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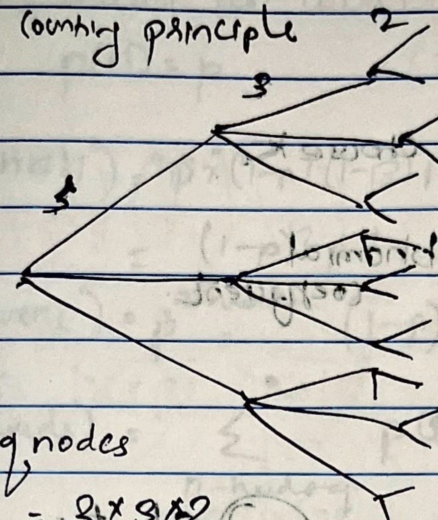
# Lecture 4. Counting

Basic counting principle



No. of Leaf nodes

$= 3 \times 3 \times 2$

$= 18$

Number of licence plates with 3 letters and 4 digits

$\underline{26}\ \underline{26}\ \underline{26} \times \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}$ with Repetition

$\underline{26}\ \underline{25}\ \underline{24} \times \underline{10}\ \underline{9}\ \underline{8}\ \underline{7}$ without Repetition

Permutations:

$n(n-1)(n-2)(n-3) \cdots 1 = n!$

Number of subsets of $\{1, 2, \ldots n\} = 2^n$

Example

Prob that six Rolls of a six-sided die all give different numbers

→ Assume Rolls Are Independent, Die is fair

~~P(Dif~~

$P(\text{one outcome}) = \left(\frac{1}{6}\right)^6$

Sample space $= (6)^6$

to have All Diff numbers is just permutation is $\Rightarrow 6!$

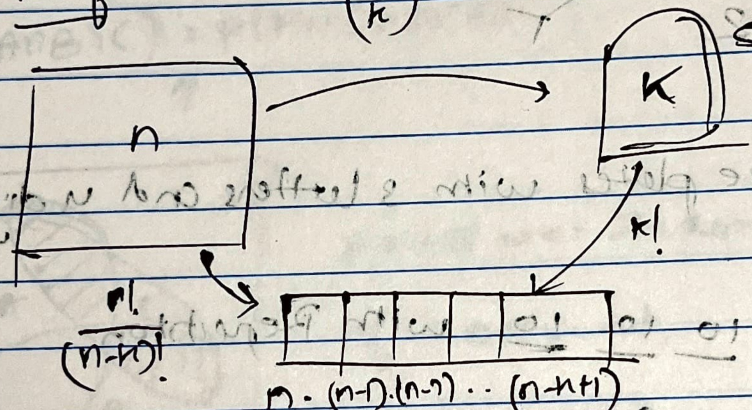
$\text{Ans} = \frac{6!}{6^6}$

# Combinations

$\binom{n}{k}$ n choose k
$\downarrow$
out of n elements choose k.

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ $\longrightarrow$ binomial coefficient.

**Proof** $\binom{n}{k}$




$0! = 1$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} \cdots \binom{n}{k}$$

$$= \frac{n!}{0!\,n!} + \frac{n!}{1!\,(n-1)!} + \frac{n!}{(n-2)!\,2!} \cdots \frac{n!}{k!\,(n-k)!}$$

$$= 1 + \frac{n(n-1)!}{1\,(n-1)!} + \frac{n \times (n-1)(n-2)!}{(n-2)!\,2!}$$

$$= 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} \cdots n(n-1)\cdots$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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**Example** n Independent coin tosses

$$p(H) = p$$

→ $p(HTTHHH) = p \times (1-p)(1-p)\, p\, p\, p$

$= (1-p)^2 p^4 = p(HHHTTHH)$

→ $p(\text{sequence}) = p^{\#\text{heads}} \times (1-p)^{\#\text{tails}}$.

→ $p(k \text{ heads}) = \sum_{k\text{-head seq.}} p(\text{seq})$

$= (\# \text{ of } k\text{-head seqs}) \cdot p^k (1-p)^{n-k}$

$= \binom{n}{k} p^k (1-p)^{n-k}$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

**Example** → 3 out of 10 tosses were heads

What is the prob that first two tosses were heads?

All outcomes of Event $= p^3(1-p)^7$.

No. of B $= {}^{10}C_3 = \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8 \times 7!}{3 \times 2 \times 1 \times 7!} = 120$

Start with HH $= {}^{8}C_1 = \frac{8!}{7!\,1!} = \frac{8 \times 7!}{7! \times 1!} = 8$

$P = \frac{8}{120} = \frac{1}{15}$

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Example

52 card deck, dealt to 4 players
Find P (each gets an ace).

outcome $= \dfrac{52!}{13!\,13!\,13!\,13!} = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$

$= \dfrac{52!}{13!\,\cancel{39!}} \times \dfrac{\cancel{39!}}{13! \times \cancel{26!}} \times \dfrac{\cancel{26!}}{13!\,\cancel{13!}} \times \dfrac{\cancel{13!}}{13!\,0!}$

$= \dfrac{52!}{13!\,13!\,13!\,13!}$

No of ways to Distribute 4 aces $= 4!$

to distribute Remaining 48 cards $=$

$$= \frac{\dfrac{4! \times 48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$