Quantitative Macroeconomics

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Week 7

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Contents

1 Ordinary Least Squares Estimation of VAR(p)

Consider the VAR(p) model with constant written in the more compact form

$$y_t = [c, A_1, ..., A_p]Z_{t-1} + u_t = AZ_{t-1} + u_t$$

where $Z_{t-1} = (1, y'_{t-1}, ..., y'_{t-p})'$ and u_t is assumed to be iid white noise with non-singular covariance matrix Σ_u. Given a sample of size T, $y_1, ..., y_T$, and p presample vectors, $y_{-p+1}, ..., y_0$, ordinary least squares for each equation separately results in efficient estimators. The OLS estimator is

$$\hat{A} = \left[\hat{c}, \hat{A}_1, ..., \hat{A}_p\right] = \left(\sum_{t=1}^{T} y_t Z'_{t-1}\right)\left(\sum_{t=1}^{T} Z_{t-1}Z'_{t-1}\right)^{-1} = YZ'(ZZ')^{-1}$$

where $Y = [y_1, ..., y_T]$ and $Z = [Z_0, ..., Z_{T-1}]$. More precisely, stacking the columns of $A = [c, A_1, ..., A_p]$ in the vector $\alpha = vec(A)$,

$$\sqrt{T}\,(\hat{\alpha} - \alpha) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\hat{\alpha}})$$

where $\Sigma_{\hat{\alpha}} = plim(\frac{1}{T}ZZ')^{-1} \otimes \Sigma_u$, if the process is stable. Under fairly general assumptions this estimator has an asymptotic normal distribution. A sufficient condition for the consistency and asymptotic normality of $\hat{A}$ would be that u_t is a continuous iid random variable with four finite moments. A consistent estimator of the innovation covariance matrix Σ_u is, for example,

$$\hat{\Sigma}_u = \frac{\hat{U}\hat{U}'}{T - Kp - 1}$$

where $\hat{U} = Y - \hat{A}Z$ are the OLS residuals. Thus, in large samples,

$$vec(\hat{A}) \overset{a}{\sim} \mathcal{N}(vec(A), (ZZ')^{-1} \otimes \hat{\Sigma}_u)$$

where $\overset{a}{\sim}$ denotes the approximate large-sample distribution. In other words, asymptotically the usual t-statistics can be used for testing restrictions on individual coefficients and for setting up confidence intervals.

1. What are the dimensions of y_t, Y, u_t, U, c, A_1, ..., A_p, A, α, Z_{t-1}, Z, Σ_u and $\Sigma_{\hat{\alpha}}$.
2. Modify your `ARpOLS` function such that it is able to estimate VAR(p) models. Save the modified function as `VARReducedForm`.
3. Consider data given in `ThreeVariableVAR.csv` for $y_t = (\Delta gnp_t, i_t, \Delta p_t)'$, where gnp_t denotes the log of U.S. real GNP, p_t the corresponding GNP deflator in logs, and i_t the federal funds rate, averaged by quarter. The estimation period is restricted to 1954q4-2007q4.
   - Load the data and visualize it. Comment whether you think the data looks stationary.
   - Estimate a VAR(4) model using the `VARReducedForm` function. Examine the stability of the estimated process and the significance of the estimated parameters at a 95% level.

Readings

- Kilian and Lütkepohl (2017, Ch. 2.3)

2 Maximum Likelihood Estimation of VAR(p)

Consider the VAR(p) model with constant written in the more compact form

$$y_t = [c, A_1, ..., A_p]Z_{t-1} + u_t = AZ_{t-1} + u_t$$

where $Z_{t-1} = (1, y'_{t-1}, ..., y'_{t-p})'$. In VAR analysis, it is common to postulate that the innovations, u_t, are iid $\mathcal{N}(0, \Sigma_u)$ random variables. This assumption implies that the y_t's are also jointly normal and, for given initial values $y_{-p+1}, ..., y_0$,

$$f_t(y_t|y_{t-1}, ..., y_{-p+1}) = \left(\frac{1}{2\pi}\right)^{K/2} det(\Sigma_u)^{-1/2} exp\left\{-\frac{1}{2}u'_t\Sigma_u^{-1}u_t\right\}$$

Conditional on the first p observations, the conditional log-likelihood becomes:

$$\log l = -\frac{KT}{2}\log(2\pi) - \frac{T}{2}log(det(\Sigma_u)) - \frac{1}{2}\sum_{t=1}^{T} u'_t\Sigma_u^{-1}u_t$$

Maximizing this function with respect to the unknown parameters yields the Gaussian ML estimators $\widetilde{A}$ and $\widetilde{\Sigma_u}$.

1. Compare the Gaussian ML estimator $\widetilde{A}$ with the OLS estimator $\widehat{A}$ from the previous exercise. Comment on the asymptotic distribution.

2. Provide an expression for the ML estimator $\widetilde{\Sigma_u}$ of the innovation covariance matrix.

3. Consider data given in `ThreeVariableVAR.csv` for $y_t = (\Delta gnp_t, i_t, \Delta p_t)'$, where gnp_t denotes the log of U.S. real GNP, p_t the corresponding GNP deflator in logs, and i_t the federal funds rate, averaged by quarter. The estimation period is restricted to 1954q4-2007q4.
   - Estimate the parameters with Maximum Likelihood.
   - Compare your estimation results to an OLS estimation.

Readings

- Kilian and Lütkepohl (2017, Ch. 2.3)

3 Information Criteria for VAR(p)

Information criteria used for VAR lag order selection have the general form:

$$C(m) = log(det(\tilde{\Sigma_u})) + c_T\varphi(m)$$

where $\tilde{\Sigma}_u = T^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}_t'$ is the residual covariance matrix estimator for a reduced-form VAR model of order m based on OLS residuals $\hat{u}_t$, $\varphi(m)$ is a function of the order m that penalizes large VAR orders and c_T is a sequence of weights that may depend on the sample size. The function $\varphi(m)$ corresponds to the total number of regressors in the system of VAR equations. Since there are one intercept and mK lagged regressors in each equation and K equations in the VAR model, $\varphi(m) = mK^2 + K$. Information criteria are based on the premise that there is a trade-off between the improved fit of the VAR model, as m increases, and the parsimony of the model. Given T, the fit of the model by construction improves with larger m, indicated by a reduction in $\log(det(\tilde{\Sigma}_u(m)))$. At the same time the penalty term, $c_T\varphi(m)$, unambiguously increases with m. We are searching for the value of m that balances the objectives of model fit and parsimony. The choice of the penalty term determines the trade-off between these conflicting objectives.

The three most commonly used information criteria for VAR models are known as the Akaike Information Criterion (AIC), Schwarz Information Criterion (SIC) and Hannan-Quinn Criterion (HQC):

$$\begin{aligned}
AIC(p) &= \log(det(\tilde{\Sigma}_u(m))) + \frac{2}{T}\varphi(m) \\
SIC(p) &= \log(det(\tilde{\Sigma}_u(m))) + \frac{\log T}{T}\varphi(m) \\
HQC(p) &= \log(det(\tilde{\Sigma}_u(m))) + \frac{2\log(\log T)}{T}\varphi(m)
\end{aligned}$$

The VAR order is chosen such that the respective criterion is minimized over the possible orders $m = p^{min}, ..., p^{max}$. A key issue in implementing information criteria is the choice of the upper and lower bounds. In the context of a model of unknown finite lag order, the default is to set $p^{min} = 0$ or sometimes $p^{min} = 1$. The value of p^{max} must be chosen long enough to allow for delays in the response of the model variables to the shocks. In practice, common choices would be 12 or 24 lags for monthly data and 4 or 8 lags for quarterly data.

1. Why is it essential that for a given lag order we compute $\tilde{\Sigma}_u(m)$ on exactly the same evaluation period, $t = p^{max} + 1, ...T$, for all m?
2. Why is it essential that the criterion function be evaluated at the ML estimator $\tilde{\Sigma}_u = T^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}_t'$ rather than the OLS estimator $\hat{\Sigma}_u = (T - Km - 1)^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}_t'$?
3. Comment on the asymptotically and finite-sample properties of the criteria.
4. Consider data given in `ThreeVariableVAR.csv` for $y_t = (\Delta gnp_t, i_t, \Delta p_t)'$, where gnp_t denotes the log of U.S. real GNP, p_t the corresponding GNP deflator in logs, and i_t the federal funds rate, averaged by quarter. The estimation period is restricted to 1954q4-2007q4. Let $m = \{0, 1, ..., 4\}$. Select the lag-order according to the information criteria for the three-variable VAR model given by y_t.

Readings

- Kilian and Lütkepohl (2017, Ch. 2.6)

References

Kilian, Lutz and Helmut Lütkepohl (2017). *Structural Vector Autoregressive Analysis*. Themes in Modern Econometrics. Cambridge: Cambridge University Press. ISBN: 978-1-107-19657-5. DOI: 10.1017/9781108164818.