Domain and range

Consider a function $f\colon \mathbb{R} \to \mathbb{R}$.

The *domain* of f is the set of all real numbers a such that $f(a)$ is *defined*.

e.g., for $f(x) = x^2$, the domain is all of $\mathbb{R}$: a^2 is defined for all $a \in \mathbb{R}$.

e.g., for $f(x) = \frac{1}{x}$, the domain is NOT $\mathbb{R}$: $\frac{1}{0}$ is undefined. However $\frac{1}{a}$ is defined for all $a \neq 0$. Hence this function has domain $\mathbb{R} \setminus \{0\}$.

The *range* of f is the set of all $f(a)$ for a in the domain of f.
(So the domain of f is the set of valid x-values for f; the range is the set of valid y-values for f.)

A function $f\colon \mathbb{R} \to \mathbb{R}$ is then a rule for assigning to each element x of its domain the element $f(x)$ in its range. Remember: given x in the domain of f, there can only be one and only one $f(x)$.

Domain and range can usually be worked out from

- the rule given for the function;
- the graph of the function.

Example. $y = f(x) = 3x - 2$ has domain = range = $\mathbb{R}$.
First: $3a - 2$ is defined for all $a \in \mathbb{R}$.
Secondly, let $b \in \mathbb{R}$. Then $b = 3(\frac{b+2}{3}) - 2$; so $b = f(a)$ where $a = \frac{b+2}{3}$. All real numbers occur as second components/y-coordinates of pairs in this f.

Example. $y = f(x) = \frac{1}{x}$ has domain $= \mathbb{R} \setminus \{0\}$. What is its range? Look at the graph. For every value on the y-axis EXCEPT $y = 0$, we can draw a horizontal line across that cuts the graph of $y = \frac{1}{x}$. Thus range $= \mathbb{R} \setminus \{0\}$.

Example. $y = f(x) = 3x^2 + 3$ has domain $= \mathbb{R}$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, and thus $x^2 + 3 \geq 3$, we see that the range of this function is all real numbers ≥ 3. (Think about why, for each $b \in \mathbb{R}$ such that $b \geq 3$, there is $a \in \mathbb{R}$ such that $f(a) = b$; try to solve $b = 3a^2 + 3$ for a.)

Example. $y = \sin x$ has domain $= \mathbb{R}$. However, from its graph, we see that $\sin x$ oscillates infinitely between $+1$ and -1, hence the range of $\sin x$ is the closed interval $[-1, 1]$; i.e., all y such that $-1 \leq y \leq 1$.

$y = \cos x$ has the same domain $= \mathbb{R}$, range $= [-1, 1]$ as $y = \sin x$.

Example. $\tan x = \frac{\sin x}{\cos x}$ is undefined when $\cos x$ is zero, i.e., when x is an odd (positive or negative) multiple of $\pi/2$.

So the domain of $f(x) = \tan x$ is all real numbers except $(2k+1)\pi/2$, $k \in \mathbb{Z}$ (here $\mathbb{Z}$ denotes the set of integers $\{\cdots, -3, -2, -1, 0, 1, 2, \ldots\}$).

What is the range of $\tan x$?

Example. $y = \sqrt{4 - x^2}$ has domain all x such that $x^2 \leq 4$ (cannot take square root of negative real number & get a real number).

Now $x^2 \leq 4$ if and only if $-2 \leq x \leq 2$, i.e., $x \in [-2, 2]$.

What is the range of this function? Look at its graph: semicircle above the x-axis, cutting y-axis at $y = 2$. Hence the range of this function is $= [0, 2]$.

Limits

The concept of limit is *absolutely fundamental* to all of Calculus.
As usual, helps to introduce the concept with examples.

Consider the function $y = f(x) = \frac{x^2-1}{x-1}$. The domain excludes $x = 1$, because $f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$, which is undefined. However, we can calculate:

- $f(0.9) = 1.9$
- $f(0.95) = 1.95$
- $f(0.99) = 1.99$
- $f(0.999) = 1.999$
- $f(0.9999) = 1.9999$

$$\vdots$$

...and

- $f(1.1) = 2.1$
- $f(1.01) = 2.01$
- $f(1.001) = 2.001$
- $f(1.0001) = 2.0001$
- $f(1.00001) = 2.00001$

$$\vdots$$

Clearly, although this $f(x)$ is undefined at x exactly equal to 1, it has a limiting behaviour as x 'approaches' 1 (from the left or the right): we can see that the limiting value of the function for such x is 2.

To repeat: $f(x) = \frac{x^2-1}{x-1}$ is undefined at $x = 1$, but it has *limit* of 2 as x approaches 1.

In symbols:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$