The *Chain Rule* is used to differentiate composite functions $f \circ g(x) = f(g(x))$.

This rule (theorem) states that

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Perhaps easier to remember as follows. Let $u = g(x)$ (g is the 'inside' function) and then $y = f \circ g(x) = f(u)$ (f is the 'outside' function). Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

First, differentiate the outside function f w.r.t. (with respect to) the inside function g. Second, differentiate the inside function g w.r.t. x. Then take the product of these two derivatives.

Example. Differentiate $\sqrt{x+x^2}$.

Solution. Let $u = x + x^2$. Let $y = \sqrt{u} = u^{1/2}$. Then

$\frac{d}{dx}(\sqrt{x+x^2}) = \frac{d}{dx}(y(u)) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}(1+2x) = \frac{1+2x}{2u^{1/2}} = \frac{1+2x}{2\sqrt{x+x^2}}$.

Example. Differentiate $\sin(4x^2 - 7)$.

Solution. Differentiate the outside function w.r.t. the inside function: $\cos(4x^2 - 7)$.

Mutiply by the derivative of the inside function w.r.t. x, namely $8x$.

Hence $\frac{d}{dx}(\sin(4x^2 - 7)) = 8x\cos(4x^2 - 7)$.

Example. Since $\frac{u}{v} = u.v^{-1}$, the quotient rule can be inferred from the product rule, differentiating the second factor in $u.v^{-1}$ with the chain rule.

That is, $\frac{d}{dx}(u/v) = \frac{d}{dx}(u.v^{-1}) = u\frac{d}{dx}(v^{-1}) + \frac{du}{dx}.v^{-1}$

$$= u(-v^{-2}\frac{dv}{dx}) + v\frac{du}{dx}.v^{-2}$$

$$= -(u\frac{dv}{dx})/v^2 + v\frac{du}{dx}/v^2$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example. Assuming that $\frac{d}{dx}(x^n) = n.x^{n-1}$ for all integers n, infer the power rule $\frac{d}{dx}(x^{1/n}) = \frac{1}{n}.x^{\frac{1}{n}-1}$ where n is an integer, $n \neq 0$.

Solution. We have $(x^{\frac{1}{n}})^n = x$. Differentiating the RHS of this equality, x, gives 1. Differentiating the LHS, $(x^{\frac{1}{n}})^n$, by the chain rule gives

$$n(x^{\frac{1}{n}})^{n-1}.\frac{d}{dx}(x^{\frac{1}{n}}) = nx^{1-\frac{1}{n}}\frac{d}{dx}(x^{\frac{1}{n}}).$$

Equating LHS with RHS:

$$nx^{1-\frac{1}{n}}.\frac{d}{dx}(x^{\frac{1}{n}}) = 1 \implies \frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{nx^{1-\frac{1}{n}}} = \frac{1}{n}(x^{1-\frac{1}{n}})^{-1} = \frac{1}{n}.x^{\frac{1}{n}-1}.$$

After the preceding example, we can use the chain rule further to prove that, for integers m and $n \neq 0$,

$$\frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^{m} = \tfrac{m}{n}x^{\frac{m}{n}-1}.$$

So now we have the power rule

$$\frac{d}{dx}(x^{a}) = ax^{a-1}$$

for all *rational numbers* (quotients of integers) a.

Actually $\frac{d}{dx}(x^{a}) = ax^{a-1}$ for *all real numbers* a.