Lab 2: Intro to Bayes

Olivier Binette

21/08/2020

Agenda

1. Introduction :)
2. Solution to Lab 2
3. Some tips for homework 1
4. Questions

Introduction

Olivier Binette

olivier.binette@gmail.com

Office hours: Wednesday 9am-10am EST

My job:

- Help you with the labs, homeworks, and content of the course.
  - Please email questions and/or come to office hours!
- Advocate for you.
  - Let me know if you have any issue, if you're not satisfied with grading, or if there's anything else.
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Task 1

Reminder:

If you have independent variables $Y_i \sim \text{Bernouilli}(\theta)$, then

$$X = \sum_{i=1}^{n} Y_i \sim \text{Binomial}(\theta, n).$$

Example:

- Y_i: indicator variable that individual i gets sick in a certain period of time.
- X: total number of people getting sick in the given period of time among the individuals $i = 1, 2, \ldots, n$.

Task 1

Assume that

$$X \mid \theta \sim \text{Binomial}(\theta, n),$$

$$\theta \sim \text{Beta}(a, b).$$

Derive the posterior distribution of θ given X.

Solution to Task 1

$$
\begin{aligned}
p(\theta \mid X) &= \frac{p(X \mid \theta)p(\theta)}{p(X)} \\
&\propto p(X \mid \theta)p(\theta) \\
&= \binom{n}{X}\theta^{X}(1-\theta)^{(n-X)} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{(a-1)}(1-\theta)^{(b-1)} \\
&\propto \theta^{X}(1-\theta)^{(n-X)} \times \theta^{(a-1)}(1-\theta)^{(b-1)} \\
&\propto \theta^{X+a-1}(1-\theta)^{(n-X+b-1)}
\end{aligned}
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Therefore

$$\theta \mid X \sim \mathrm{Beta}(X+a, n-X+b).$$

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Therefore

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Task 2

Simulate some data using the rbinom function of size $n = 100$ and probability equal to 1%. Remember to set.seed(123) so that you can replicate your results.

The data can be simulated as follows:

```
# set a seed
set.seed(123)
# create the observed data
obs.data <- rbinom(n = 100, size = 1, prob = 0.01)
head(obs.data)
```

```
## [1] 0 0 0 0 0 0
```

```
length(obs.data)
```

```
## [1] 100
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Task 3

1. Write a function with:

- **input:** simulated data and sequence of θ values.
- **output:** binomial likelihood of the data corresponding to each θ value.

```
theta = c(0.01, 0.1)
N = length(obs.data)
X = sum(obs.data)
LH = choose(N, X) * theta^(X) * (1-theta)^(N-X)
LH
```

```
## [1] 0.3697296376 0.0002951267
```

```
likelihood <- function(obs.data, theta) {
  N = length(obs.data)
  X = sum(obs.data)
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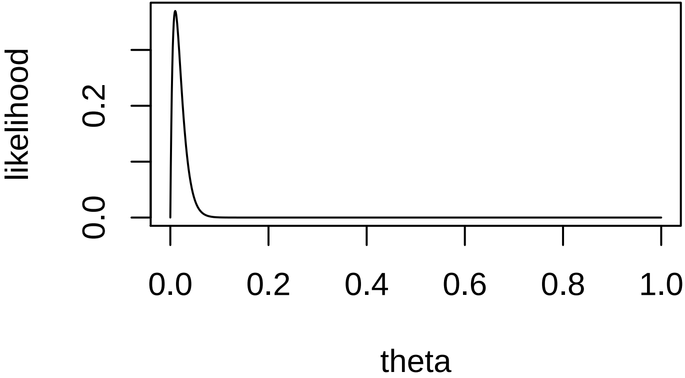
```

Task 3

2. Plot the likelihood over a grid of θ values

```
theta = seq(0,1, length.out = 1000)
plot(theta, likelihood(obs.data, theta), type="l",
     ylab="likelihood", xlab="theta")
```



Task 4

1. Write a function with:

- **input:** prior parameters *a*, *b*, and the observed data.
- **output:** parameters of the Beta posterior distribution of θ.
  takes as its inputs prior parameters a and b for the Beta-Bernoulli model and the observed data, and produces the posterior parameters you need for the model.

```r
post_parameters <- function(a, b, obs.data) {
  N = length(obs.data)
  X = sum(obs.data)
  a.post = a + X
  b.post = N - X + b
  return(c(a.post, b.post))
}
```

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Task 4

2. **Generate and print** the posterior parameters for a non-informative prior i.e. (a,b) = (1,1) and for an informative case (a,b) = (3,1).

```
post_parameters(1,1, obs.data)
```

```
## [1]   2 100
```

```
post_parameters(3,1, obs.data)
```

```
## [1]   4 100
```

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```

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```
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```

```
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```

Task 5

Create two plots, one for the informative and one for the non-informative case to show the posterior distribution and superimpose the prior distributions on each along with the likelihood. What do you see?

Task 5

I'll only do the non-informative prior part with you.

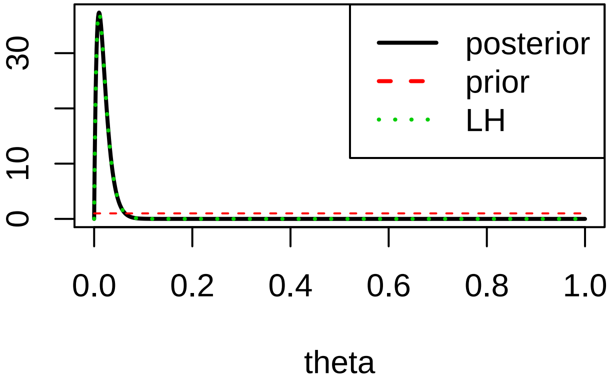
```
params1 = post_parameters(1,1, obs.data)

# Plot posterior distribution
theta = seq(0,1, length.out=1000)
plot(theta, dbeta(theta, shape1=params1[1], shape2=params1[2]),
     type="l", xlab="theta", ylab="")
# Plot prior
lines(theta, dbeta(theta, shape1=1, shape2=1), col=2, lty=2)
# Plot likelihood
LH = likelihood(obs.data, theta)
lines(theta, 1000*LH/sum(LH), col=3, lty=3)

legend("topright", legend=c("posterior", "prior", "LH"),
       lty=c(1,2,3), col=c(1,2,3))
```

Task 5

Here's the result:



Observation: The posterior is almost the same as the normalized likelihood.

Interpretation: With a non-informative prior, the *likelihood* drives inference.

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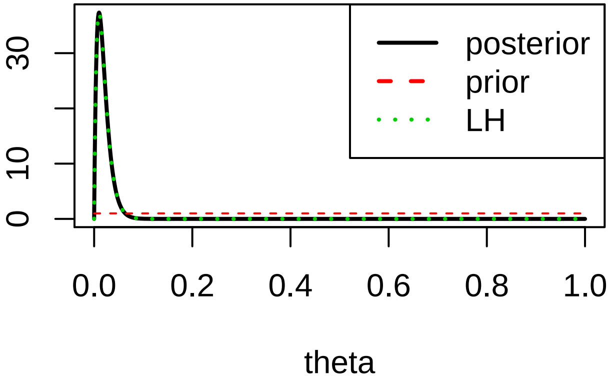

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