Example. Let $a, b \in \mathbb{R}$. If $f(x) = ax + b$ then $f'(x) = a$.

Proof. One way: $y = ax + b$ is the equation of a straight line with slope a. Since the derivative is the slope of tangent function, $\frac{dy}{dx} = a$.

Another way: we use the limit definition of derivative.

$$
\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} \\
&= \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} \\
&= \lim_{h \to 0} \frac{ah}{h} \\
&= \lim_{h \to 0} a \\
&= a.
\end{aligned}
$$

Example. If $f(x) = b$, b a constant, then $f'(x) = 0$. Reason: take $a = 0$ in the previous example. Also note that the graph of $y = b$ is a horizontal straight line, which has slope 0.

Example. If $f(x) = x$, then $f'(x) = 1 = 1.x^0 = 1.x^{1-1}$. Reason: take $a = 1$ and $b = 0$ in the example on the previous slide.

Example. If $f(x) = x^2$, then $f'(x) = 2x = 2.x^{2-1}$. Reason: proof by limit-definition of derivative (done in previous lecture).

Example. If $f(x) = x^3$, then $f'(x) = 3x^2 = 3.x^{3-1}$. Reason: proof by limit-definition of derivative.

Example. $\frac{d}{dx}(x^4) = 4x^3$, $\frac{d}{dx}(x^5) = 5x^4$, $\frac{d}{dx}(x^6) = 6x^5$...

$$\vdots$$

Power rule. For any positive integer n, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

The above power rule can be proved by the *product rule* (below) and the *Principle of Mathematical Induction*.

Constant multiple rule. If $c \in \mathbb{R}$ is a constant, and f is a differentiable function, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x).$$

e.g., $\frac{d}{dx}(2x) = 2\frac{d}{dx}(x) = 2.1 = 2.$

Sum rule. If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

e.g., $\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) = 3x^2 + 2x$, using both the sum and power rules.

For any functions $f, g\colon \mathbb{R} \to \mathbb{R}$, we have $f(x) - g(x) = f(x) + (-1.g(x))$. Thus, applying the sum and constant multiple differentiation rules gives

Difference rule.

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Now we can differentiate any polynomial function.

Example. Let $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$. Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5) \\ &= 8x^7 + 12.5x^4 - 4.4x^3 + 10.3x^2 - 6.1 + 0 \\ &= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6. \end{aligned}$$

Remember: to say that f is *differentiable at* a means that $f'(a)$ exists.

Polynomial functions are differentiable everywhere; i.e., for all $a \in \mathbb{R}$.

Next we look at a function that is not differentiable (at a point in its domain).

Example. Let $f(x) = |x|$.

We try to find the derivative of f at $x = 0$:

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{|h|}{h}$ is not defined. For

$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$, while $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$.

Left- and right-hand limits at 0 exist, but don't agree; thus limit at 0 does not exist, so $f'(0)$ does not exist.