Define the *absolute value* function $|\cdot|\colon \mathbb{R} \to \mathbb{R}$ by $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$.

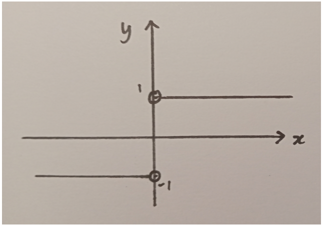
A property we use later: $|ab| = |a| \cdot |b|$.

Example. Find $\lim_{x\to 0} \frac{|x|}{x}$.

Solution. Suppose that $x > 0$. Then $\frac{|x|}{x} = \frac{x}{x} = 1$.
Suppose that $x < 0$. Then $\frac{|x|}{x} = \frac{-x}{x} = -1$.

So, no way function values can approach a fixed common value as x approaches 0 (from the left *and* the right): this function has no limit at $x = 0$. (Also 0 is not in the domain of f; but this is irrelevant as to whether or not f has a limit at 0.)

Graph of $y = |x|/x$:



(Note the holes at $(0, 1)$ and $(0, -1)$: function is undefined at $x = 0$.)

We say that this function has *left-hand* and *right-hand* limits at $x = 0$.

BUT, because these one-sided limits don't agree, the function has no limit at $x = 0$.

The ϵ-δ definition of limit

In mathematical proofs, a rigorous definition of limit is required, as follows.

$\lim_{x \to a} f(x) = l$ means that: for each positive real number ϵ, there exists a positive real number δ such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \epsilon.$$

For two real numbers a, b, the absolute value $|a - b|$ is the *distance* between a and b (always non-negative).

$|x - a|$ in the above definition accounts for x approaching a from the left (always $x < a$) and x approaching a from the right (always $x > a$).

Similarly, $|f(x) - l|$ is measuring the distance between the y-coordinate $f(x)$ and the limit l: we could have $f(x) > l$ or $f(x) < l$.

Also note $0 < |x - a|$, i.e., $x \neq a$; remember that a *need not be in the domain of* f for f to have a limit at a.