Example. Find the points at which the following function is continuous:

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x \neq -1 \\ -2 & x = -1. \end{cases}$$

Solution. If $x \neq -1$ then

$$f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x-1)(x+1)}{x+1} = x - 1$$

so that, as a polynomial function, $f(x)$ is continuous at all $x \neq -1$.

Now $f(-1) = -2$ and

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2.$$

Thus $\lim_{x \to -1} f(x) = f(-1)$, so this $f(x)$ is continuous at $x = -1$ also: it is continuous for all $x \in \mathbb{R}$.

One-sided continuity

Let $y = f(x)$ be a function. Then f is *continuous from the right at* a if

- a is in the domain of f;
- $\lim_{x \to a^+} f(x)$ exists, and
- $\lim_{x \to a^+} f(x) = f(a)$.

Here the notation $\lim_{x \to a^+} f(x)$ means 'right-hand limit'. That is, we consider the limit only for $x > a$.

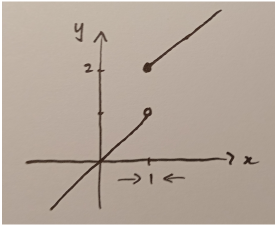
Similar notation $\lim_{x \to a^-} f(x)$ for 'left-hand limit' (only consider $x < a$) and then *continuous from the left at* a.

Thus $\lim_{x \to a} f(x) = l$ if and only both left- and right-handed limits exist, and are equal: $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$.

Example. Let

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 1 \\ x & \text{if } x < 1. \end{cases}$$

This function has graph



We see that $\lim_{x \to 1^+} f(x) = 2$, and $\lim_{x \to 1^-} f(x) = 1$.
Both left- and right-handed limits exist at 1 but are unequal: this function is discontinuous at $x = 1$.

Continuity on a closed interval

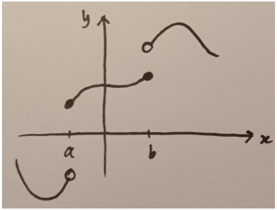
The *open interval* (a, b), where $a < b$, is all $x \in \mathbb{R}$ such that $a < x < b$; the *closed interval* $[a, b]$ is all $x \in \mathbb{R}$ such that $a \leq x \leq b$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is said to be *continuous on the closed interval* $[a, b]$ if

- f is continuous (at every point) on the open interval (a, b);
- $\lim_{x \to b^-} f(x)$ exists and is equal to $f(b)$;
- $\lim_{x \to a^+} f(x)$ exists and is equal to $f(a)$.

That is, f is continuous on $[a, b]$ if it is continuous from the left at b, continuous from the right at a, and continuous at every point in between.

Example. The following graph depicts a function that is continuous on $[a, b]$.



However, the function is discontinuous at a and at b (the left- and right-handed limits exist there, but do not agree there).

Example. Let $f(x) = \sqrt{1 - x^2}$ (semicircle centered at $(0, 0)$, radius 1). The domain of f is $[-1, 1]$ and f is continuous on $[-1, 1]$ (check).

Intermediate Value Theorem

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function continuous on $[a, b]$ with $f(a) \neq f(b)$.

Let r be *any* real number *between* $f(a)$ and $f(b)$: either $f(a) > r > f(b)$ or $f(a) < r < f(b)$ depending on f.

Then there exists $c \in (a, b)$ such that $f(c) = r$.