Solutions: Sheet 3

Question 1

a) We need to show

$$\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^c). \qquad (*)$$

By monotonicity,

$$\mu^*(S \cap A) \leq \mu^*(A) = 0$$
$$\mu^*(S \cap A^c) \leq \mu^*(S)$$

So $(*)$ holds.

b) Suppose $\mu^*(Z) = 0$ and $N \subset Z$. By monotonicity $\mu^*(N) = 0$, so by part a), $N \in \mathcal{M}$. Hence $(X, \mathcal{M}, \mu)$ is complete.

Q2&3: see printed solutions

Question 4

Put $X = \mathbb{R}$, $\mathcal{S} = \mathcal{I}$,

$$\pi(A) = \begin{cases} 0 & A = \emptyset, \\ \infty & A = \mathcal{S} \text{ otherwise}. \end{cases}$$

This has many extensions, eg:

- $\pi(A) = \infty \quad \forall A \in \mathcal{B}$ except $\emptyset$
- $\pi(A) =$ counting measure
- $\pi(A) = \begin{cases} 0 & A \in \mathcal{B} \text{ countable} \\ \infty & A \in \mathcal{B} \text{ uncountable} \end{cases}$

etc.