Markowitz Mean Variance Optimisation

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Abstract:

An introduction to Markowitz Mean Variance Optimisation in portfolio theory in financial mathematics. Mean variance analysis is a single-period analysis where we want to consider investing our capital in some assets for a single period optimally. In this study we will explore the problems that arise from portfolio optimisation and how Markowitz tries to solve them. We start with portfolios that only include risky assets, before exploring portfolios that include risk-free assets as well.

Contents

1 Introduction

Portfolio theory has significantly shaped modern financial management, equipping investors with mathematical models to construct efficient portfolios. Portfolio optimisation is the approach to optimise these models to achieve specific objectives that the investor has. For example, a risk-oriented investor would use optimisation techniques to minimise the risk whilst maximising the returns.

Markowitz Mean-Variance Optimisation model, introduced by Harry Markowitz in 1952, has become one of the most influential models in the field. Markowitz Mean-Variance model laid the foundations to the Modern Portfolio Theory (MPT), a study about diversification and the trade-off between risk and return. Before modern portfolio theory, investors would manage portfolios using subjective analysis, with little to none mathematical decision making. Additionally, investors would focus typically on maximising returns on an asset with little regard for risk. Markowitz *Portfolio Selection* Paper, demonstrated that investors could reduce portfolio risk by having a diversified selection of assets, whilst maintaining similar returns. This also pushed investors to focus on overall portfolio performance rather than an individual assets.

In this paper we will be applying Markowitz in a practical application, focusing on constructing the *Efficient Frontier curve* on a portfolio made up by different ETFs. The paper will begin with exploring the theoretical foundations of the Markowitz model and its assumptions. We then will introduce a risk-free asset to the portfolio and demonstrating how diversification can minimise risk while maximising returns. This will be followed by looking into the results of the Python script of the Markowitz model which is complementing this paper. The aim of the results is to showcase these benefactors of portfolio optimisation in financial markets in different scenarios. Finally, we will conclude this paper by briefly looking at further studies into modern portfolio theory.

2 Markowitz Mean Variance Analysis (MVA)

To begin, Markowitz theory relies on the following assumptions:

- Investors are rational and risk-averse, meaning they prefer less risk for a given level of expected return.
- The Mean Variance Analysis (MVA) is a single-period analysis, meaning that the probability distribution's parameters of the asset returns are constant and will not develop over time.
- The only risk input is the variance in the portfolio's returns.
- An investor either: wants to maximise returns for a given level of risk, or wants to minimise risk for a given level of expected return.

Markowitz also allows for short sales and the assumption of unlimited borrowing, although we will be focusing mainly on long-traded portfolio only, i.e. $w_i \in [0, 1)$, $\forall i$.

Consider the a single-period where the portfolio holds m risky assets, indexed $i = 1, ..., m$, and take the single-period returns as the m-variate random vector $\underline{\boldsymbol{R}} = [R_1, ..., R_m]^T$, then the mean and variance/covariance of returns is:

$$\mathbb{E}[\underline{\boldsymbol{R}}] = \underline{\boldsymbol{\alpha}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}, \qquad \mathbb{C}\mathrm{ov}[\underline{\boldsymbol{R}}] = \underline{\boldsymbol{\Sigma}} = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \dots & \Sigma_{1,m} \\ \Sigma_{2,1} & \Sigma_{2,2} & \dots & \Sigma_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m,1} & \Sigma_{m,2} & \dots & \Sigma_{m,m} \end{pmatrix},$$

where $\alpha_i = \mathbb{E}[\underline{\boldsymbol{R_i}}]$ represents the expected return for asset i, and $\sum_{i,j} = \mathbb{C}\mathrm{ov}(R_i, R_j)$ represents the covariance between assets i and j.

The portfolio is modelled as a m-vector of weights, $\underline{\boldsymbol{w}}$, where w_i represents the fraction of portfolio wealth held in each asset i, which the sum of all weights sum to 1:

$$\underline{\boldsymbol{w}} = (w_1, ..., w_m)^T \quad : \quad \sum_{i=1}^{m} w_i = 1$$

Portfolio return, denoted $R_{\underline{\boldsymbol{w}}}$, is the weighted sum of the individual assets:

$$R_{\underline{\boldsymbol{w}}} = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{R}} = \sum_{i=1}^{m} w_i R_i,$$

which is a random variable. The expected poertfolio return, $\alpha_{\underline{\boldsymbol{w}}}$, is the expected weighted sum of individual assets:

$$\alpha = \underline{\boldsymbol{w}} = \mathbb{E}[\underline{\boldsymbol{R_w}}] = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}}.$$

The portfolio variance, $\sigma^2_{\underline{\boldsymbol{w}}}$, is the weighted sum of the covariances between assets:

$$\sigma^2_{\underline{\boldsymbol{w}}} = \mathbb{V}\mathrm{ar}[\underline{\boldsymbol{R_w}}] = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma w}}.$$

With these definitions defined let's proceed with a simple case of a two asset portfolio, we will try to minimise the risk on the returns with constraints on the returns. We will then look at three different m-asset portfolios, with different constraint that we hope to optimise. This chapter will focus on portfolios consisting of only risky assets, whereas the next chapter consists of both risk-free and risky assets.

2.1 Example: 2 Asset Portfolio

Consider a portfolio made up of $m = 2$ assets with returns R_1, R_2, both random variables, with variances σ_1^2, σ_2^2 respectively, and $\mathrm{Corr}(R_1, R_2) = \rho$. The portfolio weighting in this example is w_1, w_2 which can be expressed as:

$$w_1 \equiv w, \quad w_2 = 1 - w_1 = 1 - w.$$

This gives a portfolio return of

$$R_{\underline{\boldsymbol{w}}} = (1 - w)R_1 + wR_2.$$

The portfolio's expected return and variance is defined as

$$\mathbb{E}[\underline{\boldsymbol{R}}] = \underline{\boldsymbol{\alpha}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (1 - w)\alpha_1 + w\alpha_2,$$

$$\mathbb{V}\mathrm{ar}[\underline{\boldsymbol{R}_{\boldsymbol{w}}}] = \sigma^2_{\underline{\boldsymbol{w}}} = (1-w)^2\sigma_1^2 + w^2\sigma_2^2 + 2(1-w)w\rho\sigma_1\sigma_2.$$
$$= (1-w)^2\sigma_1^2 + w^2\sigma_2^2$$

This gives us the feasible portfolio set:

$$\Pi^* = \{(\sigma_{\underline{\boldsymbol{w}}}, \alpha\underline{\boldsymbol{w}}) : 0 \le w \le w)\}.$$

Say the investor was risk-oriented, meaning they want to minimise risk, with no constraints on the returns, then we can compute the minimal portfolio variance by

$$\frac{\partial \sigma^2_{\underline{\boldsymbol{w}}}}{\partial \underline{\boldsymbol{w}}}| = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho}{\mathbb{V}\mathrm{ar}(R_1 - R_2)}$$

In our example, we have two assets that are perfectly uncorrelated ($\rho = 0$), as a result, we get the following optimal weight and optimal expected return of the portfolio:

$$w = \frac{\frac{1}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}, \quad \mathbb{E}[R_w] = \frac{1}{\sigma_1^2 + \sigma_2^2}(\sigma_2^2\alpha_1 + \sigma_1^2\alpha_2).$$

Now, let the parameters have the following values:

$$R_1 : \mathbb{E}[R_1] = 0.15 = \alpha_1, \quad \sqrt{\mathbb{V}\mathrm{ar}(R_1)} = 0.25 = \sigma_1$$
$$R_2 : \mathbb{E}[R_2] = 0.20 = \alpha_2, \quad \sqrt{\mathbb{V}\mathrm{ar}(R_2)} = 0.25 = \sigma_2$$

Then, the expected returns and variance of the portfolio is

$$\mathbb{E}[R_w] \approx 0.17, \quad \sigma^2_{\underline{\boldsymbol{w}}} \approx 0.037.$$

We can notice that if we invested 100% into asset 1, we would get an expected return of 0.15 and a variance 0.0625. This shows the importance of diversification in portfolio theory, as in this example, we have shown how investing in two uncorrelated assets gave us a lower portfolio risk and a higher portfolio returns than investing in just one asset.

2.2 m Asset Portfolios

We will now consider the general case for a portfolio made up of m assets. There are three problems that can be solved in a $m > 2$ asset portfolio in Markowitz Mean Variance Analysis:

1) Risk Minimisation: For a given choice of target mean return α_0, choose the portfolio $\underline{\boldsymbol{w}}$ to
   minimise: $\frac{1}{2}\underline{\boldsymbol{w}}^T\underline{\boldsymbol{\Sigma}\boldsymbol{w}}$
   subject to: $\underline{\boldsymbol{w}}^T\underline{\boldsymbol{\alpha}} = \alpha_0$
   $\underline{\boldsymbol{w}}^T\mathbf{1}_m = 1$
   Solution: Apply the method of Lagrange Multiplier to the convex optimisation (minimisation) problem subject to the linear constraints. Define the Lagrangian

$$\mathcal{L}(\underline{\boldsymbol{w}}, \lambda_1, \lambda_2) = \frac{1}{2}\underline{\boldsymbol{w}}^T\underline{\boldsymbol{\Sigma}\boldsymbol{w}} + \lambda_1(\alpha_0 - \underline{\boldsymbol{w}}^T\alpha) + \lambda_2(1 - \underline{\boldsymbol{w}}^T\mathbf{1}_m),$$

we can derive the first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \underline{\boldsymbol{w}}} &= \underline{\mathbf{0}}_m = \underline{\boldsymbol{\Sigma}\boldsymbol{w}} - \lambda_1 \alpha - \lambda_2 \mathbf{1}_m, \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= 0 = \alpha_0 - \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}}, \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= 0 = 1 - \underline{\boldsymbol{w}}^T \mathbf{1}_m, \\
\frac{\partial^2 \mathcal{L}}{\partial \underline{\boldsymbol{w}} \partial \underline{\boldsymbol{w}}^T} &= \underline{\boldsymbol{\Sigma}} \geq 0.
\end{aligned}$$

Now we solve for $\underline{\boldsymbol{w}}$ in terms of $\lambda_1 \lambda_2$:

$$\underline{\boldsymbol{w}}_0 = \lambda_1 \underline{\boldsymbol{\Sigma}}^{-1} \underline{\boldsymbol{\alpha}} + \lambda_2 \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m$$

Solve for $\lambda_1 \lambda_2$ by substituting for $\underline{\boldsymbol{w}}$:

$$\begin{aligned}
\alpha_0 &= \underline{\boldsymbol{w}}_0 \underline{\boldsymbol{\alpha}} = \lambda_1 (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \underline{\boldsymbol{\alpha}}) + \lambda_2 (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m) \\
1 &= \underline{\boldsymbol{w}}_0^T \mathbf{1}_m = \lambda_1 (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m) + \lambda_2 (\mathbf{1}_m^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m) \\
&\Rightarrow \begin{pmatrix} \alpha_0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}
\end{aligned}$$

with

$$a = (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \underline{\boldsymbol{\alpha}}), \quad b = (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m), \quad c = (\mathbf{1}_m^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m),$$

Variance of optimal portfolio with return α_0 with the given values of λ_1 and λ_2E, the solution portfolio

$$\underline{\boldsymbol{w}}_0 = \lambda_1 \underline{\boldsymbol{\Sigma}}^{-1} \underline{\boldsymbol{\alpha}} + \lambda_2 \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m$$

has minimum variance equal to

$$\begin{aligned}
\sigma_0^2 &= \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma}} \underline{\boldsymbol{w}}_0 \\
&= \lambda_1^2 (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \underline{\boldsymbol{\alpha}}) + 2\lambda_1 \lambda_2 (\underline{\boldsymbol{\alpha}}^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m) + \lambda_2^2 (\mathbf{1}_m^T \underline{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_m) \\
&= \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.
\end{aligned}$$

By substituting in $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} \alpha_0 \\ 1 \end{pmatrix}$, we get a final optimal solution of:

$$\sigma_0^2 = \begin{pmatrix} \alpha_0 \\ 1 \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} \alpha_0 \\ 1 \end{pmatrix} = \frac{1}{ac - b^2} (c\alpha_0^2 - 2b\alpha_0 + c)$$

Here our optimal portfolio has a variance of σ_0^2: parabolic in the mean.

2) <u>Expected Return Maximisation</u>: For a given choice target return variance σ_0^2, choose the portfolio $\underline{\boldsymbol{w}}$ to
maximise: $\mathbb{E}(R_w) = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}}$
subject to: $\underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma} \boldsymbol{w}} = \sigma_0^2$
$\underline{\boldsymbol{w}}^T \mathbf{1}_m = 1$

3) Risk Aversion Optimisation: Let $\lambda \geq 0$ denote the Arrow-Pratt risk aversion index gauging the trade-off between risk and return. Choose the portfolio $\underline{\boldsymbol{w}}$ to
Maximise: $[\mathbb{E}(R_w) - \frac{1}{2}\lambda \mathbb{V}\mathrm{ar}(R_{\underline{\boldsymbol{w}}})] = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}} - \frac{1}{2}\lambda \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma}} \underline{\boldsymbol{w}}$
subject to: $\underline{\boldsymbol{w}}^T \mathbf{1}_m = 1$

Note: The problems of Expected Return Maximisation and Risk Aversion Optimisation can be solved similarly to how we solved Risk Minimisation, by employing equivalent Lagranges.

When constructing two asset portfolios, we observe the parabola of the risk-return plane, the distinct combinations of risk and return trade-offs. When we add multiple two asset portfolios, these parabolas combine to form the feasible set, which is the convex set of possible solutions. The boundary of this convex set of all feasible assets defines the efficient frontier, representing the highest return given a certain level of risk.

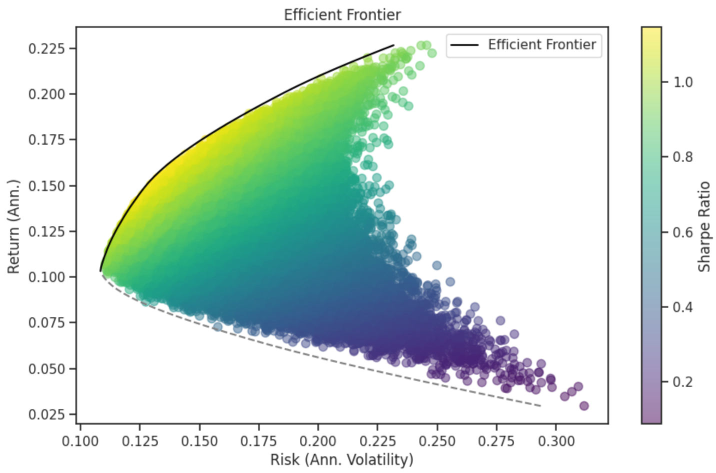

Figure 1: A plot of simulated long-only risky asset only portfolios. Efficient Frontier (black line) displays optimal returns at a given volatility.

3 Mean-Variance Optimisation with Risk-Free Assets

In addition to the m risky assets ($i = 1, ..., m$) assume there is a risk-free asset ($i = 0$) for which has a constant return, $R_0 \equiv r_0$. This implies, $\mathbb{E}[R_0] = r_0$, and $\mathbb{V}\mathrm{ar}(R_0) = 0$. r_0 here represents the known return on the risk-free asset.
Note: R_0 has a zero variance and is uncorrelated with the risky asset returns $\underline{\boldsymbol{R}}$.

3.1 Portfolio with Investment in Risk-Free Assets

Suppose the investor can now allocate wealth between m risky investments as well as in the risk-free assets. we will define our available assets for investment:

- Risky assets $i = 0, ..., m$ with returns $\underline{\boldsymbol{R}} = (R_1, ..., R_m)$ with $\mathbb{E}[\underline{\boldsymbol{R}}] = \underline{\boldsymbol{\alpha}}$, $\mathbb{C}\mathrm{ov}[\underline{\boldsymbol{R}}] = \underline{\boldsymbol{\Sigma}}$
- Risk-free asset with return $R_0 \equiv r_0$, a constant

As before, let $\underline{\boldsymbol{w}}$ denote the portfolio weights assigned to the risky assets, then:

$$\underline{\boldsymbol{w}}^T \mathbf{1}_m = \sum_{i=1}^{m} w_i \text{ is invested in risky assets and}$$

$$1 - w\mathbf{1}_m \text{ is invested in the risk-free asset.}$$

Hence,

$$R_{\underline{\boldsymbol{w}}} = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{R}} + (1 - \underline{\boldsymbol{w}}^T \mathbf{1}_m) R_0,$$

where $\underline{\boldsymbol{R}} = (R_1, .., R_m)$ is the returns of all the individual assets.

Note: If borrowing is allowed, $1 - w\mathbf{1}_m$ can be negative although when coding up our example later, we will stick to long only portfolios.

A portfolio that includes both risky and risk-free assets has the expected return $\alpha_{\underline{\boldsymbol{w}}}$ and variance $\sigma^2_{\underline{\boldsymbol{w}}}$ are calculated as follows:

$$\alpha_{\underline{\boldsymbol{w}}} = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}} + (1 - \underline{\boldsymbol{w}}^T \mathbf{1}_m) R_0$$
$$\sigma^2_{\underline{\boldsymbol{w}}} = \underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma w}}$$

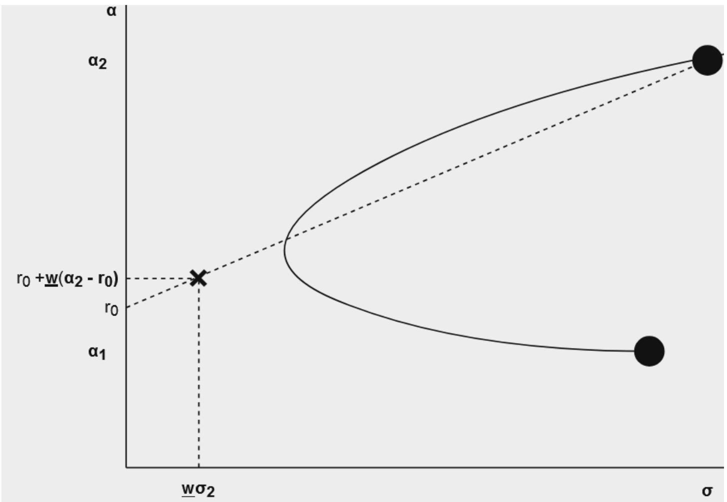

Figure 2: The point here represents the risk-free asset. The dotted line from the top of the old feasible set to the intercept r_0 represents the new feasible set of the portfolio

Similar to the previous chapter we come across the same three problems which Markowitz mean variance analysis can solve. These are risk minimisation, expected return maximisation, and risk aversion optimisation. Here however, we must take into account the risk-free asset. Like before, all solutions to each problem are very similar so we will continue to focus on risk minimisation problem.

Risk Minimisation with Risk Free Asset: For a given target mean return α_0, choose the portfolio $\underline{\boldsymbol{w}}$ to
minimise: $\frac{1}{2}\underline{\boldsymbol{w}}^T \underline{\boldsymbol{\Sigma w}}$
subject to: $\underline{\boldsymbol{w}}^T \underline{\boldsymbol{\alpha}} + (1 - \underline{\boldsymbol{w}}^T \mathbf{1}_m) r_0 = \alpha_0$

Solution: Like last chapter, we will apply the method of Lagrange multipliers to the convex optimisation problem. Define the Lagrangian:

$$\mathcal{L}(\underline{\boldsymbol{w}}, \lambda_1) = \frac{1}{2}\underline{\boldsymbol{w}}^T\underline{\boldsymbol{\Sigma}}\underline{\boldsymbol{w}} + \lambda_1[(\alpha_0 - r_0) - \underline{\boldsymbol{w}}^T(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)],$$

we can derive the first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \underline{\boldsymbol{w}}} &= \underline{\mathbf{0}}_m = \underline{\boldsymbol{\Sigma}}\underline{\boldsymbol{w}} - \lambda_1[\alpha - \mathbf{1}_m r_0],\\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= 0 = (\alpha_0 - r_0) - \underline{\boldsymbol{w}}^T(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0),\\
\frac{\partial^2 \mathcal{L}}{\partial \underline{\boldsymbol{w}}\partial \underline{\boldsymbol{w}}^T} &= \underline{\boldsymbol{\Sigma}} \geq 0.
\end{aligned}$$

We now solve for $\underline{\boldsymbol{w}}$ in terms of λ_1:

$$\begin{aligned}
\underline{\boldsymbol{w}}_0 &= \lambda_1\underline{\boldsymbol{\Sigma}}^{-1}[\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0],\\
\Rightarrow \lambda_1 &= \frac{(\alpha_0 - r_0)}{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}
\end{aligned}$$

Therefore, the optimal portfolio P has a target return α_0 and invests in risky assets according to fractional weights vector:

$$\underline{\boldsymbol{w}}_0 = \lambda_1\underline{\boldsymbol{\Sigma}}^{-1}[\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0], \quad \text{and,}$$

where,

$$\lambda_1 = \lambda_1(P) = \frac{(\alpha_0 - r_0)}{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}.$$

Here we are investing in the risk-free asset with the weight $(1 - \underline{\boldsymbol{w}}^T\mathbf{1}_m)$. As a result the **optimal portfolio return** R_P and **optimal portfolio variance** σ_P^2 is:

$$\begin{aligned}
R_P &= \underline{\boldsymbol{w}}_0^T\underline{\boldsymbol{R}} + (1 - \underline{\boldsymbol{w}}_0^T\mathbf{1}_m)r_0,\\
\mathbb{V}\text{ar}[R_P] &= \mathbb{V}\text{ar}[\underline{\boldsymbol{w}}_0^T\underline{\boldsymbol{R}} + (1 - \underline{\boldsymbol{w}}_0^T\mathbf{1}_m)r_0] = \mathbb{V}\text{ar}[\underline{\boldsymbol{w}}_0^T\underline{\boldsymbol{R}}]\\
&= \underline{\boldsymbol{w}}_0^T\underline{\boldsymbol{\Sigma}}\underline{\boldsymbol{w}}_0 = \frac{(\alpha_0 - r_0)^2}{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}
\end{aligned}$$

3.2 Tobin's Separation Theorem and the Capital Market Line

We now need to construct the **market portfolio** M, which is the portfolio of all risky assets in the market, weighted according to their market values. It is the tangency portfolio on the Capital Market Line (CML) which represents the highest Sharpe ratio achievable by combining the risk free and risky assets. The fully invested optimal portfolio with $\underline{\boldsymbol{w}}_M$ such that $\underline{\boldsymbol{w}}_M^T\mathbf{1}_m = 1$. In other words, $\underline{\boldsymbol{w}}_M = \lambda_1\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)$, where $\lambda_1 = \lambda_1(M) = [\mathbf{1}_m^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)]^{-1}$. The market portfolio return is $R_M = \underline{\boldsymbol{w}}_M^T\underline{\boldsymbol{R}} + 0 \cdot R_0 = \underline{\boldsymbol{w}}_M^T\underline{\boldsymbol{R}}$, and hence the **market portfolio expected return** and **market portfolio variance** are as followed:

$$\begin{aligned}
\mathbb{E}[R_M] &= \mathbb{E}[\underline{\boldsymbol{w}}_M^T\underline{\boldsymbol{R}}] = \underline{\boldsymbol{w}}_M^T\underline{\boldsymbol{\alpha}} = \frac{\underline{\boldsymbol{\alpha}}^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}{\mathbf{1}_m^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}\\
&= r_0 + \frac{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}{\mathbf{1}_m^T\underline{\boldsymbol{\Sigma}}^{-1}(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}
\end{aligned}$$

$$\mathbb{V}\mathrm{ar}[R_M] = \underline{\boldsymbol{w}}_M^T \underline{\boldsymbol{\Sigma}} \underline{\boldsymbol{w}}_M = \frac{(\mathbb{E}[R_M] r_0)^2}{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T \underline{\boldsymbol{\Sigma}}^{-1} (\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}$$
$$= \frac{(\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)^T \underline{\boldsymbol{\Sigma}}^{-1} (\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)}{[\mathbf{1}_m^T \underline{\boldsymbol{\Sigma}}^{-1} (\underline{\boldsymbol{\alpha}} - \mathbf{1}_m r_0)]^2}$$

We can clearly see from the above results that the optimal returns and risk can involve both risky and risk-free assets. In the case of a portfolio that contains risk free and risky assets, the optimal portfolio is achieved by maximising the returns for a given risk across all portfolios. This leads us to the tangent line that crosses the fully invested market portfolio. This is the Capital Market Line and the slope of this line represents the Sharpe ratio, indicating the best combination of risky and risk-free assets that the investor can achieve. This result leads us to the following theorem:

Theorem: *(Tobin's Separation Theorem)*
Every optimal portfolio invests in a combination of the risk-free asset and the Market Portfolio

Using this, let P be the optimal portfolio for the target return α_0 with risky investment weights $\underline{\boldsymbol{w}}_P$. The portfolio P invests in the same risky assets as the mark portfolio M, and in the same proportions. The only difference between P and M is the total weight allocated to the risky assets denoted by $w_M = \underline{\boldsymbol{w}}_P^T \mathbf{1}_m$. We can express the market portfolio weights w_M as the ratio of the Lagrangian multipliers $\lambda_1(P)$ and $\lambda_1(M)$ from the optimisation problem earlier:

$$w_M = \frac{\lambda_1(P)}{\lambda_1(M)} = \frac{\alpha_0 - r_0}{\mathbb{E}[R_M - r_0]}.$$

The optimal portfolio return is $R_P = (1 - w_M) r_0 + w_M R_M$, with an expected return of

$$\mathbb{E}[R_P] = r_0 + w_m(\mathbb{E}[R_M] - r_0),$$

which shows the risk-free rate r_0 plus the weighted excess return from the market portfolio. The variance of the optimal portfolio is proportional to the square of the market weight w_M allocated to the market portfolio:

$$\sigma_P^2 = \mathbb{V}\mathrm{ar}[R_P] = \mathbb{V}\mathrm{ar}[w_M R_M] = w_M^2 \mathbb{V}\mathrm{ar}[R_M] = w_M^2 \sigma_M^2.$$

We are now ready to write out the formal definition of the Capital Market Line (CML). The CML represents the set of optimal portfolios that combine the risk-less assets with the market portfolio. It can be represented as a straight line on the (σ_P, μ_P) plane, where μ_P is the expected return and σ_P is the standard deviation (risk) of all portfolios. CML offers the highest expected returns for each level of risk, outperforming portfolios consisting solely on risky assets. If we represent the Efficient Frontier of optimal portfolios in the (σ_P, w_P) space, then the Capital Market Line (CML) is defined as:

$$CML = \{(\sigma_P, \mathbb{E}[R_P]) : P \text{ optimal with } w_M = \underline{\boldsymbol{w}}^T \mathbf{1}_m > 0\}$$
$$= \{(\sigma_P, \mathbb{E}[R_P]) = (\sigma_P, r_0 + w_M(\mu_M - r_0),\ w_m > 0\}$$

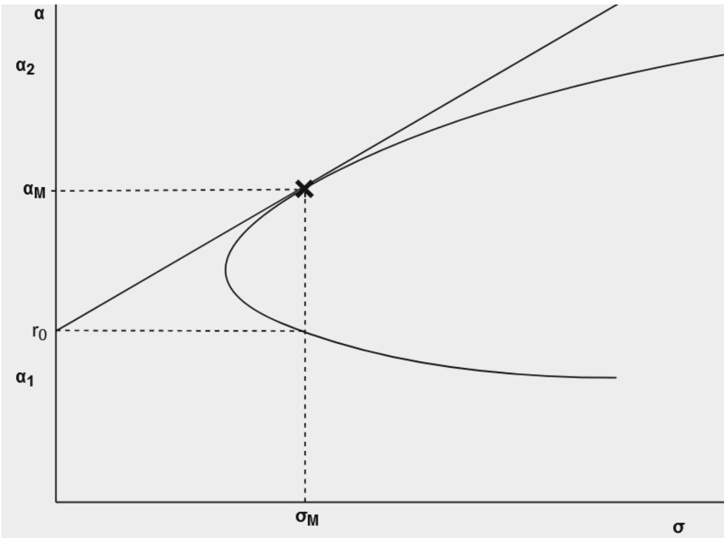

Figure 3: The slope of the line represents the Sharpe ratio, given by $\mathbb{E}[R_P] = r_0 + \sigma_P \left(\frac{\alpha_M - r_0}{\sigma_M}\right)$. $\frac{\alpha_M - r_0}{\sigma_M}$ is the Sharpe ration of the market portfolio.

4 Modelling Markowitz Mean Variance Optimisation

In this section we will go over some of the code results of the Markowitz Mean Variance Optimisation. In our examples below we have decided to use the following variety of assets as our risky assets:

- SPY - SPDR S&P 500 ETF Trust
- TLT - iShares 20+ Year Treasury Bond ETF
- AAPL - Apple Inc.
- MSFT - Microsoft Corporation
- GOOGL - Alphabet Inc.
- XOM - Exxon Mobil Corporation
- BTC-USD - Bitcoin
- USO - United States Oil Fund LP
- XLK - Technology Select Sector SPDR Fund
- SLV - iShares Silver Trust

The risk free asset we will use is the 3 month Treasury bills, DGS3MO. After collecting the data, we compute the expected log returns for the assets in the portfolio, as well as their covariance matrix. We will additionally simulate 100,000 portfolios, and then calculate the returns, variances, volatility, and Sharpe ratio of these portfolios using the formulas derived above.
We now plot the feasible set also know as the *Markowitz's Bullet*. This scatter plot

displays all combinations of returns for a given risk level (volatility level). This is shown below in 4.[1]

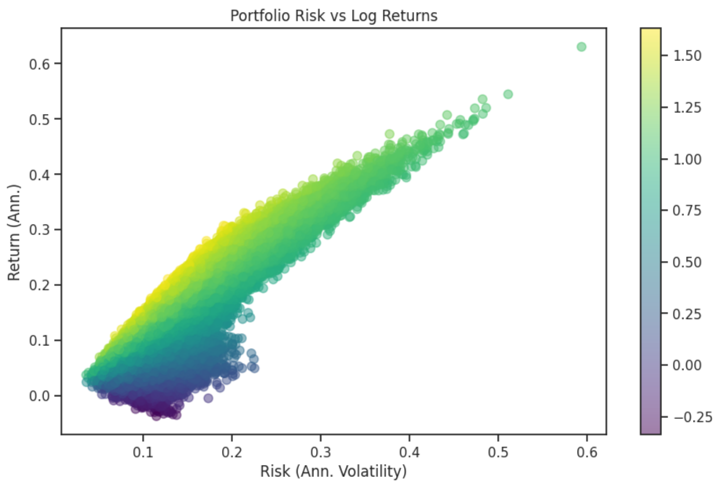

Figure 4: Feasible Set of the risk and returns of all 100,000 portfolios simulated. This is also known as the *Markowitz Bullet*.

The feasible set's boundary, minimised or maximised under a certain constraints, varies depending on the objectives of the investor, such as risk tolerance or target return. Here we have modelled the investor's objectives as to maximise return for a given risk. This gives us the efficient frontier (the upper curve of the feasible set's boundary), which is the optimal portfolios for this objective. Portfolios below this curve are considered sub-optimal, and as such, no rational investor would invest in these portfolios. No portfolio can can have a higher return above the efficient frontier for a given risk level.

The constraints are the same as the constraints mentioned in **Chapter 2.2** and **Chapter 3.1**. We use *SciPy*'s *Optimize* package to solve the minimisation problem with the given constraints. We follow this by plotting the Capital Market Line (CML), which represents the set of portfolios including both risky and risk-free assets, and is the tangent to the efficient frontier at the market portfolio. The results of both are shown in 5.

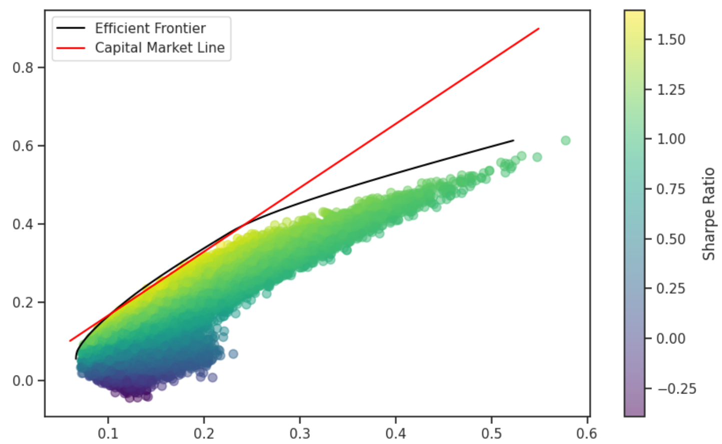

Figure 5: Capital Market Line and Efficient Frontier

[1] This is feasible set of portfolios which include both risky and risk-free assets.

Finally, we can compare the performance of a optimal and sub-optimal portfolio for this time frame. We will randomly choose a portfolio from the feasible set and choose the optimal portfolio for that given risk level. We will look at the performance difference of both portfolios investing in only risky assets as well as portfolios investing in both risky and risk-free assets. These plots are shown in 6 and 7.

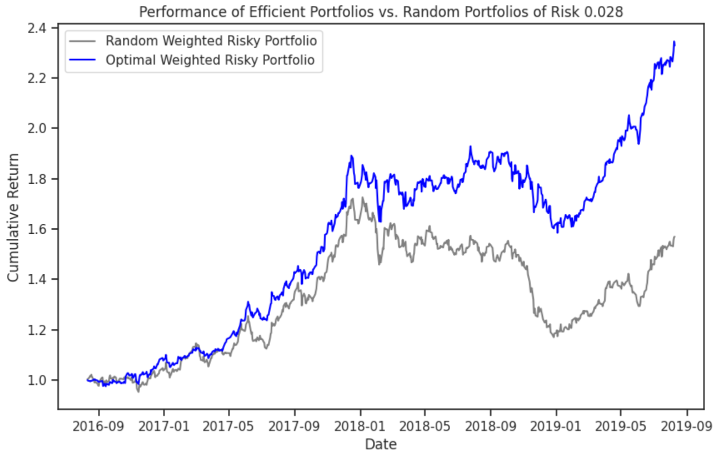

Figure 6: Pre COVID performance of the different portfolios investing in only risky assets. We can see that the optimised weights perform better than the random portfolios, despite risk is the same.

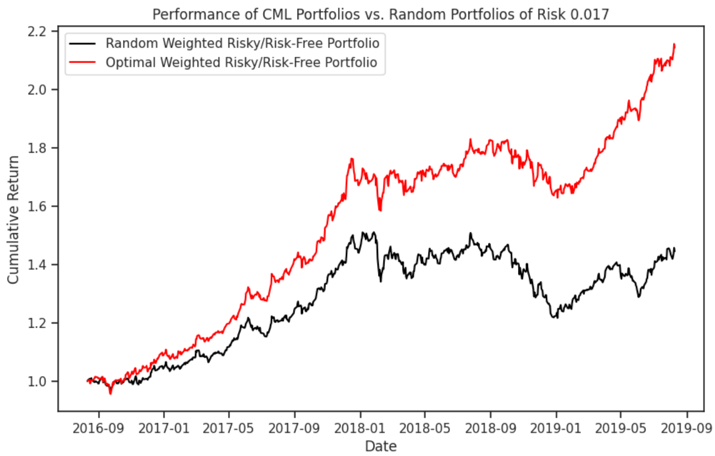

Figure 7: Pre COVID performance of the different portfolios investing in both risky and risk-free assets. We can see that the optimised weights perform better than the random portfolios, despite risk is the same.

5 Further Studies in Portfolio Theory

In this paper, we looked at the birth of Modern Portfolio Theory, with the foundational concepts of Markowitz Mean Variance Optimisation, and applied these concepts to construct Markowitz's bullet for both portfolios investing in only risky assets as well as portfolios investing in both risky and risk-free assets. We furthered our study by looking at the efficient frontier and the capital market line to highlight the optimal combination of assets for a given level of risk to maximise returns. This showed us the importance of diversified portfolios compared to non-diverse portfolios.

The framework of Markowitz's Optimisation shed light onto a new perspective of risk management. Since then, we have developed many more topics in this area of financial markets that take portfolio optimisation further. These other models take into account factors which Markowitz does not. Two of these studies that extend modern portfolio theory are the *Von Neumann-Morgenstern Utility Theorem* and *Value at Risk (VaR)* models.

5.1 Von Neumann-Morgenstern Utility Theorem

Markowitz's model assumes that investors are primarily concerned with balancing risk and return. In reality, these preferences can vary depending on their utility functions. Von Neumann-Morgenstern's Utility Theorem introduces a framework for us to evaluate portfolios based on risk-aversion and other factors.

5.2 Value at Risk (VaR)

Introduced by JPMorgan in their *RiskMetrics* paper released in 1994, the Value at Risk model (VaR) has become one of the most widely used methods for evaluating risk in financial markets. VaR estimates the maximum potential loss of a portfolio over a specific time period, under a given significance level, under normal market conditions. This discovery has allowed investors to better manage downside risks and account for extreme market conditions.