Lab 8: Data Augmentation

Olivier Binette

Friday October 9, 2020

Agenda

- Problem statement
- Go through the lab's tasks
- Office hours

Problem statement

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Data points $Y_1, Y_2, \ldots, Y_n$ coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^{n} w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3,$
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In other words:

$$p(Y_i \mid w_{1:3}, \mu_{1:3}, \varepsilon^2) = \sum_{j=1}^{3} w_j N(Y_i; \mu_j, \varepsilon^2)$$

and

$$p(Y_i \mid Z_i, w_{1:3}, \mu_{1:3}, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

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Let's see what this mixture model could look like in an example.

Let $\mu_1 = -5$, $\mu_2 = 0$ and $\mu_3 = 5$, and let $\varepsilon = 1$. Let $w_j = 1/3$ for $j = 1, 2, 3$.

Now let's generate data from the mixture model:

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n = 100
mu = c(-5, 0, 5)

Z = sample(1:3, size=n, replace=TRUE)
Y = rnorm(n, mean=mu[Z], sd=1)

hist(Y, breaks=20)
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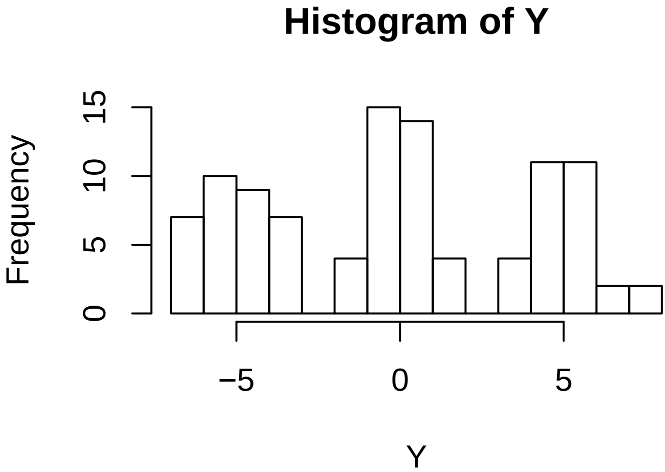

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We have the model

$$Y_i \sim \sum_{j=1}^{n} w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j, w_j and ε.

Priors

For the means:

$$\mu_j \mid \mu_0, \sigma_0 \sim N(\mu_0, \sigma_0^2)$$
$$\mu_0 \sim N(0, 3)$$
$$\sigma_0^2 \sim IG(2, 2)$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

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Priors

For the mixture weights:

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\mathbf{1})$$

which means that $p(w_1, w_2, w_3) \propto 1$.

Recall that, in general,

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$$
$$\Rightarrow p(w_1, w_2, w_3) \propto w_1^{\alpha_1 - 1} w_2^{\alpha_2 - 1} w_3^{\alpha_3 - 1}.$$

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For the variance:

$$\varepsilon^2 \sim IG(2,2)$$

This means that

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In summary,

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\begin{aligned}
p(Y_i|\mu_1, \mu_2, \mu_3, w_1, w_2, w_3, \tau) &= \sum_{j=1}^{3} w_i N(\mu_j, \tau^{-1}) \\
\mu_j|\mu_0, \sigma_0^2 &\sim N(\mu_0, \phi_0^{-1}) \\
\mu_0 &\sim N(0, 3) \\
\phi_0 &\sim \text{Gamma}(2, 2) \\
(w_1, w_2, w_3) &\sim \mathit{Dirichlet}(\mathbf{1}) \\
\tau &\sim \text{Gamma}(2, 2),
\end{aligned}
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for $i = 1, \ldots n$.

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Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

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Task 1

The full joint:

$$\left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right)\left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0)\right) p(\mu_0)p(\phi_0)p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right)\left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0)\right) p(\mu_0)p(\phi_0)p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

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$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau)$$

$$\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0)\right) p(\mu_0) p(\phi_0) p(\tau)$$

$$\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right)$$

$$\propto \prod_{i=1}^{n} \left(\sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau)\right)$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$
\begin{aligned}
& p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\
\propto & \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\
\propto & \prod_{i=1}^{n} \left(\sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau) \right)
\end{aligned}
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Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

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\begin{aligned}
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\propto & \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\
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Task 2

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\begin{aligned}
& p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\
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Task 2

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\begin{aligned}
& p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\
& \propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\
& \propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\
& \propto \left(\prod_{i=1}^{n} \left(\sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau) \right) \right) p(\mu_j \mid \mu_0, \phi_0)
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Task 2

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\begin{aligned}
&p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\
&\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\
&\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\
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Task 2

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\propto & \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\
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Task 2

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\begin{aligned}
&p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\
&\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0)\right) p(\mu_0) p(\phi_0) p(\tau) \\
&\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau)\right) p(\tau) \\
&\propto \left(\prod_{i=1}^{n} \left(\sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau)\right)\right) \tau^{2-1} \exp\{-2\tau\}.
\end{aligned}
$$

Task 2

$$p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3})$$
$$\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau)$$
$$\propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau)$$
$$\propto \left(\prod_{i=1}^{n} \left(\sum_{j=1}^{3} w_j N(Y_i; \mu_j, \tau) \right) \right) \tau^{2-1} \exp\{-2\tau\}.$$

Task 2

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\begin{aligned}
& p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\
& \propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\
& \propto \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau) \\
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Task 2

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Task 2

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\begin{aligned}
& p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\
\propto & \left(\prod_{i=1}^{n} p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\
\propto & \exp\left\{ -\frac{\phi_0}{2} \sum_{j=1}^{3} (\mu_j - \mu_0)^2 \right\} \exp\{\mu_0^2/6\} \\
\propto & \exp\left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3})\mu_0^2 - 2\mu_0\phi_0 \sum_{j=1}^{3} \mu_j \right] \right\} \\
\Rightarrow & \mu_0 \mid - \sim N\left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^{3} \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right).
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Task 2

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Task 2

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$$
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Task 2

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Task 2

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& p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\
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\propto & \left(\prod_{j=1}^{3} p(\mu_j \mid \mu_0, \phi_0)\right) p(\phi_0) \\
\propto & \left(\prod_{j=1}^{3} \sqrt{\frac{\phi_0}{2\pi}} \exp\left\{-\frac{\phi_0}{2}(\mu_0 - \mu_j)^2\right\}\right) \phi_0^{2-1} \exp\{-2\phi_0\} \\
\propto & \phi_0^{7/2-1} \exp\left\{-\phi_0 \left(2 + \frac{1}{2}\sum_{j=1}^{3}(\mu_0 - \mu_j)^2\right)\right\} \\
\Rightarrow & \phi_0 \mid - \sim \text{Gamma}\left(7/2, 2 + \frac{1}{2}\sum_{j=1}^{3}(\mu_0 - \mu_j)^2\right)
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Task 2

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Task 2

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Task 2

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\propto & \left(\prod_{j=1}^{3} \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp\{-2\phi_0\} \\
\propto & \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^{3} (\mu_0 - \mu_j)^2 \right) \right\} \\
\Rightarrow & \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^{3} (\mu_0 - \mu_j)^2 \right)
\end{aligned}
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Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim Cat(3, \boldsymbol{w})$$
$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- The marginal distribution of Y_i is **unchanged**.
- Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$
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Now the full joint becomes:

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&p(Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau)\\
=&p(Y_{1:n} \mid Z_{1:n}, \mu_{1:3}, \tau)p(Z_{1:n})p(\mu_{1:3} \mid \mu_0, \phi_0)p(\mu_0)p(\phi_0)p(\tau)\\
=&\prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1})w_{Z_i}\right) p(\mu_{1:3} \mid \mu_0, \phi_0)p(\mu_0)p(\phi_0)p(\tau)
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\propto & \prod_{i=1}^{n} w_{Z_i} \\
\propto & \, w_1^{N_1} w_2^{N_2} w_3^{N_3} \\
\Rightarrow & \, w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1)
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where $N_j = \sum_{i=1}^{n} \mathbb{I}(Z_i = j)$.

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Full conditional of μ_j:

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\begin{aligned}
&p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\
&\propto \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\
&\propto \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) \right) p(\mu_j \mid \mu_0, \phi_0) \\
&\propto \exp\left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \} \\
&\propto \exp\left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0) \mu_j \right] \right\} \\
&\Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1})
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Task 3

Full conditional for τ:

$$p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3})$$

$$\propto \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau)$$

$$\propto \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) \right) p(\tau)$$

$$\propto \tau^{n/2} \exp\left\{ \frac{-\tau}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp\{-2\tau\}$$

$$\propto \tau^{2+n/2-1} \exp\{-\tau(2 + \frac{1}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2)\}$$

$$\Rightarrow \tau \mid - \sim \text{Gamma}(2 + n/2, 2 + \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 / 2)$$

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\propto &\prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) \right) p(\tau) \\
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& p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) \right) p(\tau) \\
\propto & \tau^{n/2} \exp\left\{ \frac{-\tau}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp\{-2\tau\} \\
\propto & \tau^{2+n/2-1} \exp\{-\tau(2 + \frac{1}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2)\} \\
\Rightarrow & \tau \mid - \sim \text{Gamma}(2 + n/2, 2 + \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 / 2)
\end{aligned}
$$

Task 3

Full conditional for τ:

$$
\begin{aligned}
& p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) \right) p(\tau) \\
\propto & \tau^{n/2} \exp\left\{ \frac{-\tau}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp\{-2\tau\} \\
\propto & \tau^{2+n/2-1} \exp\{-\tau(2 + \frac{1}{2} \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2)\} \\
\Rightarrow & \tau \mid - \sim \text{Gamma}(2 + n/2, 2 + \sum_{i=1}^{n} (Y_i - \mu_{Z_i})^2 / 2)
\end{aligned}
$$

Task 3

Full conditional for Z_i:

$$
\begin{aligned}
& p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\
\Rightarrow & P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}).
\end{aligned}
$$

Task 3

Full conditional for Z_i:

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\begin{aligned}
& p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\
\propto & \prod_{i=1}^{n} \left(N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \right) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\
\propto & N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\
\Rightarrow & P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}).
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$$

Task 3

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$$
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\propto &N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\
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$$

Task 3

Full conditional for Z_i:

$$
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& p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\
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\propto & N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\
\Rightarrow & P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}).
\end{aligned}
$$

Task 3

Summary:

$$p(\mu_0 \mid \ldots) = N\left(\frac{\sigma_0^2 \sum_{i=1}^3 \mu_i}{1/3 + 3\sigma_0^{-2}}, (1/3 + 3\sigma_0^{-2})^{-1}\right),$$

$$p(\sigma_0^2 \mid \ldots) = IG\left(2 + 3/2, 2 + (1/2)\sum_{i=1}^{3}(\mu_i - \mu_0)^2\right),$$

$$p(\epsilon^2 \mid \ldots) = IG\left(2 + n/2, 2 + (1/2)\sum_{i=1}^{n}(Y_i - \mu_{Z_i})^2\right),$$

$$p(\boldsymbol{w} \mid \ldots) = Dir(3, (1 + N_1, 1 + N_2, 1 + N_3)),$$

$$p(\mu_j \mid \ldots) = N\left(\left(\mu_0\sigma_0^{-2} + \epsilon^{-2}\sum_{i:Z_i=j} y_i\right)(\sigma_0^{-2} + N_j\epsilon^{-2})^{-1}, \left(\sigma_0^{-2} + N_j\epsilon^{-2}\right)^{-1}\right),$$

$$P(Z_i = j) = \frac{wjN(y_i \mid \mu_j, \epsilon^2)}{\sum_{k=1}^3 w_k N(y_i \mid \mu_k, \epsilon^2)}.$$