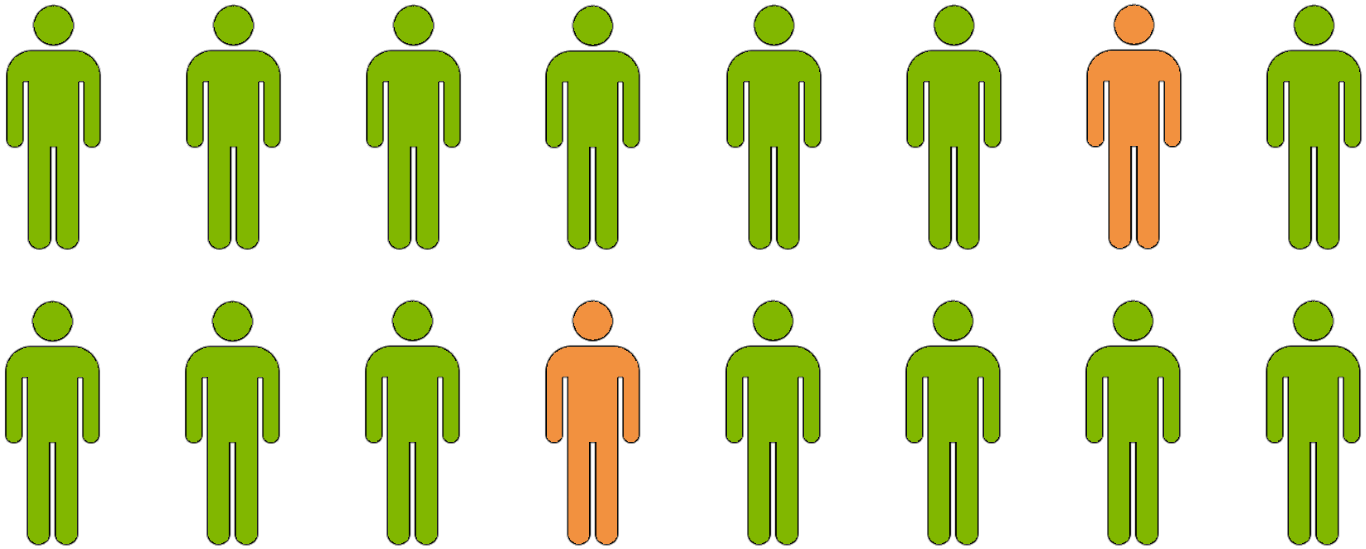
Group Testing

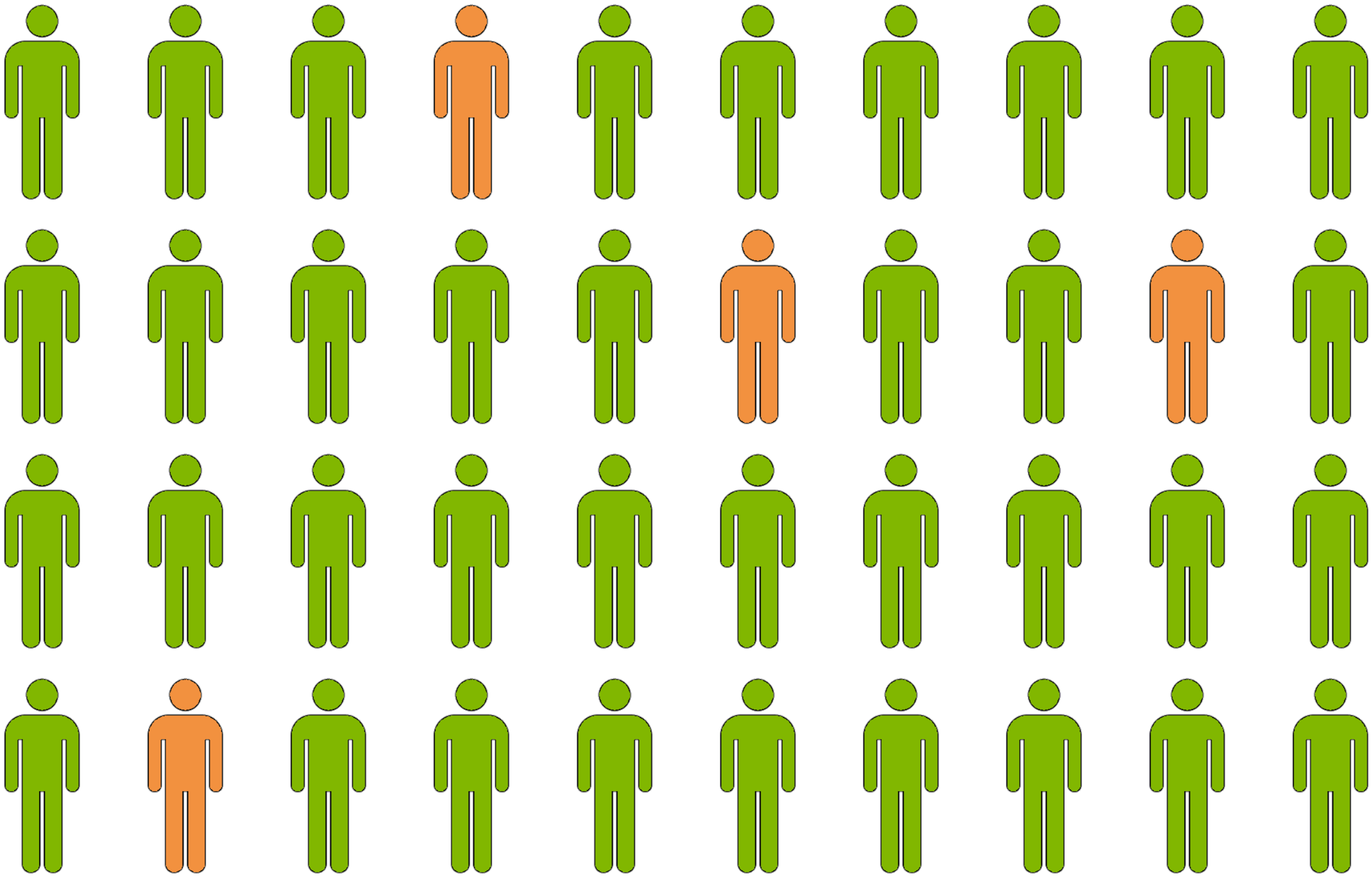
for the Coronavirus



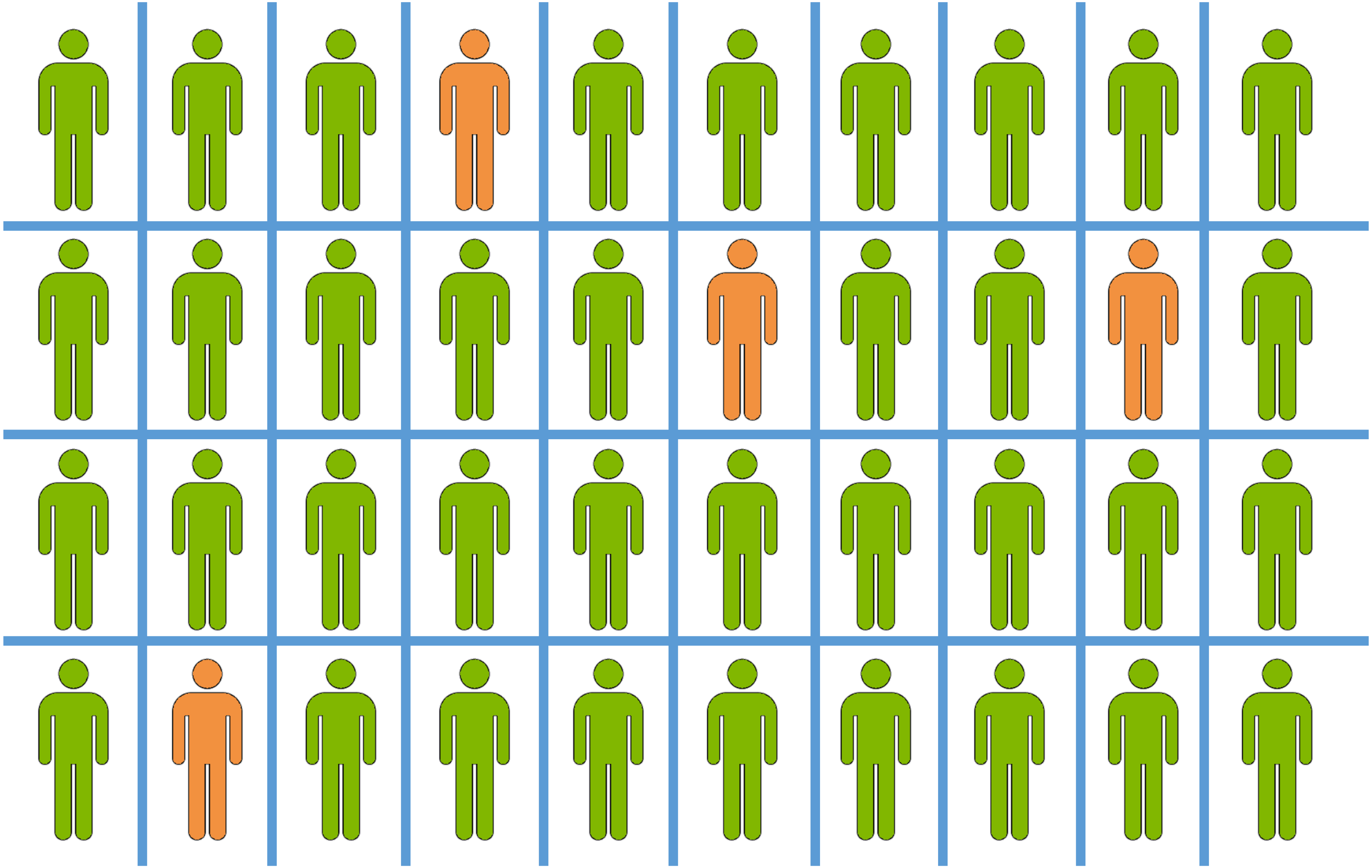
Matthew Aldridge
University of Leeds

Leeds–Liverpool Workshop
June 2020

Group testing



Group testing

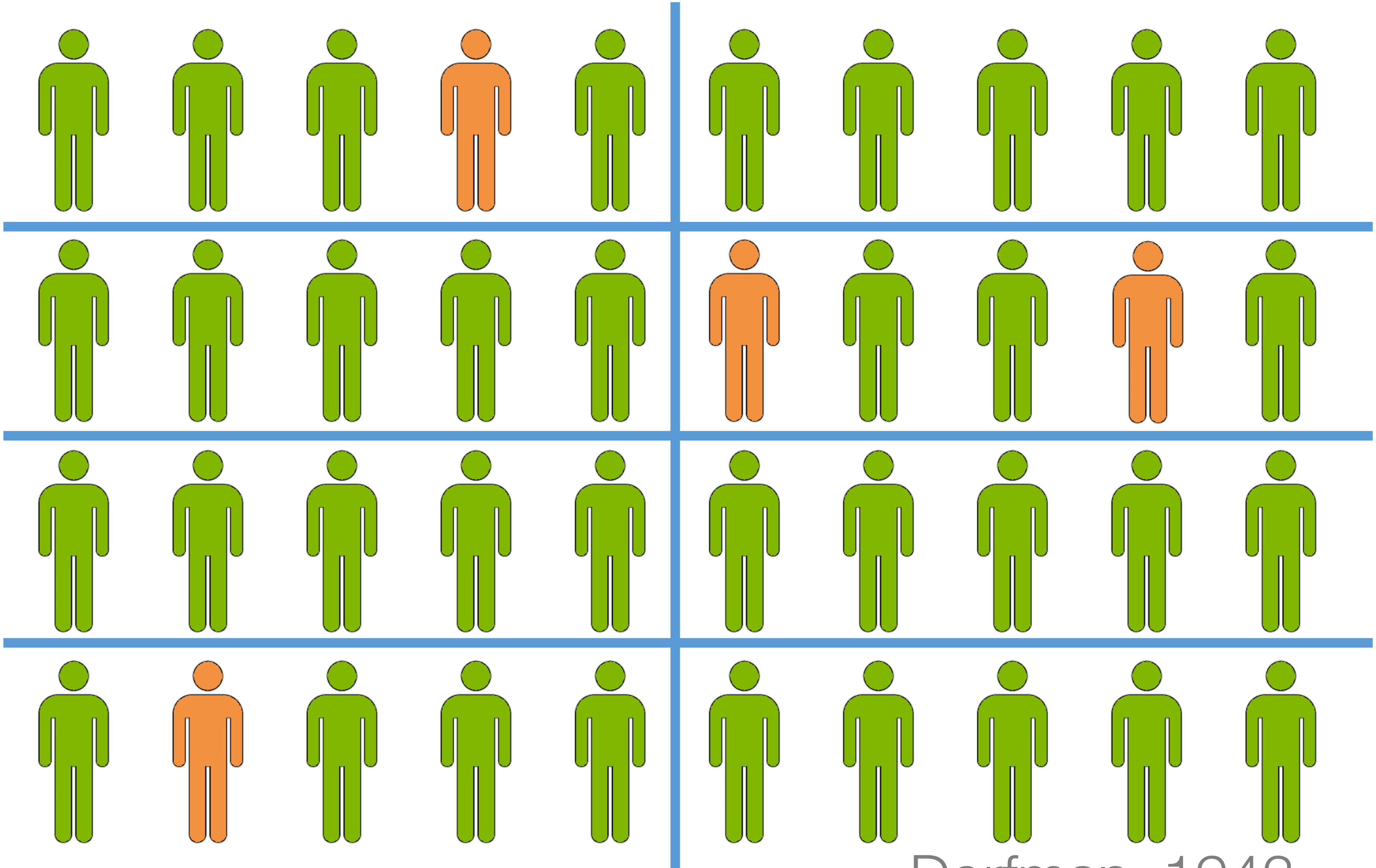


Group testing

Individual testing:

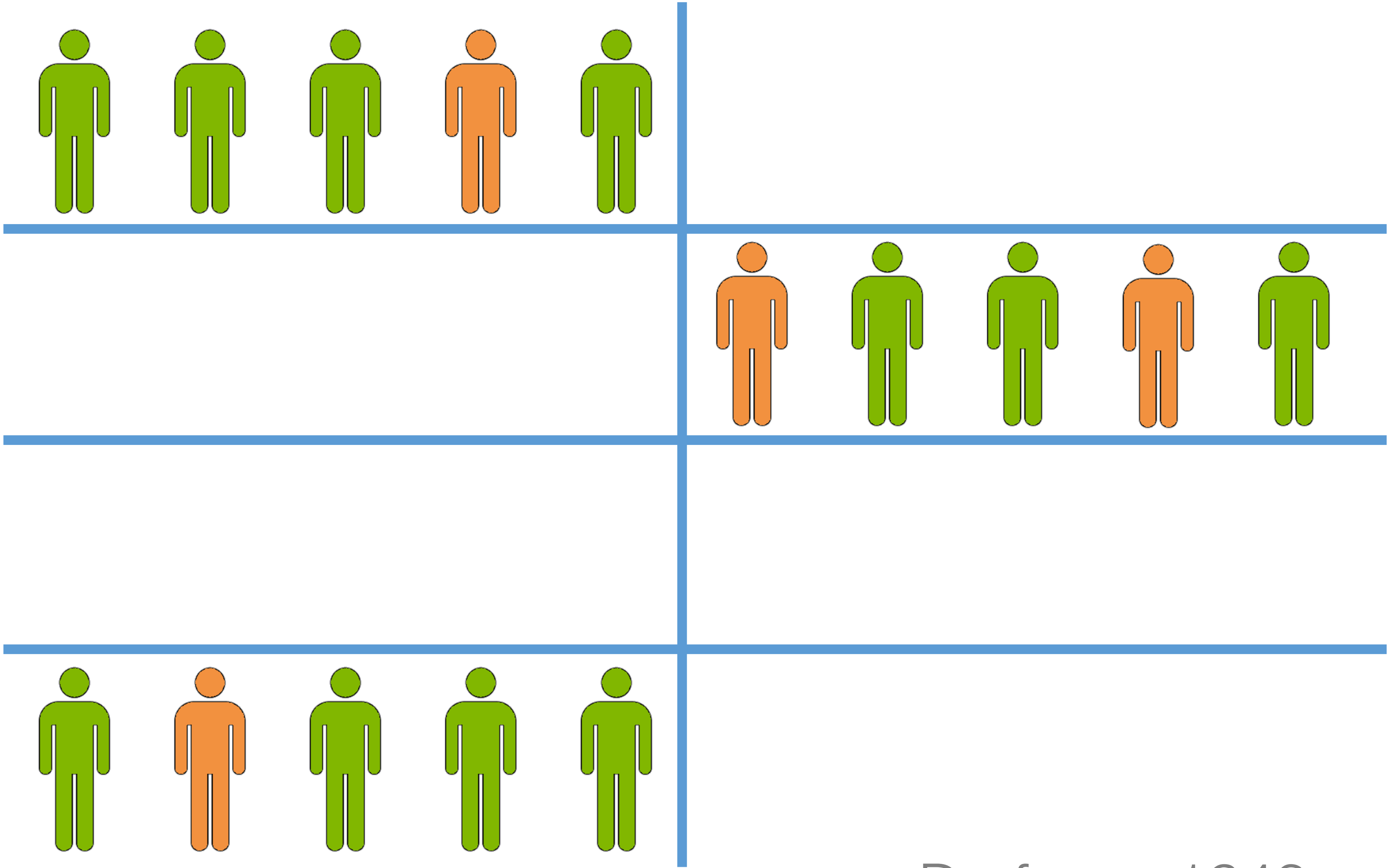
40 soldiers = **40 tests**

Group testing



Dorfman, 1943

Group testing



Dorfman, 1943

Group testing

Individual testing:

40 soldiers = **40 tests**

Dorfman's algorithm:

Stage 1: pooled testing:
8 groups of 5 soldiers = 8 tests

Stage 2: individual testing:
3 groups of 5 soldiers = 15 tests

Total: 8 + 15 = **23 tests**

Group testing

n items (soldiers)

p prevalence of "defective items"
(proportion of soldiers with syphilis)

T tests: *"Does this group of items contain at least one defective item?"* (blood tests)

Group testing

n items

p prevalence

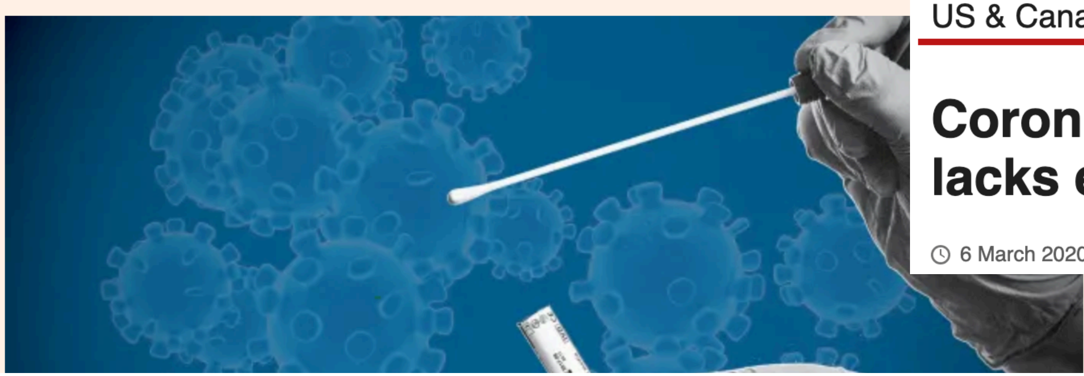
T tests

Given n and p,
how big does T have to be
(on average) to reliably work out
which items were defective?

Coronavirus + Add to myFT

Why the UK is struggling to scale up coronavirus testing

Shift in policy has left British labs facing global competition for equipment



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US & Canada

Coronavirus: White House concedes US lacks enough test kits

6 March 2020

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Coronavirus: Is the UK testing enough people?

1. How have other countries responded to the coronavirus threat?
2. How many people are being tested in the UK?
3. Why are numbers lower than might be expected?
4. What does it mean for health workers?
5. What is being done to increase testing in the UK?

Harry Cockburn | 1 day ago

WIRED

Technology | Science | Culture | Gear | Business | Politic

Biology

Why isn't the UK testing more people for coronavirus?

There are two broad types of tests for coronavirus. Manufacturers of the kits are vastly ramping up their production cycles

It seems there are fewer coronavirus tests available than the number of people it would be useful to test:

Could group testing help make better use of these limited tests?

Forbes

EDITORS' PICK | 2,004 views | Mar 29, 2020, 09:00am EDT

Group Testing Is Our Surefire Secret Weapon Against Coronavirus

Laurence Kotlikoff Cont

Taxes

The Washington Post

Democracy Dies in Darkness

Coronavirus Live updates U.S. map World map

PostEverything • Perspective

A temporary coronavirus testing fix: Use each kit on 50 people at a time.

We don't have enough tests. Group testing offers a way to make best use of them.

Shtetl-Optimized

The Blog of Scott Aaronson

If you take just two pieces of information from this blog: START HUMAN CHALLENGE TRIALS FOR VACCINES AND HOLD THE NOVEMBER US ELECTION BY MAIL

PSPACE PostBQP NP BQP P

Pooled testing for covid: Guest post by Zeph Landau

June 4th, 2020

Scott's foreword: Zeph Landau, a noted quantum computing theorist at UC Berkeley who's worked closely with my adviser Umesh Vazirani, recently asked me if he could write a guest post about pooled testing for covid—an old idea that, Zeph argues, could play a crucial role in letting universities safely reopen this fall. Seeing a small chance to do a great good, I readily agreed.

I should confess that I'm more … fatalistic than Zeph. Not that I'm proud of it: I think that Zeph's attitude is superior to mine. But, like, I'm a theoretical computer scientist with zero expertise in medical testing or statistics, and *I* knew about pooled testing and WWII origins—so imagine how thoroughly actual experts must know the idea. Just like they know all about variolation, and challenge trials, and copper fixtures, and UV light, and vitamin D supplements, and a dozen other possible tools against covid that future historians might ask why we didn't try more.

QUANTUM COMPUTING SINCE DEMOCRITUS

Archives

» June 2020

» May 2020

Zeph Landau's Guest Post

This post describes how every university could efficiently use modest testing resources to sensibly and extensively reduce the number of COVID-19 cases on their campus this fall. It is meant as a call to action to the reader — because without a concerted effort to get the right people the necessary information and take immediate consequential action, a far worse alternative will be implemented almost everywhere. It is my sincere hope, that immediately after reading this post, you will take the following steps:

1) Figure out who is part of the reopening committee at your institution.

2) Find the right people and engage with them either as a fellow faculty member or, better yet, through a connection to get them good information about the information posted here.

3) Then stay engaged and keep pushing. (See below for links to sample documents.)

OK, here we go.

Coronavirus tests

Antigen test

Tests if you currently have the virus

UK currently testing almost 100,000 per day: mostly either ill people or healthcare workers

Antibody test

Tests if you have had the virus in the past and are now immune

Capacity exists in the UK. Tests available soon?

Coronavirus tests

Antigen test

Tests if you currently have the virus

UK currently testing about 10,000 per day: either ill people in hospital or healthcare workers

Antibody test

Tests if you have had the virus in the past and are now immune

Could be available soon?

Group testing

n items

p prevalence

T tests

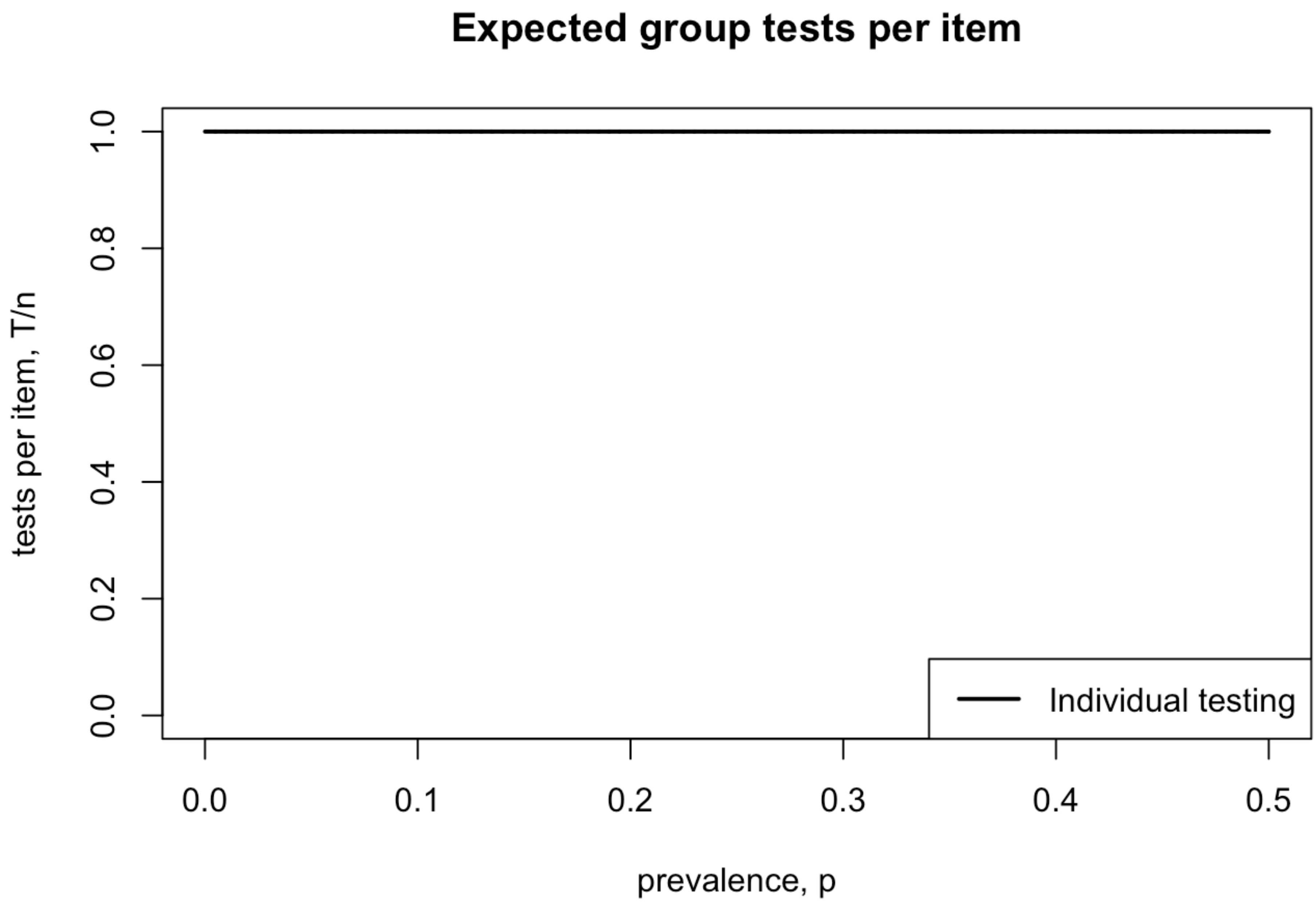
Given n and p,
how big does T have to be
(on average) to reliably work out
which items were defective?

Individual testing

Test each item individually.

n items requires n tests

1 test per item



Expected group tests per item
tests per item, T/n
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
prevalence, p
Individual testing

Lower bound

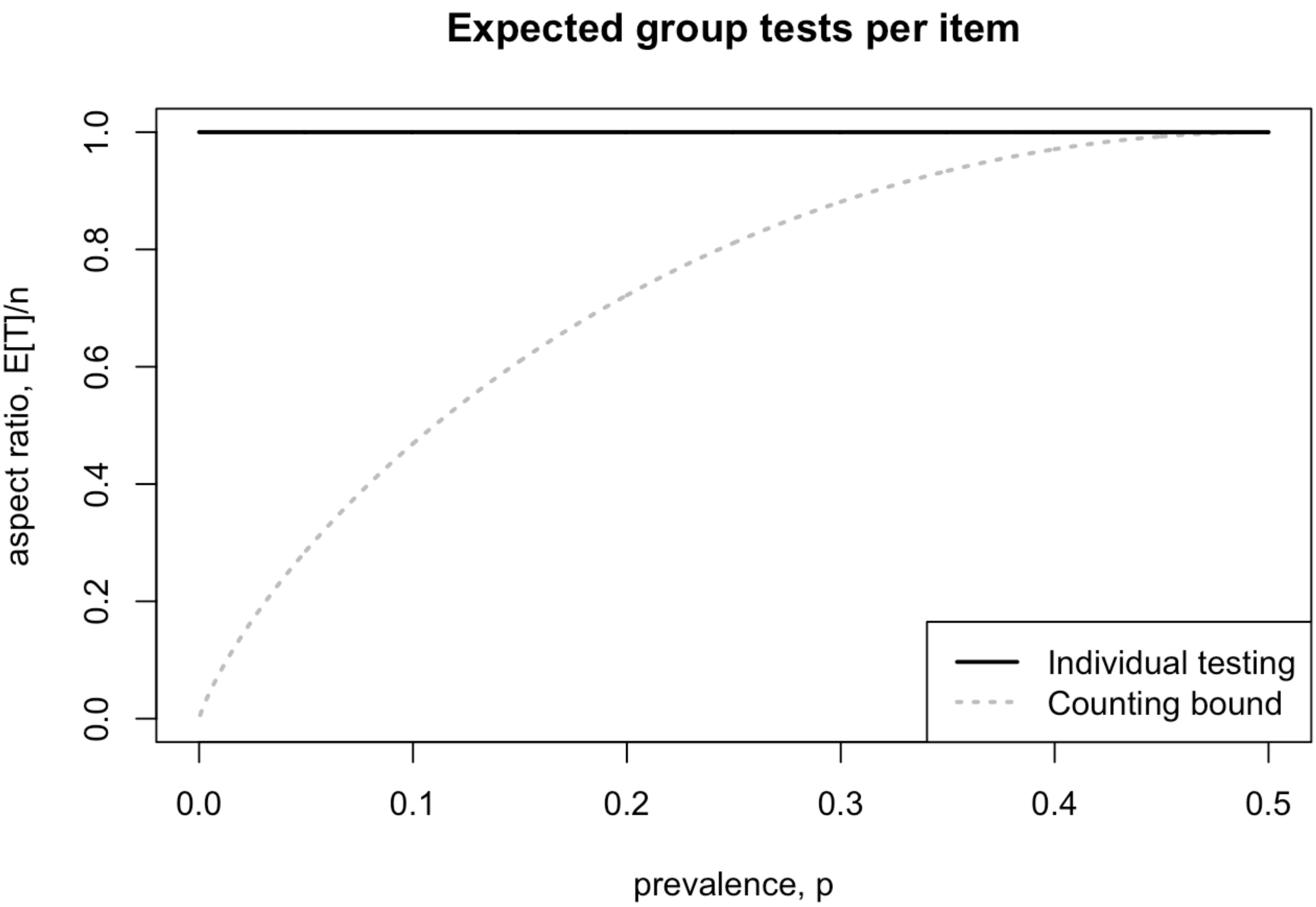
Standard combinatorial and/or information theoretic bounds tell us we need at least

$$T \geq H(p)n$$

tests, where

$$H(p) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p}$$

is the binary entropy


Expected group tests per item
aspect ratio, E[T]/n
prevalence, p
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
Individual testing
Counting bound

Dorfman's algorithm

(Dorfman, 1943)

Split into n/s sets of size s

For each set:

Test the whole set.

If the test is positive:
test each item individually to find which items are defective and nondefective

If the test is negative:
all items are nondefective.

Dorfman's algorithm

(Dorfman, 1943)

Each set is tested once together.

There are n/s sets.

Any set containing at least one defective item requires s more tests.

This happens with probability

$$1-(1-p)^s$$

Dorfman's algorithm

(Dorfman, 1943)

Each set is tested once together.

There are n/s sets.

Any set containing at least one defective item requires s more tests.

This happens with probability

$$1 - (1 - p)^s$$

Total expected number of tests:

$$\frac{n}{s}(1 + s(1 - (1 - p)^s))$$

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(Dorfman, 1943)

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number of sets

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$$1 - (1 - p)^s$$

Total expected number of tests:

$$\frac{n}{s}(1 + s(1 - (1 - p)^s))$$

first-stage tests

Dorfman's algorithm

(Dorfman, 1943)

Each set is tested once together.

There are n/s sets.

Any set containing at least one defective item requires s more tests.

This happens with probability

$$1-(1-p)^s$$

expected second-stage tests

Total expected number of tests:

$$\frac{n}{s}\left(1+s\left(1-(1-p)^s\right)\right)$$

Dorfman's algorithm

(Dorfman, 1943)

Each set is tested once together.

There are n/s sets.

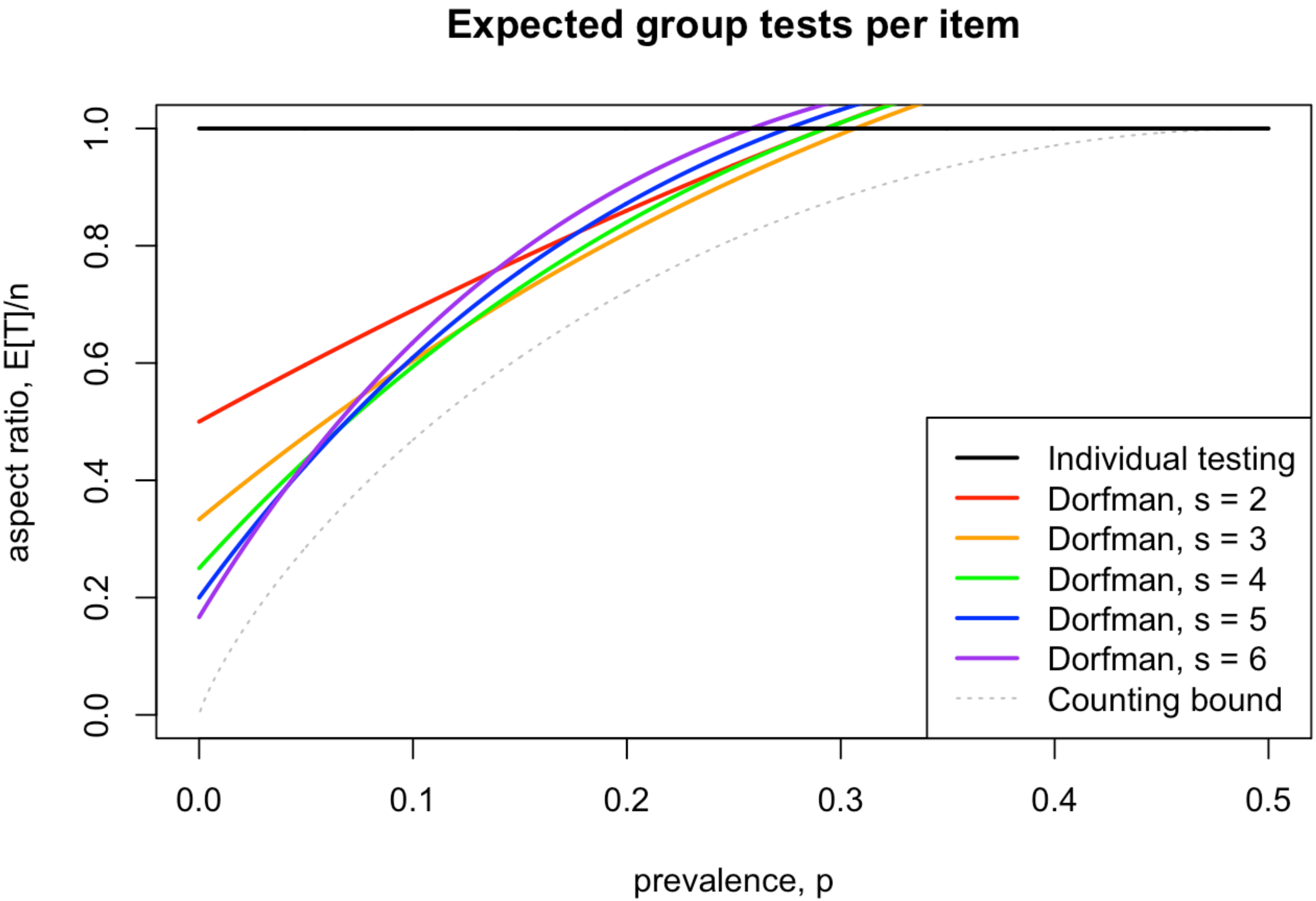
Any set containing at least one defective item requires s more tests.

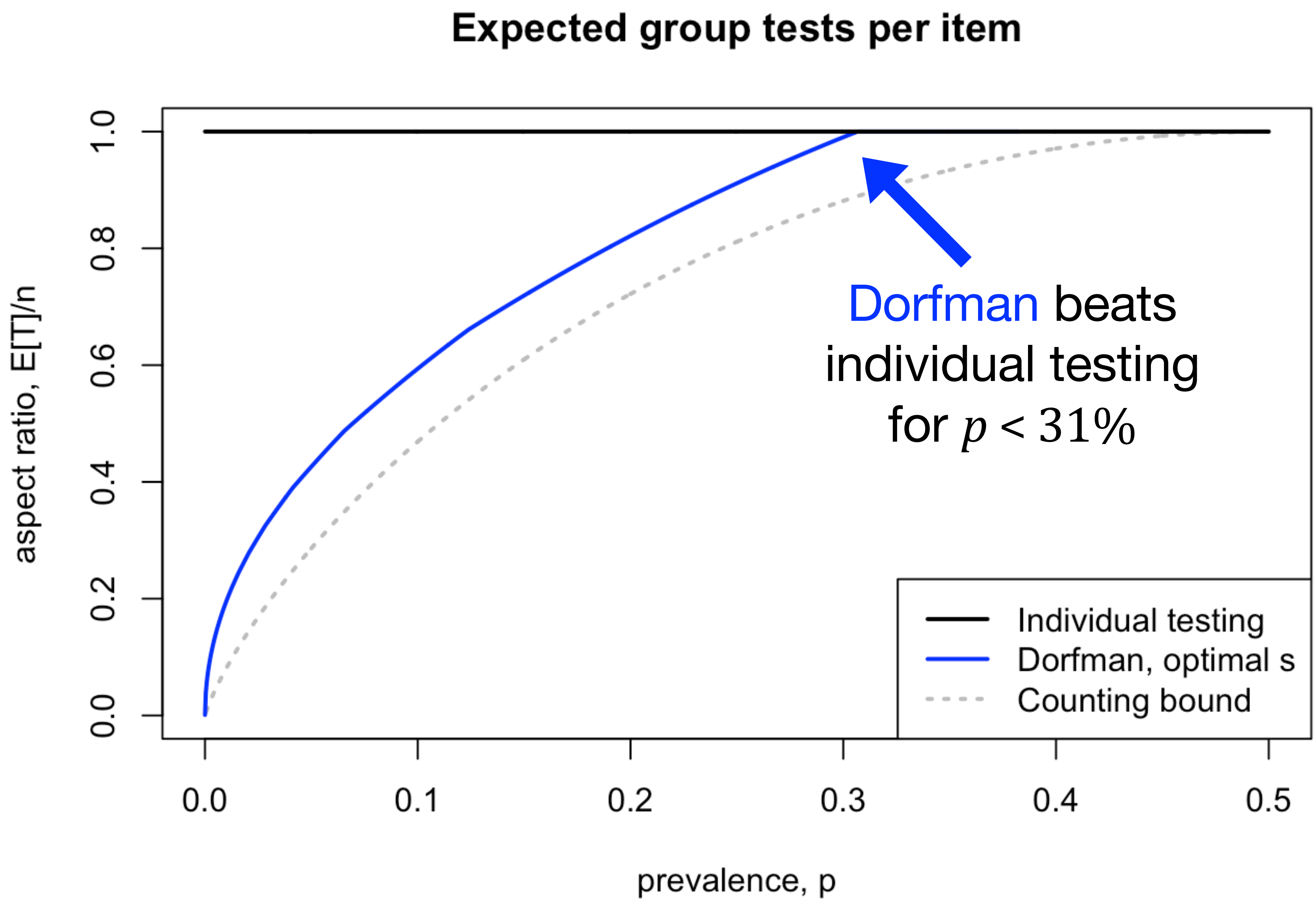
This happens with probability

$$1-(1-p)^s$$

Total expected number of tests:

$$\frac{n}{s}(1+s(1-(1-p)^s)) = n\left(\frac{1}{s}+1-(1-p)^s\right)$$


Expected group tests per item
aspect ratio, E[T]/n
prevalence, p
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
Individual testing
Dorfman, s = 2
Dorfman, s = 3
Dorfman, s = 4
Dorfman, s = 5
Dorfman, s = 6
Counting bound



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0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
Dorfman beats individual testing for $p < 31\%$
Individual testing
Dorfman, optimal s
Counting bound

Rate

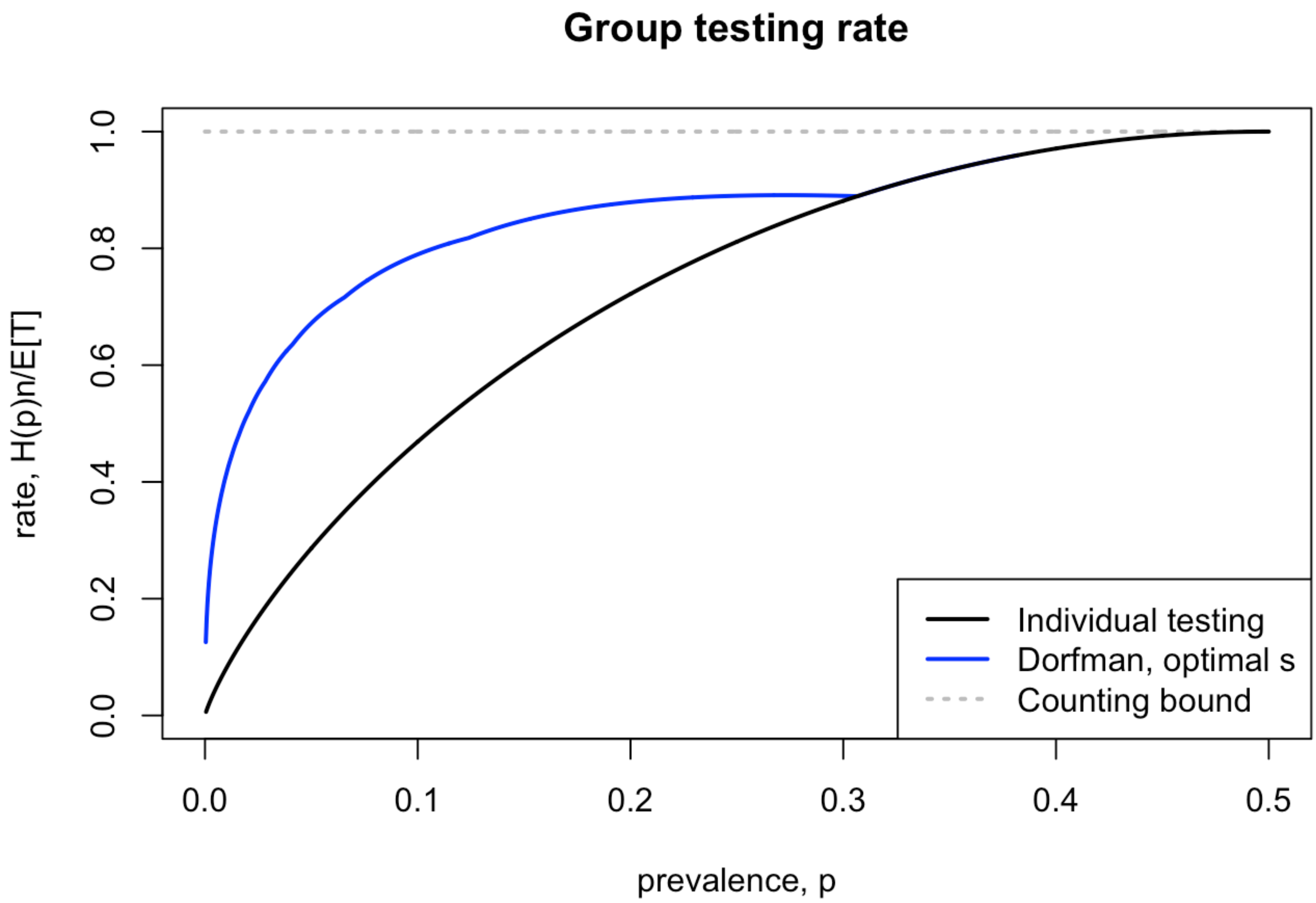
(Baldassini–Johnson–Aldridge, 2013)

We often prefer to plot the **rate** $H(p)n/\mathbb{E}T$.

The rate tells us:

How many “bits of information” we learn from each test

How close we are to the entropy bound


Group testing rate
rate, H(p)n/E[T]
0.0
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0.4
0.6
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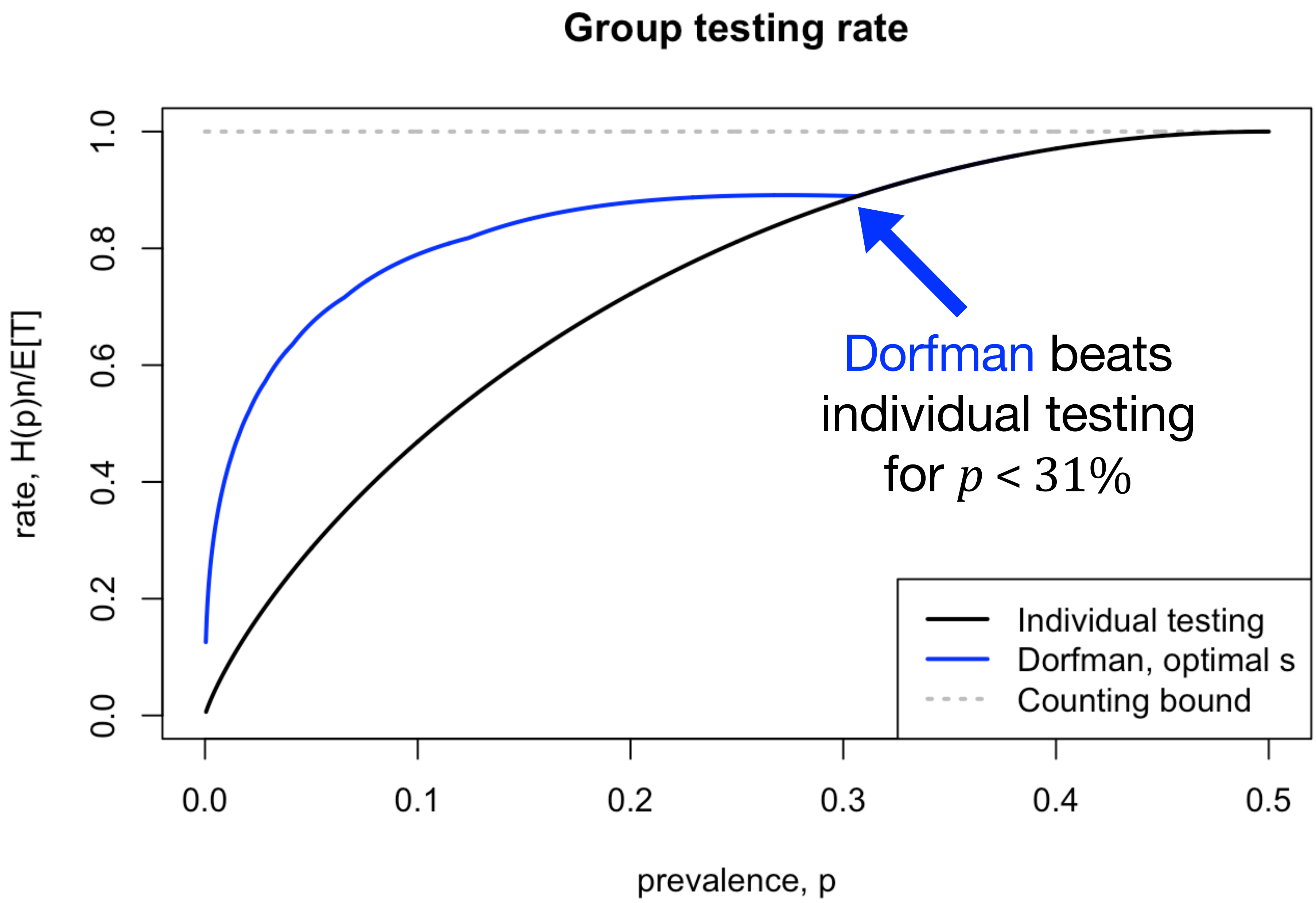
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prevalence, p
Dorfman beats
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Individual testing
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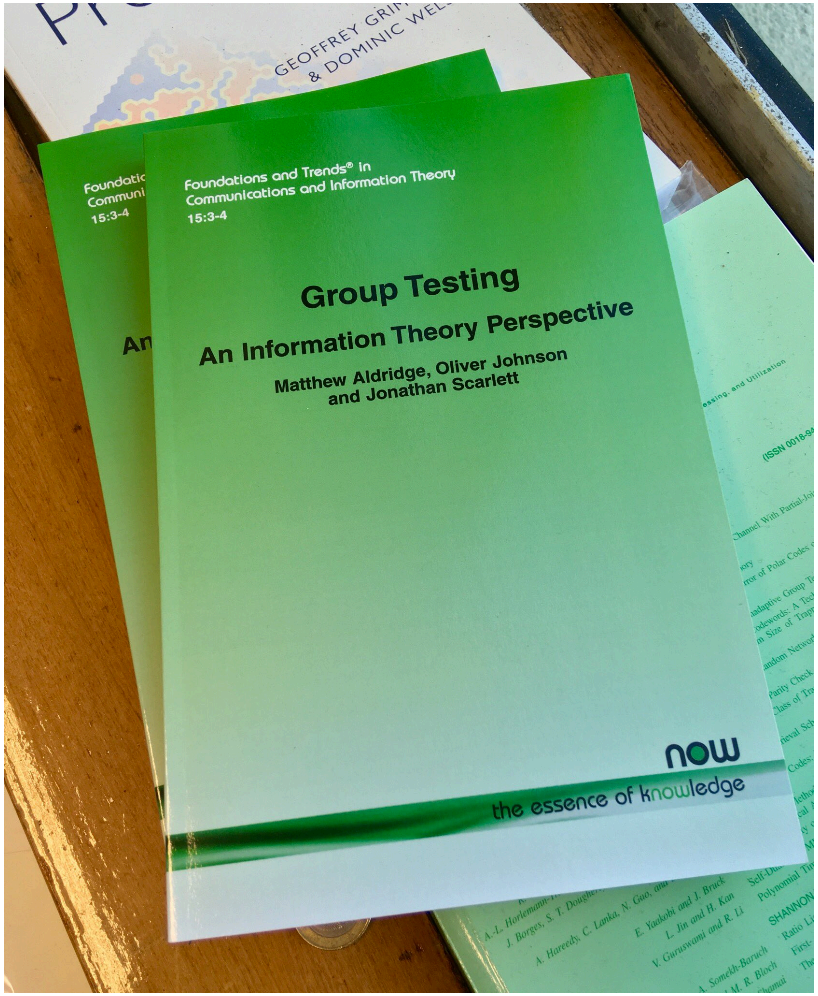
M Aldridge, O Johnson and J Scarlett
Group Testing: An Information Theory Perspective
Foundations and Trends in Communications and Information Theory, 2019

Preprint:
arXiv:1902.06002



Gov. Ricketts provides update on coronavirus testing

Last week, Ricketts said capacity was 200 a day. Now it's over 600 a day. The governor says testing capacity is ramping up because the state lab can now pool samples.

That means they can put five nasal swabs into one test tube and instead of testing them separately, do five at a time.

“If that one sample comes back negative, we know all five of those are negative, so we have just saved four tests. Now if that test comes back positive, then we will have to come back and retest all five of those again to figure out which one was positive,” Ricketts said.

Why do practitioners like Dorfman's algorithm?

Why do practitioners like Dorfman's algorithm?

Only requires two stages

Stage 1: the pooled tests

Stage 2: Individual tests from positive pools

"Two-stage group testing"

Why do practitioners like Dorfman's algorithm?

Only requires two stages

Stage 1: the pooled tests

Stage 2: Individual tests from positive pools

"Two-stage group testing"

All infected individuals definitively confirmed

with an individual test in the second stage

"Trivial two-stage group testing"

Dorfman's algorithm

(Dorfman, 1943)

Stage 1

Each item is in 1 test

Each test contains s items

Stage 2

Each item that was in a positive test

is re-tested individually

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Stage 1

Each item is in r tests

Each test contains s items

Stage 2

Each item that was in all positive tests

is re-tested individually

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

How many items need retesting?

All the defective items

All nondefective items
whose r tests were all positive

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

How many items need retesting?

All the defective items
pn on average

All nondefective items
whose r tests were all positive
$(1-p)n \times \mathbb{P}(\text{test positive})^r$ on average

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\left(1-(1-p)^{s-1}\right)^r$$

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\left(1-(1-p)^{s-1}\right)^r$$

first-stage tests

T first-stage tests
n items
r tests per item
s items per test

$Ts = nr$

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\left(1 - (1-p)^{s-1}\right)^r$$

defective items
to be re-tested

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\,(1-(1-p)^{s-1})^r$$

nondefective items

probability the other $s-1$ items in the test are nondefective also

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\left(1-(1-p)^{s-1}\right)^r$$

nondefective items

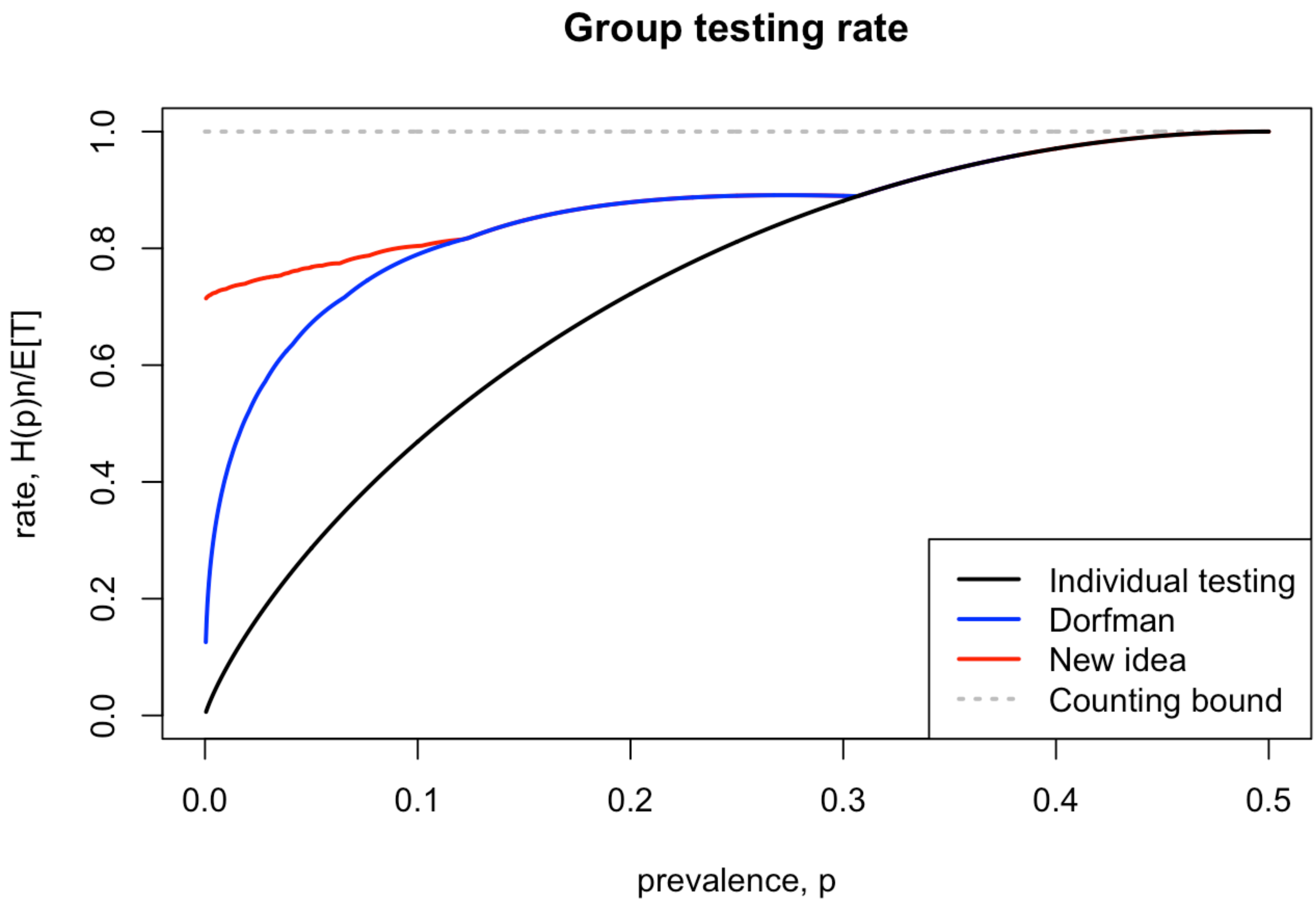
probability all r tests containing a given nondefective are negative

New idea

(Broder–Kumar, 2020; Aldridge, 2020)

Expected number of tests

$$\frac{nr}{s} + pn + (1-p)n\left(1-(1-p)^{s-1}\right)^r$$



Group testing rate
rate, H(p)n/E[T]
prevalence, p
0.0
0.2
0.4
0.6
0.8
1.0
0.0
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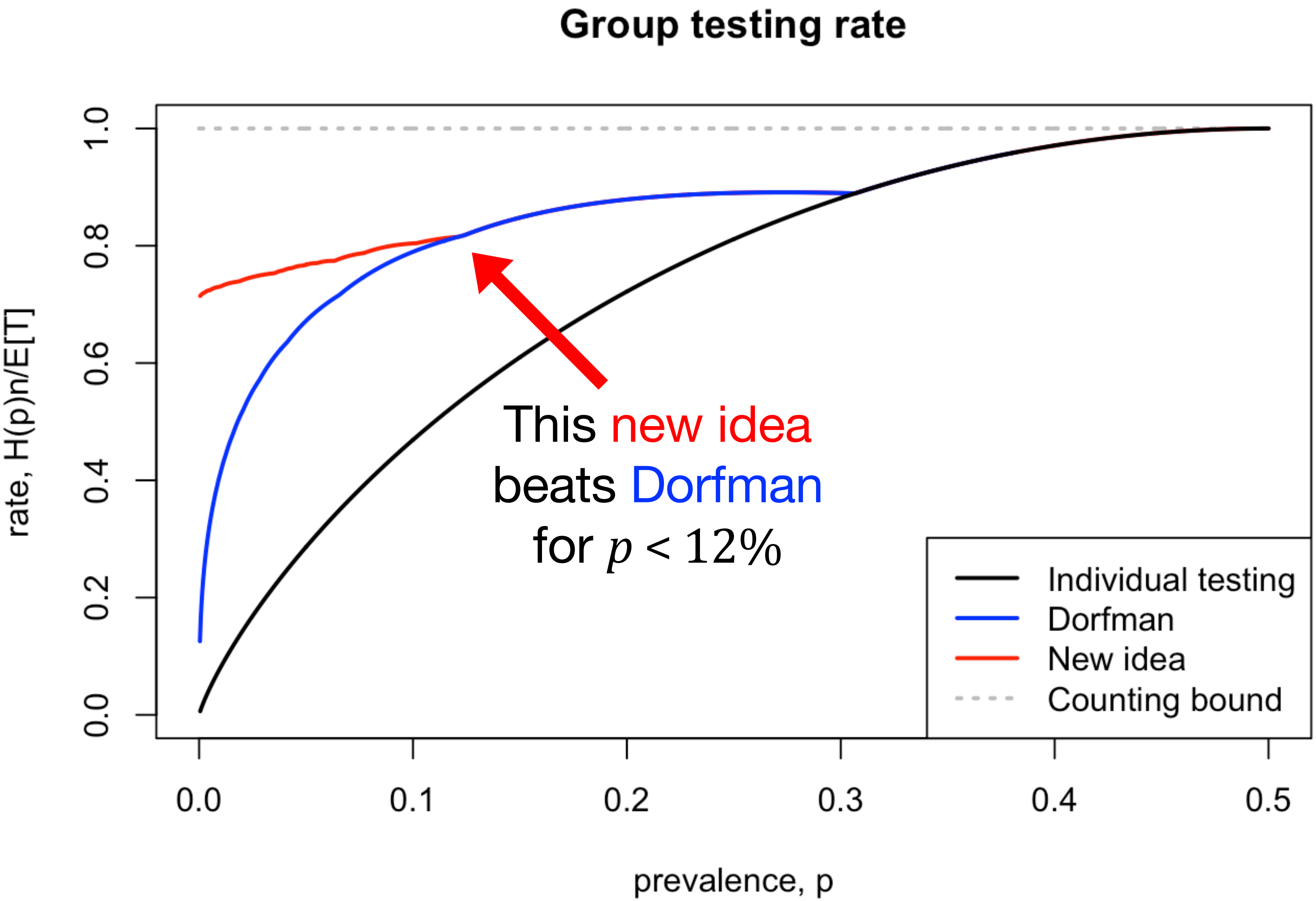
Individual testing
Dorfman
New idea
Counting bound



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0.4
0.5
This new idea
beats Dorfman
for $p < 12\%$
Individual testing
Dorfman
New idea
Counting bound

A bound for trivial two-stage testing

(Aldridge, 2020)

In the second stage, we must test any item that has been in first-stage tests that were all positive.

A bound for trivial two-stage testing

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$$\mathbb{E}(\#\text{ 2nd stage tests}) = \sum_{i=1}^{n} \mathbb{P}(\text{all item } i\text{'s tests +ve})$$

A bound for trivial two-stage testing

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$$\mathbb{E}(\#\ \text{2nd stage tests}) = \sum_{i=1}^{n} \mathbb{P}(\text{all item } i\text{'s tests +ve})$$

$$\mathbb{P}(\text{all item } i\text{'s tests +ve}) \geq \prod_{\substack{\text{tests item} \\ i \text{ is in}}} \mathbb{P}(\text{test +ve})$$

A bound for trivial two-stage testing

$$\mathbb{P}(\text{all item } i\text{'s tests +ve}) \geq \prod_{\substack{\text{tests item} \\ i \text{ is in}}} \mathbb{P}(\text{test +ve})$$

$$\mathbb{P}(\text{test +ve}) = 1 - (1 - p)^s$$

where s is the number of items in the test

A bound for trivial two-stage testing

$$\mathbb{P}(\text{all item } i\text{'s tests +ve}) \geq \prod_{\substack{\text{tests item} \\ i \text{ is in}}} \mathbb{P}(\text{test +ve})$$

$$\mathbb{P}(\text{test +ve}) = 1 - (1 - p)^s$$

where s is the number of items in the test

Sum over all items, and do some maths.

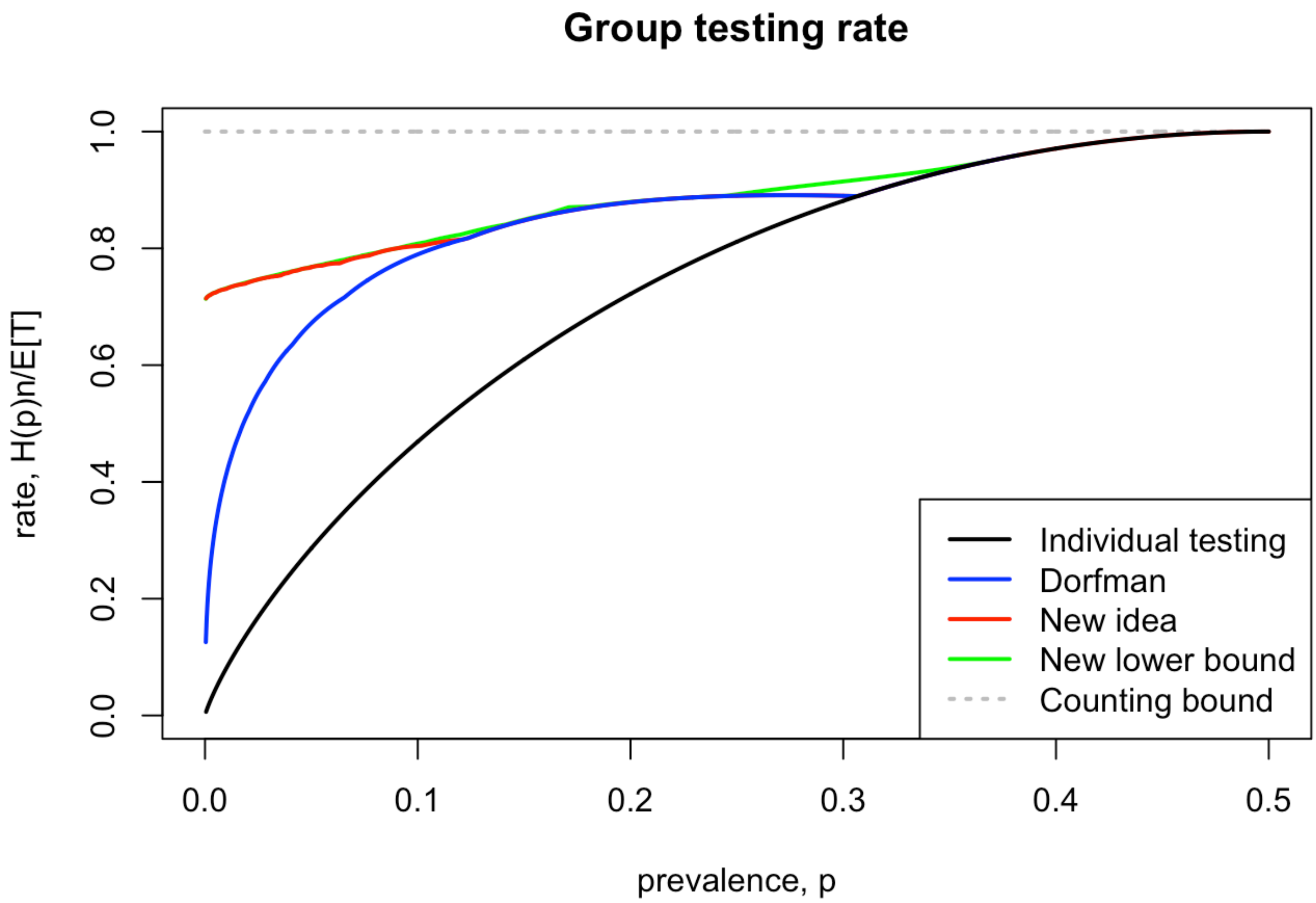
A better bound

(Aldridge, 2020)

$$\mathbb{E}T \geq n \frac{1}{g(p)} (\ln g(p) + 1)$$

where

$$g(p) = \max_{s=2,3,\ldots} \{-s \ln(1 - (1-p)^s)\}$$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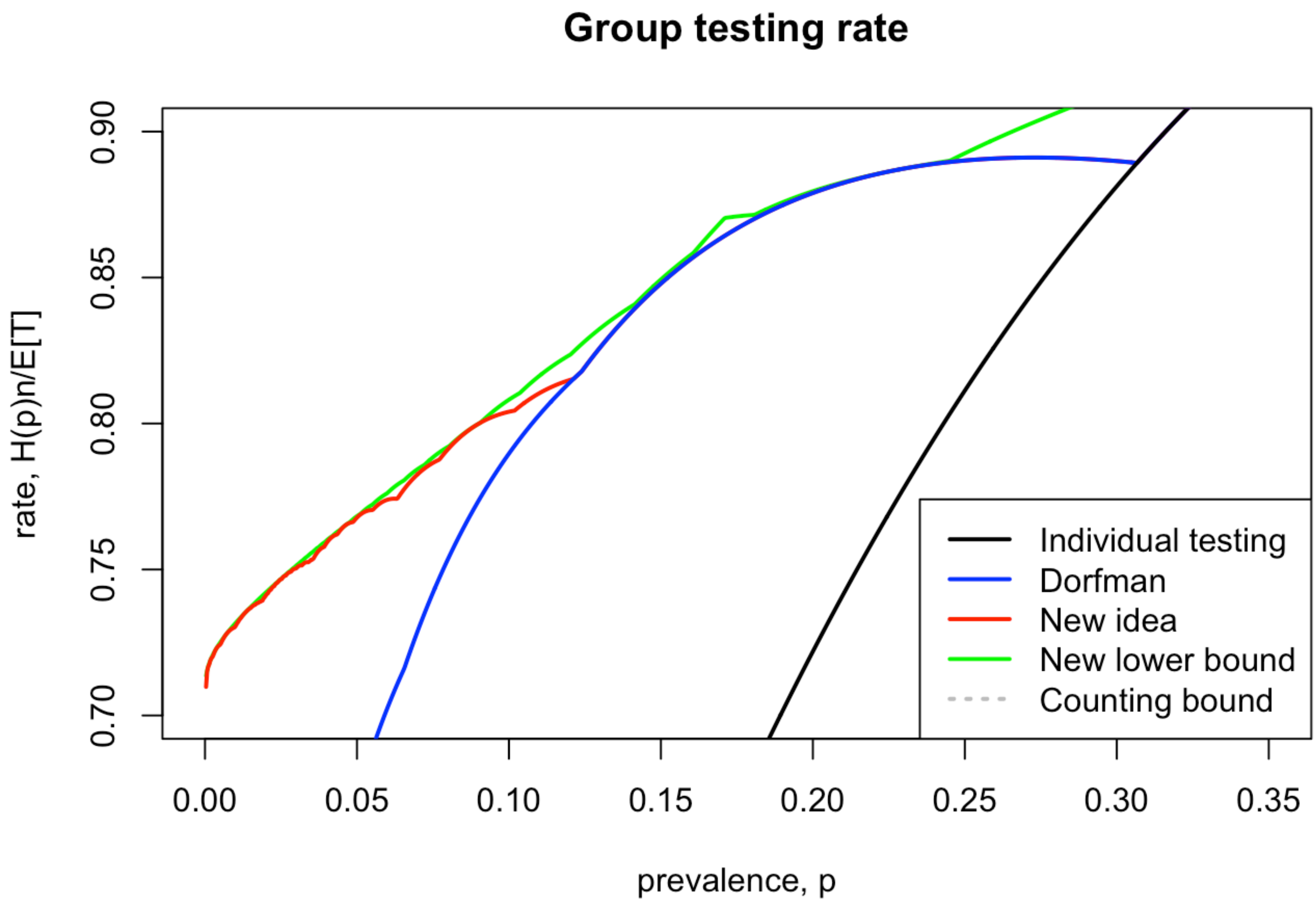
0.4
0.5
Individual testing
Dorfman
New idea
New lower bound
Counting bound


Group testing rate
rate, H(p)n/E[T]
0.70
0.75
0.80
0.85
0.90
prevalence, p
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
Individual testing
Dorfman
New idea
New lower bound
Counting bound

Conclusion

Trivial two-stage group testing:

For $p > 31\%$, use **individual testing**

For $12\% < p < 31\%$, use **Dorfman's algorithm**

For $p < 12\%$, test each item
more than once in the first stage

These algorithms are **very close to optimal**.

Open question 1

Data suggests that tests are:

Highly specific – tests with no infected samples are almost certain to be negative

Weakly sensitive – tests with an infected sample have only about a 70% chance of being positive

How does this change the best strategies?

Open question 1

Data suggests that tests are:

Highly specific – tests with no infected samples are almost certain to be negative

Weakly sensitive – tests with an infected sample have only about a 70% chance of being positive

Open question 2

Suppose we don't have enough tests
to test everybody every day.

Should we…

Open question 2

Suppose we don’t have enough tests
to test everybody every day.

Should we…

test a small proportion of people,
and find most of the infected people?

test a larger proportion of people,
but find only some of the infected people?

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