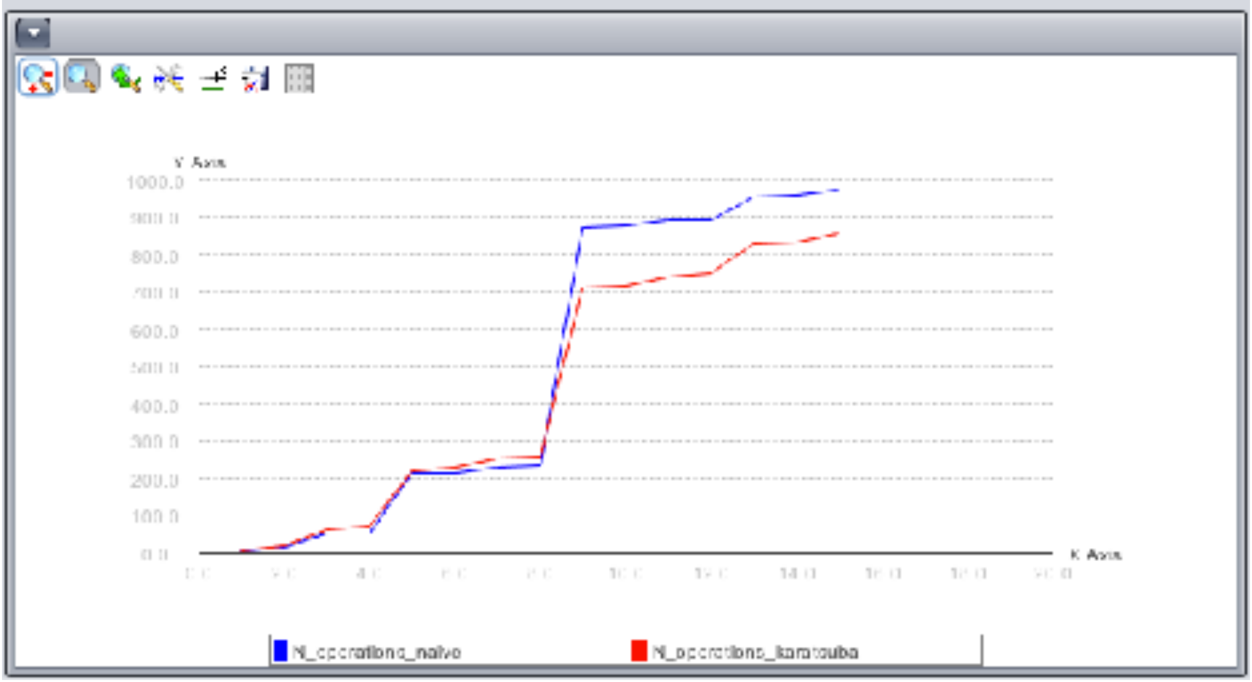

This graph shows that the naive algorithm is faster for smaller bit sizes but as the bit size grows, it becomes less efficient to use than the Karatsuba algorithm (i.e. the Karatsuba algorithm is asymptotically faster than the naive algorithm). This is because there are less recursive calls in the Karatsuba algorithm so when the bit size grows the number of recursive calls starts to add up and makes the naive algorithm slower.

Question 2

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Master Theorem:

If $T(n) = aT(n/b) + \Theta(n^k)$ where $a>0$, $b>1$, and $k \geq 0$

$$T(n) = \begin{cases} \Theta(n^k) & \text{if } k > \log_b a \\ \Theta(n^k \log n) & \text{if } k = \log_b a \\ \Theta(n^{\log_b a}) & \text{if } k < \log_b a \end{cases} \Leftarrow$$

$T(n) = aT(n/b) + f(n)$

$$T(n) = \begin{cases} \Theta(n^k) & \text{if } f(n) = O(n^{k-\varepsilon}), \varepsilon > 0 \\ \Theta(n^k \log^{p+1} n) & \text{if } f(n) = \Theta(n^k \log^p n) \\ \Theta(f(n)) & \text{if } f(n) = \Omega(n^{k+\varepsilon}), \varepsilon > 0 \text{ and if } a(f(n/b)) \leq cf(n), c<1 \end{cases}$$

(a) $T(n) = 25 \cdot T(\frac{n}{5}) + n$

$a = 25, b = 5, f(n) = n, k = \log_5 25 = 2$

$\varepsilon = 2-1$ ($f(n) = O(n^{2-(2-1)}) = O(n)$

$\therefore T(n) = \Theta(n^2)$

(b) $T(n) = 2 \cdot T(\frac{n}{3}) + n\log n$

$a = 2, b = 3, f(n) = n^1 \log^1 n, k = \log_3 2$

$\therefore T(n) = \Theta(n \log n)$

Case 3: 1st. $f(n) = \Omega(n^{\log_3 2 + (1 - \log_3 2)})$

let $\varepsilon = 1 - \log_3 2$ ✓

2nd. $2 \cdot (\frac{n}{3} \log(\frac{n}{3})) \leq c \cdot n \log n$

$c = \frac{2}{3} < 1$ ✓

(c) $T(n) = T(\frac{3n}{4}) + 1$

$a = 1, b = \frac{4}{3}, k = 0$

$\log_{4/3} 1 = 0 = k$

$\therefore T(n) = \Theta(n^0 \log n) = \Theta(\log n)$

$a = 1, b = \frac{4}{3}, f(n) = 1$

$f(n) = \Theta(n^0 \log^0 n)$

$\therefore T(n) = \Theta(n^0 \log^1 n) = \Theta(\log n)$

(d) $T(n) = 7 \cdot T(\frac{n}{3}) + n^3$

$a = 7, b = 3, k = 3$

$\log_3 7 \approx 1.77124374\ldots < k$

$\therefore T(n) = \Theta(n^3)$

$a = 7, b = 3, f(n) = n^3, k = \log_3 7$

1st. $f(n) = \Omega(n^{\log_3 7 + (2 - \log_3 7)})$

let $\varepsilon = 2 - \log_3 7$ ✓

2nd. $7 \cdot (\frac{n}{3})^3 \leq c \cdot n^3$

$\frac{7}{9} \cdot n^3 \leq c \cdot n^3$

so $c = \frac{7}{9} < 1$ ✓

$\therefore T(n) = \Theta(n^3)$

(e) $T(n) = T(n/2) + n(2 - \cos n)$

$= 2n - n\cos n$

$a = 1, b = 2, f(n) = n(2 - \cos n), k = \log_2 1 = 0$

1st. $f(n) = \Omega(n^{k+\varepsilon}) = \Omega(n^{0+1}) = \Omega\, n$ ✓

Let $\varepsilon = 1$

*note: $O(n(2-\cos(n))$ is basically $O(n)$

2nd. $1 \cdot (\frac{n}{2})(2 - \cos\frac{n}{2}) \leq c \cdot n(2 - \cos n) \Rightarrow (\frac{n}{2}) \leq c \cdot n$

$n - \frac{n}{2}\cos\frac{n}{2} \leq c \cdot (2n - n\cos n)$

$\Rightarrow c \geq \frac{3}{2} \nless 1 \therefore$ not solvable by Master Thm!!

Consider $n = 2\pi k \Rightarrow 2\pi k - \pi k \cos(\pi k) \leq c \cdot (4\pi k - 2\pi k \cos 2\pi k)$

Question 3

T_A and T_B are two functions returning running time of alg's A and B defined by the recursions $T_A(n) = 7T_A(\frac{n}{2}) + n^2$ and $T_B(n) = \alpha T_B(\frac{n}{4}) + n^2$

→ largest value of α for which alg B is asympotically faster than A

$T_A(n) = 7T_A(\frac{n}{2}) + n^2$

$a = 7, b = 2$

$\log_2 7 \approx 2.807354922057 60 > 2$

CASE 1:

$a = 7, b = 2, f(n) = n^2, K = 2.807\ldots$

$\varepsilon = 0.807\ldots$ satisfies $O(n^{k-\varepsilon})$

Then, $T(n) = \Theta(n^k)$

$K = \log_2 7$

$\therefore T(n) = \Theta(n^{\log_2 7})$

$T_B(n) = \alpha T_B(\frac{n}{4}) + n^2$

$a = \alpha, b = 4, k = 2$

$f(n)$ is the same ⇒ also case 1

So need to find α such that

$\Theta(n^{\log_4 \alpha}) < \Theta(n^{\log_2 7})$

So $\log_4 \alpha < \log_2 7$

let $\alpha = 48$

$\log_4 48 \approx 2.79248125\ldots$ which is less than $\log_2 7$

let $\alpha = 49$

$\log_4 49 \approx 2.80735492\ldots$ ← too high!

⇒ CASE 1: (check)

$a = 48, b = 4, f(n) = n^2, K = \log_4 48$

$\varepsilon = 0.79248\ldots$ satisfies $O(n^{k-\varepsilon})$

Then, $T(n) = \Theta(n^k)$

$k = \log_4 48$

$\therefore T(n) = \Theta(n^{\log_4 48})$

which is asypotically faster ~~than T_A~~

Therefore, $\alpha = 48$