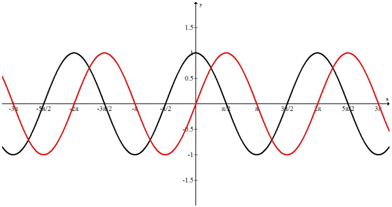
Trigonometric functions

Recall basic facts about sin, cos, and tan. Below, the graph of $y = \sin x$ is in red, $y = \cos x$ is in black.



Angular measurements are in radians: 2π radians = 360 degrees.

Trigonometric identities: $\sin^2\theta + \cos^2\theta = 1$,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

An important trigonometric limit:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \tag{1}$$

The proof uses the inequalities

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

which can be proved by geometry of triangles.

Limit rules, the trigonometric identity $\cos^2 \theta - 1 = -\sin^2 \theta$, and (1) can then be used to prove

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0. \tag{2}$$

Now we can differentiate trigonometric functions.

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \qquad \text{(trig. identity)} \\
&= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \cos x \, \frac{\sin h}{h} \right) \\
&= \sin x . \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x . \lim_{h\to 0} \frac{\sin h}{h} \qquad \text{(limit rules)} \\
&= \sin x . 0 + \cos x . 1 \qquad \text{(limits (1), (2) above)} \\
&= \cos x .
\end{aligned}
$$

This proves that

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}.$$

Similarly we can prove that

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}.$$

Example. We differentiate $\tan x$, using the above and the quotient rule.

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.\end{aligned}$$

$f(x) = \frac{1}{\cos x}$ is called the secant function, $\sec x$. So $\frac{d}{dx}(\tan x) = \sec^2 x$.

Example. Differentiate $\sin 2x$.

Solution. $\sin 2x$ is a function of a function; $\sin$ ['outside' function] of $2x$ ['inside' function]. Differentiate such composite functions by the *chain rule* (next topic).

For now, we use the trigonometric identity: $\sin 2x = 2\sin x \cos x$ (from $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$). So

$$\begin{aligned}\frac{d}{dx}(\sin 2x) &= \frac{d}{dx}(2\sin x\cos x)\\ &= 2\left(\frac{d}{dx}(\sin x)\cos x + \sin x\frac{d}{dx}(\cos x)\right)\\ &= 2(\cos^2 x - \sin^2 x) \;=\; 2\cos 2x\end{aligned}$$

because $\cos^2 x - \sin^2 x = \cos 2x$ (from the trigonometric identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$).

The chain rule

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions.

Then the *composite* of f with g, denoted $f \circ g$, is defined by

$$f \circ g(x) = f(g(x)).$$

Example. Let $f(x) = x - 2$, $g(x) = x^3$. Then $f(g(x)) = f(x^3) = x^3 - 2$. That is, $f \circ g(x) = x^3 - 2$.

However, $g \circ f(x) = g(x - 2) = (x - 2)^3$; so $f \circ g \neq g \circ f$.

Example. Let $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$. Then $f \circ g(x) = \sqrt{x^2 + 1}$ and $g \circ f(x) = x + 1$.