HW3

In [3]:
```
import numpy as np
import pandas as pd
```

In [4]:
```
data = pd.read_csv('titanic_data.csv')

y = data.Survived
X = data.loc[:,'Pclass':"Fare"]
data.head()
```

Out[4]:

	Survived	Pclass	Sex	Age	Siblings/Spouses Aboard	Parents/Children Aboard	Fare
0	0	3	0	22.0	1	0	7.2500
1	1	1	1	38.0	1	0	71.2833
2	1	3	1	26.0	0	0	7.9250
3	1	1	1	35.0	1	0	53.1000
4	0	3	0	35.0	0	0	8.0500

In [5]:
```
toy = data.head()
toy
```

Out[5]:

	Survived	Pclass	Sex	Age	Siblings/Spouses Aboard	Parents/Children Aboard	Fare
0	0	3	0	22.0	1	0	7.2500
1	1	1	1	38.0	1	0	71.2833
2	1	3	1	26.0	0	0	7.9250
3	1	1	1	35.0	1	0	53.1000
4	0	3	0	35.0	0	0	8.0500

3.1

Consider log-likelihood function:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{\mathrm{N}} \mathrm{y_i} \log\left(\frac{1}{1+e^{-\theta^\top \mathbf{x}_i}}\right) + \left(1-\mathrm{y_i}\right)\log\left(\frac{1}{1+e^{\theta^\top \mathbf{x}_i}}\right)$$

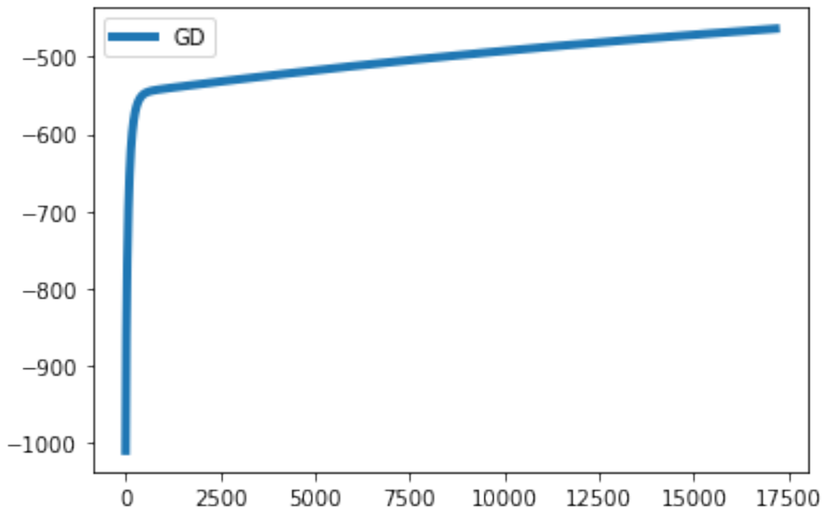
In [6]:
```
def LogLik(theta):
    return X.dot(theta).dot(y) - np.log(np.exp(1 + X.dot(theta))).sum(
)
def Gr(theta):
    return (y - 1/(1 + np.exp(-X.dot(theta)))).dot(X)
```

In [7]:
```
def GD(steps, size, tol):
    i = 0
    theta = np.linalg.inv(X.T.dot(X)).dot(X.T).dot(y)
    loglik = LogLik(theta)
    vec = [loglik]
    while i < steps:
        theta1 = theta + size*Gr(theta)
        vec.append(LogLik(theta1))

        if np.square((theta1 - theta)).sum() <= tol or np.square(vec[-
1] - vec[-2])<= tol:
            print('Total %d' %i  ,'iteration')
            return theta1, vec

        theta = theta1
        i += 1

    print('Total %d' %i  ,'iteration')
    return theta, vec
```

In [8]:
```
theta, vec = GD(20000,1e-7, 1e-10)
```

```
Total 17155 iteration
```

In [9]:
```
df = pd.DataFrame({"GD":vec})
df.plot.line(linewidth = 4)
None
```



In [10]:
```
theta
```

Out[10]:
```
Pclass                      -0.157296
Sex                          0.691840
Age                         -0.022838
Siblings/Spouses Aboard     -0.128911
Parents/Children Aboard     -0.040576
Fare                         0.013129
dtype: float64
```

(a)

I choose constant stepsize as 10^{-7}, the optimization process went pretty well. I've chosen some other stategies at the beginning, such as exact lone search/ learning rate decay, momentum method, or some other methods involving second gradient. The method involving learn search didn't get good result and usually requires more iterations. The likelihood will fluctuate in some iterations. The method methods involving second gradient cost more time and computation complexity.

In the end, the constant stepsize showed the desired result.

(b)

In [11]:
```
import time
start_time = time.time()
theta, vec = GD(20000,1e-7, 1e-10)
print("Total %s seconds elapsed" % (time.time() - start_time))
```

```
Total 17155 iteration
Total 28.299755096435547 seconds elapsed
```

It took 28.30 seconds to complete the iteration. It took about 3 seconds to converge according to the loglikelihood plot.

(c)

In [12]:
```
theta
```

Out[12]:
```
Pclass                    -0.157296
Sex                        0.691840
Age                       -0.022838
Siblings/Spouses Aboard   -0.128911
Parents/Children Aboard   -0.040576
Fare                       0.013129
dtype: float64
```

(d)

In [13]:
```
vec[-1]
```

Out[13]:
```
-463.53153422224244
```

(e)

Using $\theta^\star$ to denote ture parameter.

Consider Fisher information:

$$\mathbf{I}_{\theta^\star} := -\mathbb{E}\left[\frac{\mathrm{d}^2\ell(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}^2}\bigg|_{\theta=\theta^\star}\right] = \mathbb{E}\left[\sum_{\mathbf{i}=1}^{\mathrm{N}} \frac{e^{-\theta^\top \mathbf{x_i}}}{\left(1+e^{-\theta^\top \mathbf{x_i}}\right)^2}\mathbf{x_i}\mathbf{x_i}^\top\bigg|_{\theta=\theta^\star}\right] = \sum_{\mathbf{i}=1}^{\mathrm{N}} \frac{e^{-\theta^{\star\top} \mathbf{x_i}}}{\left(1+e^{-\theta^{\star\top} \mathbf{x_i}}\right)^2}\mathbf{x_i}\mathbf{x_i}^\top$$

Let

$$\hat{\boldsymbol{\theta}} := \underset{\theta\in\mathbb{R}^{\mathrm{D}+1}}{\arg\max}\,\ell(\boldsymbol{\theta})$$

Suppose $\frac{\mathrm{d}^2\ell(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}^2}$. Then we have:

$$\hat{\boldsymbol{\theta}} \xrightarrow{d} \mathcal{N}\left(\boldsymbol{\theta}^\star, \mathbf{I}_{\boldsymbol{\theta}^\star}^{-1}\right)$$

According to computation, we have:

```
In [14]: def I(theta):
             W = np.diag(np.exp(-X.dot(theta))/(1 + np.exp(-X.dot(theta)))**2)
             return X.T.dot(W).dot(X)
```

```
In [15]: var = np.linalg.inv(I(theta))
```

Then we have:

$$\hat{\boldsymbol{\theta}} \xrightarrow{d} \mathcal{N}\left(\boldsymbol{\theta}^\star, \mathbf{I}_{\hat{\theta}}^{-1}\right)$$

where $\mathbf{I}_{\hat{\theta}}^{-1}$ is

In [16]:
```
var
```

Out[16]:
```
array([[ 3.21359720e-03, -1.96270908e-03, -1.94319179e-04,
        -1.48733570e-03, -1.01101582e-03,  5.15191289e-05],
       [-1.96270908e-03,  2.37288881e-02, -3.72060099e-05,
         1.08457571e-04, -2.95630615e-03, -7.20737802e-05],
       [-1.94319179e-04, -3.72060099e-05,  2.19491616e-05,
         1.01031536e-04,  6.98258066e-05, -7.11340387e-06],
       [-1.48733570e-03,  1.08457571e-04,  1.01031536e-04,
         5.81895119e-03, -2.23402533e-03, -6.34861528e-05],
       [-1.01101582e-03, -2.95630615e-03,  6.98258066e-05,
        -2.23402533e-03,  1.05884034e-02, -5.48940130e-05],
       [ 5.15191289e-05, -7.20737802e-05, -7.11340387e-06,
        -6.34861528e-05, -5.48940130e-05,  6.73377556e-06]])
```

In [ ]:

3.2

Consider the the log-odds are defined as

$$\omega^\star := \log\left(\frac{\mathbb{P}(y=1 \mid \mathbf{x})}{\mathbb{P}(y=0 \mid \mathbf{x})}\right) = \boldsymbol{\theta}^{\star\top}\mathbf{x}$$

By the invariance property of the MLE, we know that the MLE of $\omega^\star$ is given by

$$\hat{\omega} := \hat{\boldsymbol{\theta}}^\top \mathbf{x}$$

We further have:

$$\hat{\omega} \xrightarrow{d} \mathcal{N}\left(\omega^\star, \mathbf{x}^\top \mathbf{I}_{\theta^\star}^{-1}\mathbf{x}\right)$$

In [ ]:

3.3

My own feature would be Plass:3, Gender:0, Age:23, Siblings:0, Parents/Children:0, Fare: 7.25.

In [17]:
```
x = np.array([3,0,23,0,0,7.25])
```

In [18]:
```
ome_hat = 1 / (1 + np.exp(theta.dot(x)))
ome_hat
```

Out[18]:
```
0.7113562614023329
```

(a)

since 0.66>0.5. According to (6.7) in lecture notes, I will survive the Titanic sinking.

(b)

According to:

$$\mathbb{P}\left(\left|\hat{\omega}-\omega^{\star}\right|>\tau\right)=2\Phi\left(\tau \mid 0, \mathbf{x}^{\top} \mathbf{I}_{\theta^{\star}}^{-1} \mathbf{x}\right)$$

In [19]:
```
np.linalg.inv(I(theta)).dot(x).dot(x)
```

Out[19]:
```
0.013940140637597352
```

$$\tau=\Phi^{-1}\left(\alpha / 2 \mid 0, \mathbf{x}^{\top} \mathbf{I}_{\theta^{\star}}^{-1} \mathbf{x}\right)$$

When $\alpha=0.05$, the τ I compute based on my estimated variance is 0.2245956.

In [20]:
```
ome_hat - 0.2245956
```

Out[20]:
```
0.4867606614023329
```

In [21]:
```
ome_hat + 0.2245956
```

Out[21]:
```
0.9359518614023329
```

Then the 95% confidence interval would be [0.487,0.936]

(c)

I think my answer from (a) is fairly certain. In (a) I compute estimated probablity larger than 1/2. In (c) I got my 95% confidence interval. Most of my interval lies in [0.5,1], so I think the interval I got coincides with the answer I got from (a).

In [ ]:

```

```

3.4

(a)

Consider LRT test:

$$\frac{\mathbb{P}\left(\hat{\theta}_{\mathrm{j}} \mid \theta_{\mathrm{j}}^{\star} \neq 0\right)}{\mathbb{P}\left(\hat{\theta}_{\mathrm{j}} \mid \theta_{\mathrm{j}}^{\star} = 0\right)} \quad \gtrless_{H_0}^{H_1} 1$$

$$\left(\frac{\hat{\theta}_{\mathrm{j}}}{\nu_{\mathrm{j}}}\right)^2 \gtrless_{H_0}^{H_1} 2 \log \tau$$

The quantile I calulated for chi-square distribution with df 1 is 3.841459.

In [22]:

```
np.square(theta/ np.diag(var)) > 3.841459
```

Out[22]:

```
Pclass                        True
Sex                           True
Age                           True
Siblings/Spouses Aboard       True
Parents/Children Aboard       True
Fare                          True
dtype: bool
```

(b)

According the result in (a), all the features are significant.

(c)

Yes, the survival prediction is based on the feature prediction. If I change most significant feature in my feature vector, my survival prediction will change.

In [ ]:

```

```

In [ ]:

```

```