Limits

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function.

$\lim_{x\to a} f(x) = l$ means that: *the values $y = f(x)$ can be made as close to l as we want by choosing x sufficiently close to a.*

In the example above, note that $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ as long as $x \neq 1$; for then we can cancel the common factor $x-1$ on top and bottom. Then

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2,$$

which agrees with our calculations of $f(x)$ for x close to 1.

Example. Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution. $\frac{x^3-1}{x-1}$ is not defined at $x = 1$:
$\lim_{x\to 1}(x^3 - 1) = 1^3 - 1 = 0 = \lim_{x\to 1}(x - 1)$.

However note that $x^3 - 1$ has 1 as a root (just seen above), so we can divide $x - 1$ exactly into $x^3 - 1$: $x^3 - 1 = (x - 1)(x^2 + x + 1)$ (check).

Thus

$$\frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$$

as long as $x \neq 1$. Since in the limit we are not interested in x exactly equal to 1, this implies that

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1}(x^2 + x + 1) = 1^2 + 1 + 1 = 3.$$

Limits may not exist.

Example. Find $\lim_{x \to 0} \frac{1}{x}$.

Solution. As can be seen from the graph of $y = 1/x$, the closer we get to zero on the x-axis (from the right or left, i.e., $x > 0$ & x closer and closer to 0, or $x < 0$ & x closer and closer to 0), the larger $1/x$ becomes in absolute value.

Hence $y = 1/x$ cannot approach a fixed limiting value: this function has no limit at 0 (but it does have a limit everywhere else; just substitute in).