$\texttt{MA40042}$ Measure Theory and Integration

Problem Sheet 3

Solutions should be submitted to the MA40042 pigeonhole by 1700 on ***Monday 24 October****.*

Work will be returned and answers discussed in the problems class on Tuesday 25 October.

1. Let X be a nonempty set, and μ^* an outer measure on X.
   (a) Show that if $A \subset X$ has outer measure 0, then A is measurable.
   (b) Show that the measure space $(X, \mathcal{M}, \mu)$ (with the usual definitions) is complete.

2. Our aim is to construct a probability space representing an infinite sequence of coin tosses. Let $\Omega = \{\mathtt{H}, \mathtt{T}\}^{\mathbb{N}}$, so $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots) \in \Omega$ represents an infinite string of heads and tails. Given $\mathbf{x} \in \{\mathtt{H}, \mathtt{T}\}^n$ for some n, write $C(\mathbf{x})$ for the *cylinder set*

$$C(\mathbf{x}) = \{\boldsymbol{\omega} \in \Omega : \omega_1 = x_1, \omega_2 = x_2, \dots, \omega_n = x_n\}.$$

Write $\mathcal{C}$ for the empty set and all cylinder sets $C(\mathbf{x})$ for $\mathbf{x} \in \{\mathtt{H}, \mathtt{T}\}^n$ for any $n \in \mathbb{N}$.

   (a) Show that $\mathcal{C}$ is a semialgebra on Ω

Define $\pi : \mathcal{C} \to [0, \infty]$ by $\pi(\varnothing) = 0$ and for $\mathbf{x} \in \{\mathtt{H}, \mathtt{T}\}^n$ put $\pi\big(C(\mathbf{x})\big) = 2^{-n}$.

   (b) Show that π is finitely additive on disjoint sets in $\mathcal{C}$.
   (c) Show that no cylinder set can be written as a countably infinite disjoint union of cylinder sets. *(This is quite hard. If you can't give a proof, try to sketch the general idea, or even just explain why one might expect this to be true.)*
   (d) Deduce that π is a premeasure on $\mathcal{C}$.

Hence, by Carathéodory's extension theorem, π can be extended to a measure $\mathbb{P}$ on $\big(\Omega, \sigma(\mathcal{C})\big)$.

   (e) Show that any such $\mathbb{P}$ is a probability measure.
   (f) Show that $\mathbb{P}$ is the unique extension of π.

3. Let (X, Σ, μ) and (Y, Π, ν) be two measure spaces. Write

$$\mathcal{S} = \{A \times B : A \in \Sigma, B \in \Pi\}$$

and put $\Sigma \otimes \Pi = \sigma(\mathcal{S})$ for the σ-algebra on $X \times Y$ generated by $\mathcal{S}$.

   (a) Suppose X and Y are countable. What is $\mathcal{P}(X) \otimes \mathcal{P}(Y)$?
   (b) Suppose $X = Y = \mathbb{R}$. Show that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$.
   (c) Show, in general, that $\mathcal{S}$ is a semialgebra on $X \times Y$.

Write $\pi(A \times B) = \mu(A)\nu(B)$. You may assume (and will probably prove later in the course) that π is a premeasure on $(X, \mathcal{S})$. Hence π extends to a measure on $\Sigma \otimes \Pi$, which is called the *product measure* and written $\mu \times \nu$.

   (d) Show that if μ and ν are both σ-finite, then the product measure is unique.
   (e) Let X and Y be countable, and endowed with their powersets and counting measures. What is the corresponding product measure.
   (f) Outline (without proofs) an alternative construction of the Lebesgue measure on $\mathbb{R}^d$ for $d \geq 2$.

4. Give an example of a set X, a semialgebra $\mathcal{S}$ and a premeasure π that does not have a unique extension.

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