GSERM 2020
Regression for Publishing

June 15, 2020 (first session)

“Regression for Publishing”

- “Regression” course
- Texts: Mostly posted readings; also Weisberg (2014) and/or Faraway (2002)
- Course materials at the github repo: `https://github.com/PrisonRodeo/GSERM-2020-git` and on CANVAS
- Software:
  - Support: R $\geq$ Stata $>$ others...
  - GSERM virtual machines at `https://vdi.unisg.ch/gserm`; more details are available at the *GSERM 2020 Software Guide*
  - “Introduction to R and RStudio” - useful?
- Grading: Two homework assignments plus a final examination

Things We Will And Won't Do

Will: "Regression":

$$Y = f(\mathbf{X})$$

Won't: Multivariate regression:

$$\mathbf{Y} = f(\mathbf{X})$$

Won't: Measurement (e.g. PCA, factor analysis, etc.):

$$\mathbf{Y} = \mathbf{W}^{\mathrm{T}}\mathbf{X}$$

Won't: Classification:

- Cluster Analysis
- Classification and Regression Trees $\rightarrow$ Random Forests.
- Pattern Recognition
- Machine Learning, Support Vector Machines, etc.

"Regression," conceptually:

$$\Pr(Y|\mathbf{X}) = f(\mathbf{X})$$

Two important things:

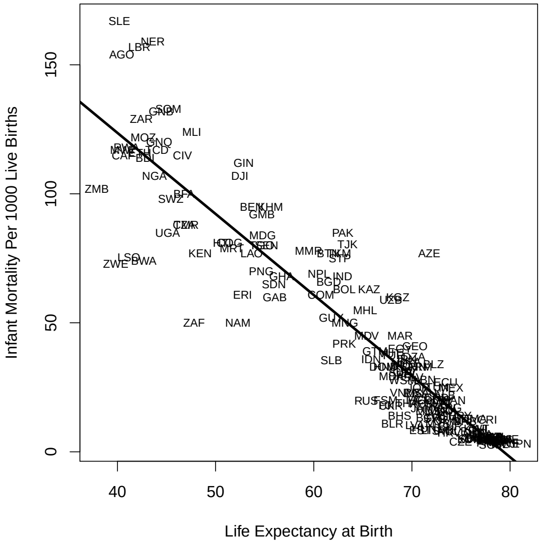
- The distribution of Y is *conditional on all variables in* $\mathbf{X}$, and
- The conditional distribution of Y is conditional on the *joint distribution* of the elements of $\mathbf{X}$.

$\rightarrow$ Regression is <u>hard</u>...

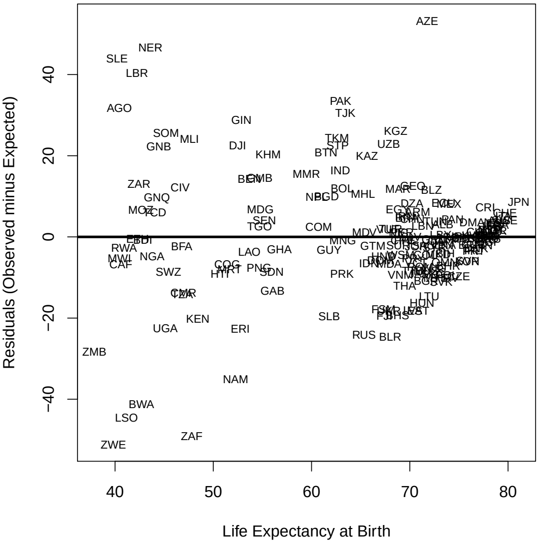
Why Regression?

	Description	**Explanation**	**Prediction**
Task	Summarize data	Correlation/causation	Forecast OOS / future data
Emphasis	Data	Theory / Hypotheses	Outcomes
Focus	Univariate	Multivariate	Multivariate
Typical Application	Summarize / “reduce” data	Discuss marginal associations between predictors and an outcome of interest	Optimize out-of-sample predictive power / minimize prediction error

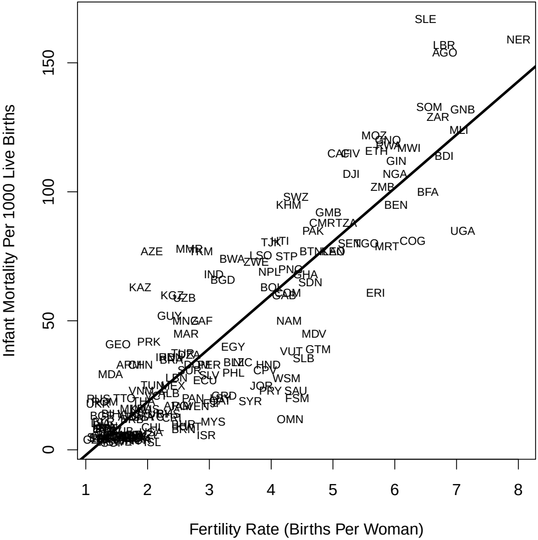
Example: Infant Mortality and Life Expectancy



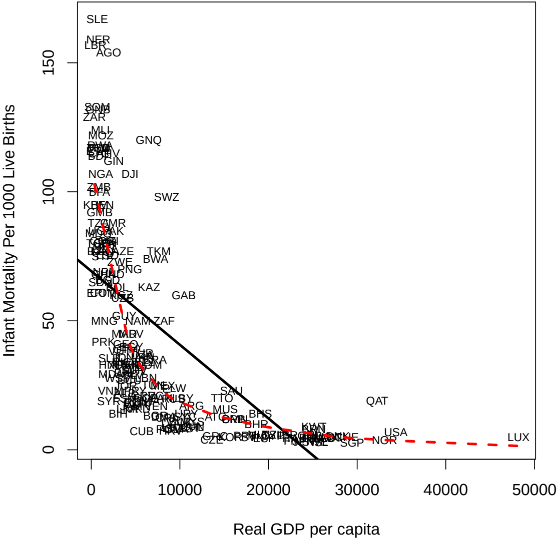
Infant Mortality and Life Expectancy: “Residuals”



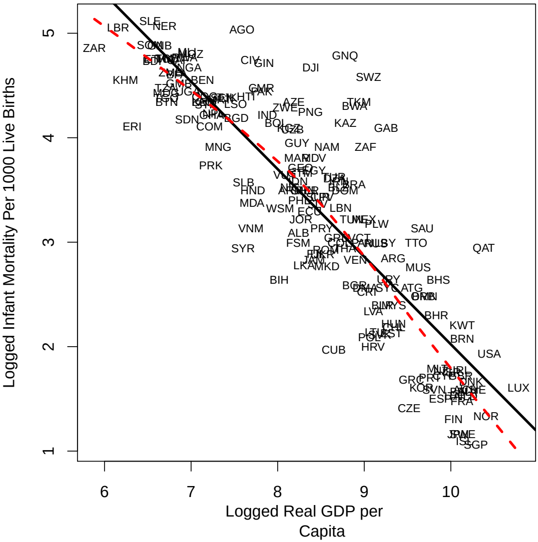
Infant Mortality and Fertility



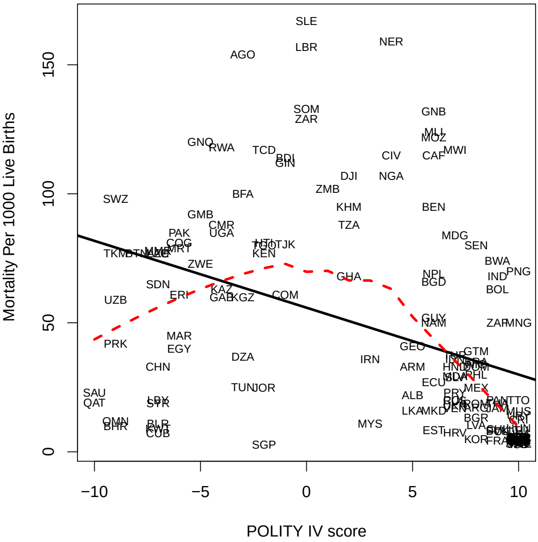
Infant Mortality and Wealth



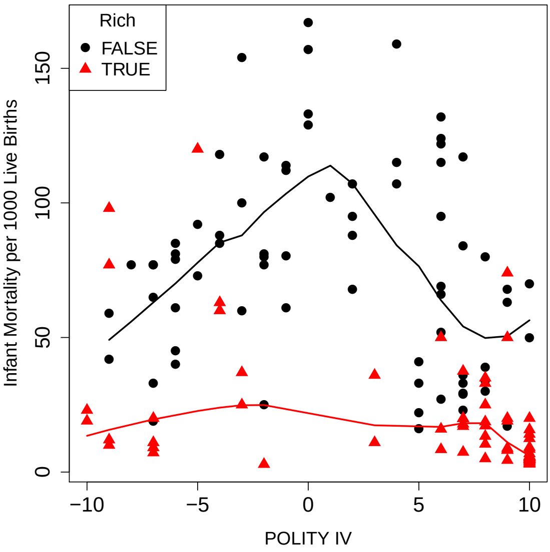
(Logged) Infant Mortality and (Logged) Wealth



Infant Mortality and Democracy



Infant Mortality, (Dichotomized) Wealth, and Democracy



$$Y_i = \mu + u_i \tag{1}$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

so:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \tag{2}$$

<u>Goals</u>:

- Estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
- Estimate the *variability* $\hat{\beta}_0$ and $\hat{\beta}_1$

Bivariate OLS - Estimation

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} \tag{3}$$

$$= \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \tag{4}$$

$$u_i \sim \text{i.i.d. } N(0, \sigma^2)$$

meaning:

$$\text{Var}(Y|X, \beta) = \sigma^2$$

so:

$$\begin{aligned}
\text{Var}(\hat{\beta}_1) &= \text{Var}\left[\frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}\right] \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\text{Var}(Y) \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\sigma^2 \\
&= \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.
\end{aligned}$$

$\text{Var}(\hat{\beta}_0)$ and $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)$

Similarly:

$$\text{Var}(\hat{\beta}_0) = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2$$

and :

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2$$

Important Things

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto \sigma^2$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -\sum(X_i - \bar{X})$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -N$
- $\text{sign}[\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)] = \text{sign}(\bar{X})$

If $u_i \sim N(0, \sigma^2)$, then:

$$\hat{\beta}_0 \sim N[\beta_0, \text{Var}(\hat{\beta}_0)]$$

and

$$\hat{\beta}_1 \sim N[\beta_1, \text{Var}(\hat{\beta}_1)]$$

Means:

$$\begin{aligned} z_{\hat{\beta}_1} &= \frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)}{\text{s.e.}(\hat{\beta}_1)} \\ &= \sim N(0,1) \end{aligned}$$

A Small Problem...

$\sigma^2 =$???

Solution: use

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{N - k}$$

Gives:

$$\widehat{\text{Var}(\hat{\beta}_1)} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2},$$

and

$$\widehat{\text{Var}(\hat{\beta}_0)} = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \hat{\sigma}^2$$

Inference (continued)

$$
\begin{aligned}
\widehat{\text{s.e.}(\hat{\beta}_1)} &= \sqrt{\widehat{\text{Var}(\hat{\beta}_1)}} \\
&= \sqrt{\frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2}} \\
&= \frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}
\end{aligned}
$$

implies:

$$
\begin{aligned}
t_{\hat{\beta}_1} \equiv \frac{(\hat{\beta}_1 - \beta_1)}{\widehat{\text{s.e.}(\hat{\beta}_1)}} &= \frac{(\hat{\beta}_1 - \beta_1)}{\frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}} \\
&= \frac{(\hat{\beta}_1 - \beta_1)\sqrt{\sum(X_i - \bar{X})^2}}{\hat{\sigma}} \\
&\sim t_{N-k}
\end{aligned}
$$

Predictions and Variance

Point prediction:

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

Y_k is unbiased:

$$\begin{aligned}
E(\hat{Y}_k) &= E(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\
&= E(\hat{\beta}_0) + X_k E(\hat{\beta}_1) \\
&= \beta_0 + \beta_1 X_k \\
&= E(Y_k)
\end{aligned}$$

Variability:

$$\begin{aligned}
\text{Var}(\hat{Y}_k) &= \text{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\
&= \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2 + \left[\frac{\sigma^2}{\sum (X_i - \bar{X})^2} \right] X_k^2 + 2 \left[\frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2 \right] X_k \\
&= \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]
\end{aligned}$$

Variability of Predictions

$$\text{Var}(\hat{Y}_k) = \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]$$

means that $\text{Var}(\hat{Y}_k)$:

- Decreases in N
- Decreases in $\text{Var}(X)$
- Increases in $|X - \bar{X}|$

Standard error of the prediction:

$$\widehat{\text{s.e.}(\hat{Y}_k)} = \sqrt{\sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]}$$

$\rightarrow$ (e.g.) confidence intervals:

$$95\% \, \text{c.i.}(\hat{Y}_k) = \hat{Y}_k \pm [1.96 \times \widehat{\text{s.e.}(\hat{Y}_k)}]$$

Variation in Y

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(\hat{Y} + \hat{u}) \\
&= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u}) + 2\,\mathrm{Cov}(\hat{Y}, \hat{u}) \\
&= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u})
\end{aligned}$$

TSS	$=$	**MSS**	$+$	**RSS**
("Total")		("Estimated," or "Model")		("Residual")

Model Fit: R^2

$$
\begin{aligned}
R^2 &= \frac{\text{MSS}}{\text{TSS}} \\
&= \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} \\
&= 1 - \frac{\text{RSS}}{\text{TSS}} \\
&= 1 - \frac{\sum \hat{u}_i^2}{\sum(Y_i - \bar{Y})^2}
\end{aligned}
$$

R^2 Is...

R-squared:

- is "the proportion of variance explained"
- $\in [0,1]$
  - $R^2 = 1.0 \equiv$ a "perfect (linear) fit"'
  - $R^2 = 0 \equiv$ no (linear) $X - Y$ association

For a single X,

$$
\begin{aligned}
R^2 &= \hat{\beta}_1^2 \frac{\sum(X_i - \bar{X})^2}{\sum(Y_i - \bar{Y})^2} \\
&= r_{XY}^2
\end{aligned}
$$

Adjusted R^2

$$R^2_{adj.} = 1 - \frac{(1 - R^2)(N - c)}{(N - k)}$$

where $c = 1$ if there is a constant in the model and $c = 0$ otherwise.

$R^2_{adj.}$:

- $R^2_{adj.} \rightarrow R^2$ as $N \rightarrow \infty$
- $R^2_{adj.}$ can be > 1, or < 0...
- $R^2_{adj.}$ increases with model "fit," but
- The extent of that increase is discounted by a factor proportional to the number of covariates.

R^2 Alternatives

- Standard Error of the Estimate:

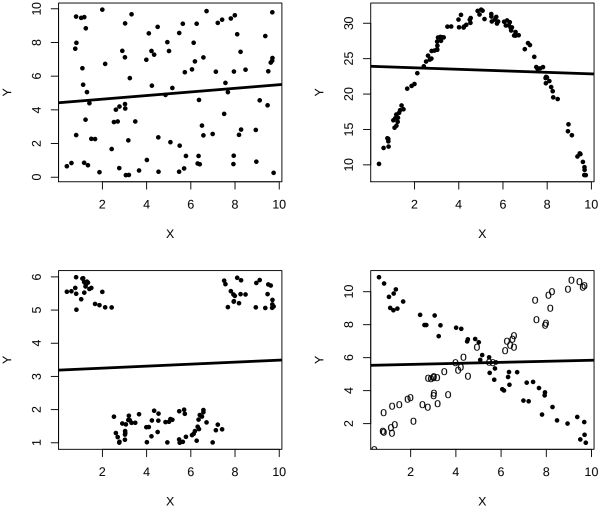
$$\text{SEE} = \sqrt{\frac{\text{RSS}}{N - k}}$$

- F-tests

- ROC / AUC

- Graphical methods

Caution: Different Ways to get $R^2 = 0$



$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\beta$$

The inner product of **u**:

$$\begin{aligned}
\mathbf{u}'\mathbf{u} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

$$\begin{aligned} \mathbf{u}'\mathbf{u} &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \mathbf{Y}'\mathbf{Y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{Y}' + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \end{aligned}$$

Now get:

$$\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}$$

Solve:

$$\begin{aligned} -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\ -\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\ \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{X}'\mathbf{Y} \\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ \boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \end{aligned}$$

$$\begin{aligned}\mathbf{V}(\hat{\beta}) &= \mathsf{E}[\hat{\beta} - \mathsf{E}(\hat{\beta})]^2 \\ &= \mathsf{E}\{[\hat{\beta} - \mathsf{E}(\hat{\beta})][\hat{\beta} - \mathsf{E}(\hat{\beta})]'\}\end{aligned}$$

Rewrite:

$$\begin{aligned}\mathbf{V}(\hat{\beta}) &= \mathsf{E}(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \\ &= \mathsf{E}\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}]'\} \\ &= \mathsf{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}$$

Taking expectations:

$$\begin{aligned} \mathbf{V}(\hat{\beta}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}(\mathbf{uu}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1} \end{aligned}$$

Estimating $\mathbf{V}(\hat{\beta})$

Empirical estimate:

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N - K}$$

Yields:

$$\widehat{\mathbf{V}(\hat{\beta})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

OLS Assumptions

1. Zero Expectation Disturbances

$$\mathsf{E}(\mathbf{u}) = \mathbf{0}$$

2. Homoscedasticity / No Error Correlation

$$\mathsf{E}(\mathbf{u}\mathbf{u}') = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

OLS Assumptions (continued)

3. "Fixed" $\mathbf{X}$...

- No *measurement error* in the $\mathbf{X}$s, and
- $\text{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$.

4. $\mathbf{X}$ is of full column rank.

Means:

- no exact linear relationship among $\mathbf{X}$, and
- $K < N$.

5. Normal Disturbances

$$\mathbf{u} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

Under these assumptions, the OLS estimate of $\hat{\boldsymbol{\beta}}$ is:

- **Unbiased**
- **Fully Efficient**

(i.e., **"BLUE"**)

Example Data: Infant Mortality

```
> url <- getURL("https://raw.githubusercontent.com/PrisonRodeo/
    GSERM-2020-git/master/Data/CountryData2000.csv")
> Data <- read.csv(text = url) # read the data
> rm(url)
>
> # Summary statistics
>
> # install.packages("psych") <- Install psych package, if necessary
> library(psych)

> with(Data, describe(infantmortalityperK))
  vars   n  mean    sd median trimmed   mad min max range skew kurtosis   se
1    1 179 43.83 40.39     29   38.38 34.26 2.9 167 164.1    1     0.06 3.02

> with(Data, describe(DPTpct))
  vars   n  mean    sd median trimmed   mad min max range  skew kurtosis   se
1    1 181 81.71 19.77     90   85.23 11.86  24  99    75 -1.31     0.57 1.47
```

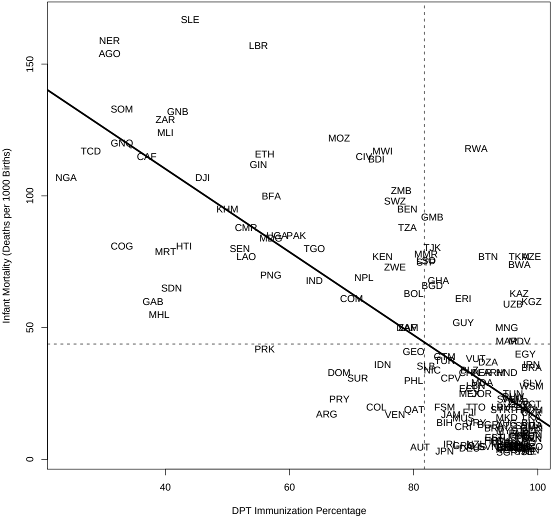
OLS Regression

```
> IMDPT<-lm(infantmortalityperK~DPTpct,data=Data,na.action=na.exclude)
> summary.lm(IMDPT)

Call:
lm(formula = infantmortalityperK ~ DPTpct, data = Data)

Residuals:
    Min      1Q  Median      3Q     Max
-56.801 -16.328  -5.105  11.777  86.590

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 173.2771     8.4893   20.41   <2e-16 ***
DPTpct       -1.5763     0.1009  -15.62   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.19 on 175 degrees of freedom
  (14 observations deleted due to missingness)
Multiple R-squared:  0.5824,Adjusted R-squared:   0.58
F-statistic: 244.1 on 1 and 175 DF,  p-value: < 2.2e-16
```

Analysis of Variance

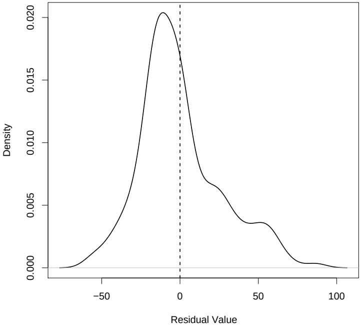
```
> anova(IMDPT)
Analysis of Variance Table

Response: infantmortalityperK
           Df Sum Sq Mean Sq F value    Pr(>F)
DPTpct      1 167423  167423  244.09 < 2.2e-16 ***
Residuals 175 120033     686
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Regression of Infant Mortality on DPT Immunization Rates



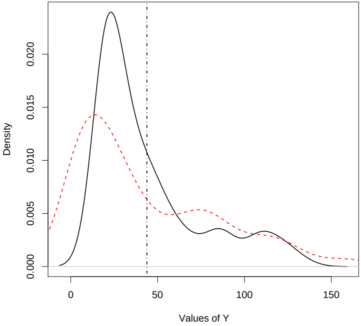
Fitted Values, Residuals, etc.

```
> # Residuals (u):
> Data$IMDPTres <- with(Data, residuals(IMDPT))
> describe(Data$IMDPTres)

  var   n mean    sd median   mad   min   max range skew kurtosis   se
1   1 177    0 26.12   -5.1 19.42 -56.8 86.59 143.4 0.75     0.44 1.96
```

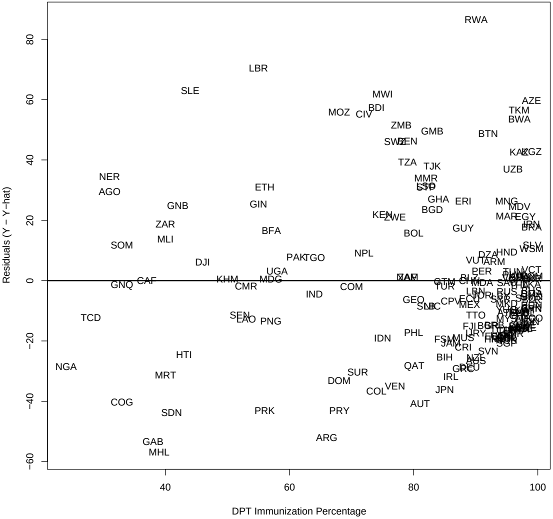


```
> # Fitted Values:
> Data$IMDPThat<-fitted.values(IMDPT)
> describe(Data$IMDPThat)

  var   n  mean    sd median  mad   min   max range skew kurtosis   se
1   1 177 44.26 30.84  31.41 18.7 17.22 135.4 118.2  1.3     0.59 2.32
```

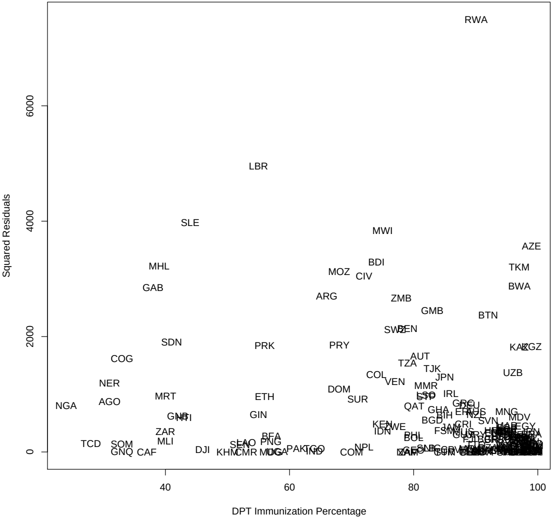
Density Plot: Actual (Y) and Fitted Values ($\hat{Y}$)



Regression Residuals ($\hat{u}$) vs. DPT Percentage



Squared Residuals vs. DPT Percentage



Var($\hat{\beta}$):

```
> vcov(lMDPT)

            (Intercept)   DPTpct
(Intercept)     72.0677 -0.83317
DPTpct          -0.8332  0.01018
```

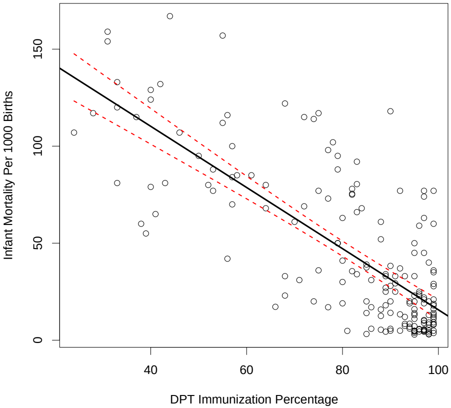
95 percent c.i.s:

```
> confint(lMDPT)

              2.5 %  97.5 %
(Intercept) 156.523 190.032
DPTpct       -1.775  -1.377
```

```
> SEs<-predict(IMDPT,interval="confidence")
> SEs
       fit    lwr    upr
1    25.10  20.53  29.68
3    17.22  12.05  22.40
4    23.53  18.84  28.21
.
.
<rows omitted>
.
.
189  21.95  17.15  26.75
190  39.29  35.36  43.23
191  17.22  12.05  22.40
```

A Plot, With CIs

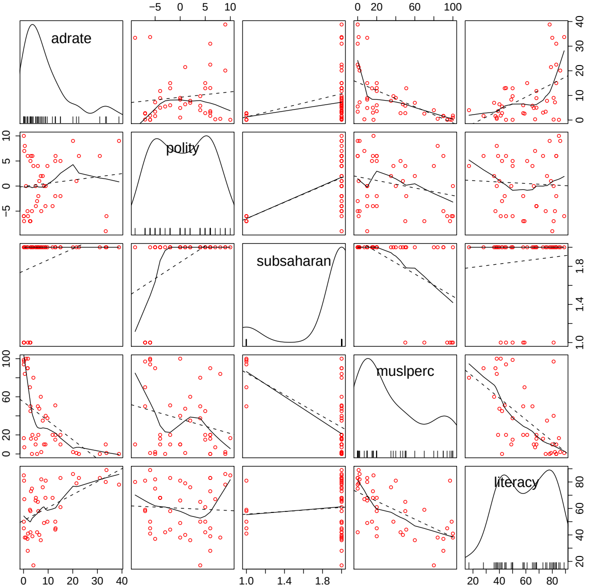
Scatterplot of Infant Mortality and DPT Immunizations, along with Least-Squares Line and 95% Prediction Confidence Intervals



Multivariate Example: Africa Data

```
> library(RCurl)
> temp<-getURL("https://raw.githubusercontent.com/PrisonRodeo/GSERM-2020-git/master/Data/africa2001.csv")
> Data<-read.csv(text=temp, header=TRUE)
> Data<-with(Data, data.frame(adrate,polity,
+           subsaharan=as.numeric(subsaharan),muslperc,literacy))

> summary(Data)
     adrate           polity           subsaharan      muslperc          literacy
 Min.   : 0.100   Min.   :-9.0000   Min.   :1.00   Min.   :  0.00   Min.   :17.00
 1st Qu.: 2.700   1st Qu.:-4.5000   1st Qu.:2.00   1st Qu.: 10.00   1st Qu.:43.00
 Median : 6.000   Median : 0.0000   Median :2.00   Median : 20.00   Median :61.00
 Mean   : 9.365   Mean   : 0.5116   Mean   :1.86   Mean   : 35.96   Mean   :60.07
 3rd Qu.:12.900   3rd Qu.: 5.5000   3rd Qu.:2.00   3rd Qu.: 55.50   3rd Qu.:78.50
 Max.   :38.800   Max.   :10.0000   Max.   :2.00   Max.   :100.00   Max.   :89.00

> cor(Data)
               adrate      polity  subsaharan   muslperc    literacy
adrate      1.0000000  0.11794182  0.33129420 -0.5709233  0.51489444
polity      0.1179418  1.00000000  0.52819844 -0.2391715 -0.05079354
subsaharan  0.3312942  0.52819844  1.00000000 -0.5772513  0.09472968
muslperc   -0.5709233 -0.23917151 -0.57725134  1.0000000 -0.61960385
literacy    0.5148944 -0.05079354  0.09472968 -0.6196039  1.00000000
```

Africa Data



adrate
polity
subsaharan
muslperc
literacy
−5
0
5
10
0
20
60
100
40
30
20
10
0
10
5
0
−5
1.8
1.4
1.0
100
60
20
0
80
60
40
20
0
10
20
30
40
1.0
1.4
1.8
20
40
60
80

A Regression

```
> model<-lm(adrate~polity+subsaharan+muslperc+literacy,data=Data)
> summary(model)

Call:
lm(formula = adrate ~ polity + subsaharan + muslperc + literacy,
    data = Data)

Residuals:
     Min       1Q   Median       3Q      Max
-15.4681  -4.3947  -0.5251   3.4246  22.9358

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.39843   14.94744  -0.294   0.7702
polity      -0.01390    0.27969  -0.050   0.9606
subsaharan   3.72969    5.43093   0.687   0.4964
muslperc    -0.08689    0.06282  -1.383   0.1747
literacy     0.16575    0.09433   1.757   0.0869 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 8.264 on 38 degrees of freedom
Multiple R-squared:  0.3771,Adjusted R-squared:  0.3115
F-statistic: 5.751 on 4 and 38 DF,  p-value: 0.001013
```

Variance-Covariance Matrix of $\hat{\boldsymbol{\beta}}$

```
> options(digits=4)
> vcov(model)

            (Intercept)    polity subsaharan  muslperc  literacy
(Intercept)    223.4259  1.088030   -72.2628 -0.771309 -1.002421
polity           1.0880  0.078229    -0.6642 -0.000293  0.001968
subsaharan     -72.2628 -0.664212    29.4950  0.206067  0.171765
muslperc        -0.7713 -0.000293     0.2061  0.003946  0.004098
literacy        -1.0024  0.001968     0.1718  0.004098  0.008898
```

Test $H_0 : \beta_{\texttt{polity}} = \beta_{\texttt{subsaharan}} = 0$:

```
> library(lmtest)
> modelsmall<-lm(adrate~muslperc+literacy,data=Data)
> waldtest(model,modelsmall)

Wald test

Model 1: adrate ~ polity + subsaharan + muslperc + literacy
Model 2: adrate ~ muslperc + literacy
  Res.Df Df    F Pr(>F)
1     38
2     40 -2 0.27   0.76
```

Test $H_0 : \beta_{\texttt{muslperc}} = 0.1$:

```
> library(car)
> linearHypothesis(model,"muslperc=0.1")

Linear hypothesis test

Hypothesis:
muslperc = 0.1

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1     39 3200
2     38 2595  1       605 8.85 0.0051 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Test $H_0 : \beta_{\texttt{literacy}} = \beta_{\texttt{muslperc}}$:

```
> linearHypothesis(model,"literacy=muslperc")

Linear hypothesis test

Hypothesis:
- muslperc  + literacy = 0

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F  Pr(>F)
1     39 3534
2     38 2595  1       938 13.7 0.00067 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Supplementary Materials

Hypothetically: If we have $\hat{\beta}_0$ and $\hat{\beta}_1$, then:

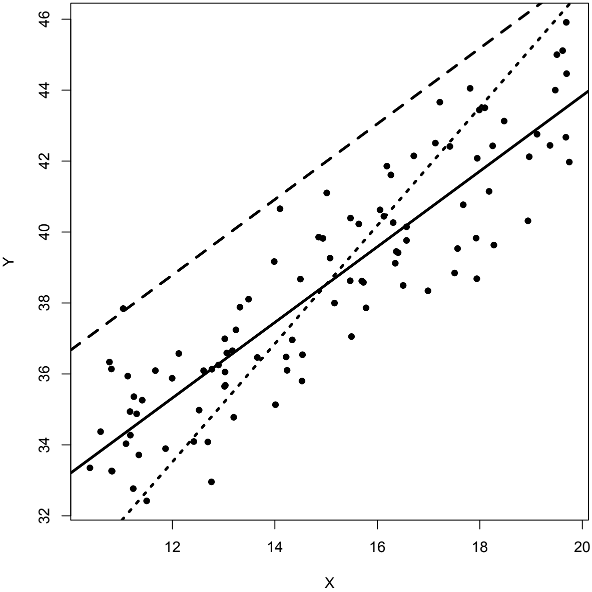
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \tag{5}$$

and

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \end{aligned} \tag{6}$$

Q: How to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$?

Scatterplot: X and Y (with regression lines)



Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize $\hat{S} = \sum_{i=1}^{N} \hat{u}_i^2$.

$$\begin{aligned}
\hat{S} &= \sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2 \\
&= \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\
&= \sum_{i=1}^{N} (Y_i^2 - 2Y_i\hat{\beta}_0 - 2Y_i\hat{\beta}_1 X_i + \hat{\beta}_0^2 + 2\hat{\beta}_0\hat{\beta}_1 X_i + \hat{\beta}_1^2 X_i^2)
\end{aligned}$$

Differentiate:

$$
\begin{aligned}
\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= \sum_{i=1}^{N}(-2Y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 X_i) \\
&= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) \\
&= -2\sum_{i=1}^{N}\hat{u}_i
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= \sum_{i=1}^{N}(-2Y_i X_i + 2\hat{\beta}_0 X_i + 2\hat{\beta}_1 X_i^2) \\
&= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) X_i \\
&= -2\sum_{i=1}^{N}\hat{u}_i X_i
\end{aligned}
$$

OLS (continued)

Yields:

$$\sum_{i=1}^{N} Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i$$

and

$$\sum_{i=1}^{N} Y_i X_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2$$

Solving yields:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} \tag{7}$$
$$= \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X}$$

and

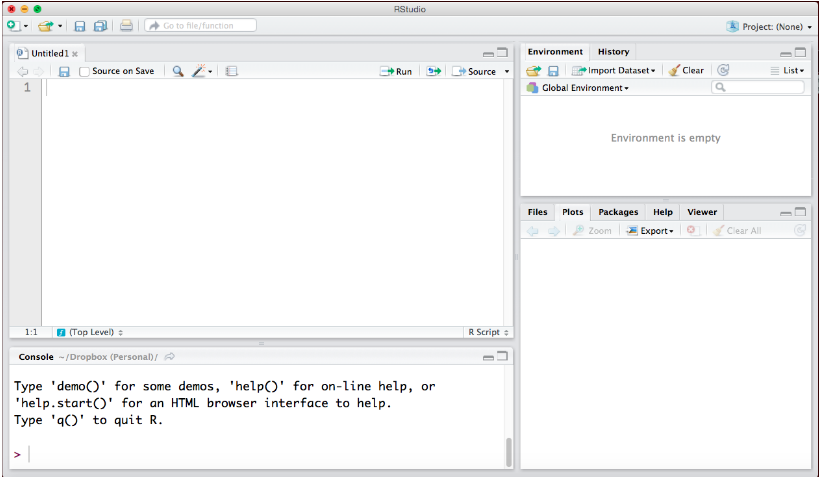
$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X} \tag{8}$$

R Things

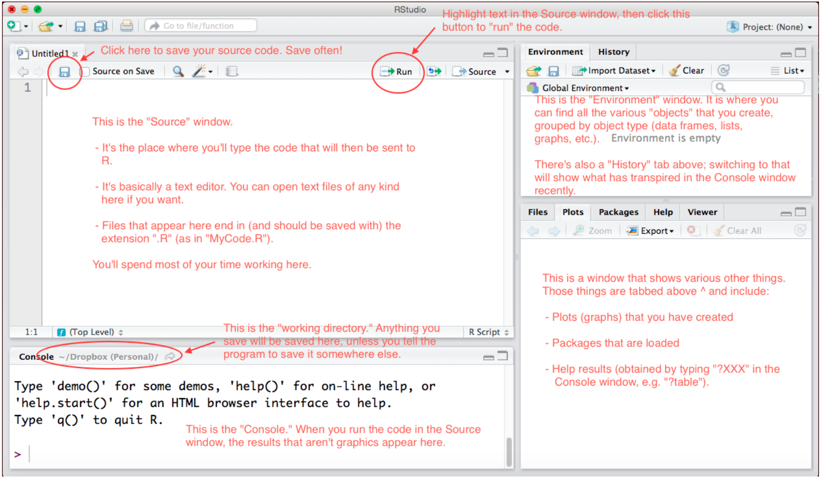
R

R Console

```
R version 3.6.3 (2020-02-29) -- "Holding the Windsock"
Copyright (C) 2020 The R Foundation for Statistical Computing
Platform: x86_64-apple-darwin15.6.0 (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.70 (7735) x86_64-apple-darwin15.6.0]

[Workspace restored from /Users/cuz10lcl/.RData]
[History restored from /Users/cuz10lcl/.Rapp.history]

>
```


RStudio
Go to file/function
Project: (None)
Untitled1
Source on Save
Run
Source
1
1:1
(Top Level)
R Script
Environment
History
Import Dataset
Clear
List
Global Environment
Environment is empty
Files
Plots
Packages
Help
Viewer
Zoom
Export
Clear All
Console ~/Dropbox (Personal)/
Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.
>

RStudio (annotated)



Inside the Source Window

This:

```
> table(df$X)
```

... means "Type the phrase 'table(df$X)' on the command line," or – equivalently – "Type the phrase 'table(df$X)' into your Source code, and then run it."

More often, you'll see:

```
with(df, plot(Y~X,pch=19,col="red")) # draw a scatterplot
abline(h=0,lty=2) # add a horizontal line at zero
abline(v=0,lty=2) # add a vertical line at zero
text(df$X,df$Y,labels=df$names,pos=1) # add labels
```

... which means "Put this block of text into your Source code, and then run it."

Note:

- R / RStudio ignores line breaks
- Anything to the right of a "#" is a comment

Very basic R examples...

(see `GSERM-2020-R-Intro.R` in the github repo)

Help For Learning R(Studio)

In rough order of preference:

- Quick-R (`http://www.statmethods.net/`)
- The "Level-Zero" R Tutorial (doesn't integrate RStudio, but is otherwise very good)
- Statistics with R
- The Do It Yourself Introduction to R
- Also be sure to consult the Regression for Publishing "Useful R Resources" guide (on GitHub).