Week 8, lecture 1:

MA180/185/190 Algebra

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Matrix transformations

Recap

Matrix transformations

Recap

Example/recap. Consider the following system of linear equations:

$$\begin{cases} 5x + 3y + 2z = 4 \\ 3x + 3y + 2z = 2 \\ y + z = 5 \end{cases}$$

1. Write the system in matrix form.
2. Solve the system by inverting the coefficient matrix.

1. $A = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$; $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $\underline{b} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

Then our system is equivalent to $A\underline{x} = \underline{b}$

so we're looking for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\underline{b}$

Example/recap

2. We find the inverse of A by applying Gaussian elimination to the augmented matrix

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 3 & 2 & -3/2 & 5/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & -3/2 & -5/2 & 3 \end{array}\right)$$

A^{-1}

Example/recap

Our solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 + (-1) + 0 \\ -6 + 5 - 10 \\ 6 - 5 + 15 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \\ 16 \end{pmatrix}$$

<u>Exercise</u> verify that $\begin{pmatrix} 1 \\ -11 \\ 16 \end{pmatrix}$ is a solution

Recap

Matrix transformations

Matrix transformations

Recall that a function is a rule f that associates with each element of a set X (domain) one and only one element in a set Y (codomain).

If f associates the element b with the element a, then we write

$$b = f(a)$$

and we say that b is the image of a under f or that $f(a)$ is the value of f at a.

The subset of the codomain that consists of all images of elements in the domain is called the range of f.

While in many applications the domain and codomain are (subsets of) the real numbers, here we will consider functions whose domain is $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$ for some positive integers m, n. We will often call these functions **transformations**.

Matrix transformations

If T is a function with domain $\mathbb{R}^n$ and codomain R^m, then we say that T is a transformation from $\mathbb{R}^n$ to R^m , which we denote by writing

$$T: \mathbb{R}^n \to \mathbb{R}^m.$$

As we have seen before, matrices (or better, matrix multiplication) can be viewed as transformations:

$n = 3, \quad m = 2$

$$\begin{aligned} w_1 &= 2x_1 + 3x_2 + x_3 \\ w_2 &= 1x_1 + 4x_2 \end{aligned} \quad \longleftrightarrow \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Matrix transformations

So multiplication by an $m \times n$ matrix A is a transfrormation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

Example 1. The matrix $\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ represents the following transformation:

here in coordinates our transformation is:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - x_2 \end{pmatrix}$$

Matrix transformations

Example 2. Multiplication by the $m \times n$ zero matrix is the transformation that maps $\mathbb{R}^n$ to $\mathbb{R}^m$, sending each vector of the domain into the vector with all zero coordinates:

E.g. If $m=2, n=2$ then $\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{T_0} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0x + 0y \\ 0x + 0y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ For all x, y

Example 3. Multiplication by the $n \times n$ identity matrix is the identity transformation: it sends each vector of the domain to itself:

Eg if $m = n = 3$ then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{T_I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 0\cdot y + 0\cdot z \\ 0\cdot x + y + 0\cdot z \\ 0\cdot x + 0\cdot y + z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Linear transformations

Linear transformation

A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if it satisfies the following:

- $T(k\mathbf{x}) = kT(\mathbf{x})$ ("homogeneity")
- $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ ("additivity")

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and scalar k.

Note.

1. The properties above imply that $T(\mathbf{0}) = \mathbf{0}$.

$$T(\underline{0}) = T(0\cdot\underline{v}) \underset{\uparrow}{=} 0\cdot T(\underline{v}) = 0$$

homogeneity

Linear transformations

Linear transformation

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for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and scalar k.

Note.

2. The arithmetic properties of matrix operations imply that **matrix transformations are linear transformations.** Let T_A denote the transformation

We want to check that the two properties above hold

$$T_A(k\underline{v}) \overset{\text{def of } T_A}{=} A(k\underline{v}) = (kA)\underline{v} = k\cdot(A\underline{v}) = k\cdot T_A(\underline{v})$$

$$T_A(\underline{v}+\underline{w}) \underset{\text{def of } T_A}{=} A(\underline{v}+\underline{w}) = A\underline{v} + A\underline{w} = T_A(\underline{v}) + T_A(\underline{w})$$

Matrix transformations are linear transformations

It turns out that the converse of the previous point is also true:

Matrix and linear transformations

Every linear transformation from $\mathbb{R}^n$ to R^m is a matrix transformation and conversely every matrix transformation from R^n to R^m is a linear transformation.

Question. If we discover that a transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, how can we find a matrix A such that $T = T_A$ (that is, T is multiplication by A)?

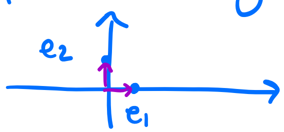
We'll see that it is enough to know the transformation on certain special vectors ("standard basis vectors") in order to determine said matrix.

Standard basis vectors

The **standard basis vectors** of $\mathbb{R}^n$ are the vectors $\mathbf{e}_i$ for $i = 1, \ldots, n$ whose ith entry is 1 and all remaining entries are 0.

Example (a) The standard basis vectors of $\mathbb{R}^2$ are $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In the Cartesian plane, these are the "length 1" vectors stemming from the origin and going along the x- and y-axis respectively



(b) The standard basis vectors of $\mathbb{R}^3$ are $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.