`MA40042` Measure Theory and Integration

Problem Sheet 2

*Solutions should be submitted to the `MA40042` pigeonhole by 1700 on **Monday 17 October**.*

Work will be returned and answers discussed in the problems class on Tuesday 18 October.

1. (a) Show that the following are measure spaces. (You may assume Σ is a σ-algebra on X.)
   i. The counting measure $\#$, where X is a nonempty set and Σ its powerset.
   ii. The Dirac measure δ_x at $x \in X$ for some measurable space (X, Σ).
   iii. A discrete measure of the form $\mu(A) = \sum_{x \in A} w_x$ for $w_x \in [0, \infty]$, where X is a countable set and Σ its powerset.
   iv. The function
   $$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ \infty & \text{otherwise,} \end{cases}$$
   where X is a nonempty set, and Σ is the σ-algebra of countable and co-countable subsets.

   (b) Under what conditions are the above measures
   i. a probability measure,
   ii. a finite measure,
   iii. a σ-finite measure?

2. Let (X, Σ, μ) be a measure space.

   (a) Show that μ is finitely additive, in that for $A_1, A_2, \ldots, A_N$ a finite sequence of disjoint sets in Σ, we have
   $$\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n).$$

   (b) Show that μ is monotone, in that for $A, B \in \Sigma$ with $A \subset B$, we have $\mu(A) \leq \mu(B)$.

   (c) Show that μ is countably subadditive, in that for $A_1, A_2, \ldots,$ a countably infinite sequence of sets in Σ, we have
   $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

   (d) Show that μ is finitely subadditive, in that for $A_1, A_2, \ldots, A_N$ a finite sequence of sets in Σ, we have
   $$\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mu(A_n).$$

   (e) Show that for $A \in \Sigma$, we have $\mu(A) + \mu(A^{\mathrm{c}}) = \mu(X)$.

   (f) Show that μ is upwardly continuous, in that for $A_1 \subset A_2 \subset \cdots$ a countably infinite 'expanding' sequence of sets in Σ, we have
   $$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

   (g) Let p_n the probability a branching process has died out by generation n, and let p_{ext} be the probability the process goes extinct. Justify the statement $p_{\text{ext}} = \lim_{n\to\infty} p_n$.

   (h) Show that μ is downwardly continuous, in that for $A_1 \supset A_2 \supset \cdots$ a countably infinite 'contracting' sequence of sets in Σ with $\mu(A_1) < \infty$, we have
   $$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

   (i) Give an example to show that we do need the condition $\mu(A_1) < \infty$ in the previous statement.

3. Given a measure space (X, Σ, μ), we write $\mathcal{Z}$ for the collection of sets with measure zero, and $\mathcal{N}$ for the collection of *null sets* – the subsets of sets with measure zero – so

$$\begin{aligned}\mathcal{Z} &:= \{Z \in \Sigma : \mu(Z) = 0\},\\ \mathcal{N} &:= \{N \subset X : N \subset Z \text{ for some } Z \in \mathcal{Z}\}.\end{aligned}$$

A measure space is called *complete* if every null set is measurable with measure zero; that is, if $\mathcal{N} = \mathcal{Z}$.

Let (X, Σ, μ) be a (not necessarily complete) measure space.

(a) Let $\overline{\Sigma} = \{A \cup N : A \in \Sigma, N \in \mathcal{N}\}$.

  i. Show that $\overline{\Sigma}$ is a σ-algebra.

  ii. Explain why $\overline{\Sigma}$ is the smallest σ-algebra containing Σ and $\mathcal{N}$.

(b) We define a function $\bar{\mu}$ on $\overline{\Sigma}$ as follows: for $A \in \Sigma$ and $N \in \mathcal{N}$, let $\bar{\mu}(A \cup N) = \mu(A)$.

  i. Explain why we need to show that $\bar{\mu}$ is 'well-defined'.

  ii. Show that $\bar{\mu}$ is well-defined.

  iii. Show that $\bar{\mu}$ is a measure on $(X, \overline{\Sigma})$.

  iv. Show that $(X, \overline{\Sigma}, \bar{\mu})$ is a complete measure space.

(c) Let μ^* be the outer measure on X constructed from $\mathcal{R} = \Sigma$ and $\rho = \mu$ in the standard way. Show that, for $A \in \overline{\Sigma}$, we have $\mu^*(A) = \bar{\mu}(A)$.

4. Let X be a nonempty set, $\mathcal{R}$ be a collection of subsets of X including the empty set, ρ a function $\rho\colon \mathcal{R} \to [0, \infty]$ with $\rho(\varnothing) = 0$, and μ^* the outer measure constructed in the standard way from them.

(a) Show that $\mu^*(R) \leq \rho(R)$ for all $R \in \mathcal{R}$.

(b) Give an example of when $\mu^*(R) \neq \rho(R)$.

(c) Show that μ^* is finitely subadditive.

(d) Give an example of when μ^* is *strictly* subadditive on disjoint sets (and hence is not a measure).

5. Show that the Lebesgue outer measure λ^* of any countable set is 0.

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