THE MODELIING HIFIMAGNET GUIDE

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Contents

1. Notations

1.1. Constants and variable.

ρ	material's density	$kg.m^{-3}$
C_p	thermal capacity	$J.K^{-1}$
r	radius	m
T	temperature	K
k	thermal conductivity	$W.m^{-1}.K^{-1}$
U	electrical potential	V
σ	electrical conductivity	$S.m^{-1}$
h	heat transfer coefficient	$W.m^{-2}.K^{-1}$
A	magnetic potential	$V.s.m^{-1}$
B	magnetic induction	T
H	magnetic field	$A.m^{-1}$

1.2. Conventions.

f**:** function
f: vector of functions

1.3. Functions.

∇f**:** gradient of f
$\nabla . \mathbf{f}$ **:** divergence of **f**
$\nabla \times \mathbf{f}$ **:** curl of **f**
$L_2(\Omega)$**:** $\{f \mid \int f^2 < \infty\}$ Lagrangian
$H_1(\Omega)$**:** $\{f \in L_2(\Omega) \mid \nabla f \in [L_2(\Omega)^d]\}$ Hilbert space

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$H_{div}(\Omega)$**:** $\{\mathbf{f} \in [L^2(\Omega \cup \Omega_c)^d | \nabla.\mathbf{f} \in L^2(\Omega \cup \Omega_c)\}]$ Divergence space
$H_{curl}(\Omega)$**:** $\{\mathbf{f} \in [L^2(\Omega \cup \Omega_c)^d | \nabla \times \mathbf{f} \in [L^2(\Omega \cup \Omega_c)]^d\}]$ Rotational space

2. **Modelisation**

The study of a magnet is a multi-physic problem. It combine electromagnetism, thermic and mechanics. We divide the modeling into 4 parts:

(1) **Thermo-electric**
   The current flow inside the magnet (copper) involves a warming of the material, due to Joule effect. The temperature in the different sections of the magnet are calculated thanks to the heat equation (more details below).
(2) **Elasticity**
   The temperature's elevation involves a dilatation of the material. It implies some constraints due to the nature of the material.
(3) **Hydraulics**
   Due to the strong current flow inside the magnet, the temperature will rise dangerously. We need to cool down the magnet, with fluid which will circulate around the material. This involves the Colburn correlation with the properties of the fluid.
(4) **Electromagnetism**
   The purpose of a magnet is to create a magnetic field, consequently, we will use the Maxwell's equations.

2.1. **Thermo-electric.**

2.1.1. *Equations.* First of all, we start with the standart heat equation:

$$\rho C_p \frac{\partial T}{\partial t} - \nabla.(k\nabla T) = P$$

The only source of heat considered is the Joule effect, translated by :

$$P = \mathbf{j}.\mathbf{E}$$

Where $\mathbf{j} = \sigma\mathbf{E}$ (from the Ohm's law). But $\exists V | \mathbf{E} = -\nabla V$

NOTE: To see more details, Thermo-Electric Toolbox We also consider that T is time independent, so, finally, we have :

$$-\nabla.(k\nabla T) = \sigma\nabla V.\nabla V$$

Coefficients σ and k are temperature-dependent as shown in this equations :

- $\sigma = \frac{\sigma_0}{1+\alpha(T-T_{ref})}$
- $k = k_0 \frac{T}{(1+\alpha(T-T_{ref}))T_{ref}}$

2.1.2. *Finite Element.*

FEM. The Finite Element Method is detailed in the chapter The Mathematics of Feel++

Finally, in our case, the variationnal formulation then consists in finding $V \in X_T = H_1(\Omega) \forall \phi_T \in X_T$:

$$\int_\Omega k(T)\nabla\phi_T + \int_{\partial\Omega_{cooled}} hT\phi_T = \int_\Omega \sigma(T)\nabla V.\nabla V + \int_{\partial\Omega_{cooled}} hT_\omega\phi_T$$

We use the Picard method to solve this non linear variationnal formulation.

Picard method. The Picard method is an iterative method which approach the solution of a differential equation, until the difference between y_{n+1} and y_n becomes lower than an user-defined tolerance.

- $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$
- $y_n = y_0 + \int_{x_0}^x f(s, y_{n-1}(s))ds$

2.1.3. *Boundary conditions.* The current is due to a difference of potential. We model this with the Dirichlet conditions :

- V=0 on V_{in}
- V=V_0 on V_{out}

We consider that the air and the cooling water around the material are electrically insulating. We model that with the Neumann conditions which set the current density flux to null :

$-\sigma(T)\nabla V.n = 0$ for all surfaces except V_{in} and V_{out}

On the surfaces cooled, we set the heat transfer coefficient h, determined as follow :

$h = \frac{k(T)N_u}{D_h}$ with N_u the Nusselt number and D_h the hydraulic diameter.

NOTE: To see some examples (with the code behind), see the chapter Examples

2.2. **Magnetostatic.**

2.2.1. *Equations.* First of all, we start with two of the four Maxwell's equations :

$$\begin{cases} \nabla \times \mathbf{H} = \mathbf{j} \\ \nabla.\mathbf{B} = 0 \end{cases}$$

We also can link the magnetic induction ($\mathbf{B}$) and the magnetic field ($\mathbf{H}$) using the permeability (μ) as :

$$\mathbf{B} = \mu\mathbf{H}$$

with $\mu = \mu_r\mu_0$

- μ_r being the permeability specific to the material
- μ_0 being the vacuum permeability

From the differential operators, we know that $\mathbf{rot}.\mathbf{div}.\mathbf{A} = 0 \ \forall \ \mathbf{A}$.

Thus, since $\nabla.\mathbf{B} = 0$ there exists $\mathbf{A}$ such as $\mathbf{B} = \nabla \times \mathbf{A}$

Finally, we can write :

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{j}$$

2.2.2. *Finite Element.* The Finite Element Method is detailed in the chapter The Mathematics of Feel++

In our case, the variational formulation consists in resolve :

$$\int_{\Omega\cup\Omega_c} \nabla \times \frac{1}{\mu} \nabla \times \mathbf{A}\varphi = \int_{\Omega} \mathbf{j}\varphi$$

with Ω being the volume of our geometry and Ω_c the volume inside (for example the air).

By integration by parts, we now consider this equation :

$$\int_{\Omega\cup\Omega_c} \frac{1}{\mu}\mathbf{A}.\nabla \times \varphi + \int_{\partial\Omega_c} \frac{1}{\mu} \left(\nabla \times \mathbf{A} \times \mathbf{n} \right).\varphi = \int_{\Omega} \mathbf{j}.\varphi$$

$\mathbf{A} \in \{\mathbf{v} \in L^2(\Omega \cup \Omega_c); \nabla \times \mathbf{v} \in L^2(\Omega \cup \Omega_c)\}$ corresponding of H_{curl} .

2.2.3. *Boundary conditions.* The current density is located in some finite region in space, this involves considering **B** as zero at infinity. But, with the Finite Element Method,we discretize the domain so we impose the domain to be of finite dimension. In fact, this domain is composed by Ω (the conductor) and a box inside, Ω_c, whose boundaries model the infinity. The size of this box can be set manually or be calculated for more precision and validity. The boundary condition is expressed with the magnetic potential (**A**) like this :

$$\mathbf{A} \times \mathbf{n} = 0 \, on \, \partial\Omega_c$$

Our materials allow us to only consider the vacuum permeability, so $\mu = \mu_0 = 4\pi.10^{-7}kg.m.A^{-2}.s^{-2}$.

NOTE: To see some examples (with the code behind), see the chapter Examples

2.3. **Linear elasticity.** The objective is to calculate the displacement vector **u** generated by the dilatation due to the temperature elevation.

2.3.1. *Conditions and suppositions.* We need some preconditions :

- The moments chosen are considered to be equilibrium
- The only forces considered are the Lorentz force and the thermal expansion.

2.3.2. *Equations.* First, we start with the equation of motion, becoming the equilibrium equation :

$$div(\bar{\bar{\sigma}}) + \mathbf{f} = 0$$

With $\bar{\bar{\sigma}}$ the stress tensor and **f** the volume forces applied on the conductor. As we search the displacement vector **u**, we introduce the tehsor of small deformation $\bar{\bar{\epsilon}}$

$$\bar{\bar{\epsilon}} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

We divide the stress tensor in two terms : $\bar{\bar{\sigma}}^E$ given by the Hooke's law for the small deformation and $\bar{\bar{\sigma}}^T$ bring by the Joule eating.

$$\bar{\bar{\sigma}}(\bar{\bar{\epsilon}}) = \bar{\bar{\sigma}}^E(\bar{\bar{\epsilon}}) + \bar{\bar{\sigma}}^T(\bar{\bar{\epsilon}})$$

This terms are define so :

- $\bar{\bar{\sigma}}^E(\bar{\bar{\epsilon}}) = \frac{E}{1+\nu}(\bar{\bar{\epsilon}} + \frac{\nu}{1+2\nu}Tr(\bar{\bar{\epsilon}})\bar{\bar{I}})$
- $\bar{\bar{\sigma}}^T(\bar{\bar{\epsilon}}) = -\frac{E}{1+\nu}\alpha_T(T - T_0)\bar{\bar{I}}$

With :

- E the Young modulus
- ν the Poisson's ratio
- $\bar{\bar{I}}$ the identity tensor
- α_T the linear dilatation coefficient
- $T and T_0$ the temperature at time $t and t_0$

2.3.3. *Finite Element.* The Finite Element Methid is detailed in the chapter The Mathematics of Fell++

In our case, the variational formulation consists in resolve :

$$-\int_\Omega (\nabla.\bar{\bar{\sigma}}) \circ \varphi = \int_\Omega \mathbf{f} \circ \varphi$$

With $\circ$ the element-wise product.

Finally, considering this with the equations, we finally have :

$$\frac{E}{1+\nu} \int_\Omega Tr \left(\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T) \right) + \frac{E\nu}{(1+\nu)(1-2\nu)} \int_\Omega (\nabla.\mathbf{u})(\nabla.\varphi) = \int_\Omega \mathbf{f}.\varphi + \int_{\partial\Omega_P} \mathbf{g}.\varphi + \int_\Omega \frac{E\alpha_T}{1-2\nu}(T-T_0)(\nabla.\varphi$$

for all $\varphi \in H^d_{1,\varphi}(\Omega)$

NOTE: To see some examples (with the code behind), see the chapter Examples