Week 6, lecture 1:
Systems of linear equations
MA180/185/190 Algebra

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Systems of linear equations

Gaussian elimination

Matrix algebra

Problems

Problem 1. Suppose that we want to find values for a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, 1)$, $(2, 4)$, and $(-1, 1)$. Find and solve a system of linear equations whose solutions provide values for a, b, and c.

The coordinates of each of the points should satisfy the eqn

$$y = ax^2 + bx + c$$

→ Through $(1,1)$ means: $1 = a\cdot(1)^2 + b\cdot 1 + c \rightsquigarrow a+b+c=1$

→ " $(2,4)$ means: $4 = a\cdot(2)^2 + b\cdot 2 + c \rightsquigarrow 4a+2b+c=4$

→ " $(-1,1)$ means: $1 = a(-1)^2 + b(-1) + c \rightsquigarrow a-b+c=1$

So our system of linear equations is

$$\begin{cases} a+b+c=1 \\ 4a+2b+c=4 \\ a-b+c=1 \end{cases}$$

Problems

$$\begin{cases} a+b+c=1 \\ 4a+2b+c=4 \\ a-b+c=1 \end{cases}$$

we can solve this with some ad. hoc tricks.

If 1^{st} and 3^{rd} eqn hold true, then this is still true if we subtract them:

$$a+b+c-(a-b+c)=1-1$$

so $\cancel{a}+b+\cancel{c}-\cancel{a}+b-\cancel{c}=0 \Rightarrow 2b=0 \Leftrightarrow \boxed{b=0}$ substitute in system

$$\begin{cases} a+c=1 \\ 4a+c=4 \\ b=0 \end{cases}$$

now subtract the first eqn from the 2nd:

$$4a+c-(a+c)=4-1$$

$$4a+\cancel{c}-a-\cancel{c}=3 \quad \text{so} \quad 3a=3 \quad \text{so} \quad \boxed{a=1}$$

substitute in 1^{st}: $\cancel{1}+c=\cancel{1} \Rightarrow \boxed{c=0}$

So our parabola is $\boxed{y=x^2}$

Problems

Problem 2. Suppose that a certain diet calls for 7 units of fat, 9 units of protein, and 16 units of carbohydrates for the main meal, and suppose that an individual has three possible foods to choose from to meet these requirements:

Food 1: Each ounce contains 2 units of fat, 2 of protein, and 4 of carbs.

Food 2: Each ounce contains 3 units of fat, 1 of protein, and 2 of carbs.

Food 3: Each ounce contains 1 unit of fat, 3 of protein, and 5 of carbs.

Let x, y, and z denote the number of ounces of the first, second, and third foods that the dieter will consume at the main meal. How can we model the problem of finding how many ounces of each food must be consumed to meet the diet requirements?

Problems

We can model the previous problem as a system of linear equations as follows: The constraint that our mix of foods should contain 7 units of fat in total becomes

units of fat per ounce in 1st food — units of fat per ounce in 3rd food

units of fat required

$$2x + 3y + 1\cdot z = 7$$

units of fat per ounce in 2nd food

Similarly, we translate the requirement re. the units of protein as

$$2x + y + 3z = 9$$

Finally, the carb requirements become

$$4x + 2y + 5z = 16$$

All these equations should be satisfied at once, giving us the system:

$$\begin{cases} 2x + 3y + z = 7 \\ 2x + y + 3z = 9 \\ 4x + 2y + 5z = 16 \end{cases}$$

Elementary row operations

We know that the following **elementary** operations on equations **do not** alter the solution (if it exists) of a system of linear equation:

1. Multiply both sides of an equation by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

For instance, the previous system yields the same solutions as this one:

$$\begin{cases} 2x + y + 3z = 9 \\ 2x + 3y + z = 7 \\ 8x + 4y + 10z = 32 \end{cases}$$

In which we swapped the first two rows and we multiplied by 2 the last equation (both sides)

Elementary row operations

We know that the following **elementary** operations on equations **do not** alter the solution (if it exists) of a system of linear equation:

1. Multiply both sides of an equation by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Idea:

- use a **matrix** defined by the data from our linear system,
- translate the operations above into **elementary row operations**,
- devise a procedure involving these operations in order to solve our system (if possible).

Elementary row operations

The following **elementary row operations** on the **augmented matrix** of a system of linear equations **do not** alter the solution (if it exists).

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

We will use these operations to transform the matrix into a "staircase" one.

From that, we will be able to find the values of each variable by back-substitution.

Let's see how this works out on our diet problem

Elementary row operations

The following **elementary row operations** on the **augmented matrix** of a system of linear equations **do not** alter the solution (if it exists).

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Example. Let's start by finding the augmented matrix of our system coming from the diet problem:

$$\begin{cases} 2x + 3y + z = 7 \\ 2x + y + 3z = 9 \\ 4x + 2y + 5z = 16 \end{cases}$$

we put all the coefficients (in order) in a grid with 3 rows (as many as the equations) and 4 column (number of variables plus 1). We get

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 9 \\ 4 & 2 & 5 & 6 \end{array}\right)$$

Gaussian elimination

Example (continued). Our augmented matrix is therefore:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 9 \\ 4 & 2 & 5 & 16 \end{array}\right)$$

Goal: transform this into a "staircase" matrix from which we'll be able to solve the system.

We call R_1, R_2, R_3 the three rows (from top to bottom) of the matrix. At each step we use elementary row operations to obtain (if possible) a matrix of the form $\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$ we will see that the numbers in the highlighted corners are particularly important.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 9 \\ 4 & 2 & 5 & 16 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & -2 \end{array}\right)$$

[Remember, each row represents an equation, so the last row already tells us: $-z = -2$ so $z = 2$]

Gaussian elimination

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & -2 \end{array}\right) \xrightarrow[R_2 - R_1]{(-1)\cdot R_3} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow[R_2\cdot\left(-\frac{1}{2}\right)]{R_1\cdot\left(\frac{1}{2}\right)} \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

At this point we can use back substitution to get a complete solution

$$\begin{cases} x + \frac{3}{2}y + \frac{1}{2}z = \frac{7}{2} \\ y - z = -1 \\ z = 2 \end{cases}$$

use $z=2$ to get y

$y - 2 = -1$ so $y = 1$

$$\begin{cases} x + \frac{3}{2} + 1 = \frac{7}{2} \quad \text{So} \quad x = \frac{7}{2} - \frac{3}{2} - 1 = 1 \\ y = 1 \\ z = 2 \end{cases}$$

use these to get x

Our final solution is:

$$\boxed{x=1,\ y=1,\ z=1}$$

Gaussian elimination

Some observations:

1. For now we will only deal with systems of linear equations in which the number of unknowns is equal to the number of equations
2. A system of n linear equations in n unknowns doesn't always admit a (unique) solution. There are cases in which there are infinitely many solutions (we will not discuss those), cases in which there is a unique solutions, and cases in which a solution does not exist.

We will consider systems of 2 equations in 2 and systems of 3 equations in 3 unknowns.

When does a solution exist?

We will consider systems of 2 equations in 2 and systems of 3 equations in 3 unknowns.

Let's look at the following. Suppose that by using elementary row operations we obtain the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 7 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

Exercises

Exercise. Use Gaussian elimination to solve **Problem 1.** (parabola through three points).

Exercise. Suppose you are asked to find three real numbers such that the sum of the numbers is 12, the sum of two times the first plus the second plus two times the third is 5, and the third number is one more than the first. Model the three conditions through a system of linear equations and use Gaussian elimination to determine the three numbers.

Exercise. Use Gaussian elimination to solve the following system

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$