`MA40042` Measure Theory and Integration

# Problem Sheet 4

*Solutions should be submitted to the `MA40042` pigeonhole*
*by **1500** on **Wednesday 2 November**.*
*(Note unusual day and time.)*

*Work will be returned and answers discussed*
*in the problems class on Thursday 3 November.*

1. (a) Let $\Sigma$ be a collection of subsets of a nonempty set $X$. Show that $\Sigma$ is a $\sigma$-algebra if and only if it is both a $\pi$-system and a $\lambda$-system.

   (b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and fix an event $B \in \mathcal{F}$. Let $\Lambda$ be the set of events independent of $B$, so

   $$\Lambda = \big\{A \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\big\}.$$

   Show that $\Lambda$ is a $\lambda$-system.

   (c) Let $(X, \Sigma)$ be a measurable space, and $\lambda, \nu$ two measures on $(X, \Sigma)$ with $\lambda(X) = \nu(X) < \infty$. Write

   $$\Lambda = \big\{A \in \Sigma : \mu(A) = \nu(A)\big\}$$

   for the collection of sets where the measures agree. Show that $\Lambda$ is a $\lambda$-system.

   (d) Give a specific example of a $\lambda$-system that is not a $\sigma$-algebra.

   (e) An alternative definition of a $\lambda$-system (see, for example, A1.2 in *Probability with Martingales* by Williams, who calls them $d$-systems) is the following: it is a collection $\Lambda$ of subsets of $X$ with

   1. $X \in \Lambda$;
   2. if $A, B \in \Lambda$ with $A \subset B$, then $B \setminus A \in \Lambda$;
   3. if $A_1 \subset A_2 \subset \cdots$ is a countably infinite 'expanding' sequence of sets in $\Lambda$, then $\bigcup_{n=1}^{\infty} A_n \in \Lambda$.

   Show that this is equivalent to the definition we gave in lectures. (Remember to show the implication in both directions.)

2. For a set $A \subset \mathbb{R}$ and a real number $\alpha$, write $\alpha A$ for the dilation

   $$\alpha A = \{\alpha x : x \in A\}.$$

   Let $\lambda$ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

   (a) Show that $\lambda(\alpha A) = |\alpha|\lambda(A)$.

   (b) Show that the Lebesgue measure is the unique measure on $(\mathbb{R}, \mathcal{B})$ with $\lambda(\alpha A) = |\alpha|\lambda(A)$ and $\lambda\big([0,1)\big) = 1$.

   (c) What is the equivalent result in $d$ dimensions? (A sketch proof is sufficient.)

3. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing, right-continuous function. Let $\mathcal{J}$ be the set of intervals of the form $(a, b]$, including $\varnothing$, and the infinite intervals $(-\infty, b]$, $(a, \infty)$ and $\mathbb{R}$. Define $\pi_F(\varnothing) = 0$ and $\pi_F\big((a,b]\big) = F(b) - F(a)$.

   (a) Very briefly explain how you know that $\mathcal{J}$ is a semialgebra that generates the Borel $\sigma$-algebra $\mathcal{B}$.

   (b) What can you say about the existence of $F(-\infty) := \lim_{x\to-\infty} F(x)$ and $F(\infty) := \lim_{x\to\infty} F(x)$?

   (c) We wish to extend $\pi_F$ to a premeasure on $(\mathbb{R}, \mathcal{J})$. How should we define $\pi_F$ for the infinite intervals?

   (d) Show that this $\pi_F$ is indeed a premeasure on $(\mathbb{R}, \mathcal{J})$. (You may wish to follow the similar proof of Lemma 6.9 from the lectures.)

   (e) Explain why $\pi_F$ extends to a measure $\lambda_F$ on $(\mathbb{R}, \mathcal{B})$. Show that this measure is the unique such extension.

   (This unique extension is called the *Lebesgue–Stieltjes measure* corresponding to $F$.)

   (f) Show that the Lebesgue measure is a special case of the Lebesgue–Stieltjes measure.

   (g) Explain how we know that the left limit $F(x-) := \lim_{y\uparrow x} F(y)$ exists for every $x \in \mathbb{R}$.

   (h) What is $\lambda_F\big(\{x\}\big)$?

   (i) What is $\lambda_F\big([a,b)\big)$?

   (j) To what extent is the function $F$ determined by its Lebesgue–Stieltjes measure $\lambda_F$. (*Hint:* To start with, assume $F(0) = 0$. Can you build $F$ from the measure $\lambda_F$?)

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