Recall:

$$\lim_{x\to a} f(x) = l$$

means that for each positive real number ϵ, there exists a positive real number δ such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \epsilon.$$

A way to remember the ϵ-δ definition:

$\lim_{x\to a} f(x) = l$ means that if we pick any positive real number ϵ (measuring closeness to l) then we are guaranteed to find a positive real number δ (measuring closeness to a) such that if x-values are within δ of a then the $f(x)$-values must be within ϵ of l.

Note the direction:

get δ *after* ϵ;

ϵ-closeness to l then *follows from* δ-closeness to a.

An epsilon-delta proof.

We prove that $\lim_{x\to 3}(4x-5)=7$ from 'first principles'.

Let $\epsilon>0$. We want to find $\delta>0$ such that

$$0<|x-3|<\delta \quad \Rightarrow \quad |4x-5-7|=|4x-12|=4|x-3|<\epsilon,$$

(using $|ab|=|a|\cdot|b|$). That is, $|x-3|<\epsilon/4$, suggesting we take $\delta=\epsilon/4$. And that works: given $\epsilon>0$, so that $\delta=\epsilon/4>0$, if

$$0<|x-3|<\delta=\tfrac{\epsilon}{4},$$

then by inequality laws,

$$|4x-5-7|=4|x-3|<4\cdot\tfrac{\epsilon}{4}=\epsilon$$

as required in the ϵ-δ definition of $\lim_{x\to 3}(4x-5)=7$.

Limit rules

Because of the difficulty of ϵ-δ proofs, we almost always never use them to prove individual limits; rather we use a combination of established rules (each of which has an ϵ-δ proof!).

1. $\lim_{x \to a} x = a$. (Try the ϵ-δ proof; should find $\delta = \epsilon$ works.)

2. If c is a constant then $\lim_{x \to a} c = c$.

3. If c is a constant then for any function $f(x)$,
$\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x)$.

e.g., $\lim_{x \to 1}(-\sqrt{2}x) = -\sqrt{2} \lim_{x \to 1}(x) = -\sqrt{2}.1 = -\sqrt{2}$.

Let $f(x)$ and $g(x)$ be any functions.

4. (Product rule.) $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a}(f(x))\cdot\lim_{x\to a}(g(x))$.

e.g., $\lim_{x\to a}(x^2) = \lim_{x\to a}(x)\cdot\lim_{x\to a}(x) = a.a = a^2$.
Similarly, $\lim_{x\to a}(x^3) = a^3$, $\lim_{x\to a}(x^4) = a^4$, etc.

5. (Sum rule.) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

e.g., $\lim_{x\to 3}(x^3 - 2x + 1) = \lim_{x\to 3}(x^3) - 2\lim_{x\to 3}(x) + 1 =$
$27 - 6 + 1 = 22$.

6. (Quotient rule.) If $\lim_{x\to a} g(x) \neq 0$, then
$\lim_{x\to a}(f(x)/g(x)) = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x))$.

e.g., $\lim_{x\to 1}\frac{3x^2-1}{x-2} = \frac{\lim_{x\to 1}(3x^2-1)}{\lim_{x\to 1}(x-2)} = \frac{3-1}{1-2} = \frac{2}{-1} = -2$.

The preceding examples illustrate the fact that for any *polynomial* function $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$, and any $a \in \mathbb{R}$,

$$\lim_{x \to a} f(x) = f(a) = c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0.$$