Quantitative Macroeconomics

Winter 2022/23

Week 4

Willi Mutschler
willi@mutschler.eu

Version: 1.0.0
Latest version available on: GitHub

Contents

1 Ordinary Least Squares Estimation Of AR(p)

Consider an AR(p) model with a constant and linear term:

$$y_t = c + d \cdot t + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + u_t = Y_{t-1}'\theta + u_t$$

where $Y_{t-1} = (1, t, y_{t-1}, ..., y_{t-p})$ and $u_t \sim WN(0, \sigma_u^2)$. The ordinary least-squares (OLS) estimator of $\theta = (c, d, \phi_1, ..., \phi_p)$ is

$$\hat{\theta} = \left(\sum_{t=1}^{T} Y_{t-1} Y_{t-1}' \right)^{-1} \sum_{t=1}^{T} Y_{t-1} y_t$$

Under the assumptions of stationarity and other standard regularity conditions one can derive that

$$\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} \tilde{U} \sim N \left(0, \sigma_u^2 \; plim \left(T^{-1} \sum_{t=1}^{T} Y_{t-1} Y_{t-1}' \right)^{-1} \right)$$

The residual variance may be estimated consistently by

$$\hat{\sigma}_u^2 = \frac{1}{T - p - 1} \sum_{t=1}^{T} \hat{u}_t^2$$

where $\hat{u}_t = y_t - Y_{t-1}'\hat{\theta}$ are the OLS residuals.

1. Write a `function OLS = ARpOLS(`y`,`p`,`$const$`,`α`)` that takes as inputs a data vector y and number of lags p. The input *const* is 1 if there is a constant in the model, 2 if there is a constant and a linear trend. The function outputs a structure OLS, which contains the OLS estimates of θ, its standard errors, t-statistics and p-values given significance value α, as well as the OLS estimate of σ_u.

2. Load simulated data for an AR(4) model given in the CSV file `AR4.csv`. Estimate an AR(4) model with a constant term using your `ARpOLS` function.

Readings

- Lütkepohl (2004)

2 Maximum Likelihood Estimation Of Gaussian AR(p)

Consider an AR(p) model with a constant and linear trend:

$$y_t = c + d \cdot t + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + u_t = Y_{t-1}\theta + u_t$$

where $Y_{t-1} = (1, t, y_{t-1}, ..., y_{t-p})$ is the matrix of regressors, $\theta = (c, d, \phi_1, ..., \phi_p)$ the parameter vector and the error terms u_t are white noise and normally distributed, i.e. $u_t \sim N(0, \sigma_u^2)$ and $E[u_t u_s] = 0$ for $t \neq s$. If the sample distribution is known to have probability density function $f(y_1, ..., y_T)$, an estimation with Maximum Likelihood (ML) is possible. To this end, we decompose the joint distribution by

$$f(y_1, ..., y_T|\theta, \sigma_u^2) = f_1(y_1|\theta, \sigma_u^2) \times f_2(y_2|y_1, \theta, \sigma_u^2) \times ... \times f_T(y_T|y_{T-1}, ..., y_1, \theta, \sigma_u^2)$$

Then the log-likelihood is

$$\log f(y_1, ..., y_T|\theta, \sigma_u^2) = \sum_{t=1}^{T} \log f_t(y_t|y_{t-1}, ..., y_1, \theta, \sigma_u^2)$$

Let's denote the values that maximize the log-likelihood as $\tilde{\theta}$ and $\tilde{\sigma}_u^2$. ML estimators have (under general assumptions) an asymptotic normal distribution

$$\sqrt{T} \begin{pmatrix} \tilde{\theta} - \theta \\ \tilde{\sigma}_u^2 - \sigma_u^2 \end{pmatrix} \xrightarrow{d} U \sim N(0, I_a(\theta, \sigma_u^2)^{-1})$$

where $I_a(\theta, \sigma_u)$ is the asymptotic information matrix. Recall that the asymptotic information matrix is the limit of minus the expectation of the Hessian of the log-likelihood divided by the sample size.

$$I_a(\theta, \sigma_u^2) = \lim_{T\to\infty} -\frac{1}{T} E \begin{pmatrix} \frac{\partial^2 \log l}{\partial \theta^2} & \frac{\partial^2 \log l}{\partial \theta \partial \sigma_u^2} \\ \frac{\partial^2 \log l}{\partial \sigma_u^2 \partial \theta} & \frac{\partial^2 \log l}{\partial (\sigma_u^2)^2} \end{pmatrix}$$

1. First consider the case of $p = 1$
   a) Derive the exact log-likelihood function for the $AR(1)$ model with $|\theta| < 1$ and $d = 0$:

   $$y_t = c + \theta y_{t-1} + u_t$$

   b) Regard the value of the first observation as deterministic or, equivalently, note that its contribution to the log-likelihood disappears asymptotically. Maximize analytically the conditional log-likelihood to get the ML estimators for θ and σ_u. Compare these to the OLS estimators.

2. Now consider the general $AR(p)$ model.
   a) Write a function `LogLikeARpNorm(`x`,`y`,`p`,`$const$`)` that computes the value of the log-likelihood conditional on the first p observations of a Gaussian $AR(p)$ model, i.e.

   $$\log l(\theta, \sigma_u) = -\frac{T-p}{2} log(2\pi) - \frac{T-p}{2} log(\sigma_u^2) - \sum_{t=p+1}^{T} \frac{u_t^2}{2\sigma_u^2}$$

   where $x = (\theta', \sigma_u)'$, y denotes the data vector, p the number of lags and $const$ is equal to 1 if there is a constant, and equal to 2 if there is a constant and linear trend in the model.

   b) Write a `function ML = ARpML(`y`,`p`,`$const$`,`α`)` that takes as inputs a data vector y, number of lags p and $const = 1$ if the model has a constant term or $const = 2$ if the model has a constant term and linear trend. α denotes the significance level. The function computes

- the maximum likelihood estimates of an AR(p) model by numerically maximizing the conditional log-likelihood function
- the standard errors by means of the asymptotic covariance matrix

Save all results into a structure "ML" containing the estimates of θ, its standard errors, t-statistics and p-values as well as the ML estimate of σ_u.

c) Load simulated data given in the CSV file `AR4.csv` and estimate an AR(4) model with constant term. Compare your results with the OLS estimators from the previous exercise.

Readings

- Lütkepohl (2004).

3 Maximum Likelihood Estimation Of Laplace AR(p)

Consider the AR(1) model with constant

$$y_t = c + \phi y_{t-1} + u_t$$

Assume that the error terms u_t are i.i.d. Laplace distributed with known density

$$f_{u_t}(u) = \frac{1}{2} \exp\left(-|u|\right)$$

Note that for the above simplified parametrization of the Laplace distribution, we have that $E(u_t) = 0$ and $Var(u_t) = 2$.

1. Derive the log-likelihood function conditional on the first observation.
2. Write a function that calculates the conditional log-likelihood of c and ϕ.
3. Load the dataset given in the CSV file `Laplace.csv`. Numerically find the maximum likelihood estimates of c and ϕ by minimizing the negative conditional log-likelihood function.
4. Compare your results with the maximum likelihood estimate under the assumption of Gaussianity. That is, redo the estimation by minimizing the negative Gaussian log-likelihood function.

Readings

- Lütkepohl (2004)

References

Lütkepohl, Helmut (2004). "Univariate Time Series Analysis". In: *Applied Time Series Econometrics*. Ed. by Helmut Lütkepohl and Markus Krätzig. First. Cambridge University Press, pp. 8–85. ISBN: 978-0-521-83919-8 978-0-521-54787-1 978-0-511-60688-5. DOI: 10.1017/CBO9780511606885.003.