MA40042 Measure Theory and Integration

Solutions: Sheet 2, Question 3

A few points about this question:

- *This was a hard question, so don't worry if you found it difficult.*
- *The measure space $(X, \overline{\Sigma}, \bar{\mu})$ we form here is called the* completion *of (X, Σ, μ). It's based on the idea that, if something is the subset of a set of measure 0, then (perhaps) it 'ought' to have measure 0 also.*
- *Take care when dealing with $N \in \mathcal{N}$. This N may not be measurable, so is very difficult to work with. The point is, though, that $N \subset Z$ for some Z that* is *measurable. So your reasoning about N should go via reasoning about Z.*

3. Given a measure space (X, Σ, μ), we write $\mathcal{Z}$ for the collection of sets with measure zero, and $\mathcal{N}$ for the collection of *null sets* – the subsets of sets with measure zero – so

$$\mathcal{Z} := \{Z \in \Sigma : \mu(Z) = 0\},$$
$$\mathcal{N} := \{N \subset X : N \subset Z \text{ for some } Z \in \mathcal{Z}\}.$$

A measure space is called *complete* if every null set is measurable with measure zero; that is, if $\mathcal{N} = \mathcal{Z}$.

Let (X, Σ, μ) be a (not necessarily complete) measure space.

(a) Let $\overline{\Sigma} = \{A \cup N : A \in \Sigma, N \in \mathcal{N}\}$.

i. Show that $\overline{\Sigma}$ is a σ-algebra.

Solution: We need to check the three usual points.

1. The empty set is in $\mathcal{Z}$ and, since $\varnothing \subset \varnothing$ and $\mu(\varnothing) = 0$, also in $\mathcal{N}$. Hence $\varnothing = \varnothing \cup \varnothing \in \overline{\Sigma}$.

2. Let $A \cup N \in \overline{\Sigma}$, with $A \in \Sigma$ and $N \subset Z \in \mathcal{Z}$. Then $(A \cup N)^{\mathsf{c}} = A^{\mathsf{c}} \cap N^{\mathsf{c}}$. Now A^{c} is in Σ; what can we say about N^{c}? Well, $N \subset Z$, with $Z \in \Sigma$, so it might be helpful to write $N^{\mathsf{c}} = Z^{\mathsf{c}} \cup (Z \setminus N)$. (Draw a picture if this equality isn't obvious.) So we have

$$(A \cup N)^{\mathsf{c}} = A^{\mathsf{c}} \cap \big(Z^{\mathsf{c}} \cup (Z \setminus N)\big) = (A^{\mathsf{c}} \cap Z^{\mathsf{c}}) \cup \big(A^{\mathsf{c}} \cap (Z \setminus N)\big).$$

First, A and Z are in Σ, so $A^{\mathsf{c}} \cap Z^{\mathsf{c}}$ is in Σ also. Second, $Z \setminus N \subset Z$, so $A^{\mathsf{c}} \cap (Z \setminus N) \subset Z$ also, meaning $A^{\mathsf{c}} \cap (Z \setminus N) \in \mathcal{N}$. Thus we have written $(A \cup N)^{\mathsf{c}}$ in the desired form.

3. Let $A_1 \cup N_1, A_2 \cup N_2, \dots$ be a countably infinite sequence of sets with $A_n \in \Sigma$ and $N_n \subset Z_n \in \mathcal{Z}$ for all n. Then

$$\bigcup_{n=1}^{\infty} (A_n \cup N_n) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} N_n.$$

For the first term, $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, since Σ is a σ-algebra. For the second term, $\bigcup_{n=1}^{\infty} N_n \subset \bigcup_{n=1}^{\infty} Z_n$. By countable subadditivity of μ, the union $\bigcup_{n=1}^{\infty} Z_n$ has measure 0, so is in $\mathcal{Z}$. Hence, $\bigcup_{n=1}^{\infty} N_n$ is a subset of a measure 0 set, so is in $\mathcal{N}$. Hence we have written $\bigcup_{n=1}^{\infty}(A_n \cup N_n)$ in the desired form.

ii. Explain why $\overline{\Sigma}$ is the smallest σ-algebra containing Σ and $\mathcal{N}$.

Solution: Any σ-algebra containing Σ and $\mathcal{N}$ clearly must contain the sets of the form $A \cup N$ for $A \in \Sigma$ and $N \in \mathcal{N}$. Any such σ-algebra will thus contain $\overline{\Sigma}$.

(b) We define a function $\bar{\mu}$ on $\overline{\Sigma}$ as follows: for $A \in \Sigma$ and $N \in \mathcal{N}$, let $\bar{\mu}(A \cup N) = \mu(A)$.

i. Explain why we need to show that $\bar{\mu}$ is 'well-defined'.

Solution: The same set might be able to be written in two different ways as $A_1 \cup N_1$ and as $A_2 \cup N_2$. It's not immediately clear that this will give the same result for $\bar{\mu}$.

ii. Show that $\bar{\mu}$ is well-defined.

Solution: Suppose $A_1 \cup N_1 = A_2 \cup N_2$, with $A_1, A_2 \in \Sigma$, and

$N_1 \subset Z_1 \in \mathcal{Z}, N_2 \subset Z_2 \in \mathcal{Z}$. Then we have

$$A_1 \subset A_1 \cup N_1 = A_2 \cup N_2 \subset A_2 \cup Z_2.$$

Hence,

$$\mu(A_1) \le \mu(A_2 \cup Z_2) \le \mu(A_2) + \mu(Z_2) = \mu(A_2),$$

where we have used monotonicity, finite subadditivity, and the fact that $\mu(Z_2) = 0$. We have shown that $\mu(A_1) \le \mu(A_2)$.

The same argument with 1s and 2s swapped over gives $\mu(A_2) \le \mu(A_1)$. Hence $\mu(A_1) = \mu(A_2)$, and we have $\bar{\mu}(A_1 \cup N_1) = \bar{\mu}(A_2 \cup N_2)$.

iii. Show that $\bar{\mu}$ is a measure on $(X, \overline{\Sigma})$.

Solution: We have two points to check.

1. Clearly $\bar{\mu}(\varnothing) = \bar{\mu}(\varnothing \cup \varnothing) = \mu(\varnothing) = 0$.
2. Let $A_1 \cup N_1, A_2 \cup N_2, \dots$ be a countably infinite disjoint sequence of sets with $A_n \in \Sigma$ and $N_n \subset Z_n \in \mathcal{Z}$ for all n. Note that the A_ns are also disjoint. First, by monotonicity,

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty}(A_n \cup N_n)\right) &\ge \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \bar{\mu}(A_n \cup N_n).\end{aligned}$$

Second, we have

$$\bigcup_{n=1}^{\infty}(A_n \cup N_n) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} N_n \subset \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} Z_n,$$

which gives

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty}(A_n \cup N_n)\right) &\le \mu\left(\bigcup_{n=1}^{\infty} A_n\right) + \mu\left(\bigcup_{n=1}^{\infty} Z_n\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \bar{\mu}(A_n \cup N_n).\end{aligned}$$

Since we have inequalities both ways, we're done.

iv. Show that $(X, \overline{\Sigma}, \bar{\mu})$ is a complete measure space.

Solution: Let $A \cup N \in \overline{\Sigma}$ with $A \in \Sigma$ and $N \subset Z$ with $Z \in \Sigma$ having μ-measure 0, and assume $\bar{\mu}(A \cup N) = 0$. Let $M \subset A \cup N$. We need to show that $M \in \overline{\Sigma}$. It will then follow by monotonicity that

$$\bar{\mu}(M) \le \bar{\mu}(A \cup N) = 0,$$

and so $\bar{\mu}(M) = 0$.

Note that $\mu(A) = \bar{\mu}(A \cup N) = 0$. Hence $A \cup N \subset A \cup Z$, and since A and Z have μ-measure 0, so does $A \cup Z$. Hence, when we write $B = \varnothing \cup (A \cup N)$, the first term $\varnothing$ is in Σ, and the second term $A \cup N \subset A \cup Z$ is a μ-null set. Thus we have written B in the necessary form for it to be in $\overline{\Sigma}$, and we are done.

(c) Let μ^* be the outer measure on X constructed from $\mathcal{R} = \Sigma$ and $\rho = \mu$ in the standard way. Show that, for $B \in \overline{\Sigma}$, we have $\mu^*(B) = \bar{\mu}(B)$.

Solution: Write $B = A \cup N$, with $A \in \Sigma$, and $N \subset Z \in \mathcal{Z}$. Clearly $\mathcal{C} = \{A, Z\}$ is covering of B. Hence

$$\mu^*(B) \le \mu(A) + \mu(Z) = \mu(A) = \bar{\mu}(A \cup N) = \bar{\mu}(B).$$

Suppose there was a strictly better covering $\mathcal{C} = \{C_1, C_2, \dots\}$ of B with $\sum_{n=1}^{N} \mu(C_n) < \bar{\mu}(B)$. This $\mathcal{C}$ would also be a covering of A, since $A \subset B$. Since $\bar{\mu}(B) = \bar{\mu}(A \cup N) = \mu(A)$, we would have

$$\bigcup_{n=1}^{N} C_n \supset A \qquad \text{and} \qquad \sum_{n=1}^{N} \mu(C_n) < \mu(A).$$

This contradicts the countable subadditivity and monotonicity of μ. Hence $\mu^*(B) = \bar{\mu}(B)$ also.