Add a further limit rule (no. 7) to the above six:

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}.$$

Example. Let $f(x) = \frac{\sqrt{x+3}-\sqrt{3}}{x}$. Does $\lim_{x \to 0} f(x)$ exist? If it exists, what is the limit?

Cannot apply quotient rule immediately: denominator function has limit 0 as x approaches 0.

In a quotient function like this, always check for factorizations of top and bottom.

Clue here is difference of two squares (remember $a^2 - b^2 = (a-b)(a+b)$):

$(\sqrt{x+3} - \sqrt{3})(\sqrt{x+3} + \sqrt{3}) = (\sqrt{x+3})^2 - (\sqrt{3})^2 = x + 3 - 3 = x.$

Example (continued).

Thus

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x} &= \lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{(\sqrt{x+3}-\sqrt{3})(\sqrt{x+3}+\sqrt{3})} \\
&= \lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}} \quad \text{(}\textbf{now}\text{ can use quotient rule)} \\
&= \frac{1}{\sqrt{3}+\sqrt{3}} \quad \text{(limit rule no. 7)} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}$$

Example. The floor function $f(x) = \lfloor x \rfloor$ rounds *down* real numbers to the nearest integer.

E.g., $\lfloor 1.875 \rfloor = 1$, $\lfloor 12.999 \rfloor = 12$, $\lfloor -3.0001 \rfloor = -4$, etc.

Consider x approaching 1. For $0 < x < 1$, $\lfloor x \rfloor = 0$. For $1 < x < 2$, $\lfloor x \rfloor = 1$.

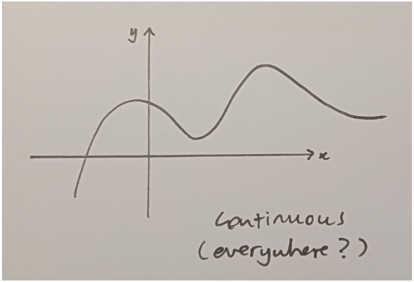
So for x approaching $a = 1$ from the left and the right, there is not a unique limiting value $f(x)$: $\lim_{x \to 1} \lfloor x \rfloor$ doesn't exist. (Exercise: draw the graph.)

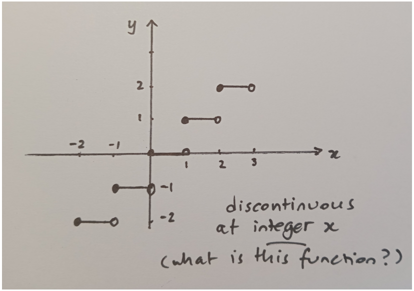
Continuity

Roughly speaking, a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* if its graph (in the x-y plane) has no gaps, jumps (discontinuities); we can draw the graph continuously without lifting pen from paper.

Definition. Let a be a real number in the domain of f. Then f is *continuous at* a if

- $\lim_{x\to a} f(x)$ exists, and
- $\lim_{x\to a} f(x) = f(a)$.


y
x
continuous
(everywhere?)


y
2
1
-2
-1
1
2
3
x
-1
-2
discontinuous
at integer x
(what is this function?)

Example. Since a polynomial function f has a limit at every point/real number a, and that limit is $f(a)$, *every polynomial function is continuous everywhere.*

Example. $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = e^x$ are other examples of functions that are continuous everywhere.

$f(x) = \tan x$ is not defined at odd integer multiples of $\pi/2$: hence, it is not continuous at such points.

Example. Find the points at which the following function is continuous:

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x \neq -1 \\ -2 & x = -1. \end{cases}$$

Solution. If $x \neq -1$ then

$$f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x-1)(x+1)}{x+1} = x - 1$$

so that, as a polynomial function, $f(x)$ is continuous at all $x \neq -1$.
Now $f(-1) = -2$ and

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2.$$

Thus $\lim_{x \to -1} f(x) = f(-1)$, so this $f(x)$ is continuous at $x = -1$ also: it is continuous for all $x \in \mathbb{R}$.