Continuous-Time Markov Chains (CTMC) in Operations Research

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1. Introduction to CTMCs

Continuous-Time Markov Chains (CTMCs) are stochastic models where changes between states happen at random times.
Unlike DTMCs, CTMCs use **exponential distributions** to model the time between transitions.
CTMCs are widely used in **queueing systems**, **reliability modeling**, and **biological systems**.

2. Transition Rate Matrix (Q Matrix)

The core of CTMC is the **transition rate matrix Q**, where each element `q(i,j)` represents the rate of transitioning from state `i` to `j`.

- Diagonal elements `q(i,i)` are **negative** and represent the total rate of leaving state `i`:
- $q(i,i) = -\sum_{j \neq i} q(i,j)$
- Off-diagonal elements are **non-negative** and show the rate of moving from one state to another.

3. Exponential Holding Times

The **time spent in each state** before transitioning is **exponentially distributed** with a parameter equal to the **absolute value of the diagonal entry** in Q.

After this time, the process jumps to a new state based on the **transition probabilities derived from Q**.

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4. Birth-Death Process Example

A common example of CTMC is the **birth-death process**.
For instance, in a simple queue:

- **Birth rate** (arrival rate): λ
- **Death rate** (service rate): μ

This results in a process where transitions only occur between **adjacent states** (e.g., from `n` to `n+1` or `n-1`) with exponential timing.

5. Applications of CTMC

- **Queueing networks** (e.g. M/M/1, M/M/c systems)
- **Machine failure and repair modeling**
- **Population dynamics and epidemiology**
- **Inventory systems with continuous review**
- **Reliability and survival analysis**

$q(i,i)=-\sum_{j\neq i} q(i,j)$