Lecture 3
Measures

- Definition of measure
- Examples of measures
- Basic properties and continuity of measure
- Probability, finite, and σ-finite measures

3.1 Definition

Recall from Lecture 0 the properties we wanted a measure to have:

- A measure is a function μ from measurable subsets of X to $[0, \infty]$. (Here, $[0, \infty] := [0, \infty) \cup \{\infty\}$.)
- $\mu(\varnothing) = 0$.
- For a finite or countably infinite sequence of disjoint measurable sets, $\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$.
- For a finite or countably infinite sequence of measurable sets, $\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n)$.
- $\mu(A) + \mu(A^{\mathsf{c}}) = \mu(X)$.

We take some of these as a definition, and prove the rest as properties later.

Definition 3.1. Let (X, Σ) be a measurable space. Then a function $\mu\colon \Sigma \to [0, \infty]$ is a *measure* on (X, Σ) if

1. $\mu(\varnothing) = 0$;
2. μ is *countably additive*, in that if $A_1, A_2, \dots,$ is a countably infinite sequence of disjoint sets in Σ, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple (X, Σ, μ) is called a *measure space*.

Recall that a sequence (A_n) of sets is disjoint (or 'pairwise disjoint') if $A_i \cap A_j$ for $i \neq j$.

3.2 Examples

The following are measures. (Checking they really are measures is a homework problem.)

- Let X be a nonempty set, and let Σ be its powerset. For $A \in \Sigma$, the *counting measure* $\#$ is given by setting $\#(A)$ to be the cardinality of A. (Recall that a set A has cardinality n if its elements can be enumerated as $A = \{a_1, a_2, \dots, a_n\}$, and has cardinality ∞ otherwise. The empty set has cardinality 0.)
- Let (X, Σ) be a measurable space, and let $x \in X$. Then the *Dirac measure* (or δ *measure* or *point measure*) at x of a set $A \in \Sigma$ is given by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

- Let X be a countable set, and let Σ be its powerset. Associate with each $x \in X$ a 'weight' $w_x \in [0, \infty]$. Then the associated *discrete measure* is

$$\mu(A) = \sum_{x \in A} w_x = \sum_{x \in X} w_x \delta_x(A).$$

- Let X be a nonempty set, and let Σ be the σ-algebra of countable and co-countable sets. Then we can set

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable}, \\ \infty & \text{otherwise} \end{cases}$$

- The *Lebesgue measure* on $\mathbb{R}$ is the unique measure λ on the measurable space $(\mathbb{R}, \mathcal{B})$ such that $\lambda\big([a, b)\big) = b - a$. Showing that this measure exists and is unique is highly nontrivial (and will take about four lectures' work).

3.3 Properties

Let's start with some simple properties.

Theorem 3.2. *Let (X, Σ, μ) be a measure space. Then we have the following properties:*

(Finite additivity) *For $A_1, A_2, \dots, A_N$ a finite sequence of disjoint sets in Σ, we have*

$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$

(Monotonicity) *For $A, B \in \Sigma$ with $A \subset B$, we have $\mu(A) \leq \mu(B)$.*

(Countable subadditivity) *For $A_1, A_2, \dots,$ a countably infinite sequence of sets in Σ, we have*

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n).$$

(Finite subadditivity) *For $A_1, A_2, \dots, A_N$ a finite sequence of sets in Σ, we have*

$$\mu\left(\bigcup_{n=1}^N A_n\right) \leq \sum_{n=1}^N \mu(A_n).$$

Also, for $A \in \Sigma$, we have $\mu(A) + \mu(A^\mathsf{c}) = \mu(X)$.

Proof. Most of these are left for homework, but let's do monotonicity (assuming finite additivity) for an example.

We can write $B = A \cup (B \setminus A)$, where the set difference $B \setminus A$ was shown to be measurable in Lecture 1. Note also that the A and $B \setminus A$ are disjoint. Hence by finite additivity we have $\mu(B) = \mu(A) + \mu(B \setminus A)$. But $\mu(B \setminus A) \geq 0$, as $\mu\colon \Sigma \to [0, \infty]$. Hence $\mu(B) \geq \mu(A)$. □

This proof illustrates a general rule: to prove a statement about measures, turn it into a statement about disjoint unions.

The following properties are known as 'continuity of measure', by analogy with the usual definition of continuity, that

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

For the 'limit' of a sequence of sets, we can think of the union of a sequence of expanding sets (an 'upward limit'), or the intersection of a sequence of contracting sets (a 'downward limit').

Theorem 3.3. *Let (X, Σ, μ) be a measure space. Then we have the following properties:*

(Upward continuity) *For $A_1 \subset A_2 \subset \cdots$ a countably infinite 'expanding' sequence of sets in Σ, we have*

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^\infty A_n\right).$$

(Downward continuity) *For $A_1 \supset A_2 \supset \cdots$ a countably infinite 'contracting' sequence of sets in Σ with $\mu(A_1) < \infty$, we have*

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^\infty A_n\right).$$

Proof. We'll prove upward continuity, and leave downward continuity for homework.

Write $B_1 = A_1$, and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then

$$\bigcup_{n=1}^N B_n = A_N \qquad \text{and} \qquad \bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n,$$

with the unions being disjoint. So by finite additivity we have

$$\mu(A_N) = \mu\left(\bigcup_{n=1}^N B_n\right) = \sum_{n=1}^N \mu(B_n).$$

Hence, by countable additivity,

$$\mu(A_N) = \sum_{n=1}^N \mu(B_n) \to \sum_{n=1}^\infty \mu(B_n) = \mu\left(\bigcup_{n=1}^\infty B_n\right) = \mu\left(\bigcup_{n=1}^\infty A_n\right)$$

as $N \to \infty$. □

3.4 Probability measures

A probability measure fulfils the definition of a measure, but also 'adds up to 1'.

Definition 3.4. Let (X, Σ, μ) be a measure space.

- If $\mu(X) = 1$, then we call μ a *probability measure*, and (X, Σ, μ) a *probability space*.
- If $\mu(X) < \infty$, then we call μ a *finite measure*.
- If X can be written as a countable union of sets in Σ with finite measure, then we call μ a *σ-finite measure*.

A probability space is usually written as $(\Omega, \mathcal{F}, \mathbb{P})$. Note that it follows from Theorem 3.2 that $\mathbb{P}(A) \leq 1$ and $\mathbb{P}(A^\mathsf{c}) = 1 - \mathbb{P}(A)$ for all $A \in \mathcal{F}$.

Probability measures are well-behaved, and your intuition is useful in dealing with them. Finite measure behave essentially the same as probability measures, and σ-finite measures are fairly well-behaved too. Measures that are not σ-finite can be pretty weird.

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