COMP 360 - Fall 2019 - Assignment 3

Due: Nov 5th 11:59pm

General rules: In solving these questions you may consult books or other available notes, but you need to provide citations in that case. You can discuss high level ideas with each other, but each student must find and write his/her own solution. Copying solutions from any source, completely or partially, allowing others to copy your work, will not be tolerated, and will be reported to the disciplinary office.
You should upload the pdf file (either typed, or a clear and readable scan) of your solution to mycourses.
There are in total 24 points, but your grade will be considered out of 20.

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1. Consider the following linear program with a missing objective function.

$$\begin{array}{lll} \max & ? & \\ \text{s.t.} & x_1 & \geq -4 \\ & x_1 & \leq 4 \\ & x_2 & \geq -4 \\ & x_2 & \leq 4 \\ & x_1 + x_2 & \leq 6 \\ & x_1 + x_2 & \geq -6 \\ & x_1 - x_2 & \leq 6 \\ & x_1 - x_2 & \geq -6 \end{array}$$

   (a) (3 points) Draw the feasible region.

   (b) (1 point) Choose the objective function so that the point $(-4, 2)$ is the unique optimal solution.

   (c) (1 point) Choose the objective function so that the points on the line segment between $(-4, 2)$ and $(-2, 4)$ are optimal solutions (and no other point is an optimal solution).

2. (3 points) We are given a linear program with 5 variables and m linear constraints in the canonical form. Describe a simple polynomial time (a.k.a. efficient) algorithm for solving such a linear program. Explain why your algorithm is a polynomial time algorithm.

3. (3 points) Formulate the following problem as a linear program (do not forget to list your variables): We are given a flow network $(G = (V, E), s, t, \{c_e\}_{e \in E})$ and a budget K for increasing the capacities of the edges, which means that we are allowed to increase the capacities of edges as long as the sum of the amounts added to the capacities is at most K. The goal is to find the maximum possible flow under these restrictions. (The new capacities need not to be integers).

4. (3 points) Explain how the following optimization problem can be solved using linear programming. Here the variables are $x_1, \ldots, x_n$, and furthermore a_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ are given constants.

$$\begin{array}{lll} \max & \frac{x_1+2x_2+3x_3+\ldots+nx_n}{x_1+\ldots+x_n} & \\ \text{s.t.} & x_1 + \ldots + x_n & > 0 \\ & \sum_{j=1}^n a_{ij}x_j & \geq 0 \quad \forall i = 1, \ldots, m \\ & x_i & \geq 0 \quad \forall i = 1, \ldots, n \end{array}$$

5. Consider the following linear program:

$$\begin{array}{lll} \max & \sum_{P \in \mathcal{P}} x_P & \\ \text{s.t.} & \sum_{P: e \in P} x_P \leq c_e & \forall e \in E \\ & x_P \geq 0 & \forall P \in \mathcal{P} \end{array}$$

(Here $\sum_{P:e \in P}$ means that the sum is over all paths P that contain the edge e).

(a) (2 points) Explain why the following linear program solves the MAX-Flow problem:

(b) (1 points) Write the dual of the above linear program.

(c) (1 points) Prove that every $s-t$-cut (A, B) provides a feasible solution to the dual linear program of Part (b) such that the value of the dual linear program equals to the capacity of the cut. Thus conclude that the solution to the dual linear program is at most the capacity of the min-cut.

6. (2 points) Use the complementary slackness to show that $x_1^* = x_3^* = 0.5$, $x_2^* = x_4^* = 0$, $x_5^* = 2$ is an optimal solution for the following Linear Program:

$$\begin{array}{ll} \max & 3.1x_1 + 10x_2 + 8x_3 - 45.2x_4 + 18x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 - x_4 + 2x_5 \leq 5 \\ & 2x_1 - 4x_2 + 1.2x_3 + 2x_4 + 7x_5 \leq 16 \\ & x_1 + x_2 - 3x_3 - x_4 - 10x_5 \leq -20 \\ & 3x_1 + x_2 + 3x_3 + \frac{3}{2}x_4 + \frac{7}{3}x_5 \leq 10 \\ & x_2 + x_3 + 6x_4 + 2x_5 \leq 4.5 \\ & 2x_2 - x_4 + x_5 \leq 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

7. (4 points) There are m raw materials, and n mixtures, each composed of different proportions of these raw materials. For each $1 \leq i \leq n$, we are given m numbers $a_{i1}, a_{i2}, \ldots, a_{im} \in [0, 1]$ meaning that a_{ij} fraction of the i-th mixture is from the raw material j (These m numbers add up to 1). The number $u_i = \max_{j=1}^m a_{ij} - \min_{j=1}^m a_{ij}$ is the *unbalancedness* of the i-th mixture.

   We would like to make a mixture of these mixtures that is as a balanced as possible (minimize its unbalancedness). In other words we want to find $x_1, \ldots, x_n \in [0, 1]$, such that if in our new mixture we take x_i fraction from the i-th mixture, then the final mixture will have the smallest possible unblanacedness.

   Explain how this problem can be modeled as a linear program.