Predicting Boston Housing Prices

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1. INTRODUCTION

1.1.1 Context

This project falls within the domains of real estate analytics, urban economics, and urban planning. It focuses on understanding the factors that influence housing prices in Boston by analyzing a dataset derived from a census survey conducted in the 1970s. The dataset includes 13 features that may impact the value of homes in different neighborhoods. Our goal is to build regression models to predict the median value of owner-occupied homes (measured in thousands of dollars) and to identify which features have the most significant effect on housing prices.

1.1.2 Problem

The problem we aim to address is the difficulty in understanding which features most influence housing prices in Boston. Without clear insights into what drives property values, it becomes challenging for prospective buyers, analysts, or stakeholders to make informed decisions. Our objective is to develop a predictive model that estimates housing prices and helps identify the key factors contributing to those predictions.

1.1.3 Challenges

The assumptions that we ran showed that the data was not normal and had heteroscedasticity. Another challenge that we faced was choosing between our full interaction model and our reduced interaction model. The ANOVA tests said that we should use the full interactive model even though the reduced had all significant terms and interactions while the full interactive model had some insignificant terms and interactions. This is supported by the respective Adjusted R squared and RSE values for both models.

1.2 OBJECTIVES

1.2.1 Overview

The housing market has been unpredictable in recent years, with prices rising in many areas and making it harder for people, especially young or first-time buyers, to afford a home. In this project, we're working with housing data from Boston to build a model that can predict house prices. By exploring which features of a home are most strongly linked to its value, we hope to better understand what drives housing prices and help future buyers know what to look for.

1.2.2 Goals & Research Questions

The primary goal of this project is to build a predictive model that accurately estimates housing prices in Boston based on various property features. In doing so, we aim to uncover which features most strongly influence a home's value.

To guide this objective, we explore the following research questions:

- Can we develop a reliable model to predict the median value of homes in Boston?
- Which features are the most important in influencing housing prices?

2. METHODOLOGY

2.1 Data

The dataset used in this project consists of housing data collected from the Boston Standard Metropolitan Statistical Area (SMSA) in the 1970s. It contains 506 entries and includes 11 qualitative independent variables, 2 quantitative independent variables, and 1 quantitative dependent variable.

The dataset was originally collected as part of a census report and is considered open data. It is publicly available at:
https://lib.stat.cmu.edu/datasets/boston

Below is a brief description of each variable:

- **CRIM**: Per capita crime rate by town. Indicates the level of crime in the area.
- **ZN**: Proportion of residential land zoned for lots over 25,000 sq.ft. Reflects residential density.
- **INDUS**: Proportion of non-retail business acres per town. Indicates commercial land usage.
- **CHAS**: Charles River dummy variable (1 if tract bounds river; 0 otherwise). Indicates proximity to the Charles River.
- **NOX**: Nitric oxides concentration (parts per 10 million). Represents industrial pollution.
- **RM**: Average number of rooms per dwelling. Suggests spaciousness.
- **AGE**: Proportion of owner-occupied units built prior to 1940. Reflects the age of buildings in the area.
- **DIS**: Weighted distances to five Boston employment centres. Measures accessibility to work locations.
- **RAD**: Index of accessibility to radial highways. Higher values indicate better road access.
- **TAX**: Full-value property-tax rate per $10,000. Indicates the annual property tax burden.
- **PTRATIO**: Pupil-teacher ratio by town. Lower values suggest better educational facilities.
- **B**: $1000(Bk - 0.63)^2$, where Bk is the proportion of Black residents by town.
- **LSTAT**: Percentage of the population considered lower status.
- **MEDV**: Median value of owner-occupied homes in $1000s. This is the dependent variable we aim to predict.[1]

2.2 Approach

In this project, we use a **predictive modeling approach** based on **multiple linear regression** to estimate the median value of homes in Boston. This method is well-suited for our goal of understanding how different housing features influence prices while also producing accurate predictions.

We believe multiple linear regression is an effective choice for this project for several reasons:

- **Interpretability**: The model provides clear and meaningful insights into how each variable affects housing prices, which is valuable for both analysis and decision-making.

- **No Multicollinearity**: VIF results confirmed the absence of multicollinearity, supporting the reliability and stability of the regression coefficients.

- **Well-Structured Data**: The dataset consists of numeric features that align well with the assumptions of linear regression, making it a natural modeling choice.

- **Proven Technique**: Linear regression is a widely accepted method in real estate analytics, with a long history of successful application in similar predictive tasks.
- **Strong Baseline**: This approach serves as a solid baseline for future comparisons with more complex models if needed, offering a balance of performance and simplicity.

2.3 Workflows

Below are the key steps:

A. Test for Multicollinearity

B. Create the Best Additive Model

C. Create the Best Interaction Model

D. Explore Higher Order Terms

E. Test Multiple Regression Assumptions (linearity, independence, equal variance, normality, outliers)

2.4 Contributions

1. Akinyemi Apampa

- Introduction writing: Context, Problem, Challenges
- Initial data preprocessing and cleaning
- Creating the additive regression model and interpreting results
- Assisting with final report compilation and proofreading

2. Ravin Jayasuriya

- Multicollinearity testing and VIF analysis
- Stepwise regression and reduced additive model selection
- Writing and interpreting results of the additive model (t-tests, F-tests)
- Assisting with final report compilation and proofreading

3. Joshua Ogunbo

- Developing and refining the full and reduced interaction models
- Conducting and interpreting ANOVA tests for interaction models
- Writing the section on interaction terms and model selection justification
- Assisting with final report compilation and proofreading

4. Prince Oloma Eworitsemoghan

- Higher-order term exploration and model building (crim, zn, nox, rm, dis, rad, tax, lstat)
- Identifying and incorporating significant polynomial terms
- Writing the section on higher-order terms and their interpretation
- Assisting with final report compilation and proofreading

5. David Fakojulo

- Addressing model assumption violations: log, Box-Cox, and WLS transformations
- Finalizing the best regression model using weighted least squares
- Performing and interpreting Shapiro-Wilk and Breusch-Pagan tests
- Assisting with final report compilation and proofreading

3. MAIN RESULT OF THE ANALYSIS

Data Import and Initial Inspection

Table 1: First Three Rows of the Boston Housing Dataset

crim	zn	indus	chas	nox	rm	age	dis	rad	tax	ptratio	b	lstat	medv
0.00632	18	2.31	0	0.538	6.575	65.2	4.0900	1	296	15.3	396.90	4.98	24.0
0.02731	0	7.07	0	0.469	6.421	78.9	4.9671	2	242	17.8	396.90	9.14	21.6
0.02729	0	7.07	0	0.469	7.185	61.1	4.9671	2	242	17.8	392.83	4.03	34.7

Recoding Categorical Variables for Model Compatibility

Table 2: First Three Rows of the Boston Housing Dataset

crim	zn	indus	chas	nox	rm	age	dis	rad	tax	ptratio	b	lstat	medv
0.00632	18	2.31	No	0.538	6.575	65.2	4.0900	1	296	15.3	396.90	4.98	24.0
0.02731	0	7.07	No	0.469	6.421	78.9	4.9671	2	242	17.8	396.90	9.14	21.6
0.02729	0	7.07	No	0.469	7.185	61.1	4.9671	2	242	17.8	392.83	4.03	34.7

A. **Test for Multicollinearity**
From the below, multicollinearity was not detected for any of the variables.

```
boston_additive <- lm(
  medv ~ crim + zn + indus + chas + nox + rm + age +
    dis + rad + tax + ptratio + b + lstat,
  data = boston_data
)
```

Table 3: TEST FOR MULTICOLINEARITY

Variable.Name	VIF	Detection
crim	1.7922	0
zn	2.2928	0
indus	3.9916	0
chas	1.0740	0
nox	4.3937	0
rm	1.9337	0
age	3.1008	0
dis	3.9559	0
rad	7.4845	0
tax	9.0086	0
ptratio	1.7991	0
b	1.3485	0
lstat	2.9415	0

The **Variance Inflation Factor (VIF)** values for all predictors in the model are below the commonly used threshold of **10**, indicating that **multicollinearity is not a concern**.

B. **Create the Best Additive Model**

We perform individual t-tests to assess the significance of each predictor in the model. The hypotheses are as follows:

- **Null Hypothesis**:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$$

- **Alternative Hypothesis**:

$$H_a : \text{At least one } \beta_i \neq 0, \quad \text{where } i = 1, 2, \dots, p$$

These tests help determine whether each predictor contributes significantly to explaining the variability in the response variable.

Table 4: Coefficient Estimates from the Additive Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	36.4595	5.1035	7.1441	0.0000
crim	-0.1080	0.0329	-3.2865	0.0011
zn	0.0464	0.0137	3.3816	0.0008
indus	0.0206	0.0615	0.3343	0.7383
chasYes	2.6867	0.8616	3.1184	0.0019
nox	-17.7666	3.8197	-4.6513	0.0000
rm	3.8099	0.4179	9.1161	0.0000
age	0.0007	0.0132	0.0524	0.9582
dis	-1.4756	0.1995	-7.3980	0.0000
rad	0.3060	0.0663	4.6129	0.0000
tax	-0.0123	0.0038	-3.2800	0.0011
ptratio	-0.9527	0.1308	-7.2825	0.0000
b	0.0093	0.0027	3.4668	0.0006
lstat	-0.5248	0.0507	-10.3471	0.0000

From the regression output, we observe:

- **Significant predictors** (p-value < 0.05): `crim`, `zn`, `chas`, `nox`, `rm`, `dis`, `rad`, `tax`, `ptratio`, `b`, `lstat`
- **Insignificant predictors**: `indus` (p = 0.7383) and `age` (p = 0.9582)

Because the p-values of `indus` and `age` are greater than 0.05, we fail to reject the null hypotheses that their coefficients are zero. Thus, these variables do not significantly contribute to the prediction of `medv` in the presence of other variables.

The fitted multiple linear regression model is:

$$\begin{aligned}
\widehat{medv} = 3.646 &- 0.108 \cdot crim + 0.04642 \cdot zn \\
&+ 0.02056 \cdot indus + 2.687 \cdot chas - 17.7666 \cdot nox \\
&+ 3.8099 \cdot rm + 0.0007 \cdot age - 1.4756 \cdot dis \\
&+ 0.3060 \cdot rad - 0.0123 \cdot tax - 0.9527 \cdot ptratio \\
&+ 0.0093 \cdot b - 0.5248 \cdot lstat
\end{aligned}$$

Building the Reduced Additive model

We remove the variables `indus` and `age`, which were found to be statistically insignificant in the full model, and test the following hypotheses:

- **Null Hypothesis**:

$$H_0 : \beta_{\text{indus}} = \beta_{\text{age}} = 0$$

- **Alternative Hypothesis**:

$$H_a : \text{At least one of } \beta_{\text{indus}}, \beta_{\text{age}} \neq 0$$

  A high p-value would indicate that removing `indus` and `age` does not significantly worsen the model, and thus the reduced model is preferred for its simplicity.

```
reduced_additive_model <- lm(
  formula = medv ~ crim + zn + factor(chas) + nox + rm +
    dis + rad + tax + ptratio + b + lstat,
  data = boston_data
)
```

Table 5: Coefficient Estimates from the Reduced Additive Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	36.3411	5.0675	7.1714	0.0000
crim	-0.1084	0.0328	-3.3074	0.0010
zn	0.0458	0.0135	3.3902	0.0008
factor(chas)Yes	2.7187	0.8542	3.1826	0.0016
nox	-17.3760	3.5352	-4.9151	0.0000
rm	3.8016	0.4063	9.3562	0.0000
dis	-1.4927	0.1857	-8.0370	0.0000
rad	0.2996	0.0634	4.7255	0.0000
tax	-0.0118	0.0034	-3.4925	0.0005
ptratio	-0.9465	0.1291	-7.3337	0.0000
b	0.0093	0.0027	3.4746	0.0006
lstat	-0.5226	0.0474	-11.0187	0.0000

The reduced model shows that **all included variables have p-values less than 0.05**, indicating that they are statistically significant.

We will now run a **global F-test (ANOVA)** to assess whether removing the variables `indus` and `age` significantly worsens the model fit.

Table 6: ANOVA Comparison of Reduced and Full Additive Model

Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
494	11081.36	NA	NA	NA	NA
492	11078.78	2	2.5794	0.0573	0.9443

Since the p-value is **0.9443**, which is much greater than the threshold of **0.05**, we **fail to reject the null hypothesis**. This indicates that `indus` and `age` do not significantly improve the model. Hence, the reduced model is more appropriate as it maintains model quality while eliminating unnecessary predictors.

Stepwise Model Selection**

To further validate the choice of predictors and identify the most parsimonious model, we used the `ols_step_both_p()` function from the **olsrr** package to perform stepwise selection based on p-values.

Table 7: Coefficient Estimates from the Stepwise-Selected Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	36.3411	5.0675	7.1714	0.0000
lstat	-0.5226	0.0474	-11.0187	0.0000
rm	3.8016	0.4063	9.3562	0.0000
ptratio	-0.9465	0.1291	-7.3337	0.0000
dis	-1.4927	0.1857	-8.0370	0.0000
nox	-17.3760	3.5352	-4.9151	0.0000
chasYes	2.7187	0.8542	3.1826	0.0016
b	0.0093	0.0027	3.4746	0.0006
zn	0.0458	0.0135	3.3902	0.0008
crim	-0.1084	0.0328	-3.3074	0.0010
rad	0.2996	0.0634	4.7255	0.0000
tax	-0.0118	0.0034	-3.4925	0.0005

The selected model is the same as the reduced model obtained earlier. Hence, the **final additive model is**:

$$\begin{aligned}\widehat{medv}_i = 36.3411 &- 0.5226 \cdot lstat_i + 3.8016 \cdot rm_i \\ &- 0.9465 \cdot ptratio_i - 1.4927 \cdot dis_i - 17.3760 \cdot nox_i + 2.7187 \cdot chas_i \\ &+ 0.0093 \cdot b_i + 0.0458 \cdot zn_i - 0.1084 \cdot crim_i + 0.2996 \cdot rad_i - 0.0118 \cdot tax_i\end{aligned}$$

Where:

$$chas_i = 1 \quad \text{if the tract bounds the Charles River, and 0 otherwise}$$

C. **Create the Best Interaction Model**

A full two-way interaction model was constructed by including all possible interaction terms among the predictors in the final additive model obtained from stepwise selection.

Table 8: Coefficient Estimates from the Full Two-Way Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-61.6850	56.6084	-1.0897	0.2765
crim	-7.4081	5.9098	-1.2535	0.2107
zn	-0.2549	0.4276	-0.5960	0.5515
chasYes	42.5330	18.6500	2.2806	0.0231
nox	-43.7416	56.4059	-0.7755	0.4385
rm	21.1759	5.1167	4.1386	0.0000
dis	-7.8290	4.0307	-1.9423	0.0527
rad	4.6098	2.1044	2.1905	0.0290
tax	-0.1399	0.0953	-1.4684	0.1427
ptratio	2.5864	2.3119	1.1188	0.2639
b	0.0897	0.0754	1.1903	0.2346
lstat	2.1385	0.8089	2.6437	0.0085

	Estimate	Std. Error	t value	Pr(>|t|)
crim:zn	0.3038	0.1600	1.8983	0.0583
crim:chasYes	2.5732	0.6020	4.2748	0.0000
crim:nox	-1.9333	0.9640	-2.0054	0.0455
crim:rm	0.1951	0.0503	3.8746	0.0001
crim:dis	-0.1912	0.0941	-2.0321	0.0427
crim:rad	-0.5192	0.1850	-2.8067	0.0052
crim:tax	0.0340	0.0108	3.1494	0.0017
crim:ptratio	-0.1428	0.2304	-0.6198	0.5357
crim:b	-0.0004	0.0002	-2.5069	0.0125
crim:lstat	0.0239	0.0069	3.4591	0.0006
zn:chasYes	-0.0448	0.0540	-0.8286	0.4078
zn:nox	-0.4167	0.3969	-1.0499	0.2943
zn:rm	0.0157	0.0246	0.6358	0.5252
zn:dis	0.0138	0.0058	2.3906	0.0172
zn:rad	-0.0050	0.0067	-0.7493	0.4541
zn:tax	0.0004	0.0002	2.4499	0.0147
zn:ptratio	0.0012	0.0062	0.1916	0.8482
zn:b	0.0004	0.0007	0.5477	0.5842
zn:lstat	-0.0054	0.0039	-1.3666	0.1725
chasYes:nox	-31.0752	12.3380	-2.5187	0.0121
chasYes:rm	-4.3065	1.1439	-3.7649	0.0002
chasYes:dis	0.5251	1.3608	0.3858	0.6998
chasYes:rad	-0.5426	0.4017	-1.3507	0.1775
chasYes:tax	0.0230	0.0260	0.8830	0.3777
chasYes:ptratio	-0.5936	0.6712	-0.8843	0.3770
chasYes:b	0.0251	0.0154	1.6298	0.1039
chasYes:lstat	-0.3566	0.1681	-2.1213	0.0345
nox:rm	3.2483	5.2601	0.6175	0.5372
nox:dis	4.1538	2.9521	1.4071	0.1601
nox:rad	-2.3556	1.2452	-1.8918	0.0592
nox:tax	0.1479	0.0692	2.1370	0.0331
nox:ptratio	-1.4380	2.0307	-0.7081	0.4793
nox:b	-0.0556	0.0356	-1.5616	0.1191
nox:lstat	0.3077	0.6089	0.5054	0.6135
rm:dis	0.5584	0.2886	1.9352	0.0536
rm:rad	-0.1058	0.1229	-0.8611	0.3896
rm:tax	-0.0102	0.0067	-1.5122	0.1312
rm:ptratio	-0.5236	0.2146	-2.4400	0.0151
rm:b	-0.0072	0.0035	-2.0529	0.0407
rm:lstat	-0.3198	0.0436	-7.3395	0.0000
dis:rad	-0.1617	0.0551	-2.9348	0.0035
dis:tax	0.0003	0.0023	0.1085	0.9137
dis:ptratio	0.1152	0.0909	1.2675	0.2056
dis:b	-0.0022	0.0055	-0.3951	0.6929
dis:lstat	0.0778	0.0388	2.0053	0.0455
rad:tax	-0.0003	0.0006	-0.5635	0.5734
rad:ptratio	-0.0274	0.0799	-0.3430	0.7318
rad:b	-0.0018	0.0022	-0.8171	0.4143
rad:lstat	-0.0196	0.0147	-1.3373	0.1818
tax:ptratio	0.0035	0.0020	1.7336	0.0837
tax:b	0.0001	0.0002	0.9948	0.3204
tax:lstat	-0.0015	0.0009	-1.6672	0.0962

	Estimate	Std. Error	t value	Pr(>|t|)
ptratio:b	-0.0014	0.0025	-0.5611	0.5750
ptratio:lstat	0.0050	0.0260	0.1909	0.8487
b:lstat	-0.0013	0.0004	-3.0059	0.0028

A full two-way interaction model was constructed by including all possible interaction terms among the predictors in the final additive model obtained from stepwise selection.

The full interaction model includes several interaction terms with **p-values greater than 0.05**, indicating that they are not statistically significant. These insignificant interactions were dropped to create a **reduced interaction model**.

Table 9: Coefficient Estimates from the Reduced Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-90.6158	17.8213	-5.0847	0.0000
crim	-9.2738	2.7963	-3.3164	0.0010
zn	-0.1636	0.0494	-3.3130	0.0010
chasYes	45.8055	9.1403	5.0114	0.0000
nox	-24.4974	13.2152	-1.8537	0.0644
rm	21.9919	2.4669	8.9148	0.0000
dis	-8.6346	1.2444	-6.9387	0.0000
rad	2.2883	0.4760	4.8077	0.0000
tax	-0.0632	0.0229	-2.7586	0.0060
ptratio	5.3595	0.7317	7.3245	0.0000
b	0.0542	0.0249	2.1811	0.0297
lstat	1.6647	0.2839	5.8644	0.0000
crim:zn	0.2722	0.1074	2.5353	0.0116
crim:chasYes	2.0022	0.3306	6.0571	0.0000
crim:nox	0.4397	0.8537	0.5150	0.6068
crim:rm	0.0882	0.0486	1.8136	0.0704
crim:dis	-0.1332	0.0849	-1.5693	0.1173
crim:rad	-0.5622	0.1381	-4.0709	0.0001
crim:tax	0.0332	0.0089	3.7340	0.0002
crim:b	-0.0003	0.0002	-1.8384	0.0666
crim:lstat	-0.0044	0.0061	-0.7090	0.4787
zn:dis	0.0165	0.0043	3.8431	0.0001
zn:tax	0.0002	0.0001	1.5833	0.1140
chasYes:nox	-30.4724	6.0576	-5.0304	0.0000
chasYes:rm	-4.1716	1.1015	-3.7871	0.0002
chasYes:lstat	-0.2236	0.1440	-1.5524	0.1212
nox:rad	-2.5610	0.7488	-3.4202	0.0007
nox:tax	0.0802	0.0416	1.9281	0.0544
rm:dis	1.0300	0.1710	6.0237	0.0000
rm:ptratio	-0.9099	0.1138	-7.9928	0.0000
rm:b	-0.0029	0.0035	-0.8188	0.4133
rm:lstat	-0.3691	0.0404	-9.1312	0.0000
dis:rad	-0.1151	0.0322	-3.5722	0.0004
dis:lstat	0.1614	0.0216	7.4692	0.0000
b:lstat	-0.0015	0.0004	-3.9483	0.0001

The output of the reduced interactive model showed more insignificant interactions, which were further dropped.

Table 10: Table: Coefficient Estimates from the Final Reduced Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-75.3241	14.9000	-5.0553	0.0000
crim	-9.5963	2.7403	-3.5019	0.0005
zn	-0.0998	0.0313	-3.1898	0.0015
chasYes	38.2243	6.8141	5.6096	0.0000
nox	-17.5242	11.4888	-1.5253	0.1278
rm	19.3263	2.0435	9.4576	0.0000
dis	-7.8883	1.1719	-6.7312	0.0000
rad	2.1207	0.3763	5.6351	0.0000
tax	-0.0516	0.0180	-2.8597	0.0044
ptratio	4.7316	0.7148	6.6191	0.0000
b	0.0292	0.0062	4.6879	0.0000
lstat	1.2139	0.2083	5.8276	0.0000
crim:zn	0.2777	0.1071	2.5937	0.0098
crim:chasYes	2.2138	0.3262	6.7860	0.0000
crim:rad	-0.5802	0.1397	-4.1522	0.0000
crim:tax	0.0351	0.0090	3.9025	0.0001
zn:dis	0.0162	0.0043	3.7206	0.0002
chasYes:nox	-33.5288	5.9660	-5.6200	0.0000
chasYes:rm	-3.1508	0.7824	-4.0269	0.0001
nox:rad	-2.3342	0.6145	-3.7983	0.0002
nox:tax	0.0627	0.0342	1.8329	0.0674
rm:dis	0.9348	0.1624	5.7565	0.0000
rm:ptratio	-0.8125	0.1115	-7.2861	0.0000
rm:lstat	-0.3117	0.0346	-9.0205	0.0000
dis:rad	-0.1304	0.0268	-4.8642	0.0000
dis:lstat	0.1592	0.0201	7.9221	0.0000
b:lstat	-0.0012	0.0003	-4.1913	0.0000

The further reduced model then showed nox:tax as insignificant, which was also dropped.

Table 11: Table: Coefficient Estimates from the Third Reduced Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-83.3729	14.2731	-5.8413	0.0000
crim	-11.1718	2.6084	-4.2829	0.0000
zn	-0.0897	0.0309	-2.9052	0.0038
chasYes	38.1964	6.8308	5.5918	0.0000
nox	1.2029	5.2663	0.2284	0.8194
rm	19.0011	2.0407	9.3109	0.0000
dis	-8.0659	1.1708	-6.8895	0.0000
rad	1.6045	0.2502	6.4124	0.0000
tax	-0.0190	0.0030	-6.4174	0.0000
ptratio	4.6552	0.7154	6.5072	0.0000
b	0.0292	0.0063	4.6741	0.0000
lstat	1.2469	0.2080	5.9935	0.0000
crim:zn	0.2484	0.1061	2.3409	0.0196
crim:chasYes	2.2155	0.3270	6.7746	0.0000

	Estimate	Std. Error	t value	Pr(>|t|)
crim:rad	-0.6602	0.1331	-4.9616	0.0000
crim:tax	0.0403	0.0085	4.7219	0.0000
zn:dis	0.0149	0.0043	3.4648	0.0006
chasYes:nox	-33.6871	5.9800	-5.6333	0.0000
chasYes:rm	-3.1157	0.7841	-3.9736	0.0001
nox:rad	-1.3857	0.3323	-4.1705	0.0000
rm:dis	0.9675	0.1618	5.9796	0.0000
rm:ptratio	-0.7967	0.1115	-7.1481	0.0000
rm:lstat	-0.3161	0.0346	-9.1449	0.0000
dis:rad	-0.1239	0.0266	-4.6515	0.0000
dis:lstat	0.1581	0.0201	7.8530	0.0000
b:lstat	-0.0012	0.0003	-4.1852	0.0000

Now that all interactions were significant, an f-test was run to compare reduced interactive model and full interaction model

Table 12: ANOVA Comparison of Third Reduced Interaction Model and Full Interaction Model

Res.Df	RSS	Df	Sum.of.Sq	F	Pr..F.
480	5573.760	NA	NA	NA	NA
439	4159.123	41	1414.638	3.641868	0

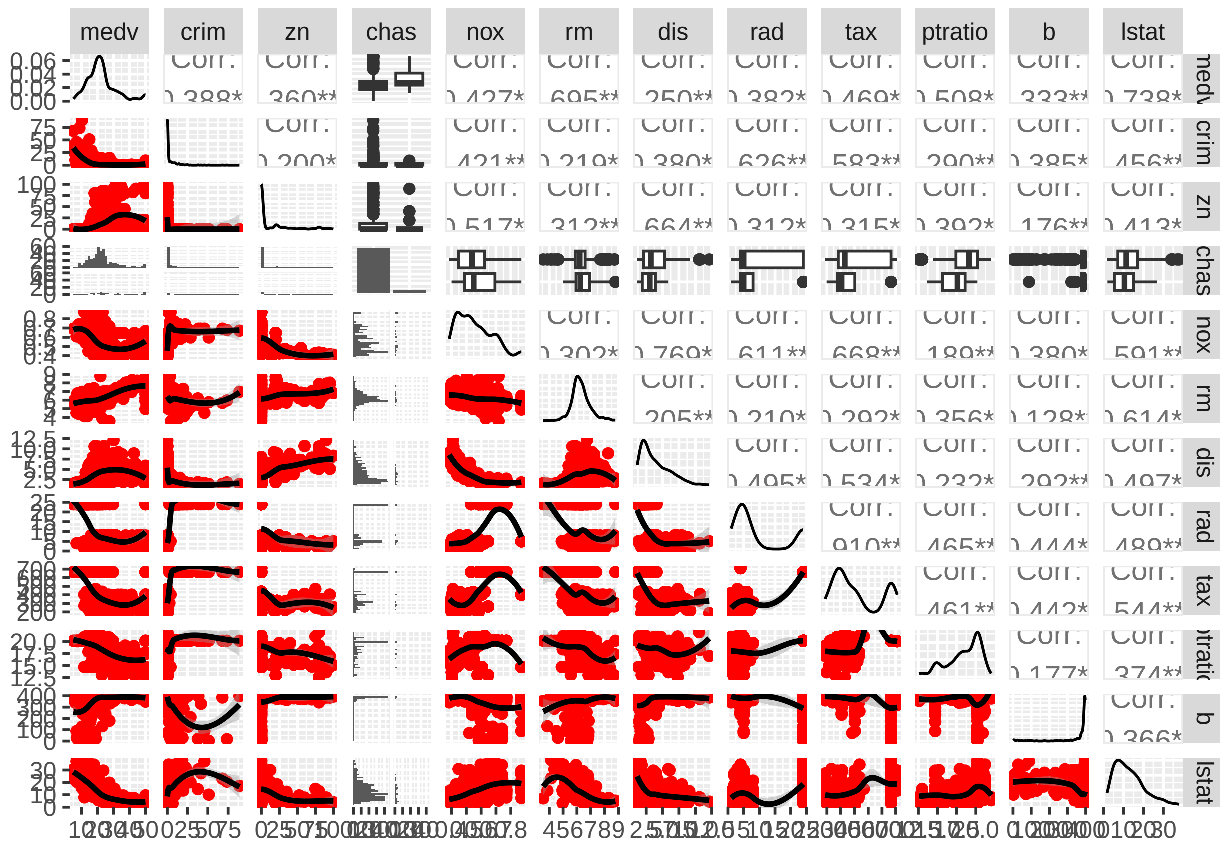
Based on the summary above, the p-value is $8.242e - 12 < 0.05$, suggesting the null hypothesis should be rejected. Also, the adjusted R^2_{adj} and RSE of the full interaction model are 0.888 and 3.078 respectively, while the adjusted R^2_{adj} and RSE of the reduced interaction model are 0.8627 and 3.408 respectively.

These suggest the full interaction model should be preferred. However, the full interaction model has a number of insignificant interactions, while the reduced interaction model has only significant interactions. Even though the anova test, adjusted R^2_{adj} and RSE suggest preferring the full interaction model, there isn't a major difference between the adjusted R^2_{adj} and RSE of the two models.

We would choose the reduced model because it retains all significant interactions while eliminating insignificant ones, ensuring better interpretability and avoiding unnecessary complexity without a substantial loss in explanatory power.

D. **Explore Higher Order Terms**

To check for possible higher order relationships, we explored all pairwise combinations of continuous variables in scatterplots to see how the response variable looked with respect to each of the continuous additive predictors



It looks like the variables that might be worth exploring for possible higher-order relationships with `medv` are: `crim`, `zn`, `nox`, `rm`, `dis`, `rad`, `tax`, and `lstat`. Each of these variables was tested using second or higher-degree polynomial terms.

Based on the exploration of potential higher-order relationships for variables such as `crim`, `zn`, `nox`, `rm`, `dis`, `rad`, `tax`, and `lstat`, only the **significant polynomial terms** were retained. These significant higher-order terms were then added to the **reduced interaction model** to form an extended model that captures potential **non-linear effects** while preserving **interpretability**.

Table 13: Coefficient Estimates from the Higher-Order Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	9.9501	20.6459	0.4819	0.6301
crim	-7.6931	2.6257	-2.9300	0.0036
I(crim^2)	0.0033	0.0010	3.4291	0.0007
zn	-0.0274	0.0449	-0.6094	0.5425
chasYes	40.8059	6.4722	6.3049	0.0000
nox	-50.9960	20.7700	-2.4553	0.0144
I(nox^2)	37.3520	15.9688	2.3391	0.0197
rm	5.1164	4.4713	1.1443	0.2531
I(rm^2)	0.8483	0.2501	3.3914	0.0008
dis	-9.5208	1.3265	-7.1776	0.0000
I(dis^2)	0.1950	0.0668	2.9183	0.0037
rad	1.4647	0.2533	5.7819	0.0000
tax	-0.0168	0.0028	-6.0234	0.0000
ptratio	4.5064	0.7377	6.1088	0.0000
b	0.0219	0.0061	3.5628	0.0004
lstat	-5.6167	1.1499	-4.8846	0.0000
I(lstat^2)	0.6351	0.1467	4.3296	0.0000
I(lstat^3)	-0.0354	0.0094	-3.7866	0.0002
I(lstat^4)	0.0009	0.0003	3.4318	0.0007
I(lstat^5)	0.0000	0.0000	-3.1448	0.0018
crim:zn	0.1002	0.1224	0.8188	0.4133
crim:chasYes	2.2145	0.3103	7.1357	0.0000
crim:rad	-0.4371	0.1336	-3.2712	0.0011
crim:tax	0.0267	0.0086	3.1043	0.0020
zn:dis	0.0025	0.0066	0.3808	0.7036
chasYes:nox	-35.3766	5.6715	-6.2377	0.0000
chasYes:rm	-3.4379	0.7404	-4.6434	0.0000
nox:rad	-1.2613	0.3207	-3.9325	0.0001
rm:dis	0.8557	0.1539	5.5598	0.0000
rm:ptratio	-0.7816	0.1137	-6.8745	0.0000
rm:lstat	-0.0730	0.0547	-1.3346	0.1826
dis:rad	-0.0913	0.0287	-3.1778	0.0016
dis:lstat	0.1575	0.0209	7.5254	0.0000
b:lstat	-0.0008	0.0003	-2.7996	0.0053

The new insignificant variables were then dropped to get a reduced higher order interaction model.

Table 14: Coefficient Estimates from the Reduced Higher-Order Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	21.1228	18.1497	1.1638	0.2451
crim	-6.8087	2.3071	-2.9512	0.0033
I(crim^2)	0.0030	0.0009	3.2112	0.0014
chasYes	40.8986	6.4563	6.3347	0.0000

	Estimate	Std. Error	t value	Pr(>|t|)
nox	-45.6887	19.3695	-2.3588	0.0187
I(nox^2)	34.0115	15.0189	2.2646	0.0240
rm	2.0706	3.8551	0.5371	0.5914
I(rm^2)	1.0438	0.2129	4.9031	0.0000
dis	-9.6278	1.2806	-7.5182	0.0000
I(dis^2)	0.2099	0.0402	5.2174	0.0000
rad	1.4358	0.2495	5.7557	0.0000
tax	-0.0167	0.0027	-6.1010	0.0000
ptratio	4.5541	0.7094	6.4197	0.0000
b	0.0207	0.0061	3.4008	0.0007
lstat	-6.2120	0.9861	-6.2998	0.0000
I(lstat^2)	0.6456	0.1438	4.4887	0.0000
I(lstat^3)	-0.0359	0.0092	-3.8818	0.0001
I(lstat^4)	0.0009	0.0003	3.4969	0.0005
I(lstat^5)	0.0000	0.0000	-3.1780	0.0016
crim:chasYes	2.2772	0.3062	7.4366	0.0000
crim:rad	-0.3893	0.1162	-3.3501	0.0009
crim:tax	0.0237	0.0075	3.1680	0.0016
chasYes:nox	-35.8644	5.6434	-6.3551	0.0000
chasYes:rm	-3.4176	0.7377	-4.6328	0.0000
nox:rad	-1.3052	0.3164	-4.1248	0.0000
rm:dis	0.8439	0.1468	5.7485	0.0000
rm:ptratio	-0.7927	0.1070	-7.4074	0.0000
dis:rad	-0.0770	0.0271	-2.8443	0.0046
dis:lstat	0.1559	0.0206	7.5497	0.0000
b:lstat	-0.0007	0.0003	-2.5465	0.0112

The higher-order interaction model with significant variables is:

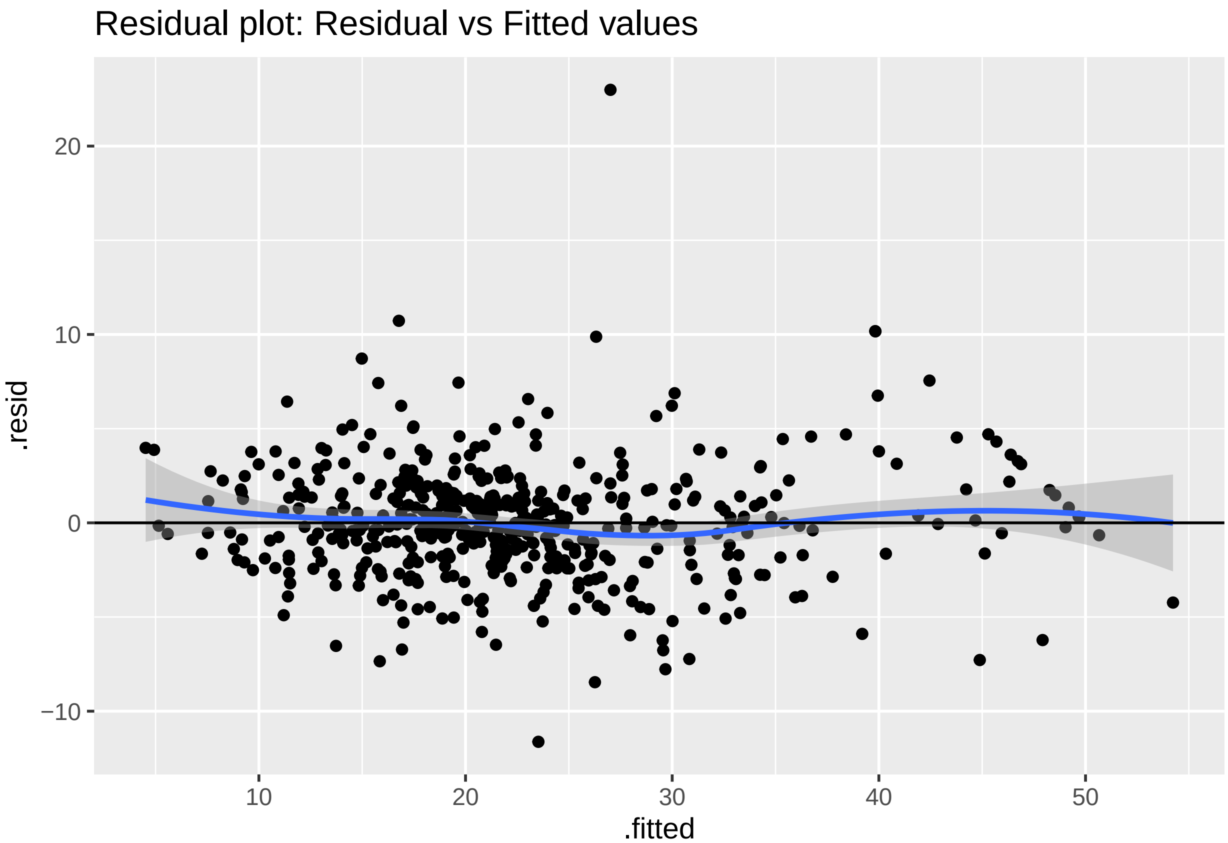
$$\begin{aligned} \widehat{medv}_i = 36.3411 - 0.1084 \cdot crim_i + 0.0458 \cdot zn_i + 2.7187 \cdot chas_i - 17.3760 \cdot nox_i \\ + 3.8016 \cdot rm_i - 1.4927 \cdot dis_i + 0.2996 \cdot rad_i - 0.0118 \cdot tax_i - 0.9465 \cdot ptratio_i \\ + 0.0093 \cdot b_i - 0.5226 \cdot lstat_i \end{aligned}$$

where $chas_i$ is 1 if the tract bounds Charles River and 0 if otherwise

E. **Testing Multiple Regression Assumptions**
The following multiple regression assumptions were tested to check if the model is trustworthy

Linearity Assumption

A scatter plot of the distribution of residuals (errors) vs fitted values (predicted values) was plotted below



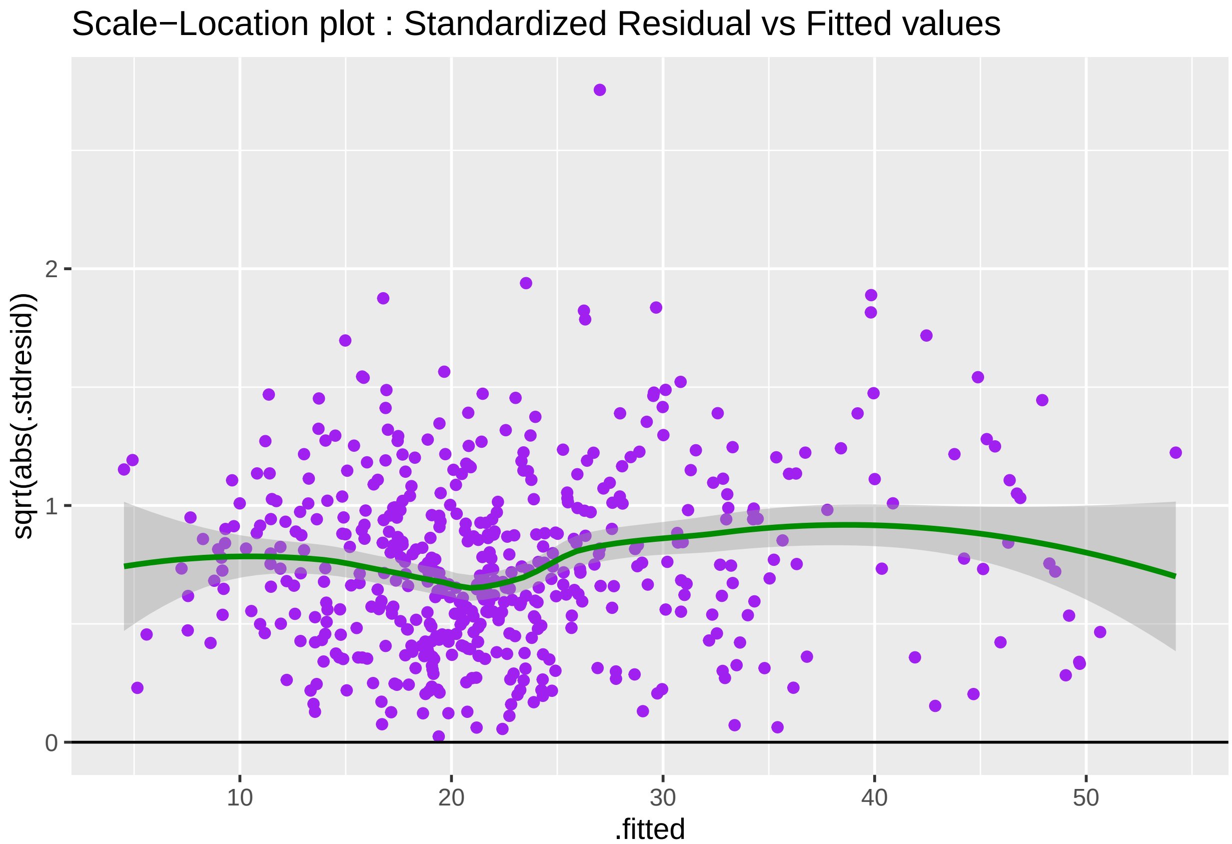
There appears to be no pattern of the residuals at all, indicating that the model passes the linearity assumption that there is a straight-line (linear) relationship between the predictors and the response.

Independence Assumption

In the Boston housing dataset, the subjects were not related to time, space, or group, so we can be pretty sure that their measurements are independent.

Equal Variance Assumption

The residuals plot in the linearity assumption section indicates a smooth fit to the residuals, which is good. In addition to the residuals plot, a scale-location plot between fitted values and standardized residuals was also plotted to show if the residuals are spread equally along the ranges of predictors



Based on the plot, we can see that the scale-location plot is quite horizontal, and there is not any funneling in the residual plot, indicating equal variance.

The Breusch-Pagan test was then run as a more formal way to assess if we have homo/heteroscedasticity using the following hypotheses:

$$
\begin{aligned}
H_0 &: \text{heteroscedasticity is not present (homoscedasticity)} \\
H_a &: \text{heteroscedasticity is present}
\end{aligned}
$$

```
bptest(higher_order_interaction_model_2)
```

```
##
##  studentized Breusch-Pagan test
##
## data:  higher_order_interaction_model_2
## BP = 70.605, df = 29, p-value = 2.505e-05
```

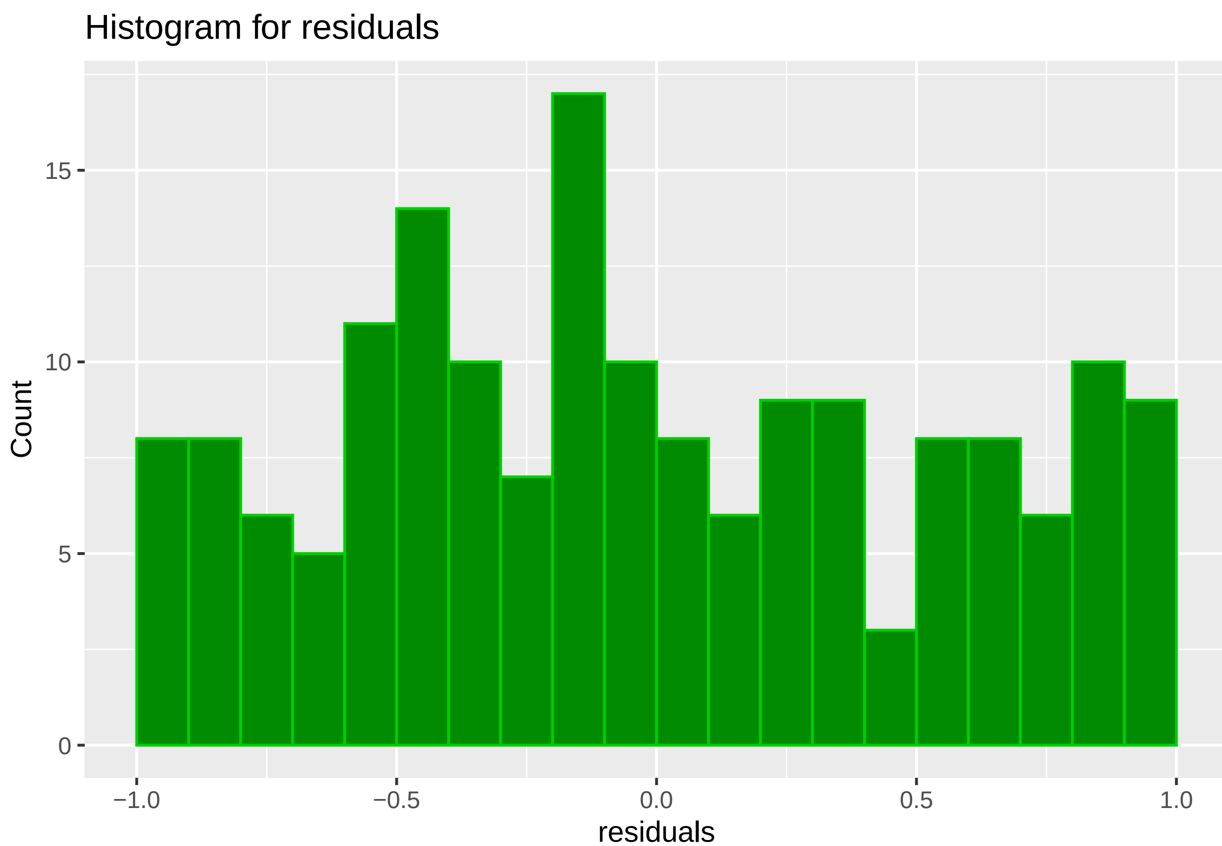
The p-value of the Breusch-Pagan test is less than 0.05 (2.505e-05), so we fail to reject the null hypothesis, indicating we do have heteroscedasticity.

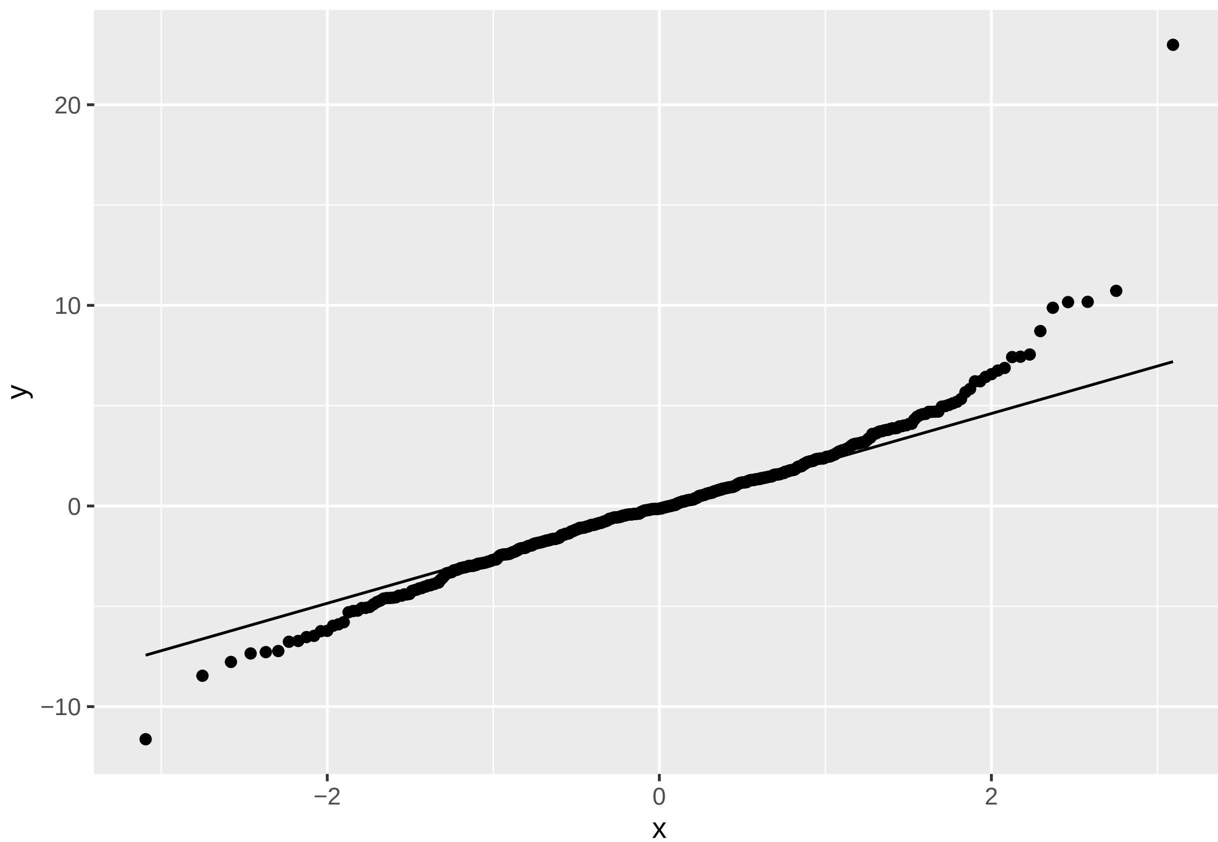
An attempt to address the heteroscedasticity, in addition to any other assumption failure will be done after testing all other assumptions.

Normality Assumption

The multiple linear regression analysis requires that the errors between observed and predicted values (i.e., the residuals of the regression) should be normally distributed.

A histogram and q-q plot were developed to check if the residuals were normally distributed or not.





Based on the plots above, it appears the residuals are not normal. A **Shapiro-Wilk test** was conducted to assess whether the residuals from the model are normally distributed.

Hypotheses

$$H_0: \text{The residuals are normally distributed}$$
$$H_a: \text{The residuals are not normally distributed}$$

```
##
##  Shapiro-Wilk normality test
##
## data:  residuals(higher_order_interaction_model_2)
## W = 0.94232, p-value = 4.088e-13
```

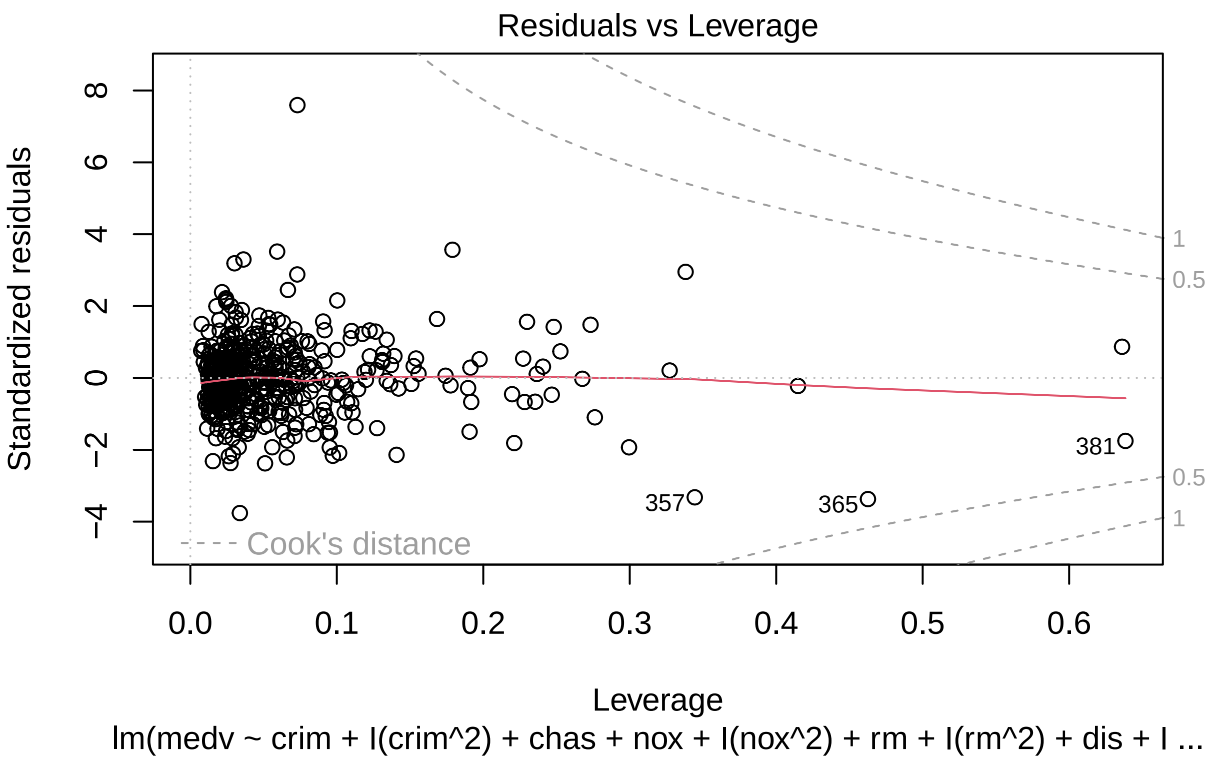
The p-value of the Shapiro-Wilk normality test is less than 0.05 (4.088e-13), so we fail to reject the null hypothesis, indicating we do not have normality.

An attempt to address the normality will be done after testing all other assumptions.

Outliers

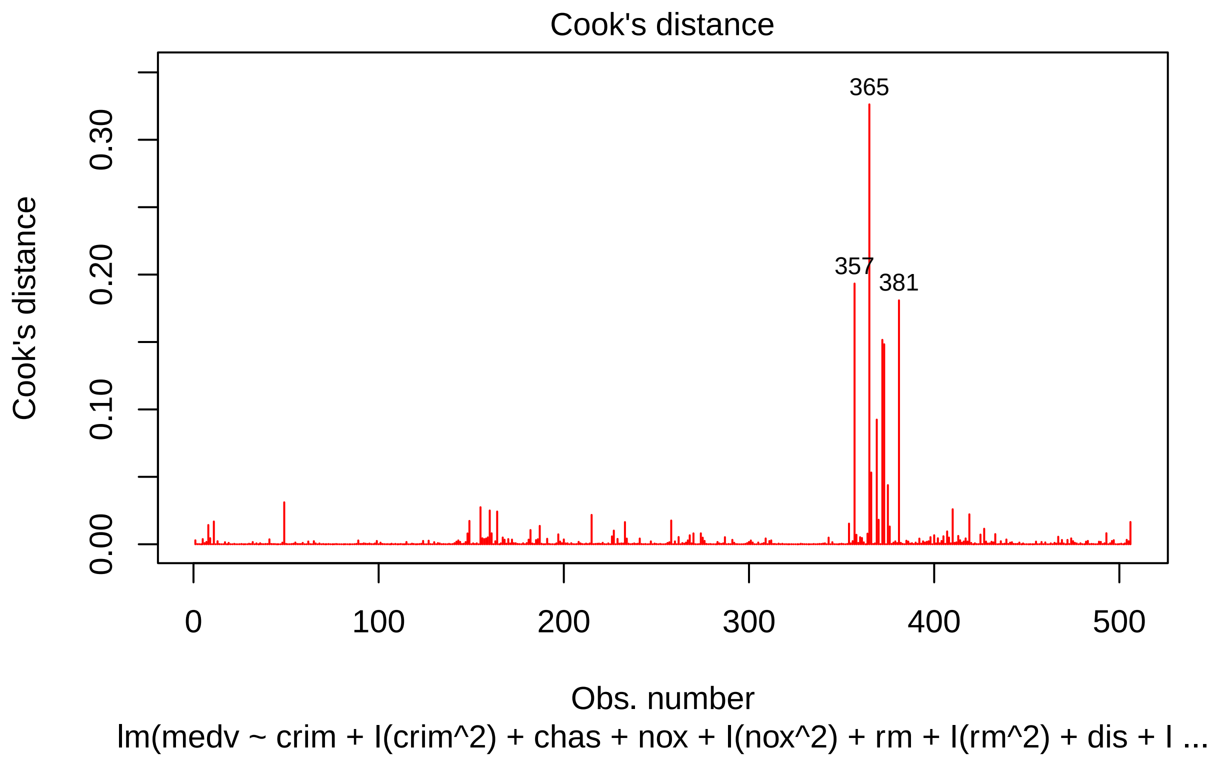
The below approaches were explored to find and evaluate outliers or influential points.

1. Residuals vs Leverage Plot



The plot above shows that all cases are well inside of the Cook's distance lines, indicating no outliers or no influential points.

2. Cook's Distance



Based on the consensus that a value of more than 1 indicates an influential value, the cook's distance plot above indicates there are no influential values.

3. Leverage points

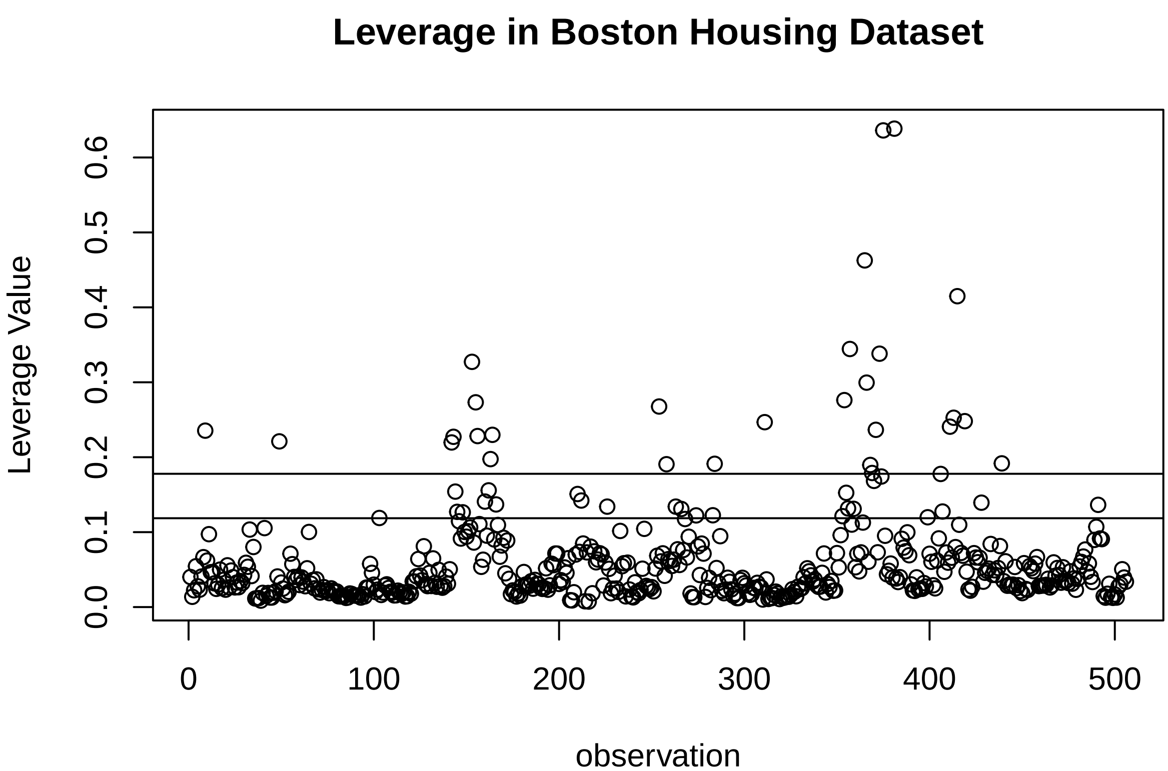
```
## [1] "h_I>2p/n, outliers are"
```

```
##         9        49       103       142       143       144       145       148
## 0.2354513 0.2211416 0.1189061 0.2196870 0.2272326 0.1541038 0.1270919 0.1264748
##       153       155       156       160       162       163       164       166
## 0.3272566 0.2732434 0.2282326 0.1407945 0.1557776 0.1974755 0.2298756 0.1369647
##       210       212       226       254       258       263       266       274
## 0.1509071 0.1421784 0.1340614 0.2676467 0.1906601 0.1340792 0.1309452 0.1223700
##       283       284       311       353       354       355       356       357
## 0.1225426 0.1912266 0.2467671 0.1213298 0.2761866 0.1525040 0.1316790 0.3444005
##       359       365       366       368       369       370       371       373
## 0.1311273 0.4626355 0.2995252 0.1896458 0.1789528 0.1684122 0.2365689 0.3381414
##       374       375       381       399       406       407       411       413
## 0.1742488 0.6361285 0.6384477 0.1198824 0.1776729 0.1275039 0.2407394 0.2526555
##       415       419       428       439       491
## 0.4148876 0.2481005 0.1393357 0.1917905 0.1364389
```

```
## [1] "h_I>3p/n, outliers are"
```

```
##         9        49       142       143       153       155       156       163
```

```
## 0.2354513 0.2211416 0.2196870 0.2272326 0.3272566 0.2732434 0.2282326 0.1974755
##       164       254       258       284       311       354       357       365
## 0.2298756 0.2676467 0.1906601 0.1912266 0.2467671 0.2761866 0.3444005 0.4626355
##       366       368       369       371       373       375       381       411
## 0.2995252 0.1896458 0.1789528 0.2365689 0.3381414 0.6361285 0.6384477 0.2407394
##       413       415       419       439
## 0.2526555 0.4148876 0.2481005 0.1917905
```



Based on the above plot, it appears we have high leverage points but none of them appear to be particularly influential (no points with a concerning cooks distance). Hence, it appears we do not have any outliers that could pose problems.

Dealing with Heteroscedasticity and Normality

Now that all multiple regression assumptions have been tested to check if the model is trustworthy, we then made attempts to address the heteroscedasticity and lack of normality.

We started by making a log transformation of the predictor so the difference between big and small numbers relatively becomes small.

Table 15: Coefficient Estimates from the Log-Transformed Higher-Order Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.4823	0.8708	5.1471	0.0000

	Estimate	Std. Error	t value	Pr(>|t|)
crim	-0.2790	0.1107	-2.5207	0.0120
I(crim^2)	0.0002	0.0000	4.4711	0.0000
chasYes	1.5587	0.3098	5.0315	0.0000
nox	-1.8645	0.9294	-2.0063	0.0454
I(nox^2)	1.3414	0.7206	1.8614	0.0633
rm	-0.1186	0.1850	-0.6412	0.5217
I(rm^2)	0.0342	0.0102	3.3530	0.0009
dis	-0.4052	0.0614	-6.5949	0.0000
I(dis^2)	0.0081	0.0019	4.2099	0.0000
rad	0.0666	0.0120	5.5602	0.0000
tax	-0.0008	0.0001	-5.9526	0.0000
ptratio	0.0917	0.0340	2.6942	0.0073
b	0.0007	0.0003	2.5487	0.0111
lstat	-0.1671	0.0473	-3.5310	0.0005
I(lstat^2)	0.0160	0.0069	2.3226	0.0206
I(lstat^3)	-0.0009	0.0004	-1.9904	0.0471
I(lstat^4)	0.0000	0.0000	1.6681	0.0960
I(lstat^5)	0.0000	0.0000	-1.3179	0.1882
crim:chasYes	0.0801	0.0147	5.4500	0.0000
crim:rad	-0.0158	0.0056	-2.8256	0.0049
crim:tax	0.0009	0.0004	2.6413	0.0085
chasYes:nox	-1.3425	0.2708	-4.9581	0.0000
chasYes:rm	-0.1302	0.0354	-3.6790	0.0003
nox:rad	-0.0613	0.0152	-4.0392	0.0001
rm:dis	0.0375	0.0070	5.3256	0.0000
rm:ptratio	-0.0179	0.0051	-3.4928	0.0005
dis:rad	-0.0026	0.0013	-2.0157	0.0444
dis:lstat	0.0060	0.0010	6.0399	0.0000
b:lstat	0.0000	0.0000	-1.5860	0.1134

Some insignificant variables were observed after the log transformation, which were subsequently dropped from the model

Table 16: Coefficient Estimates from the Final Reduced Log-Transformed Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.7592	0.8451	5.6317	0.0000
crim	-0.3149	0.1078	-2.9224	0.0036
I(crim^2)	0.0002	0.0000	4.1092	0.0000
chasYes	1.5244	0.3111	4.8996	0.0000
nox	-0.3748	0.2194	-1.7085	0.0882
rm	-0.2175	0.1789	-1.2154	0.2248
I(rm^2)	0.0374	0.0101	3.6981	0.0002
dis	-0.4104	0.0585	-7.0146	0.0000
I(dis^2)	0.0085	0.0018	4.7124	0.0000
rad	0.0459	0.0081	5.6678	0.0000
tax	-0.0008	0.0001	-5.9706	0.0000
ptratio	0.0682	0.0325	2.0953	0.0367
b	0.0003	0.0001	3.3344	0.0009
lstat	-0.1217	0.0243	-4.9996	0.0000

	Estimate	Std. Error	t value	Pr(>|t|)
I(lstat^2)	0.0074	0.0024	3.1177	0.0019
I(lstat^3)	-0.0003	0.0001	-3.2375	0.0013
I(lstat^4)	0.0000	0.0000	3.5399	0.0004
crim:chasYes	0.0835	0.0147	5.6817	0.0000
crim:rad	-0.0174	0.0053	-3.2674	0.0012
crim:tax	0.0011	0.0003	3.0806	0.0022
chasYes:nox	-1.3024	0.2711	-4.8047	0.0000
chasYes:rm	-0.1282	0.0354	-3.6239	0.0003
nox:rad	-0.0408	0.0129	-3.1648	0.0017
rm:dis	0.0370	0.0070	5.2671	0.0000
rm:ptratio	-0.0149	0.0050	-2.9978	0.0029
dis:lstat	0.0056	0.0009	5.9387	0.0000

The log-transformed model was tested for homoscedasticity and normality.

Table 17: Coefficient Estimates from the Reduced Log-Transformed Higher-Order Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.7592	0.8451	5.6317	0.0000
crim	-0.3149	0.1078	-2.9224	0.0036
I(crim^2)	0.0002	0.0000	4.1092	0.0000
chasYes	1.5244	0.3111	4.8996	0.0000
nox	-0.3748	0.2194	-1.7085	0.0882
rm	-0.2175	0.1789	-1.2154	0.2248
I(rm^2)	0.0374	0.0101	3.6981	0.0002
dis	-0.4104	0.0585	-7.0146	0.0000
I(dis^2)	0.0085	0.0018	4.7124	0.0000
rad	0.0459	0.0081	5.6678	0.0000
tax	-0.0008	0.0001	-5.9706	0.0000
ptratio	0.0682	0.0325	2.0953	0.0367
b	0.0003	0.0001	3.3344	0.0009
lstat	-0.1217	0.0243	-4.9996	0.0000
I(lstat^2)	0.0074	0.0024	3.1177	0.0019
I(lstat^3)	-0.0003	0.0001	-3.2375	0.0013
I(lstat^4)	0.0000	0.0000	3.5399	0.0004
crim:chasYes	0.0835	0.0147	5.6817	0.0000
crim:rad	-0.0174	0.0053	-3.2674	0.0012
crim:tax	0.0011	0.0003	3.0806	0.0022
chasYes:nox	-1.3024	0.2711	-4.8047	0.0000
chasYes:rm	-0.1282	0.0354	-3.6239	0.0003
nox:rad	-0.0408	0.0129	-3.1648	0.0017
rm:dis	0.0370	0.0070	5.2671	0.0000
rm:ptratio	-0.0149	0.0050	-2.9978	0.0029
dis:lstat	0.0056	0.0009	5.9387	0.0000

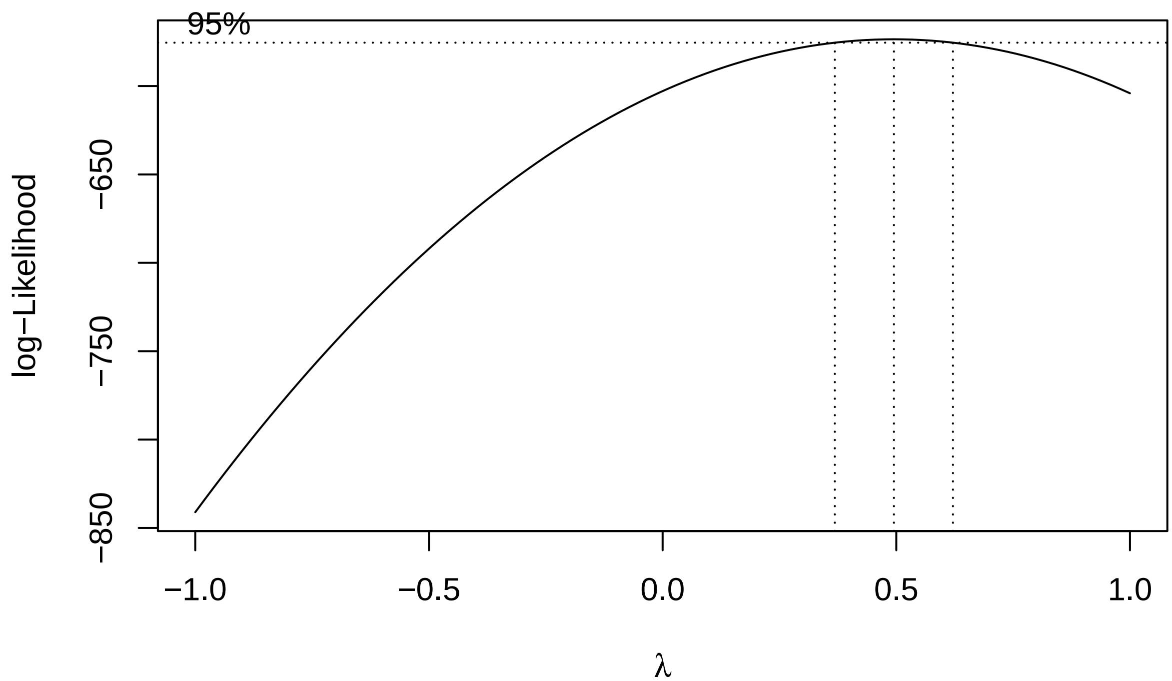
The p-value of the Breusch-Pagan test is less than 0.05 (4.085e-07), indicating we still have heteroscedasticity.

```
##
##  Shapiro-Wilk normality test
```

```
##
## data:  residuals(higher_order_interaction_model_log_2)
## W = 0.95209, p-value = 9.614e-12
```

The p-value of the Shapiro-Wilk normality test is less than 0.05 (9.614e-12), so we fail to reject the null hypothesis, indicating we do not have normality.

Since the log-transformation approach did not resolve the heteroscedasticity or lack of normality, we then explored the Box-Cox transformations approach by identifying $\hat{\lambda}$, the maximum likelihood estimate of λ to use in the power transformation




```
## [1] 0.4949495
```

The best lambda estimate was then used to transform the model as follows:

Table 18: Coefficient Estimates from the Box-Cox Transformed Higher-Order Interaction Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	10.4380	3.6864	2.8315	0.0048
crim	-1.3006	0.4686	-2.7754	0.0057
I(crim^2)	0.0007	0.0002	3.8987	0.0001
chasYes	7.6844	1.3113	5.8600	0.0000
nox	-8.8470	3.9341	-2.2488	0.0250
I(nox^2)	6.3975	3.0505	2.0972	0.0365

	Estimate	Std. Error	t value	Pr(>|t|)
rm	-0.0319	0.7830	-0.0407	0.9675
I(rm^2)	0.1788	0.0432	4.1351	0.0000
dis	-1.9261	0.2601	-7.4052	0.0000
I(dis^2)	0.0400	0.0082	4.8998	0.0000
rad	0.2896	0.0507	5.7164	0.0000
tax	-0.0035	0.0006	-6.2155	0.0000
ptratio	0.6587	0.1441	4.5718	0.0000
b	0.0039	0.0012	3.1498	0.0017
lstat	-0.9961	0.2003	-4.9734	0.0000
I(lstat^2)	0.1004	0.0292	3.4383	0.0006
I(lstat^3)	-0.0056	0.0019	-2.9802	0.0030
I(lstat^4)	0.0001	0.0001	2.6395	0.0086
I(lstat^5)	0.0000	0.0000	-2.3135	0.0211
crim:chasYes	0.4094	0.0622	6.5834	0.0000
crim:rad	-0.0734	0.0236	-3.1094	0.0020
crim:tax	0.0045	0.0015	2.9339	0.0035
chasYes:nox	-6.6660	1.1462	-5.8156	0.0000
chasYes:rm	-0.6423	0.1498	-4.2870	0.0000
nox:rad	-0.2647	0.0643	-4.1180	0.0000
rm:dis	0.1742	0.0298	5.8435	0.0000
rm:ptratio	-0.1198	0.0217	-5.5096	0.0000
dis:rad	-0.0131	0.0055	-2.3742	0.0180
dis:lstat	0.0292	0.0042	6.9726	0.0000
b:lstat	-0.0001	0.0001	-2.1840	0.0295

The best lambda transformed model was then tested for homoscedasticity and normality.

```
##
##  studentized Breusch-Pagan test
##
## data:  higher_order_interaction_model_2_box
## BP = 68.228, df = 29, p-value = 5.277e-05
```

The p-value of the Breusch-Pagan test is less than 0.05 (5.277e-05), indicating we still have heteroscedasticity.

```
##
##  Shapiro-Wilk normality test
##
## data:  residuals(higher_order_interaction_model_2_box)
## W = 0.96285, p-value = 5.297e-10
```

The p-value of the Shapiro-Wilk normality test is less than 0.05 (5.297e-10), so we fail to reject the null hypothesis, indicating we do not have normality.

Since the box-cox transformation approach did not resolve the heteroscedasticity or lack of normality also, we then explored the weighted least squares regression method, which is an application of the more general concept of generalized least squares. This method gives less weight to observations with high variance as follows:

Table 19: Coefficient Estimates from the Weighted Least Squares (WLS) Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	62.6407	18.5326	3.3800	0.0008
crim	-5.8644	1.8633	-3.1473	0.0018
I(crim^2)	0.0033	0.0011	3.0125	0.0027
chasYes	25.9603	7.1304	3.6408	0.0003
nox	-32.3226	14.7937	-2.1849	0.0294
I(nox^2)	19.5057	12.3357	1.5812	0.1145
rm	-10.0911	4.2774	-2.3592	0.0187
I(rm^2)	1.7307	0.2554	6.7774	0.0000
dis	-7.4567	1.1012	-6.7717	0.0000
I(dis^2)	0.1581	0.0337	4.6951	0.0000
rad	0.8730	0.2617	3.3355	0.0009
tax	-0.0130	0.0021	-6.1216	0.0000
ptratio	3.1064	0.6968	4.4583	0.0000
b	0.0199	0.0056	3.5426	0.0004
lstat	-5.0624	0.8233	-6.1488	0.0000
I(lstat^2)	0.5210	0.1195	4.3607	0.0000
I(lstat^3)	-0.0281	0.0077	-3.6491	0.0003
I(lstat^4)	0.0007	0.0002	3.1929	0.0015
I(lstat^5)	0.0000	0.0000	-2.8414	0.0047
crim:chasYes	1.6158	0.3828	4.2208	0.0000
crim:rad	-0.3063	0.1045	-2.9308	0.0035
crim:tax	0.0193	0.0064	3.0024	0.0028
chasYes:nox	-25.0754	5.9957	-4.1823	0.0000
chasYes:rm	-1.9067	0.8498	-2.2437	0.0253
nox:rad	-0.6934	0.3302	-2.1002	0.0362
rm:dis	0.6555	0.1270	5.1618	0.0000
rm:ptratio	-0.5768	0.1088	-5.3033	0.0000
dis:rad	-0.0335	0.0245	-1.3692	0.1716
dis:lstat	0.1010	0.0165	6.1227	0.0000
b:lstat	-0.0007	0.0003	-2.5663	0.0106

Some insignificant variables were observed, which were subsequently dropped from the model

Table 20: Coefficient Estimates from the Final Weighted Least Squares (WLS) Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	68.5135	18.2808	3.7478	0.0002
crim	-6.2359	1.8136	-3.4385	0.0006
I(crim^2)	0.0028	0.0010	2.7280	0.0066
chasYes	24.9904	7.1315	3.5042	0.0005
nox	-12.3056	3.9348	-3.1274	0.0019
rm	-12.0553	4.1541	-2.9020	0.0039
I(rm^2)	1.8107	0.2524	7.1752	0.0000
dis	-7.5744	1.0829	-6.9944	0.0000
I(dis^2)	0.1672	0.0329	5.0777	0.0000
rad	0.5139	0.1532	3.3537	0.0009
tax	-0.0128	0.0021	-6.1063	0.0000

	Estimate	Std. Error	t value	Pr(>|t|)
ptratio	2.7413	0.6697	4.0932	0.0000
b	0.0194	0.0056	3.4682	0.0006
lstat	-4.9962	0.8246	-6.0589	0.0000
I(lstat^2)	0.5105	0.1196	4.2674	0.0000
I(lstat^3)	-0.0274	0.0077	-3.5548	0.0004
I(lstat^4)	0.0007	0.0002	3.1045	0.0020
I(lstat^5)	0.0000	0.0000	-2.7640	0.0059
crim:chasYes	1.5647	0.3825	4.0912	0.0001
crim:rad	-0.3414	0.0995	-3.4308	0.0007
crim:tax	0.0211	0.0062	3.4308	0.0007
chasYes:nox	-23.3711	5.9264	-3.9436	0.0001
chasYes:rm	-1.8831	0.8507	-2.2135	0.0273
nox:rad	-0.2886	0.2507	-1.1515	0.2501
rm:dis	0.6478	0.1272	5.0910	0.0000
rm:ptratio	-0.5280	0.1056	-4.9986	0.0000
dis:lstat	0.0968	0.0164	5.9095	0.0000
b:lstat	-0.0007	0.0003	-2.5246	0.0119

A further insignificant variables was observed, which was also dropped from the model

Table 21: Coefficient Estimates from the Final Reduced WLS Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	72.0267	18.0305	3.9947	0.0001
crim	-6.2812	1.8137	-3.4631	0.0006
I(crim^2)	0.0030	0.0010	2.9893	0.0029
chasYes	24.6121	7.1263	3.4537	0.0006
nox	-15.3681	2.9009	-5.2978	0.0000
rm	-12.4565	4.1409	-3.0082	0.0028
I(rm^2)	1.8274	0.2520	7.2507	0.0000
dis	-7.7389	1.0738	-7.2068	0.0000
I(dis^2)	0.1715	0.0327	5.2422	0.0000
rad	0.3466	0.0488	7.1037	0.0000
tax	-0.0130	0.0021	-6.2272	0.0000
ptratio	2.7039	0.6692	4.0408	0.0001
b	0.0199	0.0056	3.5703	0.0004
lstat	-4.9514	0.8240	-6.0091	0.0000
I(lstat^2)	0.5050	0.1196	4.2230	0.0000
I(lstat^3)	-0.0271	0.0077	-3.5133	0.0005
I(lstat^4)	0.0007	0.0002	3.0642	0.0023
I(lstat^5)	0.0000	0.0000	-2.7247	0.0067
crim:chasYes	1.4847	0.3762	3.9463	0.0001
crim:rad	-0.3624	0.0979	-3.7035	0.0002
crim:tax	0.0219	0.0061	3.5816	0.0004
chasYes:nox	-22.7484	5.9036	-3.8533	0.0001
chasYes:rm	-1.8725	0.8510	-2.2004	0.0283
rm:dis	0.6587	0.1269	5.1899	0.0000
rm:ptratio	-0.5220	0.1055	-4.9456	0.0000
dis:lstat	0.0983	0.0163	6.0181	0.0000
b:lstat	-0.0007	0.0003	-2.6107	0.0093

The model derived from the weighted least squares regression method was tested for homoscedasticity and normality.

```
##
##  studentized Breusch-Pagan test
##
## data:  higher_order_interaction_model_wls_3
## BP = 11.552, df = 26, p-value = 0.9934
```

The p-value of the Breusch-Pagan test is greater than 0.05 (0.9934), indicating we now finally have homoscedasticity.

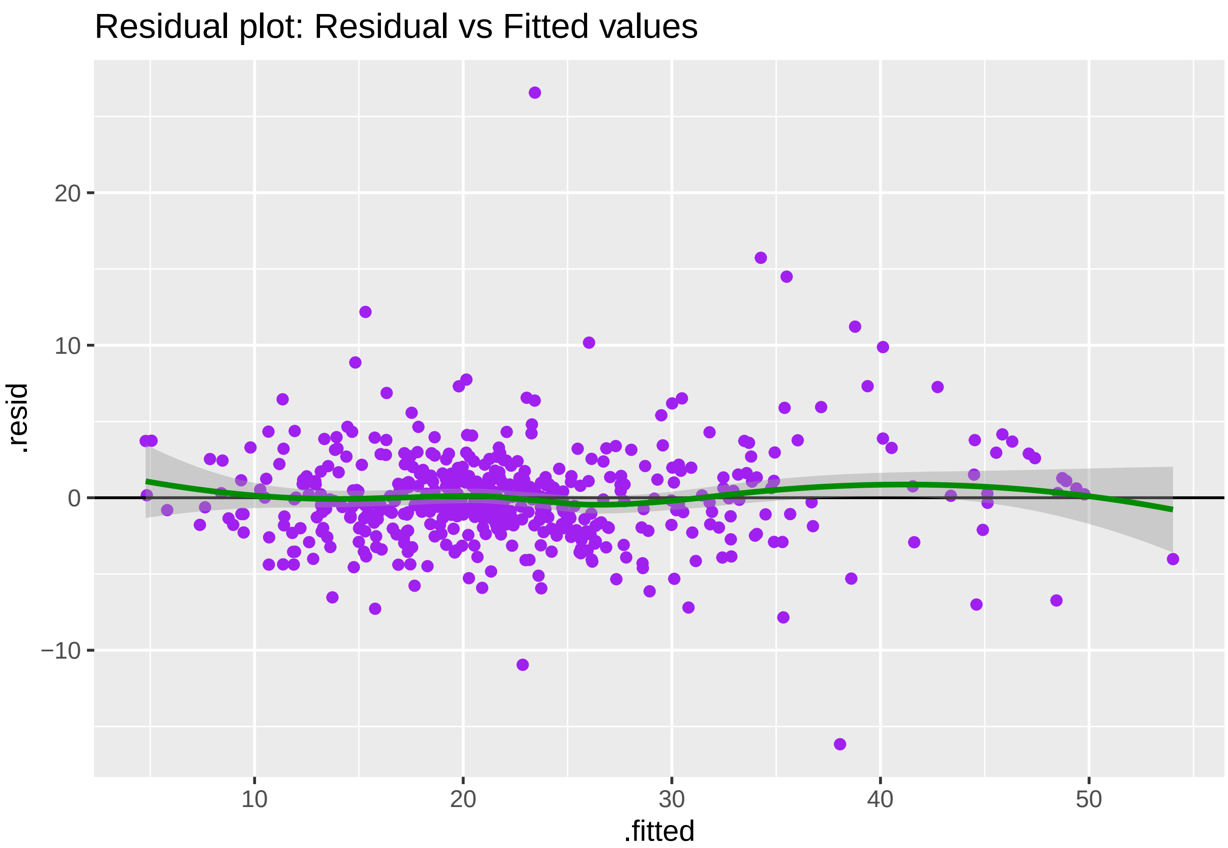
```
##
##  Shapiro-Wilk normality test
##
## data:  residuals(higher_order_interaction_model_wls_3)
## W = 0.88798, p-value < 2.2e-16
```

The p-value of the Shapiro-Wilk normality test is less than 0.05 (9.614e-12), indicating we still do not have normality.

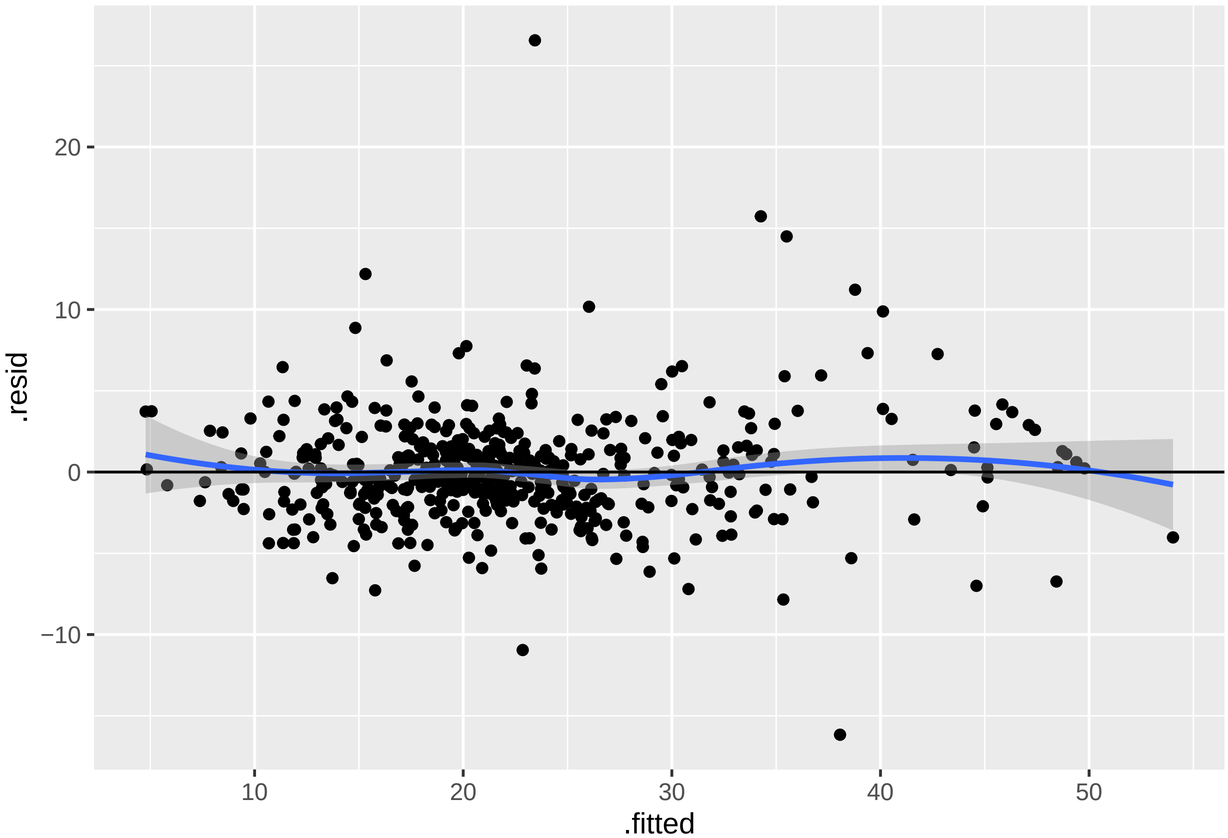
Even though we do not have normality, due to our dataset having a large sample size, the Central Limit Theorem can compensate for the non-normality in the residuals. The Central Limit Theorem states that, for a sufficiently large sample from a population, the sampling distribution of the estimates tends to be normal, regardless of the underlying population distribution. This suggests that the residuals do not need to be perfectly normal.

The linearity and outliers assumptions would be tested again to confirm they are still met

Linearity Assumption

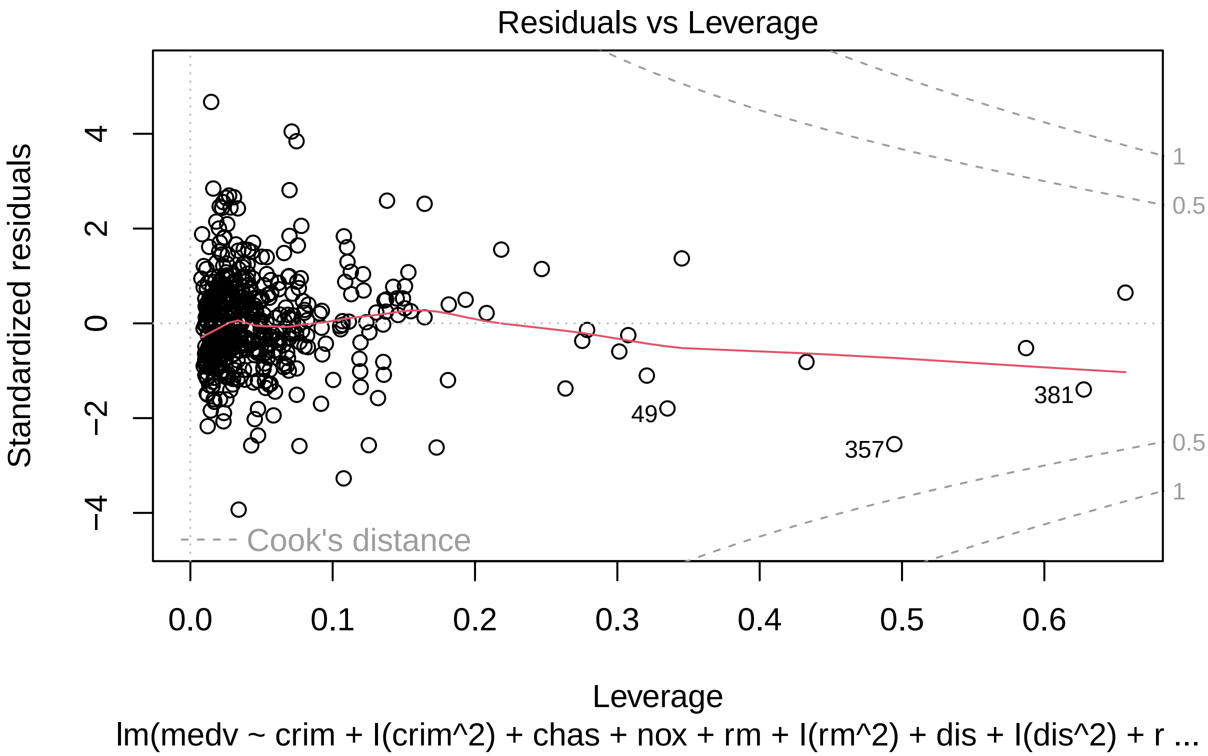


Based on the updated model, there appears to be no pattern of the residuals at all, indicating that the model still passes the linearity assumption that there is a straight-line (linear) relationship between the predictors and the response.



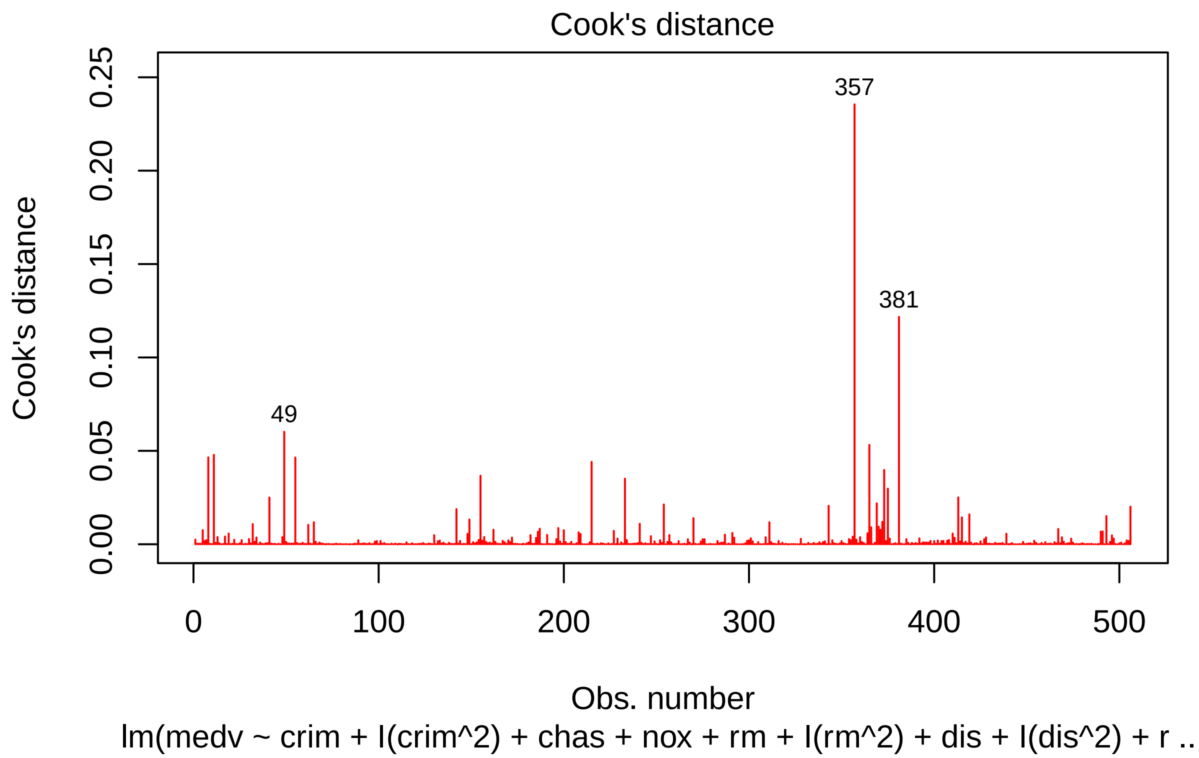
Outliers

1. Residuals vs Leverage Plot



The plot above shows that all cases are well inside of the Cook's distance lines, indicating no outliers or no influential points.

2. Cook's Distance



Based on the consensus that a value of more than 1 indicates an influential value, the cook's distance plot above indicates there are still no influential values.

3. Leverage points

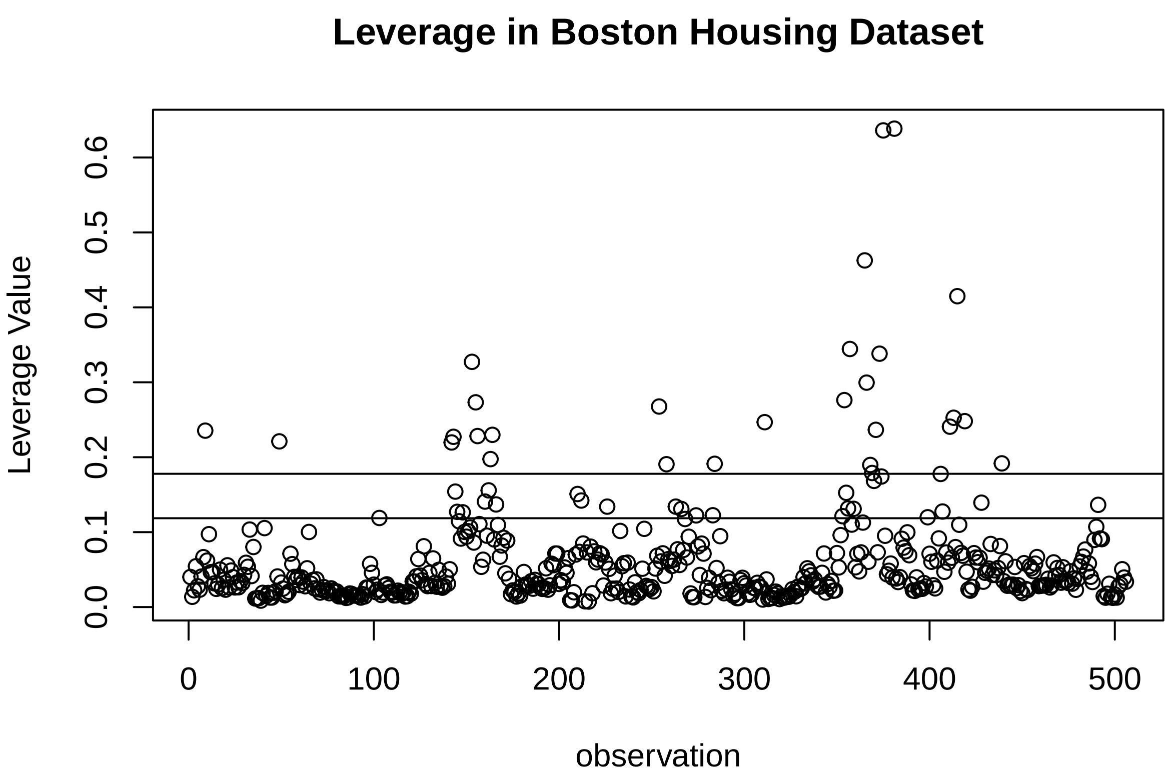
```
## [1] "h_I>2p/n, outliers are"
```

```
##         9        49       103       142       143       144       145       148
## 0.2354513 0.2211416 0.1189061 0.2196870 0.2272326 0.1541038 0.1270919 0.1264748
##       153       155       156       160       162       163       164       166
## 0.3272566 0.2732434 0.2282326 0.1407945 0.1557776 0.1974755 0.2298756 0.1369647
##       210       212       226       254       258       263       266       274
## 0.1509071 0.1421784 0.1340614 0.2676467 0.1906601 0.1340792 0.1309452 0.1223700
##       283       284       311       353       354       355       356       357
## 0.1225426 0.1912266 0.2467671 0.1213298 0.2761866 0.1525040 0.1316790 0.3444005
##       359       365       366       368       369       370       371       373
## 0.1311273 0.4626355 0.2995252 0.1896458 0.1789528 0.1684122 0.2365689 0.3381414
##       374       375       381       399       406       407       411       413
## 0.1742488 0.6361285 0.6384477 0.1198824 0.1776729 0.1275039 0.2407394 0.2526555
##       415       419       428       439       491
## 0.4148876 0.2481005 0.1393357 0.1917905 0.1364389
```

```
## [1] "h_I>3p/n, outliers are"
```

```
##         9        49       142       143       153       155       156       163
```

```
## 0.2354513 0.2211416 0.2196870 0.2272326 0.3272566 0.2732434 0.2282326 0.1974755
##       164       254       258       284       311       354       357       365
## 0.2298756 0.2676467 0.1906601 0.1912266 0.2467671 0.2761866 0.3444005 0.4626355
##       366       368       369       371       373       375       381       411
## 0.2995252 0.1896458 0.1789528 0.2365689 0.3381414 0.6361285 0.6384477 0.2407394
##       413       415       419       439
## 0.2526555 0.4148876 0.2481005 0.1917905
```



Based on the above plot, it appears we still have high leverage points but none of them appear to be particularly influential (no points with a concerning cooks distance). Hence, it appears we do not have any outliers that could pose problems.

Predicting the Median Value

Table 22: Predicted Median Home Value with 95% Prediction Interval

fit	lwr	upr
17.4323	12.558	22.3065

Based on the prediction using the `predict()` function, we are **95% confident** that the median value (`medv`) of owner-occupied homes, given the specified predictor values, lies between **$12,558** and **$22,307**.

4. CONCLUSION AND DISCUSSION

We developed a predictive model using **multiple linear regression** to estimate the median value of owner-occupied homes in Boston based on 13 housing and neighborhood features. After loading the data, we tested for **multicollinearity**, and none was detected.

We then built a **baseline additive model** and refined it by removing insignificant predictors. Interaction effects were explored, and a **reduced interaction model** was selected based on predictor significance and model performance. Next, we examined **higher-order terms** to capture non-linear relationships, iteratively refining the model to retain only statistically significant predictors.

Finally, we tested the **multiple regression assumptions**—linearity, independence, equal variance, normality, and presence of outliers. While the final model satisfied most assumptions, the **residuals remained non-normal** despite attempts to correct this using log transformation, Box-Cox transformation, and **weighted least squares (WLS)** regression.

Ultimately, the **WLS model** was selected as the final model for its ability to address **heteroscedasticity**, improve model **interpretability**, and maintain both **linearity** and **robustness to outliers**. This comprehensive modeling process revealed key factors influencing housing prices.

The final predictive model is:

$$\begin{aligned}
\hat{medv_i} = {} & 72.03 - 6.281 \cdot crim_i + 0.003005 \cdot I(crim_i^2) + 24.61 \cdot chas_i - 15.37 \cdot nox_i \\
& - 12.46 \cdot rm_i + 1.827 \cdot I(rm_i^2) - 7.739 \cdot dis_i + 0.1715 \cdot I(dis_i^2) \\
& + 0.3466 \cdot rad_i - 0.01298 \cdot tax_i + 2.704 \cdot ptratio_i + 0.01990 \cdot b_i \\
& - 4.951 \cdot lstat_i + 0.5050 \cdot I(lstat_i^2) - 0.02711 \cdot I(lstat_i^3) + 0.0006835 \cdot I(lstat_i^4) \\
& - 0.000006415 \cdot I(lstat_i^5) + 1.485 \cdot crim_i \cdot chas_i - 0.3624 \cdot crim_i \cdot rad_i \\
& + 0.02194 \cdot crim_i \cdot tax_i - 22.75 \cdot chas_i \cdot nox_i - 1.872 \cdot chas_i \cdot rm_i \\
& + 0.6587 \cdot rm_i \cdot dis_i - 0.5220 \cdot rm_i \cdot ptratio_i + 0.09832 \cdot dis_i \cdot lstat_i \\
& - 0.0006878 \cdot b_i \cdot lstat_i
\end{aligned}$$

where $chas_i$ is 1 if the tract bounds Charles River and 0 if otherwise

Key predictors of the median home value ($1000s) include:
`crim`, `I(crim^2)`, `chas`, `nox`, `rm`, `I(rm^2)`, `dis`, `I(dis^2)`, `rad`, `tax`, `ptratio`, `b`, `lstat`, `I(lstat^2)`, `I(lstat^3)`, `I(lstat^4)`, `I(lstat^5)`, `crim:chas`, `crim:rad`, `crim:tax`, `chas:nox`, `chas:rm`, `rm:dis`, `rm:ptratio`, `dis:lstat`, and `b:lstat`.

4.1 Approach

The approach we took does seem logically sound. Starting with a basic multiple linear regression model allowed us to build a strong baseline and interpret the effects of various predictors on housing prices. Looking at interaction terms and higher-order relationships enabled us to improve model complexity effectively. Additionally, testing each of the regression assumptions step-by-step ensured that our final model would be valid statistically. Some issues we had were that there was consistent violation of the normality assumption, even with various transformations. Possibilities for improvement are possibly running non-linear or machine learning models. They may be able to handle normality and allow us to create a better predictive model.

4.2 Future Work

In the future, we could explore using machine learning models to improve prediction accuracy. These models may be able to capture more complex patterns in the data and might perform better than traditional linear regression. All in all, they could be a valuable next step in our analysis.

5. REFERENCES

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[2] Lipman, Danika. Data603: Statistical Modelling with Data. University of Calgary, 2025, https://d2l.ucalgary.ca/d2l/le/content/648044/viewContent/6885027/View.