Notes on Category Theory

José Ramírez-Gómez

June 2024

Contents

1 Basic Concepts

1.1 Categories

Definition 1.1 (Category)**.** A category $\mathcal{C}$ consists of:

- A collection $Ob(\mathcal{C})$ of objects $A, B, C, \ldots$
- A collection $Ar(\mathcal{C})$ of arrows $f, g, h, \ldots$
- Two morphisms $dom, cod : Ar(\mathcal{C}) \to Ob(\mathcal{C})$ which assign to each arrow f its domain $dom(f)$ and codomain $cod(f)$. For each arrow f with domain A and codomain B we write $f : A \to B$. And for each pair of objects A, B we define the set

  $$Hom_{\mathcal{C}}(A,B) := \{f \in Ar(\mathcal{C}) | f : A \to B\}$$

  which we call Hom-set.
- For any three objects A, B, C the arrow composition

  $$C_{A,B,C} : Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$$

  for given arrows f, g we write $g \circ f$ to denote $C_{A,B,C}(f,g)$
- For each object A an identity arrow $\mathbf{1}_A : A \to A$

Such that the following axioms are satisfied

- **Associativity:** for any arrows $f : A \to B$, $g : B \to C$, $h : C \to D$

  $$h \circ (g \circ f) = (h \circ g) \circ f$$

  $$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

- **Identities:** for any arrow $f : A \to B$

  $$f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f$$

  $$\begin{tikzcd} A \arrow[loop left, "\mathbf{1}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathbf{1}_B"] \end{tikzcd}$$

▲

To have an idea of what we mean when we talk about objects and arrows lets see some examples of categories

Example 1.1 (Set)**.** The category of sets and functions, usually called Set, consists of sets as objects and functions as arrows between them. This is, if A and B are sets then an arrow $f : A \to B$ is a correspondence between those sets which assigns to each element a of A one and only one element b in B.

The composition on this category is given by the composition between functions, i.e., given $f : A \to B$ and $g : B \to C$, we can form the arrow $g \circ f : A \to C$ which is a mapping of the form $x \mapsto g(f(x))$. For each set A the identity arrow is the usual identity function which, i.e., $a \mapsto a$ for every a in A.

To understand de next example we need to recall the definition of a partial ordered set (A.1)

Example 1.2 (Poset)**.** Let $(P, \leq)$ be a poset. We can make $(P, \leq)$ into a category using the reflexivity, transitivity and antisymmetry properties. We say that if p and q are elements of P then, there's an arrow $f : p \to q$ if $p \leq q$, the identity arrow is given by reflexivity and the composition is given by transitivity. The way that this category is defined let us ensure that there is at most one arrow between any two objects. The axioms hold since there is at most one arrow between two objects.

Example 1.3 (Pos)**.** Previously we saw a single poset as a category, however, there is also a category called Pos whose objects are partial ordered sets and arrows are monotone functions (A.2).

Example 1.4 (Discrete category)**.** A category $\mathcal{C}$ is discrete when every arrow in it is an identity. Every set X is the set of objects of a discrete category, and every discrete category is so determined by its set of objects. Thus, discrete categories are sets.

Example 1.5 (Monoids as categories)**.** A monoid $(M, *, e)$ (B.1) can be seen as a category with one object. Let $\mathbf{M}$ be a category defined by the following data:

- There is just one object, say $\star$.
- Any object $a \in M$ is an arrow $a : \star \to \star$ in $\mathbf{M}$.
- The composition of arrows a and b is defined to be the product of the monoid, i.e., $a \circ b = a * b$.
- The identity arrow $1_\star$ is defined to be the monoid identity e.

If $a, b, c \in M$ we can represent the idea of the monoid M as a category $\mathbf{M}$ like in the following diagram

$$\begin{tikzcd}
\star \arrow[loop left, "1_\star"] \arrow[r, "a"] \arrow[dr, "b*a"'] & \star \arrow[d, "b"] \arrow[dr, "c*b"] & \\
& \star \arrow[r, "c"'] & \star \arrow[loop right, "1_\star"]
\end{tikzcd}$$

Definition 1.2 (Isomorphism)**.** In any category, an isomorphism is a map $f : A \to B$ such that there exists $g : B \to A$ such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. Diagrammatically this is

$$\begin{tikzcd}
A \arrow[loop left, "\mathbf{1}_A"] \arrow[rr, bend right, "f"'] & & B \arrow[loop right, "\mathbf{1}_B"] \arrow[ll, bend right, "g"']
\end{tikzcd}$$

$\blacktriangle$

Definition 1.3 (Small and locally small categories)**.** A category $\mathcal{C}$ is called small if the collections $Ob(\mathcal{C})$ and $Ar(\mathcal{C})$ are sets. A category C is called locally small if each of its hom-sets is a set. $\blacktriangle$

1.2 Functors

If there's arrows between objects then a natural question appears. Is there some kind of arrows between categories? and it turns out that there's such arrows. They're called functors

Definition 1.4 (Functor)**.** For categories $\mathcal{C}$ and $\mathcal{B}$ a functor $T : \mathcal{C} \to \mathcal{B}$ with domain $\mathcal{C}$ and codomain $\mathcal{B}$ consists of two morphisms: The object morphism T, which assigns to each arrow $f : c \to c'$ of $\mathcal{C}$ an arrow $Tf : Tc \to Tc'$ of $\mathcal{B}$ in such way that

$$T(1_C) = 1_{Tc} \quad \text{and} \quad T(g \circ f) = Tg \circ Tf$$

the latter whenever the composite $g \circ f$ is defined in $\mathcal{C}$. ▲

There another way of defining functors just using arrows

Definition 1.5 (Functor)**.** A functor T is a morphism from arrows f of $\mathcal{C}$ to arrows Tf of $\mathcal{B}$, carrying each identity of $\mathcal{C}$ to an identity of $\mathcal{B}$ and each composible pair $\langle g, f \rangle$ in $\mathcal{C}$ to a composible pair $\langle Tg, Tf \rangle$ in $\mathcal{B}$, with $Tg \circ Tf = T(g \circ f)$. ▲

Definition 1.6 (Composition of functors)**.** Functors may be composed. Explicitly, given functors $T : \mathcal{C} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$, there is a functor $S \circ T : \mathcal{C} \to \mathcal{A}$ called the composite of S with T whose objects morphism is defined by $c \mapsto S(Tc)$ and arrows morphism $f \mapsto S(Tf)$. This composition is associative.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathcal{B} \arrow[r, "S"] & \mathcal{A} \end{tikzcd}$$

▲

Lets see an example

Example 1.6 (Power set functor)**.** The power set functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is the functor whose objects morphism assigns to each set X the usual power set $\mathcal{P}X$ (A.3), whose elements are all the subsets of X. It's arrows morphism assigns to each $f : X \to Y$ the map $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ which sends each subset $S \subset X$ to its image $fS \subset Y$. Since both $\mathcal{P}(1_X) = 1_{\mathcal{P}X}$ and $\mathcal{P}(g \circ f) = \mathcal{P}g \circ \mathcal{P}f$, this clearly defines a functor.

Now we'll see some specific functors which are very useful

Definition 1.7 (Forgetful functor)**.** A functor which simply "forgets" some or all the structure of an algebraic object is commonly called a forgetful functor or underlying functor. ▲

Example 1.7. The forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$, assigns to each group G the set UG of its elements and assign to each morphism $f : G \to G'$ of groups the same function f, regarded just as a function between sets.

Definition 1.8 (Isomorphism between functors)**.** A functor $T : \mathcal{C} \to \mathcal{B}$ is an isomorphism if and only if there is a functor $S : \mathcal{B} \to \mathcal{C}$ for which both composites $S \circ T$ and $T \circ S$ are identity functors. ▲

Now we'll see some properties that are weaker than isomorphism

Definition 1.9 (Full)**.** A functor $T : \mathcal{C} \to \mathcal{B}$ is full when to every pair c, c' of objects of $\mathcal{C}$ and to every arrow $g : Tc \to Tc'$ of $\mathcal{B}$, there is an arrow $f : c \to c'$ of $\mathcal{C}$ with $g = Tf$. ▲

Definition 1.10 (Faithful)**.** A functor $T : \mathcal{C} \to \mathcal{B}$ is faithful or an embedding when to every pair c, c' of objects of $\mathcal{C}$ and to every pair $f_1, f_2 : c \to c'$ if parallel arrows of $\mathcal{C}$ the equality $Tf_1 = Tf_2 : Tc \to Tc'$ implies $f_1 = f_2$. ▲

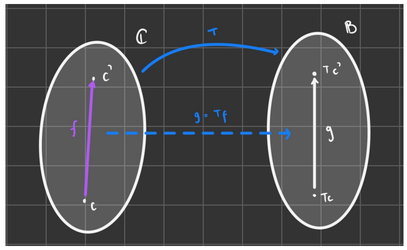

Figure 1: Graphic idea

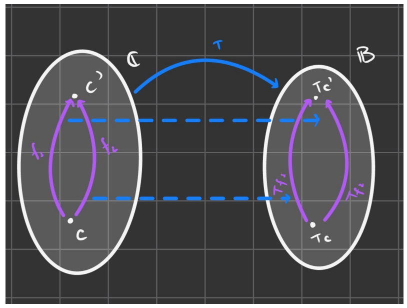

Figure 2: Graphic idea

This two properties can be visualized in terms if Hom-sets. Given a pair of objects $c, c' \in \mathcal{C}$, the arrow morphism of $T : \mathcal{C} \to \mathcal{B}$ assigns to each $f : c \to c'$ an arrow $Tf : Tc \to Tc'$ and so defines a function

$$
\begin{gathered}
T_{c,c'} : Hom(c, c') \to Hom(Tc, Tc') \\
f \mapsto Tf
\end{gathered}
$$

Then T is full when every such function is surjective and faithful when every such function is injective. For a functor which is both full and faithful every such function is a bijection which need not mean that the fully faithful functor itself is an isomorphism of categories.

Definition 1.11 (Subcategory)**.** A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is a collection of the objects and some of the arrows of $\mathcal{C}$, which includes with each arrow f both the objects $dom(f)$ and $cod(f)$, with each object s its identity arrow 1_s and with each pair of composable arrows $s \to s' \to s''$ their composite. ▲

We now are allowed to define the injection $\mathcal{S} \to \mathcal{C}$ which sends each object and each arrow of $\mathcal{S}$ to itself in $\mathcal{C}$. Such injection is a functor that we will call the inclusion functor. This functor is automatically faithful since all the arrows of $\mathcal{S}$ are in $\mathcal{C}$.

We will say that $\mathcal{S}$ is a full subcategory of $\mathcal{C}$ when the inclusion functor is full. Then a full subcategory $\mathcal{S}$ of some category $\mathcal{C}$ is determined just by it set of objects since for every pair of objects s, s' all the morphisms $s \to s'$ are in $\mathcal{C}$.

Example 1.8. The category $\mathsf{Set}_{\mathsf{fin}}$ of all finite sets is a full subcategory of the category Set.

1.3 Natural transformations

Definition 1.12 (Natural transformation)**.** Given two functors $S, T : \mathcal{C} \to \mathcal{B}$, a natural transformation $\alpha : S \Rightarrow T$ is a morphism wich assigns to each object c of $\mathcal{C}$ an arrow $\alpha_c = \alpha c : Sc \to Tc$ of $\mathcal{B}$ in such way that every arrow $f : c \to c'$ in $\mathcal{C}$ yields a diagram1

$$\begin{tikzcd} c \arrow[d, "f"'] & Sc \arrow[r, "\alpha_c"] \arrow[d, "Sf"'] & Tc \arrow[d, "Tf"] \\ c' & Sc' \arrow[r, "\alpha_{c'}"'] & Tc' \end{tikzcd}$$

which is commutative. ▲

We will call α_i a component of the natural transformation. In some cases, the natural transformations are called morphisms of functors.

Definition 1.13 (Natural isomorphism)**.** A natural transformation α with evert component α_c invertible in $\mathcal{B}$ is called a natural equivalence or a natural isomorphism. We'll write $\alpha : S \cong T$ for the natural isomorphism $\alpha : S \Rightarrow T$. ▲

Definition 1.14 (Equivalence of categories)**.** An equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a pair of functors $S : \mathcal{C} \to \mathcal{D}$, $T : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $I_{\mathcal{C}} \cong T \circ S$, $I_{\mathcal{D}} \cong S \to T$. ▲

Now we'll define a binary operation between a functor an a natural transformaion

Definition 1.15 (Whiskering)**.** If $\eta : F \Rightarrow G$ is a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$, and $H : \mathcal{D} \to \mathcal{B}$ is another functor, then we can form the natural transformation $H\eta : H \circ F \Rightarrow H \circ G$ by defining $(H\eta)_x = H(\eta_x)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "H"] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "H\circ G"'] \arrow[r, phantom, "\Downarrow H\eta"] & \mathcal{B} \end{tikzcd}$$

In the other hand, if $K : \mathcal{A} \to \mathcal{C}$ is a functor, the natural transformation $\eta K : F \circ K \Rightarrow G \circ K$ is defined by $(\eta K)_x = \eta_{Kx}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "K"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F\circ K"] \arrow[r, bend right, "G\circ K"'] \arrow[r, phantom, "\Downarrow \eta K"] & \mathcal{D} \end{tikzcd}$$

It is also a horizontal composition where one of the natural transformations is the identity natural transformation $H\eta = id_H * \eta$ and $\eta K = \eta * id_K$. ▲

2 Adjoints

2.1 Adjunctions

Definition 2.1 (Adjunction)**.** Let $\mathcal{A}$ and $\mathcal{X}$ be categories. An adjunction from $\mathcal{X}$ to $\mathcal{A}$ is a triple $\langle F, G, \varphi \rangle$ where $F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ while φ is a function which assigns to each pair of objects $x \in \mathcal{X}$, $a \in \mathcal{A}$ a bijection of sets

$$\varphi = \varphi_{x,a} : \mathcal{A}(Fx, a) \cong \mathcal{X}(x, Ga)$$

which is natural in x and a. ▲

Here, the left-hand side is the bifunction is the bifunctor

$$\mathcal{X}^{OP} \times \mathcal{A} \xrightarrow{F^{OP} \times Id_{\mathcal{A}}} \mathcal{A}^{OP} \times \mathcal{A} \xrightarrow{hom} \mathsf{Set}$$

which sends each pair of objects $\langle x, a\rangle$ to the hom-set $\mathcal{A}(Fx, a)$ an the right-hand side is a similar bifunctor

$$\mathcal{X}^{OP} \times \mathcal{A} \longrightarrow \mathsf{Set}$$

The naturality of the bijection φ means that for all $k : a \to a'$ and $h : x' \to x$ both the diagrams

$$\begin{tikzcd}
\mathcal{A}(Fx,a) \arrow[r, "\varphi"] \arrow[d, "k_*"'] & \mathcal{X}(x,Ga) \arrow[d, "(Gk)_*"] \\
\mathcal{A}(Fx,a') \arrow[r, "\varphi"'] & \mathcal{X}(x,Ga')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(Fx,a) \arrow[r, "\varphi"] \arrow[d, "(Fh)_*"'] & \mathcal{X}(x,Ga) \arrow[d, "h^*"] \\
\mathcal{A}(Fx',a) \arrow[r, "\varphi"'] & \mathcal{X}(x',Ga)
\end{tikzcd}$$

will commute. Given sucha an adjunction, the functor F is said to be the left adjoint for G while G is called the right adjoint for F. We'll usually write $F \dashv G$ to denote and adjunction.

Lets see an example

Example 2.1 (Vct_K and Set)**.** For a fixed field K consider the following functors $V : \mathsf{Set} \to \mathsf{Vct}_K$ and $U : \mathsf{Vct}_K \to \mathsf{Set}$ where for each vector space W, $U(W)$ is the set of all vextors in W, so U is the forgetful functor, while for any set X, $V(X)$ is the vector space with basis X. The vectors of $V(X)$ are thus the formal finite linear combinations $\sum r_i x_i$ with scalar coeficients $r_i \in K$ and with each $x_i \in X$, with the evident vector operations.

Each function $g : X \to U(X)$ extends to a unique linear transformation $f : V(X) \to W$, given explicitly by $f(\sum r_i x_i) = \sum r_i(gx_i)$. This correspondence $\psi : g \mapsto f$ has an invrse $\varphi : f \mapsto f|x$, the restriction of f to X, hence is a bijection

$$\varphi : \mathsf{Vct}_K(V(X), W) \cong \mathsf{Set}(X, U(W))$$

This bijection $\varphi = \varphi_{X,W}$ is defined "in the same way" for all sets X and all vector spaces W. This means that the $\varphi_{X,W}$ are the components of a natural transformation φ when both sides above are regarded as functors of X and W. It suffices to verify naturality in X and W separately. Naturality in X means that for each arrow $h : X' \to X$ the diagram

$$\begin{tikzcd}
\mathsf{Vct}_K(V(X),W) \arrow[r, "\varphi"] \arrow[d, "(Vh)^*"'] & \mathsf{Set}(X,U(W)) \arrow[d, "h^*"] \\
\mathsf{Vct}_K(V(X'),W) \arrow[r, "\varphi"'] & \mathsf{Set}(X',U(W))
\end{tikzcd}$$

where $h^*(g) = g \circ h$ will commute. Thid commutativity follos from the definition of φ by a routine calculation, as does also naturality in W.

There are two important natural tranformations which are linked to every adjunction

Definition 2.2 (Unit and Counit)**.** Every adjuntion $\langle F, G, \varphi\rangle : \mathcal{X} \to \mathcal{A}$ determines:

- A natural transformation $\eta : I_{\mathcal{X}} \Rightarrow GF$ sich that for each object x the arrow η_x is universal to G from x, while the right adjunction of each $f : Fx \to a$ is

$$\varphi f = Gf \circ \eta_x : x \to Ga$$

- Anatural transformation $\varepsilon : FG \Rightarrow I_A$ such that each arrow ε_a is universal to a from F, while each $g : x \to Ga$ has left adjunct

$$\varphi^{-1} g = \varepsilon_a \circ Fg : Fx \to a$$

Moreover, both the following composites are the identities (of G respect to F)

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G \qquad\qquad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

▲

3 Topics in Category Theory

3.1 Monoidal categories

Definition 3.1 (Strict monoidal category)**.** A *strict* monoidal category $(\mathbf{B}, \otimes, e)$ is a category $\mathbf{B}$ with a bifunctor $\otimes : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$ which is associative, i.e., $\otimes(\otimes \times 1) = \otimes(1 \times \otimes) : \mathbf{B} \times \mathbf{B} \times \mathbf{B} \to \mathbf{B}$ or diagrammatically

$$\begin{tikzcd}
\mathbf{B} \times \mathbf{B} \times \mathbf{B} \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] & \mathbf{B} \times (\mathbf{B} \times \mathbf{B}) \arrow[d, "\otimes"] \\
(\mathbf{B} \times \mathbf{B}) \times \mathbf{B} \arrow[r, "\otimes"'] & \mathbf{B}
\end{tikzcd}$$

and an object e which is a left and right unit for $\otimes$, i.e., $\otimes(e \times 1) = id_{\mathbf{B}} = \otimes(1 \times e)$ or diagrammatically

$$\begin{tikzcd}
\mathbf{B} \times \mathbf{B} \arrow[dr, "\otimes"'] & \mathbf{B} \arrow[l, "(e,1)"'] \arrow[r, "(1,e)"] \arrow[d, "id_{\mathbf{B}}"] & \mathbf{B} \times \mathbf{B} \arrow[dl, "\otimes"] \\
& \mathbf{B} &
\end{tikzcd}$$

▲

In the previous definition, the functor $(e, 1)$ assigns to each object b in $\mathbf{B}$ an object (e, b) in $\mathbf{B} \times \mathbf{B}$ and the functor $(1 \times e)$ assigns to each object b in $\mathbf{B}$ an object (b, e) in $\mathbf{B} \times \mathbf{B}$. The bifunctor $\otimes$ assigns to each object pair of objects a, b in $\mathbf{B}$ an object $a \otimes b$ of $\mathbf{B}$ and to each pair of arrows $f : a \to a'$ and $g : b \to b'$ an arrow $f \otimes g : a \otimes b \to a' \otimes b'$. Then $\otimes$ being a bifunctor means that the interchange law $1_a \otimes 1_b$, $(f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g)$ holds whenever ht composites $f'f$ and $g'g$ are defined. This can be represented diagrammatically as follows:

$$\begin{tikzcd}
a \arrow[r, "f"'] \arrow[rr, bend left, "f'f"] & a' \arrow[r, "f'"'] & a'' \\
b \arrow[r, "g"] \arrow[rr, bend right, "g'g"'] & b' \arrow[r, "g'"] & b''
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes b \arrow[d, "f\otimes g"] \arrow[dd, bend right, "(ff')\otimes(gg')"'] \\
a' \otimes b' \arrow[d, "f'\otimes g'"] \\
a'' \otimes b''
\end{tikzcd}$$

Remark. The associative law states that the binary operation $\otimes$ is associative both for objects and for arrows. Similarly, the unit law means that $e \otimes c = c = c \otimes e$ for objects c and that $1_e \otimes f = f = f \otimes 1_e$ for arrows f.

Example 3.1 (Monoids)**.** Any monoid $(M, *, e)$ regarded as a category is a strict monoidal one with $\otimes$ the multiplication $*$ of the elements of M.

Example 3.2 (Endofunctors)**.** If $\mathbf{X}$ is a category, the category $\mathsf{End}(X)$ whose objects are all endofunctors $S : \mathbf{X} \to \mathbf{X}$ and arrows all natural transformations $\theta : S \Rightarrow T$ is strict monoidal with $\otimes$ the composition of functors.

The condition that the bifunctor $\otimes$ must be associative is too strict. That is why there is a "relaxed" definition of strict monoidal category

Definition 3.2 (Monoidal category)**.** A monoidal category $\mathbf{B} = \langle \mathbf{B}, \otimes, e, \alpha, \lambda, \rho \rangle$ is a category $\mathbf{B}$, a bifunctor $\otimes$, an object $e \in \mathbf{B}$ and three natural isomorphisms α, λ, ρ. Explicitly

$$\alpha = \alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$$

is natural for all $a, b, c \in \mathbf{B}$, and the pentagonal diagram

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"] & (a \otimes b) \otimes (c \otimes d) \arrow[r, "\alpha"] & ((a \otimes b) \otimes c) \otimes d \\
a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha"'] & & (a \otimes (b \otimes c)) \otimes d \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

commutes for all $a, b, c, d \in \mathbf{B}$. The natural transformations λ and ρ are natural

$$\lambda_a : e \otimes a \cong a, \qquad \rho_a : a \otimes e \cong a$$

and for all objects, the triangular diagram commutes

$$\begin{tikzcd}
a \otimes (e \otimes c) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\lambda_c"'] & & (a \otimes e) \otimes c \arrow[dl, "\rho_a\otimes 1"] \\
& a \otimes c &
\end{tikzcd}$$

and also $\lambda_e = \rho_e : e \otimes e \to e$. ▲

The pentagon diagram establishes a commutativity condition for the natural transformation α, which is defined between the following functors $-\otimes(-\otimes-) : \mathbf{B}\times\mathbf{B}\times\mathbf{B} \to \mathbf{B}$ and $(-\otimes-)\otimes- : \mathbf{B}\times\mathbf{B}\times\mathbf{B} \to \mathbf{B}$. This means that we're using the bifunctor $\otimes$ in two different ways and we're not obtaining the same result but isomorphic ones. Precisely when using this two functors we're obtaining the isomorphic objects $a\otimes(b\otimes c) \cong (a\otimes b)\otimes c$.

In other words, for each triple of objects $a, b, c \in \mathbf{B}$ there are objects $a\otimes(b\otimes c)$, $(a\otimes b)\otimes c$ and arrows $f : a\otimes(b\otimes c) \to (a\otimes b)\otimes c$, $g : (a\otimes b)\otimes c \to a\otimes(b\otimes c)$ in $\mathbf{B}$ such that $g\circ f = 1_{a\otimes(b\otimes c)}$ and $f\circ g = 1_{(a\otimes b)\otimes c}$.

Remark. If $\mathbf{B}$ is a strict monoidal category, then $a\otimes(b\otimes c) = (a\otimes b)\otimes c$ since α would be the identity natural transformation.

Example 3.3. Any category with finite products is monoidal if we take $a\otimes b$ as the categorical product of a, b an e as the terminal object. Then α, λ and ρ must be the unique natural transformations which commutes with the projection maps.

Similarly, any category with finite coproducts is monoidal if we take $\otimes$ as the coproduct and e as the initial object.

Definition 3.3 (Morphism of monoidal categories)**.** A strict morphism between the monoidal categories $T : (\mathbf{B}, \otimes, e, \alpha, \lambda, \rho) \to (\mathbf{B}', \otimes', e', \alpha', \lambda', \rho')$ is a functor $T : \mathbf{B} \to \mathbf{B}$ such that for all a, b, c, f and g

$$\begin{aligned} &T(a\otimes b) = Ta\otimes' Tb, && T(f\otimes g) = f\otimes' g, && Te = e' \\ &T\alpha_{a,b,c} = \alpha'_{Ta,Tb,Tc}, && T\lambda_a = \lambda'_{Ta}, && T\rho_a = \rho'_{Ta} \end{aligned}$$

▲

With the previous morphisms we can form Moncat which is the category with small monoidal categories as objects and strict morphisms as arrows. There's also a full subcategory with objects all the strict monoidal categories.

Definition 3.4 (Symmetric monoidal category)**.** A monoidal category $\mathbf{B}$ is said to be symmetric if it's equipped with isomorphisms

$$\gamma_{a,b} : a\otimes b \cong b\otimes a$$

natural in $a, b \in \mathbf{B}$ such that the following diagrams commute

$$\gamma_{a,b}\circ\gamma_{b,a} = 1, \qquad \rho_b = \lambda_b\circ\gamma_{b,e} : b\otimes e \cong b$$

$$\begin{tikzcd}
a\otimes(b\otimes c) \arrow[r, "\alpha"] \arrow[d, "1\otimes\gamma"'] & (a\otimes b)\otimes c \arrow[r, "\gamma"] & c\otimes(a\otimes b) \arrow[d, "\alpha"] \\
a\otimes(c\otimes b) \arrow[r, "\alpha"'] & (a\otimes c)\otimes b \arrow[r, "\gamma\otimes 1"'] & (c\otimes a)\otimes b
\end{tikzcd}$$

▲

Example 3.4. The monoidal categories $(\mathbf{B}, \otimes, e, \alpha, \lambda, \rho)$ where $\otimes$ is the product or the coproduct are automatically symmetric when $\gamma : a\times b \cong b\times a$ is the canonical isomorphism which commutes with the projection morphisms.

3.2 Monads

Definition 3.5 (Monad)**.** A monad $T = (T, \eta, \mu)$ in a category $\mathcal{X}$ consist of a functor $T : \mathcal{X} \to \mathcal{X}$ and two natural transformations $\eta : 1_{\mathcal{X}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ which make the following diagrams commute

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} 1 \circ T \arrow[r, "\eta T"] \arrow[dr, "1"'] & T^2 \arrow[d, "\mu"] & T \circ 1 \arrow[l, "T\eta"'] \arrow[dl, "1"] \\ & T & \end{tikzcd}$$

▲

3.3 Monoids

Definition 3.6 (Monoid)**.** A monoid c in a monoidal category $\mathcal{B}$ is an object $c \in \mathcal{B}$ together with two arrows $\mu : c \otimes c \to c$ and $\eta : e \to c$ such that the following diagrams

$$\begin{tikzcd} c \otimes (c \otimes c) \arrow[r, "\alpha"] \arrow[d, "1\otimes\mu"'] & (c \otimes c) \otimes c \arrow[r, "\mu\otimes 1"] & c \otimes c \arrow[d, "\mu"] \\ c \otimes c \arrow[rr, "\mu"'] & & c \end{tikzcd} \qquad \begin{tikzcd} e \otimes c \arrow[r, "1\otimes\eta"] \arrow[dr, "\lambda"'] & c \otimes c \arrow[d, "\mu"] & c \otimes e \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\rho"] \\ & c & \end{tikzcd}$$

commute. ▲

Definition 3.7 (Monoids morphism)**.** A monoid morphism $f : (c, \mu, \eta) \to (c', \mu', \eta')$ is an arrow $f : c \to c'$such that $f\mu = \mu'(f \otimes f) : c \otimes c \to c$ and $f\eta = \eta' : e \to c'$. ▲

Those morphisms and the monoids in $\mathcal{B}$ form the category $\mathsf{Mon}_{\mathcal{B}}$ and $(c, \mu, \eta) \mapsto c$ defines a forgetful functor $U : \mathsf{Mon}_{\mathcal{B}} \to \mathcal{B}$.

If we carefully watch the definition of a monoid, we can see the similarities with that of a monad. Then the following question arises naturally: Is there a way to define a monad in terms of a monoid? Indeed, it is possible. If $\mathcal{X}$ is any category, we know that there is a monoidal category whose objects are endofunctors $T : \mathcal{X} \to \mathcal{X}$, where the arrows are natural transformations between those endofunctors. The operation $\otimes$ is the composition $\circ$, and the identity element e is the identity functor $\mathbf{1}$. We also know that a monad in a category $\mathcal{X}$ consists of an endofunctor $T : \mathcal{X} \to \mathcal{X}$ and two natural transformations $\mu : T^2 \to T$ and $\eta : \mathbf{I}_{\mathcal{X}} \to T$ which satisfy the respective commuting diagrams. [TERMINAR]

3.4 Bicategories

A bicategory is a particular algebraic notion of weak 2-category. The idea is that a bicategory is a category weakly enriched[1] over Cat, that is, the *hom-objects* of a bicategory are *hom-categories*, but the commutativity and unity laws of enriched categories hold up to coherent isomorphism.

Definition 3.8 (Bicategory)**.** Formally, a bicategory $\mathcal{B}$ consists of:

- **0-cells:** $a, b, c, \ldots$.
- **1-cells:** $f, g, h, \ldots$.

[1] Read appendix for more information

- **2-cells:** $\alpha, \beta, \gamma, \ldots$.

With sources and targets as in the diagram

$$
\begin{tikzcd}
a \arrow[r, shift left=3, "f"] \arrow[r, shift right=3, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, shift left=3, "g"] \arrow[r, shift right=3, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & c \arrow[r, shift left=3, "h"] \arrow[r, shift right=3, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & d
\end{tikzcd}
$$

Specifically, each 1-cell $f : a \to b$ has 0-cells a and b as domain and codomain respectively, while each 2-cell $\alpha : f \to f'$ has parallel 1-cells f and f' as its domain and codomain respectively.

Moreover, to each pair of 0-cells (a, b) there is a category $\mathcal{B}(a, b)$ in which the objects are all the 1-cells $f, f', \ldots$ with source a and target b while the arrows are the 2-cells between such 1-cells. In this category, the vertical composition of 2-cells is associative and has for each object $f : a \to b$ a 2-cell $1_f : f \Rightarrow f$ which acts as an identity for this vertical composition, which we denote by $\circ$.

Also, for each ordered triple of 0-cells a, b, c there is a bifunctor:

$$* : \mathcal{B}(b, c) \times \mathcal{B}(a, b) \to \mathcal{B}(a, c)$$

called horizontal composition. Therefore, given the diagram above, there are composite 2-cells $\beta * \alpha$, $\gamma * \beta$ and composite 1-cells $g * f$ as in the diagram

$$
\begin{tikzcd}
a \arrow[r, shift left=3, "g*f"] \arrow[r, shift right=3, "g'*f'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & c & & b \arrow[r, shift left=3, "h*g"] \arrow[r, shift right=3, "h'*g'"'] \arrow[r, phantom, "\Downarrow\gamma*\beta"] & d
\end{tikzcd}
$$

▲

The composition $*$ is not strictly associative, but is associative "up to" isomorphism σ between iterated composite functors as follows

$$
\begin{tikzcd}
& \mathcal{B}(c,d)\times\mathcal{B}(b,c)\times\mathcal{B}(a,b) \arrow[dl, "1\times *"'] \arrow[dr, "*\times 1"] \arrow[dd, shift right=5, "*.(1\times *)"'] \arrow[dd, shift left=5, "*.(*\times 1)"] \arrow[dd, phantom, "\xRightarrow[\sigma]{}"] & \\
\mathcal{B}(c,d)\times\mathcal{B}(a,c) \arrow[dr, "*"'] & & \mathcal{B}(b,d)\times\mathcal{B}(a,b) \arrow[dl, "*"] \\
& \mathcal{B}(a,d) &
\end{tikzcd}
$$

The requirement that α be a natural transformation of functors amounts to the following commutativity for the 2-cells:

$$
\begin{tikzcd}
h*(g*f) \arrow[r, "\sigma_{hgf}"] \arrow[d, "\gamma*(\beta*\alpha)"'] & (h*g)*f \arrow[d, "(\gamma*\beta)*\alpha"] \\
h'*(g'*f') \arrow[r, "\sigma_{h'g'f'}"'] & (h'*g')*f'
\end{tikzcd}
$$

The arrows 1_a are also required to act as identities for the horizontal composition only up to the following isomorphisms

$$\rho_{a,b} : f * 1_a \Rightarrow f, \qquad \lambda_{a,b} : 1_b * f \Rightarrow f$$

3.5 Spans

Definition 3.9 (Span)**.** Given a category $\mathcal{C}$, the span from the object X to the object Y is a diagram of the form

$$\begin{tikzcd} & S \arrow[dl, "s_x"'] \arrow[dr, "s_y"] & \\ X & & Y \end{tikzcd}$$

Where S is the apex, X the domain, Y the codomain, s_x the left leg and s_y the right leg. ▲

It turns out that $Span(\mathsf{Set})$ is a bicategory where the 0-cells are sets $A, B, C, \ldots$, the 1-cells are Spans $A \xleftarrow{f} C \xrightarrow{g} B$ which we will write as $C : A \nrightarrow B$ and the 2-cells are morphisms between spans $\alpha : C \Rightarrow C'$ which have the following form

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ A & & B \\ & C' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

Remark. Note that since C and C' are sets, the arrow α is a function between sets.

Example 3.5 (Monads in $Span(\mathsf{Set})$)**.** Since $Span(\mathsf{Set})$ is a bicategory, we must define the monad in terms of 0-cells, 1-cells and 2-cells. We will construct the monad for a 0-cell $\mathbf{X}_0$, the 1-cell $\mathbf{X}_1 : \mathbf{X}_0 \nrightarrow \mathbf{X}_0$ works as the endofunctor T and the natural transformations μ and η are

- η :

$$\begin{tikzcd} & \mathbf{X}_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\eta"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

- μ :

$$\begin{tikzcd} \mathbf{X}_0 & & \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"'] \arrow[ur, "t"] & & \mathbf{X}_1 \arrow[ul, "s"'] \arrow[ur, "t"] & \\ & & \mathbf{X}_1 \, {}_t\!\times_s \mathbf{X}_1 \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] \arrow[uull, bend left, "s"] \arrow[uurr, bend right, "t"'] & & \end{tikzcd}$$

Looking carefully at the information above one can realize that a monad in $Span(\mathsf{Set})$ correspond to a small category. Precisely the object of the corresponding category is the 0-cell $\mathbf{X}_0$, the arrows are the 1-cells $\mathbf{X}_1 : \mathbf{X}_0 \nrightarrow \mathbf{X}_0$, the identity morphism is the unit η and the composition is given by the multiplication μ.

4 Restriction structures

In this section we will describe a first approach on what restriction monads are. Here we'll define the restriction operator as a family of functions within each hom-category of $\mathcal{B}$. In order to do this, we'll have to assume that the bicategory $\mathcal{B}$ is locally small.
[ACÁ DEBERÍA IR UNA DESCRIPICIÓN MÁS ADECUADA DE LO QUE CONTIENE LA SECCIÓN]

4.1 Restriction categories

Definition 4.1 (Restriction structure)**.** A restriction structure on a category $\mathbf{X}$ is an assignment of an arrow $\bar{f} : A \to A$ to each arrow $f : A \to B$, such that the following four conditions are satisfied

1. (R1) $f\bar{f} = f$ for all f.
2. (R2) $\bar{f}\bar{g} = \bar{g}\bar{f}$ whenever $dom(f) = dom(g)$.
3. (R3) $\overline{g\bar{f}} = \bar{g}\bar{f}$ whenever $dom(f) = dom(g)$.
4. (R4) $\bar{g}f = f\overline{gf}$ whenever $cod(f) = dom(g)$.

▲

Definition 4.2 (Restriction category)**.** A restriction category is a category together with a restriction structure. ▲

4.2 Restriction monads

Definition 4.3 (General elements)**.** Suppose that $\mathcal{C}$ is any category and consider the objects A, T of $\mathcal{C}$. A T-valued element of A is a morphism $p : T \to A$. We call T the shape of the element p. ▲

Generally speaking, knowing about the points of an object is not sufficient to know about its elements. In the particular case of Set most of the things that can be said and done about the elements of a set X can more generally be said and done for morphisms $x : U \to X$. The fact that the product $X \times Y$ must be pairs (x, y) where x belong to X and y belong to Y is not a coincidence. When they are expressed as morphism out of some set U, they express the universal property of a categorical product.

Instead if defining a 2-cell on the entirety of the monad, we define the restriction operator as a family of functions within each hom-category of $\mathcal{B}$. This is reasonably easy to do in $Span(\mathsf{Set})$ since we can choose elements: given a small restriction category $\mathbf{X}$, its objects can be obtained from the hom-category $Span(\mathsf{Set})(\{*\}, \mathbf{X}_0)$. Concretely, each object A of $\mathbf{X}$ corresponds to a span $\vec{A}$:

$$\begin{tikzcd} & \{*\} \arrow[dl, "id"'] \arrow[dr, "\tilde{A}"] & \\ \{*\} & & \mathbf{X}_0 \end{tikzcd}$$

More generally, we will consider the category $\mathcal{B}(E, x)$ of E-shaped elements of x in $\mathcal{B}$. In order to motivate the definition of restriction monad lets take a look at an example considering the case when $\mathcal{B} = Span(\mathsf{Set})$ with $E = \{*\}$.

Example 4.1. Suppose that $\mathbf{X}$ is a small restriction category. Its corresponding monad in $\mathit{Span}(\mathsf{Set})$ is of the form

$$\begin{tikzcd}[column sep=small] & \mathbf{X}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathbf{X}_1 \,{}_t\times_s \mathbf{X}_1 \arrow[dl, "s\pi_1"'] \arrow[dr, "t\pi_2"] \arrow[dd, "\mu"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathbf{X}_0 \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[dd, "\eta"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

The compose of the spans $\mathbf{X}_0 \xleftarrow{s} \mathbf{X}_1 \xrightarrow{t} \mathbf{X}_0$ and $\{*\} \xleftarrow{id} \{*\} \xrightarrow{\tilde{A}} \vec{A}$ is the span $T\vec{A}$ which contains as data all arrows of $\mathbf{X}$ with sorce A. This compose has the form

$$\begin{tikzcd}[column sep=small] & & \{*\} \,{}_{\tilde{A}}\times_s \mathbf{X}_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[ddll, bend right, "id\circ\pi_1"'] \arrow[ddrr, bend left, "t\circ\pi_2"] & & \\ & \{*\} \arrow[dl, "id"'] \arrow[dr, "\tilde{A}"] & & \mathbf{X}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \{*\} & & \mathbf{X}_0 & & \mathbf{X}_0 \end{tikzcd} \quad = \quad \begin{tikzcd} & \{*\} \,{}_{\tilde{A}}\times_s \mathbf{X}_1 \arrow[dl, "id\circ\pi_1"'] \arrow[dr, "t\circ\pi_2"] & \\ \{*\} & & \mathbf{X}_0 \end{tikzcd}$$

Given another object $B \in \mathbf{X}_0$, we can have the span morphism of the form $\tilde{f} : \vec{B} \nrightarrow T\vec{A}$

$$\begin{tikzcd} & \{*\} \arrow[dl, "id"'] \arrow[dr, "\tilde{B}"] \arrow[dd, "\tilde{f}"] & \\ \{*\} & & \mathbf{X}_0 \\ & \{*\} \,{}_{\tilde{A}}\times_s \mathbf{X}_1 \arrow[ul, "\pi_1"] \arrow[ur, "t\pi_2"'] & \end{tikzcd}$$

which is equivalent to the choice of an arrow f in $\mathbf{X}$ whose source is A and whose target is B. This idea lead us to the following correspondence

$$\mathit{Span}(\mathsf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \leftrightarrow \mathbf{X}(A, B)$$

Definition 4.4 (Restriction monad)**.** Suppose that $\mathcal{B}$ is a locally small bicategory with a 0-cell E. A restriction E-monad in $\mathcal{B}$ is a monad (T, η, μ) on x in $\mathcal{B}$ together with a family of functions

$$\rho_{A,B} : \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) \to \mathcal{B}(E, x)(C, TA)$$

indexed by E-shaped elements $A, B : E \to x$ of x. Two 2-cells between E-shaped elements of x are composed via multiplication map

$$\tilde{\mu}_{A,B,C} : \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) \to \mathcal{B}(E, x)(C, TA)$$

▲

defined by the composite

A Set theory

Definition A.1 (Partial ordered set)**.** Given a set S, a partial order on S is a binary relation $\leq$ with the following properties

- **Reflexivity:** $x \leq x$ always.
- **Transitivity:** if $x \leq y$ and $y \leq z$ then $x \leq z$.
- **Antisymmetry:** if $x \leq y$ and $y \leq x$ then $x = y$.

A *poset* is a set equipped with a partial order.

Definition A.2. A monotone function f from a poset S to a poset T is a function from S to T that preserves $\leq$, that is, $x \leq y \implies f(x) \leq f(y)$.

Definition A.3 (Power set)**.**

B Algebra

Definition B.1 (Monoid)**.** A monoid is a set M together with an operation $*$ in M which satisfies the following axioms:

- **Associativity:** $a * (b * c) = (a * b) * c$ for all $a, b, c \in M$.
- **Identity existence:** for each $a \in M$ there exists an element $e \in M$ such that $a * e = e * a = a$.

C Further studies

References

[1] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. *New structures for physics*, pages 3–94, 2011.

[2] Steve Awodey. *Category theory*, volume 52. OUP Oxford, 2010.

[3] Tom Leinster. *Basic category theory*, volume 143. Cambridge University Press, 2014.

[4] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.