Universität Zürich
Institut für Informatik

FS 2019

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Formale Grundlagen der Informatik I - Assignment 3

Hand out: 28.03.2019 - Due to: 18.04.2019

Please upload your solutions to the Olat system.

3.1 Sequences and Sums

a) Do the following sequences converge or diverge for $\lim_{i\to\infty}$? If they converge, please give the converging value. In any case, give a short reasoning (instead of a full proof).

i. $a_i = \frac{5-i}{5+i}$, $i \in \mathbb{N}$.

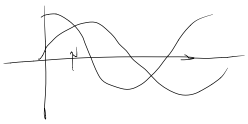
$$\lim_{i\to\infty} \frac{5-i}{5+i} = \lim_{i\to\infty} \frac{5}{5+i} + \lim_{i\to\infty} \frac{-i}{5+i} = 0 - \lim \frac{i}{5+i} \overset{\text{l'hopital}}{=} -\left(\frac{1}{1}\right) = -1$$ converges

ii. $b_i = \frac{i^3+5}{4-i^2}$, $i \in \mathbb{N}_{\geq 3}$.

$$\lim \frac{i^3}{4-i^2} + \lim \frac{5}{4-i^2} = \lim \frac{i^3}{4-i^2} = \lim \frac{\cancel{i^2} \cdot i}{\cancel{i^2}(\underbrace{\frac{4}{i^2}}_{0}-1)} = \lim_{i\to\infty} -i = -\infty$$

diverging

iii. $c_i = \sin(i) + \cos(i)$, $i \in \mathbb{N}$.



oscillating between $-\sqrt{2}$ and $\sqrt{2}$, diverging

3.2 Binomial Coefficients

a) Without proof: What is the relation between Pascal's triangle and the binomial coefficients?

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

b) With respect to the relation you found before, please give a recursive formula to calculate $\binom{n}{k}$ with $n, k \in \mathbb{N}$ that appears reasonable. Please give some reasoning.

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \Rightarrow \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

3.3 Mathematical Induction and Proofs

a) Please describe, how induction works as a proof. For what kind of problems is it well suited and for what kind of problems is it badly applicable and why?

$\Rightarrow$ infer a conclusion from general principles

well suited e.g. for: trees, correctness of algorithms, recursion

not well suited for: expressing general principles

b) Name the four steps that you have to perform in every induction (including the ones in the exam!)

Given a statement $P(n)$, $\forall n \in \mathbb{Z}$, let a be $\in \mathbb{Z}$ fixed

1. show: $P(a)$ true
2. $\forall k \in \mathbb{Z} \geq a$, $P(k) \rightarrow P(k+1)$
3. suppose $P(k)$ true for arbitrary chosen k
4. show $P(k+1)$ true

$$\frac{n(n+1)}{2}$$

c) Prove the following statements using induction:

i. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}^+$.

1\. $a = 1$, $\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$ is true

2\. suppose $\forall k \in \mathbb{Z}, k \geq a$, $P(k)$ true $\rightarrow$ $P(k) = \frac{k(k+1)}{2}$

3\. $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)\cdot(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2} \Rightarrow \frac{n(n+1)}{2}$ thus true

ii. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \forall n \in \mathbb{N}^+$.

1. $a=1 \quad \sum_{i=1}^{1} i^2 = \frac{1(2)(2+1)}{6} = 1$ true

2. suppose $\forall k \in \mathbb{Z},\ k \geq a$, assume $P(k)$ true $\rightarrow \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

3. show $P(k+1)$ true

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} \text{ mit } n=k+1 = \sum_{i=1}^{k+1} i^2 \Rightarrow \underline{\underline{\text{true}}}$$

d) Now prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$ without induction.

$$S = 1+2+3+4 \dots (n-3)+(n-2)+(n-1)+n$$

$$2S = \underbrace{(n+1)+(n+1) \dots (n+1)}_{n \text{ times}}$$

$$S = \frac{(n+1)\cdot n}{2}$$

e) Given $P(n) = (2^n < (n+1)!) \forall n \in \mathbb{N}^+$. ($P$ takes a positive integer and returns a boolean.)

i. Write $P(2)$, is $P(2)$ true?

$2^2 < (2+1)! = 3 \cdot 2 \cdot 1 = 6 \qquad 4 < 6$ is true

ii. Write $P(k)$.

$2^k < (k+1)! = (k+1) \cdot k!$

iii. Write $P(k+1)$.

$2^{k+1} < (k+1+1)! = (k+2)(k+1) \cdot k!$

iv. In a proof by mathematical induction that this inequality holds for all integers $n \geq 2$, what must be shown in the inductive step?

Assume $P(k)$ true

show $P(k+1)$ true

$2^{k+1} < (k+2)(k+1) \cdot k!$

$2 \cdot 2^k < (k+2)(k+1) \cdot k!$ and we know $2^k < (k+1) \cdot k!$

we thus need to proof:

$$2 < k + 2$$

which is true because we know $k > 2$