6.2.2

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(a)

$$\begin{aligned}
P(I=1) &= \int_0^1 P(I=1|\Theta=\theta) * f_\Theta(\theta) d\theta \\
&= \int_0^1 \theta * f_\Theta(\theta) d\theta \\
&= E[\Theta]
\end{aligned}$$

$$\begin{aligned}
P(I_i=1, I_j=1) &= \int_0^1 P(I_i=1, I_j=1|\Theta=\theta) * f_\Theta(\theta) d\theta \\
&= \int_0^1 P(I_i=1|\Theta=\theta) P(I_j=1|\Theta=\theta) * f_\Theta(\theta) d\theta \\
&= \int_0^1 \theta^2 * f_\Theta(\theta) d\theta \\
&= E[\Theta^2]
\end{aligned}$$

(b)

$$\begin{aligned}
P(N_n=k) &= \int_0^1 P(N_n=k|\Theta=\theta) * f_\Theta(\theta) d\theta \\
&= \int_0^1 P(\sum_{i=1}^n I_i = k|\Theta=\theta) * f_\Theta(\theta) d\theta
\end{aligned}$$

On trouve la forme de la somme :

$$\begin{aligned}
M_{\sum I|\Theta}(t) &= E[e^{t*\sum I}|\Theta] \\
&= \prod^n E[e^{tI}|\Theta] \\
&= (\theta t + 1 - \theta)^n \\
&\sim Binom(n, \theta)
\end{aligned}$$

On substitue :

$$\begin{aligned}
&= \int_0^1 \binom{n}{k} \theta^k (1-\theta)^{n-k} \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{I(\alpha,\beta)} d\theta \\
&= \binom{n}{k} \frac{I(\alpha+k, \beta+n-k)}{I(\alpha,\beta)}
\end{aligned}$$

(c)

```
k <- 0:3
a <- 1
b <- 3
n <- 3

fv <- function(k) choose(n,k)*beta(a+k,b+n-k)/beta(a,b)
v <- fv(k)
names(v) <- k
v
```

```
##    0    1    2    3
## 0.50 0.30 0.15 0.05
```

(d)

```
fv(3)*1000
```

```
## [1] 50
```

(e)

$\tau_1 = \frac{\alpha}{\alpha+\beta} = 0.25$

$$\begin{aligned} P(S_3' = 1000k) &= P(N_3' = k) \\ &= \begin{cases} 0.421875, k &= 0 \\ 0.421875, k &= 1 \\ 0.140625, k &= 2 \\ 0.015625, k &= 3 \end{cases} \end{aligned}$$

$$\begin{aligned} E[max(S_3' - 2000, 0)] &= P(N_3' = 3) * (3000 - 2000) \\ &= 15.625 \end{aligned}$$

La supposition ne réflète pas correctement le comportement de N pour les plus grandes valeurs.