Maxima and minima

Many applications of differential calculus involve *optimisation*: finding the best way to complete a task under constraints.

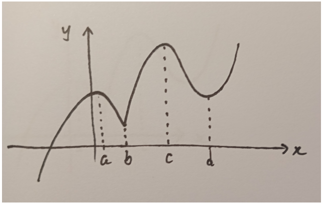
E.g.: a company makes cylindrical cans, each of volume 380 cm^3. Find the dimensions of each can that minimise the amount of material (hence cost) needed.

E.g.: we have 6 m^2 of cardboard to construct a box with a square base. Find the dimensions of the box with maximum possible volume.

Such optimisation problems can be solved by calculus.

Key words: *maximum*, *minimum*.

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ has a *local minimum* at $x = c$ if $f(x) \geq f(c)$ for all x 'near' c, i.e., all x in an open interval containing c. The function f has a *local maximum* at c if $f(x) \leq f(c)$ for all x 'near' c.

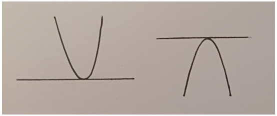


A function with this graph has local minima at b, d; local maxima at a, c.

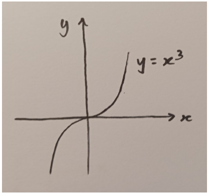
Note that there are function values $f(x)$ less than $f(b) < f(d)$; and there could be $f(x)$ greater than $f(c) > f(a)$.

Theorem (Fermat). If f has a local maximum or local minimum at c, and f is differentiable at c, then $f'(c) = 0$.

The following picture illustrates the theorem:



N.B. converse is false. E.g., if $f(x) = x^3$ then $f'(x) = 3x^2$, so $f'(0) = 0$. But graph clearly shows no local max or min at $x = 0$.



Example. Let $f(x) = |x|$. Then f has a local minimum at $x = 0$. However, f is not differentiable at 0.

Definition. A *critical point* of a function f is a number c such that either $f'(c) = 0$ or f is not differentiable at c.

So Fermat's theorem states: *if f has a local maximum or local minimum at c, then c is a critical point of f.*

Example. Let $f(x) = x^{\frac{3}{5}}(4-x)$. Find all critical points of f.

Solution. By the product rule we have

$$\frac{d}{dx}(x^{\frac{3}{5}}(4-x)) = \frac{3}{5}x^{-\frac{2}{5}}(4-x) + x^{\frac{3}{5}}(-1) = \tfrac{3(4-x)}{5x^{2/5}} - \tfrac{5x}{5x^{2/5}}.$$

Thus

$$f'(x) = \frac{12-8x}{5x^{\frac{2}{5}}}.$$

At $x = 0$, $f'(x)$ is undefined; at $x = \frac{12}{8} = \frac{3}{2}$, $f'(x) = 0$.

So the critical points of this function are 0 and $3/2$.

Absolute extrema

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function defined on a set D (usually the domain of f).

Then f has an *absolute* (or *global*) *maximum* at a point c in D if $f(x) \leq f(c)$ for all $x \in D$.

Similarly, f has an *absolute* (or *global*) *minimum* at c in D if $f(x) \geq f(c)$ for all $x \in D$.

Example. Let $f(x) = x^2$. Then $f(x) \geq 0$ for all x. Also $f(0) = 0$. Thus f has an absolute minimum value, 0, at $x = 0$.

Example. Let $f(x) = x^3$. Then $f(x)$ has no absolute maximum nor minimum on its domain $\mathbb{R}$ (no local max or min: look at the graph again).

Absolute extrema are local extrema, so by Fermat's theorem they occur at critical points.

Extreme Value Theorem. A function continuous on a closed interval $[a, b]$ has an absolute maximum and an absolute minimum in $[a, b]$.

Method to find the absolute extrema of a continuous function f on a closed interval $[a, b]$.

1. Find all critical points of f.
2. For each critical point c, determine $f(c)$.
3. Determine $f(a)$ and $f(b)$.
4. The greatest function value found in steps 2 and 3 is an absolute maximum; the least function value found is an absolute minimum.