Investigating the Dynamics of Solitons and Other Nonlinear Wave Phenomena using Numerical Methods

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Abstract

Numerical methods were used to investigate the propagation of different waveforms under two nonlinear wave equation. The equations modelled where the Korteweg-de Vries (KDV) and Burgers' equations, both of which are nonlinear partial differential equations. To model these equations a central difference schemes for the spacial derivatives and a 4^{th}order Runge-Kutta method for temporal derivatives was used. Primarily the propagation and properties of solitons solution was investigated.

1 Introduction

The first recorded sighting and investigations of solitons was by the Victorian civil engineer John Scott Russell. In 1834, Russell was experimenting with canal boat designs in Scotland's Union canal when he observed the emission of a singular water wave pulse caused when a fast moving canal boat was brought to a sudden halt. Russell noticed that this pulse travelled for miles with almost no change in shape and only a very gradual change in height[1]. At the time of this discovery most water wave theories were based on work by d'Alembert and Bernoulli and had no solutions that were consistent with a non-dispersive wave pulse of this type. This led to the dismissal of Russell's finding by many prominent applied mathematicians and physicists of the day. Nevertheless, this did not deter Russell, who continued to investigate the properties of these miraculous pulses, which he termed "waves of translation" due to their shape retaining nature, using water channels in his lab. It was not until the end of the 19th century with the advent of the KDV equation that Russell's solitons were backed by a theoretical model.

Since solitons were found to be the solution to the KDV equation it has been shown that they are also solutions to a number of other nonlinear equations. These days solitons have many applications in a wide range of physical research areas: everything from field theories to biophysics. The most important application of solitons is to nonlinear optics where they can be used to solve the nonlinear Schrödinger equation. This allows for soliton wave packets to be used to send data down fibre optic cables, a technique that has been used to send data at over 1 Terabit per second[2].

2 Theory

2.1 Solitons

A soliton is a single wave pulse that propagates without an change in shape. The analytical equation of a soliton has been known to scientists for over a hundred years and takes the following form:

$$U(x,t) = 12\alpha^2 \mathrm{sech}^2(\alpha(x - 4\alpha^2 t)) \tag{1}$$

where α is an arbitrary amplitude and all other symbols have their usual meaning.

The two main properties of the solution are as follows. Firstly the amplitude of a soliton will determine its propagation speed, which can be seen directly from the phase velocity, v_p, of the waveform described in equation (1). This gives a phase velocity of:

$$v_p = \frac{\omega}{k} = \frac{4\alpha^3}{\alpha} = 4\alpha^2 \tag{2}$$

where ω is angular frequency and k is wave number.

The other important thing to note from equation (1) is that the width of the soliton is proportional to $\frac{1}{\alpha}$, which can again be seen directly from its analytical form with a basic understanding of functional scaling.

2.2 KDV and Burgers' Equation

The KDV equation is a 3^{rd} order, dispersive nonlinear partial differential equation from the family of weakly nonlinear, strongly disruptive equations. A dimensionless expression for the KDV equation is as follows:

$$\partial_t U + U\partial_x U + \partial_{xxx} U = 0 \tag{3}$$

where ∂_t denotes $\frac{\partial}{\partial t}$ and so on. This equation has the property that its solutions do not disperse with propagation. This is because the 3^{rd} order diffusive term, $\partial_{xxx}U$, is exactly canceled out by the nonlinear, $U\partial_x U$, term. These conditions mean that a soliton is an exact solution to the KDV equation Another interesting property of the KDV equation is that when expressed as a Lagrange density it has an infinite number of symmetries with respect to transformation. This in turn means that the KDV equation will conserve an infinite number of quantities. Two of these quantities are the so called "mass" and "momentum" which are described using:

$$\text{mass} = \int U dx \tag{4a}$$

$$\text{momentum} = \int U^2 dx \tag{4b}$$

Although it is important to note that these are only the actual mass and momentum of the system if the units of the solutions are chosen correctly.

The Burgers' equation can be used to describe the propagation of a shockwave through a viscous medium. It is a 2^{nd} order partial nonlinear diffusive differential equation of the form:

$$\partial_t U + U\partial_x U - \hat{D}\partial_{xx} U = 0 \tag{5}$$

where $\hat{D}$ is the dimensionless diffusion term and is proportional to the viscosity of the fluid. When $\hat{D} = 0$ this equation is called the inviscid Burgers' equation describing the propagation of a shock wave through an idealised fluid with no viscosity.

Both differential equations have propagation speed that are proportional to the amplitude of the waveform. This is a common theme among equations describing shockwaves.

3 Methods

3.1 Implementation of Numerical method

In this project all numerical methods were implemented in the python programming language using the object orientated paradigm. Solver classes were created for both the KDV and Burgers' equations that could integrate forward in time to propagate the initial conditions supplied. A `For` loop was used to iterate the numerical method forward in time while a `numpy` array was used to vectorise the spacial coordinates allowing for large decreases in computational operations needed. The spacial data obtained from every time iteration was then appended to a list for manipulation and analysis. To validate that the code created worked correctly plots and an animation were compared to known physical results. It was found that in trivial cases the model replicated the expected behaviour indicating that it was giving valid results.

Initially a central difference scheme for all spatial derivatives and a forward difference scheme for the time derivatives was used to discretise the KDV equation. This led to the following discretisation of equation (3):

$$\frac{U_i^{j+1} - U_i^j}{\Delta t} = -\frac{1}{\Delta x}((U_{i+1}^j)^2 - (U_{i-1}^j)^2) - \frac{1}{\Delta x^3}(U_{i+2}^j - 2U_{i+1}^j + 2U_{i-1}^j - U_{i-2}^j) \tag{6}$$

where U_i^j denotes the value of U at the j^{th} time step and i^{th} spacial point and Δx and Δt is the size of the special and time steps respectively. However, after applying Von Neumann stability analysis it was found that this approach was unconditionally unstable. For this reason an RK4 method was chosen for the temporal derivative as it offered greater stability. In the discretisation of Burgers' another central difference scheme was used for the diffusive term taking the form:

$$+\hat{D}\frac{1}{\Delta x^2}(U_{i+1}^j - 2U_i^j + U_{i-1}^j) \tag{7}$$

which replaces the final term in equation (6).

3.2 Stability and Accuracy

For the discretised KDV equation it was found that the stability condition was $\Delta t \leq \Delta x^3$. This can be derived directly from the discretised differential equation where leading the error comes from the third order spacial derivative which has the stability condition stated above. For this reason throughout this this project data obtained was created with the condition $\frac{\Delta t}{\Delta x^3} = 1$ to ensure fair comparison between results obtained.

In the case of the inviscid Burgers' equation it was found that the solution was also unconditionally unstable due to the nature in which the wave is propagated. However, with the addition the diffusive term it was found that the method was stable as long as $\hat{D}$ was not ≈ 0

It was noted that for larger α solitons the stability would also be bounded from above with larger α solitons needing a smaller Δx. A test was therefore devised that would loop over different solitons of different amplitudes and different Δx value to find the upper limit of the Δx term. This upper limit was denoted Δx_{crit}. This was found that there was a near perfect reciprocal relation between Δx_{crit} and the initial amplitude of the soliton. This can be seen in figure 1.

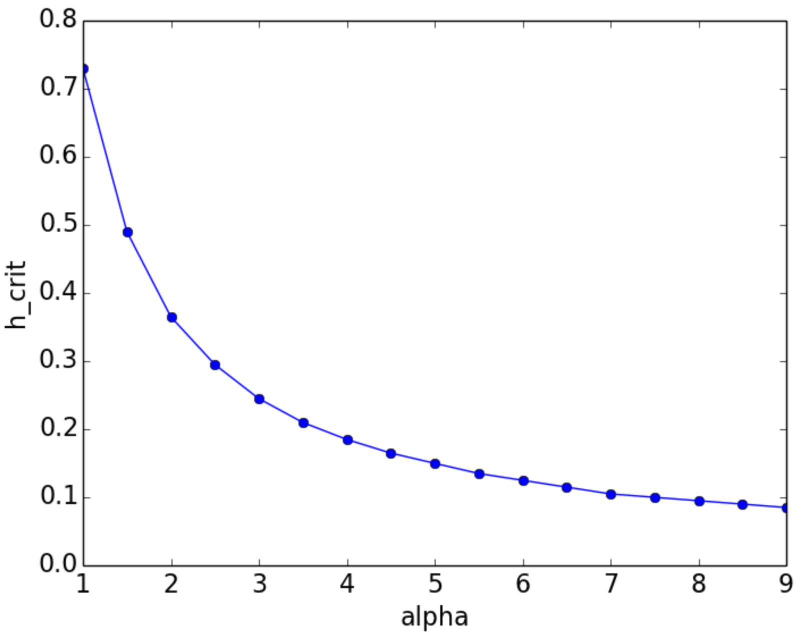

Figure 1: Graph showing relationship between dx_{crit} and α

By plotting a log-log graph and applying a linear fitting it was found the gradient was -0.98 implying that the relation was indeed $\Delta x_{crit} \propto \frac{1}{\alpha}$. The log-log graph can be seen in figure 2.

This relation is expected due to the scaling nature of the soliton mentioned in section 2.1 meaning that as α increases its width will decrease meaning that a smaller spacial resolution is needed to resolve the soliton correctly.

To test the accuracy of the numerical method the velocities of solitons were recoreded with different initial amplitudes. This was then plotted to see over what range the velocities agreed with the theoretical phase velocities calculated in equation (2), see figure 3.

It can be seen from this plot that at values of $\alpha < 3$ the solutions are very physically accurate. Once α increases over this threshold the numerical solutions quickly start to deviate from the physical solution and the velocity of increasingly larger solitons starts to plateau and tends towards a specific velocity. This is due to the spacial resolution only allowing solitons above certain height to be correctly resolved. Anything above this height will break into a solution comprised of the maximum resolvable soliton and much smaller solitons. Therefore as Δx is decreased larger solitons become resolvable and can be supported by the numerical method. This can be seen by the raising of the maximum velocity with a smaller spacial resolution in figure 3.

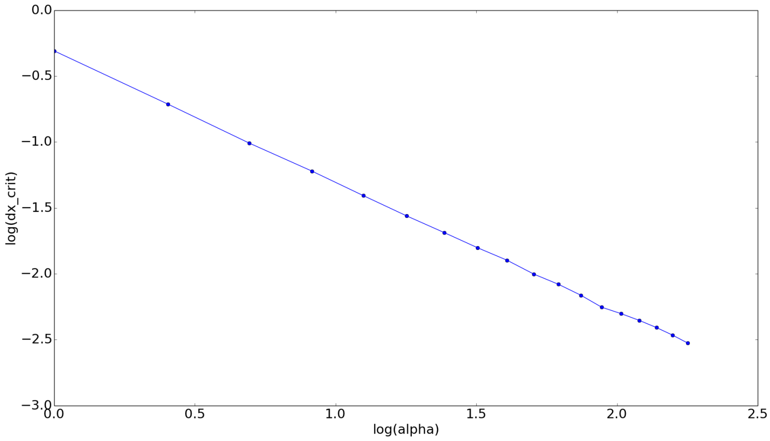

Figure 2: Log-log graph showing relationship between $log(dx_{crit})$ and $log(\alpha)$ used to find power relation between dx_{crit} and α.

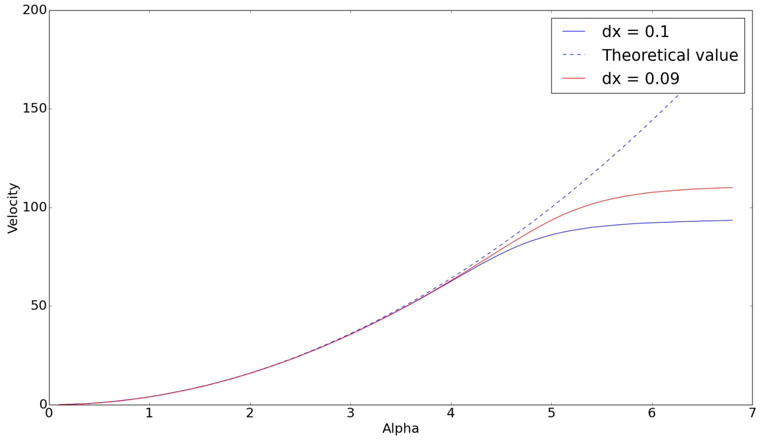

Figure 3: Graph plotting velocity against α with theoretical velocity plot to assess accuracy of numerical method at different Δx.

4 Results and Analysis

As previously discussed in section 3.2 section all plots and data obtained in this section are using a normalised ratio of $\frac{\Delta t}{\Delta x^3} = 1$. To do this Δx and Δt were set to 0.1 and 0.001 respectively.

4.1 Solitons Iteration and Conserved Quantities

Investigations were carried out into the 2-soliton solutions to the KDV equation. During the propagation of the 2-soliton solution the larger soliton would catch and collide with the smaller soliton due to its larger velocity. It was found that during these collisions two characteristic behaviours were observed that were dependent on the relative heights of the two initial solitons. If the solitons are of very different heights then during a collision the maxima of both solitons can no longer be resolved and the larger soliton seems to swamp and move straight over the smaller one. This can be seen in figures 4 and 5 where the larger maximum remains with the original largest soliton.

This is in contrast with the behaviour observed when both solitons are of similar heights. During these collisions the maxima of both solitons are resolvable at all times. An example of this can be seen in figures 6 and 7 where it can be seen that the maxima of the two solitons switch with a saddle point in waves displacement during the

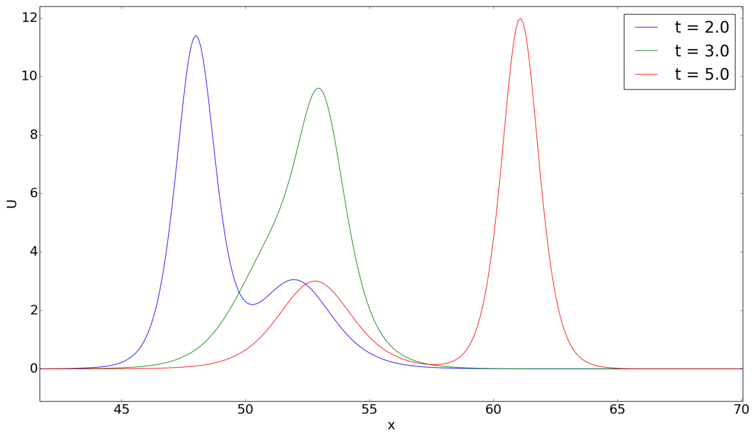

Figure 4: Graph showing amplitude against position of a large soliton interacting with a smaller soliton.

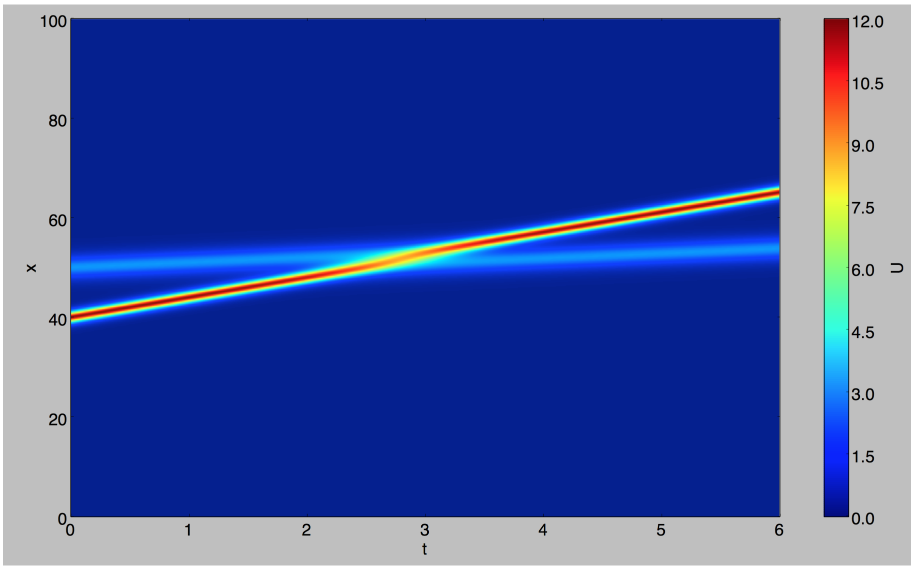

Figure 5: Space-time diagram showing amplitude of a large soliton interacting with a smaller soliton.

collision.

In both situations a phase shift of the maxima can be observed by the discontinuity in the maxima gradient on the space-time diagrams. This can be seen much more clearly in the interaction of the similar sized solitons. This implies that the mass of the larger soliton is passed to the smaller one which then continues to propagate at the same velocity as the initially larger soliton. This is a similar interaction to what is seen during the interaction of two similarly charged particles when the momentum of the first particle is passed to the other without an actually colliding.

During the interaction of the two solitons it was found that both the mass and momentum were conserved. This is as expected from the symmetry of the KDV equation as discussed in section 2.2.

4.2 Wave Breaking

During the investigation of the KDV equations arbitrary functions were used as initial conditions to see how these would be propagated under the KDV equation. It was found that for functions with no infinite derivatives, as the numerical method cannot deal with an infinite derivative, would break up into a train of solitons. This implies that this initial condition is in fact an N-soliton solution to the KDV equation. An example of this can be seen in

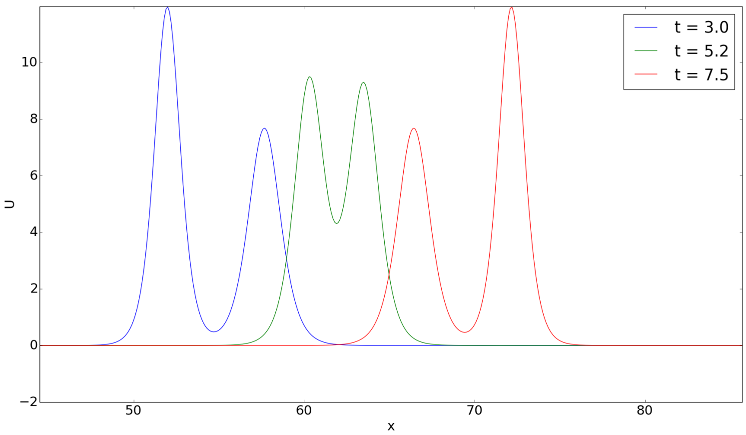

Figure 6: Graph showing amplitude against position of two similar sized solitons.

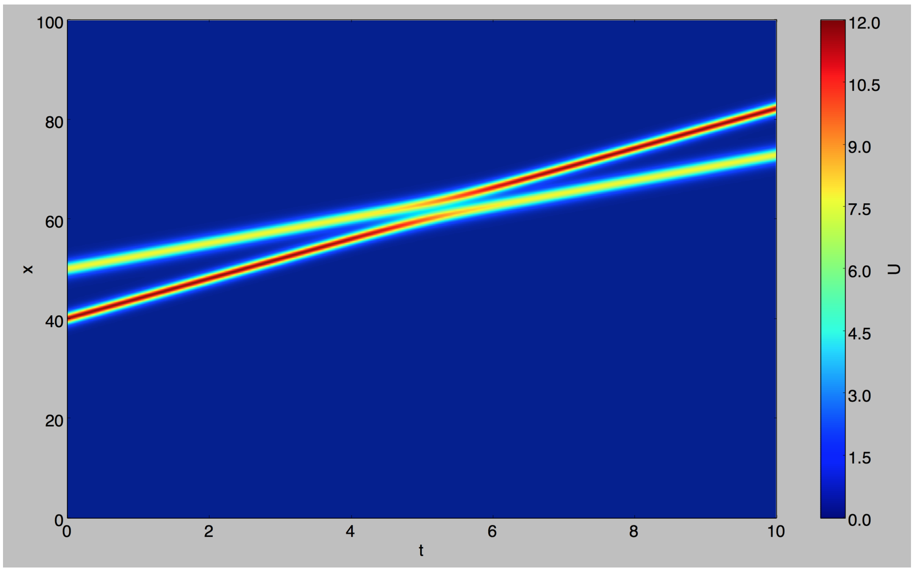

Figure 7: Space-time diagram showing amplitude of two similar sized solitons.

figures 8 and 9, where a sinusoid was used as the initial condition.

This behaviour is due to solitons of different amplitudes being normal modes of the KDV equation allowing for any arbitrary function to be composed from some superposition of solitons. This is analogous to different fundamental harmonics being the normal modes of the standard wave equation.However, this analogy is crucially different as in the case of the standard wave equation the superposition is linear, which is not the case with solitons which are superposing in a nonlinear manner.

4.3 Shockwaves

When solitons were propagated under the inviscid Burgers' equation it was found that they exhibit the standard shockwave behaviour, as can be seen in figure 10, with the parts of the wave with higher amplitude travelling at a higher velocity. However, due to the lack of diffusion in the system this causes the higher amplitude part of the wave to catch up with the front part of the waveform. This creates an infinite derivative with respect to space in the equation which this simulation cannot handle causing instabilities somewhat akin to the Gibbs phenomena being formed and propagating which can be seen in figure 10 at $t = 5$ and $t = 8$. If this simulation was physically accurate then the function would become multivalued creating a shape not dissimilar to an ocean wave breaking.

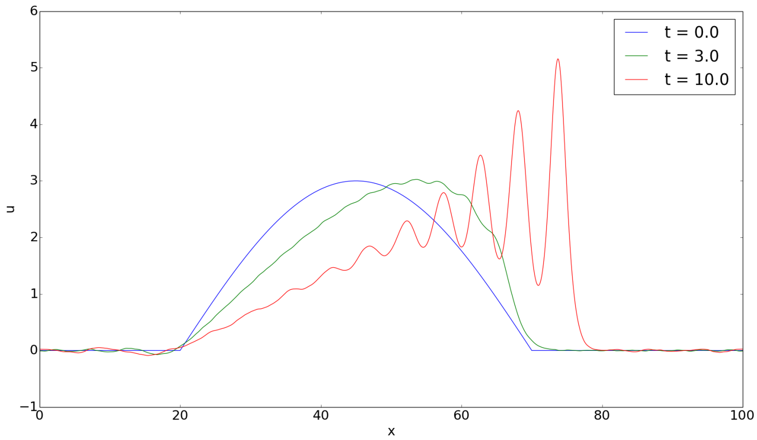

Figure 8: Sinusoidal initial condition breaking up into a soliton train.

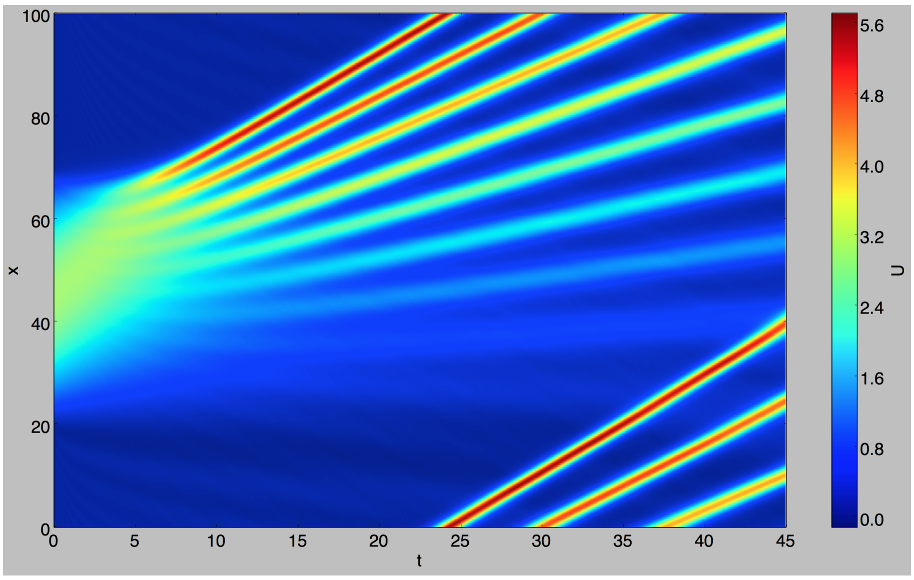

Figure 9: Space-time plot showing the propagation of a soliton train from an initial sinusoid.

When a diffusion is added to the system this models the propagation of a shockwave through a viscous medium. The soliton now propagates with characteristic shockwave shape as expected with diffusion acting to spread the waveform so that the special derivative can never become infinite. An example of this can be seen in figure 11.

In the case of the diffuse shockwave it was found that the mass of the solution was indeed conserved, however the momentum was not. The momentum was found to drop over time at exactly the same rate at which the amplitude drops. This shows that the change in amplitude is directly proportional to the change in momentum of the diffuse shockwave. One can see this by comparison between figures 12 and 13.

5 Conclution

The physics of solitons and shockwaves has very many important application all areas of physics. Numerical methods were devised to solve the KDV and Burgers' equation and study there solutions. It was found that solitons of different heights where in fact normal mode of the KDV equation allowing arbitrary function to be composed into an N-soliton solution of the KDV equation. It was shown that the inviscid case of the equation could not be solved by finite difference method and would need a much more complicated numerical method to be

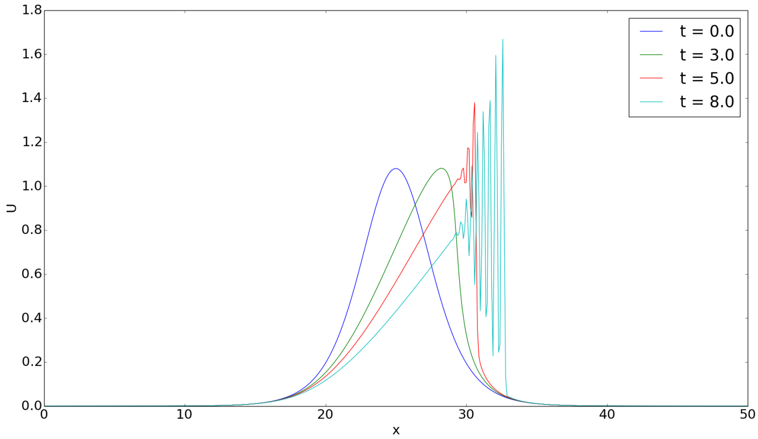

Figure 10: Propagation of soliton under invscid Burgers' equation.

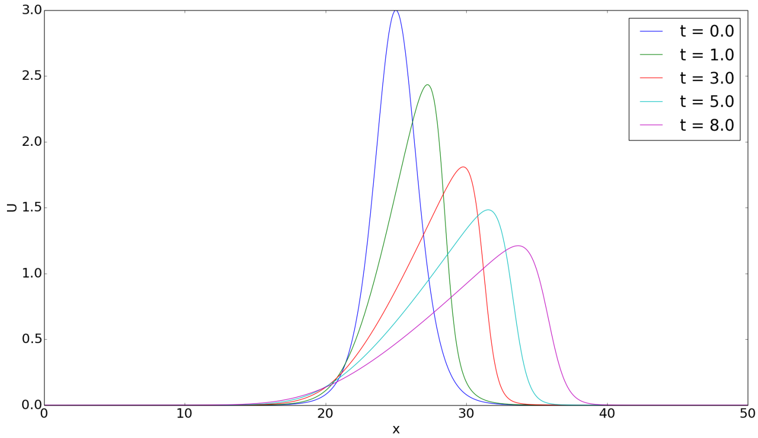

Figure 11: Propagation of a soliton under the diffusive Burgers' equation.

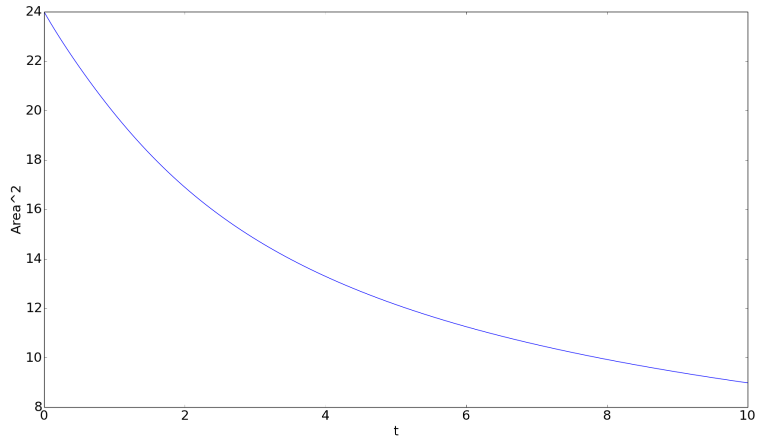

Figure 12: Diffusion of momenta from the diffuse shockwave.

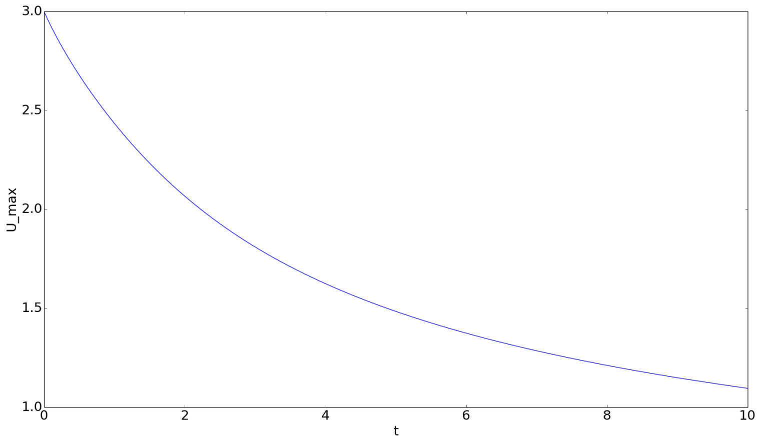

Figure 13: Change in amplitude of the diffuse shockwave.

solved computationally. In future it would be interesting to quantify the phase shift exhibited during the collation of solitons of different amplitudes.

References

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