METHOD OF VARIATION OF PARAMETERS:

This is one of the methods for finding the Particular Integral (P.I.) of a linear differential equation whose Complimentary function (C.F.) is known. Though the method is general, we will illustrate it by applying it to a second order and third order differential equation.

(1) Consider the linear equation of second order with constant coefficients. $aD^2y + bDy + cy = X$

i.e. $(aD^2 + bD + c)\,y = X$

Let Complementary function $= c_1\,y_1 + c_2y_2$ then Particular Integral $= uy_1 + vy_2$ where

$u = -\int \frac{y_2X}{W}dx \quad \& \ V = \int \frac{y_1X}{W}dx \quad \& \ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$\therefore$ General solution = Complementary function + Particular Integral.

(2) Consider the linear equation of third order with constants coefficient

$aD^3y + bD^2y + cDy + dy = X$

i.e. $(aD^3 + bD^2 + cD + d)\,y = X$

Let Complementary function $= c_1\,y_1 + c_2y_2 + c_3\,y_3$ then

Particular Integral $= uy_1 + vy_2 + wy_3$ where

$u = \int \frac{(y_2y_3' - y_3y_2')X}{W}dx, \quad v = \int \frac{(y_3y_1' - y_1y_3')X}{W}dx, \quad w = \int \frac{(y_1y_2' - y_2y_1')X}{W}dx$

Where $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$\therefore$ General Solution = Complementary function + Particular Integral.

(3) **Method of variation of parameters for n^{th} order differential equation.**

Consider $\frac{1}{f(D)}y = X$ Find Complementary function $= y_c = \sum_{j=1}^{n} c_jy_j = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

Let $w(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & & & \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix}$ = wronskian of $(y_1, y_2, \ldots\ldots y_n)$

Write $v_j(x) = \int \frac{X \cdot w_j(x)}{w(x)}dx$

Where $w_j(x)$ $(1 \le j \le n)$ is the determinant obtained by replacing j^{th} column of $w(x)$ by $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Then Particular Integral $= y_P = \sum_{j=1}^{n} v_jy_j = v_1y_1 + v_2y_2 + \cdots + v_ny_n$

EXERCISE

Apply the method of variation of parameters to solve.

1. $\frac{d^2y}{dx^2} + k^2y = tan\,kx$

2. $(D^2 - 1)y = \frac{2}{1+e^x}$

3. $(D^2 - 1)y = \frac{2}{\sqrt{1-e^{-2x}}}$

4. $(D^2 + D)y = \frac{1}{1+e^x}$

5. $\frac{d^2y}{dx^2} + a^2y = \sec ax$

6. $(D^2 - 2D + 2)y = e^x \tan x$

7. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

8. $\frac{d^2y}{dx^2} - y = e^{-x}sin(e^{-x}) + cos(e^{-x})$

9. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

10. $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

11. $(D^2 - 4D + 4)y = e^{2x}sec^2x$

12. $(D^3 + D)y = cosec\ x$

13. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \cdot \sec^2 x\,(1 + 2\tan x)$

14. $(D^3 + 4D)y = 4\cot 2x$

15. $(D^2 + 1)y = \cot x$

ANSWERS

1. $y = c_1 cos\ kx + c_2 sin\ kx - \frac{1}{k^2} cos\ kx.log(sec\ kx + tan\ kx)$

2. $y = c_1 e^x + c_2 e^{-x} - 1 + log(1 + e^{-x})e^x - [log(1 + e^{-x})]e^{-x}$ **or**

$y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\log(1 + e^x)$

3. $y = c_1 e^x + c_2 e^{-x} - e^x sin(e^{-x}) - \sqrt{1 - e^{-2x}}$

4. $y = c_1 + c_2 e^{-x} - log(1 + e^{-x})$ *or* $y = c_1 + c_2 e^{-x} - (1 + e^{-x})log(1 + e^x) + x$

5. $y = (c_1 cos\ ax + c_2 sin\ ax) + \frac{1}{a^2}(log\ cos\ ax)cos\ ax + \frac{x}{a} sin\ ax$

6. $y = e^x(c_1 cos\ x + c_2 sin\ x) - e^x cos\ xlog(sec\ x + tan\ x)$

7. $y = c_1 e^x + c_2 e^{-2x} + e^{-2x} e^{e^x}$

8. $y = c_1 e^x + c_2 e^{-x} - e^x sin(e^{-x})$

9. $y = c_1 cos\ x + c_2 sin\ x - [1 - sin\ x + xcos\ x] + sinx.log(1 + sin\ x)$

10. $y = c_1 e^{3x} + c_2 xe^{3x} - e^{3x}(logx + 1)$

11. $y = c_1 e^{2x} + c_2 xe^{2x} + e^{2x} log\ sec\ x$

12. $y = c_1 + c_2 cos\ x + c_3 sin\ x + log(cosec\ x - cot\ x) - (logsin\ x)cos\ x - x\ sin\ x$

13. $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1 + 2\tan x)^2 - \frac{e^x}{2}(1 + 2\tan x)$

14. $y = c + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2}\log\sin 2x - \frac{1}{2}\log\tan x \cdot \cos 2x \left[\because c_1 - \frac{1}{2} = c\right]$

15. $y = c_1 \cos x + c_2 \sin x + \sin x \cdot \log(\csc x - \cot x)$

SOME SOLVED EXAMPLES:

3. $(D^2 - 1)y = \frac{2}{\sqrt{1-e^{-2x}}}$

Solution: The auxiliary equation is $D^2 - 1 = 0$ $\therefore D = +1, -1$

$\therefore$ The C.F. $y = c_1 e^x + c_2 e^{-x}$

$\therefore y_1 = e^x, y_2 = e^{-x}, X = \frac{2}{\sqrt{1-e^{-2x}}}$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

$\therefore u = -\int \frac{y_2 X}{W} dx = -\int e^{-x} \cdot \frac{2}{\sqrt{1-e^{-2x}}} \cdot \frac{1}{-2} dx$

$= \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx = \int \frac{-dt}{\sqrt{1-t^2}}$ Putting $e^{-x} = t$, $-e^{-x}dx = dt$

$= -\sin^{-1}(t) = -\sin^{-1}(e^{-x})$

$\therefore uy_1 = -e^x \sin^{-1}(e^{-x})$

$v = \int \frac{y_1 X}{W} dx = \int e^x \cdot \frac{2}{\sqrt{1-e^{-2x}}} \cdot \frac{1}{-2} dx = \int \frac{e^x}{\sqrt{1-e^{-2x}}} dx = \int \frac{e^x \cdot e^x}{\sqrt{e^{2x}+1}} dx$

(multiply by e^x in the numerator and denominator)

Put $e^x = t$, $e^x dx = dt$ $\therefore v = \int \frac{tdt}{\sqrt{t^2+1}} = \sqrt{t^2 + 1} = \sqrt{e^{2x} + 1}$

$v \cdot y_2 = e^{-x}\sqrt{e^{2x} + 1} = \sqrt{1 + e^{-2x}}$

$\therefore$ The complete soltuion is $y = c_1 e^x + c_2 e^{-x} - e^x \sin^{-1}(e^{-x}) + \sqrt{1 + e^{-2x}}$

5. $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Solution: The auxiliary equation is $D^2 + a^2 = 0$ $\therefore D = ai, -ai$

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax$

Here, $y_1 = \cos ax\,, y_2 = \sin ax\,, X = \sec ax$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a$

$\therefore u = -\int \frac{y_2X}{W} dx = -\frac{1}{a}\int \sin ax \cdot \sec ax\, dx = -\frac{1}{a}\int \tan ax\, dx = \frac{1}{a^2}\log \cos ax$

and $v = \int \frac{y_1X}{W} dx = \frac{1}{a}\int \cos x \cdot \sec ax\, dx = \frac{1}{a}\int dx = \frac{x}{a}$

$\therefore P.I. = \frac{1}{a^2}\log \cos ax \cdot \cos ax + \frac{x}{a} \cdot \sin ax$

$\therefore$ The complete solution is $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}\log \cos ax \cdot \cos ax + \frac{x}{a} \cdot \sin ax$

6. $(D^2 - 2D + 2)y = e^x \tan \mathrm{x}$

Solution: The auxiliary equation is $D^2 - 2D + 2 = 0$ $\therefore D = 1, \pm i$

$\therefore$ The C.F. is $y = e^x(c_1 \cos x + c_2 \sin x)$

Here, $y_1 = e^x \cos x\,, y_2 = e^x \sin x\,, X = e^x \tan x$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$

$= e^x \cos x \cdot e^x(\sin x + \cos x) - e^x \sin x \cdot e^x(\cos x - \sin x)$

$= e^{2x}(\sin^2 x + \cos^2 x) = e^{2x}$

$\therefore u = -\int \frac{y_2X}{W} dx$

$= -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} dx = -\int \frac{\sin^2 x}{\cos x} dx$

$= -\int \frac{(1 - \cos^2 x)}{\cos x} dx = -\int \sec x\, dx + \int \cos x\, dx$

$= -\log(\sec x + \tan x) + \sin x$

and $v = \int \frac{y_1X}{W} dx$

$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx$

$= \int \sin x\, dx = -\cos x$

$\therefore P.I. = -\log(\sec x + \tan x) \cdot e^x \cos x + e^x \sin x \cos x - e^x \cos x \sin x$

$\therefore$ The complete solution is $y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \cdot \log(\sec x + \tan x)$

7. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Solution: The auxiliary equation is $D^2 + 3D + 2 = 0$

$\therefore (D + 1)(D + 2) = 0$ $\therefore D = -1, 2$

$\therefore$ The C.F. is $y = c_1e^{-x} + c_2e^{-2x}$

Here, $y_1 = e^{-x}, y_2 = e^{-2x}, X = e^{e^x}$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$

$\therefore u = -\int \frac{y_2X}{W} dx = -\int \frac{e^{-2x} \cdot e^{e^x}}{-e^{-3x}} dx = \int e^{e^x} \cdot e^x\, dx$

Put $e^x = t,\ e^x dx = dt$ $\therefore u = \int e^t\, dt = e^t = e^{e^x}$

and $v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-x} \cdot e^{e^x}}{e^{-3x}} dx = \int e^{2x} \cdot e^{e^x}\, dx$

Putting $e^x = t, v = \int e^t \cdot t\, dt = te^t - e^t$

$\therefore v = e^x e^{e^x} - e^{e^x}$

$\therefore P.I. = e^{e^x} \cdot e^{-x} - \left(e^x e^{e^x} - e^{e^x}\right) \cdot e^{-2x} = e^{-2x} \cdot e^{e^x}$

$\therefore$ The complete solution is $y = c_1 e^x + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$

8. $\frac{d^2y}{dx^2} - y = e^{-x} sin(e^{-x}) + cos(e^{-x})$

Solution: The auxiliary equation is $D^2 - 1 = 0$

$\therefore D = -1, 1$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^x$

Here $y_1 = e^{-x}, y_2 = e^x, X = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = e^0 + e^0 = 2$

$\therefore u \quad = -\int \frac{y_2 X}{W} dx = -\frac{1}{2} \int e^x [\cos(e^{-x}) + e^{-x} \sin(e^{-x})]\, dx$

$= -\frac{1}{2} e^x \cos(e^{-x}) \qquad\qquad [\because \int e^x\left(f(x) + f'(x)\right) = e^x f(x)]$

and $v = \int \frac{y_1 X}{W} dx = \frac{1}{2} \int e^{-x} [e^{-x} \sin(e^{-x}) + \cos(e^{-x})]\, dx$

For integration, put $e^{-x} = t \qquad \therefore -e^{-x} dx = dt$

$\therefore v \quad = -\frac{1}{2} \int (t \sin t + \cos t)\, dt$

$= -\frac{1}{2} [t(-\cos t) - (1)(-\sin t) + \sin t]$

$= \frac{1}{2} t \cos t - \sin t = \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$

$\therefore P.I. = -\frac{1}{2} e^x \cos(e^{-x}) \cdot e^{-x} + \left[\frac{1}{2} \cdot e^{-x} \cos(e^{-x}) - \sin(e^{-x})\right] e^x$

$= -e^x \cdot \sin(e^{-x})$

$\therefore$ The complete solution is $y = c_1 e^x + c_2 e^{-x} - e^x \cdot \sin(e^{-x})$

9. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

Solution: The auxiliary equation is $D^2 + 1 = 0$

$\therefore D = i, -i$

$\therefore$ The C.F. is $y = c_1 \cos x + c_2 \sin x$

Here $y_1 = \cos x\,, y_2 = \sin x\,, X = \frac{1}{1+\sin x}$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore u \quad = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin x}{1} \cdot \frac{1}{1+\sin x} dx$

$= -\int \frac{\sin x}{1+\sin x} \cdot \frac{(1-\sin x)}{(1-\sin x)} dx = -\int \frac{\sin x(1-\sin x)}{\cos^2 x} dx$

$= -\int (\sec x \tan x - \tan^2 x)\, dx$

$= -\int (\sec x \tan x - \sec^2 x + 1)\, dx$

$= -[\sec x - \tan x + x]$

and $v = \int \frac{y_1 X}{W} dx = \int \frac{\cos x}{1} \cdot \frac{1}{(1+\sin x)} dx = \log(1 + \sin x)$

$\therefore P.I. = -[\sec x - \tan x + x]\cos x + \log(1+\sin x)\cdot\sin x$

$\therefore$ The complete solution is $y = c_1\cos x + c_2\sin x - [1 - \sin x + x\cos x] + \sin x\cdot\log(1+\sin x)$

10. $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

Solution: The auxiliary equation is $(D-3)^2 = 0$ $\therefore D = 3, 3$

$\therefore$ The C.F. is $y = (c_1 + c_2x)e^{3x} = c_1e^{3x} + c_2xe^{3x}$

Here $y_1 = e^{3x}, y_2 = xe^{3x}, X = e^{3x}/x^2$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$

$\therefore u = -\int \frac{y_2X}{W}dx = -\int \frac{xe^{3x}\cdot(e^{3x}/x^2)}{e^{6x}}dx = -\int\frac{dx}{x} = -\log x$

and $v = \int\frac{y_1X}{W}dx = \int\frac{e^{3x}\cdot(e^{3x}/x^2)}{e^{6x}}dx = \int\frac{dx}{x^2} = -\frac{1}{x}$

$\therefore P.I. = -e^{3x}\cdot\log x - xe^{3x}\cdot\frac{1}{x} = -e^{3x}(\log x + 1)$

$\therefore$ The complete solution is $y = c_1e^{3x} + c_2xe^{3x} - e^{3x}(\log x + 1)$

11. $(D^2 - 4D + 4)y = e^{2x}sec^2x$

Solution: The auxiliary equation is $(D-2)^2 = 0$ $\therefore D = 2, 2$

$\therefore$ The C.F. is $y = (c_1 + c_2x)e^{2x} = c_1e^{2x} + c_2xe^{2x}$

Here $y_1 = e^{2x}, y_2 = xe^{2x}, X = e^{2x}\sec^2 x$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}$

$\therefore u \quad = -\int\frac{y_2X}{W}dx = -\int\frac{xe^{2x}\cdot e^{2x}\sec^2 x}{e^{4x}}dx = -\int x\sec^2 x\,dx$

$\quad = -[x\tan x - \int\tan x\cdot 1\cdot dx] = -x\tan x + \log\sec x$

and $v = \int\frac{y_1X}{W}dx = \int\frac{e^{2x}\cdot e^{2x}\sec^2 x}{e^{4x}}dx = \int\sec^2 x\,dx = \tan x$

$\therefore P.I. = -xe^{2x}\tan x + e^{2x}\cdot\log\sec x - xe^{2x}\tan x = e^{2x}\cdot\log\sec x$

$\therefore$ The complete solution is $y = c_1e^{2x} + c_2xe^{2x} + e^{2x}\cdot\log\sec x$

12. $(D^3 + D)y = cosec\ x$

Solution: The auxiliary equation is $D(D^2+1) = 0$

$\therefore D = 0, i, -i$

$\therefore$ The C.F. is $y = c_1 + c_2\cos x + c_3\sin x$

Here $y_1 = 1, y_2 = \cos x, y_3 = \sin x, X = \csc x$

Let $P.I.$be $y = uy_1 + vy_2 + wy_3$

Now, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y"_1 & y"_2 & y"_3 \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$

$\therefore u \quad = \int\frac{(y_2y'_3 - y_3y'_2)X}{W}dx$

$\quad = \int(\cos^2 x + \sin^2 x)\csc x\,dx$

$\quad = \int\csc x\,dx = \log(\csc x - \cot x)$

$\therefore v \quad = \int\frac{(y_3y'_1 - y_1y'_3)X}{W}dx$

$\quad = \int(\sin x\cdot 0 - 1\cdot\cos x)\cdot\csc x\,dx$

$= -\int \cot x \, dx = -\log \sin x$

& $w = \int \frac{(y_1 y'_2 - y_2 y'_1)X}{W} dx$

$= \int [1 \cdot (-\sin x) - 0 \cdot \cos x] \csc x \, dx$

$= \int -dx = -x$

$\therefore P.I. = \log(\csc x - \cot x)\, 1 - \log \sin x \cdot \cos x - x \sin x$

$\therefore$ The complete solution is

$y = c_1 + c_2 \cos x + c_3 \sin x + \log(\csc x - \cot x) - \log \sin x \cdot \cos x - x \sin x$

13. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \cdot \sec^2 x\,(1 + 2\tan x)$

Solution: The auxiliary equation is $D^2 + 5D + 6 = 0$

$\therefore (D+2)(D+3) = 0 \qquad \therefore D = -2, -3$

$\therefore$ The C.F. is $y = c_1 e^{-2x} + c_2 e^{-3x}$

$\therefore y_1 = e^{-2x}, y_2 = e^{-3x}, X = e^{-2x} \sec^2 x\,(1 + 2\tan x)$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x}$

$\therefore u = -\int \frac{y_2 X}{W} dx$

$= -\int \frac{e^{-3x} \cdot e^{-2x}}{-e^{-5x}} \cdot \sec^2 x \cdot (1 + 2\tan x)\, dx$

$= \int (1 + 2\tan x) \sec^2 x \, dx$

$= \frac{1}{4}(1 + 2\tan x)^2$

$v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-2x} \cdot e^{-2x} \cdot \sec^2 x (1+2\tan x)}{-e^{-5x}} = -\int e^{-x} \cdot (1 + 2\tan x) \cdot \sec^2 x \, dx$

Let $f(x) = \left(\frac{1+2\tan x}{2}\right) \qquad \therefore f'(x) = \sec^2 x$

$\therefore \int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

$\therefore v = -e^x \cdot \frac{(1+2\tan x)}{2}$

$\therefore$ The complete solution is $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1 + 2\tan x)^2 - \frac{e^x}{2}(1 + 2\tan x)$

14. $(D^3 + 4D)y = 4\cot 2x$

Solution: The auxiliary equation is $D^3 + 4D = 0$

$\therefore D(D^2 + 4) = 0 \qquad \therefore D = 0, 2i, -2i$

$\therefore$ C.F is $y = c_1 + c_2 \cos 2x + c_3 \sin 2x$

Here, $y_1 = 1, y_2 = \cos 2x\,, y_3 = \sin 2x\,, X = 4\cot 2x$

Let $P.I.$be $y = uy_1 + vy_2 + wy_3$

Now, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8(\sin^2 2x + \cos^2 2x) = 8$

$\therefore u = \int \frac{(y_2 y'_3 - y_3 y'_2)}{W} \cdot X dx$

$= \int \frac{2(\cos^2 2x + \sin^2 2x)}{8} \cdot 4\cot 2x \, dx$

$= \int \cot 2x \, dx = \frac{1}{2} \log \sin 2x$

& $v = \int \frac{(y_3 y'_1 - y_1 y'_3)X}{W} dx$

$= \int \frac{(\sin 2x \cdot 0 - 1 \cdot 2\cos 2x)}{8} \cdot 4\cot 2x \, dx$

$= -\int \frac{\cos^2 2x}{\sin 2x} dx = -\int \frac{(1 - \sin^2 2x)}{\sin 2x} dx$

$= -\int(\csc 2x - \sin 2x)\,dx$

$= -\left(\frac{1}{2}\log\tan x + \frac{1}{2}\cos 2x\right)$

& $w \quad = \int \frac{(y_1y'_2 - y_2y'_1)X}{W}\,dx$

$= \int \frac{(-2\sin 2x - 0)}{8} \cdot 4\cot 2x\,dx$

$= -\int \cos 2x\,dx = -\frac{1}{2}\sin 2x$

∴ The complete solution is

$y \quad = c_1 + c_2\cos 2x + c_3\sin 2x + \frac{1}{2}\cdot 1\cdot\log\sin 2x - \frac{1}{2}\log\tan x\cdot\cos 2x - \frac{1}{2}\cos^2 2x - \frac{1}{2}\sin^2 2x$

$= c_1 + c_2\cos 2x + c_3\sin 2x + \frac{1}{2}\log\sin 2x - \frac{1}{2}\log\tan x\cdot\cos 2x - \frac{1}{2}$

$= c + c_2\cos 2x + c_3\sin 2x + \frac{1}{2}\log\sin 2x - \frac{1}{2}\log\tan x\cdot\cos 2x \qquad \left[\because c_1 - \frac{1}{2} = c\right]$

15. $(D^2+1)y = \cot x$

Solution: The auxiliary equation is $D^2 + 1 = 0$ $\quad\therefore D = i, -i$

∴ C.F is $y = c_1\cos x + c_2\sin x$

Here, $y_1 = \cos x\,, y_2 = \sin x\,, X = \cot x$

Let $P.I.$be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore u = -\int \frac{y_2X}{W}\,dx = -\int \frac{\sin x}{1}\cdot\cot x\,dx = -\int\cos x\,dx = -\sin x$

& $v \quad = \int\frac{y_1X}{W}\,dx = \int\frac{\cos x}{1}\cdot\cot x\,dx = \int\frac{\cos^2 x}{\sin x}\,dx$

$= \int\frac{(1-\sin^2 x)}{\sin x}\,dx = \int(\csc x - \sin x)\,dx$

$= \log(\csc x - \cot x) + \cos x$

$\therefore P.I. = -\sin x\cos x + \log(\csc x - \cot x)\cdot\sin x + \sin x\cos x$

∴ The complete solution is $y = c_1\cos x + c_2\sin x + \sin x\cdot\log(\csc x - \cot x)$