The Capacity of Adaptive Group Testing

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Abstract—We define capacity for group testing problems and deduce bounds for the capacity of a variety of noisy models, based on the capacity of equivalent noisy communication channels. For noiseless adaptive group testing we prove an information-theoretic lower bound which tightens a bound of Chan et al. This can be combined with a performance analysis of a version of Hwang's adaptive group testing algorithm, in order to deduce the capacity of noiseless and erasure group testing models.

I. INTRODUCTION AND NOTATION

We consider the problem of noiseless group testing, as introduced by Dorfman [1] in the context of testing populations of soldiers for syphilis. Group testing has recently been used to screen DNA samples for rare alleles by a pooling strategy, and relates to compressive sensing – see [2]–[4]. It also offers advantages for spectrum sharing in cognitive radio models [5].

Suppose we have a population of N items, K of which are defective. At the ith stage, we pick a subset $\mathcal{S}_i$ of the population, and test all the items in $\mathcal{S}_i$ together. If $\mathcal{S}_i$ contains at least one defective item, the test is said to be positive. If $\mathcal{S}_i$ contains no defective item, the test is said to be negative.

The group testing problem requires us to infer which items are defective, using the smallest number of tests. We have the freedom to design the tests in a way that minimises the (expected) number of tests required. For simplicity, this paper will assume that the number of defectives K is known.

The problem as described above is referred to as noiseless, since the test output is a known deterministic function of the items tested. In this case, positive and negative tests allow us to make complementary inferences. If the ith test is negative, we know that all the items in $\mathcal{S}_i$ are not defective, so we need not test them again. If the ith test is positive, at least one item in $\mathcal{S}_i$ is defective, but it will usually require further testing to discover which. We refer to an item which has not yet appeared in a negative test as a Possible Defective.

It is natural to consider noisy models of group testing. For example, [6] considers an "additive model" (only false positives occur) where a negative test may erroneously be reported positive with probability p. The paper [7] considers a "symmetric error model" where the output of the tests are transmitted through memoryless binary symmetric channels with error probability p. Finally, we consider an "erasure model", where with probability p, the test simply fails, and returns an erasure symbol.

In this paper we consider adaptive testing, where the test design depends on the outcome of previous tests, in contrast to the non-adaptive case, where the entire sequence of tests is specified in advance. In other contexts (see for example [8]), the ability to perform adaptive measurements can provide significant improvements (for example, moving problems from polynomial accuracy to exponential accuracy). Perhaps surprisingly (see [9]), in the context of group testing, the improvements are smaller, in that allowing adaptive algorithms only provides an improvement by a constant factor in the bounds (see for example Theorems 2.2 and 2.4).

We follow the notation of Atia and Saligrama [6] who formulated group testing as a channel coding problem. We make analogies in this sense between the number of tests T required and the code block length, and between the number of possible defective sets $\binom{N}{K}$ and the number of possible messages. This suggests that in the spirit of Shannon [10] we should consider the rate as $\log_2 \binom{N}{K}/T$. This prompts us to introduce the capacity of a group testing problem:

Definition 1.1: Consider a sequence of group testing problems, indexed by the number of items $N = 1, 2, \ldots$. The Nth problem has $K = K(N)$ defective items, where $1 \leq K \leq N$. and $T = T(N)$ tests are available. We refer to a constant C as the capacity if for any $\epsilon > 0$:

1) any sequence of algorithms with

$$\liminf_{N \to \infty} \frac{\log_2 \binom{N}{K(N)}}{T(N)} \geq C + \epsilon, \tag{1}$$

has success probability tending to 0,

2) and there exists a sequence of algorithms with

$$\liminf_{N \to \infty} \frac{\log_2 \binom{N}{K(N)}}{T(N)} \geq C - \epsilon \tag{2}$$

with success probability tending to 1.

Malyutov [4, Section 2] gives a different definition of capacity, based on the defintion of the rate (in our notation) as $(\log N)/T$ for given K. Malyutov's survey paper [4] gives a historical perspective on problems of this kind.

Group testing makes particular gains when the set of defectives is sparse, in that $K = o(N)$. Otherwise, we could be accurate to within a multiple of the optimal answer simply by testing all items individually. For benchmarking purposes, it

is plausible to use the parameterisation used by [8], [11] and others, that is, to take $K = N^{1-\beta}$ for some $0 < \beta < 1$. The main result of the paper is the following:

Theorem 1.2: The capacity of the adaptive noiseless group testing problem is $C = 1$, in any regime such that $K = o(N)$.

Theorem 1.2 is proved in Section III by combining existing performance guarantees (summarised in Section II) with a new information theoretic lower bound, Theorem 3.1.

Remark 1.3: As discussed below, our Theorem 3.1 improves on a result of Chan et al [7] – see (6) below - which does not give the capacity in the sense of Definition 1.1. [7] shows that if $\log_2 \binom{N}{K}/T \geq 1+\epsilon$, then the success probability is bounded above by $1/(1+\epsilon)$, rather than tending to zero. To be precise, Definition 1.1 requires a strong converse (in the sense discussed in [12]), whereas [7] only proves a weak converse.

It is natural to ask what is the capacity of noisy group testing problems under different noise models. The sequence of outcomes of tests can be considered as a binary codeword encoding the defective set. However, tests can only verify whether a subset intersects the defective set, whereas in a general communication channel the encoding being agreed by the receiver and the sender may incorporate more information. Hence we can state a new key principle:

> if it exists, the capacity of a noisy group testing problem can never exceed the capacity of the equivalent communication channel.

To derive these capacity upper bounds we must use the strong converse to Shannon's channel coding theorem [12]. We then state the following bounds on group testing capacities:

Theorem 1.4: 1) In any regime with $K = o(N)$, for the "erasure model" with probability p, $C = 1 - p$.
2) For the "symmetric error model" with probability p of [7], if it exists $C \leq 1 - h(p)$, with h the binary entropy. It is natural to conjecture $C = 1 - h(p)$.
3) For the "additive model" with probability p of [6], if it exists $C \leq \log(1 + (1-p)p^{p/(1-p)})$, the capacity of a Z-channel. Again we conjecture equality holds.

Proof: In each case, the lower bound follows from the key principle above, using standard values of channel capacities. The exact value for 1) is derived in Section III below. ∎

The structure of the remainder of the paper is as follows. In Section II we review existing approaches to the group testing problem, describing upper bounds on the number of the tests required, including Hwang's Generalized Binary Splitting Algorithm. In Section III we prove lower bounds on the number of tests required, giving an upper bound on the success probability, and proving the capacity theorem, Theorem 1.2. In Section IV we give simulation evidence which shows that these bounds are tight for a range of success probabilities, and prove Theorem 4.3 which shows we can improve on Hwang's algorithm, and can be adapted to suggest bounds on success probabilities in general.

II. Existing performance guarantees

We first describe some simple bounds on the performance of group testing algorithms. We will describe these as bounds on the number of tests required, though in Figure 1 such bounds are plotted in terms of success probabilities. Clearly, an upper bound on the number of tests required will relate to a lower bound on the success probability, and vice versa.

Early approaches to the group testing problem involved deterministic (combinatorial) test designs, as reviewed in [13]. In particular, at least $\Omega(K^2 \log N/ \log K)$ tests are required for such designs in the non-adaptive case to guarantee success.

More recent work has focussed on the use of randomised test designs. As a result, Atia and Saligrama [6] were able to show that $O(K \log N)$ tests would suffice with high probability, for adaptive group testing, even in the noisy case. This work was developed by Chan et al [7], who provided explicit algorithms, and corresponding bounds on their performance, in the non-adaptive case. In a paper [14] being prepared in parallel to this one we strengthen some of the results of [7].

Performance of many adaptive algorithms is guaranteed by the following simple lemma:

Lemma 2.1: Given a set of b items that is known to contain at least one defective item, label its elements $1, \ldots, b$ and let L be the smallest label of a defective item in the set. Then in $\lceil \log_2 b \rceil$ tests we can discover with certainty that item L is defective and that items $1, \ldots, L-1$ are all non-defective.

Proof: Use the following recursive procedure, which we refer to as 'binary search'. If necessary, we add 'dummy items' to create a set of size $2^{\lceil \log_2 b \rceil}$. At each stage, given a set of size S which is guaranteed to contain a defective, we label its items with integers $\{1, 2, \ldots, S\}$. We test the items with labels $\{1, 2, \ldots, S/2\}$.

1) If the test is positive, we have a set of half the previous size, guaranteed to contain a defective.
2) If the test is negative, we know that items $\{S/2 + 1, \ldots, S\}$ must contain a defective item.

Each test therefore halves the size of the set, with it remaining guaranteed to contain a defective item. The property of finding the defective with the smallest label follows easily by induction and by the fact that at each step we always test items $\{1, 2, \ldots, S/2\}$ before $\{S/2+1, \ldots, S\}$, and discard the latter if the former is found to be positive. ∎

A simple adaptive algorithm with guaranteed performance bounds is given by Repeated Binary Testing – see for example [13, p24–5]. The algorithm simply performs binary search on the set of size N, in order to find a defective item. This item is then removed from consideration, and the next round of testing carries out binary testing on a set of size $N - 1$. Repeatedly using Lemma 2.1, it is clear that this algorithm provides a performance guarantee for adaptive testing.

Theorem 2.2 ([13]): Repeated Binary Testing is guaranteed to succeed in $K \lceil \log_2 N \rceil \leq K \log_2 N + K$ tests.

The Repeated Binary Testing algorithm is inefficient, in that each round of tests starts by testing large sets, which are very

likely to contain at least one defective. In that sense, the early tests in each round are very uninformative.

Hwang's Generalized Binary Splitting Algorithm [15] is designed to overcome this problem. Hwang suggests testing groups of size 2^α, where α is an integer chosen to ensure that the probability of the test being positive is close to $1/2$. If the test is negative, all the items in it can be immediately classified as non-defective. If the test is positive, it must contain a defective, which can be found in α tests using the binary search procedure of Lemma 2.1 above.

Using this procedure, Hwang [15, Theorem 1] deduces an upper bound on the number of tests required, which further analysis (see [13, Corollary 2.2.2]) shows is close to optimal in the sense discussed in the proof of Theorem 1.2.

Theorem 2.3 ([13]): Given a problem with K defectives in a population of size N, Hwang's adaptive Generalized Binary Splitting Algorithm is guaranteed to succeed using

$$T = \log_2 \binom{N}{K} + K \text{ tests.} \tag{3}$$

In order to see the gain due to adaptivity, this bound can be compared with the following result, Theorem 4 of [7].

Theorem 2.4 ([7]): The Combinatorial Orthogonal Matching Pursuit (COMP) algorithm recovers the defective set with error probability $\leq N^{-\delta}$, using $T = (2(1+\delta)e)K \ln N$ tests.

Theorems 2.3 and 2.4 can be compared using well-known bounds on the binomial coefficients, that is for all K and N:

$$K \log_2(N/K) \leq \log_2 \binom{N}{K} \leq K \log_2(Ne/K), \tag{4}$$

In the regime $K(N) = N^{1-\beta}$ considered by [8], [11], we can see that (4) and Theorem 2.3 can be combined to show that Hwang's adaptive algorithm is guaranteed to succeed in

$$\beta K \log_2 N + K(\log_2 e + 1)$$

tests. In contrast, Theorem 2.4 shows that the (non-adaptive) COMP algorithm [7] requires

$$\left(\frac{2(1+\delta)e}{\log_2 e} \right) K \log_2 N \approx 3.76(1+\delta)K \log_2 N,$$

showing that adaptivity offers asymptotic gains over the bounds of Theorem 2.4 which are greatest when β is small – which is when there are relatively many defectives.

III. NEW INFORMATION-THEORETIC LOWER BOUND

We can contrast the performance guarantees of Section II with a new information-theoretic lower bound, Theorem 3.1. Note that Theorem 3.1 applies to both adaptive and non-adaptive group testing. It is possible that a tighter lower bound could be found if we restrict ourselves to non-adaptive tests, but we are not aware of such a result.

Theorem 3.1: Consider testing a set of N items with K defectives. Any algorithm to recover the defective set with T tests has success probability $\mathbb{P}(\text{suc})$ satisfying

$$\mathbb{P}(\text{suc}) \leq \frac{2^T}{\binom{N}{K}}. \tag{5}$$

Proof: Given a population of N objects, we write $\Sigma_{N,K}$ for the collection of subsets of size K from the population. Further, we write $\mathcal{D}$ for the true defective set.

The testing procedure naturally defines a mapping $\theta : \Sigma_{N,K} \to \{0,1\}^T$. That is, given a putative defective set $S \in \Sigma_{N,K}$, write $\theta(S)$ to be the vector of test outcomes, with positive tests represented as 1s and negative ones represented as 0s. For each vector $\mathbf{y} \in \{0,1\}^T$, write $\mathcal{A}_\mathbf{y} \subseteq \Sigma_{N,K}$ for the inverse image of $\mathbf{y}$ under θ,

$$\mathcal{A}_\mathbf{y} = \theta^{-1}(\mathbf{y}) = \{S \in \Sigma_{N,K} : \theta(S) = \mathbf{y}\},$$

and write $A_\mathbf{y} = |\mathcal{A}_\mathbf{y}|$ for the size of $\mathcal{A}_\mathbf{y}$.

The role of an algorithm which decodes the outcome of the tests is to mimic the effect of the inverse image map θ^{-1}. Given a test output $\mathbf{y}$, the optimal decoding algorithm would use a lookup table to find the inverse image $\mathcal{A}_\mathbf{y}$. If this inverse image $\mathcal{A}_\mathbf{y} = \{S\}$ has size $A_\mathbf{y} = 1$, we can be certain that the defective set was S. In general, if size $A_\mathbf{y} \geq 1$, we cannot do better than to pick uniformly among $\mathcal{A}_\mathbf{y}$, with success probability $1/A_\mathbf{y}$. (We can ignore empty $\mathcal{A}_\mathbf{y}$, since we are only concerned with vectors $\mathbf{y}$ which occur as a test output).

Hence overall, the probability of recovering a defective set S is $1/A_{\theta(S)}$, depending only on $\theta(S)$. We can write the following expression for the success probability, conditioning over all the equiprobable values of the defective set:

$$\mathbb{P}(\text{suc}) = \sum_{S \in \Sigma_{N,K}} \mathbb{P}(\text{suc} \mid \mathcal{D} = S) \frac{1}{\binom{N}{K}}$$

$$= \frac{1}{\binom{N}{K}} \sum_{S \in \Sigma_{N,K}} \sum_{\mathbf{y} \in \{0,1\}^T} \mathbb{I}(\theta(S) = \mathbf{y}) \mathbb{P}(\text{suc} \mid \mathcal{D} = S)$$

$$= \frac{1}{\binom{N}{K}} \sum_{S \in \Sigma_{N,K}} \sum_{\mathbf{y} \in \{0,1\}^T : A_\mathbf{y} \geq 1} \mathbb{I}(\theta(S) = \mathbf{y}) \frac{1}{A_\mathbf{y}}$$

$$= \frac{1}{\binom{N}{K}} \sum_{\mathbf{y} \in \{0,1\}^T : A_\mathbf{y} \geq 1} \frac{1}{A_\mathbf{y}} \left(\sum_{S \in \Sigma_{N,K}} \mathbb{I}(\theta(S) = \mathbf{y}) \right)$$

$$= \frac{1}{\binom{N}{K}} \sum_{\mathbf{y} \in \{0,1\}^T : A_\mathbf{y} \geq 1} \frac{1}{A_\mathbf{y}} A_\mathbf{y}$$

$$= \frac{|\mathbf{y} \in \{0,1\}^T : A_\mathbf{y} \geq 1|}{\binom{N}{K}} \leq \frac{2^T}{\binom{N}{K}},$$

since $\{0,1\}^T$, a set of size 2^T. ∎

The fact that $\log_2 \binom{N}{K}$ is the 'magic number' of tests providing a lower bound on the number of tests required for recovery with success probability 1 is folklore – see for example [15]. However, the exponential scaling of success probability for lower numbers of tests which we provide here is new. Theorem 3.1 is a strengthening of Theorem 1 of [7], which implies that

$$\mathbb{P}(\text{suc}) \leq \frac{T}{\log_2 \binom{N}{K}}. \tag{6}$$

In fact, Theorem 1 of [7] is stated with $K \log_2(N/K)$ in the denominator – the stronger form given by (6) is given within their proof. The differing form of (5) and (6) is plotted in
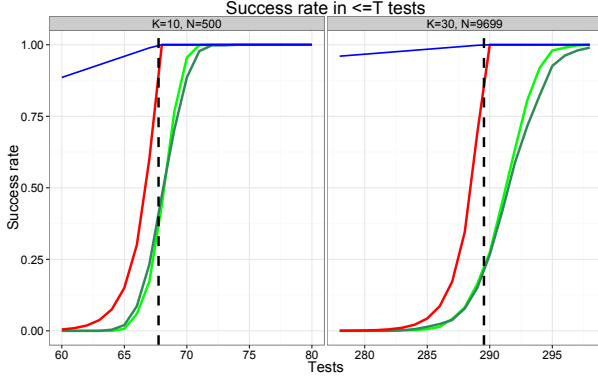
Fig. 1. Plot of success probability of Hwang's group testing algorithm for $(K, N) = (10, 500)$ and $(30, 9699)$. The simulated success probability of Hwang's algorithm is plotted as a bright green line, and the related algorithm analysed in Section IV is plotted in dark green. The upper bound on success probability of Theorem 3.1 is plotted in red, and the upper bound of [7] Equation (6) in blue. The dotted vertical line is at $\log_2 \binom{N}{K}$.

Figure 1, emphasising the point made in Remark 1.3 that (5) is significantly stronger.

Observe that (using the fact that for any random variable $\mathbb{E}T = \sum_{t=0}^{\infty}(1 - \mathbb{P}(T \le t))$, (5) implies that for any algorithm that uses a random number of tests T to detect the defective set with certainty, the expected success time

$$\mathbb{E}T \ge \log_2 \binom{N}{K} - 2. \qquad (7)$$

We now prove the main result of the paper, Theorem 1.2:

Proof of Theorem 1.2: The result is obtained using the binomial coefficient bounds (4), with the lower bound meaning that in the regime $K = o(N)$, then

$$\lim_{N \to \infty} \frac{\log_2 \binom{N}{K(N)}}{K(N)} = \infty, \qquad (8)$$

since we also have $K \ge 1$. Now fix $\epsilon > 0$. First if, as assumed in (1) for N sufficiently large, $T(N) \le \frac{1}{1+\epsilon} \log_2 \binom{N}{K(N)}$ then Theorem 3.1 shows that $\mathbb{P}(\mathrm{suc}) \le \binom{N}{K(N)}^{-\epsilon/(1+\epsilon)}$. We deduce the strong converse, i.e. that $\mathbb{P}(\mathrm{suc})$ tends to zero by (8).

Theorem 2.3 shows that Hwang's Generalized Binary Splitting Algorithm is guaranteed to succeed using $T(N) = \log_2 \binom{N}{K(N)} + K(N)$ tests. We can deduce by (8) that $\log_2 \binom{N}{K(N)}/T(N)$ is greater than $1 - \epsilon$, for all N sufficiently large, so (2) follows with $C = 1$. ∎

Proof of Theorem 1.4: Extending this, we deduce the capacity of the erasure model for adaptive group testing in the $K = o(N)$ regime is exactly $1 - p$. We simply repeat any erased test until erasure fails to happen and then use Hwang's algorithm. We need to have a number of non-erased tests greater than the bound of Theorem 2.3. The probability that this doesn't happen goes to 0 exponentially, hence with $\log \binom{N}{K}/(1 - p - \varepsilon)$ tests the probability of success approaches 1 exponentially fast. ∎

IV. A TIGHTER UPPER BOUND

We briefly illustrate the performance of Hwang's algorithm by simulation. For example, in the two cases of Figure 1 we keep the parameter $\beta = 0.63$ fixed and plot the success probability of the algorithm compared with the lower bound of Theorem 3.1 for different problem sizes. Notice that the (bright green) simulated success curve of Hwang's algorithm is very close to the (red) theoretical upper bound on success probability (lower bound on tests required). The dark green line corresponds to the performance of the same algorithm using the sample size discussed in Section IV. The dotted vertical line is at $\log_2 \binom{N}{K}$, close to the rapid transition of success probability.

Figure 1 clearly shows that the bound of Theorem 3.1 is close to the true performance of the algorithm in these cases. We therefore prove Theorem 4.3, an upper bound on the performance of a version of Hwang's algorithm, which reduces the gap between the upper bound of Theorem 2.3 and the information-theoretic lower bound, from K to under $K/2$. We group the tests in K rounds, each of which comprises a sequence of negative tests before a positive test. Once a positive test occurs, we can find a defective using binary search as in Lemma 2.1. We introduce the following notation, which allows us to keep track of the size of each subproblem (round).

Definition 4.1: We write $N_j^{(i)}$ the number of Possible Defectives left after the j-th consecutive negative test of the i-th round and $N_0^{(i)}$ for the number of Possible Defectives at the start of the i-th round, and likewise we write $K^{(i)} = K - i + 1$ for the number of defectives in the i-th round. We write $S_j^{(i)}$ for the indicator that the j-th test of the i-th round is positive. Finally define T_i as the random time of the last negative test of the round.

After every negative test the set of Possible Defectives will be reduced. In particular, if we denote by $b_j^{(i)}$ the size of the j-th test in the i-th round, then

$$N_{j+1}^{(i)} = N_j^{(i)} - b_{j+1}^{(i)}, \ 1 \le i \le K, \ 0 \le j \le T_i . \qquad (9)$$

As described previously, we use maximally informative tests by making the probability of observing a negative test (just) less than $1/2$. By truncating binomial coefficients, we prove:

Lemma 4.2: Conditional on having observed $j - 1$ consecutive negative tests in the i-th round of Hwang's algorithm, any random sample of size

$$b_j^{(i)} \ge N_{j-1}^{(i)} \left(1 - 2^{-1/K^{(i)}}\right) . \qquad (10)$$

has probability less than $1/2$ of being negative, that is

$$\mathbb{P}(S_j^{(i)} = 0 \mid S_0^{(i)} = 0, \ldots, S_{j-1}^{(i)} = 0) \le 1/2.$$

If the test is negative, we update $N_j^{(i)}$ using (9). If the test is positive, the only upper bound we can derive for sure for the new size of the set of possible defectives is $N_j^{(i)} \le N_{j-1}^{(i)} - 1$, as we're only sure that one (defective) item will be removed from the set. By induction, within each round the following

formulae for $b_j^{(i)}$ and $N_j^{(i)}$ hold:

$$b_j^{(i)} = 2^{-\frac{j-1}{K^{(i)}}}\left[N(1 - 2^{-1/K^{(i)}}) - (K^{(i)} - 1)\right] \quad (11)$$

$$N_j^{(i)} = 2^{-\frac{j}{K^{(i)}}}N_0^{(i)} + (K^{(i)} - 1)\sum_{h=0}^{j-1}2^{-\frac{h}{K^{(i)}}} \ . \quad (12)$$

Here $N_0^{(i)}$ clearly depends on the previous round; it equals N for $i = 1$ and it can be suitably upper bounded for $i \geq 2$. With these results we can produce an upper bound on the average number of tests for this version of Hwang's algorithm.

Theorem 4.3: Our version of Hwang's algorithm is guaranteed to succeed with number of tests satisfying:

$$T_{tot} \leq K \log N + (1 + \log \ln 2)K - \log K! + R \ . \quad (13)$$

where R is a negative random term.

Proof: We observe that

$$b_j^{(i)} = \left\lceil N_{j-1}^{(i)}\left(1 - 2^{-1/K^{(i)}}\right)\right\rceil \leq 2N_{j-1}^{(i)}\left(1 - 2^{-1/K^{(i)}}\right) \ . \quad (14)$$

Similarly, repeatedly substituting (10) in (9) we obtain

$$N_j^{(i)} \leq N_0^{(i)}2^{-j/K^{(i)}} \ . \quad (15)$$

This can be plugged back into (14), giving

$$b_j^{(i)} \leq 2N_0^{(i)}2^{-(j-1)/K^{(i)}}(1 - 2^{-1/K^{(i)}}) \ . \quad (16)$$

We can then compute the upper bound on the total number of tests using Lemma 2.1. The number of tests in the ith round satisfies:

$$T_{tot}^{(i)} = T_i + \log b_{T_i+1}^{(i)} \leq T_i + 1 + \log N_0^{(i)}$$
$$- \frac{T_i}{K^{(i)}} + \log(1 - 2^{-1/K^{(i)}}) \ . \quad (17)$$

In order to obtain a bound on the total number of tests we have to relate $N_0^{(i)}$ with $N_{T_{i-1}}^{(i-1)}$. Recalling Lemma 2.1, we define L_i by making $1 + L_i$ be the position of the leftmost defective in the $(T_i + 1)$-th sample, thus obtaining the equality

$$N_0^{(i+1)} = N_{T_i}^{(i)} - L_i \ . \quad (18)$$

Using iteratively the update formulae $N_j^{(i)} \leq N_0^{(i)}2^{-j/K^{(i)}}$ and (18), we can deduce that $\log N_0^{(i+1)}$ equals

$$\log N - \sum_{j=1}^{i}\frac{T_j}{K^{(j)}} + \log\left[1 - \frac{1}{N}\sum_{h=1}^{i}\frac{L_h}{\prod_{j=1}^{h}2^{-T_j/K^{(j)}}}\right] \ .$$

Summing together the bounds for the number of tests in each round and calling $R = \sum_{i=1}^{n}C_i$, we obtain:

$$T_{tot} \leq K \log N + K + \sum_{i=1}^{K}\log(1 - 2^{-1/K^{(i)}}) + R. \quad (19)$$

To gain a more manageable expression, we bound the penultimate term of (19). Calling $f(x) := 2^{-x} - 1 + x \ln 2$, notice that $f'(x) \geq 0$ for $x \geq 0$ and $f(0) = 0$; in particular, $1 - 2^{-1/i} \leq \frac{1}{i}\ln 2$. Substituting $K^{(i)} = K - i + 1$, reversing the order of the sum and taking logs, we deduce the result. ∎

It is hard to say more about the random terms C_i than that they are negative, though we hope that future simulation and probabilistic bounds will give us further insights into the resulting number of tests required. Notice the slightly suprising feature that the number of negative tests T_i has no effect on the final bound (19). This can be explained by the fact that summing (17) over i creates a double sum over i and j, and that the coefficients of T_i exactly cancel.

V. Conclusion

Using a sharper information-theoretic lower bound, we have shown that in the noiseless adaptive case, the capacity of group testing is 1, and that for an erasure channel, the capacity is $1 - p$. For other noise models, we have found lower bounds on the capacity. It remains of interest to find exact values of the capacity in other cases, including non-adaptive problems.

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