Recall: the tangent to the graph of $y = f(x)$ at $(a, f(a))$ has slope

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example. Find the equation of the tangent to the graph of $y = 3/x$ at the point $(3, 1)$.

Solution. Graph of $y = f(x) = 3/x$ is a hyperbola. Tangent at $(a, 3/a)$ has slope

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{3/x - 3/a}{x - a} \\
&= \lim_{x \to a} \frac{\frac{3a-3x}{ax}}{x - a} \\
&= \lim_{x \to a} \left(\frac{3(a - x)}{ax} \cdot \frac{1}{x - a} \right)
\end{aligned}$$

Solution (continued).

$$
\begin{aligned}
&= \lim_{x \to a} \frac{-3}{ax} \\
&= \frac{-3}{a^2}.
\end{aligned}
$$

Therefore, at $(3, 1)$, the tangent has slope $-3/9 = -1/3$.

The equation of the tangent is

$$y - 1 = -\tfrac{1}{3}(x - 3) = 1 - \tfrac{1}{3}x \implies y = 2 - \tfrac{1}{3}x.$$

The derivative

We rewrite the slope-of-tangent formula

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Let $h = x - a$, i.e., $x = a + h$. Note: $x \to a \Leftrightarrow h \to 0$. Thus

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

is an equivalent definition of the slope of the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$.

This number $f'(a)$ is called the *derivative of f at a.*

Derivative as function

Now consider tangents at all (possible) points $(a, f(a))$ on the graph of $y = f(x)$.

That is, after replacing 'a' above by 'x', we see that for each point $(x, f(x))$ on the graph, there is a tangent with slope

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This defines a new function of x, denoted $f'(x)$ or $\frac{dy}{dx}$, *the derivative of f.*

To repeat the definition (because it's so important):

$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

Example. Let $f(x) = x^2$. Find $f'(x)$. Compare the graphs of f and f'.

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

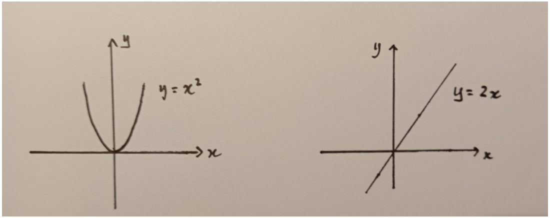
Now $(x+h)^2 - x^2 = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$. Then

$$\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$

provided $h \neq 0$. Therefore

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

Solution (continued). Here are the familiar graphs of $y = x^2$ and $y = 2x$:



For $x < 0$, tangents to the graph of $y = x^2$ have *negative* slope. The function values of the derivative function $y = 2x$ for $x < 0$ are *negative*.

At $x = 0$, the tangent to the graph of $y = x^2$ is horizontal, i.e., has *zero* slope. The derivative function has value *zero* at $x = 0$.

For $x > 0$, tangent to the graph of $y = x^2$ have *positive* slope. The function values of the derivative function for $x > 0$ are *positive*.

Differentiation

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, $y = f(x)$. If the derivative

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, then f is said to be *differentiable* (at x).

The process of calculating derivatives (if they exist) is called *differentiation*.

Earlier we saw a list of seven *limit rules*. These give rise to *differentiation rules* (because a derivative is a limit).

Example. Let $a, b \in \mathbb{R}$. If $f(x) = ax + b$ then $f'(x) = a$.

Proof. One way: $y = ax + b$ is the equation of a straight line with slope a. Since the derivative is the slope of tangent function, $\frac{dy}{dx} = a$.

Another way: we use the limit definition of derivative.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{a(x+h) + b - (ax + b)}{h} \\
&= \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} \\
&= \lim_{h \to 0} \frac{ah}{h} \\
&= \lim_{h \to 0} a \\
&= a.
\end{aligned}$$