Product rule. If $u = u(x)$ and $v = v(x)$ are differentiable functions, then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

or more briefly in dash notation

$$(uv)' = u'v + uv'.$$

Example. Differentiate $y = (6x^3)(7x^4)$.

Solution. Applying the product rule:

$$\frac{dy}{dx} = (18x^2)(7x^4) + (6x^3)(28x^3) = 126x^6 + 168x^6 = 294x^6.$$

Also, $y = 42x^7$, and differentiating directly: $\frac{dy}{dx} = 7.42x^6 = 294x^6$.

Quotient rule. If $u = u(x)$ and $v = v(x)$ are differentiable functions, and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example. Differentiate $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$.

Solution. Let $u = x^2 + x - 2$ and $v = x^3 + 6$. Then

$$\begin{aligned}
\frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2).3x^2}{(x^3 + 6)^2} \\
&= \frac{2x^4 + 12x + x^3 + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3 + 6)^2} \\
&= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}.
\end{aligned}$$

Recall we have the *power rule*: for any positive integer n,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Now we can use the quotient rule to deduce that for *any* integer n (positive, negative, zero),

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof: exercise.

Example. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = -x^{-1-1} = -x^{-2} = -\dfrac{1}{x^2}.$

In fact, it can be proved that the power rule holds for all *rational* powers (proof uses the Chain Rule; to be covered later).
Remember rational powers such as $x^{1/2} = \sqrt{x}$, $x^{1/3} = \sqrt[3]{x}$, $x^{1/n} = \sqrt[n]{x}$ for positive integer n, $x^{3/2} = (x^{1/2})^3 = (\sqrt{x})^3 = \sqrt{x^3}$, etc.

Example.

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

Example.

$$\frac{d}{dx}(\sqrt[3]{x^2}) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{\frac{2}{3}-1} = \tfrac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}.$$

There is a strong connection between continuity and differentiability.

Theorem. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. If f is differentiable at a, then f is continuous at a.

Caution: although differentiable $\implies$ continuous, the converse does **not** hold; i.e., a continuous function need not be differentiable.

For example, saw earlier that $f(x) = |x|$, which is continuous everywhere, is not differentiable at $x = 0$.

Proof. By the sum and product limit rules,

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}\big(f(x)-f(a)+f(a)\big) \\
&= \lim_{x\to a}\big(f(x)-f(a)\big)+\lim_{x\to a} f(a) \\
&= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)+f(a) \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)+f(a) \\
&= f'(a)\cdot 0+f(a) \qquad\qquad \because f'(a) \text{ exists!} \\
&= f(a).
\end{aligned}$$

This proves that $\lim_{x\to a} f(x) = f(a)$; by definition, f is continuous at a.