Example. Let $f(x) = x^3 - 3x^2 + 1$. Find the absolute extrema of f on $[-\frac{1}{2}, 4]$.

Solution.

Step 1. $f'(x) = 3x^2 - 6x$ is defined everywhere. So the critical points are where $3x^2 - 6x = 3x(x-2) = 0$, i.e., $x = 0$ and $x = 2$.

Step 2. $f(0) = 1$ and $f(2) = 8 - 12 + 1 = -3$.

Step 3. $f(-1/2) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$ and $f(4) = 64 - 48 + 1 = 17$.

Step 4. Selecting from $-3, \frac{1}{8}, 1, 17$, we see that the absolute minimum value is -3, at $x = 2$, and the absolute maximum value is 17, at $x = 4$.

Derivatives and graphs

Proceeding from the interpretation of derivative as slope of tangent line, differentiation can reveal much information about the graph of a function.

Definition. A function $f \colon \mathbb{R} \to \mathbb{R}$ is *increasing* on an interval if $x_1 < x_2 \implies f(x_1) < f(x_2)$ for all x_1, x_2 in the interval.

A function $f \colon \mathbb{R} \to \mathbb{R}$ is *decreasing* on an interval if $x_1 < x_2 \implies f(x_1) > f(x_2)$ for all x_1, x_2 in the interval.

With regard to the graph of f: if f is increasing then its graph *rises* as we move from left to right along the x-axis; if f is decreasing then its graph *descends* as we move from left to right.

Graphs of increasing functions (with tangent lines):

Tangents all have *positive* slope.

Graphs of decreasing functions (with tangent lines):

Tangents all have *negative* slope.

The above pictures suggest a test to determine where a function is increasing or decreasing:

if $f'(x) > 0$ on an interval, then $f(x)$ is increasing on the interval;

if $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on the interval.

Example. Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing, and where it is decreasing.

Solution.
$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$.

Thus f' has roots $-1, 0, 2$. We work out the sign of $f'(x)$ as follows.

(i) $x < -1$: x negative, $x - 2$ negative, $(x+1)$ negative.

(ii) $-1 < x < 0$: x negative, $x - 2$ negative, $(x+1)$ positive.

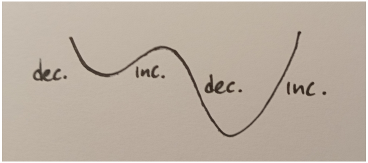
(iii) $0 < x < 2$: x positive, $x - 2$ negative, $(x+1)$ positive.

(iv) $x > 2$: x positive, $x - 2$ positive, $(x+1)$ positive.

Solution, continued. Taking products of signs (if even number of '$-$' then product is positive; if odd number of '$-$' then product is negative):

(i) $x < -1$: $f'(x)$ is $- - -$, which is $-$: decreasing.
(ii) $-1 < x < 0$: $f'(x)$ is $- - +$, which is $+$: increasing.
(iii) $0 < x < 2$: $f'(x)$ is $+ - +$, which is $-$: decreasing.
(iv) $x > 2$: $f'(x)$ is $+ + +$, which is $+$: increasing.

Graph of f looks as below, confirming (i)–(iv). (Graph sketching: later.)



The *First Derivative Test* is a consequence of the increasing/decreasing test; it determines the nature of a critical point; i.e., whether the point is max, or min, or neither. (A *Second Derivative Test* will be defined later.)

Let c be a critical point of a continuous function f.

- If f' changes from positive to negative at c then f has a local maximum at c: $/\backslash$.
- If f' changes from negative to positive at c then f has a local minimum at c: $\backslash/$.
- If the sign of f' does not change through c, then f has neither a local maximum nor local minimum at c: ${}_{/}{}^{/}$ or ${}^{\backslash}{}_{\backslash}$.

Example. Previous example: $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ has critical points -1, 0, 2. Also calculated:

Critical point		-1		0		2	
Sign of f'	$-$		$+$		$-$		$+$

Thus, by the first derivative test:

- local min at -1;
- local max at 0;
- local min at 2.

Again, graph of f looks like