Proof that the analytic solution with sin-wave boundary is correct

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Intro

The partial differential equation for the trainsient flow in the aquifer with spatially constant trasnmissivity kD and storage coefficient S was derived to be

$$kD\frac{\partial^2\phi}{\partial x^2} = S\frac{\partial\phi}{\partial t}$$

The proposed solution was

$$\phi\left(x,t\right) = Ae^{-ax}\sin\left(\omega t - bx\right)$$

To prove this and find out how what the parameters a and b are, we fill this equation into the partial differential equation. For this we have to differentiate the solution twice with respect to x and once with respect to time t:

$$\frac{\partial\phi}{\partial x} = -aA^{-ax}\sin\left(\omega t - bx\right) - bAe^{-ax}\cos\left(\omega t - bx\right)$$

Differentiating once more with respect to x gives

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= a^2A^{-ax}\sin\left(\omega t - bx\right) + abA^{-ax}\cos\left(\omega t - bx\right) + abAe^{-ax}\cos\left(\omega t - bx\right) - b^2Ae^{-ax}\sin\left(\omega t - bx\right)\\ &= Ae^{-ax}\left[\left(a^2-b^2\right)\sin\left(\omega t - bx\right) + 2ab\cos\left(\omega t - bx\right)\right]\end{aligned}$$

Now we must take the derivative of the solution with respect to time

$$\frac{\partial\phi}{\partial t} = \omega Ae^{-ax}\cos\left(\omega t - bx\right)$$

Then filling the obtained expressions for $\frac{\partial^2\phi}{\partial x^2}$ and $\frac{\partial\phi}{\partial x}$ in the partial differential equation yields, after dividing out A^{-ax}

$$kD\left[\left(a^2-b^2\right)\sin\left(\omega t - bx\right) + 2ab\cos\left(\omega t - bx\right)\right] = S\omega\cos\left(\omega t - bx\right)$$

This equation can only be true if the term with the sin is zero, which is the case when $a^2 - b^2 = 0$. Therefore, we have

$$a^2 - b^2 = 0 \rightarrow a = +b \ \ or \ \ a = -b$$

Because $a = -b$ is not physically possible, we obtain $a = b$. And, therefore, with $b = a$, the equation reduces to

$$kD2a^2\cos\left(\omega t - ax\right) = S\omega\cos\left(\omega t - ax\right)$$

After dividing out $\cos\left(\omega t - ax\right)$ on both sides we get

$$kD2a^2 = \omega S$$

or

$$a = \sqrt{\frac{\omega S}{2kD}}$$

We have now derived how the damping and delay factor a is constructed from the ω, the storage coefficient S and the transmissivity kD, which shows on what the damping factor and delay factor a depends.

We check its dimension, which should be $1/L$ because ax should be dimensionless.

Here is the check:

$$a = \sqrt{\frac{\omega S}{2kD}} \propto \left[\sqrt{\frac{1}{T}\frac{1}{\frac{L^2}{T}}}\right] = \left[\frac{1}{L}\right]$$

With this expression we know that the damping is propertional to the root of ω, the root of S and inversely proportional to the root of the transmissivity kD.

It may be more natural to think in terms of $L = \frac{1}{a}$ as a characteristic length of this groundwater setting

$$L = \sqrt{\frac{2kD}{\omega S}}$$

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