Entanglement Entropy & Ads/CFT Correspondence

1 Entropy

Here we introduce two kinds of "Normal" entropy, which can be used later to define entanglement entropy.

1.1 Von Neumann entropy

First we consider the density matrix of a given state Ψ,

$$\rho = |\Psi\rangle\langle\Psi|.$$

The matrix element is given by,

$$\rho_{ij} = \langle i|\Psi\rangle\langle\Psi|j\rangle$$

for some states $|i\rangle, |j\rangle$. For simplicity we first consider diagonalised density matrix, say that,

$$\rho = \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_n \end{pmatrix} \tag{1.1}$$

and the trace is normalised so that, $\mathrm{Tr}\,\rho = 1$. If there is one of the $\rho_i = 1$ for $i = 1 \ldots, n$, i.e.

$$\rho = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

then ρ is said to be *pure*. Another extreme case is when $\rho_1 = \cdots = \rho_i = \cdots = \rho_n = \frac{1}{n}$. In matrix form,

$$\rho = \begin{pmatrix} \frac{1}{n} & & & \\ & \ddots & & \\ & & \frac{1}{n} & \\ & & & \frac{1}{n} \end{pmatrix}$$

In this case, ρ is said to be *maximally mixed.* Now we formally define the Von Neumann entropy.

Von Neumann entropy For a given density matrix, the Von Neumann entropy S is given by,

$$S = -\operatorname{Tr} \rho \log \rho \tag{1.2}$$

The Von Neumann entropy, is the quantum version of the thermodynamics entropy, and quantifies the lack of information about a system. For example, we can understand this by applying the Von Neumann entropy to pure and maximally mixed states respectively.

For pure state, $\log \rho_{\text{pure}}$ is simply 0. therefore,

$$S(\rho_{\text{pure}}) = 0.$$

For the maximally mixed state,

$$\log \rho_{\max} = \begin{pmatrix} -\log n & & \\ & \ddots & \\ & & -\log n \end{pmatrix}$$

so we have,

$$\rho \log \rho = \begin{pmatrix} \frac{-\log n}{n} & & \\ & \ddots & \\ & & \frac{-\log n}{n} \end{pmatrix}.$$

Therefore,

$$S(\rho_{\max}) = \log n.$$

We can picture this using the 1 dimensional function $f(x) = -x \log x$, $x \in [0, 1]$. Since $f'(x) = -\log x + 1$, $f(x)$ has a maximum at $x = \frac{1}{e}$, where $f_{\max} = \frac{1}{e}$. The graph of $f(x)$ is shown bellow,

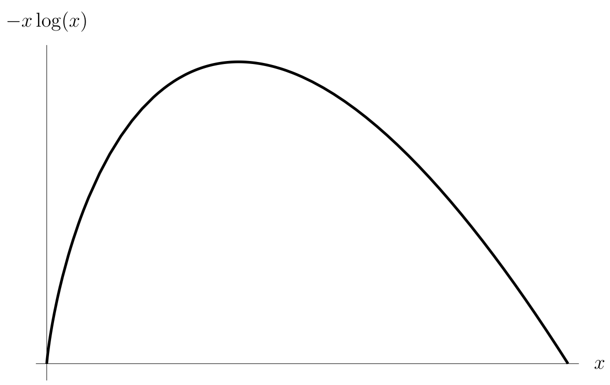

Figure 1

Clearly, the information stored in the pure state can be completely recovered, since there is only one state. For the maximally mixed state, the probability to get each state is equal, the "lack of information" reaches its maximum, so does the Von Neumann entropy.

1.2 Rényi Entropy

Von Neumann entropy is well enough to quantify the entropy of a given system though, it is not always easy to evaluate it due to the logarithm in the expression. Now we introduce the Rényi entropy, which cannot be interpreted as the quantum version of the thermodynamics entropy.

Let us return to the two examples we considered, and consider an arbitrary power α of the density matrix ρ^α. For the pure state, nothing happens, and $S(\rho_{\text{pure}}) = 0$. As for the Maximally mixed case,

$$\rho^\alpha = \begin{pmatrix} \frac{1}{n^\alpha} & & & \\ & \ddots & & \\ & & \frac{1}{n^\alpha} & \\ & & & \frac{1}{n^\alpha} \end{pmatrix},$$

then we have $\operatorname{Tr} \rho^\alpha = \frac{1}{n^{\alpha-1}}$, and therefore,

$$-\log \operatorname{Tr} \rho^\alpha = (\alpha - 1) \log n.$$

Dividing both sides with the extra factor $(\alpha - 1)$,

$$-\frac{1}{\alpha - 1} \log \operatorname{Tr} \rho^\alpha = \log n = S(\rho_{\max})$$

we recover the Von Neumann entropy. Inspired by this, we define the Rényi entropy,

Rényi entropy

$$S_\alpha = -\frac{1}{\alpha - 1} \log \operatorname{Tr} \rho^\alpha, \tag{1.3}$$

and by taking the limit $\alpha \to 1$, we retrieve the Von Neumann entropy,

$$S = \lim_{\alpha \to 1} S_\alpha = \lim_{\alpha \to 1} -\frac{1}{\alpha - 1} \log \operatorname{Tr} \rho^\alpha = -\operatorname{Tr} \rho \log \rho. \tag{1.4}$$

2 Entanglement Entropy

Now we consider a system consists of two part A and B, the Hilbert space can be written as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The entanglement entropy of either of the subsystem is defined using the reduced density matrix.

Reduced density matrix The reduced density matrix ρ_A of a composite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is defined as,

$$\rho_A := \operatorname{Tr}_B \rho. \tag{2.1}$$

More explicitly, let $|a\rangle \in \mathcal{H}_A$, $|b\rangle \in \mathcal{H}_B$, the states of the whole system is given by,

$$|a\rangle \otimes |b\rangle \equiv |a, b\rangle.$$

The partial trace over B is given by,

$$\rho_A = \sum_b \langle a', b|\rho|a, b\rangle. \tag{2.2}$$

Entanglement entropy The entanglement entropy of the subsystem A is the same as the Von Neumann entropy, but tracing over the reduced density matrix ρ_A,

$$S_A = -\lim_{\alpha\to 1}\frac{1}{\alpha-1}\log \operatorname{Tr}\rho_A^\alpha = -\operatorname{Tr}\rho_A\log\rho_A \tag{2.3}$$

2.1 Path Integral Formulation

Here I skip the derivation of the partition function using the path integral, (may be write in other notes, some other day) but merely state the results. The partition function without source Z is given by,

$$Z = \int \mathcal{D}\varphi e^{iS[\varphi]} \tag{2.4}$$

in the associated Euclidean theory,

$$Z = \operatorname{Tr} e^{-\beta\hat{H}} \equiv \int \mathcal{D}\varphi e^{-S_E[\varphi]},$$

where S_E is the Euclidean action of the original action S. The density matrix can be formally written as, normalised by the partition function

$$\rho(\varphi'(x,0),\varphi''(x,\beta)) = \langle\varphi''(x,\beta)|\frac{e^{-H\tau}}{Z}|\varphi'(x,0)\rangle \tag{2.5}$$

$$= \frac{1}{Z}\int \mathcal{D}\varphi \prod_x \delta(\varphi(x,0)-\varphi'(x'))\prod_x \delta(\varphi(x,\beta)-\varphi''(x''))e^{-S_E} \tag{2.6}$$

where τ is the Euclidean time, defined by Wick rotation $t=-i\tau$. In the thermodynamical theory, τ can be identified as the inverse temperature β. Consider the path integral on the interval $\tau\in[0,\beta]$, the path integral corresponds to integrating all field configurations from $\tau=0$ to $\tau=\beta$, this is schematically shown bellow,

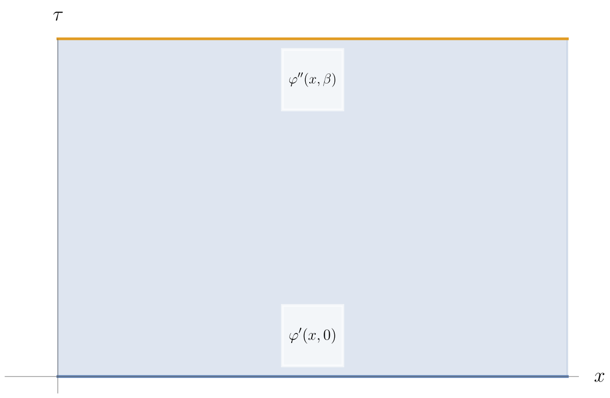

Figure 2: Path integral representation of density matrix

$\operatorname{Tr}\rho=1$ is normalised by Z. This can be found by setting $\{\varphi'\}=\{\varphi''\}$ and integrating over them,

$$\begin{aligned}\operatorname{Tr}\rho &= \frac{1}{Z}\int \mathcal{D}\varphi' \int \mathcal{D}\varphi \prod_x \delta(\varphi(x,0)-\varphi'(x')) \prod_x \delta(\varphi(x,\beta)-\varphi'(x'))e^{-S_E} \\ &= \frac{1}{Z}\int \mathcal{D}\varphi e^{-S_E} \\ &= 1.\end{aligned}$$

Geometrically, tracing over $\{\varphi'\}$ has the effect of sewing $\tau = 0$ and $\tau = \beta$ together, and integrate over the surface of a cylinder of radius β.

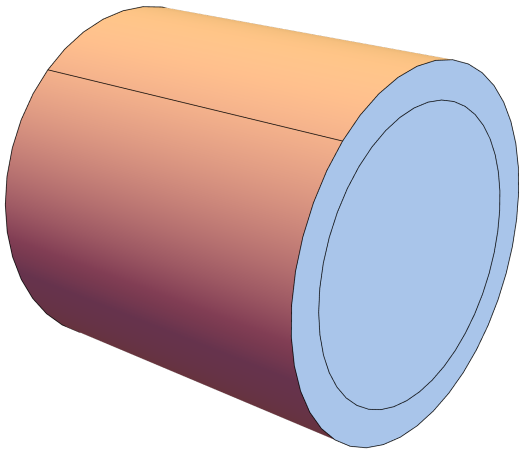

Figure 3: Tracing over φ'

In order to calculate Rényi entropy, we need to know what is the effect of tracing over arbitrary power of the density matrix, i.e $\operatorname{Tr}\rho^n$. To start with, consider $\operatorname{Tr}\rho^2$. The matrix element of ρ^2 can be written as,

$$\langle\varphi''(x'')|\rho^2|\varphi'(x')\rangle = \int \mathcal{D}\varphi''' \langle\varphi''(x'')|\rho|\varphi'''\rangle\langle\varphi'''|\rho|\varphi'(x')\rangle.$$

Therefore $\operatorname{Tr}\rho^2$ is to identify $\varphi'(x')$ and $\varphi''(x'')$,and integrating over all φ'. This has the effect of identifying the top of the first rectangle with the bottom of the second rectangle, and sewing them together into a cylinder with radius 2β. So we can conclude that, for general power n, $\operatorname{Tr}\rho^n$ is equivalent to integrating over all field configurations on a cylinder of radius $n\beta$. If we define path integral on the n-patch cylinder as $Z_n(\rho)$, we can write,

$$\operatorname{Tr}\rho^n = \frac{Z_n(\rho)}{Z^n}. \tag{2.7}$$

For entanglement entropy, the reduced density matrix ρ_A can be obtained by identifying the field configurations of $\mathcal{H}_B$ at $(x,0)$ and (x,β), and sewing the fields of B together with A part left open.

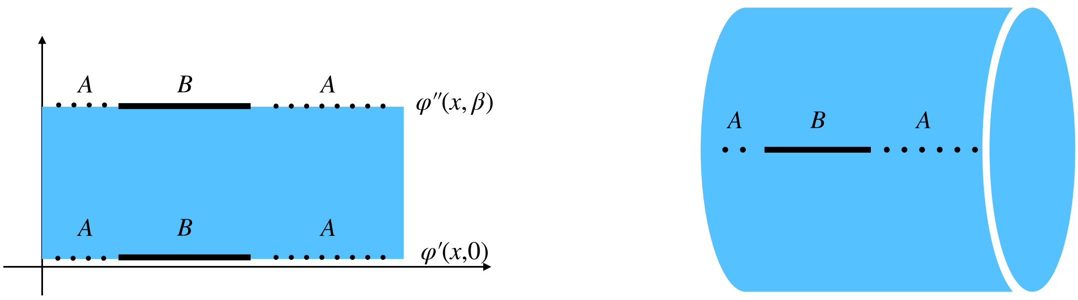

Figure 4: Reduced density matrix ρ_A

To consider $\operatorname{Tr}\rho_A^n$, again, we start with $\operatorname{Tr}\rho_A^2$. The matrix element simply reads,

$$\int \mathcal{D}A' \langle A''(x,\beta)|\rho_A|A'\rangle\langle A'|\rho_A|A(x,0)\rangle$$

For a infinite system, let $\beta \to \infty$, the cylinder becomes the whole (τ, x) plane. Then $\operatorname{Tr}\rho^2$ can be identified as two sheets of $\mathbb{R}^2$ sewn together.

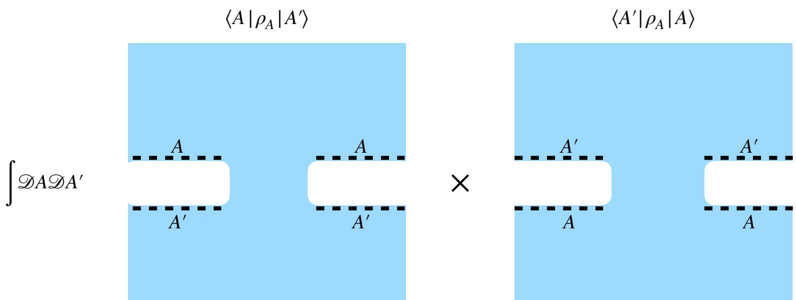

Figure 5: $\operatorname{Tr}\rho_A^2$ showed schematically

For general power n, the n-sheeted $\mathbb{R}$ emerges – n sheets of $\mathbb{R}$ sewed together sequentially, and the last one is sewed back to the first one.

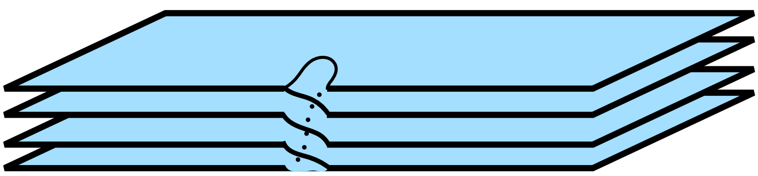

Figure 6: $\operatorname{Tr}\rho_A^n$ as a n-sheeted plane sewed together.

The entanglement entropy in this case is

$$S_A = -\lim_{n\to 1} \log \mathrm{Tr}\, \rho_A^n = -\lim_{n\to} \log \frac{Z_n(A)}{Z^n}$$

2.2 Entanglement entropy of 2D CFT

With the discussion of multi-sheeted $\mathbb{R}$ done, we have come to the point to introduce a specific example. Here we consider a 1+1 dimensional CFT with central charge c, and at zero temperature, i.e. $\beta \to \infty$. The lattice spacing is denoted as a, the total length of the system L, and the length of the subsystem A is l.

Infinite system First we consider a infinitely long system, i.e. $L \to \infty$. Let us consider the following conformal transformation,

$$\zeta : w \mapsto \zeta(w) = \frac{w-u}{w-v}$$

where w denotes the point on the original n-sheeted manifold, here we call it $\mathcal{M}_{n,a,b}$, and $[a, b]$ is the segment of the branch cut. So ζ maps the branch cut from $[u, v]$ to $[0, \infty]$.

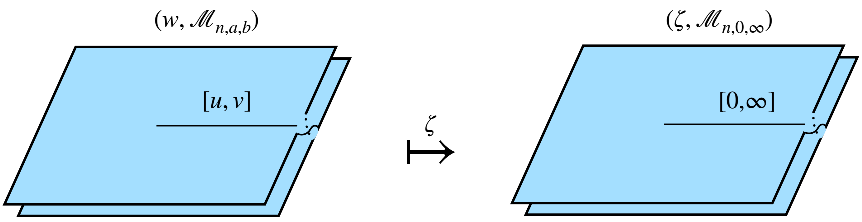

Figure 7: conformal mapping ζ

Next we consider $z : \zeta \mapsto z(\zeta) = \zeta^{\frac{1}{n}}$. This mapping normalises the n-sheets onto the complex plane $\mathbb{C}$. Thus, The composite mapping $z \circ \zeta$ maps w into the complex plane. Let the holomorphic part of the stress-energy tensor on the n-sheeted manifold be $T(w)$. under the finite conformal transformation $z \circ \zeta(w)$, $T(w)$ transforms as,

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\}, \tag{2.8}$$

and the Schwarzian $\{z, w\}$ is defined as,

$$\{z, w\} = \frac{z'''z' - \frac{3}{2}z''^2}{z'^2}.$$

Taking the expectation value of $T(w)$, due to dimensional reason – the expectation value should be dimensionless, the dimensionful part on the right hand side of (2.8) should vanish, which means

$\langle T(z)\rangle_{\mathbb{C}} = 0$. The remaining part is the Schwarzian.

$$\begin{aligned}\langle T(w)\rangle_{\mathcal{M}_{n,u,v}} &= \frac{c}{12}\{z,w\} \\ &= \frac{c}{12} \times \frac{\left(n^2-1\right)(u-v)^2}{2n^2(u-w)^2(v-w)^2} \\ &= \frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{(u-v)^2}{(w-u)^2(w-v)^2}.\end{aligned} \tag{2.9}$$

To understand the meaning of (2.9), we need to introduce a set of artificial fields $\mathcal{T}(u,0), \tilde{\mathcal{T}}(v,0)$ called the twist field. We can consider twist fields as conformal primary operators. These fields are defined on the branch cut $[u,v]$ of $\mathcal{M}_{n,u,v}$. For now, we only use the fact that twist fields are primary operators, and we will summarise the meaning of twist fields later.

Consider the expectation value of the product of the stress-energy tensor $T(w)$ and two twist fields $\mathcal{T}(u,0), \tilde{\mathcal{T}}(v,0)$ on $\mathbb{C}$. The result can be obtained using the Ward identity.

$$\langle T(w)\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}} = \left(\frac{h}{(w-u)^2} + \frac{\tilde{h}}{(w-v)^2} + \frac{\partial_u}{w-u} + \frac{\partial_v}{w-v}\right)\langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}, \tag{2.10}$$

where, $h, \tilde{h}$ are the conformal weights of $\mathcal{T}(u,0), \tilde{\mathcal{T}}(v,0)$ respectively. The correlation function of two primary operators can be determined using conformal invariance. The argument goes as follows:

Correlation function of two primary fields Consider the correlation function of two arbitrary primary fields $\phi_1(x_1), \phi_2(x_2)$, with conformal scaling dimensions Δ_1, Δ_2. Under general conformal transformation $x \mapsto x'$, the correlation function transforms as

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_1/d}\left|\frac{\partial x_2'}{\partial x_2}\right|^{\Delta_2/d}\langle\phi_1(x_1')\phi_2(x_2')\rangle, \tag{2.11}$$

where d is the dimension of the theory. Specifically, for scale transformation $x \mapsto \lambda x$,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\phi_1(x_1')\phi_2(x_2')\rangle. \tag{2.12}$$

Rotation and translation invariance requires the correlation function depends only on relative positions,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1-x_2|), \tag{2.13}$$

combining (2.12) and (2.13), we have,

$$f(|x_1-x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1-x_2|). \tag{2.14}$$

Therefore, the correlation function has the form,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}. \tag{2.15}$$

The remaining conformal transformation is the specical conformal transformation (SCT),

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2\boldsymbol{b}\cdot\boldsymbol{x}+b^2x^2)^d}. \tag{2.16}$$

The distance between two points $|x_i - x_j|$ transforms under SCT as,

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{\gamma_i^{1/2}\gamma_j^{1/2}}, \tag{2.17}$$

where, $\gamma_{i,j} = 1 - 2\boldsymbol{b}\cdot\boldsymbol{x}_{i,j} + b^2 x_{i,j}^2$. To check this, recall that under finite SCT, $\boldsymbol{x}$ transforms as,

$$\boldsymbol{x}' = \frac{\boldsymbol{x} - \boldsymbol{b}x^2}{\gamma}, \tag{2.18}$$

again, γ is defined as before with the subscript suppressed. The square norm of $\boldsymbol{x}$ can be computed,

$$|x'|^2 = \frac{x^2 - 2\boldsymbol{b}\cdot\boldsymbol{x}x^2 + b^2x^4}{\gamma^2} = \frac{|x|^2\gamma}{\gamma^2} = \frac{|x|^2}{\gamma}. \tag{2.19}$$

Also, we can compute the inner product of two vectors,

$$\boldsymbol{x}_i' \cdot \boldsymbol{x}_j' = \frac{\boldsymbol{x}_i\cdot\boldsymbol{x}_j - \boldsymbol{x}_j\cdot\boldsymbol{b}x_i^2 - \boldsymbol{x}_i\cdot\boldsymbol{b}x_j^2 + b^2x_i^2x_j^2}{\gamma_j\gamma_j} \tag{2.20}$$

Similarly, consider the distance of two points $|x_i - x_j|$,

$$|x_i' - x_j'|^2 = |x_i'|^2 + |x_j'|^2 - 2\boldsymbol{x}_j'\cdot\boldsymbol{x}_j'. \tag{2.21}$$

Applying (2.19),(2.20), we have,

$$|x_i' - x_j'|^2 = \frac{|x_i|^2}{\gamma_i} + \frac{|x_j|^2}{\gamma_j} - 2\frac{\boldsymbol{x}_i\cdot\boldsymbol{x}_j - \boldsymbol{x}_j\cdot\boldsymbol{b}x_i^2 - \boldsymbol{x}_i\cdot\boldsymbol{b}x_j^2 + b^2x_i^2x_j^2}{\gamma_j\gamma_j}. \tag{2.22}$$

After rearranging terms, finally we have,

$$|x_i' - x_j'|^2 = \frac{|x_i - x_j|^2}{\gamma_i\gamma_j}. \tag{2.23}$$

Combining (2.11)-(2.15) and (2.23), and using the invariance under SCT, we have,

$$\begin{aligned} \langle\phi_1(x_1)\phi_2(x_2)\rangle &= \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \\ = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_1/d}\left|\frac{\partial x_2'}{\partial x_2}\right|^{\Delta_2/d}\langle\phi_1(x_1')\phi_2(x_2')\rangle &= \frac{C_{12}(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \end{aligned} \tag{2.24}$$

The only choice to obtain non-vanishing $\langle\phi_1(x_1)\phi_2(x_2)\rangle$ is to set $\Delta_1 = \Delta_2$. So we have,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}}. \tag{2.25}$$

After a somewhat long discussion about the correlation function of two primary fields, we can now go back on track to compute (2.10). Since we are considering 2 dimensional case, if we normalise $\mathcal{T}(u,0), \tilde{\mathcal{T}}(v,0)$ such that $\langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}} = \frac{1}{|u-v|^{2\Delta}}$, where Δ is the scaling dimension of twist fields, for spinless fields Δ satisfies $\Delta = 2h = 2\tilde{h}$. Therefore we can write,

$$\langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}} = \frac{1}{|u-v|^{4h}} = \frac{1}{(u-v)^{2h}(\bar{u}-\bar{v})^{2h}}. \tag{2.26}$$

By Ward identity,

$$\langle T(w)\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}} = \left(\frac{h}{(w-u)^2} + \frac{\tilde{h}}{(w-v)^2} + \frac{\partial_u}{w-u} + \frac{\partial_v}{w-v}\right)\frac{1}{(u-v)^{2h}(\bar{u}-\bar{v})^{2h}}. \quad (2.27)$$

The result turns out to be,

$$\langle T(w)\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}} = \frac{h(\bar{u}-\bar{v})^{-2h}(u-v)^{2-2h}}{(u-w)^2(v-w)^2}. \quad (2.28)$$

Note that,

$$\frac{\langle T(w)\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}}{\langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}} = \frac{h(u-v)^2}{(w-u)^2(w-v)^2}. \quad (2.29)$$

Comparing (2.9) to (2.29), $\frac{\langle T(w)\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}}{\langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}}$ coincides with the expectation value $\langle T(w)\rangle_{\mathcal{M}_{n,u,v}}$, when

$$h = \frac{c}{24}\left(1 - \frac{1}{n^2}\right). \quad (2.30)$$

Due to the property of the twist field, the partition function on $\mathcal{M}_{n,u,v}$ can be expressed as,

$$Z[\mathcal{M}_{n,u,v}] = \langle\mathcal{T}(u,0)\tilde{\mathcal{T}}(v,0)\rangle_{\mathbb{C}}. \quad (2.31)$$

Thus, the trace of the n-th power of the reduced density matrix ρ_A can be determined as,

$$\operatorname{Tr}\rho_A^n = \frac{Z_n(A)}{Z^n} \propto Z^n[\mathcal{M}_{n,u,v}] = c_n\left(\frac{u-v}{a}\right)^{-\frac{c}{6}(n-\frac{1}{n})}. \quad (2.32)$$

The lattice spacing a appears because we have to make the whole thing dimensionless. The power is just $2n\Delta = 4nh$. The dimensionless parameter $l := \frac{u-v}{a}$ is the size of the subsystem A. The entanglement entropy can be obtained by taking the logarithm and the limit $n \to 1$.

$$S_A = \lim_{n\to} S_A^{(n)} \propto -\lim_{n\to}\frac{1}{n-1}\frac{c}{6}\left(n - \frac{1}{n}\right)\log l = \frac{c}{3}\log l \quad (2.33)$$

Finite system Since the partition function can be written in the form (2.31), we can map the two point function into any other geometry by a conformal map. The two point function of two primary field transforms as follows,

$$\langle\Phi(x_1)\Phi(x_2)\rangle_{\mathcal{U}} = \left|\frac{dy_1}{dx_1}\right|^{2\Delta}\left|\frac{dy_2}{dx_2}\right|^{2\Delta}\langle\Phi(y_1)\Phi(y_2)\rangle_{\mathcal{V}}, \quad (2.34)$$

where Δ is the scaling dimension of $\Phi(x_1)$ and $\Phi(x_2)$. To compute the entanglement entropy of a subsystem of length l for a finite 1d system with length L, we consider the following conformal mapping,

$$w = \frac{iL}{2\pi}\log z, \quad (2.35)$$

which maps the complex plane to a cylinder with radius $L/2\pi$. Write $z = re^{i\theta}$, we have,

$$w = \frac{iL}{2\pi}\log r - \frac{L}{2\pi}\theta.$$

Constant radius circles are mapped to segments parallel to the real axis of w-plane.

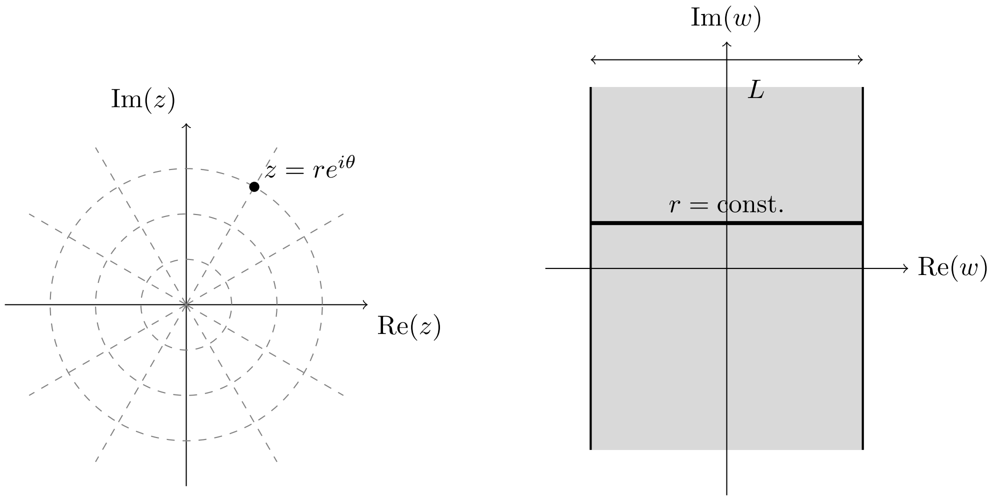

Figure 8: z and w plane.

For two point function, we compute the relation (2.34) for two point function of twist fields,

$$\begin{aligned}\langle \mathcal{T}(w_1)\tilde{\mathcal{T}}(w_2)\rangle_{cyl} &= |z_1'(w_1)|^{4h}\,|z_2'(w_2)|^{4h}\frac{1}{|z_1-z_2|^{4h}}\\ &= \left(\frac{2\pi}{L}\right)^{4h}\frac{e^{-i\frac{2h\pi}{L}(w_1-\bar{w}_1+w_2-\bar{w}_2)}}{(e^{-i\frac{2\pi}{L}(w_1-\bar{w}_1)}+e^{-i\frac{2\pi}{L}(w_2-\bar{w}_2)}-e^{-i\frac{2\pi}{L}(w_1-\bar{w}_2)}-e^{-i\frac{2\pi}{L}(w_2-\bar{w}_1)})^{2h}}.\end{aligned}$$

Writing $w = x + iy$, we have,

$$\begin{aligned} w_j - \bar{w}_j &= 2iy_j,\\ w_1 - \bar{w}_2 &= (x_1 - x_2) + i(y_1 + y_2),\\ w_2 - \bar{w}_1 &= (x_2 - x_1) + i(y_1 + y_2).\end{aligned}$$

Simplifying the two point function on the cylinder,

$$\begin{aligned} &= \left(\frac{2\pi}{L}\right)^{4h}\frac{e^{\frac{2\pi}{L}(y_1+y_2)(2h)}}{[e^{\frac{4\pi}{L}y_1}+e^{\frac{4\pi}{L}y_2}-e^{\frac{2\pi}{L}(y_1+y_2)}(e^{i\frac{2\pi}{L}(x_1-x_2)}+e^{i\frac{2\pi}{L}(x_1-x_2)})]^{2h}}\\ &= \left(\frac{2\pi}{L}\right)^{4h}\frac{e^{\frac{2\pi}{L}(y_1+y_2)(2h)}}{[e^{\frac{4\pi}{L}y_1}+e^{\frac{4\pi}{L}y_2}-e^{\frac{2\pi}{L}(y_1+y_2)}(2\cos(\frac{2\pi}{L}(x_1-x_2)))]^{2h}}.\end{aligned}$$

We choose the image of the branch cut $w_1 - w_2$ lies perpendicular to the axis of the cylinder, which means $y_1 = y_2$. Then we have,

$$\begin{aligned} &= \left(\frac{2\pi}{L}\right)^{4h}\frac{1}{\left(2-2\cos\left(\frac{2\pi}{L}(x_1-x_2)\right)\right)^{2h}}\\ &= \left(\frac{\pi}{L}\right)^{4h}\frac{1}{(\sin(\frac{\pi l}{L}))^{4h}}\end{aligned}$$

So the Renyí entropy is obtained as,

$$-\frac{1}{n-1}\log\rho_A^n = -\frac{1}{n-1}(-4nh)\log\left(\frac{L}{a\pi}\sin\left(\frac{\pi l}{L}\right)\right) + c_1', \tag{2.36}$$

with $h = \frac{c}{24}\left(1-\frac{1}{n^2}\right)$. Taking the limit $n \to 1$, we obtain the entanglement entropy,

$$S_A = \frac{c}{3}\log\left(\frac{L}{a\pi}\sin\left(\frac{\pi l}{L}\right)\right) + c_1' \tag{2.37}$$