ASSIGNMENT 6

QUESTION NUMBER 1

$$\textit{To Find}: \frac{\partial f}{\partial x} \text{ for a forward difference of second order}$$

$$\textit{SOLUTION}$$

$$f(x+\Delta x) \;=\; f(x) \;+\; \frac{\partial f}{\partial x}*(\Delta x) \;+\; \frac{\partial^2 f}{\partial x^2}*(\Delta x)^2 \;+\; \frac{\partial^3 f}{\partial x^3}*(\Delta x)^3 + \ldots \;\; (\textit{Equation I})$$

$$f(x+2\Delta x) \;=\; f(x) \;+\; \frac{\partial f}{\partial x}*(2\Delta x) \;+\; \frac{\partial^2 f}{\partial x^2}*(2\Delta x)^2 \;+\; \frac{\partial^3 f}{\partial x^3}*(2\Delta x)^3 + \ldots \;\; (\textit{Equation II})$$

$$\textit{we need a second order formula}$$

$$\textit{Hence we need to eliminate the } \frac{\partial^2 f}{\partial x^2} \textit{ term}$$

$$4*\textit{Equation I} \;-\; \textit{Equation II gives}$$

$$4*(f(x+\Delta x)) \;-\; f(x+2\Delta x) \;=\; 3*f(x) \;+2*\frac{\partial f}{\partial x}*(\Delta x) \;-\; 2*\frac{\partial^3 f}{\partial x^3}*(\Delta x)^3 \;+\; \textit{other terms}$$

$$\Longrightarrow \frac{4*(f(x+\Delta x)) \;-\; f(x+2\Delta x) \;-\; 3*f(x)}{2\Delta x} \;=\; \frac{\partial f}{\partial x} \;-\; \left(\frac{\partial^3 f}{\partial x^3}*(\Delta x)^2 + \;\textit{other terms}\right)$$

$$\textit{The bracketed part is a second order error}$$

$$\textit{Hence, a second order formula is}:$$

$$\frac{\partial f}{\partial x} \;=\; \frac{4*(f(x+\Delta x)) \;-\; f(x+2\Delta x) \;-\; 3*f(x)}{2\Delta x}$$

Question 2: Write a short note on Van Leer

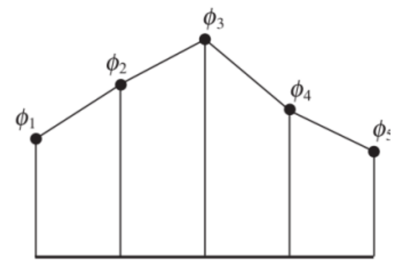
Answer:

Higher-order methods cause oscillations while the lower order methods are not very accurate. So, we need a trick to get the accuracy of the higher-order methods and the steadiness of lower order methods. In other words, we need to minimize total variation.

TV is can be understood from this figure:

- For monotonicity to be satisfied, TV must not increase

$$\boxed{\begin{aligned} TV(\phi) &= |\phi_2 - \phi_1| + |\phi_3 - \phi_2| + |\phi_4 - \phi_3| + |\phi_5 - \phi_4| \\ &= |\phi_3 - \phi_1| + |\phi_5 - \phi_3| \end{aligned}}$$



We need a scheme in which TV is continuously decreased or in other words, the TV should be diminished. Hence, these schemes are called Total Variation Diminishing Schemes or TVD schemes. The mathematical condition is:

$$\boxed{TV(\phi^{n+1}) \leq TV(\phi^n)}$$

Now let's go back and look at the higher-order schemes. All of them have a coefficient ψ which is a function of the parameter r. This relation changes for changing order. The list is given below:

$$\boxed{\phi_e = \phi_P + \frac{1}{2}\psi(r)(\phi_E - \phi_P)}$$

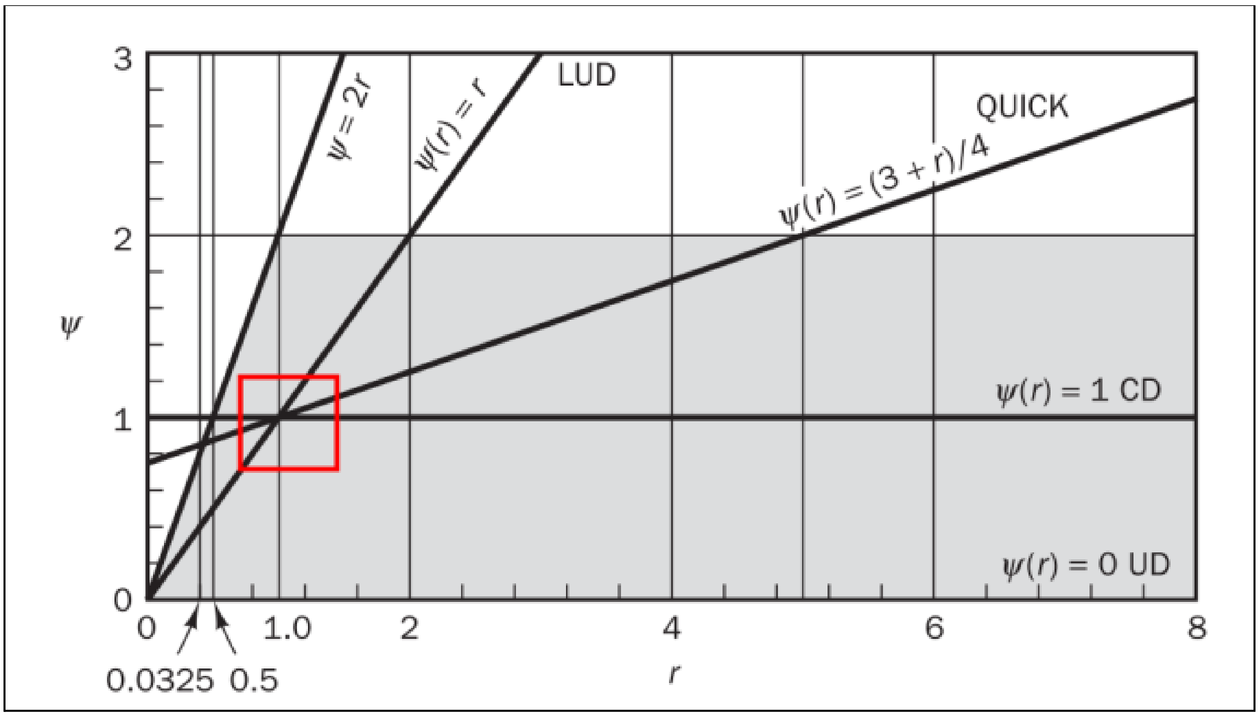
For the UD scheme $\psi(r) = 0$
For the CD scheme $\psi(r) = 1$
For the LUD scheme $\psi(r) = r$
For the QUICK scheme $\psi(r) = (3 + r)/4$

All of this can be represented on a ψ - r curve, also known as the Sweby Diagrams. Sweby gave a condition for the discretization schemes to be TVD. It says:

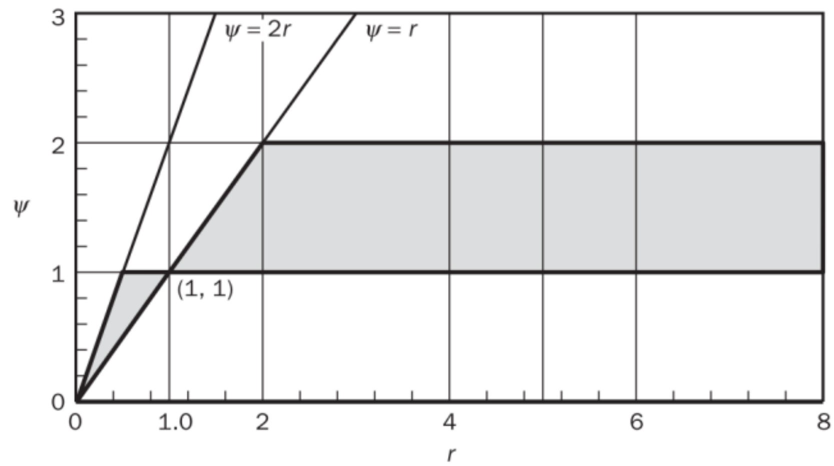
$$\text{If } 0 < r < 1 \text{ the upper limit is } \psi(r) = 2r, \text{ so for TVD schemes } \psi(r) \leqslant 2r$$
$$\text{If } r \geqslant 1 \text{ the upper limit is } \psi(r) = r, \text{ so for TVD schemes } \psi(r) \leqslant r$$

Thus, everything mentioned above except the UD is not TVD in some range or the other. Hence, we need a method to change ψ as r changes.

The acceptable range can be visualised in the following figure as the shaded region:



Sweby then introduced another condition that ψ should pass through the point (1,1) to be second-order accurate. This changes our allowed region to this:



If $0 < r < 1$ the lower limit is $\psi(r) = r$, the upper limit is $\psi(r) = 1$, so for TVD schemes $r \le \psi(r) \le 1$

If $r \ge 1$ the lower limit is $\psi(r) = 1$, the upper limit is $\psi(r) = r$, so for TVD schemes $1 \le \psi(r) \le r$

Then he finally put another condition on the flux limiter functions:

$$\boxed{\frac{\psi(r)}{r} = \psi(1/r)}$$

With all these constraints, if we try to get a smooth function for ψ, we get Van leer as one of the possibilities:

$$\boxed{\psi(r) = \frac{r + |r|}{1 + r}}$$

We then use Van Leer for convection and Central difference for Diffusion to find our solution.

THE END
