Martingales & Applications in Simulation

1. Introduction to Martingales

A martingale is a stochastic process (X_t) where the conditional expectation of the next value, given all past values, equals the current value: $E[X_{t+1} | F_t] = X_t$. Martingales model 'fair game' processes.

2. Key Properties

- Martingale property (fairness): future expected value equals present value.
- Submartingale: $E[X_{t+1}|F_t] >= X_t$ (tendency to increase).
- Supermartingale: $E[X_{t+1}|F_t] <= X_t$ (tendency to decrease).
- Stopping time theorem and optional sampling theorem (under conditions).

3. Examples

- Simple symmetric random walk (starting at 0) is a martingale.
- Discounted asset prices under risk-neutral measure (financial martingales).
- Martingale differences in time series.

4. Applications in Simulation & OR

- Modeling fair queues and gambling processes.
- Variance reduction techniques in Monte Carlo (control variates using martingales).
- Sequential analysis and stopping rules.
- Pricing and risk-neutral valuation in finance.

5. Simulation Goals

- Simulate simple symmetric random walks and verify martingale property empirically.
- Visualize sample paths and compute running averages.
- Demonstrate optional sampling on bounded stopping times.