Lecture 4

Constructing measures I: Outer measure

- How to construct measures
- The Lebesgue measure on intervals and boxes
- Constructing an outer measure
- Properties of outer measures

4.1 How to construct measures: a roadmap

So far we have only seen simple examples of measures, such as the counting measure or some discrete measures. To create more powerful measures, such as the Lebesgue measure of length/area/volume, we have to build them up from basic pieces.

Suppose we are working on a set X.

0. We begin with a collection $\mathcal{R}$ of subsets of X, and a function ρ on $\mathcal{R}$. The idea is that $\mathcal{R}$ is the collection of sets we 'know', and, for $R \in \mathcal{R}$, we 'want' the measure of R to be $\rho(R)$.
1. The first step is to construct an **outer measure** μ^* from $\mathcal{R}$ and ρ. This gives a 'size' to *all* subsets of X, and has many of the properties of a measure.
2. The outer measure μ^* is (in general) not a measure. But if we restrict ourselves to the σ-algebra of **measurable sets** satisfying a certain 'splitting condition', this does give a measure μ on those measurable sets.
3. The measure μ does not necessarily 'extend' (that is, agree with) our original set-up: we can't guarantee that every $R \in \mathcal{R}$ is a measurable set, and even if R is measurable, there's no guarantee that $\mu(R) = \rho(R)$. However, **Carathéodory's extension theorem** tells us that if $\mathcal{R}$ and ρ have certain properties, then μ is a proper extension of ρ. In certain other conditions, the μ is the unique extension of ρ.

4.2 Set-up

Let X be a nonempty set. We start with

- A collection $\mathcal{R}$ of subsets of X. We shall assume throughout that $\varnothing \in \mathcal{R}$.
- A function $\rho\colon \mathcal{R} \to [0, \infty]$. We shall assume throughout that $\rho(\varnothing) = 0$.

The idea is that $\mathcal{R}$ is a collection of 'simple' sets, and we want the measure of $R \in \mathcal{R}$ to be $\rho(R)$. We then need to 'grow' ρ into a measure, by defining a measure on some σ-algebra on X that has been built up from $\mathcal{R}$ and ρ.

An important case is that of the Lebesgue measure. On $X = \mathbb{R}$, we take $\mathcal{R}$ to be the collection of intervals $\mathcal{I}$, and define the length ρ by

$$\rho(\varnothing) = 0 \qquad \rho\big([a, b)\big) = b - a \qquad \rho\big([-\infty, b)\big) = \rho\big([a, \infty)\big) = \rho(\mathbb{R}) = \infty.$$

On $X = \mathbb{R}^d$, we take $\mathcal{R}$ to be the collection of interval boxes $\mathcal{I}_d$ and ρ_d to be the (hyper)volume

$$\rho_d(I_1 \times I_2 \times \cdots \times I_d) = \rho(I_1)\rho(I_2)\cdots\rho(I_d).$$

(Remember our convention that $0 \times \infty = 0$.) In particular, a Cartesian product of semi-open intervals has volume

$$\rho_d\left(\prod_{i=1}^{d} [a_i, b_i)\right) = \prod_{i=1}^{d} (b_i - a_i).$$

(As ever, we suppress the dimension d when it's clear by context.)

4.3 Constructing μ^*

Given X, $\mathcal{R}$, and ρ, we will construct a function $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$, which gives a concept of size to all subsets of X. We shall see later that μ^* has many of the properties of a measure.

The idea is that, for any set $A \subset X$, we approximate A from the outside by a countable union of sets from $\mathcal{R}$, and we know what we want their sizes to be.

Definition 4.1. Let X be a nonempty set, $\mathcal{R}$ a collection of subsets containing the empty set, and ρ a function from $\mathcal{R}$ to $[0, \infty]$ satisfying $\rho(\varnothing) = 0$.

Given a countable subcollection $\mathcal{C} \subset \mathcal{R}$, we write

$$\rho(\mathcal{C}) = \sum_{R \in \mathcal{C}} \rho(R).$$

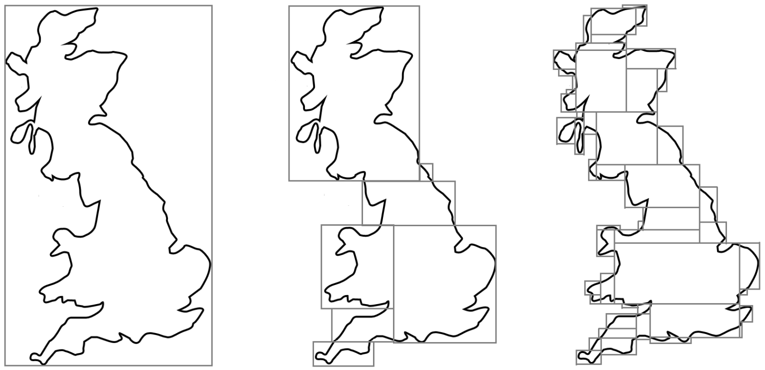
We call $\mathcal{C}$ a *covering* of a set $A \subset X$ if $A \subset \bigcup_{R \in \mathcal{C}} R$.

We define a function $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(A) := \inf \{\rho(\mathcal{C}) : \mathcal{C} \text{ is a covering of } A\}.$$

(By convention, $\inf \varnothing = \infty$, meaning we take $\mu^*(A) = \infty$ if there is no covering of A.)

The infimum in the definition means we want to take the tightest outer approximation – or at least the limit of increasingly tight approximations – of the set A.



When $\mathcal{R} = \mathcal{I}$ and ρ is the length/volume as above, we write λ^* for the function obtained from this covering construction, and call it the *Lebesgue outer measure.*

4.4 Properties of outer measures

The function μ^* has many of the properties of a measure.

Theorem 4.2. *Let X, $\mathcal{R}$, ρ, and μ^* be as above. Then*

1. $\mu^*(\varnothing) = 0$
2. μ^* *is* monotone, *in that for $A \subset B \subset X$ we have $\mu^*(A) \leq \mu^*(B)$.*
3. μ^* *is* countably subadditive, *in that for a countably infinite sequence A_1, $A_2, \dots$ of subsets of X, we have*
$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n).$$

Definition 4.3. Any function that satisfies points 1, 2, and 3 from the previous theorem is called an *outer measure* on X.

Note that 'an outer measure' is any function satisfying these properties, and the construction of μ^* above is one way of forming an outer measure.

Proof of Theorem 4.2. For the first point, note that one covering of $\varnothing$ is $\mathcal{C} = \{\varnothing\}$, and $\rho(\mathcal{C}) = 0$. No covering has a negative size, so $\mu^*(\varnothing) = 0$.

Monotonicity is immediate, since any covering of B is also a covering of A.

Now for countable subadditivity. First note that if for some A_n we have $\mu^*(A_n) = \infty$, then there's nothing to prove. So assume $\mu^*(A_n)$ is finite for all n.

Since the definition of μ^* includes an infimum, we will want to show that, for arbitrary $\epsilon > 0$,
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_n \mu^*(A_n) + \epsilon,$$
which would prove the theorem.

Let's pause for a moment, and look at a failed attempt to prove the statement.

By definition, for each A_n there exists a covering $\mathcal{C}_n$ of A_n with
$$\rho(\mathcal{C}_n) \leq \mu^*(A_n) + \epsilon.$$
Now the union $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is countable and is a covering of $\bigcup_{n=1}^{\infty} A_n$. Hence
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \rho(\mathcal{C}) \leq \sum_{n=1}^{\infty} \rho(\mathcal{C}_n) \leq \sum_{n=1}^{\infty} \big(\mu^*(A_n) + \epsilon\big) = \sum_{n=1}^{\infty} \mu^*(A_n) + \infty \times \epsilon.$$

Had we been left with '5ϵ' at the end, we go back to the proof and replace all our 'ϵ's with '$\epsilon/5$'s. But we can't go back and divide through by ∞. But, we can use 'the $\epsilon/2^n$ trick'.

Proof continued. By definition, for each A_n there exists a covering $\mathcal{C}_n$ of A_n with
$$\rho(\mathcal{C}_n) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}.$$
Now the union $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is countable and is a covering of $\bigcup_{n=1}^{\infty} A_n$. Hence
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \rho(\mathcal{C}) \leq \sum_{n=1}^{\infty} \rho(\mathcal{C}_n) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$
since
$$\sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = \epsilon \times 1 = \epsilon.$$
The statement is proved. ∎

Matthew Aldridge
m.aldridge@bath.ac.uk