Module 2

Rank of Matrix and System of Equations

Unit 2.1

Types of Matrices

Review

- **Matrix**: A set of mn elements (real or complex) arranged in a rectangular array of m rows and n columns enclosed by a pair of square brackets is called a matrix of order m by n written as $m\times n$. A $m\times n$ matrix is usually written as

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The matrix can also be expressed in the form $A=\left[a_{ij}\right]_{m\times n}, 1\le i\le m \;\; \& \;\; 1\le j\le n$ where

a_{ij} is the element in the i^{th} row and j^{th} column, written as element of the matrix.

- **Determinant of A** is denoted by $|A|$.
- **TYPES OF A MATRIX:**

1. **Row Matrix:** A matrix having only one row and any number of columns is called a row matrix or a row vector. For example, $A=\left[2\;5\;3\;4\right]$
2. **Column Matrix:** A matrix having only one column and any number of rows is called a column matrix or a column vector. For example, $A=\begin{bmatrix}1\\5\\3\end{bmatrix}$
3. **Zero or Null Matrix :** A matrix whose all elements are zero is called zero matrix or null matrix and is denoted by 0 . For example, $\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
4. **Square Matrix :** A matrix in which the number of rows is equal to the number of columns is called a square matrix. For example, $\begin{bmatrix}2&3\\4&1\end{bmatrix}, \begin{bmatrix}1&2&3\\-1&4&-5\\2&6&8\end{bmatrix}$

5. **Diagonal Matrix:** A square matrix whose all non-diagonal elements are zero and at least one diagonal element is non-zero is called a diagonal matrix.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

6. **Scalar Matrix**: A diagonal matrix whose all diagonal elements are equal is called a scalar matrix. For example, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Unit or Identity Matrix:** A scalar matrix whose all diagonal elements are unity is called a unit matrix and is denoted by I. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. **Upper Triangular Matrix :** A square matrix in which all the elements below the diagonal are zero is called an upper triangular matrix. For example, $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$

9. **Lower Triangular Matrix:** A square matrix in which all the elements above the diagonal are zero, is called an upper triangular matrix. For example, $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 2 \end{bmatrix}$

10. **Trace of Matrix:** The sum of all the diagonal elements of a square matrix is called the trace of a matrix. For example, if $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$ **,** Trace of A=2+6+3=11

11. **Transpose of a Matrix:** A matrix obtained by interchanging rows and columns of a matrix is called transpose of that matrix and is denoted by A' or A^T.

For example, if $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -4 & 1 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \\ 3 & 6 & 5 \end{bmatrix}$

i.e. if $A = \left[a_{ij}\right]_{m\times n}$, then $A^T = \left[a_{ji}\right]_{n\times m}$

12. Symmetric Matrix: A square matrix $A=\left[a_{ij}\right]_{m\times n}$ is called symmetric

if $a_{ij}=a_{ji}$ for all i and j. i.e. $A=A^T$. For example, $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & i & -3i \\ i & -2 & 4 \\ -3i & 4 & 3 \end{bmatrix}$

13. Skew Symmetric Matrix: A square matrix $A=\left[a_{ij}\right]_{m\times n}$ is called skew symmetric

if $a_{ij}=-a_{ji}$ for all i and j. i.e. $A=-A^T$.

Thus, the diagonal elements of a skew symmetric matrix are all zero.

For example, $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ is a skew symmetric matrix .

14. Conjugate of a matrix: A matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and it is denoted by $\overline{A}$.

For example, if $A=\begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}$, then $\overline{A}=\begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$

15. Transposed Conjugate of a matrix: The conjugate of the transpose of a matrix A is called the transposed conjugate of A and is denoted by A^θ & $A^\theta=\left(\overline{A}\right)^T=\overline{\left(A^T\right)}$

For example, if $A=\begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}$, then $A^\theta=\begin{bmatrix} 1-3i & i \\ 2-5i & 6 \\ 8 & 9+i \end{bmatrix}$

16. Hermitian Matrix: A square matrix $A=\left[a_{ij}\right]$ is called Hermitian if

$a_{ij}=\overline{a_{ji}}$ for all i and j. i.e. $A=A^\theta$. For example, $\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$

17. Skew Hermitian Matrix: A square matrix $A=\left[a_{ij}\right]$ is called skew Hermitian if

$a_{ij}=-\overline{a_{ji}}$ for all i and j. i.e. $A=-A^\theta$.

The diagonal elements of a skew Hermitian matrix are either purely imaginary or zero.

For example, $\begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$

18. Singular and Non-singular Matrices: A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

19. Minor of an Element : The minor of an element of a square matrix A is the determinant obtained from $|A|$ by deleting the row and column in which the element lies.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then, minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$, minor of $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, etc.

20. Cofactor of an Element :

Cofactor of an element a_{ij} is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is denoted by A_{ij}.

e.g., cofactor of $a_{12} = A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, cofactor of $a_{32} = A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, etc.

21. Adjoint of a Matrix:

The transpose of the matrix of cofactors is called the Adjoint of the matrix.

22. Inverse of a matrix: If A is a square matrix and $|A| \neq 0$ then $AA^{-1} = I = A^{-1}A$

where A^{-1} is called inverse of a matrix A.

Also, $A^{-1} = \dfrac{1}{|A|} adj\ A$

Some Important Theorems:

1. Every square matrix can be uniquely expressed as the sum of a Symmetric and a Skew - Symmetric matrix.
2. Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian matrix.
3. Every square matrix A can be uniquely expressed as P + iQ where P and Q are Hermitian matrices.
4. Every Hermitian matrix A can be written as P + iQ where P is real Symmetric and Q is real Skew-Symmetric matrix.
5. Every Skew Hermitian matrix can be uniquely expressed as P + iQ, where P is real Skew Symmetric and Q is real Symmetric matrix.

Matrix	**Expressed As**	**Unique Representation**
Square	Symmetric + Skew Symmetric	$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
Square	Hermitian + Skew Hermitian	$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$
Square	P + iQ, P and Q both Hermitian	$A = \frac{1}{2}(A + A^\theta) + i\left[\frac{1}{2i}(A - A^\theta)\right]$
Hermitian	P + iQ, P Real Symmetric Q Real Skew Symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$
Skew Hermitian	P + iQ, P Real Skew Symmetric Q Real Symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

SOME SOLVED EXAMPLES

1. Express following Matrix as sum of Hermitian and skew Hermitian Matrices.

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix}$$

Solution:

As we know the unique representation, $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$ say, A = P + Q,

Where, $P = \frac{1}{2}(A + A^{\theta})$ *and* $Q = \frac{1}{2}(A - A^{\theta})$

Now, Consider $A^{\theta} = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}$

$$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} + \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix}$$

Clearly, P is Hermitian as $a_{ij} = \overline{a_{ji}}$, $\forall\, i, j$.

$$Q = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} - \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$$

Clearly, Q is skew Hermitian as $a_{ij} = -\overline{a_{ji}}$, $\forall\, i, j$.

Hence we get the unique expression,

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$$

2. Express following skew Hermitian Matrix as P + iQ, where P is real skew symmetric and Q is real symmetric matrix. $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$

Solution:

As we know the unique representation, $A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

say, A = P + iQ, Where, $P = \frac{1}{2}(A + \bar{A})$ *and* $Q = \frac{1}{2i}(A - \bar{A})$

Now, Consider $\bar{A} = \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}$

$$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} + \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix}$$

Clearly, P is real skew symmetric as all elements of P are real and $\boldsymbol{a_{ij}} = -\boldsymbol{a_{ji}}$.

$$and\ \ Q = \frac{1}{2i}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2i}\begin{bmatrix} 6i & 2i & -4i \\ 2i & -2i & 4i \\ -4i & 4i & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

Clearly, Q is real symmetric as all elements of Q are real and $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$.

Hence we get the unique expression, A = P + iQ,

$$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} + i\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

3. Express the matrix $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$ as $P + iQ$ where P and Q are Hermitian matrices.

Solution:

As we know the unique representation, $A = \frac{1}{2}(A + A^{\theta}) + i\frac{1}{2i}(A - A^{\theta})$ say, A = P + iQ,

Where, $P = \frac{1}{2}(A + A^{\theta})$ *and* $Q = \frac{1}{2i}(A - A^{\theta})$

Now, Consider $A^{\theta} = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$

$$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} + \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$$

$$\therefore Q = \frac{1}{2i}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} - \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$$

For all elements P & Q, $\boldsymbol{a_{ij}} = \overline{\boldsymbol{a_{ji}}}$. Hence P and Q are Hermitian.

Hence we get the unique expression,

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix} + i\frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$$

4. Express following Matrix as P + iQ, where P is real symmetric and Q as real skew symmetric matrix.

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$$

Solution:

As we know the unique representation, $A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

Say, A = P + iQ, Where, $P = \frac{1}{2}(A + \bar{A})$ *and* $Q = \frac{1}{2i}(A - \bar{A})$

Now, Consider $\bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}$

$$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}\right\} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$$

All elements of P are real and $a_{ij} = a_{ji}$. Hence P is symmetric.

$$Q = \frac{1}{2i}\left\{\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & -2i \\ 2i & 2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

All elements of Q are real and $\boldsymbol{a_{ij} = -a_{ji}}$. So, Q is real skew symmetric.
Hence we get the unique expression, A = P + iQ,

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} + i\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

❖ Orthogonal and Unitary Matrices

1. Orthogonal Matrix

Definition: A real square matrix A is called orthogonal if $AA^T = A^T A = I$.
Properties:

- If A is orthogonal then $\boldsymbol{A^{-1} = A^T}$.
- If A is orthogonal matrix then $|\boldsymbol{A}| = \pm\boldsymbol{1}$.
- If A is orthogonal then A^{-1}, A^T are also orthogonal.

2. Unitary Matrix

Definition: A square matrix A is called unitary if $AA^\theta = A^\theta A = I$.
Properties:

- If A is Unitary then $\boldsymbol{A^{-1} = A^\theta}$
- If A is unitary matrix then Its determinant is of unit modulus.
- If A and B are unitary matrices of order n then $A^{-1},\ A^\theta,\ AB\ and\ BA$ are also unitary.

SOME SOLVED EXAMPLES

1. Prove that following matrix is orthogonal and hence find A^{-1}, $A = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$$\therefore \ AA^T = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & -\cos\alpha\sin\alpha + \cos\alpha\sin\alpha \\ 0 & 1 & 0 \\ -\cos\alpha\sin\alpha + \cos\alpha\sin\alpha & 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also, $A^TA = \begin{bmatrix} cos\,\alpha & 0 & -sin\,\alpha \\ 0 & 1 & 0 \\ sin\,\alpha & 0 & cos\,\alpha \end{bmatrix}\begin{bmatrix} cos\,\alpha & 0 & sin\,\alpha \\ 0 & 1 & 0 \\ -sin\,\alpha & 0 & cos\,\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Since, $A^TA = AA^T = I$, A is orthogonal.

For orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} cos\,\alpha & 0 & -sin\,\alpha \\ 0 & 1 & 0 \\ sin\,\alpha & 0 & cos\,\alpha \end{bmatrix}$$

2. Is the following matrix orthogonal? If not, can it be converted into orthogonal matrix?

$$A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$$

Solution: $A^T = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$

$$\therefore \ AA^T = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+3 & 2-2 & 2+1-3 \\ 2-2 & 2+4 & 2-2 \\ 2+1-3 & 2-2 & 2+1+3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I \neq I$$

Thus Given matrix A is not orthogonal.

But it can be converted into an orthogonal matrix as follow

$$\therefore AA^T = 6I \quad \therefore \tfrac{1}{6}AA^T = I$$

$$\therefore \left(\tfrac{1}{\sqrt{6}}A\right).\left(\tfrac{1}{\sqrt{6}}A^T\right) = I$$

Hence $\frac{1}{\sqrt{6}}A = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ is the orthogonal matrix.

3. Is the matrix $A = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ orthogonal? If not, can it be converted to an orthogonal matrix? If yes, how ?

Solution: A is orthogonal if $AA' = I$

$$AA' = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$\therefore AA' = 81I \neq I$

$\therefore A$ is not orthogonal

$AA' = 81I \quad \therefore \frac{1}{81}AA' = I$

$i.e. \left(\frac{1}{9}A\right)\left(\frac{1}{9}A'\right) = I$

Let $B = \frac{1}{9}A$, then $BB' = I$

$\therefore B = \frac{1}{9}A$ is orthogonal.

4. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal, then find a, b, c

Solution: Consider $A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$

Since A is orthogonal we have, $AA^T = I$

$$\therefore AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ -2+ac & 2+bc & 8+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing we get total 6 distinct equations,

$$\frac{5+a^2}{9} = 1 \qquad \therefore a^2 = 9-5 = 4 \quad \therefore a = \pm 2$$

$$\frac{5+b^2}{9} = 1 \qquad \therefore b^2 = 9-5 = 4 \quad \therefore b = \pm 2$$

$$\frac{8+c^2}{9}=1 \qquad \therefore c^2=9-8=1 \quad \therefore c=\pm 1$$

$4+ab=0 \ \ \therefore ab=-4 \ \ \to when\ a=+2,\ \ b=-2 \ \ \&$

$when\ a=-2,\ \ b=+2$

Also, $-2+ac=0 \ \ \therefore ac=2 \ \ \to when\ a=+2,\ \ c=+1$

$and\ when\ a=-2,\ \ c=-1$

Hence (2, -2, 1) and (-2, 2, -1) are the required pairs.

5. Show matrix A is unitary and hence find A^{-1} where $A=\begin{bmatrix}\frac{2+i}{3} & \frac{2i}{3}\\ \frac{2i}{3} & \frac{2-i}{3}\end{bmatrix}$

Solution: T. S. T. A is unitary, we have to show $AA^{\theta}=I$

Consider $A=\frac{1}{3}\begin{bmatrix}2+i & 2i\\ 2i & 2-i\end{bmatrix} \ \ \therefore A^{\theta}=\frac{1}{3}\begin{bmatrix}2-i & -2i\\ -2i & 2+i\end{bmatrix}$

$$AA^{\theta}=\frac{1}{9}\begin{bmatrix}2+i & 2i\\ 2i & 2-i\end{bmatrix}\begin{bmatrix}2-i & -2i\\ -2i & 2+i\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}(2+i)(2-i)-(2i)(2i) & -2i(2+i)+2i(2+i)\\ 2i(2+i)-2i(2+i) & -(2i)(2i)+(2-i)(2+i)\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}5+4 & 0\\ 0 & 5+4\end{bmatrix}=I$$

$$A^{\theta}A=\frac{1}{9}\begin{bmatrix}2-i & -2i\\ -2i & 2+i\end{bmatrix}\begin{bmatrix}2+i & 2i\\ 2i & 2-i\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}(2-i)(2+i)-(2i)(2i) & 2i(2-i)-2i(2-i)\\ -2i(2+i)+2i(2+i) & -(2i)(2i)+(2+i)(2-i)\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}5+4 & 0\\ 0 & 5+4\end{bmatrix}=I$$

As $AA^{\theta}=A^{\theta}A=I$

Hence the given matrix is unitary.

For unitary matrix $A^{-1}=A^{\theta}=\frac{1}{3}\begin{bmatrix}2-i & -2i\\ -2i & 2+i\end{bmatrix}$

6. Show matrix A is unitary and hence find A^{-1} where, $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

Solution: $A^{\theta}=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}=A$ and

$$AA^{\theta}=\frac{1}{4}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4 & -2i+2i & 0\\ 2i-2i & 4 & 0\\ 0 & 0 & 4\end{bmatrix}=\frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}=I$$

$$A^\theta A = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = I$$

As $AA^\theta = A^\theta A = I$

Hence the given matrix is unitary.

For unitary matrix $A^{-1} = A^\theta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. Show that $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

Solution: Since, $A^\theta = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$ Let us check,

$$AA^\theta = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}\begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & (\alpha + i\gamma)(\beta - i\delta) - (\beta - i\delta)(\alpha + i\gamma) \\ (\beta + i\delta)(\alpha - i\gamma) - (\beta + i\delta)(\alpha - i\gamma) & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By comparing, we get the required condition,

that A is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

8. If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, then show that $(I - N)(I + N)^{-1}$ is a unitary matrix.

Solution: $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$\therefore I - N = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \quad \text{and} \quad I + N = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - \left(4i^2 - 1\right) = 1 + 4 + 1 = 6$$

$$\therefore adj.(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore (I + N)^{-1} = \frac{1}{|I + N|} adj.(I + N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore (I - N)(I + N)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Let $A = (I - N)(I + N)^{-1}$

Then, $A^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

Then, $AA^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$

$\therefore AA^{\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore A = (I-N)(I+N)^{-1}$ is unitary.

9. Prove that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary.

Solution: Let given matrix be $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ then, $A' = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\therefore A^{\theta} = \overline{(A')} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

Consider $A^{\theta}A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore A^{\theta}A = I \qquad \therefore A$ is unitary.

Unit 2.2
Rank of Matrices

❖ Elementary Transformations:

(i) Interchanging any two rows or any two columns:
R_{ij} denotes the interchange of i^{th} and j^{th} rows and
C_{ij} denotes the interchange of i^{th} and j^{th} columns.

(ii) Multiplication of each element of i^{th} row by non zero k, i. e. kR_i
Multiplication of each element of i^{th} column by non zero k, kC_i

(iii) Adding row $(R_i + kR_j)$ / Adding columns $\left(C_i + kC_j\right)$.
These are only valid transformations.
Two matrices A and B are said to be **Equivalent Matrices** if the matrix B is obtained by performing elementary transformations on the matrix A.
Denoted by, $A \sim B$(A is equivalent to B).

❖ Rank of Matrix:

- **Minor of order r/ sub-matrix of order r –** If we select any r rows and r columns in Given $m \times n$ matrix then a matrix formed by these r rows and r columns is called a square sub-matrix of order r.
Determinant of this square sub-matrix of order r is called Minor of order r.

- **Definition of rank of 'A':** A number 'r' is said to be the rank of matrix A, if
(i) There exists at least one sub – matrix of A of order r whose determinant is non – zero
(ii) Every sub- matrix of A whose determinant with order (r + 1), if it exists, should be zero.

Note:
a) The rank of matrix is the order of any highest order non - vanishing minor.
b) The rank 'r' of a matrix A is denoted by $\rho(A)$.

- **Properties:**
(i) The rank of a null matrix is always zero.
(ii) If A is a non-zero square matrix of order n, then $1 \leq \rho(A) \leq n$.
(iii) If A is a matrix of order $m \times n$, then $1 \leq \rho(A) \leq$min (m, n)
(iv) Rank of a non - singular matrix is always equal to its order.
i.e. If $|A| \neq 0 \; then \; \rho(A) = n$
(v) Rank of a matrix is always unique.
(vi) $\rho(A) = \rho(A')$
(vii) $\rho(AB) \leq \rho(A) \; and \; \rho(AB) \leq \rho(B)$
(viii) Rank is invariant under elementary transformations. i.e. If $A \sim B \; then \; \rho(A) = \rho(B)$
(ix) Rank of A = Rank of (kA), where k is any scalar
(x) If $A_{n\times n}$ is non – singular i.e., $|A| \neq 0$ then rank of $A = n$ and rank of $A^2 = n$
Since $|A^2| = |A.A| = |A|.|A| \neq 0$

SOME SOLVED EXAMPLES

- **Determine the ranks of the following matrices by finding minors.**

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

Solution:

We have

$$\begin{aligned} |A| &= 1(6-8) - 2(4-0) + 3(4-0) \\ &= -2 - 8 + 12 \\ &= 2 \neq 0 \end{aligned}$$

Thus A is non – singular matrix,
i.e., $|A|$ is the highest order non - vanishing minor of order 3.
Hence rank of A is 3.

2. $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

Solution:

$|A| = 1(28+2) - (-2)(-14-1) + 3(-4+4) \ = 0$

Here the minor of order 3 is zero.
Let us check if there is at least one minor of order 2 which is non zero.

$$\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0,$$

$$but \begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = -10 \neq 0$$

i.e., at least one minor of order 2 is non -zero. Hence rank of A is 2.

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

Solution:

Since we have $|A| = 0$ i.e., the minor of order 3 is zero (Since all rows are identical).
All minors of order 2 are also zero (Since all rows are identical).
Minor of order one is not zero.(Say [1] or any single element of matrix)
Hence rank of A is 1.

4. $A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 1 & -1 & 0 & 3 \\ 3 & 5 & 1 & 6 \end{bmatrix}_{3\times 4}$

Solution:

Here, A is the matrix of order 3×4.
Therefore $1 \leq \rho(A) \leq \min(3,4),\ i.e\ 3.$

Now, consider the minor. $\begin{vmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \\ 3 & 5 & 1 \end{vmatrix} = 2(-1-0) - 4(1-0) + 3(5+3)$

$$= -2 - 4 + 24 = 18 \neq 0$$

Hence rank of A is 3.

- **Rank of matrix using Elementary Row Transformations.**

1. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Solution:

$$R_4 - (R_1 + R_3), \quad \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 - (R_1 + R_2), \quad \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$\therefore$ Minor of order 4 is zero. All minors of order 3 are zero.

Consider the minor of order two $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$

Hence, the rank of matrix B is 2.

As $A \sim B \quad \therefore \rho(A) = \rho(B) = 2$

2. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

Solutions:

$$\left.\begin{matrix} R_4 - R_1 \\ R_3 - R_1 \\ R_2 - R_1 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 7 & 7 & 7 & 7 \end{bmatrix}$$

$$\left.\begin{matrix} R_4 - 7R_2 \\ R_3 - 2R_2 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$\therefore$ Minor of order 4 is zero. All minors of order 3 are zero .

Consider the minor of order two $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1 \neq 0$

Hence, the rank of matrix B is 2.

As $A \sim B \quad \therefore \rho(A) = \rho(B) = 2$

❖ **Row Equivalent Matrix**

Definition: If a matrix A is reduced to a matrix B by using elementary row transformations alone, then B is said to be **row equivalent** to A.

❖ **Echelon form** (**Row Echelon form**) or **Canonical form**

Definition: The **Echelon form** or **Canonical form** of a matrix A is a row equivalent matrix of rank 'r' in which

(a) One or more elements of each of the first r rows are non – zero while all other rows have only zero elements, (i.e all zero rows, if any, are placed at the bottom of the matrix so that the first r rows form an upper triangular matrix).

(b) The number of zero before the first non – zero element in a row is less than the number of such zeros in the next row.

Note:

1. By performing only row transformations, a given matrix that is reduced to an **upper triangular form** is called its **Echelon form**.
2. Rank of a given matrix is equal to the number of non – zero rows in the Echelon form.

Ex. Reduce the matrix $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ to Echelon Forms and hence find the ranks.

Solution:
$$R_{14} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

$$\left.\begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -1 & 12 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix}$$

$$R_2 - R_4 \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix}$$

$$\left.\begin{matrix} R_3 - 3R_2 \\ R_4 + 7R_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 23 & 24 \end{bmatrix} \quad R_4 + \frac{23}{4}R_3 \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & -19/4 \end{bmatrix}$$

This is Echelon form of the given matrix, in which the number of non – zero rows is 4. Hence the rank of the matrix is 4.

Normal Form:

Definition: By performing elementary row and column transformations, every non – zero matrix can be reduced to one of the four forms, called the normal form of A:

(i) $[I_r]$ (ii) $[I_r \quad 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Note: Rank of A = Rank of the normal form of $A = r$.

Method to Reduce a Given Matrix to its Normal Form by Applying Elementary Transformations:

Step 1: Reduce the first diagonal element a_{11}, which is called a leading element (or a pivot), to 1 by applying any (row or column) transformation.

Step 2: Apply row – transformation to reduce all other elements in first column to zero.

Step 3: Apply column – transformation to reduce all other elements in first row to zero.

Step 4: Reduce the second diagonal element a_{22}, which is then called the leading element, to1 by applying any (row or column) transformation without disturbing the elements of the first row and first column.

Step 5: Applying row – transformation clear off all other non – zero elements of the second column and reduce them to zero without disturbing the first row.

Step 6: Applying column – transformation clear off all other non – zero elements of the second row and reduce them to zero without disturbing the first column.

Continuing the above procedure with the successive rows and columns, we can reduce a given matrix to its normal form.

Note: Application of elementary transformation on any matrix A may differ but rank of A is unique.

SOME SOLVED EXAMPLES

- **Reduce the matrix to Normal Form. Find the rank of following matrices.**

1. $\begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$

Solution: $R_1 - R_2 \quad \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$

$$\left.\begin{matrix} R_2 - 3R_1 \\ R_3 - 7R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$$

$$\left.\begin{matrix} C_2 + C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$$

$$\frac{C_2}{7} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 2 & -8 & -12 \end{bmatrix}$$

$$R_3 - 2R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} C_3 + 4C_2 \\ C_4 + 6C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the rank of matrix is 2.

2. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Solution: $R_{12} \quad \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$\left.\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 6R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 03 & 07 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\left.\begin{matrix} C_2 + C_1 \\ C_3 + 2C_1 \\ C_4 + 4C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_2 - R_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\left.\begin{matrix} R_3 - 4R_2 \\ R_4 - 9R_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$\left.\begin{matrix} C_3 + 6C_2 \\ C_4 + 3C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$\frac{C_3}{33} \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 2 & 44 \end{bmatrix}$$

$$R_4 - 2R_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 - 22C_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the rank of matrix is 3.

3. $\begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Solution:

$$R_2 - 4R_1 \sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$C_2 + C_1, C_3 + 2C_1, C_4 + 3C_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_2 \longleftrightarrow R_4 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & 8 & 14 \end{bmatrix}$$

$$R_3 - 3R_2 \ , R_4 - 5R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix}$$

$$C_4 - 2C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix}$$

$$R_4 - 8R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$C_4+2C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix} \qquad \frac{1}{12}R_4 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$\therefore$ Rank of A = 4.

4. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Solution:

$$R_2 \longleftrightarrow R_1 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2-2R_1, R_3-3R_1, R_4-6R_1 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$C_2+C_1, C_3+2C_1, C_4+4C_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_2-R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad R_3-4R_2\ , R_4-9R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$C_3+6C_2\ , C_4+3C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} \qquad R_4-2R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{33}C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4-22C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A=3.

5. $\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$

Solution:

$$R_2 \longleftrightarrow R_1 \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix} \qquad R_2 - 2R_1, R_3 - 3R_1, R_4 - R_1 \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

$$C_3 - C_1, C_4 - 2C_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix} \qquad (-1)R_2\ , R_5 + R_4 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 - 3R_2\ , R_4 - 4R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 - R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 - C_2\ , C_4 - 3C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left(-\frac{1}{3}\right)C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 + 14C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A=3.

6. If A and B are as given below, find the rank of A by reducing it to the normal form. Find 3A – B, hence or otherwise , show that $3A^2 - AB = 2A$ also find the rank of $3A^2 - AB$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix} \qquad \left.\begin{matrix} R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} C_2 - 2C_1 \\ C_3 - C_1 \\ C_4 - 2C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{C_2}{2} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{matrix} C_3 - C_2 \\ C_4 - C_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2.$

$$3A - B = 3\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 3 & 6 \\ 0 & 6 & 3 & 3 \\ 6 & 18 & 9 & 15 \\ 6 & 12 & 6 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2I$$

$\therefore 3A^2 - AB = A(3A - B) = A(2I) = 2A$

Since $\rho(A) = \rho(2A) = \rho(3A^2 - AB)$

Hence $\rho(3A^2 - AB) = 2$

7. Find the values of P for which the following matrix A will have (i) rank 1 (ii) rank 2 (iii) rank 3, where $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$

Solution:

Let us first find the determinant of A.

$$|A| = \begin{vmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{vmatrix} = 3(9 - P^2) - P(3P - P^2) + P(P^2 - 3P)$$

$$= 3(3 - P)(3 + P) - P^2(3 - P) + P^2(P - 3)$$

$$= (3 - P)[3(3 + P) - P^2 - P^2]$$
$$= (3 - P)[9 + 3P - 2P^2]$$
$$= (3 - P)^2(3 + 2P)$$

If $|A| = 0$, i.e if $P = 3$ or $-3/2$,

then the rank of A is either 1 or 2

Consider, if $P = 3$, then $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ all minors of order 2 are zero.

Hence rank of A is 1, when $P = 3$, ………….(i)

If $P = -3/2$, then $A = \begin{bmatrix} 3 & -3/2 & -3/2 \\ -3/2 & 3 & -3/2 \\ -3/2 & -3/2 & 3 \end{bmatrix}$

Consider the minor of order of 2, $\begin{vmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & 3 \end{vmatrix} = 9 - \frac{9}{4} = \frac{27}{4} \neq 0$

Hence rank of A is 2, when $P = -3/2$ ……(ii)

For rank 3, $|A| \neq 0$. When P can take any value other than 3 or $-3/2$ ………(iii)

8. If $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$ is the given square matrix of order 3, find the values of k for which rank of A is less than 3. Also find the ranks for those values of k.

Solution: $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$ $\quad R_{12} \sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$

$$\left.\begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \text{……………..(i)}$$

For the matrix A to be of rank less than 3, we must have $|A| = 0$

i.e., $(k-8)(-k+2)-(-5k)(k-2)=0$

i.e., $-k^2+10k-16+5k^2-10k=0$

i.e., $4k^2-16=0$ i.e $k^2=4$ i.e., $k=\pm 2$

Now three cases arise.

Case (i) If $k \neq \pm 2$ then A has $rank = 3$.

Case (ii) If $k = 2$, then (i) $\Rightarrow A \sim \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}$

$$\left.\frac{C_2}{6}, \frac{C_3}{10}\right\} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(-1)R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 - C_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

Case (iii) If $k = -2$, then (i) $\Rightarrow A \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix}$

$$\left.\frac{R_2}{-10}, \frac{R_3}{-4}\right\} \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} C_2 - 2C_1 \\ C_3 + 6C_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 + C_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 2$

9. Find the value of p for which the following matrix A will have

(i) Rank 1 , (ii) Rank 2, (iii) Rank 3. $A=\begin{bmatrix} p & p & 2 \\ 2 & p & p \\ p & 2 & p \end{bmatrix}$

Solution:

The determinant of the matrix A is

$$|A| = p\left(p^2-2p\right)-p\left(2p-p^2\right)+2\left(4-p^2\right)$$
$$= 2p^3-6p^2+8$$
$$= 2(p^3+p^2-4p^2-4p+4p+4)$$
$$= 2(p+1)(p^2-4p+4)$$
$$= 2(p+1)(p-2)^2$$

$\therefore |A| = 0 \ iff \ p=-1 \, or \ p=2.$

1) Rank of $A=3$ if $|A|\neq 0 \ i.e. \ if \ p\neq -1 \, or \ p\neq 2$

2) If $p=-1$,

$$A=\begin{bmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

$Now, \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} \neq 0 \ \ But \ |A| = 0$

$\therefore$ When $p=-1$, Rank of $A=2$.

3) If $p=2$,

$$A=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$By \ R_2-R_1, \ R_3-R_1$

$$A=\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ When $p=2$, Rank of $A=1$

SOME PRACTICE PROBLEMS

1. Express the following matrices as the sum of a symmetric and a skew-symmetric matrix.

(i) $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 8 & 6 \\ 0 & 4 & 4 \\ 2 & 10 & 12 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3a & 2b & 2c \\ b & c & a \\ 3c & 3a & 3b \end{bmatrix}$

(iv) $\begin{bmatrix} 2a & 3b & 2c \\ -b & c & 3a \\ 3c & 3a & 2b \end{bmatrix}$ **(v)** $\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix}$

2. Express the following matrices as the sum of a Hermitian and a skew-Hermitian matrix.

(i) $\begin{bmatrix} 2+i & -i & 3+i \\ 1+i & 3 & 6-2i \\ 3-2i & 6i & 4-3i \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$

(iii) $\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$

3. Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $P + iQ$ where P and Q are Hermitian matrices.

4. Express the Hermitian matrix $A = \begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$ as $P + iQ$ where P is real symmetric and Q is real skew-symmetric matrix.

5. Express the skew-Hermitian matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$ where P is real skew-symmetric and Q is real symmetric matrix.

6. Express the Hermitian matrix $\begin{bmatrix} 4 & 3-2i & -1+i \\ 3+2i & 2 & 5+4i \\ -1-i & 5-4i & 7 \end{bmatrix}$ as B + iC where B is real symmetric and C is real skew symmetric.

7. Express the skew - Hermitian matrix $\begin{bmatrix} 2\mathrm{i} & 3+\mathrm{i} & 2-\mathrm{i} \\ -3+\mathrm{i} & 0 & 6\mathrm{i} \\ -2-\mathrm{i} & 6\mathrm{i} & -2\mathrm{i} \end{bmatrix}$ as P + iQ where P is real skew – symmetric and Q is real symmetric.

8. Verify that the matrix A is orthogonal, where $\mathrm{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ and find A^{-1} .

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