If there exists a point $\mathbf{x}$ such that $f(\mathbf{x}) = -\infty$, then f is not real-valued anywhere – it equals either $-\infty$ or $+\infty$ everywhere. [5pts]

(ii) If, $\forall \mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x}) \leq M$, for some constant $M < \infty$, then f must be a constant function (i.e., taking the same value for all $\mathbf{x} \in \mathbb{R}^n$). [5pts]

Q 5 (Jensen's Inequality)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that for any sequence of numbers $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ and any sequence of non-negative scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0$ such that $\sum_{i=1}^k \alpha_i = 1$ we have:

$$f\Big(\sum_{i=1}^k \alpha_i \mathbf{x}_i\Big) \leq \sum_{i=1}^k \alpha_i f(\mathbf{x}_i). \quad \text{[8pts]}$$

Q 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex continuously differentiable function. Using the definition of convexity from the class and properties of directional derivatives, prove that it must be $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle . \quad \text{[10pts]}$$

Q 7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Prove that if f is convex, then it must be

$$\nabla^2 f(\mathbf{x}) \succeq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

that is, the Hessian of f is positive semidefinite. [10pts]

Q 8. Let $f(\mathbf{x}) := \mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x}$, where $\mathbf{b}$ is the vector of all 1's and $\mathbf{A}$ is an $n \times n$ matrix defined by: $A_{ii} = 2$ for $1 \leq i \leq n$, $A_{i,i+1} = -1$, for $1 \leq i \leq n-1$ and $A_{n,1} = A_{1,n} = -1$. That is, $\mathbf{A}$ is defined as:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

Is f convex? Justify your answer. [15pts]

Q 9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function that satisfies the following:

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n): \quad f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{m}{2}\|\mathbf{y} - \mathbf{x}\|_2^2,$$

where $m > 0$ is some constant.

Prove that f cannot be Lipschitz continuous on the entire $\mathbb{R}^n$. Would it be possible for f to be Lipschitz continuous if we take the domain to be the unit Euclidean ball (i.e., the set: $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq 1\}$)? Explain. [15pts]