Week 7, lecture 1: Matrix algebra

MA180/185/190 Algebra

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Matrix algebra

Arithmetic properties of matrix operations

Matrices

Given a matrix A we refer to the entry in row i and column j as a_{ij}.

Example. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Then $a_{11} = 2$, $a_{12} = 3$, $a_{21} = 1$, and $a_{22} = -1$.

Two matrices A and B are **equal** if they have the **same size** and their corresponding entries are equal.

Matrix operations

Addition/subtraction

If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A. The difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A.

Note. Matrices of different sizes cannot be added or subtracted.

Example. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -5 \end{pmatrix}$$

Matrix operations

Scalar multiplication

If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying **each entry** of the matrix A by c. We say that the matrix cA is a scalar multiple of A.

Example. Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Then $3A = \begin{pmatrix} 6 & 9 & 3 \\ 3 & -3 & 0 \end{pmatrix}$

Example. Let $B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -1 & 0 \\ -5 & 6 & 2 \end{pmatrix}$. Then $(-1)B = \begin{pmatrix} -1 & -3 & -1 \\ -1 & 1 & 0 \\ 5 & -6 & -2 \end{pmatrix}$

Matrix operations

The following operation is the “natural” product between matrices. Note the requirement about the number of columns of the first matrix and the number of rows of the second matrix.

Matrix multiplication

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows:
To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add the resulting products.

Matrix multiplication: examples

Example. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}$. Since A is a 3×2 matrix and B is a 2×2 matrix, we can perform the product AB. The resulting matrix is

$$AB = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 2\cdot1+3\cdot4 & 2\cdot3+3\cdot(-2) \\ 1\cdot1+1\cdot4 & 1\cdot3+1\cdot(-2) \\ -1\cdot1+0\cdot1 & -1\cdot3+0\cdot(-2) \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 5 & 1 \\ -1 & -3 \end{pmatrix}$$

Note that the two matrices above cannot be multiplied in the opposite order. Indeed, since B is 2×2 and A is 3×2, the product BA cannot be computed.

In general $AB \neq BA$

Consider now two square matrices A and B of the same size. Then we can compute both AB and BA. These two matrices are, in general, different!

Example. Let $A = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2+(-3) & 1+3 \\ 8+2 & 4+(-2) \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 10 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 3 & -5 \end{pmatrix}$$

So $AB \neq BA$!

Exercises

Exercise. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 2\pi & 6 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 & 1/2 \\ 1 & 2 & -1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \\ 3 & -1 \\ \sqrt{5} & 0.6 \end{pmatrix}$.

Decide which of the following can be evaluated. If an expression can be evaluated, do so. If not, explain why:

$A + B, \quad A + C, \quad AB, \quad BC, \quad CB, \quad AD, \quad DA.$

Properties of matrix operations

Provided that the sizes allow for the operations to be performed, the following hold

$$A+B=B+A$$

$$(A+B)+C = A+(B+C)$$

$$(AB)C = A(BC)$$ ⚠ careful to the order!

$$A(B+C) = AB+AC$$

$$(A+B)C = AC+BC$$