Week 6, lecture 2:
Systems of linear equations. Matrix algebra
MA180/185/190 Algebra

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Systems of linear equations

Matrix algebra

Example

Use Gaussian elimination to find a solution (if it exists) to:

$$\begin{cases} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -3 \\ a + 4b + 2c &= 0 \end{cases}$$

First, let's find the augmented matrix of this system. (Remember: we put the coefficients in order in each row. If a variable does not appear in an equation, then its coefficient is 0).

Here:

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -3 \\ 1 & 4 & 2 & 0 \end{array}\right)$$

Our goal is to bring this to a staircase form, so we start by swapping R_1 and R_3 which brings us one step closer to the staircase shape - - - - >

Example

this is because all coefficients are divisible by 3, so we can work with smaller numbers

Use Gaussian elimination to find a solution (if it exists) to:

$$\begin{cases} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -3 \\ a + 4b + 2c &= 0 \end{cases}$$

this is to get a 0 at the beginning of the 2nd row.

$$\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & 6 & -3 & -3 \\ 0 & -2 & 3 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}\cdot R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -2 & -3 & -1 \\ 0 & -2 & 3 & 1 \end{array}\right) \xrightarrow{R_3 - R_2}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 6 & 2 \end{array}\right)$$

This is now in staircase form and we can solve it by backsubstitution: the third row gives us $6c = 2$ $\boxed{c = 1/3}$. The second row (plus the fact that $c = 1/3$) tells us that $-2b - 3 \cdot \frac{1}{3} = -1$ so $-2b = 0$ so $\boxed{b = 0}$

The first equation becomes $a + 4 \cdot 0 + 2 \cdot \frac{1}{3} = 0$ so $\boxed{a = -\frac{2}{3}}$

Gaussian elimination

Some observations:

1. For now we will only deal with systems of linear equations in which the number of unknowns is equal to the number of equations
2. A system of n linear equations in n unknowns doesn't always admit a (unique) solution. There are cases in which there are infinitely many solutions (we will not discuss those), cases in which there is a unique solutions, and cases in which a solution does not exist.

We will consider systems of 2 equations in 2 and systems of 3 equations in 3 unknowns.

When does a solution exist?

We will consider systems of 2 equations in 2 and systems of 3 equations in 3 unknowns.

Let's look at the following. Suppose that by using elementary row operations we obtain the following augmented matrix:

$$\left(\begin{array}{ccc:c} 1 & 3 & 1 & 7 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & \boxed{0} & 6 \end{array}\right)$$

this pivot being 0 is the problem.

This is equivalent to the following system in unknowns x, y, z:

$$\begin{cases} x + 3y + z = 7 \\ y + 3z = 9 \\ 0 = 6 \end{cases}$$

This tells us that the system has no solution.

When does a solution exist?

The goal of Gaussian elimination is to bring the augmented matrix to this form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

(this is an example of a system of 3 equations in 3 unknowns).

The highlighted numbers are called PIVOTS.

If (up to reordering the rows) we cannot bring an augmented matrix to the above shape where ALL THREE PIVOTS are non-zero, then the system does not admit a unique solution.

Exercises

Exercise. Use Gaussian elimination to solve **Problem 1.** (parabola through three points).

Exercise. Suppose you are asked to find three real numbers such that the sum of the numbers is 12, the sum of two times the first plus the second plus two times the third is 5, and the third number is one more than the first. Model the three conditions through a system of linear equations and use Gaussian elimination to determine the three numbers.

Exercise. Use Gaussian elimination to solve the following system

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

Systems of linear equations

Matrix algebra

Motivation

We have used matrices (and Gaussian elimination) to model and solve systems of linear equations. Wee will now discuss matrix algebra more systematically. This will help us learn

- A new approach to solving systems of linear equations.
- Describe and study linear transformations (in the plane and in 3-dimensional space.
- Eigenvalues and eigenvectors.

We'll also see new applications of (variants of) Gaussian elimination.

Matrices

"m by n"

A real $m \times n$ matrix is a "grid" or an array consisting of m rows and n columns filled with real numbers. We call **entries** the numbers appearing in a matrix.

Example. $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ is a 2×2 matrix.

$B = \begin{pmatrix} -1 & 1 \\ 2\pi & 6 \end{pmatrix}$ is also a 2×2 matrix.

$C = \begin{pmatrix} 2 & 0 & 1/2 \\ 1 & 2 & -1 \end{pmatrix}$ is a 2×3 matrix.

$D = \begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \\ 3 & -1 \\ \sqrt{5} & 0.6 \end{pmatrix}$ is a 4×2 matrix.

Matrices

Given a matrix A we refer to the entry in row i and column j as a_{ij}.

Example. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Then $a_{11} = 2$, $a_{12} = 3$, $a_{21} = 1$, and $a_{22} = -1$.

Two matrices A and B are **equal** if they have the **same size** and their corresponding entries are equal.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$ then

$A = B$ if $a_{11} = b_{11}$, $a_{21} = b_{21}$, $a_{12} = b_{12}$, $a_{22} = b_{22}$