Quantitative Macroeconomics

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Week 3

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Contents

1 Properties Of AR(1)

Let $\{\varepsilon_t\}$ be a white noise process with variance σ_ε^2.

1. Consider the univariate first-order autoregressive process AR(1):

$$y_t = \phi y_{t-1} + \varepsilon_t$$

   Derive the conditional and unconditional first and second moments.

2. Simulate different AR(1) processes and plot the corresponding autocorrelation function (ACF). To this end write a function `ACFPlots(`y`,`p^{max}`,`α`)` to plot the ACF of the data vector y with maximum number of lags p^{max}. Also include an approximate (1-α)% confidence interval around zero in your plot. Hints:

   - The empirical autocorrelation function at lag k is defined as $r_k = c_k/c_0$ where

$$c_k = \frac{1}{T}\sum_{t=k+1}^{T}(y_t - \bar{y})(y_{t-k} - \bar{y})$$

     and

$$c_0 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})(y_t - \bar{y})$$

   - You can either use a for-loop to compute the sum or use vectors: $(y - \bar{y})'(y - \bar{y})$.
   - The sample autocorrelation function is an estimate of the actual autocorrelation only if the process is stationary. If the process is purely random, that is, all members are mutually independent and identically distributed so that y_t and y_{t-k} are stochastically independent for $k \neq 0$, then the normalized estimated autocorrelations are asymptotically standard normally distributed, i.e. $\sqrt{T}r_k \to U \sim N(0,1)$ and thus $r_k \to \tilde{U} \sim N(0, 1/T)$.

Hints:

- If $|\phi| < 1$, then $\sum_{j=0}^{\infty} \phi^j = \frac{1}{(1-\phi)}$

Readings

- Bjørnland and Thorsrud (2015, Ch.2)
- Lütkepohl (2004)

2 Properties AR(1) With Time Trend

Consider the univariate AR(1) model with a constant and time trend

$$y_t = c + d \cdot t + \phi y_{t-1} + u_t$$

where $u_t \sim WN(0, \sigma_u^2)$, $|\phi| < 1$, $c \in \mathbb{R}$ and $d \in \mathbb{R}$.

1. Compute the unconditional first and second moments, i.e. the unconditional mean, variance, autocovariance and autocorrelation of y_t.

2. Why is this process not covariance-stationary? How could one proceed to make it covariance-stationary?

Hints:

- If $|\phi| < 1$, then $\sum_{j=0}^{\infty} \phi^j j = \frac{\phi}{(1-\phi^2)}$

Readings

- Lütkepohl (2004)

3 Law Of Large Numbers

Let $Y_1, Y_2, \ldots$ be an i.i.d. sequence of arbitrarily distributed random variables with finite variance σ_Y^2 and expectation μ. Define the sequence of random variables

$$\overline{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

1. Briefly outline the intuition behind the "law of large numbers".
2. Write a program to illustrate the law of large numbers for uniformly distributed random variables (you may also try different distributions such as normal, gamma, geometric, student's t with finite or infinite variance). To this end, do the following:
   - Set $T = 10000$ and initialize the $T \times 1$ output vector u.
   - Choose values for the parameters of the uniform distribution. Note that $E[u] = (a+b)/2$, where a is the lower and b the upper bound of the uniform distribution.
   - For $t = 1, ..., T$ do the following computations:
     - Draw t random variables from the uniform distribution with lower bound a and upper bound b.
     - Compute and store the mean of the drawn values in your output vector at position t.
   - Plot your output vector and add a line to indicate the theoretical mean of the uniform distribution.
3. Now suppose that the sequence $Y_1, Y_2, \ldots$ is an $AR(1)$ process:

   $$Y_t = \phi Y_{t-1} + \varepsilon_t$$

   where $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$ is not necessarily normally distributed and $|\phi| < 1$. Illustrate numerically that the law of large numbers still holds despite the intertemporal dependence.

Readings

- Lütkepohl (2005, App. C)
- Neusser (2016, App. C)
- Ploberger (2010)
- White (2001, Ch. 3)

4 Central Limit Theorem For Dependent Data

Suppose that the sequence $Y_1, Y_2, \ldots$ is an $AR(1)$ process, i.e.

$$Y_t - \mu = \phi \left(Y_{t-1} - \mu\right) + \varepsilon_t$$

where $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$ is (not necessarily but in our case) normally distributed and $|\phi| < 1$.

1. Briefly describe the intuition and result of the Lindeberg-Levy Central Limit Theorem for iid random variables. Why does it not hold for the $AR(1)$ process?

2. Show that Y_t has mean equal to μ and finite variance equal to $\sigma_\varepsilon^2/(1-\phi^2)$.

3. To derive the asymptotic distribution of the sample mean, do the following steps:
   a) Derive the asymptotic distribution of $\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_t$.
   b) Show that $\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_t = \sqrt{T}\left[(1-\phi)(\hat{\mu}-\mu) + \phi\left(\frac{Y_T - Y_0}{T}\right)\right]$ with $\hat{\mu} = \frac{1}{T}\sum_{t=1}^T Y_t$.
   c) Show that $plim\left[\frac{\phi}{1-\phi}\left(\frac{Y_T - Y_0}{\sqrt{T}}\right)\right] = 0$.
   Hint: Use Tchebychev's Inequality, i.e. for a random variable X with expectation μ_x and finite variance σ_x^2:
   $$Pr(|X - \mu_x| > \delta) \leq \frac{\sigma_x^2}{\delta^2}$$
   for any small real number $\delta > 0$.
   d) Put your results of (a),(b) and (c) together and derive the asymptotic distribution of the sample mean. That is, show that
   $$Z_T = \sqrt{T}\frac{\hat{\mu}-\mu}{\sigma_Z} \xrightarrow{d} U \sim N(0,1)$$
   for $\sigma_Z = \sqrt{\sigma_\varepsilon^2/(1-\phi)^2}$.

4. Write a program to demonstrate the central limit theorem for the AR(1) process. To this end:
   - Simulate $B = 5000$ stationary (e.g. $\phi = 0.8$) AR(1) processes with each $T = 10000$ observations. Store these in a $T \times B$ matrix Y.
   - Compute $\hat{\mu}$ for each column of Y.
   - Plot the histograms of the (naive) standardized variables
   $$\widetilde{Z}_T = \sqrt{T}\frac{\hat{\mu}-\mu}{\sigma_\varepsilon/\sqrt{1-\phi^2}}$$
   and of the (correct) standardized variables
   $$Z_T = \sqrt{T}\frac{\hat{\mu}-\mu}{\sigma_\varepsilon/(1-\phi)}$$
   Compare these to the standard normal distribution.

Readings

- Crack and Ledoit (2010)
- Lütkepohl (2005, App. C)
- Neusser (2016, App. C)
- White (2001, Ch. 5)

References

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