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Midterm 2 Formale Grundlagen der Informatik I (AINF1110, FS18)

03. Mai 2018
Version A

- You have **60** Minutes time for solving the exercise.
- In this mandatory exercise you can achieve up to **60** points. You are admitted to the final exam if you can achieve either more than 30 points in a single mandatory exercise or 36 points in both mandatory exercises together.

The following rules apply for this mandatory exercises:

- Answer the questions directly on the exercises sheets.
- Check for completeness at the end of the exercise.
- Usa a pen with blue or black color. No pens with red or green color nor pencils will be considered for correction.
- Always use the notation used in the lecture. However, you can write your answers in **English** or **German**.
- No additional material and supporting devices are allowed. If spotted while cheating, the mandatory exercise will be counted with 0 points.
- Put the student id ("Legi") in front on your desk.

With this signature I confirm that I have read and understood this rules.

Signature: ______________________________

1 (10 P.)	2 (12 P.)	3 (13 P.)	4 (13 P.)	5 (12 P.)	Total

1 True or False? (10 P.)

Check, if the following statements are true or false.

Every correct answer adds a point to your score. Every wrong answer deducts a point from your score. Every answer left blank is not considered for point counting. Allover this task gives a least 0 points.

If a function is surjective (onto), then it is also injective (one-to-one) False	☐ true	☐ false
The Euclidean norm ($f(x, y) = \sqrt{x^2 + y^2}$)is injective. False	☐ true	☐ false
The smaller-or-equal-than-operator '$\leq$' defines a partial order of the real numbers. True	☐ true	☐ false
If A is a set, then $\{A\}$ is a partition of A. True	☐ true	☐ false
Let $\mathcal{S}$ be the set of all surjective functions and $\mathcal{I}$ the set of all injective functions. $\{\mathcal{S}, \mathcal{I}\}$ is a partition of the set of all functions. False	☐ true	☐ false
A logarithmic and an exponential function are always inverse to each other. False	☐ true	☐ false
$\log_2(5^5) = (\log_2 5)^5$. False	☐ true	☐ false
$\ln 10 = \frac{\log_2 e}{\log_2 10}$. False	☐ true	☐ false
If $f : X \mapsto Y$ is a function, then $\|X\| \leq \|Y\|$. False	☐ true	☐ false
$\mathscr{P}(\emptyset) = \{\emptyset\}$. True	☐ true	☐ false

2 Set theory (12 P.)

Prove or disprove the following statements for the sets A, B and C.

a) (3) $\forall x \in A$, $x \in B$, then $A = B$.
Die Aussage ist falsch, z.B.:
$A = \{0\}$, $B = \{0, 1\}$

b) (3) Proof $(A \cup B) - (C - A) = A \cup (B - C)$ using the set properties you learned in class.

$$
\begin{aligned}
(A \cup B) - (C - A) &= (A \cup B) \cap (C - A)^c && (1)\\
&= (A \cup B) \cap (C \cap A^c)^c && (2)\\
&= (A \cup B) \cap (A^c \cap C)^c && (3)\\
&= (A \cup B) \cap ((A^c)^c \cap C^c) && (4)\\
&= (A \cup B) \cap (A \cap C^c) && (5)\\
&= A \cup (B \cap C^c) && (6)\\
&= A \cup (B - C) && (7)
\end{aligned}
$$

c) (3) For arbitrary sets A, B and C, it holds that $(A \cap B) \cup C = (A \cap C) \cup (B \cap C)$.
Die Aussage ist falsch, wähle die Mengen wiefolgt,
$A = \{\}$, $B = \{0\}$, $C = \{1\}$, dann gilt
$(A \cap B) \cup C = \{1\} \neq \{\} = (A \cap C) \cup (B \cap C)$

d) (3) If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
Die Aussage ist richtig. Direkter Beweis mit Fallunterscheidung:

Fall 1 Falls $A = \emptyset$ oder $C = \emptyset$, dann gilt auch $A \times C = \emptyset$. Da die leere Menge eine Untermenge jeder beliebigen Menge ist, gilt die Aussage.

Fall 2 Seien beide Mengen A und C nicht leer. Sei (a, c) ein beliebiges Element von $A \times C$. Dann gilt $a \in A$ und $c \in C$. Da $A \subseteq B$ und $C \subseteq D$, gilt auch $a \in B$ und $c \in D$. Das Element (a, c) kommt also auch im kartesischen Produkt $B \times D$ vor. $A \times C \subseteq B \times D$ gilt.

3 Sequences and series (13 P.)

a) (6) Given is the following recursive equation

$$a_k = 6a_{k-2} - a_{k-1}$$

and additionally one knows the following values:

$$a_2 = 15 \text{ und } a_4 = 75.$$

Which are the initial values a_0 and a_1? Derive a closed formula for a_k and compute a_0, a_1 and a_3.

The characteristic equation is $t^2 + t - 6 = (t-2)(t+3) = 0$. The distinct roots are $r_1 = 2$ und $r_2 = -3$. The explicit formula is therefore $a_k = A \cdot 2^k + B \cdot (-3)^k$. A and B can be derived from $a_2 = 15$ and $a_4 = 75$:

$$\begin{aligned} a_2 &= 4A + 9B = 15 \\ a_4 &= 16A + 81B = 75, \end{aligned}$$

and therefore $A = 3$ and $B = \frac{1}{3}$. We can write the explicit formula as $a_k = 3 \cdot 2^{k-2} + \frac{1}{3} \cdot (-3)^k$ and $a_0 = \frac{10}{3}$, $a_1 = 5$, and $a_3 = 15$.

b) (7) Given is the recursive equation

$$c_k = c_{k-1} + 2c_{k-2} + 1$$

and the initial values

$$c_0 = 1 \text{ and } c_1 = 1.$$

Prove that $c_k = 2^k$ if k is even and $c_k = 2^k - 1$ if k is odd for all $k \in \{0, 1, 2, \dots\}$.

Hint: Prove the statement using mathematical induction. Consider both cases (k even and k odd) in the induction hypothesis and in the induction step.

Die Induktionsverankerung für gerade k ist gegeben durch $c_0 = 2^0 = 1$. Die Induktionsverankerung für ungerade k ist gegeben durch $c_1 = 2^1 - 1 = 1$. Wir nehmen nun an, dass die Behauptung stimmt für alle (geraden oder ungeraden) Werte von 0 bis k. Der Induktionsschritt für ein gerades $k+1$ ist gegeben durch

$$\begin{aligned} c_{k+1} &= c_k + 2c_{k-1} + 1 \\ &\overset{I.A.}{=} 2^k - 1 + 2 \cdot 2^{k-1} + 1 \\ &= 2^k + 2^k = 2^{k+1}. \end{aligned}$$

Der Induktionsschritt für ein ungerades $k+1$ ist gegeben durch

$$\begin{aligned} c_{k+1} &= c_k + 2c_{k-1} + 1 \\ &\overset{I.A.}{=} 2^k + 2(\cdot 2^{k-1} - 1) + 1 \\ &= 2^k + 2^k - 2 + 1 = 2^{k+1} - 1. \end{aligned}$$

Der Induktionsschritt gelingt also in beiden Fällen uns somit ist die Behauptung bewiesen.

4 Functions (13 P.)

Prove or disprove the following statements.

a) (4) Let $\mathbb{B} = \{1, 2, 3, 4, 5, 6\}$ and f a function defined as

$$f(i) = 3^i mod\ 7, \quad i \in \mathbb{B} \tag{8}$$

Then it follows that f is a function of $\mathbb{B}$ to $\mathbb{B}$.

We can easily write out the function for the set $\mathbb{B}$

i	3^i	$3^i \mod 7$
1	3	3
2	9	2
3	27	6
4	81	4
5	243	5
6	729	1

Therefore f is a function, since all elements of B map exactly to one other element of B and all elements are covered (non are left out). Points: 1 for table or other derivation, 1 for correct conclusion, 2 for argumentation (one-to-one and completeness).

b) (3) Similar to a), let $\mathbb{B} = \{1, 2, 3, 4, 5, 6\}$ and g a function defined as

$$g(i) = 3^i mod\ 5, \quad i \in \mathbb{B} \tag{9}$$

Then it follows that the inverse of g is still a function.

We can easily write out the function for the set $\mathbb{B}$:
For the inverse, both function values 3 and 4 map to multiple values of i. Therefore the inverse is not a function anymore. Points: 1 for table or other derivation, 1 for correct conclusion, 1 for argumentation.

i	3^i	$3^i \mod 5$
1	3	3
2	9	4
3	27	2
4	81	1
5	243	3
6	729	4

c) (2) What do you have to change in the set definition B from the previous statement (task b) to make the inverse of g a function on the set B?

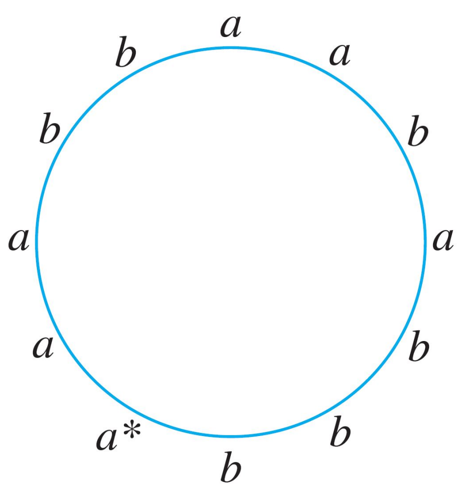
Example: $\mathbb{B} = \{1, 2, 3, 4\}$, because then the set does not result in any values twice (one-to-one) and the inverse is still a function. Points: 1 for set, 1 for argument.

d) (4) We define a function $f(x) = tan(x)$ with $\mathbb{W} = \{x \in \mathbb{R} \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. What numbers do $\mathbb{W}$ and $f(x)$ span? Use it to show that $\mathbb{W}$ and $\mathbb{R}$ have the same cardinality.

The graph of the tangens function is monotonically increasing from $-\infty$ to ∞ on $\mathbb{W}$, which both shows that it is one-to-one and onto. Therefore the cardinality of both sets must be the same. Points: 1 for graph, 1 for span, 1 for one-to-one, 1 for onto.

5 Mathematical Induction (12 P.)

a) (12) Suppose that n a's and n b's are distributed around the outside of a circle. Use mathematical induction to prove that for all integers $n \geq 1$, given any such arrangement, it is possible to find a starting point so that if one travels around the circle in a clockwise direction, the number of a's one has passed is never less than the number of b's one has passed. For example, in the diagram shown below, one could start at the a with an asterisk.



Given $k+1$ a's and $k+1$ b's arrayed around the outside of the circle, there has to be at least one location where an a is followed by a b as one travels in the clockwise direction. In the inductive step, temporarily remove such an a and the b that follows it, and apply the inductive hypothesis.