DE MOIVRE'S THEOREM

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Statement : For any rational number n the value or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\ \sin n\,\theta$$

1. If $z = cos\ \theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta +\ i\ sin\ \theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta -\ i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta -\ i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\ n\,\theta - i\ \sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\ (-\theta) + i\ \sin\ (-\theta)\}^n$

$= \cos(-n\theta) + isin(-n\theta).$

$= \cos n\,\theta - i\sin\ n\ \theta$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\ n\left(\frac{\pi}{2} - \theta\right) + i\sin n\ \left(\frac{\pi}{2} - \theta\right)$

(2) $(\cos\ \theta + \mathrm{i}\sin\Phi\ )^n \neq \cos\ n\,\theta +\ i\sin n\ \Phi.$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

Solution: $cos2\theta - i\sin 2\theta = (cos\theta + i\sin\theta)^{-2}$

$cos3\theta + i\sin 3\theta = (cos\theta + i\sin\theta)^3$

$cos5\theta - i\sin 5\theta = (cos\theta + i\sin\theta)^{-5}$

$\therefore$ Expression $= \frac{(cos\theta + i\sin\theta)^{-14}(cos\theta + i\sin\theta)^{15}}{(cos\theta + i\sin\theta)^{36}(cos\theta + i\sin\theta)^{-35}} = \frac{(cos\theta + i\sin\theta)^1}{(cos\theta + i\sin\theta)^1} = 1$

2. Prove that $\frac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8} = -\frac{1}{4}$

Solution: $\frac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8}$

$$(1+i)^8 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]^8 = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8 = \left\{\sqrt{2}e^{i\pi/4}\right\}^8 = 2^4.e^{i\,2\pi}$$

$$(1-i)^4 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]^4 = \left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^4 = \left\{\sqrt{2}e^{-i\pi/4}\right\}^4 = 2^2.e^{-i\,\pi}$$

$$\left(\sqrt{3}-i\right)^4 = \left[2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\right]^4 = \left[2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^4 = \left\{2e^{-i\pi/6}\right\}^4 = 2^4.e^{-i\,2\pi/3}$$

$$\left(\sqrt{3}+i\right)^8 = \left[2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]^8 = \left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^8 = \left\{2e^{i\pi/6}\right\}^8 = 2^8.e^{i\,4\pi/3}$$

$$\text{Expression} = \frac{(2^4.e^{i\,2\pi}).(2^4.e^{-i\,2\pi/3})}{(2^2.e^{-i\,\pi}).(2^8.e^{i\,4\pi/3})} = \frac{1}{2^2}.\frac{e^{i\,3\pi}}{e^{i\,2\pi}} = \frac{1}{4}e^{i\,\pi} = \frac{1}{4}(\cos\pi + i\sin\pi) = \frac{-1}{4}$$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$

Solution: We have $1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)$

$$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = 2\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)$$

$$\therefore \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{2^{16}[cos(\pi/3)+i\sin(\pi/3)]^{16}}{2^{17}[cos(\pi/6)-i\sin(\pi/6)]^{17}}$$

$$= \frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)^{-17}$$

$$\therefore \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left[cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]^{-17}$$

$$= \frac{1}{2}\left(cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)\left[cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[cos\left(\frac{16}{3}+\frac{17}{6}\right)\pi + i\sin\left(\frac{16}{3}+\frac{17}{6}\right)\pi\right]$$

$$= \frac{1}{2}\left[cos\left(\frac{49}{6}\right)\pi + i\,sin\left(\frac{49}{6}\right)\pi\right]$$

$$= \frac{1}{2}\left[cos\left(8\pi+\frac{\pi}{6}\right)+i\sin\left(8\pi+\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

Hence, the modulus is $\frac{1}{2}$ and principal value of the argument is $\frac{\pi}{6}$

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

Solution: We have $1 = sin^2\alpha + cos^2\alpha = sin^2\alpha - i^2cos^2\alpha$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha)$$

$$\therefore 1 + \sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha + 1)$$

$$\therefore \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha} = \sin\alpha + i\cos\alpha = cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\therefore \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \left\{cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)\right\}^n$$

$$= \cos n\left(\frac{\pi}{2} - \alpha\right) + i\sin n\left(\frac{\pi}{2} - \alpha\right)$$

5. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ *and* $\overline{z}$ is the conjugate of z prove that $(z)^{10} + (\overline{z})^{10} = 0$.

Solution: $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ $\therefore \bar{z} = \cos\frac{\pi}{4} - isin\,\frac{\pi}{4}$

$$\therefore (z)^{10} + (\bar{z})^{10} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= \left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) + \left(\cos\frac{10\pi}{4} - i\sin\frac{10\pi}{4}\right)$$

$$= 2\cos\frac{10\pi}{4} = 2\cos\left(\frac{5\pi}{2}\right) = 0$$

(ii) $\left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n = 2^{n+1}cos(n\pi/3)$.

Solution: $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$1 - i\sqrt{3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$\therefore \left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n$$

$$= 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + 2^n\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$
$$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$
$$= 2^n\left(2\cos\frac{n\pi}{3}\right)$$
$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2}\cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Solution: The given equation is $x^2 - 2x + 2 = 0$

$$\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$
$$\therefore \alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$\beta = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$
$$\therefore \alpha^n + \beta^n = \left[\sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$
$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{n/2}\left(cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$
$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$
$$= \left(\sqrt{2}\right)^n\left(2cos\frac{n\pi}{4}\right)$$
$$= 2 \cdot 2^{n/2} cos\frac{n\pi}{4}$$

Putting $n = 8$ $\quad \alpha^8 + \beta^8 = 2 \cdot 2^4 \cos 2\pi = 2^5 = 32$

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$

Solution: The given equation is $x^2 - 2\sqrt{3}x + 4 = 0$

$\therefore x = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i = 2\left(\frac{\sqrt{3}}{2} \pm i.\frac{1}{2}\right) = 2\left(cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right)$ are the roots

Let $\alpha = 2\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),\ \beta = 2\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$\therefore \alpha^3 + \beta^3 = 2^3\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$$

$$= 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - i\ sin\frac{\pi}{2}\right] = 2^3.2\ cos\frac{\pi}{2} = 0$$

Similarly, $\alpha^3 - \beta^3 = 2^3\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$

$$= 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{\pi}{2} + i\ sin\frac{\pi}{2}\right] = 2^3.2\ i\ sin\ \frac{\pi}{2} = 16\ i$$

8. If $a = \cos 2\alpha + i \sin 2\alpha\,, b = \cos 2\beta + i \sin 2\beta\,, c = \cos 2\gamma + i \sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

Solution: $\frac{ab}{c} = \frac{(\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + \sin 2\beta)}{(\cos 2\gamma + i \sin 2\gamma)}$

$$= \cos\ (2\alpha + 2\beta - 2\gamma) + i \sin(2\alpha + 2\beta - 2\gamma)$$

$$= \cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)$$

$$\sqrt{\frac{ab}{c}} = [\cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)]^{1/2}$$

$$= \cos(\alpha + \beta - \gamma) + i \sin(\alpha + \beta - \gamma)$$

Similarly, $\sqrt{\frac{c}{ab}} = \cos(\alpha + \beta - \gamma) - i \sin(\alpha + \beta - \gamma)$

By addition we get $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

9. If $x - \frac{1}{x} = 2i \sin\theta\,,\quad y - \frac{1}{y} = 2\, i \sin\Phi\,,\quad z - \frac{1}{z} = 2i \sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Solution: Since $x - \frac{1}{x} = 2i \sin\theta \quad \therefore x^2 - 2ix \sin\theta - 1 = 0$

Solving the quadratic for x, we get,

$$x = \frac{2i \sin\theta \pm \sqrt{4i^2 \sin^2\theta - 4(1)(-1)}}{2(1)} = i \sin\theta \pm \sqrt{1 - \sin^2\theta} = i \sin\theta \pm \cos\theta$$

consider $x = \cos\theta + i \sin\theta$

Similarly, $= \cos\Phi + i \sin\Phi\,,\qquad z = \cos\psi + i \sin\psi$

(i) $xyz = (\cos\theta + i \sin\theta)(\cos\Phi + i \sin\Phi)(\cos\psi + i \sin\psi)$

$$= \cos(\theta + \Phi + \psi) + i \sin(\theta + \Phi + \psi)$$

$$\therefore \frac{1}{xyz} = \cos(\theta + \Phi + \psi) - i\sin(\theta + \Phi + \psi)$$

Adding we get $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{(\cos\theta + i\sin\theta)^{1/m}}{(\cos\Phi + i\sin\Phi)^{1/n}} = \frac{\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right)}{\left(\cos\frac{\Phi}{n} + i\sin\frac{\Phi}{n}\right)} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Similarly, $\frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) - i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Adding we get $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$.

Solution: We have $cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

$$\therefore (cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$$

$$\therefore (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$$

Let $x = cos\ \alpha + i\sin\alpha, y = 2(\cos\beta + i\sin\beta), z = 3(\cos\gamma + i\sin\gamma)$

$$\therefore x + y + z = 0$$

$$\therefore (x + y + z)^3 = 0$$

$$\therefore x^3 + y^3 + z^3 + 3(x + y + z)(xy + yz + zx) - 3xyz = 0$$

$$\therefore x^3 + y^3 + z^3 = 3\,xyz$$

$$\therefore (\cos\alpha + i\sin\alpha)^3 + 2^3(\cos\beta + i\sin\beta)^3 + 3^3(\cos\gamma + i\sin\gamma)^3$$

$$= 3(\cos\alpha + i\sin\alpha).2.(\cos\beta + i\sin\beta).3.(\cos\gamma + i\sin\gamma)$$

$\therefore$ By De Moivre's Theorem,

$$(\cos 3\alpha + i\sin 3\alpha) + 8.(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$$

$$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$$

$$(\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$$

$$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$$

Equating imaginary parts, we get the required result.

11. If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \ldots ad.inf. = i$

(ii) $x_0x_1x_2 \ldots ad.inf. = -i$

Solution: We have $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$

Putting $r = 0, 1, 2, 3 \ldots\ldots\ldots$ we get $x_0 = \cos\frac{\pi}{3^0} + i\sin\frac{\pi}{3^0} = \cos\pi + i\sin\pi = -1$

$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, $x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2} \ldots\ldots\ldots\ldots$ and so on

$x_1\, x_2\, x_3 \ldots\ldots\ldots\ldots$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\ldots\ldots\ldots$$

$$= \cos\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \ldots\ldots\right)\pi + i\sin\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \ldots\ldots\right)\pi$$

But $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \ldots\ldots\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

$x_1\, x_2\, x_3 \ldots\ldots\ldots\ldots = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

Also $x_0\, x_1\, x_2\, x_3 \ldots\ldots\ldots\ldots = x_0(i) = (-1)(i) = -i$

12. If $(cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots[cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2r\pi}{n^2}$

Solution:

L.H.S $= (cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots[cos(2n-1)\theta + i\sin(2n-1)\theta]$

$$= cos[1 + 3 + \cdots + (2n-1)]\theta + i\sin[1 + 3 + \cdots + (2n-1)]\theta$$

But $1 + 3 + \cdots + (2n-1)$ is an A.P. with first term 1, the number of terms n and common difference 2.

$\therefore$ The Sum, $S_n = \frac{n}{2}[2a + (n-1).d] = \frac{n}{2}[2 + (n-1).2] = n^2$

$\therefore$ L.H.S = $cos(n^2\theta) + \ i\sin(n^2\theta)$

R.H.S $= 1 = \cos 2\,r\,\pi + i\sin 2\,r\,\pi$ where $r = 0,1,2 \ldots\ldots$

Equating the two sides, we get $n^2\theta = 2\,r\,\pi$ $\therefore \theta = \frac{2\,r\,\pi}{n^2}$

13. By using De Moivre's Theorem show that

$$\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{sin\alpha/2}$$

Solution: $\frac{1-z^6}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5$ ....(i)

Let $z = \cos\alpha + i\sin\alpha$, then by De Moivre's theorem, $z^n = \cos n\alpha + i \sin n\,\alpha$

$$\therefore 1 + z + z^2 + z^3 + z^4 + z^5 = 1 + (cos\alpha + i\sin\alpha) + (\cos 2\,\alpha + i \sin 2\alpha)$$

$$+(cos3\alpha + i\sin 3\alpha) + (cos4\alpha + i\sin 4\alpha) + (cos5\alpha + i\sin 5\alpha)$$

$$= (1 + cos\alpha + \cos 2\,\alpha + cos3\alpha + cos4\alpha + cos5\alpha)$$

$$+i\,(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin\alpha\, 4 + \sin 5\alpha) \text{ ....(ii)}$$

Now, $\frac{1-z^6}{1-z} = \frac{1-(cos\alpha+i\sin\alpha)^6}{1-(cos\alpha+i\sin\alpha)} = \frac{1-\cos 6\alpha - i\sin 6\alpha}{1-cos\alpha - i\sin\alpha} = \frac{2sin^2 3\alpha - 2i\,sin3\alpha\cos 3\alpha}{2sin^2(\alpha/2) - 2isin(\alpha/2)\cos(\alpha/2)}$

$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)\,[sin(\alpha/2) + i\cos(\alpha/2)]}{sin(\alpha/2)[sin(\alpha/2) - icos(\alpha/2)][\sin(\alpha/2) + i\cos(\alpha/2)]}$$

$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]}{sin(\alpha/2)[\sin^2(\alpha/2) + \cos^2(\alpha/2)]}$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{\pi}{2} - 3\alpha\right) - \; isin\left(\frac{\pi}{2} - 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \; i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(-\frac{\pi}{2} + 3\alpha\right) + \; i\sin\left(-\frac{\pi}{2} + 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$\therefore \frac{1-z^6}{1-z} = \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(3\alpha - \frac{\alpha}{2}\right) + \; i\sin\left(3\alpha - \frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{5\alpha}{2}\right) + i\;sin\left(\frac{5\alpha}{2}\right)\right] \text{ ....(iii)}$$

Using (i) equating real parts, from (ii) and (iii), we get

$$1 + cos\alpha + \cos 2\,\alpha + \cdots. + cos5\alpha = \frac{\sin 3\alpha.cos(5\alpha/2)}{sin(\alpha/2)}$$

And equating imaginary parts, we get

$$\sin\alpha + \sin 2\alpha + \cdots. + \sin 5\alpha = \frac{\sin 3\alpha.sin(5\alpha/2)}{sin(\alpha/2)}$$

PRACTICE PROBLEMS:

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

5. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

6. Simplify $(1 + cos\,\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$

7. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

8. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ *and* $\overline{z}$ is the conjugate of z find the value of $(z)^{15} + (\overline{z})^{15}$.

9. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $\left(\sqrt{3}+i\right)^{120} + \left(\sqrt{3}-i\right)^{120} = 2^{121}$

10. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$

and 2 if $n = 3k$ where k is an integer.

12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,

prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15}+\beta^{15}=-2^{16}$ **(ii)** Deduce that $\alpha^{6}+\beta^{6}=128$

13. If α,β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that $\alpha^n+\beta^n = 2\cos n\,\theta\, cosec^n\theta$

14. If $a=\cos 3\alpha + i\sin 3\alpha, b=\cos 3\beta + i\sin 3\beta, c=\cos 3\gamma + i\sin 3\gamma$,
prove that $\sqrt[3]{\frac{ab}{c}}+\sqrt[3]{\frac{c}{ab}}=2cos(\alpha+\beta-\gamma)$

15. If $x+\frac{1}{x}=2\cos\theta$, $y+\frac{1}{y}=2\cos\emptyset$, $z+\frac{1}{z}=2\cos\psi$, prove that

(i) $xyz+\frac{1}{xyz}=2\cos(\theta+\Phi+\psi)$ **(ii)** $\sqrt{xyz}+\frac{1}{\sqrt{xyz}}=2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}}+\frac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\,cos\left(\frac{\theta}{m}-\frac{\emptyset}{n}\right)$

16. If $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, $c=\cos\gamma+i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc}=8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}$.

17. If a, b, c are three complex numbers such that $a+b+c=0$, prove that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and **(ii)** $a^2+b^2+c^2=0$

18. If $\cos\alpha+\cos\beta+\cos\gamma=0$ *and* $\sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$, $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$.

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$

(iii) $cos(\alpha+\beta)+cos(\beta+\gamma)+cos(\gamma+\alpha)=0$.

(iv) $\sin(\alpha+\beta)+sin(\beta+\gamma)+sin(\gamma+\alpha)=0$.

(v) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

19. If $a\cos\alpha+b\cos\beta+c\cos\gamma=a\sin\alpha+b\sin\beta+c\sin\gamma=0$, Prove that
$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$ and
$a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3\,abc\,sin(\alpha+\beta+\gamma)$

20. If $x_r=cos\left(\frac{2}{3}\right)^r\pi+i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3\ldots\infty=1$, **(ii)** $x_0x_1x_2\ldots\infty=-1$

COMPLEX NUMBERS

INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers x and y. It is written as $z = (x, y)$ **or** $z = x + i\,y$, where i = $\sqrt{-1}$ is known as **imaginary unit**. Here, x is called the Real part of z and is written as "**Re (z)**" and y is called the Imaginary part of z and is written as **"Im(z)"**.

If x = 0 and y ≠ 0, then $z = 0 + i\,y$ which is purely imaginary.

If x ≠ 0 and y = 0, then $z = x + i\,0 = x$ which is purely real

Hence z is **purely imaginary** if its real part is zero and is **real** if its imaginary part is zero.

This shows that every real number can be written in the form of a complex number by taking its imaginary part as zero. Hence the set of real numbers is contained in the set of complex numbers.

POWERS OF i:

We know that $i = \sqrt{-1}$,

$$i^2 = i \times i = -1, \qquad i^3 = i^2 \times i = -i,$$
$$i^4 = (i^2)^2 = (-1)^2 = 1, \; i^5 = i \times i^4 = i \quad \text{etc.}$$

Even power of i is either 1 or -1 and odd power of i is either i or $-i$.

EQUALITY OF COMPLEX NUMBER:

If $z_1 = z_2$ then, $x_1 + i\,y_1 = x_2 + i\,y_2$ Comparing real and imaginary parts $x_1 = x_2$ and $y_1 = y_2$

This shows that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

CONJUGATE OF COMPLEX NUMBER:

If $z = x + i\,y$ is a complex number then its conjugate or complex conjugate is defined as

$\bar{z} = x - i\,y$. Also $z\,\bar{z} = (x + i\,y)\,(x - i\,y) = x^2 + y^2$

Note: To write the conjugate of a complex number, replace i by $-\,i$ in the complex number.

ALGEBRA OF COMPLEX NUMBER:

Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are two complex numbers. Then

(a) Addition: $z_1 + z_2 = (x_1 + i\,y_1) + (x_2 + i\,y_2)$

$$= (x_1 + x_2) + i\,(y_1 + y_2)$$

(b) Subtraction: $z_1 - z_2 = (x_1 + i\,y_1) - (x_2 + i\,y_2$
$$= (x_1 - x_2) + i\,(y_1 - y_2)$$

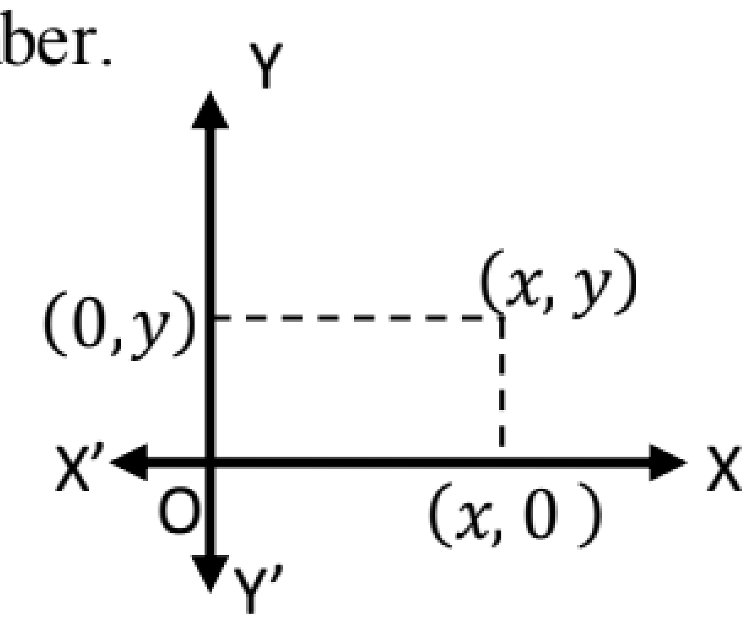
(c) Multiplication: $z_1 \cdot z_2 = (x_1 + i\,y_1) \cdot (x_2 + i\,y_2)$
$$= (x_1 x_2 - y_1\,y_2) + i\,(x_2\,y_1 + y_2\,x_1) \qquad [\,i^2 = -1]$$

(d) Division: $\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)}{(x_2+iy_2)} \cdot \frac{(x_2-iy_2)}{(x_2-iy_2)} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{(y_1x_2-x_1y_2)}{(x_2^2+y_2^2)}$

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER:

1. Argand's Diagram:

We know that the real numbers can be represented by point on a line in such a way that corresponding to every real number, there is one and only one point on the line and corresponding to every point on the line, there is one and only one real number.



Similarly, we can represent a complex number as follows:

Consider a complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.
Draw the coordinate axes. Since $x \in R, x$ can be represented by the point $(x, 0)$ on the X-axis.

Since $y \in R, y$ can be represented by the point $(0, y)$ on the Y-axis.

Then the point (x, y) represents the complex number $x + iy$, i.e., $x + iy = (x, y)$

Such a representation of complex numbers by points in a plane is called **Argand's diagram**.

The horizontal x-axis is called the real axis, vertical y-axis is called the imaginary axis and xy-plane is called complex plane.

We observe that the point representing a complex number can be on any of the coordinate axes or in any of the quadrants. The complex number $z = 0 + 0i$ i.e., the zero complex number is represented by the origin.

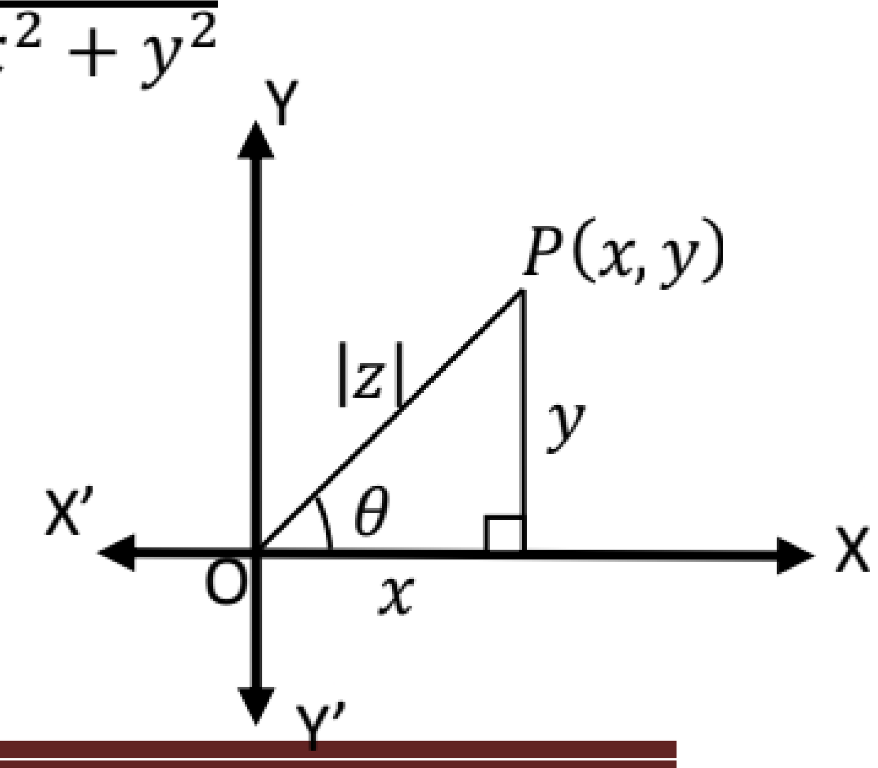
2. Geometrical Meaning of Modulus and Argument:

If $z = x + yi$ is a complex number, then the modulus of z is $|z| = \sqrt{x^2 + y^2}$

Let the point $P(x, y)$ represents the complex number $z = x + yi$

$\therefore |z| = \sqrt{x^2 + y^2} = OP$

Hence, modulus of z is the distance of the point P from the origin where P represents the complex number z in the plane.



And is denoted by $|z|$ or mod (z)

Again, OP makes an angle θ with the positive direction of X-axis.

θ is called the amplitude or argument of the complex number $z = x + yi$ and is denoted by arg (z) or amp (z).

$\therefore \sin\theta = \frac{y}{|z|}$ and $\cos\theta = \frac{x}{|z|}, |z| \neq 0$ $\quad\quad \therefore \tan\theta = \frac{y}{x}$, if $x \neq 0$

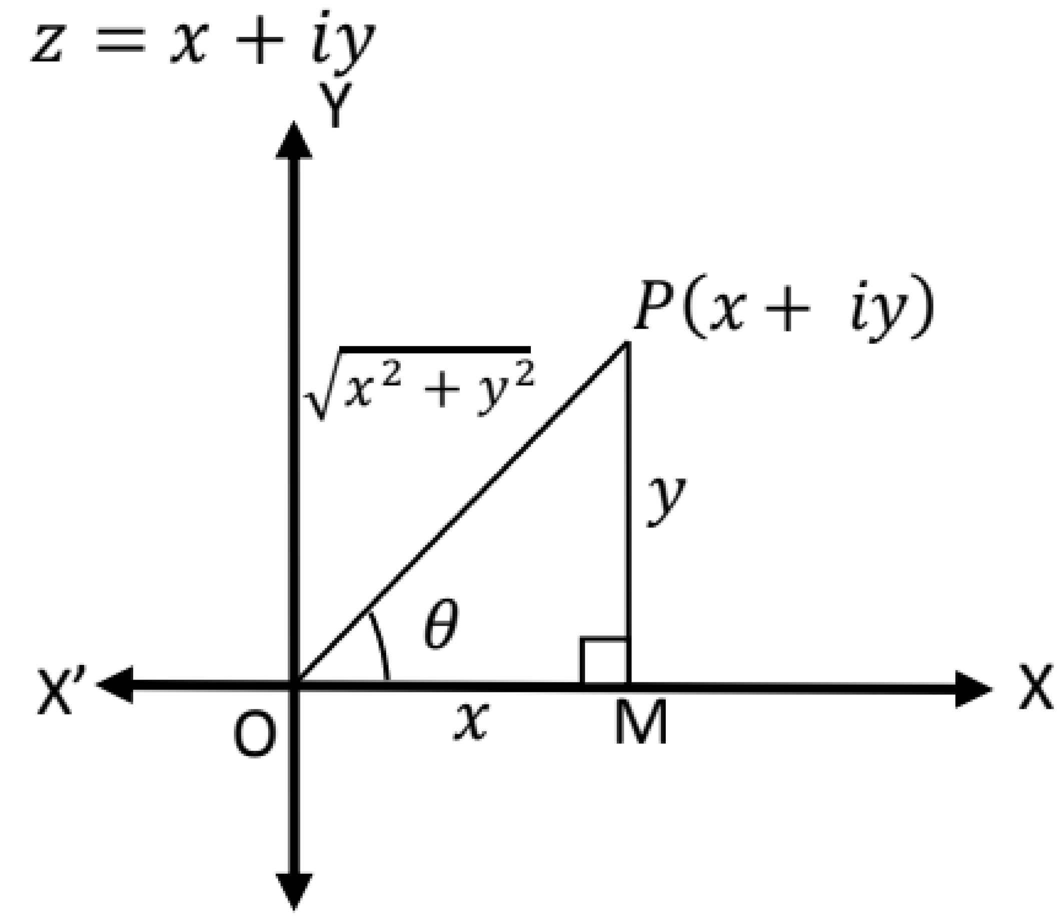
Hence, $|z| = r = \sqrt{x^2 + y^2}$ & $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Note: The value of θ which satisfies both the equation x = r cos θ and y = r sin θ, gives the argument of z. Argument θ has infinite number of values. The value of θ lying between $-\pi$ and π is called the **principal value** of Argument.

POLAR FORM OF A COMPLEX NUMBER:

Let a complex number $z = x + iy$ be represented by the point $P(x, y)$.

Then $OP = r = \sqrt{x^2 + y^2}$ is called the modulus of the complex number $z = x + iy$ and is denoted by $|z|$.



Thus $|z| = |x + iy| = \sqrt{x^2 + y^2}$

If the ray OP makes an angle θ with the positive X-axis, then this angle θ is called the argument of z and is denoted by $arg\ z$ (if $z \neq 0$)

From the figure $arg\ z$ is given by the equations:

$\cos\theta = \frac{x}{|z|} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$ and $\sin\theta = \frac{y}{|z|} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$

$\therefore x = r\cos\theta$ and $y = r\sin\theta$

Hence $z = x + iy$ can be written as $z = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$

This is called polar representation of z, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

Polar form of $x + iy$ for different sings of x, y :

Let $z = x + iy$ be a complex number

Let $z = r(\cos\theta + i\sin\theta)$ be its polar form. Then modulus of z, i.e., $|z| = \sqrt{x^2 + y^2}$ and amplitude of z is θ.

We have find to the value of θ for different signs of x and y, i.e., for different position of the point $P(x, y)$. The point $P(x, y)$ may be in the first, second, third or fourth quadrant.

Remark:

1. To find the amplitude of $z = x + iy$, if $P(x, y)$ does not lie in the quadrant but it lies on either of the coordinate axes

(i) If $P(x,y)$ lies on positive side of X-axis, i.e., $x > 0, y = 0$ (e.g. $z = 3$), then amp $(z) = 0$

(ii) If $P(x,y)$ lies on negative side of X-axis, i.e., $x < 0, y = 0$ (e.g. $z = -3$), then amp $(z) = \pi$

(iii) If $P(x,y)$ lies on positive side of Y-axis, i.e., $x = 0, y > 0$ (e.g. $z = 5i$), then amp $(z) = \pi/2$

(iv) If $P(x,y)$ lies on negative side of Y-axis, i.e., $x = 0, y < 0$ (e.g. $z = -5i$), then amp $(z) = \frac{3\pi}{2}$

(v) If $z = 0$, then $r = 0$ and amp (z) is not defined

2. We may be tempted to take $\tan\theta = y/x$, if $x \neq 0$. But this will not give us unique value of θ. If $\tan\theta > 0$, then θ may be in the first or third quadrant and if $\tan\theta < 0$, then θ may be in the second or fourth quadrant. Hence, it is not advisable to take $\tan\theta = y/x$ **or** $\theta = \tan^{-1}(y/x)$.

Therefore, we have to obtain the value of θ using the equations $\cos\theta = \frac{x}{r} = \frac{x}{|z|}$ and $\sin\theta = \frac{y}{r} = \frac{y}{|z|}$

Case (i): If $x > 0,\ y > 0$, then $P(x,\ y)$ lies in the first quadrant.

In this case $0 < \theta < \frac{\pi}{2}$

Case (ii): If $x < 0,\ y > 0$, then $P(x,\ y)$ lies in the second quadrant.

In this case $\frac{\pi}{2} < \theta < \pi$

Case (iii): If $x < 0,\ y < 0$, then $P(x,\ y)$ lies in the third quadrant.

In this case $\pi < \theta < \frac{3\pi}{2}$

Case (iv): If $x > 0,\ y < 0$, then $P(x,\ y)$ lies in the fourth quadrant.

In this case $\frac{3\pi}{2} < \theta < 2\pi$

EXPONENTIAL FORM OF A COMPLEX NUMBER:

We know $e^{i\theta} = \cos\theta + i\sin\theta$. Using polar form , $z = r\,(cos\,\theta + i\,sin\,\theta) = re^{i\theta}$

This is called exponential form or Euler's form of a complex number z. $z = re^{i\theta}$

Thus we have three forms of a complex number

$z = x + iy$ **(Cartesian form)** $z = r\,(cos\,\theta + i\,sin\,\theta)$ **(Polar form)**

$z = re^{i\theta}$ **(Exponential form)**

Note: $e^{i\theta} = \cos\theta + i\,\sin\theta,\ e^{-i\theta} = \cos\theta - i\,sin\,\theta$

Hence, $cos\,\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $sin\,\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

PROPERTIES OF COMPLEX NUMBER:

Let $z = x + iy$ and $\bar{z} = x - iy$

(a) $Re(z) = x = \frac{1}{2}(z + \bar{z})$

(b) $Im(z) = y = \frac{1}{2i}(z - \bar{z})$

(c) $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$

(d) $\overline{(z_1 \,.\, z_2)} = \overline{z_1}\,.\overline{z_2}$

(e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(f) $z\bar{z} = |z|^2 = |\bar{z}|^2$ since $|z| = |\bar{z}| = \sqrt{x^2 + y^2}$

(g) $|z_1 z_2| = |z_1||z_2|$ & $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 e^{i\theta_1}.\, r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$

Comparing with exponential form

$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ And $\arg.(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(h) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ & $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_1}\right) e^{i(\theta_1 - \theta_2)}$

Comparing with exponential form

$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_1} = \frac{|z_1|}{|z_2|}$ And $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

SOME SOLVED EXAMPLES:

1. Express $\alpha = \frac{(1+i)^3}{(2+i)(1+2i)}$ in the form $a + ib$. Also find α^2.

Solution: $\alpha = \frac{(1+i)^3}{(2+i)(1+2i)} = \frac{1+3i+3i^2+i^3}{2+4i+i+2i^2} = \frac{1+3i-3-i}{2+4i+i-2} = \frac{-2+2i}{5i}$

$$= \frac{-2+2i}{5i}.\frac{i}{i} = \frac{-2i+2i^2}{5\,i^2} = \frac{-2-2i}{-5} = \frac{2}{5} + \frac{2}{5}i$$

$$\alpha^2 = \left(\frac{2}{5}(1+i)\right)^2 = \frac{4}{25}(1+2i+i^2) = \frac{8i}{25}$$

2. Find the value of $z^4 - 4z^3 + 6z^2 - 4z - 12$ when $z = 1 + 2i$

Solution: Since $z = 1 + 2i$ i.e $z - 1 = 2i$

$\therefore (z-1)^2 = 4i^2$ $\therefore z^2 - 2z + 1 = -4$

$\therefore z^2 - 2z + 5 = 0$

When express the give expressions in terms of $z^2 - 2z + 5$.

For this we divide the given expressions by $z^2 - 2z + 5$

Expressions $= (z^2 - 2z + 5)(z^2 - 2z - 3) + 3$

$$= 0(z^2 - 2z - 3) + 3 = 0 + 3 = 3$$

3. Find the modulus and the principal argument of

$$\frac{\left(1+i\sqrt{3}\right)^3(1+i)^{-2}\left(\sqrt{3}+i\right)^{-1}}{2}$$

Solution:

$$z = \frac{\left(1+i\sqrt{3}\right)^3(1+i)^{-2}\left(\sqrt{3}+i\right)^{-1}}{2} = \frac{\left(1+i\sqrt{3}\right)^3}{2(1+i)^2\left(\sqrt{3}+i\right)} = \frac{1+i3\sqrt{3}-3(3)-i3\sqrt{3}}{2(1+2i-1)(\sqrt{3}+i)}$$

$$\therefore z = -\frac{8}{2(2i)\left(\sqrt{3}+i\right)} = -\frac{2}{i\left(\sqrt{3}+i\right)} = -\frac{2}{i\sqrt{3}-1} = \frac{2}{1-i\sqrt{3}}$$

$$= \frac{2}{1-i\sqrt{3}} \cdot \frac{1+i\sqrt{3}}{1+i\sqrt{3}} = \frac{2(1+i\sqrt{3})}{4} = \frac{(1+i\sqrt{3})}{2}$$

$$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$\therefore x = \frac{1}{2}, y = \frac{\sqrt{3}}{2},\ r = \sqrt{x^2 + y^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \quad \therefore\ \theta = \frac{\pi}{3}$

$\therefore$ Modulus $z = 1$, Amplitude $z = \frac{\pi}{3}$

4. Find the square root of $21 - 20i$

Solution: Let $x + iy = \sqrt{21 - 20i}$

$\therefore (x + iy)^2 = 21 - 20i \qquad \therefore x^2 - y^2 + 2ixy = 21 - 20i$

Equating real and imaginary parts $x^2 - y^2 = 21$ and $xy = -10$

Putting $y = \frac{-10}{x}$ in $x^2 - y^2 = 21$ We get, $x^2 - \left(\frac{-10}{x}\right)^2 = 21$

$\therefore x^2 - \frac{100}{x^2} = 21 \qquad \therefore x^4 - 100 = 21x^2$

$\therefore x^4 - 21x^2 - 100 = 0$

$\therefore (x^2 - 25)(x^2 + 4) = 0 \qquad \therefore x^2 = 25$ or $x^2 = 4$

Since x is real $x^2 = 25 \qquad \therefore x = \pm 5$

When $x = 5,\quad y = \frac{-10}{x} = \frac{-10}{5} = -2$

When $x = -5,\quad y = \frac{-10}{x} = \frac{-10}{-5} = 2$

$\therefore \sqrt{21 - 20i}$ is $5 - 2i$ or $-5 + 2i$

5. If $x + iy = \sqrt[3]{a + ib}$, prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$

Solution: $x + iy = \sqrt[3]{a + ib} \qquad \therefore (x + iy)^3 = a + ib$

$\therefore x^3 - 3ix^2y - 3xy^2 - iy^3 = a + ib$

$(x^3 - 3xy^2) + i(3x^2y - y^3) = a + ib$

Comparing real and imaginary parts $a = x^3 - 3xy^2,\quad b = 3x^2y - y^3$

$\frac{a}{x} = x^2 - 3y^2, \quad \frac{b}{y} = 3x^2 - y^2$

$\therefore \frac{a}{x} + \frac{b}{y} = (x^2 - 3y^2) + (3x^2 - y^2)$

$= 4x^2 - 4y^2 = 4(x^2 - y^2)$

6. Find the complex number z if

$\arg(z+1) = \frac{\pi}{6}$ *and* $\arg(z-1) = \frac{2\pi}{3}$

Solution: Let $z = x + iy$

$\therefore z + 1 = (x+1) + iy$ and $z - 1 = (x-1) + iy$

We are given that, $\arg(z+1) = \frac{\pi}{6}$ $\therefore tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{6}$

$\therefore \frac{y}{x+1} = \tan 30° = \frac{1}{\sqrt{3}}$ $\therefore \sqrt{3} \cdot y = x + 1$

We are given that $\arg(z-1) = \frac{2\pi}{3}$ $\therefore tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$

$\therefore \frac{y}{x-1} = \tan 120° = -\sqrt{3}$ $\therefore y = -\sqrt{3}x + \sqrt{3}$

We solve the two equations to get x and y

$\therefore \sqrt{3}.y = x + 1$ ....(1) $-\sqrt{3}.y = 3x - 3$ ....(2)

Adding both equations, we get, $0 = 4x - 2$ $\therefore x = 1/2$

Now $\sqrt{3}.y = x + 1$ gives $\sqrt{3}.y = \frac{3}{2}$ $\therefore y = \frac{\sqrt{3}}{2}$

$\therefore z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

7. Find two complex numbers such that their difference in 10i and their product is 29.

Solution: Let z_1 and z_2 are two complex numbers such that,

$z_1 - z_2 = 10i$ and $z_1 z_2 = 29$

$(z_1 + z_2)^2 = (z_1 - z_2)^2 - 4z_1 z_2 = (10i)^2 + 4(29)$

$= -100 + 116 = 16$

$|z_1 + z_2| = 4$

$\therefore z_1$ and z_2 are roots of quadratic equation

$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

i.e. $x^2 - (z_1 + z_2)x + z_1 z_2 = 0$ i.e., $x^2 - 4x + 29 = 0$

Solving $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)} = \frac{4 \pm \sqrt{-100}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i$

$\therefore z_1 = 2 + 5i$ and $z_2 = 2 - 5i$

8. If $z_1 = \cos\alpha + i\sin\alpha\ ,\ z_2 = \cos\beta + i\sin\beta\ \ show\ that$

$\frac{1}{2i}\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = sin(\alpha - \beta)$.

Solution: We have $\frac{z_1}{z_2} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \frac{e^{i\alpha}}{e^{i\beta}} = e^{i(\alpha-\beta)}$

$= cos(\alpha - \beta) + isin(\alpha - \beta)$

$\therefore \frac{z_2}{z_1} = \frac{\cos\beta + i\sin\beta}{\cos\alpha + i\sin\alpha} = \frac{e^{i\beta}}{e^{i\alpha}} = e^{i(\beta-\alpha)} = e^{-i(\alpha-\beta)}$

$= cos(\alpha - \beta) - i\sin(\alpha - \beta)$

$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = 2\,i\sin(\alpha - \beta)$. Hence, the result

9. If $z = \cos\theta + i\sin\theta$, prove that **(i)** $\frac{2}{1+z} = 1 - i\tan(\theta/2)$.

(ii) $\frac{1+z}{1-z} = i\cot\left(\frac{\theta}{2}\right)$.

Solution: **(i)** $\frac{2}{1+z} = \frac{2}{1+\cos\theta + i\sin\theta} = \frac{2}{2cos^2(\theta/2) + 2\,i\sin(\theta/2)\cos(\theta/2)}$

$= \frac{1}{cos(\theta/2)\cdot(cos(\theta/2) + i\sin(\theta/2))} = \frac{1}{cos(\theta/2)\cdot e^{i\left(\frac{\theta}{2}\right)}}$

$= \frac{e^{-i\left(\frac{\theta}{2}\right)}}{cos(\theta/2)} = \frac{cos(\theta/2) - i\sin(\theta/2)}{cos(\theta/2)} = 1 - i\tan(\theta/2)$

(ii) $\frac{1+z}{1-z} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}$

$$= \frac{2\cos^2(\theta/2)+2\,i\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)-2\,i\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{\cos(\theta/2)+i\sin(\theta/2)}{\sin(\theta/2)-i\cos(\theta/2)}$$

$$= \cot\left(\frac{\theta}{2}\right) . \frac{\cos(\theta/2)+i\sin(\theta/2)}{\sin(\theta/2)-i\cos(\theta/2)}$$

$$= \cot\left(\frac{\theta}{2}\right) . \frac{\cos(\theta/2)+i\sin(\theta/2)}{-i^2\sin(\theta/2)-i\cos(\theta/2)}$$

$$= \cot\left(\frac{\theta}{2}\right) . \frac{1}{-i}\left[\frac{\cos(\theta/2)+i\sin(\theta/2)}{\cos(\theta/2)+i\sin(\theta/2)}\right]$$

$$= \frac{1}{-i} . \cot\left(\frac{\theta}{2}\right) = \frac{1}{-i} . \frac{i}{i}\cot\left(\frac{\theta}{2}\right) = i\ \cot\left(\frac{\theta}{2}\right)$$

10. If $(1+\cos\theta+i\sin\theta)(1+\cos 2\theta+i\sin 2\theta) = u+iv$, prove that **(i)** $u^2+v^2 = 16\cos^2\left(\frac{\theta}{2}\right)\cos^2\theta$ **(ii)** $\frac{v}{u} = \tan\left(\frac{3\theta}{2}\right)$

Solution: We have to find u and v.

Now from data $(1+\cos\theta+i\sin\theta)(1+\cos 2\theta+i\sin 2\theta) = u+iv$,

$$\therefore \left[2\cos^2\left(\frac{\theta}{2}\right)+2\,i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right][2\cos^2\theta+2\,i\sin\theta\cos\theta] = u+iv$$

$$\therefore 2\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)+\ i\sin\left(\frac{\theta}{2}\right)\right].2\cos\theta[\cos\theta+i\sin\theta] = u+iv$$

$$\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta\cdot e^{i\left(\frac{\theta}{2}\right)}\cdot e^{i\theta} = u+iv$$

$$\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta e^{i\left(\frac{3\theta}{2}\right)} = u+iv$$

$$\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta\left[\cos\left(\frac{3\theta}{2}\right)+\ i\sin\left(\frac{3\theta}{2}\right)\right] = u+iv$$

Equating real and imaginary parts $u = 4\cos\left(\frac{\theta}{2}\right)\cos\theta\cos\left(\frac{3\theta}{2}\right)$ and $v = 4\cos\left(\frac{\theta}{2}\right)\cos\theta\ \sin\left(\frac{3\theta}{2}\right)$

$$\therefore u^2 + v^2 = 16\, cos^2\left(\frac{\theta}{2}\right) cos^2\theta \quad and \quad \frac{v}{u} = tan\left(\frac{3\theta}{2}\right)$$

11. If $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ *where* $0 < \alpha, \beta < \pi/2$ find the polar form of $\frac{1+{z_1}^2}{1-i\, z_1 z_2}$

Solution: Expression $= \frac{1+{z_1}^2}{1-i\, z_1 z_2}$

Dividing the numerator and denominator by z_1 we get Expression $= \frac{(1/z_1)+z_1}{(1/z_1)-iz_2}$

Putting $z_1 = \cos\alpha + i\sin\alpha$ *and* $\frac{1}{z_1} = cos\alpha - i\, sin\alpha$

$$\text{Expression} = \frac{(\cos\alpha - i\sin\alpha)+(\cos\alpha + i\sin\alpha)}{(\cos\alpha - i\sin\alpha) - i(\cos\beta - i\sin\beta)} = \frac{2\cos\alpha}{(\cos\alpha + \sin\beta) - i(\sin\alpha + \cos\beta)}$$

$$= \frac{2\cos\alpha}{\left[\cos\alpha + \cos\left(\frac{\pi}{2}-\beta\right)\right] - i\left[\sin\alpha + \sin\left(\frac{\pi}{2}-\beta\right)\right]}$$

$$= \frac{2\cos\alpha}{2\cos\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right) - i2\, sin\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)}$$

But $\cos\left(-\frac{\pi}{4} + \frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\alpha+\beta}{2}\right)$

$$\therefore \text{Expression} = \frac{\cos\alpha}{\cos\left(\frac{\pi}{4}-\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right) - isin\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\right]}$$

$$= \left[cos\alpha.\sec\left(\frac{\pi}{4} - \frac{\alpha+\beta}{2}\right)\right]\left[cos\left(\frac{\pi}{4} + \frac{\alpha-\beta}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\alpha-\beta}{2}\right)\right]$$

$$= r[\cos\theta + i\sin\theta]$$

Where $r = \cos\alpha sec\left(\frac{\pi}{4} - \frac{\alpha+\beta}{2}\right)$ and $\theta = \frac{\pi}{4} + \frac{\alpha-\beta}{2}$

12. If z_1 *and* z_2 are any two complex numbers, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

Solution: Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

$\therefore |z_1 + z_2|^2 = |(x_1 + iy_1) + (x_2 + iy_2)|^2$

$= |(x_1 + x_2) + i(y_1 + y_2)|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

Similarly, $|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

and $|z_1|^2 = x_1^2 + y_1^2 \qquad |z_2|^2 = x_2^2 + y_2^2$

l.h.s. $= |z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= (x_1^2 + x_2^2 + 2x_1x_2) + (y_1^2 + y_2^2 + 2y_1y_2) +$

$(x_1^2 + x_2^2 - 2x_1x_2) + (y_1^2 + y_2^2 - 2y_1y_2)$

$= 2[x_1^2 + x_2^2 + y_1^2 + y_2^2]$

Now, r.h.s$= 2[|z_1|^2 + |z_2|^2] = 2[x_1^2 + x_2^2 + y_1^2 + y_2^2]$

$\therefore \ l.h.s = r.h.s$

13. If $|z - 1| < |z + 1|$, prove that Re $z > 0$.

Solution: We have $|z - 1| < |z + 1|$

$\therefore |x + iy - 1| < |x + iy + 1|$

$\therefore |(x - 1) + iy| < |(x + 1) + iy|$

$\therefore \sqrt{(x - 1)^2 + y^2} < \sqrt{(x + 1)^2 + y^2}$

$\therefore (x - 1)^2 + y^2 < (x + 1)^2 + y^2$

$\therefore x^2 - 2x + 1 + y^2 < x^2 + 2x + 1 + y^2$

$\therefore -2x < 2x \qquad \therefore -4x < 0 \qquad \therefore 4x > 0 \qquad \therefore x > 0$

$\therefore$ The real Part of $z > 0$

14. If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$ Prove that

$\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$.

Solution: By data, $z = \frac{(b+ic)}{(1+a)} \qquad \therefore iz = \frac{ib-c}{1+a}$

$\therefore$ By componendo and dividendo,

$$\frac{1+iz}{1-iz}=\frac{1+a+ib-c}{1+a-ib+c}=\frac{(1+a-c)+ib}{(1+a+c)-ib}\cdot\frac{(1+a+c)+ib}{(1+a+c)+ib}$$

$$=\frac{[(1+a+ib)-c].[(1+a+ib)+c]}{[(1+a+c)-ib][(1+a+c)+ib]}=\frac{1+a^2-b^2+2a+2ib+2aib-c^2}{1+a^2+c^2+2a+2c+2ac+b^2}$$

Since by data $a^2+b^2+c^2=1$,

in the numerator, we put $1-b^2-c^2=a^2$

and in the denominator, we put $a^2+b^2+c^2=1$

$$\frac{1+iz}{1-iz}=\frac{2a^2+2a+2ib+2aib}{2+2a+2c+2ac}=\frac{a(a+1)+ib(1+a)}{1(a+1)+c(1+a)}=\frac{(1+a)(a+ib)}{(1+a)(1+c)}=\frac{a+ib}{1+c}=l.h.s$$

SOME PRACTISE PROBLEMS:

1. Express the following in the form $x+iy$

(i) $\frac{(2+i)(1+2i)}{3+4i}$ **(ii)** $\frac{(2+3i)^2}{1+i}$

2. Find the complex conjugate of **(i)** $\frac{3+5i}{1+2i}$ **(ii)** $\frac{1+i}{1-i}$

3. Find the value of $x^4-4x^3+4x^2+8x+46\ when\ x=3+2i$

4. Find the modulus and the principal argument of

(i) $-1+\sqrt{3}.i$ **(ii)** $\frac{(2-3i)(5+3i)}{3-2i}$

5. Find the real part, imaginary part, modulus and argument of

$(4+2i)\left(-3+\sqrt{2}i\right)$

6. Express the following in polar form and find their arguments

(i) $\sqrt{3}+i$ **(ii)** $\sin\theta+\mathrm{i}\cos\theta$

7. If $z_1=1+i,\ z_2=2-i,\ z_3=3+2i$, find

(i) $\left|\frac{z_1-z_2-i}{z_1+z_2+i}\right|$ **(ii)** $|\overline{z}_2-z_1|^2+|\overline{z}_3-z_1|^2$ **(iii)** $\frac{z_3}{z_1}+\frac{z_2}{z_3}$

(iv) $\frac{z_1}{\overline{z}_1}-\frac{\overline{z}_1}{z_1}$ **(v)** $\frac{1}{(z_2+z_3)(z_2-z_3)}$ **(vi)** $(z_2-\overline{z}_2)^5$

8. If $z=x+iy$, prove that $\left(\frac{z}{\overline{z}}+\frac{\overline{z}}{z}\right)=2\left(\frac{x^2-y^2}{x^2+y^2}\right)$

9. If $z = a\cos\theta + ia\sin\theta$, prove that $\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = 2\cos 2\theta$

10. Prove that $\left|\frac{z-1}{\overline{z}-1}\right| = 1$

11. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1.$

12. If $\frac{1}{\alpha+i\beta} + \frac{1}{a+ib} = 1, where\ \alpha, \beta, a, b$ are real, express b in terms of α, β

13. If $x + iy = \sqrt{a+ib}$, prove that $(x^2 + y^2)^2 = a^2 + b^2$

14. If $arg.(z + 2i) = \pi/4\ and\ arg.(z - 2i) = 3\pi/4, find\ z$

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