Lab 6: Monte Carlo

Olivier Binette and Michael Christensen

Friday September 18, 2020

Agenda

1. Review of Monte Carlo and Importance Sampling
2. Lab 6 Tasks 1-3
3. Questions / Office Hours

Review of Monte Carlo and Importance Sampling

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Goal: Approximate an integral

$$I = \int_{\mathcal{X}} h(x) f(x)\, dx = \mathbb{E}_f[h(x)]$$

which is intractable, where $f(x)$ is a probability density function.

What's the problem?

- Typically $h(x)$ is messy or high-dimensional. We need numerical techniques.
- In dimension $d = 3$ or higher (e.g. $\mathcal{X} = \mathbb{R}^3$), Monte Carlo typically improves upon numerical integration techniques.

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We want to approximate $I = \int_{\mathcal{X}} h(x) f(x)\, dx = \mathbb{E}_f[h(x)] \ldots$

Monte-Carlo solution: Sample $X_1, X_2, X_3, \ldots, X_n$ from f, and estimate I by the empitical average

$$\overline{h}_n = \frac{1}{n} \sum_{i=1}^{n} h(x_i).$$

- The estimate $\overline{h}_n$ converges almost surely to I as $n \to \infty$ by the strong law of large numbers.

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Importance Sampling: Maybe it's hard to sample from f, and instead you'd like to take your samples from a density g. So divide and multiply by $g(x)$ to write

$$I = \int_{\mathcal{X}} h(x) f(x)\, dx = \int_{\mathcal{X}} h(x) \frac{f(x)}{g(x)} g(x)\, dx = \mathbb{E}_g\left[h(x)\frac{f(x)}{g(x)}\right].$$

Now use a Monte-Carlo estimate of I with respect to g: sample $x_1, x_2, \ldots, x_n$ from g and estimate I by

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)\frac{f(x_i)}{g(x_i)}.$$

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$$I = \int_{-\infty}^{\infty} e^{-x^4} \, dx.$$

1. Find a closed form solution to I and evaluate this.
2. Approximate I using Monte-Carlo.
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Task 1: Find a closed form solution to I.

How can we compute $I = \int_{-\infty}^{\infty} e^{-x^4}\, dx$?

- Change of variable (u-substitution)!
- First write $I = 2\int_0^{\infty} e^{-x^4}\, dx$.
- Set $u = x^4$, $x = u^{1/4}$.Then $du = 4x^3 dx$,$dx = \frac{du}{4x^3} = \frac{du}{4u^{3/4}}$.

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Remember that the Gamma density is $\frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$.

So $\int_0^\infty \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}\, dx = 1$ and $\int_0^\infty x^{a-1} e^{-bx}\, dx = \frac{\Gamma(a)}{b^a}$.

In our context, we therefore have

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Task 2: Estimate $I = \int_{-\infty}^{\infty} e^{-x^4}\, dx$ using Monte-Carlo.

Beka suggests the substitution $y = \sqrt{2}x^2$.Then

$$I = 2^{-5/4} \int_0^\infty \sqrt{\frac{2\pi}{y}} 2\phi(y)\, dy$$

where $2\phi(y)$ is the density of the normal distribution truncated to $[0, \infty)$.

So if $Y \sim N(0,1)$, then $Y = |X|$ has density 2ϕ.

Monte-Carlo algorithm:

1. Sample $X_1, X_2, \ldots, X_n \sim N(0,1)$.
2. Approximate I by

$$\hat{I}_{MC} = \frac{1}{n} \sum_{i=1}^{n} 2^{-5/4} \sqrt{\frac{2\pi}{|X_i|}}.$$

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where $2\phi(y)$ is the density of the normal distribution truncated to $[0, \infty)$.

So if $Y \sim N(0,1)$, then $Y = |X|$ has density 2ϕ.

Monte-Carlo algorithm:

1. Sample $X_1, X_2, \ldots, X_n \sim N(0,1)$.
2. Approximate I by

$$\hat{I}_{MC} = \frac{1}{n} \sum_{i=1}^{n} 2^{-5/4} \sqrt{\frac{2\pi}{|X_i|}}.$$

Lab 6

Task 2: Estimate $I = \int_{-\infty}^{\infty} e^{-x^4}\, dx$ using Monte-Carlo.

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Lab 6

Task 2: Estimate $I = \int_{-\infty}^{\infty} e^{-x^4}\, dx$ using Monte-Carlo.

Let's implement this:

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integrand <- function(x) {
  2^{-5/4} * sqrt(2*pi/abs(x))
}

n = 10^6
X = rnorm(n)
values = integrand(X)
mean(values)
```

```
## [1] 1.810505
```

```
sd(values)/sqrt(n)
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Lab 6

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Lab 6

Task 3: Estimate I using importance sampling.

Let's use a *Normal* instrumental distribution:

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-x^4}\, dx \\
&= \int_{-\infty}^{\infty} \frac{e^{-x^4}}{\phi(x)} \phi(x)\, dx
\end{aligned}
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where ϕ is the normal density.

Importance sampling algorithm:

1. Sample $X_1, X_2, \ldots, X_n \sim N(0,1)$.
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