Pricing Theory: Black Scholes Model & Heston Model

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Abstract:

This paper explores the mathematical foundations and practical applications of two famous pricing models: Black-Scholes model and the Heston model. The Black-Scholes is renowned for its simplicity and closed-formed solutions for option pricing, although facing practical limitations due to its assumptions such as constant volatility. In contrast, the Heston model introduces stochastic volatility to simulate the volatility smile, leverage effect, and fat-tailed distributions. The paper starts with an overview of the stochastic processes that build these models, including Brownian motion, martingales, and stochastic calculus. Detailed derivations of the Black-Scholes and Heston models are provided, before moving onto numerical implementations and simulations to show the strengths and weaknesses of both models, with more focus on how the Heston model better captures the market dynamics. Additionally, we look at calibration techniques for the Heston model before concluding by reflecting on the models' contributions and potential extensions to more complex frameworks.

Contents

1 Introduction

Pricing Theory is a vital aspect of financial markets. It provides a mathematical foundation for the valuation of derivative securities, risk-management, and portfolio theory. As a result, pricing theory is the building blocks of modern finance.

The *Black-Scholes model*, introduced in 1973, remains of the most well-known tools for pricing financial derivatives, offering a closed-form solution for pricing. The model's simplicity was so elegant that Robert C. Merton and Mryon S. Scholes won the Noble Prize in Economic Sciences in 1997 for the pricing model [1]. However, the Black-Scholes model is built on several assumptions that limit its real-world applicability, the most notable assumption being that the model assumes constant volatility.

In practice, volatility is not constant but dynamic, and varies based on factors like liquidity, and macroeconomic events for example. Real-world volatility is a function of the strike price and maturity, which when plotted, we get a *volatility smile*. These smiles state that deep-in-the-money and deep-out-the-money options tend to have higher implied volatility than at-the-money options, this is due to greater uncertainty about future price movements, as they are more sensitive to factors such as interest rates, dividends, and underlying asset volatility. If we use the Black-Scholes to calculate the option prices, the implied volatility derived from market prices is assumed to be constant, and not a function of the strike or maturity. Hence, the implied volatility curve is flat, suggesting the model is too simplistic to reflect the market dynamics.

Additionally, empirical evidence shows that return distributions in financial markets have high peaks and fat tails, meaning that the existence of extreme values occur more often than a normal distribution suggests.

Steven Heston addressed these limitations with the Heston model, which incorporates stochastic volatility by modeling the asset price and variance as two processes. The asset price process follows *Geometric Brownian Motion*, while the variance process follows a *mean-reverting Cox-Ingersoll-Ross process*. The mean reversion ensures that volatility fluctuates around a long-term average. Furthermore, the correlation between the two processes allows for correlated shocks between volatility and asset price, which captures the market behaviour where volatility increases during downturns. This correlation enables the model to reproduce skew effects seen in distributions, such as *kurtosis* (fat tails) and asymmetry.

Another major difference with the Heston model, compared to the Black-Scholes model, is the hedging strategy. The Black-Scholes model, relies on *delta hedging* to reduce risk due to its single source of randomness. The Heston model however, has two sources of randomness and hence uses a *delta-gamma hedging* strategy, which manages both changes in the underlying asset and changes in volatility. As a result, the Hence model takes into account the incomplete nature of the market, despite adding complexity. Hence this model offers more robust hedging possibilities.

Moreover, unlike the Black-Scholes, Heston's model has multiple martingale measures,

[1] Unfortunately, Fischer Black died in 1995 and was ineligible for the award.

meaning that there are many ways to represent the model under risk-neutral measure. This gives flexibility in derivative pricing and allowing it to represent market conditions more accurately. As a result, Heston model has been a very popular tool since its discovery, especially for pricing options in conditions where volatility is highly uncertain and evolves dynamically.

In this paper we begin by laying the mathematical foundation required to understand both models. Part 1 is a preliminary section, covering stochastic processes and stochastic calculus, aims to give a rigorous mathematical basis for the models discussed later. We then move into part 2, where we dive into the pricing model theory, presenting both the Black-Scholes and Heston models, and the key properties and assumptions. Finally, in part 3, we turn to empirical analysis implementing and calibrating the models, comparing their behaviours, and discussing the implication of our findings in context of the financial markets.

2 Stochastic Processes

In this section on Stochastic Processes, we will describe all stochastic processes as a continuous time-series. The main objective here is to give a quick overview on stochastic processes and not a whole course on it. To start, we will define Brownian Motion, then we will look at adaptive processes and martingales, before finally looking at the quadratic variation.

Recall: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\Omega = \{-1, 1\}^{\infty}$, $\mathcal{F}$ is the σ-algebra generated by the sequence of random variables that are i.i.d. X_t, $t \geq 1$, with the probability measure $\mathbb{P}$. We say a simple random walk is then following:

$$X_t - X_{t-1} = \begin{cases} 1 & p = \frac{1}{2}, \\ -1 & p = \frac{1}{2}. \end{cases}$$

2.1 Brownian Motion

Brownian Motion, also known as *Wiener-Process*, is described as the following:
There exists a probability distribution over the set of continuous functions $B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that the following hold:

i) $\mathbb{P}(B(0) = 0) = 1$,

ii) (stationary) $\forall 0 \leq s \leq t$,

$$B(t) - B(s) \backsim \mathcal{N}(0, t - s),$$

iii) (Independent increments) If intervals $[s_i, t_i]$ are not overlapping, then $B(t_i) - B(s_i)$ are independent,

iv) B has continuous sample paths almost surely.

The time scale intervals are a collection of normal variables with mean 0 and variance is the length of the interval. Furthermore, Brownian motion is the limit of simple random walks.

The properties of Brownian Motion are as followed:

- Crosses the t-axis infinitely often,
- Does no deviate to much from $t = y^2$,
- Brownian Motion is not differentiable anywhere using classical calculus, i.e., for each $t \geq 0$, the Brownian Motion is not differentiable at t with probability 1. [2],
- $-B$ is also a Brownian Motion (symmetry).

We can denote $M(t)$ as the maximum value of Brownian Motion $B(s)$ over the interval $[0, t]$,

$$M(t) = \max\{B(s) : s \leq t\},$$

This leads us to the reflection principle in Brownian motion:

Let $M(t)$ be maximum value of Brownian Motion $B(t)$ defined above, up to time t. Then, $\forall a > 0$,

$$\mathbb{P}(M(t) > a) = 2\mathbb{P}(B(t) > a)$$

2.2 Martingales

Before we can define a martingale, we need to define *Filtration* and *Adaptive Processes*:

A **Filtration** is a family of sub σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$, such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. We can think of filtration as the flow of information available to us through time. The latter part of this definition means that σ-algebra is increase, i.e, we are gain more information over time, and we never forget information.

Let X be a stochastic process, then we say X is **adapted** to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\mathcal{F}_t$-measurable for every $t \geq 0$.
If X is an adaptive process, the decision at time t is only based on past information, in other words, we are not looking into the future. Note: the value of X_t is known at time t. In our case, all portfolios in financial markets are adapted (portfolio value is not determined by future stock prices).

Now we can formally write the definition of *Martingales*:
Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, and let $X = (X_t)_{t\geq 0}$ be a stochastic process which is adapted to $(\mathcal{F}_t)_{t\geq 0}$, and $\mathbb{E}[|X_t|] < \infty$, $\forall t \geq 0$. X_t is a **martingale** if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad \forall s \leq t$$

Note: We can see that if X is martingale and so is Y, with respect to a given filtration $(\mathcal{F}_t)_{t\geq 0}$, then $X + Y$ is a martingale and so is aX, $a \in \mathbb{R}$.

For example, the balance of a roulette player is not martingale since the game is not a fair game, since it is designed for you to lose your money in the long run. On the other

[2]Later we will why this is the case and that Brownian Motion is differentiable but using Ito's Calculus

hand, a simple random walk is martingale.

Finally, given a stochastic process X_t a non negative integer random variable τ is called a **stopping time** if $\forall k \geq$, $k \in \mathbb{Z}$, $\tau \leq k$ depends only on $X_1, X_2, X_3, ..., X_k$. For example, for a coin toss game, the stopping time τ could be described as the point which the balance becomes 100 or -100, where you gain a 1 for heads -1 for tails. However, in relation to markets, we cannot describe tau a stopping time if tau is the time of the first price peak for the day. This is because this is not an adaptive stochastic process.

2.3 Quadratic Variation

An important theorem which we will require for stochastic calculus is the *Quadratic Variation.* We can describe the quadratic variation of Brownian motion as the following:
Theorem: *(Quadratic Variation)*
Given the Brownian motion B over the time interval $[0, T]$, $T > 0$, where t describes the t^{th} sub-interval of n equally spaced sub-intervals, the **quadratic variation** is defined as:

$$\lim_{n\to\infty} \sum_{t=1}^{n} \left(B\left(\frac{t}{n}T\right) - B\left(\frac{(t-1)}{n}T\right)\right)^2 = T,$$

with probability 1.

If we look at the incremental behaviour of Brownian motion, $\Delta B = B(t + \Delta t) - B(t)$, then we can see that it normally distributed $\Delta B \backsim \mathcal{N}(0, \Delta t)$. Therefore we know for small time increments, the variance of the increment is proportional to the length of the time interval, in other words, $\mathbb{E}[(\Delta B)^2] = \Delta t$. By using the quadratic variation, we know that for small increments of the Brownian motion of the n sub-intervals, we have

$$\mathbb{E}\left[\left(B\left(\frac{t}{N}T\right) - B\left(\frac{t-1}{N}T\right)\right)^2\right] = \frac{T}{N} = \Delta t.$$

As the time steps become infinitesimally small, we get

$$\lim_{\Delta t\to\infty} \frac{(B(t+\Delta t) - B(t))^2}{\Delta} = 1.$$

This leaves us with an important formula which we will use for stochastic calculus:

$$(\mathrm{d}B)^2 = \mathrm{d}t.$$

2.4 Multidimensional Brownian Motion

Up to this point, we have only looked at one-dimensional Brownian motion. When we get to Heston models, we will require knowledge of two-dimensional Brownian motion. *d-dimensional Brownian motion* is defined as the following:

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, and let $d \in \mathbb{N}$. An $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t\geq 0}$, which is adapted to $(\mathcal{F}_t)_{t\geq 0}$ is called ***d*-dimensional Brownian motion** (with respect to $(\mathcal{F}_t)_{t\geq 0}$) if

- $B_0 = 0 \in \mathbb{R}^d$,
- for each $0 \le s < t$, the random variable $B_t - B_s$ has a multivariate normal distribution with mean vector 0 and covariance $(t-s)I$, where I is the dxd identity matrix,
- for each $o \le s < t$, the random variable $B_t - B_s$ is independent of $(\mathcal{F}_t)_{t\ge 0}$,
- B has continuous $\mathbb{R}^d$-valued sample paths almost surely.

A d-dimensional Brownian motion can be described as $\boldsymbol{B} = (B^{(1)}, B^{(2)}, ..., B^{(d)})^T$, such that $B^{(i)}$ is a one dimensional Brownian motion, $i = 1, ..., d$, and each i is independent to each other.

3 Stochastic Calculus

Unfortunately, as we will discover in this chapter, stochastic calculus doesn't operate the same as classic calculus due to the Brownian motion's nature. For example, in classic calculus there existed a *Riemannian sum* description to explain the area under a smooth function f, through n intervals. For example:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{t} f\left(\frac{tb}{n}\right)$$

This sum is unique in classic calculus. However, this is not the case in Brownian motion, i.e. the limits at each end points are different due to quadratic variation. This is because the variance can accumulate over time. An *Itô integral*[3] is the limit of Riemannian sums when taking the leftmost point of each interval. In this paper we will only consider this limit since it is the most relevant for financial markets, since they are adaptive processes.

3.1 Stochastic Integrals

Let B_t, $t \ge 0$, be a Brownian motion that is adapted to the filtration $(\mathcal{F}_t)_{t\ge 0}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$. Let X_t be a stochastic process that is adapted to $(\mathcal{F}_t)_{t\ge 0}$, then the **Itô integral** of the process X_t with respect to the Brownian motion B_t over the interval $[0,t]$ is defined as:

$$\int_0^t X_s dB_s := \sum_{i=0}^{m-1} X_i (B_{t_{i+1}} - B_{t_i}),$$

where t_i are intervals in $[0,t]$, $0 < t_0 < t_1 < ... < t_m = t$.
We define a Itô integral is square integrable adapted to the process $X = (X_t)_{t\ge 0}$ in the space $L^2(B)$ if

$$X = \mathbb{E}\left[\int_0^\infty X_s ds\right] < \infty$$

This is also referred to as X is *reasonable.*

The Itô integrand X are always adapted to $(\mathcal{F}_t)_{t\ge 0}$. Itô integrals have a few properties one of which is the *Itô isometry*:

[3] This stochastic field is called Itô Calculus named after Kiyosi Itô.

Lemma: *(Itô isometry)* Let $\int_0^t X_s dB_s$ be a stochastic integral of X against a Brownian motion B, then for each $t \geq 0$,

$$\mathbb{E}\left[\left(\int_0^t X_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t X_s^2 ds\right].$$

Also, note that if X_s is a process that depends only on t then

$$\int_0^t X_s dB(t) \backsim \mathcal{N}\left(0, \int_0^t X_s^2 ds\right),$$

meaning it has normal distribution at all time.

3.2 Itô Calculus

For stock prices, we want the percentile difference to be normally distributed i.e.

$$\frac{dS_t}{S_t} = dB_t,$$

where S_t represents the stock price at time t, and B_t is a standard Brownian motion (with no drift function). One might assume the answer to this would be $S_t = e^{B_t}$, however this assumption is incorrect. The reason being is that the differential equation describes relative change in the stock price, not absolute change. In other words, $\frac{dS_t}{S_t}$ gives the proportional change in S_t, but is also driven by the stochastic process of B_t, which complicates the solution due to the properties of Brownian motion.

Suppose we want to compute $f(B_t)$ for some smooth function f, say a call option, where B_t is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We want to compute $f(B_t)$ for some differentiable function $f : \mathbb{R} \to \mathbb{R}$, where f is smooth. One might assume, $\mathrm{d}f = dB_t \cdot f'(B_t)$, which is not valid due to the quadratic variance, which introduces an additional term in the differential equation. However, we can solve this using Taylor series:

$$f(t+x) = f(t) + f'(t)x + \frac{f''(t)}{2}x^2 + ...$$
$$f(t+x) - f(t) = f'(t)x + \frac{f''(t)x^2}{2} + ...$$

We know that in the classical world, $f(t+x) - f(t) \approx f'(t)x$. Hence,

$$f(B_{t+x}) - f(B_t) = f'(B_t) \cdot (B_{t+x} - B_t) + \frac{f''(B_t)}{2}(\mathrm{d}B_t)^2,$$
$$f(B_{t+x}) - f(B_t) = f'(B_t)dB_t + \frac{f''(B_t)}{2}(dt),$$

by Taylor expansion. This leads us to the important lemma, and introduction to Itô Calculus:

Lemma: *(Simple Itô Lemma)*
For a standard Brownian motion $\{B_t\}_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $f : \mathbb{R} \to \mathbb{R}$

be a continuously differentiable smooth function, then the stochastic differential of $f(B_t)$, which is a function of the Brownian motion B_t, is given by

$$df(B_t) = f'(B_t)dB_t + \frac{f''(B_t)dt}{2}.$$

Now assume $f(t,x)$ and we want to evaluate $f(t,B_t)$, then we would want to know what df is. We can first look at small increment changes in $f(t,x)$:

$$\begin{aligned} f(t,+\Delta t, x+\Delta x) = f(x,t) + \frac{\partial f(t,x)}{\partial t}\Delta t + \frac{\partial f(t,x)}{\partial x}\Delta x + \\ \frac{1}{2}\left(\frac{\partial^2 f(t,x)}{\partial t^2}(\Delta t)^2 + 2\frac{\partial^2 f(t,x)}{\partial t \partial x}\Delta x \Delta t + \frac{\partial^2 f(t,x)}{\partial x^2}(\Delta x)^2\right) + ... \\ = f + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial t \partial x}dtdx + \frac{\partial^2 f}{\partial x^2}(dx)^2\right) + ... \end{aligned}$$

Now if we look at $f(t,B_t)$, we can solve for dt in a similar method as before for $f(B_t)$:

$$f(t+dt, B_t+dB_t) = f(t,B_t) + \frac{\partial^2 f}{\partial t^2}dt + \frac{\partial f}{\partial x}dB_t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial t \partial B_t} + \frac{\partial^2 f}{\partial B_t^2}(B_t)^2\right) + ...$$

By using the quadratic variation $(dB_t)^2 = dt$ and noting the fact that $\frac{\partial^2 f}{\partial t^2}dt = 0$ and $2\frac{\partial^2 f}{\partial t \partial B_t} = 0$, we get the following:

$$\begin{aligned} f(t+dt, B_t+dB_t) - f(t,B_t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B_t}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2}dt \\ &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2}\right)dt + \frac{\partial f}{\partial B_t}dB_t. \end{aligned}$$

This leads us to a more polished version of Itô's Lemma (simple):

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2}\right)dt + \frac{\partial f}{\partial B_t}dB_t.$$

Now consider a stochastic process X_t, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, such that

$$dX_t = \mu dt + \sigma dB_t,$$

where μdt is the *drift term*, and μ, σ are constants. Then we can conclude with the final gerneral version of Itô's Lemma:

Theorem: *(Itô's Lemma)*
Let X_t be a stochastic process defined by the stochastic differential equation $dX_t = \mu dt + \sigma dB_t$, where μ is the drift term, and σ is the volatility term, both of which are constants. Additionally, B_t is a one dimensional standard Brownian motion. Assume X_t is adapted to filtration $(\mathcal{F}_t)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $f : \mathbb{R}\text{x}\mathbb{R} \to \mathbb{R}$ be a continuously differentiable smooth function depending on time t and the stochastic process X_t, then the stochastic differential of $f(t,B_t)$ is

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X_t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X_t^2}\right)dt + \sigma\frac{\partial f}{\partial X_t}dX_t. \tag{1}$$

This leads us to the following definition:

$$F(t, B_t) = \int f(t, B_t) dB_t + \int g(t, Bt) dt,$$

where $dF = f dB_t + g dt$

3.3 Itô Integrals and Martingales

In financial mathematics, we are mostly interested in processes that are martingales, since this best reflects the nature of the underlying asset's price. Recall from earlier a stochastic process $X = (X_t)_{t\geq 0}$ which is adapted to $(\mathcal{F}_t)_{t\geq 0}$, is martingale if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \; \forall s \leq t$$

Itô integrals that are martingale hold the following property:
Theorem: If $g(t, B_t)$ is adapted to the Brownian motion B_t, then

$$\int g(t, B_t) dB_t \quad \text{is a martingale}$$

as long as g is reasonable, i.e,

$$\iint g^2(t, B_t) dt dB_t < \infty$$

If $dX_t = \mu(t, B_t)dt + \sigma(t, B_t)dB_t$ then X_t is martingale if $\mu = 0$, $\forall t$. If $\mu \neq 0$, then you have drift. The drift contributes to the "tendency" of which what will happen in the future. Below is a diagram of the difference between a stochastic process with and without drift

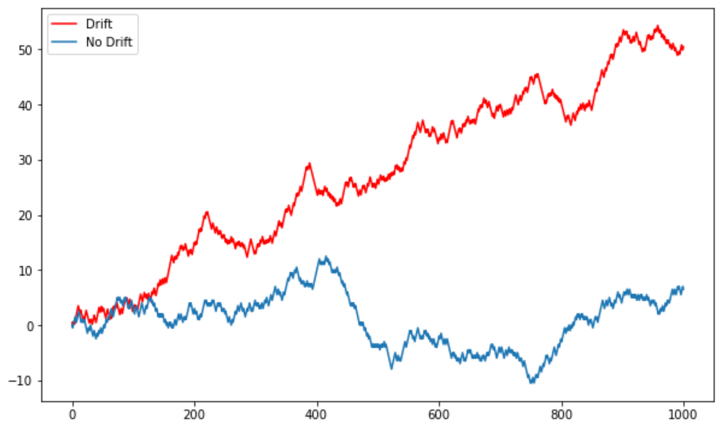

Figure 1: A plot displaying two stochastic processes, one with drift (red) and one without drift (blue).

$\sigma(t, B_t)dB_t$ is the volatility term, representing the random behaviour of Brownian motion in dX_t. This stochastic differential equation (SDE) is know as an *Itô diffusion.*

If we go back to our example for the stock price percentile difference, but this time we will include drift and a volatility term:

$$\frac{dS_t}{S_t} = dX_t,$$

where $dX_t = \mu dt + \sigma dB_t$. We can now write this formula as a *geometric Brownian motion*,

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where μ is the drift term, σ is the volatility term, and B_t is the standard Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$. To solve this we can use Itô's lemma from before. Let $f(t, S_t) = \log(S_t)$, then

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} \right) dt + \sigma \frac{\partial f}{\partial S_t} dS_t.$$

By subbing the following into the above,

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial S_t} = \frac{1}{S_t}, \qquad \frac{\partial^2 f}{\partial S_t^2} = -\frac{1}{(S_t)^2}$$

we get:

$$df = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB_t$$

We can now integrate both sides to give us:

$$f = f(t, S_0) + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t.$$

If we sub back in $f(t, S_t) = S_t$, we will get the final answer for S_t

$$\begin{aligned} \log S_t &= \log S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \\ S_t &= S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right). \end{aligned} \tag{2}$$

We will come back to this in *Black Scholes Model* later in this paper. It is important to note that if X_t is a diffusion process (includes a drift term and has Brownian motion), then its quadratic variation can be used with Itô's lemma to derive results.

Before we move on we will propose the stochastic exponential. If we let B be a Brownian motion, and let θ be a locally bounded process. The solution to the stochastic differential equation

$$dZ_t = \theta_t Z_t dB_t$$

with the initial condition $Z_0 = 1$, is given by

$$Z_t = \exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right), \quad \forall t \geq .0$$

This stochastic exponential is also martingale, and can be proven by using Itô lemma.

3.4 Change In Measure

Stock prices are typically modeled with drift term. However, the drift term causes problems for us when we attempt to work with martingales, which is crucial for pricing theory.

We want to some how model the stock price without the drift term, to become a martingale.

Suppose B_t is a Brownian motion without drift, with a probability measure $\mathbb{P}(w)$. Also suppose $\tilde{B}_t$ is a Brownian motion but with drift μ, with a probability measure $\tilde{\mathbb{P}}(w)$, both probability measures in the same measurable space $(\Omega, \mathcal{F})$. Then the question is: can switch between the two probability measures by a change of measure? In other words, does there exist $Z_T = Z_T(w)$ such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(w) = Z_T(w)$?

Recall: If $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}, \tilde{\mathbb{P}}$ are both probability measures in this space, then if $\mathbb{P}, \tilde{\mathbb{P}}$ are equivalent, then $\mathbb{P}(A) > 0 \iff \tilde{\mathbb{P}} > 0$, $\forall A \subseteq \Omega$.

We say Z is a *Radon-Mikadga derivative* if $\exists Z_T$ such that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(w) = Z_T(w) = e^{-\mu B_T - \mu^2 \frac{T}{2}} \qquad \iff \qquad \mathbb{P} \text{ and } \tilde{\mathbb{P}} \text{ are equivalent.}$$

Thereom: *(Giranov's Theorem)*
Let $\mathbb{P}$ be a probability measure over $[0,T]^\infty$ defined by Brownian motion B_t with filtration $(\mathcal{F}_t)_{t\geq 0}$ with drift μ. Define

$$Z(w) = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(w) = e^{-\mu B_T - \mu^2 \frac{T}{2}}.$$

and

$$\tilde{B}_t = B_t + \int_0^t \mu ds, \quad t \in [0,T],$$

Then under the probability measure $\tilde{\mathbb{P}}$ with the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z$, is the process $(\tilde{B}_t)_{t\geq 0}$, and is a standard Brownian motion.

As we will later see, change in measure is the foundations of risk-neutral pricing.

4 Black Scholes Model

Assume a risky asset (stock) with a price of S. S is modeled as stochastic since the future trajectory of the stock is unknown. Also assume that we have access to a risk-free asset, which is deterministic, denoted β.[4] Additionally, the interest on this risk-free assets is at a constant rate $r \in \mathbb{R}$. We can model the price of the risk-less asset as a differential equation:

$$d\beta_t = r\beta_t dt,$$

which has a solution

$$\beta_t = \beta_0 e^{rt}.$$

[4] B is sometimes used for risk-less money market account, however, I have decided to use β to prevent confusion with Brownian motion.

Assume that the stock has log-normal dynamics, then the Black Scholes model supposes that the stock price S satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \quad t \geq 0,$$

where dB_t is a standard one dimensional Brownian motion, μ is the drift term, and $\sigma > 0$ is the volatility term. From (2) earlier:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

Here, we assume that the initial value of the stock S_0 is strictly positive. Additionally, we define $(a_t, b_t)_{t\in[0,T]}$ to represent the number of stocks and bonds we have at time t respectively. We can define *Value Process* of a portfolio as the following:

$$V_t = a_t S_t + b_t \beta_t = a_t S_t + b_t \beta_0 e^{rt}$$

To add to this, we can say it is self-financing if the value process satisfies

$$dV_t = a_t dS_t + b_t d\beta_t.$$

This means that the changes to the value of the portfolio come only from the changes in the price of the stocks and bonds, and not from external inflow or outflow of portfolio value.

4.1 Replicating Portfolio

Before we can start looking at pricing options using Black Scholes, we best assume a few assumptions:

- There are no transaction (or other) costs, i.e. markets are *frictionless*.
- The stock price follows a geometric Brownian motion with a constant drift term and volatility.
- The market allows continuous trading
- We can borrow and lend cash at the constant risk-free rate.
- Underlying stocks don't pay dividends.

For us to determine the price (rational) of a stock, we need to assume the assumption that the market is arbitrage free. If there exist a portfolio (a, b), then if value process V satisfies the following:

$$V_0 = 0, \quad \mathbb{P}(V_T \geq 0) = 1, \quad \text{and } \mathbb{P}(V_T > 0) > 0,$$

then arbitrage opportunity exists. This is known as the *principle of no arbitrage.*

We can think of the following example as well to define arbitrage: if we consider a forward contract which has the payoff $S_T - K$ at the maturity T, and r is the risk-free rate, then consider the following strategy:

- Borrow $\$S_0$ to buy the stock. Enter the forward contract at a strike price of K_0
- At time T deliver the stock in exchange for K_0 and repay the loan $\$S_0e^{r(T)}$

One can see that if $K_0 > S_0e^{r(T)}$, then we make a risk-less profit and if $K_0 < S_0e^{r(T)}$, then we are guaranteed to make a loss. So by the principle of no arbitrage, $K_0 = S_0e^{r(T)}$. We can extend this example to the following important statement:

Let f be a financial derivative claim for a replicating portfolio (a, b), with a value process V. Under *principle of no arbitrage*, the rational price of the derivative at a given time $t \in [0, T]$, is equal to V_t, i.e, $X = V_t$, $\forall t \in [0, T]$.

4.2 Black Scholes' Formula

Let a stock have a log-normal dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The goal here is we want to replicate the payoff of a general derivative claim f such that

$$df(t, S_t) = a_t dS_t + b_t d\beta_t,$$

where (a_t, b_t) replicates the portfolio of the number of stock invested at time t, a_t, and number of bonds invested at time t, b_t.[5] We would like to find such coefficients a_t and b_t, such that small changes in the portfolio replicate the changes in the derivative over infinitely small time t. To do this, we use Itô's Lemma (1):

$$df(t, S_t) = \frac{\partial f(t, S_t)}{\partial t} dt + \frac{\partial F(t, S_t)}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial S^2}(dS)^2 \tag{3}$$

$$(dS_t)^2 = \sigma^2 S_t^2 dt$$

Now we want to substitute $dS, df(t, S_t), d\beta_t = r\beta dt$, and $(dS_t)^2$ into $df(t, S_t) = a_t dS_t + b_t d\beta_t$

$$\left(\frac{\partial f(t, S_t)}{\partial t} + \frac{\partial f(t, S_t)}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2\right) dt + \frac{\partial f(t, S_t)}{\partial S_t}\sigma S_t dB_t = (a_t \mu S_t + b_t r\beta)dt + a_t \sigma S_t dB_t$$

When we equate terms, get

$$a = \frac{\partial f(t, S_t)}{\partial S_t}, \quad b_t r\beta = \frac{\partial f(t, S_t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial S_t^2}\sigma^2 S_t^2$$

We know the bond price is growing at the risk-free rare, and hence know $d\beta = r\beta dt$, r being the risk-free rate. Additionally, since $b_t\beta_t$ is invested in risk-free bond, the change must be deterministic, hence $b_t\beta = f(t, S_t) - a_t S_t$ is deterministic, we can derive

$$d(f(t, S_t) - a_t S_t) = r(f(t, S_t) - a_t S_t)dt.$$

[5] Note that the price of a derivative claim at a given time t should continuously dependent on both t and the current stock price S_t.

We now want substitute into this equation Itô's Lemma 3 and $a = \frac{\partial f(t,S_t}{\partial S}$ to get the final form:

$$\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + rS_t \frac{\partial f(t,S_t)}{\partial S_t} - rf = 0$$

<u>**Theorem:**</u> *(Black-Scholes Equation)*
Let $f(t, S_t)$ denote the price of a derivative as a function of time t and the underlying asset price S_t, where

- t is time to maturity, $t \in [0, T]$
- S_t is the price of the underlying asset at time t
- σ is the volatility of the asset
- r is the risk-free rate

The price function $f(t, S_t)$ satisfies the following partial differential equation:

$$\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + rS_t \frac{\partial f(t,S_t)}{\partial S_t} - rf = 0 \tag{4}$$

The boundary condition and final conditions for this equation at time $t = T$ is given by the specific payoff of the option contract. For example, for a European call option, the boundary condition would be:

$$f(T, S_T)) = (S_T - K)^+,$$

where K is the strike price of the option.

<u>**Note:**</u> Any tradable derivative satisfies the equation and there is no dependence on the drift term μ.

Interestingly enough, we can compute a change of variables to transform the Black Scholes equation into the heat equation [**?**]:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

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Recall that for a European call we have the following boundary conditions:

$$\begin{aligned} C(S_T, T) &= (S_T - K)^+ \\ C(0, t) &= 0 \\ C(\infty, t) &\cong S_t. \end{aligned}$$

For European put options, the boundary conditions are as followed:

$$\begin{aligned} C(S_T, T) &= (K - S_T)^+ \\ C(0, t) &= Ke^{-r(T-t)} \\ C(\infty, t) &= 0. \end{aligned}$$

[6] The working out for this transformation is in the appendix

We can analytically solve the *Black Scholes* equation for European call/puts:

$$C_t = e^{-r(T-t)} \left(e^{r(T-t)} S_t \mathcal{N}(d_1) - K\mathcal{N}(d_2)\right)$$
$$P_t = e^{-r(T-t)} \left(K\mathcal{N}(-d_2) - e^{-r(T-t)} S_t \mathcal{N}(-d_1)\right)$$

where,

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$\mathcal{N} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} e^{\frac{-u^2}{2}} du.$$

[7] The standard normal distribution above represent the probabilities in a risk-neutral world.

4.3 Risk Neutral Pricing

Say you hope to find a ration price of a financial derivative, with a payoff f. We would want to consider do this by considering the value of the portfolios under a probability measure $\mathbb{Q}$, which is equivalent to $\mathbb{P}$, such that the drift term of the stock price is equal to the risk-free rate r. This is the idea of *Risk-Neutral Pricing*, where we consider the *risk-neutral measure* as $\mathbb{Q}$. To do this we first assume that under the real-world measure $\mathbb{P}$, the stock price S_t, follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Additionally, we let r be the risk-free rate, and define $\tilde{\mathbb{P}}$ as the risk-neutral measure. Let the Radon-Nikodym derivative Z_t be defined as

$$Z_t = \exp\left(-\frac{\mu - r}{\sigma} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right)$$

for $t \in [0, T]$. By applying Girsanov's Theorem, we express the Brownian motion B_t under $\mathbb{P}$ as the new Brownian motion $\tilde{B}_t$ under $\tilde{\mathbb{P}}$ given by:

$$\tilde{B}_t = B_t - \frac{(\mu - r)t}{\sigma}$$

This leads us to

$$dS_t = \mu S_t dt + \sigma S_t \left(d\tilde{B}_t - \frac{\mu - r}{\sigma} dt\right)$$
$$dS_t = rS_t dt + \sigma d\tilde{B}_t.$$

Here the mean rate of return on the stock S_t is now equivalent to the risk-free asset. In fact we can now note that the discounted stock price $e^{-rt}S_t$ is also martingale under the risk-neutral measure $\tilde{\mathbb{P}}$. Similarly, if we consider (a, b) to be a portfolio, and let V be the value process associated with this portfolio, then the discounted value process $e^{-rt}V_t$ is

[7] The proof of this is in the appendix

also a martingale under $\tilde{\mathbb{P}}$, assuming that the portfolio is self-financing.
We mentioned earlier that the replicating portfolio value is always equal to the rational price of the derivative, under principle of no arbitrage. We can obtain this value using the following theorem:

Theorem: Let f be the payoff of a financial derivative claim, and suppose that (a, b) is a replicating portfolio for f, with value process V. Then

$$V_t = e^{-r(T-t)}\, \mathbb{E}_{\tilde{\mathbb{P}}}[f(t, S_t)|\mathcal{F}_t], \quad \forall t \in [0, T]$$

For example, consider a European call options, with the strike price K. The payoff of this option is

$$f = (S_T - K)^+$$

We can recall that the dynamics of S_t under the probability measure $\tilde{\mathbb{P}}$ are

$$dS_t = rS_t dt + \sigma d\tilde{B}_t$$

Additionally, recall that the solution to this stochastic differential equation, under the probability measure $\mathbb{P}$, was given in (2)

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma B_t\right)$$

We can change the probability measure of this solution to be under $\tilde{\mathbb{P}}$:

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma \tilde{B}_t\right).$$

The price of the option, under the principle of no arbitrage, is given by

$$V_0 = e^{-rT}\, \mathbb{E}_{\tilde{\mathbb{P}}}[(S_T - K)^+|\mathcal{F}_0]$$
$$V_0 = e^{-rT}\, \mathbb{E}_{\tilde{\mathbb{P}}}\left[\left(S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma \tilde{B}_T\right) - K\right)^+\right]$$

It is important to note that it is typical for an investor to consider *historical volatility*, or *implied volatility* to determine the prices of the derivative. Historical volatility is the estimation of the past volatility over a recent period of time, whereas implied volatility is the volatility

5 Heston Model

As mentioned in the introduction, the asset stock prices have behaviours that are not captured by the assumptions of the Black-Scholes. The most notable limitation of the Black-Scholes is the assumption of constant volatility, which is unrealistic. In this section we will look the Heston model which allows for variation in both the price and volatility of a asset.

In fact in financial markets, one can observe *volatility clusters*, meaning that the tendency for periods of high volatility are followed by high volatility, and vice versa. Additionally, one can observe a negative correlation between the returns and volatility changes, which is known as *leverage effect*. These features are not captured with Black-Scholes and require models consisting of stochastic volatility.

The Heston model begins with two Stochastic Differential Equations,

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\nu_t} S_t dB_t^{(1)} \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dB_t^{(2)} \end{aligned} \tag{5}$$

The two Brownian motions $B_t^{(1)}, B_t^{(2)}$ are correlated, with

$$d\langle B_t^{(1)}, B_t^{(2)} \rangle = \rho dt,$$

where $\rho \in [-1, 1]$ is the correlation coefficient representing the correlation between the asset price and variance. The other parameters of the model are as follows:

- $S_t > 0$ - the asset price at time t
- $r > 0$- risk free rate. This is the theoretical rate that carries no risk, such as a bond or a bank account appreciating over time.
- $\sqrt{\nu_t} > 0$ - volatility (standard deviation) of the asset price
- $\sigma > 0$ - the volatility of the variance process $\sqrt{\nu_t}$
- $\kappa > 0$ - the mean-reversion rate. This captures the speed at which the volatility process will move back towards its long-term mean after deviating away from this mean.
- $\theta > 0$ - long-term mean level of volatility

Heston's model utilises the principle of no-arbitrage to model the motion of the price and volatility of the asset. Heston also assumes that the asset variance ν_t follows a mean reverting *Cox-Ingersoll-Ross (CIR) process*, with the drift term $\kappa(\theta - \nu_t)$. One problem which arises is that CIR maintains ν_t to be positive, we require volatility to be strictly positive to prevent issues such as undefined volatility, $\sqrt{0} = 0$. Fortunately, *The Feller Condition* provides the criterion under which the CIR process is strictly positive:

$$2\kappa\theta \geq \sigma^2,$$

stating that the square of the magnitude of the volatility of volatility must be less than or equal to the strength of the mean reversion pulling ν_t back to θ. Otherwise, if $2\kappa\theta < \sigma^2$ then there is a positive probability that ν_t can reach 0.

Before moving on, we will model the price of the risk-less asset as before:

$$d\beta_t = r\beta_t dt$$

5.1 Risk-Neutral Pricing

Like Black-Scholes, to price the derivative, we need to consider the dynamics under a risk-neutral measure $\tilde{\mathbb{P}}$, where all assets are expected to grow at the risk free rate r. Under $\tilde{\mathbb{P}}$ the discounted asset prices are martingales. We first want to assume under the real-world measure $\mathbb{P}$, the Heston model is characterised by the stochastic differential equations in (5):

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\nu_t} S_t dB_t^{(1)} \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dB_t^{(2)} \\ d\langle B_t^{(1)}, B_t^{(2)}\rangle &= \rho dt \end{aligned}$$

By using Girsanov's theorem, we get an adjusted price process of

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t d\tilde{B}_t^{(1)},$$

where $\tilde{B}_t$ is the Brownian motion under $\tilde{\mathbb{P}}$. Comparing it to $\mathbb{P}$, we get:

$$dS_t = \left(\mu - \frac{\mu - r}{\sqrt{\nu}}\sqrt{\nu}\right) dt + \sqrt{\nu} d\tilde{B}_t^{(1)}, \quad S_0 > 0,$$

where $\frac{\mu-r}{\sqrt{\nu}}$ is the market price of risk associated with the asset. Unlike the price process, the volatility process is not a traded asset, so the adjustment of the variance process under $\tilde{\mathbb{P}}$ involves introducing a market price of constant volatility risk λ. Under $\tilde{\mathbb{P}}$, the variance process is:

$$d\nu_t = \left(\kappa(\theta - r) - \lambda\sigma\sqrt{\nu_t}\right) dt + \sigma\sqrt{\nu_t} d\tilde{B}_t, \quad \nu_0 > 0,$$

The correlation coefficient ρ is preserved unchanged under the risk-neutral measure $\tilde{\mathbb{P}}$.

If we let $f(T, S_T, \nu_T)$ be the payoff of a financial derivative claim at maturity $T - t$. Then from the risk-neutral theorem, under the risk neutral measure $\tilde{\mathbb{P}}$, the price at time t is:

$$f(t, S_t, \nu_t) = e^{-r(T-t)}\, \mathbb{E}_{\tilde{\mathbb{P}}}[f(T, S_T, \nu_T)|S_0, \nu_0],$$

under the principle of no arbitrage.

5.2 The Heston PDE

We want to be able to derive the partial differential equation that explains the price of a derivative under the Heston model. This derivation is a special case of a PDE for general stochastic models [1], and extends the approach used in the Black-Scholes model. In the Black-Scholes model, a portfolio is formed with the underlying stock, plus a single derivative which is used to hedge the stock and render the portfolio riskless. However, in the Heston model we have to also hedge the volatility, which is an additional derivative in the replicating portfolio.

We want to derive the PDE for the derivative price $V(t, S_t, \nu_t)$, with a maturity T_1, by constructing a self-financing portfolio and eliminating the randomness through appropriate hedging strategies.

Consider a portfolio with the following:

- β_t is our risk-less asset such as a bond or a bank savings account,
- Δ represents the units of the stock S_t,
- Σ represents the units of another option $U(t, S_t, \nu_t)$, with maturity $T_2 > T_1$
- α the units of the risk-less asset β_t.

The value process of the portfolio at time t is:

$$V_t = \Delta S_t + \Sigma U_t + \alpha \beta_t,$$

Including the derivative $U(t, S_t, \nu_t)$ allows us to hedge against the volatility risk. We are trying to hedge the risk of the option of maturity T_1 using the stock price and a longer maturity option, and by choosing $T_2 > T_1$, we ensure that the hedging instrument U is available throughout the life of the primary derivative maturity at T_1. The change in the value of the portfolio is as followed:

$$dV_t = \Delta dS_t + \Sigma dU_t + \alpha d\beta_t.$$

We now need to apply Itô lemma in two dimensions to determine the dynamics of both $V(t, S_t, \nu_t)$ and $U(t, S_t, \nu_t)$. Recall from section 2.4, we specified that a d-dimensional Brownian motion can be described as $\boldsymbol{B} = (B^{(1)}, B^{(2)}, ..., B^{(d)})^T$, such that $B^{(i)}$ is a one dimensional Brownian motion, $i = 1, ..., d$, and each i is independent to each other. Hence

$$dV = \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}dS_t^2 + \frac{\partial V_t}{\partial \nu_t}d\nu_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial \nu_t^2}d\nu_t^2 + \frac{\partial^2 V_t}{\partial \nu_t \partial S_t}d\nu_t dS_t$$

$$dV = \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\nu_t S_t^2 dt + \frac{\partial V_t}{\partial \nu_t}d\nu_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial \nu_t^2}\sigma^2 \nu dt + \frac{\partial^2 V_t}{\partial \nu_t \partial S_t}\rho\sigma\nu_t S_t dt$$

$$dV = \left(\frac{\partial V_t}{\partial t} + \frac{1}{2}\nu S_t^2\frac{\partial^2 V_t}{\partial S_t^2} + \frac{1}{2}\sigma^2\nu_t\frac{\partial^2 V_t}{\partial \nu_t^2} + \rho\sigma\nu_t S_t\frac{\partial^2 V_t}{\partial \nu_t \partial S_t}\right)dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{\partial V_t}{\partial \nu_t}d\nu_t$$

Let the differential operation $\mathcal{L}$ be defined as

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\nu_t S_t^2\frac{\partial^2}{\partial S_t^2} + \frac{1}{2}\sigma^2\nu_t\frac{\partial^2}{\partial \nu_t^2} + \rho\sigma\nu_t S_t\frac{\partial^2}{\partial \nu_t \partial S_t},$$

then we can now represent both V and U drift terms as

$$(\mathcal{L}V)(t, S_t, \nu_t)dt \qquad (\mathcal{L}U)(t, S_t, \nu_t)dt.$$

If we substitute the Itô lemma for both options, and the differential equation of the price of the risk-less asset $d\beta_t = r\beta_t dt$ into the replicating portfolio, we get:

$$(\mathcal{L}V)(t, S_t, \nu_t)dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{\partial V_t}{\partial \nu_t}d\nu_t = \Delta dS_t + \Sigma\left((\mathcal{L}U)(t, S_t, \nu_t)dt + \frac{\partial U_t}{\partial S_t}dS_t + \frac{\partial U_t}{\partial \nu_t}d\nu_t\right) + \alpha r\beta_t dt$$

By combining the S_t terms:

$$(\mathcal{L}V)(t, S_t, \nu_t)dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{\partial V_t}{\partial \nu_t}d\nu_t = \Sigma(\mathcal{L}U)(t, S_t, \nu_t)dt + \left(\Delta + \Sigma\frac{\partial U}{\partial S_t}\right)dS_t + \Sigma\frac{\partial U}{\partial \nu_t}d\nu_t + \alpha r\beta_t dt \tag{6}$$

We want to remove the $dS_t d\nu_t$ terms. To do this we need to equate the coefficients of $\frac{\partial V_t}{\partial S_t}, \frac{\partial V_t}{\partial \nu_t}$ respectively and solve for Σ, Δ:

$$\frac{\partial V_t}{\partial S_t} = \Delta + \Sigma \frac{\partial U_t}{\partial S_t}, \quad \frac{\partial V_t}{\partial \nu_t} = \Sigma \frac{\partial U_t}{\partial \nu_t}$$

$$\Rightarrow \Sigma = \left(\frac{\frac{\partial V_t}{\partial \nu_t}}{\frac{\partial U_t}{\partial \nu_t}} \right), \quad \Delta = \frac{\partial V_t}{\partial S_t} - \Sigma \frac{\partial U_t}{\partial S_t}$$

If we substitute Σ, Δ into the right hand side of (6), and canceling terms we are left with:

$$(\mathcal{L}V)(t, S_t, \nu_t)dt = \Sigma(\mathcal{L}U)(t, S_t, \nu_t)dt + \alpha r \beta_t dt$$

By using the fact that $\alpha\beta_t = V_t - \Delta S_t - \Sigma U_t = \alpha\beta_t = V_t - \left(\frac{\partial V_t}{\partial S_t} - \Sigma \frac{\partial U_t}{\partial S_t} \right) S_t - \Sigma U_t$, we can remove the $\alpha\beta_t$ term and the dt term:

$$(\mathcal{L}V)(t, S_t, \nu_t)dt = \Sigma(\mathcal{L}U)(t, S_t, \nu_t)dt + r \left(V_t - \left(\frac{\partial V_t}{\partial S_t} - \Sigma \frac{\partial U_t}{\partial S_t} \right) S_t - \Sigma U_t \right) dt$$

$$(\mathcal{L}V)(t, S_t, \nu_t) - rV + r\frac{\partial V_t}{\partial S_t} dS_t = \left(\frac{\frac{\partial V_t}{\partial \nu_t}}{\frac{\partial U_t}{\partial \nu_t}} \right) \left(\mathcal{L}U)(t, S_t, \nu_t) - rU_t + r\frac{\partial U_t}{\partial S_t} S_t \right)$$

$$\frac{(\mathcal{L}V)(t, S_t, \nu_t) - rV + r\frac{\partial V_t}{\partial S_t} S_t}{\frac{\partial V_t}{\partial \nu_t}} = \frac{\mathcal{L}U)(t, S_t, \nu_t) - rU_t + r\frac{\partial U_t}{\partial S_t} S_t}{\frac{\partial U_t}{\partial \nu_t}}$$

One can see that the expression on both sides are the same, left side contains V_t, right side contains U_t. The fraction must not dependent on V_t and U_t but depends on the parameters. Hence we can say

$$\frac{(\mathcal{L}V)(t, S_t, \nu_t) - rV + r\frac{\partial V_t}{\partial S_t} S_t}{\frac{\partial V_t}{\partial \nu_t}} = -h(t, S_t, \nu_t)$$

$$\Rightarrow (\mathcal{L}V)(t, S_t, \nu_t) - rV = -rS_t \frac{\partial V}{\partial S_t} - h(t, S_t, \nu_t) \frac{\partial V}{\partial \nu_t}$$

Recall rS_t is the drift of the stock price stochastic differential equation under the risk-neutral measure. As a result, $h(t, S_t, \nu_t)$ corresponds to the adjusted drift term of the variance process under the risk-neutral measure:

$$h(t, S_t, \nu_t) = \kappa(\theta - \nu_t) - \lambda\sigma\sqrt{\nu_t}$$

By expanding out the linear operator, we get

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\nu_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{1}{2}\sigma^2 \nu_t \frac{\partial^2 V_t}{\partial \nu_t^2} + \rho\sigma\nu_t S_t \frac{\partial^2 V_t}{\partial \nu_t \partial S_t} - rV_t = -rS_t \frac{\partial V_t}{\partial S_t} - (\kappa(\theta - \nu_t) - \lambda\sigma\sqrt{\nu_t}) \frac{\partial V_t}{\partial \nu_t}$$

Finally we get the pricing PDE for the price, $V(t, S_t, \nu_t)$, of any derivative claim in the Heston model:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2} \left(\nu_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma^2 \nu_t \frac{\partial^2 V_t}{\partial \nu_t^2} \right) + \rho\sigma\nu_t S_t \frac{\partial^2 V_t}{\partial \nu_t \partial S_t} + rS_t \frac{\partial V_t}{\partial S_t} + (\kappa(\theta - \nu_t) - \lambda\sigma\sqrt{\nu_t}) \frac{\partial V_t}{\partial \nu_t} - rV_t = 0, \tag{7}$$

where

- $\nu_t S_t^2$ is the variance of the strock price SDE,
- $\sigma^2 \nu_t$ is the variance of the volatility SDE,
- $\rho\sigma\nu_t S_t$ is the covariance of the two SDEs,
- rS_t is the drift of the stock price SDE,
- $(\kappa(\theta - \nu_t) - \lambda\sqrt{\nu_t})$ is the variance of the volatility SDE.

This equation follows the fact that $V(t, S_t, \nu_t) = e^{r(T-t)}\,\mathbb{E}_{\tilde{\mathbb{P}}}[f(T, S_T, \nu_T)|S_t, \nu_t]$, where f is the derivative payoff. For a European call we can extend the boundary conditions as the following

$$\begin{aligned}
&f(t, S_t = 0, \nu_t) = 0, \\
&\lim_{S_t \to \infty} f(t, S_t, \nu_t) = S_t - Ke^{-r(T-t)}, \\
&f(t, S_t, \nu_t = 0) = (S_t e^{r(T-t)} - Ke^{-r(T-t)}, 0)^+, \\
&\lim_{\nu_t \to \infty} f(t, S_t, \nu_t) = S_t.
\end{aligned}$$

5.3 The Heston Characteristic Function

We can express the European call price in the Heston model in a similar manner to the Black-Scholes model. If we take the time t of a European call of a non-dividend paying asset with spot price S_t, strike price K, and the time to maturity $T - t$, then the call is equal to the discounted expected payoff under the risk-neutral measure $\tilde{\mathbb{P}}$:

$$\begin{aligned}
C(K) &= e^{-r(T-t)}\,\mathbb{E}_{\tilde{\mathbb{P}}}[(S_T - K)^+|S_t, \nu_t] \\
&= e^{-r(T-t)}\,\mathbb{E}_{\tilde{\mathbb{P}}}[(S_T - K)\mathbf{1}_{\{S_T > K\}}] \\
&= e^{-r(T-t)T}\,\mathbb{E}_{\tilde{\mathbb{P}}}[S_T \mathbf{1}_{\{S_T > K\}}] - Ke^{-r(T-t)}\,\mathbb{E}_{\tilde{\mathbb{P}}}[\mathbf{1}_{\{S_T > K\}}] \\
&= S_t P_1 - Ke^{-(T-t)} P_2,
\end{aligned}$$

where $\mathbf{1}$ is the indicator function and P_1 and P_2 represent the probability of the call expiring in-the-money, conditioned on the value S_t of the stock and on the value ν_t of the volatility at time t:

$$\begin{aligned}
P_1 &= \mathbb{E}_{\tilde{\mathbb{P}}}[\frac{S_T}{S_t}\mathbf{1}_{\{S_T > K\}}|S_t, \nu_t] \\
P_2 &= \mathbb{E}_{\tilde{\mathbb{P}}}[\mathbf{1}_{\{S_T > K\}}|S_t, \nu_t]
\end{aligned}$$

We can see that this is similar to the Black Scholes call price formula, difference being, the $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$ have been replaced with P_1 and P_2 respectively. Let $S_t = e^{x_t}$, where x_t is the logarithmic price of the stock at time t, we can say P_j is the probability $x_t > \ln K$ for $j = 1, 2$.

Heston showed that the Fourier transform of the log-stock price is a very useful for pricing options under stochastic volatility. When the *characteristic functions* $f(\phi, x, \nu, T)$ are known, each in-the-money probability P_j can be expressed in terms of Fourier integrals, allowing for a straightforward computation of the option price. This approach is effective because it converts the complex integration involved in expected payoff calculations into Fourier terms.

The characteristic functions for the log-stock price in the Heston model, $f(\phi, x, \nu, T)$, allows us to determine the values of P_j:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-i\phi\ln(K)}f(\phi; x, \nu, T)}{i\phi}\right) d\phi. \tag{8}$$

At maturity, the probabilities are subject to the terminal condition

$$P_j(x, \nu, T, \ln(K))$$

The characteristic function $f(\phi, x, \nu, T)$ describes the distribution of the log stock price $x = \ln S_t$ under the risk-neutral measure $\tilde{\mathbb{P}}$. For a given $x = x_t, \nu = \nu_t$, the characteristic function is defined as

$$f(\phi; x, \nu, T) = \mathbb{E}_{\tilde{\mathbb{P}}}[e^{i\phi x_T}|x = x_t, \nu = \nu_t].$$

Heston stated that the characteristic functions for the logarithm of the terminal stock price is

$$f(\phi; x, \nu, t) = \exp(C_j(T-t, \phi) + D_j(T-t, \phi)\nu + i\phi x), \tag{9}$$

where i is the imaginary unit $\sqrt{-1}$, and $\phi \in \mathbb{R}$. This formulation gives us a closed-form representation of the characteristic function, with C_j and D_j are complex-valued functions, which are obtained by solving a system of *Riccati equations*, given by:

$$\begin{aligned}\frac{dD(\tau, \phi)}{d\tau} &= \frac{-\sigma^2}{2}D(\tau, \phi)^2 - (\kappa - \rho\sigma i\phi)D(\tau, \phi) + \frac{i\phi}{2}, \\ \frac{dC(\tau, \phi)}{d\tau} &= ri\phi + \kappa\theta D(\tau, \phi),\end{aligned} \tag{10}$$

where $\tau = T - t$. The solution to the set of equations for a European option is given by:

$$\begin{aligned}D(\tau, \phi) &= \frac{b - \rho\sigma i\phi + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right) \\ C(\tau, \phi) &= ri\phi\tau + \frac{a}{\sigma^2}\left((b - \rho\sigma i\phi + d)\tau - 2\ln\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right),\end{aligned} \tag{11}$$

where

$$\begin{aligned}a &= \kappa\theta, \\ b &= \kappa + \lambda, \\ d &= \sqrt{(\rho\sigma i\phi - b)^2 - \sigma^2(2u_j i\phi + \phi^2)}, \quad u_1 = 0.5, \quad u_2 = -0.5, \\ g &= \frac{b - \rho\sigma i\phi + d}{b - \rho\sigma i\phi - d}.\end{aligned}$$

If we want to get the price of a European put, $P(K)$, we would use the put-call parity

$$P(K) = C(K) + Ke^{-r\tau} - S_t.$$

[8] Additionally, if we want to include dividend payments as a continuous yield, q, into the model, we can replace r by $r - q$ in

$$dS_t = rS_t dt + \sqrt{\nu_t}S_t dB_t^{(1)},$$

which is the risk-neutral process for the stock price.

[8]Note: $C(K)$ is the call price not the complex function of the Riccati equations $c(\tau, \phi)$

6 Modeling Pricing Models

6.1 Black-Scholes Model

To simulate the Black-Scholes model, we want to model the Brownian motion as a random standard normal distribution $\mathcal{N}(0,1)$ of $N \times M$ pairs, where N is the number of time steps, and M is the number of simulations. These time increments will be modeled as $\Delta t = \frac{T}{N}$. We will use, for both Black-Scholes and for the Heston model later, the Euler discretisation method for the numerical simulation. This iterative formula for S_{t+1} can be written as:

$$S_{t+\Delta t} = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}B_t\right).$$

We will define the parameters under a risk-neutral measure, $S_0 = 100.0, r = 0.03, \sigma = 0.5$, where r is the risk-free rate and σ is the volatility of the price process.

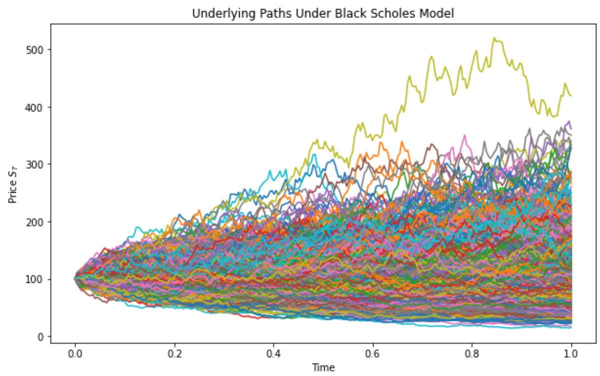


Figure 2: Monte Carlo simulations of the Underlying path

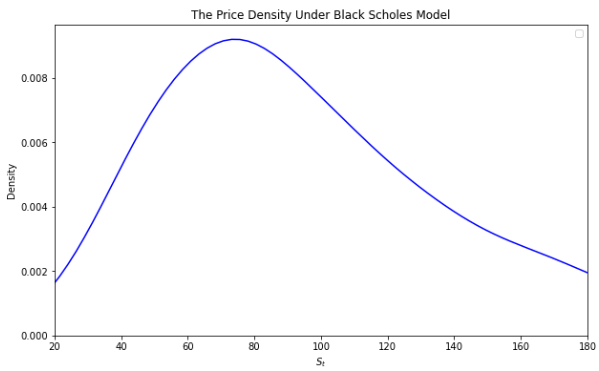


Figure 3: Density distributions of the Black Scholes

In figure 3, we can see that the Black-Scholes model follows a lognormal distribution, which is true due to the fact that we have modeled the asset price as a Brownian motion with drift:

$$S_t \sim \text{Lognormal}\left(\ln(S_0) + \left(r - \frac{\sigma}{2}\right)T, \sigma^2 T\right)$$

6.2 Heston Model

We first we will be simulating the price of stock using the Heston Model under risk-neutral dynamics. We will run the Milstein Scheme [5][4], which assumes an Euler and Milstein discretisation. In other words we assume that we can get negative variances. Here we will run Monte Carlo simulation using these processes on stock price and volatility:

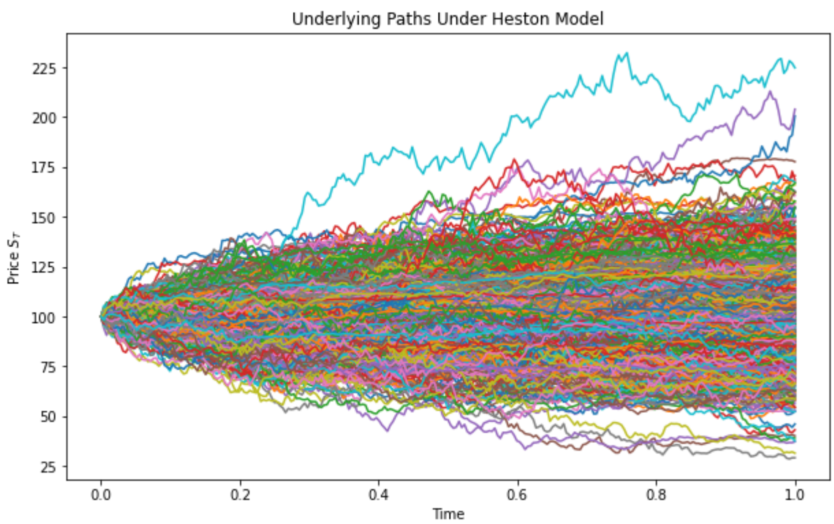
$$dS_{t+\Delta t} = S_t \exp\left((r - \frac{\nu_t}{2})\Delta t + \sqrt{\nu_t}\Delta t W^S_{t+\Delta t}\right)$$
$$\nu_{t+\Delta t} = \nu_t + \kappa(\theta - \nu_t)\Delta t + \sigma\sqrt{\nu_t}\Delta t W^{\nu}_{t+\Delta t},$$

where $\Delta t = \frac{T}{N}$ is a time steps. Like the Black-Scholes model, T will be take as the time until expiry of the contract, and N as the number of time steps we simulate over, which in this case will be 252 trading days.

We then generate $N \times M$ pairs of correlated Brownian motion increments from the bivariate normal distribution, where M is the number of Monte Carlo simulations. This will give us an $(N, M, 2)$ array which we can split up to give us two Brownian motion paths for the underlying process and volatility process, correlated by ρ.

We will define the parameters of the model under risk-neutral dynamics, being $S_0 = 100.0, r = 0.02, \kappa = 2, \theta = 0.04, v_0 = 0.0625, \sigma = 0.5, \rho = -0.4$, using similar parameters to the previous code for the Black-Scholes model.

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$



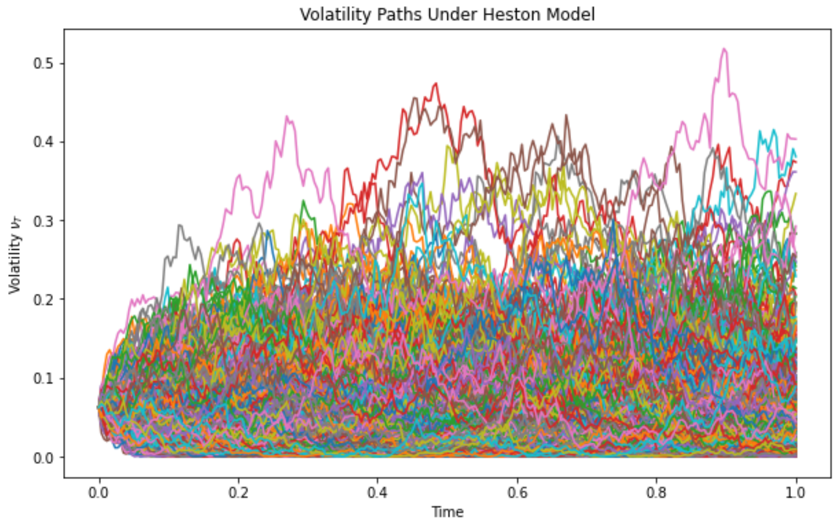


Figure 4: Monte Carlo simulations of the Underlying and volatility paths.

We can additionally compare the distribution of the underlying price at maturity under the Heston model of two different correlations. Here we will use $\rho_1 = 0.95$ and $\rho_2 = -0.95$.

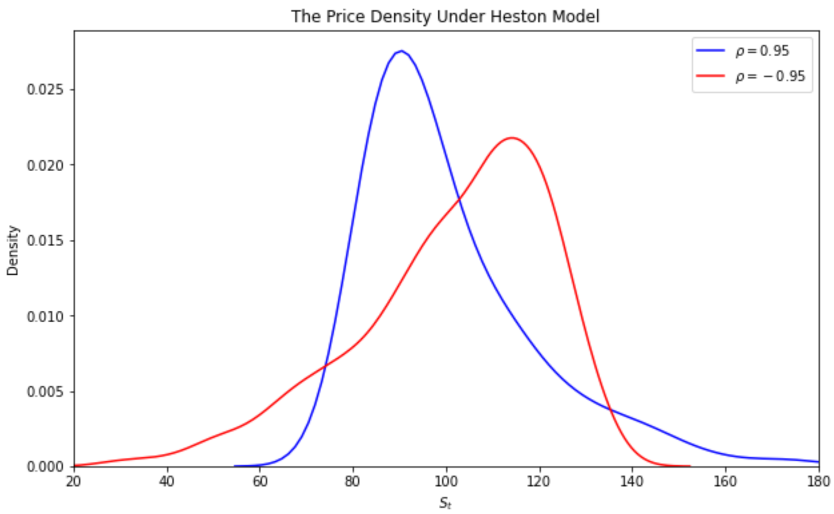


Figure 5: Density distributions of the positively correlated underlying asset (blue) to its volatility and negatively correlated underlying asset (red) to its volatility.

In figure 5, we can notice that the positive and negative correlation both have long tails in the positive and negative direction respectively. This skew distribution is due to the leverage effect, which is an opposite effect, that is where large high volatility leads to negative asset price returns. [9]

We can capture the volatility smile of the . We will use the *py_vollib_vectorized* package on Python to create the implied volatility for the call and puts.

[9] All other predefined parameters are the same in this figure, i.e., we only changed the correlation parameter.

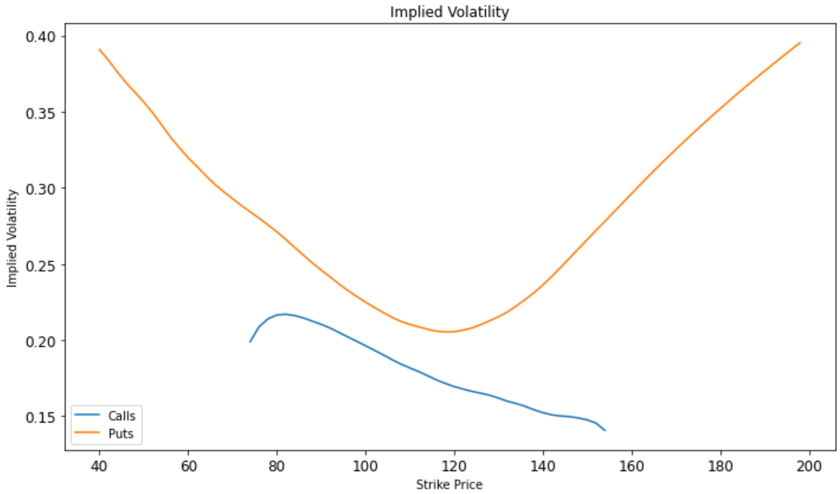


Figure 6: Implied volatility for call and put options as a function of strike prices under the Heston Model with $\rho = -0.65$. The time to maturity of these options is 1 year.

Above we used some predefined parameters. We can calibrate the Heston Model to get better parameters that are indicative of current market prices.

Recall that a non-dividend paying European call option at time t, with strike price K, spot price S_t, and the time to maturity $T - t = \tau$, is equal to the discounted expected payoff under the risk-neutral measure. Also recall that we said

$$C(K) = S_t P_1 - Ke^{-\tau} P_2$$

where P_1 and P_2 are the probabilities expiring in-the-money. If we subsitute in (8), we get the following:

$$C(K) = C(K, S_0, \nu_0, \tau) = \frac{1}{2}(S_0 - Ke^{-r\tau}) + \frac{1}{\pi}\int_0^\infty \Re\left(e^{r\tau}\frac{f(\phi - i; x, \nu, T)}{i\phi K^{i\phi}} - K\frac{f(\phi; x, \nu, T)}{i\phi K^{i\phi}}\right) d\phi$$

and by substituting (10) into (9), we get

$$f(\phi; x, \nu, \tau) = e^{r\phi i\tau} S^{i\phi}\left(\frac{1 - ge^{d\tau}}{1 - g}\right)^{\frac{-2a}{\sigma^2}} \exp\left(\frac{a\tau}{\sigma^2}(b_2 - \rho\sigma i\phi + d) + \frac{\nu_0}{\sigma^2}\frac{(b_2 - \rho\sigma i\rho + d)(1 - e^{d\tau})}{1 - ge^{d\tau}}\right),$$

where a, g, b are defined in (10). We rearrange d term to remove b_j and u_j:

$$\rho\sigma(\phi - i)i - b = \rho\sigma(i\phi + 1) - (\kappa + \lambda) = \rho\sigma i\phi - (\kappa + \lambda - \rho\sigma) = \rho\sigma i\phi - b_1$$

$$-\sigma^2(2 * -0.5 * i\phi - \phi^2) = \sigma(i\phi + \phi^2)$$

$$\Rightarrow d = \sqrt{(\rho\sigma i\phi - b)^2 + \sigma^2(i\phi + \phi^2)}.$$

which is also true for b_2, u_2 calculations respectively.
We use the *scipy.integrate* package to numerically integrate the characteristic function.

To calibrate the Heston model parameters, we minimize an objective (or loss) function, $L(\Theta)$, that quantifies the discrepancy between market-observed option prices and the prices generated by the Heston model under a given parameter set Θ. The objective function is defined as follows:

$$L(\Theta(\kappa, \theta, \sigma, \rho, \nu_0)) = \sum_{i=1}^{N}\sum_{j=1}^{M} w_{i,j}(C_{MP}(K_i, \tau_j) - C_{SV}(S_\tau, K_j, \tau_j, r_j, \Theta))^2 + Penalty(\Theta, \Theta_0),$$

where N is the number of strike prices, M the number of maturities, C_{MP} is the market price of an option with strike K_i and maturity τ_i. C_{SV} is the theoretical price computed using the Heston model, and $w_{i,j}$ is the weight assigned to each data point to reflect its importance. The penalty function will be the euclidean distance to the initial parameter $Penalty(\Theta, \Theta_0) = ||\Theta - \Theta_0||^2$.[6] We want to minimise this function between the Heston estimated price with a certain parameter set to the market prices to get the value $\hat{\Theta}$:

$$\hat{\Theta} = \arg\min_{\Theta \in U_\Theta} L(\Theta)$$

where the set of possible combinations of parameters $U_\Theta \subset \mathbb{R}^5$ is compact, so that the optimisation problem is well-posed and has a solution. We will use *scipy.optimize* package to minimise this objective function before we plot a 3 dimensional graph of the maturities, strike prices, and option prices to the calibrated Heston prices.

For our model, we will use the *Daily Treasury Par Yeild Curve Rates*[10] as our risk-free rates for each maturity. Additionally, we will use *NVIDIA* as our risky asset. The results are shown in figures 7 & 8:

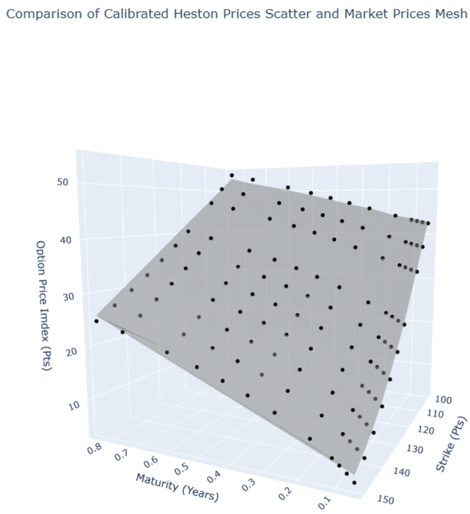


Figure 7: Heston Model calibrated estimates (markers) of the European call prices to the market prices (mesh).

Figure 7

[10]https://home.treasury.gov/resource-center/data-chart-center/interest-rates/TextView?type=daily_treasury_yield_curve&field_tdr_date_value_month=202411

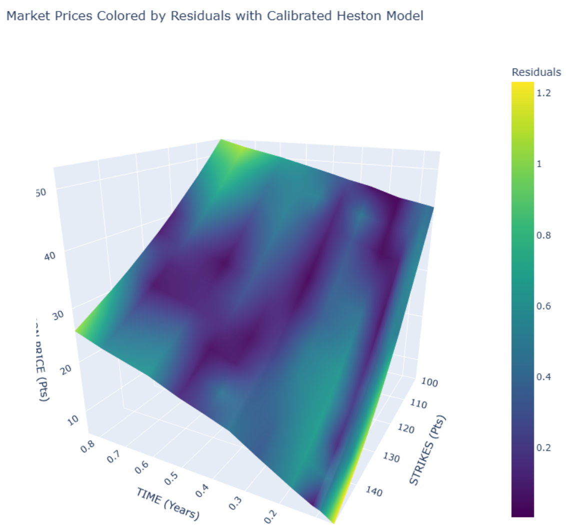


Figure 8: The residuals of the Heston Model estimates to the market prices (European call prices). The z-axis is the option price index (pts).

After running the codes for both models above, we can conclude a few key differences, giving their strenghts and weaknesses. The Black-Scholes model produces a return distribution that follows a lognormal path due to the constant volatility. This model fails to capture the empirical characteristics of the markets like the high peaks and fat tails that we observe in real return distributions. We see this limitation in figure 3, which shows a smoother distribution and lacks the extreme kurtosis present in markets.

In comparison the Heston model varies volatility over time and across different strikes, which produces a return distribution that better match the market data that show skewness and kurtosis. As a result, the Heston model can capture volatility smiles from the stochastic volatility assumption. Additionally, the volatility of the volatility, σ, allows us to control the kurtosis of the return distribution.

We can also fit the data better using calibrated parameters of the Heston model. For example, a negative correlation allows the Heston model to replicate the empirically observed increase in market volatility during downturns, something that the Black-Scholes model lacks. This is a valuable feature in the model for pricing derivatives during turbulent market conditions.

One difference with the Heston model and the Black-Scholes is the hedging strategies. In the Black-Scholes model, delta hedging allows us to reduce the impact of randomness that arises from the underlying asset's diffusion. In contrast, the Heston model has two sources of randomness; One from of uncertainty is in the asset price and the other is from the stochastic volatility process. Hence, we use the *delta-vega* hedging (also known as *dela-sigma hedging*. This hedging strategy hedges the delta risk by trading the underlying asset and simultaneously hedges the volatility risk by using another financial instrument, such as a *variance swap*, that is sensitive to changes in volatility. This naturally implies that the market is incomplete since the volatility is not a traded asset (although the abscene of arbitrage still holds).

7 Other Pricing Models

In financial mathematics, the holy grail is to have a model that produces an analytical formula for the prices of options and also captures what is observed in the markets. Whilst Heston managed to achieve this for his model, by incorporating stochastic volatility, the framework of his model doesn't take into account every possible pattern in behaviour. For example, models based on continuous diffusions, like the Heston, will struggle to explain the relatively steep scheme in the volatility observed in shorter maturity options. Also, the model has a very limited number of parameters, therefore won't be able to fit the entire surface.

In order to model the steep skew at short maturities and model the local volatility to fit the initial prices, we would need to include a jump process. The type of models that integrate both a local volatility mechanism and stochastic componenets with jumps are known as *local stochastic models.*

7.1 Merton Jump Diffusion Model

Robert C Merton extended the Black-Scholes model by removing the assumption of liquid markets by adding volatility jumps into the equation. The jumps mathematically allow for discontinuities per time step. For each infinitesimal time interval, we add a stochastic jump component consistent of a Poisson process N_t and a jump size ξ_j, which is normal distributed. Here, the Poisson process will model whether we get a volatility spike and the jump size models the size of the spike. Mathematically speaking, we define the Merton model, on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with asset price S_t by the stochastic differential equation:

$$\begin{gathered}\frac{dS_t}{dS_t} = \mu dt + \sigma dB_t + d\left\{\sum_{j=1}^{N_t}(\xi_j - 1)\right\} \\ N_t \backsim Poisson(\lambda_J), \quad \ln \xi_j \backsim \mathcal{N}(\mu_J, \sigma_J^2)\end{gathered} \tag{12}$$

As before, μ and σ, denote the drift and volatility of the model respectively. Additionally, μ_J and σ_J, denote the drift and volatility of the diffusion component respectively. We can also define the expected jump size as:

$$\kappa = \mathbb{E}[\xi_j - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$$

To transform the Merton model into the risk neutral space $(\Omega, \mathcal{F}, \mathbb{Q})$, we first derive the compensated jump process $d\tilde{J}$ by differencing the actual jump increment dJ_t by its compensator $\lambda\kappa dt$:

$$d\tilde{J}_t = dJ_t - \lambda\kappa dt,$$

where dJ_t is the original jump process:

$$dj = (\xi - 1)dN_t.$$

This ensures $d\tilde{J}_t$ is a martingale. The disounted asset price is given by

$$\tilde{S}_t = e^{-rt}S_t,$$

where r is the risk-free rate. Applying Itô's formula we get

$$d\tilde{S}_t = e^{-rt} dS_t - re^{rt} S_t dt. \tag{13}$$

By subsituting in (12) into (13), we have

$$d\tilde{S}_t = e^{-rt} S_{t^-} \left\{ [\mu + \lambda\kappa - r - \sigma\theta] \, dt + \sigma d\tilde{B}_t + d\tilde{J}_t \right\},$$

where $\tilde{B}$ is Brownian motion under $\mathbb{Q}$ probability measure, and $S_{t^-} = \lim_{s\uparrow t} S_s$. Since S_t and S_{t^-} are the same outside of jump times, we get

$$d\tilde{S}_t = e^{-rt} S_{t^-} \left\{ [\mu + \lambda\kappa - r] \, dt + \sigma dB_t + d\tilde{J}_t \right\}. \tag{14}$$

By Girsanov's theorem we change the drift of the Brownian motion to move into $\mathbb{Q}$ probability space. Define the new Brownian motion under $\mathbb{Q}$ by

$$d\tilde{B}_t = dB_t + \theta dt,$$

where θ is a constant. Hence,

$$\sigma dB_t = \sigma(d\tilde{B}_t - \theta dt),$$

and we can substitute this into (14) to get

$$d\tilde{S}_t = e^{-rt} S_{t^-} \left\{ [\mu + \lambda\kappa - r - \sigma\theta] \, dt + \sigma dB_t + d\tilde{J}_t \right\}.$$

Now, since for the discounted asset to be a martingale under $\mathbb{Q}$, we must have no drift. In other words:

$$\mu + \lambda\kappa - r - \sigma\theta = 0,$$
$$\Rightarrow \theta = \frac{\mu + \lambda\kappa - r}{\sigma}.$$

Hence we define $d\tilde{B}_t$ as

$$d\tilde{B}_t = dB_t + \frac{\mu + \lambda\kappa - r}{\sigma} dt. \tag{15}$$

By substituting into (12), we get the final form of the risk neutral metron model:

$$\frac{dS_t}{S_{t^-}} = (r - \lambda\kappa)dt + \sigma d\tilde{B}_t + dJ_t.$$

8 Conclusion

Our primary aim of this paper has been to explore the analytical understanding of both the Black-Scholes and Heston models. A second goal of this has consisted in testing the models using numerical and calibration methods, comparing the difference between the two models. Numerical simulations and calibration of the Heston model was more accurate to fitting market data compared to Black-Scholes model. Although, Heston model is more computationally complex due to the calibration methods required for parameter estimation.

Overall, we can conclude from our studies the significant value the models have in financial theory and practice. The Black-Scholes model remains a starting point in understanding basic option pricing mechanism, while Heston model is an improvement to address the real-world market dynamics. One can take this further and explore hybrid models that integrate stochastic volatility with other features, such as jump processes, to refine the accuracy and flexibility of pricing frameworks.

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A Appendix

A.1 Black-Scholes transformation into the Heat Equations

Let:

$$\tau = T - \frac{t}{\sigma^2}, \quad S_t = e^x, \quad f(t, S_t) = v(\tau, x) = v\left(\frac{\sigma^2}{2}(T-t), \ln(S_t)\right)$$

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \\
\frac{\partial f}{\partial S_t} &= \frac{\partial v}{\partial x}\frac{\partial \tau}{\partial t} = \frac{1}{S_t}\frac{\partial v}{\partial \tau} \\
\frac{\partial^2 f}{\partial S_t^2} &= \frac{\partial}{\partial S_t}\left(\frac{\partial v}{\partial S_t}\right) = \frac{\partial}{\partial S_t}\left(\frac{1}{S_t}\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S_t^2}\frac{\partial v}{\partial x} + \frac{1}{S_t}\frac{\partial}{\partial x}\frac{\partial x}{\partial S_t}\frac{\partial v}{\partial x} = -\frac{1}{S_t^2}\frac{\partial v}{\partial x} + \frac{1}{S_t^2}\frac{\partial}{\partial x}\frac{\partial v}{\partial x} \\
&= -\frac{1}{S_t^2}\frac{\partial v}{\partial x} + \frac{1}{S_t^2}\frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

Substitute into Black Scholes' formula (4):

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v. \tag{16}$$

Let $\kappa = \frac{2r}{\sigma^2}$, and replace τ with t:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1)\frac{\partial v}{\partial x} - \kappa v, \quad x \in (-\infty, \infty), \quad t \in [0, \frac{\sigma^2}{2}T],$$

where

$$v(0, x) = f(T, e^x) = f(e^x) = (e^x - K)^+,$$

and K is the strike price of the forward contract. Let α, β be arbitrary constants,

$$u(t, x) = e^{\alpha x + \beta t}(t, x) = \phi v,$$

where $\phi = e^{\alpha x + \beta t}$.

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \beta\phi u + \phi\frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial x} &= \alpha\phi u + \phi\frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(\alpha\phi v + \phi\frac{\partial u}{\partial x}\right) = \alpha^2\phi u + 2\alpha\phi\frac{\partial u}{\partial x} + \phi\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Substitute the above into partial differential equations (16):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + [2\alpha + (\kappa - 1)]\frac{\partial u}{\partial x} + [\alpha^2 + (-1)\alpha - \kappa - \beta]u.$$

By setting $\alpha = -\frac{\kappa - 1}{2}$, $\beta = \alpha^2 + (\kappa - 1)\alpha - \kappa = -\frac{(\kappa+1)^2}{4}$, we cancel out the $\frac{\partial u}{\partial x}$ and u coefficients, leaving us with

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t \in [0, \frac{\sigma^2}{2}T] \\
u(0, x) &= e^{-\alpha x}v(0, x) = e^{-\alpha x}f(e^x)
\end{aligned}$$