Week 7 challenge

Introduction to OpenFoam Development

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GRADIENT SCHEMES

Firstly, lets discuss why do we need a gradient scheme. We need to find gradient of pressure to use that term in the NS equation. Also, we need the same for other terms in the expression. But, calculation of this gradient is not straight forward. As we know, the values of the variables are stored at the cell centroid. The gradient too, is needed at the cell centroids. Thus, we need some kind of interpolation or a round about trick to get values nearby and then find the gradient. There are two approaches to this:

1) Using the values at face centres and then use the following formula:

$$(\nabla\phi)_p = \frac{1}{V_p}\sum \boldsymbol{n}_f \phi_f dS$$

where ϕ_f can be calculated either by Cell based or Node based method

This is essentially an application of the divergence theorem.

2) There is another approach which doesn't need the values at the face centres. In other words, it doesn't use the divergence theorem.

To understand this algorithm, lets start with an example of square cells. Then,

$$\phi_N = \phi_P + \boldsymbol{d}_{PN}.(\nabla\phi)_P$$

where $\boldsymbol{d}_{PN}$ is the distance vector joining the present and the neighbouring cell

DO the same for all the neighbouring cells. We get a system of equations.

$$\begin{aligned} \phi_{N1} - \phi_p &= \boldsymbol{d}_{PN1}.(\nabla\phi)_P \quad (1)\\ \phi_{N2} - \phi_p &= \boldsymbol{d}_{PN1}.(\nabla\phi)_P \quad (2)\\ &\vdots\\ \phi_N - \phi_p &= \boldsymbol{d}_{PN}.(\nabla\phi)_P \end{aligned}$$

This can be converted to a matrix

$$[\boldsymbol{d}_{PN}][(\nabla\phi)_P] = [\phi_N - \phi_p] \quad (3)$$

But, there is a problem in this solution. Let's consider a quadrilateral cell

$$\begin{bmatrix} \boldsymbol{d}_{PN1x} & \boldsymbol{d}_{PN1y} & \boldsymbol{d}_{PN1z} \\ \boldsymbol{d}_{PN2x} & \boldsymbol{d}_{PN2y} & \boldsymbol{d}_{PN2z} \\ \boldsymbol{d}_{PN3x} & \boldsymbol{d}_{PN3y} & \boldsymbol{d}_{PN3z} \\ \boldsymbol{d}_{PN4x} & \boldsymbol{d}_{PN4y} & \boldsymbol{d}_{PN4z} \end{bmatrix} * \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{bmatrix} = \begin{bmatrix} \phi_{N1} - \phi_p \\ \phi_{N2} - \phi_p \\ \phi_{N3} - \phi_p \\ \phi_{N4} - \phi_p \end{bmatrix} \tag{4}$$

Observe that Eq 4 cannot be solved directly since the d matrix is not square

Thus, we assume a value of $(\nabla\phi)_P$ *and minimize the error*:

$$\begin{aligned} e_1 &= \phi_{N1} - (\phi_p + \boldsymbol{d}_{PN1}.(\nabla\phi)_P) \\ e2 &= \phi_{N1} - (\phi_p + \boldsymbol{d}_{PN1}.(\nabla\phi)_P) \\ &\vdots \\ \text{Square error} &= \sum e_i^2 \end{aligned}$$

The target is to minimize square error. We get the best $\nabla\phi$ *as*:

$$(\nabla\phi)_P = \left(d^T d\right)^{-1} d^T(\phi_N - \phi_p) \tag{5}$$

Observe that for a stationary matrix, $\left(d^T d\right)^{-1} d^T$ *needs to be calculated only once then it can be multiplied by the* $[\phi_N - \phi_p]$ *matrix at each cell to get* $\nabla\phi$

Let's do an example:

Consider the following cell. Assume that the distance between centroids of neighbouring cells is one metre. The values of ϕ are given as shown:

$$\begin{array}{ccc} & 300 & \\ 100 & 200 & 300 \\ & 100 & \end{array}$$

then

$$d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left(d^T d\right)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\phi_N - \phi_P = \begin{bmatrix} -100 \\ -100 \\ 100 \\ 100 \end{bmatrix}$$

$$this\ gives,\ (\nabla\phi)_P = (100, 100, 0)$$

You can check the same by substituting values given in the grid.

SPECIAL CASE:

If we consider flow near walls, we generally prefer the cells to be shorter in the direction normal to the wall, to catch the boundary layer and the cells are longer in the streamwise direction. But if Eq.5 is used, the gradient is dominated by streamwise lengths. Thus, we can have a weighted Least square error.

$$w = 1/\ |d|$$
$$W = w_i \delta_{ii}$$

$$then$$
$$(\nabla\phi)_P = \left(d^T W^T W d\right)^{-1} d^T W^T W(\phi_N - \phi_P) \tag{6}$$

THE END