Interference Alignment-Based Sum Capacity Bounds for Random Dense Gaussian Interference Networks

Oliver Johnson, Matthew Aldridge, and Robert Piechocki

***Abstract*—We consider a dense K user Gaussian interference network formed by paired transmitters and receivers placed independently at random in a fixed spatial region. Under natural conditions on the node position distributions and signal attenuation, we prove convergence in probability of the average per-user capacity C_Σ/K to $\frac{1}{2}\mathbb{E}\log(1+2\mathrm{SNR})$. The achievability result follows directly from results based on an interference alignment scheme presented in recent work of Nazer *et al.* Our main contribution comes through an upper bound, motivated by ideas of "bottleneck capacity" developed in recent work of Jafar. By controlling the physical location of transmitter–receiver pairs, we can match a large proportion of these pairs to form so-called ϵ-bottleneck links, with consequent control of the sum capacity.**

***Index Terms*—Bottleneck states, dense networks, Gaussian interference networks, interference alignment, matching, sum capacity.**

I. Introduction and Main Result

A. Interference Networks and Bottleneck States

RECENT work of Jafar [1] made significant progress towards what is referred to as 'the holy grail of network information theory', namely the calculation of the capacity of arbitrary Gaussian interference networks. Jafar proves convergence in probability of the averaged sum capacity of certain dense Gaussian interference networks. Although results contained in the paper [1] made significant progress with this problem, the results were described under the constraint that each direct link had the same fading coefficient $\sqrt{\mathrm{SNR}}$ – a constraint that we relax in this paper.

In [1, Lemma 1], Jafar showed that a two-user Gaussian interference channel with one of the cross-link strengths $\mathrm{INR} = \mathrm{SNR}$ has sum capacity exactly equal to $\log(1+2\mathrm{SNR})$. Jafar described such a configuration as an example of a 'bottleneck state', in that altering the other cross-link strength does not affect the capacity. Jafar went on to define the concept of an ϵ-bottleneck link; that is, a cross-link in a two-user channel with capacity within ϵ of $\log(1+2\mathrm{SNR})$. He considered a model of large networks, where each value of SNR is fixed, and each INR is sampled independently from a fixed distribution. Jafar argues that with probabilty δ, each INR lies in the range such that the corresponding two-user channel becomes an ϵ-bottleneck. He

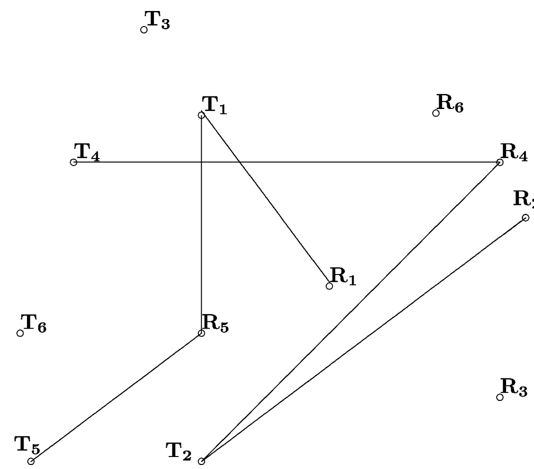

Fig. 1. Dense network with $K = 6$ transmitter–receiver pairs placed on the square $[0,1]^2$, with ϵ-bottleneck links emphasised. The distances from T_5 to R_5, R_5 to T_1 and T_1 to R_1 are all approximately equal, similarly for T_4 to R_4, R_4 to T_2 and T_2 to R_2. Transmitter-receiver pairs (T_3, R_3) and (T_6, R_6) are not matched into ϵ-bottleneck links.

uses an argument based on Chebyshev's inequality to deduce that

$$\mathbb{P}\left(\left|\frac{C_\Sigma}{K} - \frac{1}{2}\log(1+2\mathrm{SNR})\right| > \epsilon\right) = O(K^{-2}). \quad (1)$$

It is perhaps surprising that the existence of a positive proportion of ϵ-bottleneck links implies accurate probabilistic bounds on the sum capacity C_Σ of the whole network. However, we might regard it as analogous to the so-called "birthday paradox" – although each cross-link has a probability δ of being in an ϵ-bottleneck, there are $K(K-1)$ cross-links that can have this property, so as K tends to infinity, the number of cross-links with this property becomes much larger than K.

In this paper, we show that results such as (1) in fact hold more generally, in cases where transmitter node positions $T_1, \ldots, T_K$ and corresponding receiver node positions $R_1, \ldots, R_K$ are chosen independently at random in a region of space $\mathcal{D}$. While exact expressions for the capacity of networks with arbitrarily placed nodes remain elusive, in Theorem 1.5 we prove convergence in probability for the average per-user capacity C_Σ/K of such dense network configurations. The intuition is that in a dense network of points, a large proportion of nodes can be put together pairwise to form ϵ-bottleneck links (see Fig. 1).

While it would be possible to adjust individual transmitters' powers to force each SNR to become equal (as assumed by [1]), individual power constraints make this undesirable. Further, Jafar assumed that the INR are independent and identically distributed, a property that would be lost if user powers were scaled in this way.

Manuscript received July 29, 2009; revised July 09, 2010; accepted July 09, 2010. Date of current version December 27, 2010.

O. Johnson and M. Aldridge are with the Department of Mathematics, University of Bristol, Bristol BS8 1TW, U.K. (e-mail: O.Johnson@bristol.ac.uk; m.aldridge@bristol.ac.uk).

R. Piechocki is with the Centre for Communications Research, University of Bristol, Bristol BS8 1UB, U.K. (e-mail: r.j.piechocki@bristol.ac.uk).

Communicated by R. Berry, Associate Editor for Communication Networks.

Digital Object Identifier 10.1109/TIT.2010.2090242

B. Interference Alignment

The concept of interference alignment first appeared in [2] and then [3], and represents a departure from the recent paradigm of random (or rather pseudo-random) code construction. Pseudo-random codes (e.g., Turbo and LDPC codes) have revolutionized point-to-point communications. However, since multi-terminal networks are interference-limited rather than noise-limited, unstructured (pseudo-random) codes are not suitable.

Interference alignment advocates a collaborative solution. Each receiver divides its signalling space (space, time, frequency or scale resources) into two parts; one for the signal from the intended transmitter, and the second acts as a waste bin. The encoding is structured is such a way that the transmitted signal from each of the K transmitters is seen in the clear space for the intended receiver, and at the same time, it falls into the waste bin for all other receivers. In such a scenario, the network is no longer interference limited. For example Cadambe and Jafar [3] showed that spatio/temporal beamforming in the $\mathrm{SNR} \to \infty$ regime allows to achieve for each user pair a "capacity" equal to half that of the single user channel. A different idea recently appeared in [4], which does not require $\mathrm{SNR} \to \infty$. Consider the following channel pair:

$$\mathbf{H_a} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{H_b} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}. \quad (2)$$

The channels are constructed in such a way that the simple sum of the received signals leads to interference alignment since $\mathbf{H_a} + \mathbf{H_b} = 2\mathbf{I}$. In more general channels, such as those with Rayleigh fading, one needs to code over sufficiently long time intervals to observe and match complementary matrix pairs.

C. Node Positioning Model

We believe that our techniques should work in a variety of models for the node positions. We outline one very natural scenario here.

Definition 1.1: Consider a fixed spatial region $\mathcal{D} \in \mathbb{R}^D$, with two probability distributions $\mathbb{P}_T$ and $\mathbb{P}_R$ supported on $\mathcal{D}$. Given an integer K, we sample the K transmitter node positions $T_1, \ldots, T_K$ independently from the distribution $\mathbb{P}_T$. Similarly, we sample the K receiver node positions $R_1, \ldots, R_K$ independently from distribution $\mathbb{P}_R$. We refer to such a model of node placement as an "IID network".

Equivalently, we could state that transmitter and receiver positions are distributed according to two independent (nonhomogeneous) Poisson processes, conditioned such that there are K points of each type in $\mathcal{D}$. We pair the transmitter and receiver nodes up so that T_i wishes to communicate with R_i for each i. We make the following definition:

Definition 1.2: We say that transmitter and receiver distributions $\mathbb{P}_T$ and $\mathbb{P}_R$ are 'spatially separated' if there exist constants $C_{\mathrm{sep}} < \infty$ and $D_{\mathrm{sep}} < \infty$ such that for $T \sim \mathbb{P}_T$ and $R \sim \mathbb{P}_R$ the Euclidean distance $d(T, R)$ satisfies

$$\mathbb{P}(d(T,R) \leq s) \leq C_{\mathrm{sep}} s^{D_{\mathrm{sep}}} \quad \text{for all } s. \quad (3)$$

We argue in Lemma 2.2 below that a wide range of node distributions $\mathbb{P}_T$ and $\mathbb{P}_R$ have this spatial separation property, which allows us to control the tails of the distribution of SNR, and hence the maximum value of SNR in Lemma 2.3.

D. Transmission Models

For simplicity, we first describe our results in the context of so-called "line of sight" communication models, without multipath interference. That is, we consider a model where signal strengths decay deterministically with Euclidean distance d according to some monotonically decreasing continuous function $f(d)$. We make the following definition, which complements the definition of spatial separation given in Definition 1.2.

Definition 1.3: We say that the signal is "decaying at rate α" if there exists $C_{\mathrm{dec}} < \infty$ such that for all d

$$f(d) \leq C_{\mathrm{dec}} d^{-\alpha}. \quad (4)$$

Standard physical considerations imply that all signals must be decaying at some rate $\alpha \geq D$, where D is the dimension of the underlying space. Tse and Viswanath [5, Section 2.1] discuss a variety of models under which this condition holds for different exponents α. For example, signals with $f(d) = (d + d_0)^{-\alpha}$ for some fixed $d_0 \geq 0$ are decaying at rate α.

We define the full action of the Gaussian interference network:

Definition 1.4: Fix transmitter positions $\{T_1, \ldots, T_K\} \in \mathcal{D}$ and receiver positions $\{R_1, \ldots, R_K\} \in \mathcal{D}$, and consider Euclidean distance d and attenuation function f. For each i and j, define $\mathrm{INR}_{ij} = f(d(T_i, R_j))$. For emphasis, for each i we write SNR_i for INR_{ii}.

We consider the K user Gaussian interference network defined so that transmitter i sends a message encoded as a sequence of complex numbers $\mathbf{X}_i = (X_i[1], \ldots, X_i[N])$ to receiver i, under a power constraint $\frac{1}{N}\sum_{n=1}^{N} |X_i[n]|^2 \leq 1$ for each i. The nth symbol received at receiver j is given as

$$Y_j[n] = \sum_{i=1}^{K} \exp(i\phi_{ij}[n])\sqrt{\mathrm{INR}_{ij}} X_i[n] + Z_j[n] \quad (5)$$

where $Z_j[n] \sim CN(0,1)$ are independent standard complex Gaussians, and $\phi_{ij}[n]$ are $U[0, 2\pi]$ random variables independent of each other and all other terms. The INR_{ij} remain fixed, since the node positions themselves are fixed, but the phases are fast fading. It is assumed that full channel state information is available to each transmitter.

We write S_{ij} for the random variables $\frac{1}{2}\log(1 + 2\mathrm{INR}_{ij})$, which are functions of the distance between T_i and R_j. In particular, since the nodes are positioned independently in Definition 1.1, under this model the random variables $S_{ii} = \frac{1}{2}\log(1 + 2\mathrm{SNR}_i)$ are independent and identically distributed.

In Section IV-A, we explain how our techniques can be extended to apply to more general models, in the presence of random fading amplitudes.

E. Main Result: Convergence in Probability of C_Σ/K

We now state the main theorem of this paper, which proves convergence in probability of the averaged capacity, under the model of node placement described in Definition 1.1 and the model for signal attenuation described in Definition 1.4. For the sake of clarity, we restrict our attention to the case where $\mathbb{P}_R$ and $\mathbb{P}_T$ are uniform, though we discuss later to what extent this assumption is necessary.

We restrict to bounded regions $\mathcal{D}$ with a smooth boundary – the sense of this smoothness will be made precise in the proof of Theorem 1.5. Theorem 1.5 will certainly hold for squares and balls $\mathcal{D} = [0,1]^D$ and $\mathcal{D} = \{\mathbf{x} : d(\mathbf{x}, \mathbf{0}) \leq 1\}$, and indeed for any convex and bounded polytopes (with finite surface area). Essentially we require that the boundary of $\mathcal{D} \times \mathcal{D}$ has Hausdorff dimension $\leq 2D - 1$, which is very natural.

Theorem 1.5: Consider a Gaussian interference network formed by K pairs of nodes placed in an IID network, with the signal decaying at some rate $\alpha \geq D$. If distributions $\mathbb{P}_T$ and $\mathbb{P}_R$ are both uniform on a bounded region $\mathcal{D}$ with smooth boundary, then the average per-user capacity C_Σ/K converges in probability to $\frac{1}{2}\mathbb{E}\log(1+2\mathrm{SNR})$, that is for all $\epsilon > 0$

$$\lim_{K\to\infty} \mathbb{P}\left(\left|\frac{C_\Sigma}{K} - \frac{1}{2}\mathbb{E}\log(1+2\mathrm{SNR})\right| > \epsilon\right) = 0.$$

Proof: We break the probability into two terms which we deal with separately. That is, writing $E = \mathbb{E}S_{ii} = \frac{1}{2}\mathbb{E}\log(1+2\mathrm{SNR})$,

$$\mathbb{P}\left(\left|\frac{C_\Sigma}{K} - E\right| > \epsilon\right) = \mathbb{P}\left(\frac{C_\Sigma}{K} - E < -\epsilon\right) + \mathbb{P}\left(\frac{C_\Sigma}{K} - E > \epsilon\right). \quad (6)$$

The first term of (6) can be bounded relatively simply, using an achievability argument based on an interference alignment scheme presented by Nazer *et al.* [4]. In [4, Theorem 3], it is implied that the rates $R[i] = 1/2\log(1+2\mathrm{SNR}_i) = S_{ii}$ are achievable. This implies that $C_\Sigma \geq \sum_{i=1}^K S_{ii}$. This allows us to bound the first term in (6) as

$$\mathbb{P}\left(\frac{C_\Sigma}{K} - E < -\epsilon\right) \leq \mathbb{P}\left(\frac{\sum_{i=1}^K (S_{ii} - E)}{K} < -\epsilon\right). \quad (7)$$

Note that Lemma 2.2 below implies that the spatial separation condition holds in the setting of Theorem 1.5. Hence the conditions of Lemma 2.3 hold, implying that S_{ii} has finite variance by (12). This means that (7) can be bounded by $\mathrm{Var}\,(S_{ii})/(K\epsilon^2)$, and tends to zero at rate $O(1/K)$.

The major contribution of this paper comes through the converse part, in which we consider the properties of the second term of (6), completing the proof of the theorem at the end of Section III. ∎

Theorem 1.5 can be interpreted in the same way as Theorem 1 of [1], that "each user is able to achieve the same rate that he would achieve if he had the channel to himself with no interferers, half the time".

The theorem is presented under the assumptions of uniform $\mathbb{P}_R$ and $\mathbb{P}_T$ with fading with random phases and deterministic amplitude, for the sake of simplicity of exposition. We believe that the main result will be robust to relaxation of these conditions. In Section IV-C, we consider whether the $\mathbb{P}_R$ and $\mathbb{P}_T$ need necessarily be uniform. In Section IV-A. we introduce a variant of the model with a random fading amplitude term. Although the theorem is based on probabilistic arguments, in Section IV-B we describe an associated algorithm which bounds the sum capacity of arbitrary Gaussian interference networks.

F. Relation to Previous Work

As reviewed in more detail by Jafar [1], recently progress has been made in several directions towards understanding the capacity of Gaussian interference networks.

In problems concerning networks with a large number of nodes, work of Gupta and Kumar [6] uses techniques based on Voronoi tesselations to establish scaling laws (see also Xue and Kumar [7] for a review of the information theoretical techniques that can be applied to this problem).

Özgür, Lévêque, and Tse [8], [9] use a similar model of dense random network placements, though using the same points as both transmitters and receivers. They describe a hierarchical scheme, where nodes are successively assembled into groups of increasing size, each group collectively acting as a MIMO transmitter or receiver, and restrict to transmissions at a common rate. In [9, Ths. 3.1 and 3.2], it is shown that for any $\epsilon > 0$ there exists a constant c_ϵ and a fixed constant c_1 such that

$$c_\epsilon K^{1-\epsilon} \leq C_\Sigma \leq c_1 K \log K. \quad (8)$$

These bounds are close to stating that C_Σ grows like K, but without the explicit constant that Jafar [1] and Theorem 1.5 of this paper achieve. In this paper, we produce a version of the upper bound of (8) without the logarithmic factor and being explicit about the constant c_1. Note that this result is proved under a model that differs from that of [9] in the fact that we have a total of $2K$ nodes rather than K. Further, in [9], local collaboration is allowed, meaning that the true rate in their scenario could indeed be $K\log K$.

An alternative approach to Gaussian interference networks is to consider the limit of the capacity as the SNR tends to infinity, with a fixed number of users. Cadambe and Jafar [3] used interference alignment to deduce the limiting behaviour within $o(\log(\mathrm{SNR}))$. These techniques were extended by the same authors [10] to more general models in the presence of feedback and other effects.

For small networks, the classical bounds due to Han and Kobayashi [11] for the two-user Gaussian interference network have recently been extended and refined. For example Etkin, Tse and Wang [12] have produced a characterization of capacity accurate to within one bit. These results were extended by Bresler, Parekh and Tse [13], using insights based on a deterministic channel which approximates the Gaussian channel with sufficient accuracy, to prove results for many-to-one and one-to-many Gaussian interference channels.

We briefly mention an alternative proof of Theorem 5 of [1], which provides a faster rate of convergence than that achieved in (1). We first review some facts from graph theory, concerning random bipartite graphs formed by two sets of vertices of size N, with edges present independently with probability δ. Erdös and Rényi [14] prove that the probability of a complete matching failing to exist tends to 0 for any $\delta = \delta(N) = (\log N + c_N)/N$, where $c_N \to \infty$. We recall the argument where δ is fixed, so that we can be precise about the bounds, rather than just working asymptotically. As in, for example, Walkup [15], we say that a subset $\mathcal{A}_S \subset \mathcal{A}$ of size k and a subset $\mathcal{B}_S \subset \mathcal{B}$ of size $N-k+1$ form a blocking pair of size k if no edge of the graph connects $\mathcal{A}_S$ to $\mathcal{B}_S$. [15, Equation (1)] uses König's theorem to deduce that

$$\begin{aligned}
&\mathbb{P}\left(\text{no matching } \mathcal{A} \text{ to } \mathcal{B}\right) \\
&\leq \sum_{k=1}^{N} \sum_{|\mathcal{A}_S|=k, |\mathcal{B}_S|=N-k+1} \mathbb{P}\left((\mathcal{A}_S, \mathcal{B}_S) \text{ blocking pair}\right) \\
&= 2 \sum_{k=1}^{(N+1)/2} \binom{N}{k} \binom{N}{k-1} (1-p)^{k(N-k+1)}. \qquad (9)
\end{aligned}$$

By splitting the sum into terms where $k \leq \sqrt{N}$ and $k \geq \sqrt{N}$, so that $(1-p)^{k(N-k+1)}$ is bounded by $\exp(-p(N+1)/2)$ and $\exp(-pN^{3/2}/2)$ respectively, a bound of

$$2\sqrt{N} N^{2\sqrt{N}} \exp(-p(N+1)/2) + 2^{2N} \exp(-pN^{3/2}/2)$$

can be obtained. We deduce that the probability of a complete matching failing to exist decays at an exponential rate in N.

To prove [1, Theorem 5], we divide the receiver-transmitter links into two groups $\mathcal{A}$ and $\mathcal{B}$ of size $N = \lfloor K/2 \rfloor$, and consider complete matchings on the bipartite graph between them. Each edge is present in the bipartite graph if the corresponding INR lies in a particular range (see Lemma 2.1 below for details), which occurs independently with probability δ. For each pair that is successfully matched up, the corresponding two-user channel becomes an ϵ-bottleneck, and contributes $\leq \log(1 + 2\mathrm{SNR}) + \epsilon$ to the sum capacity. Hence, the high probability of a complete matching implies exponential decay of $\mathbb{P}(|C_\Sigma/K - \frac{1}{2}\log(1+2\mathrm{SNR})| > \epsilon)$, improving the $O(K^{-2})$ rate in (1).

II. Technical Lemmas

We continue to work towards our proof of Theorem 1.5, by establishing some technical results. First in Section II-A, we identify a condition under which the sum capacity of the two user interference channel can be bounded. Next, in Section II-B, we show that the spatial separation condition of Definition 1.2 holds under a variety of conditions. Further we show that spatial separation and the decaying condition Definition 1.3 together imply that no SNR can be "too large".

A. Bounds on Two User Channel

First, we identify a condition on the values of SNR and INR under which the capacity of the two user interference channel can be bounded. The proof of the following result is based on Lemma 1 of [1], which gave the key definition of a "bottleneck state", deducing that in the case $\mathrm{SNR}_i = \mathrm{SNR}_j = \mathrm{INR}_{ji} = \mathrm{SNR}$, the sum capacity equals $\log(1 + 2\mathrm{SNR})$. We reproduce the argument used there, to deduce a stability result that allows us to deduce when an ϵ-bottleneck state occurs.

Lemma 2.1: For any i, j, consider the two user inferference channel defined by

$$\begin{aligned}
Y_i &= \exp(i\phi_{ii})\sqrt{\mathrm{SNR}_i} X_i + \exp(i\phi_{ji})\sqrt{\mathrm{INR}_{ji}} X_j + Z_i, \\
Y_j &= \exp(i\phi_{ij})\sqrt{\mathrm{INR}_{ij}} X_i + \exp(i\phi_{jj})\sqrt{\mathrm{SNR}_j} X_j + Z_j,
\end{aligned}$$

where Z_i and Z_j are IID standard complex Gaussians, and ϕ_{ij} are independent $U[0, 2\pi]$ random variables independent of all other terms.

If $\mathrm{INR}_{ji} \geq \mathrm{SNR}_j$ then any reliable transmission rates satisfy

$$R[i] + R[j] \leq \log(1 + \mathrm{INR}_{ji} + \mathrm{SNR}_i). \qquad (10)$$

Proof: We adapt the argument of Lemma 1 of [1]. That is, consider reliable transmission rates $R[i]$, $R[j]$. Since the transmissions are reliable, then receiver i can determine X_i with an arbitrarily low probability of error.

Again, reliable transmission rates would remain reliable if receiver j was presented with X_i by a genie. In that case, it is easier for receiver i to determine X_j than it is for receiver j to do so (since the weighting INR_{ji} is larger than SNR_j). However, we know that receiver j can recover X_j, since the rate $R[j]$ is reliable, so we deduce that receiver i must be able to do the same.

Since receiver i can determine X_i and X_j reliably, then these messages must have been transmitted at a sum rate lower than the sum capacity of a two-user multiple access channel, see for example [5, Equation (6.6)], which is $\log(1 + \mathrm{SNR}_i + \mathrm{INR}_{ji})$. ∎

B. Decay of Tails

Recall that the node positioning model given in Definition 1.1 involves independent positions of nodes sampled from identical distributions $\mathbb{P}_T$ and $\mathbb{P}_R$. We give examples of conditions under which the spatial separation property of Definition 1.2 holds.

Lemma 2.2:

(i) If either $\mathbb{P}_T$ or $\mathbb{P}_R$ has a density with respect to Lebesgue measure which is bounded above on $\mathcal{D}$ then $\mathbb{P}_T$ and $\mathbb{P}_R$ are spatially separated.

(ii) If $\mathbb{P}_T$ and $\mathbb{P}_R$ are supported on sets $\mathcal{T}$ and $\mathcal{R}$ that are physically separated, in that

$$d_* = \inf\{d(t,r) : t \in \mathcal{T}, r \in \mathcal{R}\} > 0$$

then $\mathbb{P}_T$ and $\mathbb{P}_R$ are spatially separated.

Proof:

(i) If $\mathbb{P}_T$ has a density with respect to Lebesgue measure which is bounded above by C, then for any ball $B_s(y)$ of radius s centred on y, the probability $\mathbb{P}_T(B_s(y)) \leq C V_D s^D$, where V_D is the volume of a Euclidean ball of unit radius in $\mathbb{R}^D$. Hence

$$\begin{aligned}
\mathbb{P}(d(T,R) \leq s) &= \int \mathbb{P}_T(B_s(y)) d\mathbb{P}_R(y) \\
&\leq C V_D s^D \int d\mathbb{P}_R(y) = C V_D s^D
\end{aligned}$$

and the result follows, taking $C_{\mathrm{sep}} = C V_D$ and $D_{\mathrm{sep}} = D$. The corresponding result for $\mathbb{P}_R$ follows on exchanging $\mathbb{P}_R$ and $\mathbb{P}_T$ in the displayed equation above.

(ii) If $s < d_*$, then $\mathbb{P}(d(T,R) \leq s) = 0$. If $s \geq d_*$, then we have $\mathbb{P}(d(T,R) \leq s) \leq 1 \leq s^D/d_*^D$, so that we can take $C_{\mathrm{sep}} = 1/d_*^D$ and $D_{\mathrm{sep}} = D$. ∎

Next we show that combining the spatial separation condition of Definition 1.2 and the decaying condition Definition 1.3 gives us good control of the maximum of S_{ii}. The argument is similar to that given in Theorem 3.1 of [9].

Lemma 2.3: Consider an IID network with spatially separated $\mathbb{P}_T$ and $\mathbb{P}_R$, with signals decaying at rate α. The probability that the maximum of the K random variables S_{ii} is large tends to zero

$$\lim_{K\to\infty} \mathbb{P}\left(\max_{1\le i\le K} S_{ii} \ge \frac{\alpha \log K}{D_{\text{sep}}}\right) = 0 \tag{11}$$

where D_{sep} is the separation exponent from Definition 1.2.

Proof: We combine Definitions 1.2 and 1.3. Since all S_{ij} have the same marginal distribution, it is enough to deduce that, for any $u \ge 1$

$$\begin{aligned}\mathbb{P}(S_{ii} \ge u) &= \mathbb{P}(1/2\log(1+2\mathrm{SNR}_i) \ge u)\\ &= \mathbb{P}(\mathrm{SNR}_i \ge (\exp(2u)-1)/2)\\ &\le \mathbb{P}(\mathrm{SNR}_i \ge \exp(2u)/3)\\ &= \mathbb{P}(f(d(T_i,R_i)) \ge \exp(2u)/3)\\ &\le \mathbb{P}(c_{\text{dec}}/(d(T_i,R_i))^\alpha \ge \exp(2u)/3)\\ &= \mathbb{P}(d(T_i,R_i) \le (3\,c_{\text{dec}})^{1/\alpha}\exp(-2u/\alpha))\\ &\le c_{\text{sep}}\,(3\,c_{\text{dec}}\exp(-2u))^{D_{\text{sep}}/\alpha} \end{aligned} \tag{12}$$

where c_{sep} is the separation constant from (3) and c_{dec} is the decay constant from (4). The result follows from (12) using the union bound

$$\begin{aligned}\mathbb{P}\left(\max_{1\le i\le K} S_{ii} \ge \frac{\alpha\log K}{d_{\text{sep}}}\right) &\le K\mathbb{P}\left(S_{ii} \ge \frac{\alpha \log K}{d_{\text{sep}}}\right)\\ &\le K\frac{c_{\text{sep}}(3\,c_{\text{dec}})^{d_{\text{sep}}/\alpha}}{K^2}\end{aligned}$$

which tends to zero as $K \to \infty$. ■

III. Proof of Theorem 1.5

In this section, we complete the proof of the upper bound in Theorem 1.5. We calculate bounds on sum capacity using a strategy suggested by the work of Jafar [1], who proves that the presence of a large number of disjoint two-user channels, each close to being a "bottleneck state", allows good control of the sum capacity.

First in Section III-A, we partition the joint receiver–transmitter domain $\mathcal{D}\times\mathcal{D}$. into regions $B_{\mathbf{u},\mathbf{v}}$. Lemma 3.2 gives an upper bound on the sum capacity of a two-user channel made up of points in neighbouring regions. Further, Lemma 3.3 tells us that we can control the number of links in each region. In Section III-B, we complete the argument by matching elements of box $B_{\mathbf{u},\mathbf{v}}$ with elements of $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$. This allows us to control the overall sum capacity of the K users, and to complete the proof of Theorem 1.5.

A. Spatial Partitioning Model

Each receiver–transmitter pair $(T_i, R_i) \in \mathcal{D}\times\mathcal{D}$ can be placed in well-defined disjoint regions $B_{\mathbf{u},\mathbf{v}}$, allowing us to control the performance of the corresponding link. We write $x^{(l)}$ for the lth coordinate of the vector $\mathbf{x} = (x^{(1)},\ldots,x^{(D)})$.

Definition 3.1: Given M, we partition the space $\mathbb{R}^{2D}$ and hence the joint receiver-transmitter domain $\mathcal{D}\times\mathcal{D}$ by a regular grid of spacings $1/M$. For each $\mathbf{u}\in\mathbb{Z}^D$ and $\mathbf{v}\in\mathbb{Z}^D$, we define boxes labelled by their "bottom-left" corner as

$$B_{\mathbf{u},\mathbf{v}} = \left\{(\mathbf{x},\mathbf{y}) : \frac{u^{(l)}}{M} \le x^{(l)} < \frac{(u^{(l)}+1)}{M}, \right.\\ \left. \frac{v^{(l)}}{M} \le y^{(l)} < \frac{(v^{(l)}+1)}{M},\ \text{for all } l\right\}. \tag{13}$$

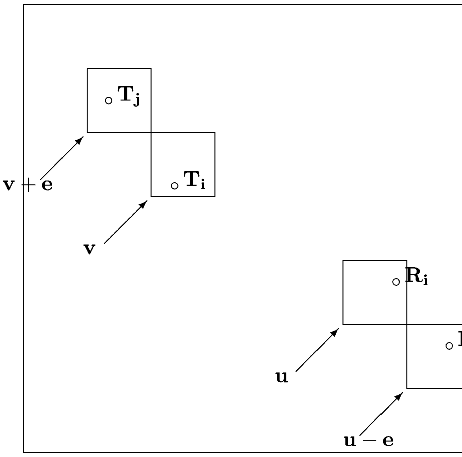

Fig. 2. Schematic plot of matched boxes, where boxes are labelled according to their bottom-left corner. We consider the case where $\mathcal{D} = [0,1]^2$, and plot the receiver and transmitter positions on the same square, with $\mathbf{e} = E(\mathbf{u},\mathbf{v}) = (-1,1)$. The key property is that we can observe that $d(T_j,R_j) \ge d(T_j,R_i) \ge d(T_i,R_i)$, as proved in Lemma 3.2.

We write $\mathcal{S} = \{(\mathbf{u},\mathbf{v}) : B_{\mathbf{u},\mathbf{v}} \bigcap (\mathcal{D}\times\mathcal{D}) \ne \emptyset\}$ for the set of possible labels $(\mathbf{u},\mathbf{v})$, and split $\mathcal{D}\times\mathcal{D}$ into orthants, indexed by vectors $\mathbf{E}(\mathbf{u},\mathbf{v}) \in \{-1,0,1\}^D$, with coordinate

$$E^{(l)}(\mathbf{u},\mathbf{v}) = \begin{cases} 1, & \text{if } v^{(l)}-u^{(l)} > 0\\ 0, & \text{if } v^{(l)}-u^{(l)} = 0\\ -1, & \text{if } v^{(l)}-u^{(l)} < 0.\end{cases}$$

We introduce two subsets of $\mathcal{S}$ that we will not attempt to match, according to the rule that $B_{\mathbf{u},\mathbf{v}}$ is matched with $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$, where $\mathbf{e} = E(\mathbf{u},\mathbf{v})$.

(i) $\mathcal{S}_{\text{spine}} = \{(\mathbf{u},\mathbf{v}) : E^{(l)}(\mathbf{u},\mathbf{v}) = 0 \text{ for some } l\}$.

(ii) $\mathcal{S}_{\text{edge}} = \{(\mathbf{u},\mathbf{v}) : (B_{\mathbf{u},\mathbf{v}} \bigcup B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}) \not\subseteq \mathcal{D}\times\mathcal{D}\}$. That is, the regions which overlap the boundary of $\mathcal{D}\times\mathcal{D}$, or which are matched with a region that overlaps the boundary.

The control of position obtained by matching links in two neighboring regions converts into control of the values of SNR and INR, allowing the sum capacity of the two pairs to be bounded using Lemma 2.1. Fig. 2 gives a schematic diagram of a pair of boxes which we attempt to match using the construction in Lemma 3.2, plotting both transmitter and receiver positions singly on $\mathcal{D}$ rather than jointly on $\mathcal{D}\times\mathcal{D}$.

Lemma 3.2: Suppose the receiver-transmitter pair (R_i, T_i) appears in region $B_{\mathbf{u},\mathbf{v}}$ and the receiver-transmitter pair (R_j, T_j) appears in region $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$, where $(\mathbf{u},\mathbf{v}) \in \mathcal{S}\backslash\mathcal{S}_{\text{spine}}$ and $\mathbf{e} = E(\mathbf{u},\mathbf{v})$, then any reliable rates for those two links satisfy

$$R[i] + R[j] \le \log(1 + 2f(d(\mathbf{u},\mathbf{v})))$$

where $d(\mathbf{u},\mathbf{v})$ is the minimum transmitter–receiver distance between $(R_i, T_i) \in B_{\mathbf{u},\mathbf{v}}$..

Proof: The rates are only improved by being presented with the messages of all the other users, reducing the situation to that of Lemma 2.1. For each l such that coordinate $e^{(l)} = 1$, by construction (a) $T_i^{(l)} - R_i^{(l)} > 0$, (b) $R_j^{(l)} < u^{(l)} \le R_i^{(l)}$, (c) $T_j^{(l)} \ge v^{(l)} > T_i^{(l)}$. Hence, by (b)

$$(T_j^{(l)} - R_i^{(l)}) = (T_j^{(l)} - R_j^{(l)}) + (R_j^{(l)} - R_i^{(l)}) < (T_j^{(l)} - R_j^{(l)})$$

and by (c)

$$(T_j^{(l)} - R_i^{(l)}) = (T_i^{(l)} - R_i^{(l)}) + (T_j^{(l)} - T_i^{(l)}) > (T_i^{(l)} - R_i^{(l)}).$$

Overall then, in the case $e^{(l)} = 1$

$$0 < (T_i^{(l)} - R_i^{(l)}) < (T_j^{(l)} - R_i^{(l)}) < (T_j^{(l)} - R_j^{(l)}).$$

A similar argument applies for each l with $e^{(l)} = -1$, with the order of the signs reversed. Overall, we deduce that

$$d(T_j, R_j) \ge d(T_j, R_i) \ge d(T_i, R_i) \ge d(\mathbf{u},\mathbf{v}),$$

or that $f(d(\mathbf{u},\mathbf{v})) \ge \mathrm{SNR}_i \ge \mathrm{INR}_{ji} \ge \mathrm{SNR}_j$, so that Lemma 2.1 applies, allowing us to bound

$$R[i] + R[j] \le \log(1 + \mathrm{INR}_{ji} + \mathrm{SNR}_i) \le \log(1 + 2f(d(\mathbf{u},\mathbf{v})))$$

as required. ■

Each of the K receiver–transmitter links are placed independently in the regions $B_{\mathbf{u},\mathbf{v}}$, where $(\mathbf{u},\mathbf{v}) \in \mathcal{S}$, with probability $p_{\mathbf{u},\mathbf{v}}$. We write $N_{\mathbf{u},\mathbf{v}}$ for the total number of links placed in region $B_{\mathbf{u},\mathbf{v}}$, noting that the marginal distribution of each $N_{\mathbf{u},\mathbf{v}}$ is $\mathrm{Bin}(K, p_{\mathbf{u},\mathbf{v}})$.

Lemma 3.3: We can bound the probability that any of the regions contain a significantly different number of links to that expected at random

$$\mathbb{P}\left(\max_{(\mathbf{u},\mathbf{v})\in\mathcal{S}} |N_{\mathbf{u},\mathbf{v}} - Kp_{\mathbf{u},\mathbf{v}}| \ge K^{\eta}\right) \le K^{1-2\eta}.$$

Proof: A standard argument using the union bound and Chebyshev gives

$$\begin{aligned}
&\mathbb{P}\left(\max_{(\mathbf{u},\mathbf{v})\in\mathcal{S}} |N_{\mathbf{u},\mathbf{v}} - Kp_{\mathbf{u},\mathbf{v}}| \ge K^{\eta}\right)\\
&\le \sum_{(\mathbf{u},\mathbf{v})\in\mathcal{S}} \mathbb{P}\left(|N_{\mathbf{u},\mathbf{v}} - Kp_{\mathbf{u},\mathbf{v}}| \ge K^{\eta}\right)\\
&\le \sum_{(\mathbf{u},\mathbf{v})\in\mathcal{S}} \frac{\mathrm{Var}\,(N_{\mathbf{u},\mathbf{v}})}{K^{2\eta}} \le K^{1-2\eta}
\end{aligned}$$

since $\mathrm{Var}\,(N_{\mathbf{u},\mathbf{v}}) = Kp_{\mathbf{u},\mathbf{v}}(1 - p_{\mathbf{u},\mathbf{v}}) \le Kp_{\mathbf{u},\mathbf{v}}$, so that $\sum_{(\mathbf{u},\mathbf{v})\in\mathcal{S}} \mathrm{Var}\,(N_{\mathbf{u},\mathbf{v}}) \le K$. ■

B. Matching Links

We now complete the proof of Theorem 1.5 – recall that we consider uniform node distributions $\mathbb{P}_T$ and $\mathbb{P}_R$ on a bounded domain $\mathcal{D}$ with smooth boundary.

Proof of Theorem 1.5: The total sum capacity $C_\Sigma \le I_M + J_M$, where I_M is the contribution from matched pairs of links and J_M is the contribution from unmatched links. We will consider M growing as a power of K, but for now, it is enough to regard M as fixed.

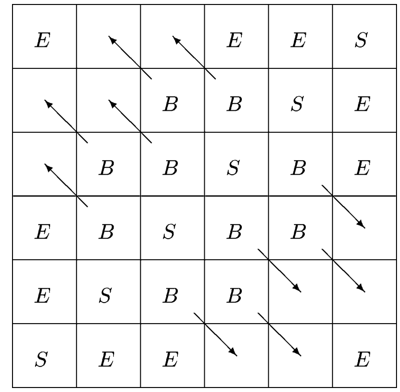

Fig. 3. Partition of regions into $\mathcal{S}_{\mathrm{spine}}$, $\mathcal{S}_{\mathrm{edge}}$ and $\mathcal{S}_{\mathrm{body}}$. This illustrates the case where $\mathcal{D} = [0,1]$ and $M = 6$, and we plot the regions as squares on $\mathcal{D}\times\mathcal{D} = [0,1]^2$. We label regions in $\mathcal{S}_{\mathrm{spine}}$ by S, regions in $\mathcal{S}_{\mathrm{edge}}$ by E and regions in $\mathcal{S}_{\mathrm{body}}$ by B, with an arrow into the region they are matched with.

We pair up the remaining edges in $\mathcal{S} \setminus (\mathcal{S}_{\mathrm{spine}} \cup \mathcal{S}_{\mathrm{edge}})$, working orthant by orthant. In particular, the matching between $\mathbf{u},\mathbf{v}$ and $\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}$ is one-to-one, and each region is counted at most once.

For each $\mathbf{e} \in \{-1,1\}^D$, we can define the function $\Pi_{\mathbf{e}}$ by $\Pi_{\mathbf{e}}(\mathbf{w}) = \mathbf{w}\cdot\mathbf{e}$. The key observation is that for each $(\mathbf{u},\mathbf{v}) \notin \mathcal{S}_{\mathrm{spine}}$, if $\mathbf{e} = E(\mathbf{u},\mathbf{v})$ then the inner product $0 \le \Pi_{\mathbf{e}}(\mathbf{v}-\mathbf{u}) \le \Pi_{\mathbf{e}}(\mathbf{v}-\mathbf{u}) + 2D = \Pi_{\mathbf{e}}((\mathbf{v}+\mathbf{e}) - (\mathbf{u}-\mathbf{e}))$, so that $(\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}) \notin \mathcal{S}_{\mathrm{spine}}$. We can sort the regions $B_{\mathbf{u},\mathbf{v}}$ by value of $\Pi_{\mathbf{e}}(\mathbf{u}-\mathbf{v})$, at each stage adding some $(\mathbf{u},\mathbf{v})$ with the lowest value of $\Pi_{\mathbf{e}}(\mathbf{u}-\mathbf{v})$ that has not yet been matched to the set $\mathcal{S}_{\mathrm{body}}$. We depict this matching in Fig. 3.

This means that we can select a set of regions $\mathcal{S}_{\mathrm{body}}$ such that for $\mathbf{u},\mathbf{v} \in \mathcal{S}_{\mathrm{body}}$, the $B_{\mathbf{u},\mathbf{v}}$ and $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$ between them cover $\mathcal{S} \setminus (\mathcal{S}_{\mathrm{spine}} \cup \mathcal{S}_{\mathrm{edge}})$, that is

$$\bigcup_{\mathbf{u},\mathbf{v}\in\mathcal{S}_{\mathrm{body}}} (B_{\mathbf{u},\mathbf{v}} \cup B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}) = \mathcal{S} \setminus (\mathcal{S}_{\mathrm{spine}} \cup \mathcal{S}_{\mathrm{edge}}).$$

For $\mathcal{D}$ with volume V and each $(\mathbf{u},\mathbf{v}) \in \mathcal{S}_{\mathrm{body}}$ the probabilities $p_{\mathbf{u},\mathbf{v}} = p_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}} = 1/(V^2M^{2D})$, since the two boxes do not intersect the boundary of $\mathcal{D}\times\mathcal{D}$. By assumption there are at least $K/(V^2M^{2D}) - K^{\eta}$ links in each of the regions $B_{\mathbf{u},\mathbf{v}}$ and $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$, since Lemma 3.3 shows that the probability that this does not occur is $\le K^{1-2\eta}$. We choose $K/(V^2M^{2D}) - K^{\eta}$ links randomly from each pair of sets and match them, with Lemma 3.2 implying that each pair of matched links contributes at most $\log(1 + 2f(d(\mathbf{u},\mathbf{v})))$ to the sum capacity. Overall, the total contribution to the sum capacity from all the matched links satisfies

$$I_M \le \sum_{\mathbf{u},\mathbf{v}\in\mathcal{S}_{\mathrm{body}}} \frac{K}{V^2M^{2D}} \log(1 + 2f(d(\mathbf{u},\mathbf{v}))). \tag{14}$$

By the definition of Riemann integration, by picking M sufficiently large, the term $I_M/K \le \frac{1}{2}\mathbb{E}\log(1 + 2\mathrm{SNR}) + \epsilon/2$, for ϵ arbitrarily small.

Next, we control J_M, the contribution from the unmatched links. Specifically, without loss of generality, if $\mathcal{D}$ is bounded with volume V, we can assume $\mathcal{D} \times \mathcal{D} \subseteq [0, L]^{2D}$ for some L.

(i) We do not attempt to match some regions because they belong in $\mathcal{S}_{\text{edge}}$ or $\mathcal{S}_{\text{spine}}$.
   a) We assume that the boundary of $\mathcal{D}$ is sufficiently smooth that there exists a finite A ("surface area") such that $|\mathcal{S}_{\text{edge}}| \leq AM^{2D-1}$ for all M. (For example if $\mathcal{D} = [0,1]^D$ then $|\mathcal{S}_{\text{edge}}| \leq 4DM^{2D-1}$, since there are $2D$ coordinates that can take values 0 or $M-1$, and then M^{2D-1} values for the remaining coordinates).
   b) The number of regions in $|\mathcal{S}_{\text{spine}}| \leq D(LM)^{2D-1}$, since there are D coordinates which can agree, and at most LM possible values the remaining coordinates can take.

   Overall, there are at most $(A + DL^{2D-1})M^{2D-1}$ regions we do not attempt to match. Each region we do not attempt to match contains at most $K/(V^2M^{2D}) + K^\eta$ links.

(ii) We attempt to perform matching between at most $(LM)^{2D}$ regions, with at most $2K^\eta$ unmatched links remaining from each.

In total, we deduce there are $(A + DL^{2D-1})(K/(V^2M) + K^\eta M^{2D-1}) + 2(LM)^{2D}K^\eta$ unmatched links. Writing $\beta = 1/(3(2D+1))$, and choosing $M = K^{3\beta(1-\eta)}$, and for example taking $\eta = 2/3$, there are $O\left(K^{1-\beta}\right)$ unmatched links.

The single user capacity bound (see for example [5, eq. (6.4)]) tells us that reliable rates satisfy

$$R[i] \leq \log(1 + \text{SNR}_i) \leq \log(1 + 2\text{SNR}_i) = 2S_{ii} \leq 2 \max_i S_{ii} \tag{15}$$

so that $J_M = c_1 K^{1-\beta} \max_{1\leq i\leq K} S_{ii}$ for some c_1. Hence overall, the probability

$$\mathbb{P}(J_M/K \geq \epsilon/2) \leq \mathbb{P}\left(\max_{1\leq i\leq K} S_{ii} \geq \frac{\epsilon K^\beta}{2c_1}\right)$$

which tends to zero by (11). ■

Note that the upper bound in (8) obtained by Özgür et al. [9] essentially has the extra factor of $\log K$ since the bound is only made up of the term J_M. It is precisely the matching argument that gives rise to the I_M term which has reduced the order of the bound, as we take advantage of the extra randomness provided by placing transmitter and receiver nodes separately.

Note that (11) and (15) together give probabilistic bounds on $\max_{1\leq i\leq K} R[i]$. Specifically, since they prove that $\max_{1\leq i\leq K} R[i] = O_{\mathbb{P}}(\log K)$, they control the extent to which a large sum capacity can be achieved by a small number of links that operate at particularly high capacity. This suggests that, in this case, the sum capacity is not too unfair a measure of network performance.

IV. Future Work and Extensions

We briefly comment on some extensions of Theorem 1.5 to more general models.

A. Random Fading Amplitudes

We briefly remark on an adaption of Definition 1.4 that would have the same independence structure, while introducing random fading amplitudes into the model. That is, we could set

$$\text{INR}_{ij} = M_{ij} f(d(t_i, r_j))$$

where the M_{ij} are i.i.d. random variables with a density. Under our node placement model, the SNR_i will again be i.i.d., making this model tractable in much the same way. In particular we can extend the tail behavior bounds given in Lemma 2.3 to this case, adjusting the constant to take account of the random fading term.

Lemma 4.1: Consider an IID network with spatially separated $\mathbb{P}_T$ and $\mathbb{P}_R$, with signals decaying at rate α. If the random variables M_{ij} have finite mean then the probability that the maximum of the K random variables S_{ii} is large tends to zero

$$\lim_{K\to\infty} \mathbb{P}\left(\max_{1\leq i\leq K} S_{ii} \geq \max\left(\frac{2\alpha}{D_{\text{sep}}}, 1\right)\log K\right) = 0.$$

Proof: An equivalent of (12) holds, since as before, for any $u \geq 1$

$$\begin{aligned}
&\mathbb{P}(S_{ii} \geq u) \\
&\leq \mathbb{P}(M_{ii} f(d(T_i, R_i)) \geq \exp(2u)/3) \\
&\leq \mathbb{P}\left(M_{ii} \geq \frac{\exp(u)}{\sqrt{3}}\right) + \mathbb{P}\left(\frac{C_{\text{dec}}}{d(T_i, R_i)^\alpha} \geq \frac{\exp(u)}{\sqrt{3}}\right) \\
&\leq \sqrt{3}\mathbb{E}M_{ii}\exp(-u) + C_{\text{sep}}(C_{\text{dec}}\sqrt{3}\exp(-u))^{D_{\text{sep}}/\alpha}
\end{aligned}$$

so the result follows as before, using the union bound. ■

The key to proving convergence in probability of C_Σ/K is to show that matching is possible between elements of $B_{\mathbf{u},\mathbf{v}}$ and $B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$. For deterministic fading amplitudes, any links $(R_i, T_i) \in B_{\mathbf{u},\mathbf{v}}$ and $(R_j, T_j) \in B_{\mathbf{u}-\mathbf{e},\mathbf{v}+\mathbf{e}}$ could be matched, since the proof of Lemma 3.2 showed that in this case

$$d(T_j, R_j) \geq d(T_j, R_i) \geq d(T_i, R_i). \tag{16}$$

In the case of random fading amplitudes, this is not enough to control the relevant values of INR. However, Lemma 4.2 shows that we can match a high proportion of links, by looking for (i, j) such that

$$M_{jj} \leq M_{ji} \leq M_{ii} \tag{17}$$

which can be combined with (16) to deduce that $\text{SNR}_j \leq \text{INR}_{ji} \leq \text{SNR}_i$, so that again the sum capacity of the relevant two user channel $\leq \log(1 + 2\text{SNR}_i)$. Note that it is enough for our purposes to consider the case of uniform M_{ij} with densities, since only the ordering between random variables matters in (17). We give a technical lemma that will imply the control that we require.

Lemma 4.2: Consider a bipartite graph with n vertices in each part which we refer to as $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ respectively. All the vertices are labelled with independent $U[0,1]$ random variables, with A_i labelled by U_i and B_j labelled by V_j. The bipartite graph has an edge from A_i to B_j iff $U_i \leq W_{ij} \leq V_j$, where W_{ij} are $U[0,1]$, independent of $(\mathbf{U}, \mathbf{V})$ and each other.

For any $\gamma \geq 2/3$, there exists a matching of all but $O(n^\gamma)$ vertices, with probability $\geq 1 - 5\exp(-2n^{2\gamma-1})$.

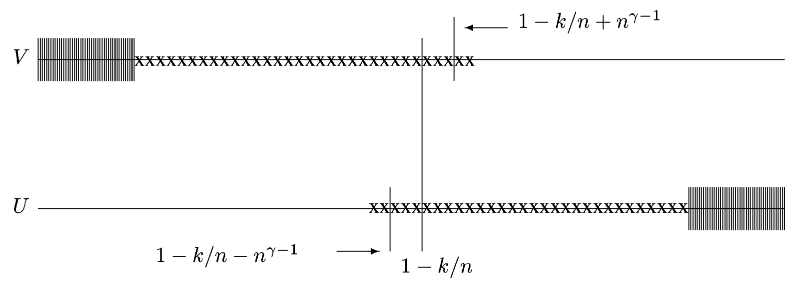

Fig. 4. Position of vertices in subsets $\mathcal{A}_S$ and $\mathcal{B}_S$ forced by throwing away $3n^\gamma$ largest values of U and $3n^\gamma$ smallest values of V.

Proof: We throw away the $3n^\gamma$ vertices with the biggest values of U_i and the $3n^\gamma$ vertices with the lowest values of V_j. This leaves new sets $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ each of size $N = N(n) = n - 3n^\gamma$. We will show that there exists a matching between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ with high probability, again using (9) and controlling the probability of blocking pairs. We condition on the event

$$\left\{\sup_t \left|\frac{\#\{i : U_i \leq t\}}{n} - t\right| \leq n^{\gamma-1}\right\} \bigcup \left\{\sup_t \left|\frac{\#\{i : V_i \leq t\}}{n} - t\right| \leq n^{\gamma-1}\right\} \quad (18)$$

since Massart's form of the Dvoretzky–Kiefer–Wolfowitz theorem [17] tells us that this does not take place with probability $\leq 4\exp(-2n^{2\gamma-1})$.

Conditional on the event (18), for any k, we know there are more than $n - k - 2n^\gamma$ values of U_i, which are less than $1 - k/n - n^{\gamma-1}$. Equivalently, there are fewer than $k + 2n^\gamma$ values of U_i larger than $1 - k/n - n^{\gamma-1}$. Since we throw away the largest $3n^\gamma$ values of $\mathcal{A}$, any subset of $\mathcal{A}_S \subset \overline{\mathcal{A}}$ of size k has at least n^γ vertices with U_i values less than $1 - k/n - n^{\gamma-1}$ ("small vertices").

By a similar argument, any subset $\mathcal{B}_S \subset \overline{\mathcal{B}}$ of size $N - k + 1$ has at least n^γ vertices with V_j values greater than $1 - k/n + n^{\gamma+1}$ ("large vertices"). See Fig. 4 for a depiction of these events.

There is an edge between each of these small vertices in $\mathcal{A}_S$ and large vertices in $\mathcal{B}_S$ independently with probability at least $2n^{\gamma-1}$. Hence, the probability that a particular $\mathcal{A}_S$ and $\mathcal{B}_S$ form a blocking pair is less than $(1 - 2n^{\gamma-1})^{n^{2\gamma}} \leq \exp(-2n^{3\gamma-1})$. Substituting in (9), the probability of no matching is

$$\begin{aligned} &\leq \sum_{k=1}^{N} \binom{N}{k}\binom{N}{N-k+1} \exp(-2n^{3\gamma-1}) \\ &= \binom{2N}{N+1} \exp(-2n^{3\gamma-1}) \leq 2^n \exp(-2n^{3\gamma-1}) \end{aligned}$$

and the result follows since $3\gamma - 1 \geq 1$, combining with the probability of (18) failing to occur. ∎

The remainder of the proof of Theorem 1.5 carries over as before. We need only alter Section III-B, and this can be done since the increase in numbers of unmatched vertices remains sublinear in K. In future work, we would like to extend this to the case where the M_{ij} vary randomly in time.

B. Constructive Algorithm

Although Theorem 1.5 only gives a result concerning average performance of large networks, it does suggest some techniques that can be used to approximate the sum capacity of any particular Gaussian interference network. If the network is created via a spatial model, then we can attempt to match cross-links into ϵ-bottleneck channels using the constraints on spatial position described in the proof of Theorem 1.5, deducing bounds as a result.

However, even if we are only presented with the values of SNR_i and INR_{ij}, it may be possible to find bounds on sum capacity using the insights given by Lemma 2.1. Using the interference alignment scheme of [4], we know that a lower bound on C_Σ is given by $\sum_i \frac{1}{2}\log(1 + 2\mathrm{SNR}_i)$.

One possible algorithm to find an upper bound works as follows.

(i) Sort the indices by value of SNR_i, and for some M, partition the transmitter– receiver links into $2M$ categories B_r of approximately equal size $K/(2M)$.

(ii) For each $1 \leq m \leq M$, we attempt to match links between B_{2m-1} and B_{2m}, considering the bipartite graph between them.

  a) We add an edge to the bipartite graph between $j \in B_{2m-1}$ and $i \in B_{2m}$ if $\mathrm{SNR}_j \leq \mathrm{INR}_{ji} \leq \mathrm{SNR}_i$.

  b) We look for a maximal matching on the bipartite graph, using (for example) the Hopcroft-Karp algorithm [18], which has complexity $\sqrt{V}E$, where V is the number of vertices and E the number of edges.

(iii) By Lemma 2.1 each edge (j, i) in each maximal matching contributes $\log(1 + \mathrm{INR}_{ji} + \mathrm{SNR}_i)$ as an upper bound on the sum capacity, and each unmatched vertex i simply contributes the single user upper bound of $\log(1 + \mathrm{SNR}_i)$

By varying the size of M, this algorithm will find a range of upper bounds, of which we can choose the tightest. We want M large enough that categories B_r each contain a narrow range of SNR values, but M small enough that there are plenty of points in each range B_r to ensure a large maximal matching.

C. Nonuniform Node Distributions

In the paper [16], we obtain a similar result to Theorem 1.5 for nonuniform distributions $\mathbb{P}_R$ and $\mathbb{P}_T$. The proof technique used in [16] is rather different, being more closer in spirit to the original work of Jafar [1]. We regard the current paper as complementary to [16], since it offers different insights into the reasons that convergence occurs, including the algorithm described in Section IV-B. Further, the matching approach has the potential to establish faster convergence, as in the way that (9) gives faster convergence than (1).

V. Conclusions

In this paper, we have deduced sharp bounds for the sum capacity of a Gaussian interference network. Our main contribution comes through the upper bound, which uses arguments based on controlling the position of pairs of vertices to match them into bottleneck states. Although our main result is proved under the assumption of random phase fading, with signal strength decaying as a function of distance, in Section IV-A we describe an extension to a model with random fading amplitudes.

Acknowledgment

M. Aldridge and R. Piechocki would like to thank Toshiba Telecommunications Research Laboratory and its directors for supporting this work. The authors would like to thank J. Coon and M. Sandell of Toshiba for their advice and support with this research.

References

[1] S. A. Jafar, The Ergodic Capacity of Interference Networks 2009 [Online]. Available: arXiv:0902.0838v1, submitted for publication

[2] M. Maddah-Ali, A. Motahari, and A. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3457–3470, Aug. 2008.

[3] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[4] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, "Ergodic interference alignment," in *Proc. IEEE Int. Symp. Information Theory (ISIT 2009)*, Seoul, Korea, Jun. 2009, pp. 1769–1773.

[5] D. N. C. Tse and P. Viswanath*, Fundamentals of Wireless Communication*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[6] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Inf. Theory*, vol. 46, no. 3, pp. 388–404, Mar. 2000.

[7] F. Xue and P. R. Kumar, "Scaling laws for ad hoc wireless networks: An information theoretic approach," *Found. Trends Netw.*, vol. 1, no. 2, pp. 145–170, 2006.

[8] A. Özgür and O. Lévêque, "Throughput-delay trade-off for hierarchical cooperation in ad hoc wireless networks," in *Proc. Int. Conf. Telecommunications*, 2008, pp. 1–5.

[9] A. Özgür, O. Lévêque, and D. N. C. Tse, "Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3549–3572, Oct. 2007.

[10] V. R. Cadambe and S. A. Jafar, "Degrees of freedom of wireless networks with relays, feedback, cooperation, and full duplex operation," *IEEE Trans. Inf. Theory*, vol. 55, no. 5, pp. 2334–2344, May 2009.

[11] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inf. Theory*, vol. IT-27, no. 1, pp. 49–60, Jan. 1981.

[12] R. H. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[13] G. Bresler, A. Parekh, and D. N. C. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4566–4592, Sep. 2010.

[14] P. Erdős and A. Rényi, "On random matrices," *Magyar Tud. Akad. Mat. Kutató Int. Közl*, vol. 8, pp. 455–461, 1964.

[15] D. W. Walkup, "Matchings in random regular bipartite digraphs," *Discr. Math.*, vol. 31, no. 1, pp. 59–64, 1980.

[16] M. P. Aldridge, O. T. Johnson, and R. Piechocki, "Asymptotic sum-capacity of random Gaussian interference networks using interference alignment," in *Proc. ISIT 2010*, Austin, TX, Jun. 2010, pp. 410–414.

[17] P. Massart, "The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality," *Ann. Probab.*, vol. 18, no. 3, pp. 1269–1283, 1990.

[18] J. E. Hopcroft and R. M. Karp, "An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs," *SIAM J. Comput.*, vol. 2, no. 4, pp. 225–231, 1973.

Oliver Johnson received the B.A. degree in 1995, Part III Mathematics in 1996, and the Ph.D. degree in 2000, all from the University of Cambridge, Cambridge, U.K.

He was Clayton Research Fellow at Christ's College and Max Newman Research Fellow at Cambridge University until 2006, during which time he published the book *Information Theory and the Central Limit Theorem* (Singapore: World Scientific, 2004). Since 2006, he has been a Lecturer in statistics at Bristol University, Bristol, U.K.

Matthew Aldridge received the B.A. degree in 2006 and Part III in 2007, both from the University of Cambridge, Cambridge, U.K. He is currently pursuing the Ph.D. degree in the Department of Mathematics, University of Bristol, Bristol, U.K.

Robert Piechocki received the M.Sc. degree from the Technical University of Wroclaw, Wroclaw, Poland, in 1997, and the Ph.D. degree from the University of Bristol, Bristol, U.K., in 2002.

He is an RCUK Senior Research Fellow at the University of Bristol. His research interests span the areas of signal processing, information, and communication theory, with an emphasis on wireless communications. He has led and participated in a number of industrial (Toshiba, IHP, QinetiQ, Fujitsu, EADS), U.K., and EU funded projects. He has authored and co-authored over 60 international journal and conference papers and holds 12 patents.