STA310 HW3

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```
if (!requireNamespace("ggplot2", quietly = TRUE)) {
  install.packages("ggplot2")
}
library(ggplot2)
```

Exercise 1

The probability mass function of a Poisson random variable is given by:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, ...$$

The likelihood function is:

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} P(X_i = x_i) \\ &= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\ &= \lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda} \cdot \prod_{i=1}^{n} \frac{1}{x_i!} \end{aligned}$$

Exercise 2

The log-likelihood function is:

$$
\begin{aligned}
\ell(\lambda) &= \log L(\lambda) \\
&= \log\left(\lambda^{\sum_{i=1}^n x_i} \cdot e^{-n\lambda} \cdot \prod_{i=1}^{n} \frac{1}{x_i!}\right) \\
&= \log(\lambda^{\sum_{i=1}^n x_i}) + \log(e^{-n\lambda}) + \log(\prod_{i=1}^{n} \frac{1}{x_i!}) \\
&= \sum_{i=1}^{n} x_i \cdot \log\lambda - n\lambda + \sum_{i=1}^{n} \log\frac{1}{x_i!}
\end{aligned}
$$

The first derivative of the log-likelihood function is:

$$
\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^n x_i}{\lambda} - n
$$

We then set the derivative equal to zero to find the MLE:

$$
\begin{aligned}
\frac{\sum_{i=1}^n x_i}{\hat{\lambda}} - n &= 0 \\
\hat{\lambda} &= \frac{\sum_{i=1}^n x_i}{n}
\end{aligned}
$$

The second derivative of the log-likelihood function is:

$$
\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^n x_i}{\lambda^2}
$$

Since $\lambda > 0$ and $x_i \geq 0$, the second derivative is negative (exclude the trivial case where all $x_i = 0$), confirming that the critical point corresponds to a maximum.

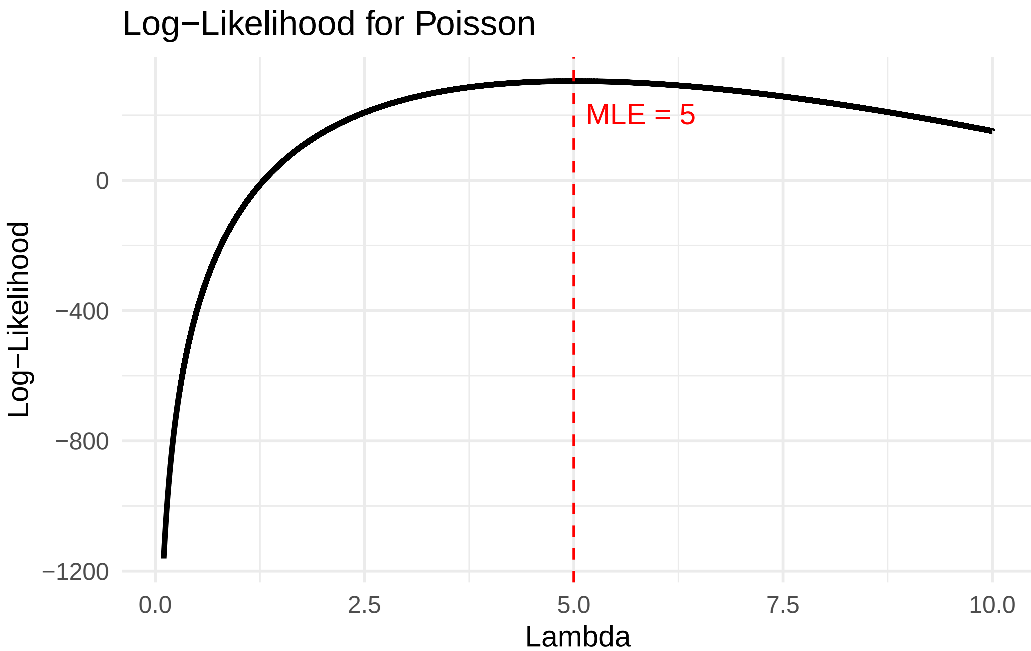
Exercise 3

```r
# Given data
n <- 100
sum_x <- 500

# Log-likelihood function for Poisson distribution
log_likelihood_function <- function(lambda) {
  if (lambda <= 0) {
    return(-Inf)
  } else {
    # The constant term can be ignored as
    # it does not affect the maximization of the log-likelihood
    return(sum_x * log(lambda) - n * lambda)
  }
}

# Set up a sequence of lambda values
# Start from 0.1 to avoid lambda = 0 resulting in -Inf log-likelihood
lambda_values <- seq(0.1, 10, length.out = 100000)

# Calculate the log-likelihood for each lambda
log_likelihood_values <- sapply(lambda_values, log_likelihood_function)

# Find the lambda that maximizes the log-likelihood
mle_lambda <- lambda_values[which.max(log_likelihood_values)]

# Print the MLE for lambda
cat("MLE for lambda:", round(mle_lambda, 3), "\n")
```

```
MLE for lambda: 5
```

```r
# Create a dataframe
data <- data.frame(lambda = lambda_values,
                   log_likelihood = log_likelihood_values)

# Plot
ggplot(data, aes(x = lambda, y = log_likelihood)) +
  geom_line(size = 1) +
  geom_vline(xintercept = mle_lambda, color = "red", linetype = "dashed") +
  annotate("text",
```

```
           x = mle_lambda + 0.8,
           y = max(log_likelihood_values) - 100,
           label = paste("MLE =", round(mle_lambda, 3)),
           color = "red") +
  labs(title = "Log-Likelihood for Poisson",
       x = "Lambda",
       y = "Log-Likelihood") +
  theme_minimal()
```



Therefore, the approximate MLE is 5, which matches the formula we derived in Exercise 2.

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{500}{100} = 5$$

Exercise 4

(a)

Game	First five shots	Likelihood (no hot hand)	Likelihood (hot hand)
1	BMMBB	$p_B^3(1-p_B)^2$	$(p_B)(1-p_{B\|B})(1-p_B)(p_B)(p_{B\|B}) = (p_B)^2(1-p_{B\|B})(1-p_B)(p_{B\|B})$
2	MBMBM	$p_B^2(1-p_B)^3$	$(1-p_B)(p_B)(1-p_{B\|B})(p_B)(1-p_{B\|B}) = (p_B)^2(1-p_{B\|B})^2(1-p_B)$
3	MMBBB	$p_B^3(1-p_B)^2$	$(1-p_B)(1-p_B)(p_B)(p_{B\|B})(p_{B\|B}) = (p_B)(1-p_B)^2(p_{B\|B})^2$
4	BMMMB	$p_B^2(1-p_B)^3$	$(p_B)(1-p_{B\|B})(1-p_B)(1-p_B)(p_B) = (p_B)^2(1-p_{B\|B})(1-p_B)^2$
5	MMMMM	$(1-p_B)^5$	$(1-p_B)(1-p_B)(1-p_B)(1-p_B)(1-p_B) = (1-p_B)^5$
Total		$p_B^{10}(1-p_B)^{15}$	$(p_B)^7(1-p_B)^{11}(p_{B\|B})^3(1-p_{B\|B})^4$

(b)

```r
likelihood <- function(pb){
  pb^(10) * (1 - pb)^(15)
}
likelihood(0.4)
```

```
[1] 4.930247e-08
```

```r
likelihood(0.3)
```

```
[1] 2.803388e-08
```

When we substitute 0.4 and 0.3 for p_B in the likelihood function, we find that $p_B = 0.4$ produces a higher likelihood than $p_B = 0.3$ (4.93×10^{-8} vs. 2.80×10^{-8}). This means that the data is more consistent with $p_B = 0.4$.

Furthermore, intuitively, if the data shows that the player made 10 baskets out of 25 attempts, it makes sense to estimate the probability of making a basket close to 0.4 (10/25), rather than lower values like 0.3, which are less aligned with the observed data.

(c)

MLE for No Hot Hand Model

The likelihood and log-likelihood functions are:

$$\begin{aligned}
Lik(p_B) &= p_B^{10}(1-p_B)^{15} \\
\log(Lik(p_B)) &= \log(p_B^{10}) + \log((1-p_B)^{15}) \\
&= 10\log(p_B) + 15\log(1-p_B)
\end{aligned}$$

We then take the first derivative of the log-likelihood function and set it to zero:

$$\begin{aligned}
\frac{d}{dp_B} log(Lik(p_B)) &= \frac{10}{p_B} - \frac{15}{1-p_B} = 0 \\
&\Rightarrow \frac{10}{p_B} = \frac{15}{1-p_B} \\
&\Rightarrow 10(1-p_B) = 15p_B \\
&\Rightarrow 25p_B = 10 \\
\hat{p_B} &= \frac{10}{25} = 0.4
\end{aligned}$$

MLE for Hot Hand Model

The likelihood and log-likelihood functions are:

$$\begin{aligned}
Lik(p_B, p_{B|B}) &= (p_B)^7(1-p_B)^{11}(p_{B|B})^3(1-p_{B|B})^4 \\
\log(Lik(p_B, p_{B|B})) &= \log((p_B)^7) + \log((1-p_B)^{11}) + \log((p_{B|B})^3) + \log((1-p_{B|B})^4) \\
&= 7\log((p_B)) + 11\log((1-p_B)) + 3\log((p_{B|B})) + 4\log((1-p_{B|B}))
\end{aligned}$$

We then take the first derivative of the log-likelihood function (with respect to p_B and $p_{B|B}$ separately) and set them to zero:

$$
\begin{aligned}
\frac{d}{dp_B} log(Lik(p_B, p_{B|B})) &= \frac{7}{p_B} - \frac{11}{1-p_B} = 0 \\
&\Rightarrow \frac{7}{p_B} = \frac{11}{1-p_B} \\
&\Rightarrow 7(1-p_B) = 11p_B \\
&\Rightarrow 18p_B = 7 \\
\hat{p_B} &= \frac{7}{18} = 0.3889
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dp_{B|B}} log(Lik(p_B, p_{B|B})) &= \frac{3}{p_{B|B}} - \frac{4}{1-p_{B|B}} = 0 \\
&\Rightarrow \frac{3}{p_{B|B}} = \frac{4}{1-p_{B|B}} \\
&\Rightarrow 3(1-p_{B|B}) = 4p_{B|B} \\
&\Rightarrow 7p_{B|B} = 3 \\
\hat{p_{B|B}} &= \frac{3}{7} = 0.4286
\end{aligned}
$$

Likelihood Ratio Test (LRT)

No Hot Hand Model: p_B

Hot Hand Model: $p_B, p_{B|B}$

Hypothesis:

$$
\begin{aligned}
H_0 &: p_B = p_{B|B} \\
H_a &: p_B \neq p_{B|B}
\end{aligned}
$$

Firstly, we need to plug the MLEs into the log-likelihood function for each model to get the maximum value of the log-likelihood for each model.

```
loglik1 <- function(pb){
  log(pb^(10) * (1 - pb)^(15))
}
loglik1(10/25)
```

```
[1] -16.82529
```

```
loglik2 <- function(pb, pbb) {
  log(pb^7 * (1-pb)^(11) * pbb^3 * (1-pbb)^4)
}
loglik2(7/18, 3/7)
```

```
[1] -16.80883
```

Then, we use the Likelihood Ratio Test to determine if the difference is statistically significant

```
LRT <- 2 * (loglik2(7/18, 3/7) - loglik1(10/25))
LRT
```

```
[1] 0.03292469
```

Therefore, LRT = 0.03292.

The test statistic follows a χ^2 distribution with 1 degrees of freedom (the difference in the number of parameters between the two models). Therefore, the p-value is $P(\chi^2 > LRT)$.

```
pchisq(LRT, 1, lower.tail = FALSE)
```

```
[1] 0.8560131
```

Therefore, p-value = 0.8560.

The p-value is very large ($0.8560 > 0.05$), so we fail to reject H_0. We don't have convincing evidence that the hot hand model (conditional model) is an improvement over the no hot hand model (unconditional model). Thus, **there is no significant evidence that the hot hand exists**.