K. J. Somaiya College of Engineering, Mumbai-77
(A Constituent college of Somaiya Vidyavihar University)

F.Y. Btech SEM-I

APPLIED MATHEMATICS-I

QUESTION BANK -1

TOPIC – COMPLEX NUMBERS

Type -1: Review

1. Express the following in the form $x + iy$

(i) $\frac{(2+i)(1+2i)}{3+4i}$ **(ii)** $\frac{(2+3i)^2}{1+i}$

2. Find the complex conjugate of **(i)** $\frac{3+5i}{1+2i}$ **(ii)** $\frac{1+i}{1-i}$

3. Find the value of $x^4 - 4x^3 + 4x^2 + 8x + 46 \; when \; x = 3 + 2i$

4. Find the modulus and the principal argument of

(i) $-1 + \sqrt{3}.i$ **(ii)** $\frac{(2-3i)(5+3i)}{3-2i}$

5. Find the real part, imaginary part, modulus and argument of $(4 + 2i)(-3 + \sqrt{2}i)$

6. Express the following in polar form and find their arguments

(i) $\sqrt{3} + i$ **(ii)** $\sin\theta + \mathrm{i}\cos\theta$

7. Find the square root of $-5 + 12i$

8. If $z_1 = 1 + i,\ z_2 = 2 - i,\ z_3 = 3 + 2i$, find

(i) $\left|\frac{z_1 - z_2 - i}{z_1 + z_2 + i}\right|$ **(ii)** $|\overline{z}_2 - z_1|^2 + |\overline{z}_3 - z_1|^2$ **(iii)** $\frac{z_3}{z_1} + \frac{z_2}{z_3}$

(iv) $\frac{z_1}{\overline{z}_1} - \frac{\overline{z}_1}{z_1}$ **(v)** $\frac{1}{(z_2+z_3)(z_2-z_3)}$ **(vi)** $(z_2 - \overline{z}_2)^5$

9. If $z = x + iy$, prove that $\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = 2\left(\frac{x^2-y^2}{x^2+y^2}\right)$

10. If $z = a\cos\theta + ia\sin\theta$, prove that $\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = 2\cos 2\,\theta$

11. Prove that $\left|\frac{z-1}{\overline{z}-1}\right| = 1$

12. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1.$

13. If p is real and the complex number $\frac{1+i}{2+pi} + \frac{2+3i}{3+i}$ represents a point on the line $y = x$, prove that $p = -5 \pm \sqrt{21}$

14. If $x + iy = \sqrt{a + ib}$, prove that $(x^2 + y^2)^2 = a^2 + b^2$

15. If $arg.(z + 2i) = \pi/4 \; and \; arg.(z - 2i) = 3\pi/4, find \; z$

16. If $|z+i| = |z|$ and $arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$, find z

17. Find two complex numbers such that their sum is 6 and their product is 13.

18. If $sin\psi = i\tan\theta$, prove that $\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$

19. Prove that $\frac{1+\cos\alpha+i\sin\alpha}{1-\cos\alpha+i\sin\alpha} = cot\left(\frac{\alpha}{2}\right).e^{i(\alpha-\pi/2)}$

20. If $p = \cos\theta + i\sin\theta$, $q = \cos\Phi + i\sin\Phi$, Show that $\frac{(p+q)(pq-1)}{(p-q)(pq+1)} = \frac{\sin\theta+\sin\Phi}{\sin\theta-\sin\Phi}$.

21. If $(a_1+i\,b_1)(a_2+i\,b_2)\ldots\ldots\ldots\ldots(a_n+i\,b_n) = A+i\,B$, prove that
$\left(a_1{}^2+b_1{}^2\right)\left(a_2{}^2+b_2{}^2\right)\ldots\ldots\ldots\ldots\left(a_n{}^2+b_n{}^2\right) = A^2+B^2$ and
$tan^{-1}\frac{b_1}{a_1} + tan^{-1}\frac{b_2}{a_2} + \cdots\ldots\ldots\ldots + tan^{-1}\frac{b_n}{a_n} = tan^{-1}\frac{B}{A}$.

22. If z_1 and z_2 are two complex numbers such that $|z_1+z_2| = |z_1-z_2|$,
prove that $arg.z_1 - arg.z_2 = \frac{\pi}{2}$

23. Prove that, if $|z-i| > |z+i|$ $then$ $Im(z) < 0$.

24. If $|z-1| = |z+1|$ then prove that Re $z = 0$

25. If $x^2+y^2 = 1$, prove that $\frac{1+x+iy}{1+x-iy} = x+iy$

26. If $x+iy = \frac{3}{2+\cos\theta+i\sin\theta}$, prove that $(x-1)(x-3)+y^2 = 0$.

27. If z_1, z_2 are non – zero complex numbers of equal modulus and $z_1 \neq z_2$
then prove that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

28. If $z_1+z_2+z_3 = 0$ and $|z_1| = |z_2| = |z_3| = k$ show that $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3} = 0$.

29. If $z = x+iy$, prove that
(i) If $\frac{z+i}{z+2}$ is real, then locus of (x,y) is a straight line.
(ii) If it is pure imaginary, then the locus of a point (x,y) is a circle. Also find radius and centre.

Type – 2: De-Moivre's Theorem

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8\left(1-i\sqrt{3}\right)^3}{(1-i)^6\left(1+i\sqrt{3}\right)^9} = \frac{i}{32}$ **(ii)** $\frac{\left(1+i\sqrt{3}\right)^9(1-i)^4}{\left(\sqrt{3}+i\right)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{\left(1+i\sqrt{3}\right)^{17}}{\left(\sqrt{3}-i\right)^{15}}$

4. Express in the form $a+ib$, $\frac{(1+i)^{10}}{\left(1+i\sqrt{3}\right)^5}$

5. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

6. If $x_n + iy_n = \left(1 + i\sqrt{3}\right)^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

7. Simplify

(i) $(1 + cos\,\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$ **(ii)** $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n$

8. Prove that $\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

9. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \overline{z}$ is the conjugate of z find the value of $(z)^{15} + (\overline{z})^{15}$.

10. Prove that, if n is a positive integer, then

(i) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2\left(\sqrt{a^2 + b^2}\right)^{m/n} cos\left(\frac{m}{n} tan^{-1}\frac{b}{a}\right)$

(ii) $\left(\sqrt{3} + i\right)^{120} + \left(\sqrt{3} - i\right)^{120} = 2^{121}$

11. If n is a positive integer, prove that $(1 + i)^n + (1 - i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1 + i)^{10} + (1 - i)^{10} = 0$

12. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$ and 2 if $n = 3k$ where k is an integer.

13. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

14. If α, β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos n\,\theta\ cosec^n\theta$

15. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$, prove that

$\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2cos(\alpha + \beta - \gamma)$

16. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2\ cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\ cos\left(\frac{\theta}{m} - \frac{\emptyset}{n}\right)$

17. If $x + \frac{1}{x} = 2\cos\theta$ then prove that $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$ and $\frac{x^{2n}-1}{x^{2n-1}-x} = \frac{\sin n\theta}{\sin(n-1)\theta}$

18. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that

$\frac{(b+c)(c+a)(a+b)}{abc} = 8\ cos\frac{(\alpha-\beta)}{2} cos\frac{(\beta-\gamma)}{2} cos\frac{(\gamma-\alpha)}{2}.$

19. If a, b, c are three complex numbers such that $a+b+c=0$, prove that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and **(ii)** $a^2+b^2+c^2=0$

20. If $\cos\alpha+\cos\beta+\cos\gamma=0$ *and* $\sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0\,,\ \sin 2\alpha+\sin 2\beta+\sin 2\gamma=0.$

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$

(iii) $cos(\alpha+\beta)+cos(\beta+\gamma)+cos(\gamma+\alpha)=0.$

(iv) $\sin(\alpha+\beta)+sin(\beta+\gamma)+sin(\gamma+\alpha)=0.$

(v) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

21. If $a\cos\alpha+b\cos\beta+c\cos\gamma=a\sin\alpha+b\sin\beta+c\sin\gamma=0$, Prove that

$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$ and

$a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3\,abc\,sin(\alpha+\beta+\gamma)$

22. If $x_r=cos\left(\frac{2}{3}\right)^r\pi+i\,sin\left(\frac{2}{3}\right)^r\pi,$ prove that

(i) $x_1x_2x_3\ldots\infty=1,$ **(ii)** $x_0x_1x_2\ldots\infty=-1$

23. If $(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\ldots\ldots\ldots\ldots(\cos n\,\theta+i\sin n\,\theta)=i,$ then show that the general value of $\theta=\left[2r+\frac{1}{n(n+1)}\right]\pi$

Type -3: Roots of Complex numbers

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\,cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6},k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$are given by $\pm i\,cot\frac{r\pi}{7},r=1,2,3.$

11. If $\alpha, \alpha^2, \alpha^3, \dots \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that $(1-\alpha)(1-\alpha^2) \dots \dots \dots \dots (1-\alpha^6) = 7.$

12. Prove that $x^5 - 1 = (x-1)\left(x^2 + 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{5} + 1\right) = 0.$

13. Solve the equation $z^n = (z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a = e^{i\,2\pi/7}$ and $b = a + a^2 + a^4,\ c = a^3 + a^5 + a^6$**.** then prove that b & c are roots of quadratic equation $x^2 + x + 2 = 0.$

15. Prove that **(i)** $\sqrt{1 - cosce(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} - \left(1 - e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1 - sce(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} - \left(1 + e^{-i\theta}\right)^{-1/2}$

16. If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.

17. Find the roots common to $x^{12} - 1 = 0$ and $x^4 - x^2 + 1 = 0$

LOGARITHMS OF COMPLEX NUMBERS

Let $z = x + iy$ and also let $x = r\cos\theta\,, y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log(\,r.e^{i\theta})$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$\therefore \quad \log(x + iy) = \log r + i\theta$

$\therefore \quad \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore$ Log (x + iy) = 2nπi + log(x + iy)

$\therefore$ Log (x + iy) = $2n\pi i + \frac{1}{2}\log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$

$\mathrm{Log}(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1}\,y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta\,, y = r\sin\theta\,, -\pi \le \theta \le \pi$.

SOLVED EXAMPLES:

$\log(-1) = ?$

$z = -1 = cos\pi + isin\pi$

$\therefore \log(-1) = \log(1) + i\pi = i\pi$

$\log(i) = ?$

$$\log(i) = \log(1) + \frac{i\pi}{2} = \frac{i\pi}{2}$$

$Log\ (\ -100) = ?$

$\mathrm{Log}(-100) = \log(100) + i(\pi + 2n\pi)$

$Log(-40i) = ?$

$$\mathrm{Log}(-40i) = \log(40) + i(-\frac{\pi}{2} + 2n\pi)$$

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\,\pi}{\log 2}$

Solution: Since $log(x + iy) = \frac{1}{2}log(x^2 + y^2) + i\,tan^{-1}\frac{y}{x}$

Putting $x = -3, y = 0$

we have $\log(-3) = \frac{1}{2}(9) + i\,tan^{-1}\left(\frac{0}{-3}\right) = \frac{1}{2}\log 3^2 + i\pi = \log 3 + i\pi$

$$log_2(-3) = \frac{log_e(-3)}{log_e 2} = \frac{\log 3 + i\pi}{\log 2}$$

2. Find the general value of $\text{Log}(1+\text{i})+\text{Log}\,(1-\text{i})$

Solution: $\log(1+i)=\frac{1}{2}\log 2+i\,\frac{\pi}{4}=log\sqrt{2}+i\frac{\pi}{4}$

$\therefore\ \text{Log}(1+i)=log\sqrt{2}+i\left(2n\pi+\frac{\pi}{4}\right)$ (General value)

Changing the sign of i,

$\text{Log}(1-i)=log\sqrt{2}-i\left(2n\pi+\frac{\pi}{4}\right)$

By addition, we get $\text{Log}(1+i)+\text{Log}(1-i)=2\,log\sqrt{2}=2.\frac{1}{2}\log 2=\log 2$

3. Prove that $\log\left(1+e^{2i\theta}\right)=\log(2\cos\theta)+i\,\theta$

Solution: $\log\left(1+e^{2i\theta}\right)=\log(1+\cos 2\theta+i\sin 2\theta)$

$$=\log(2\cos^2\theta+i2\sin\theta\cos\theta)$$
$$=\log\left(2\cos\theta\,(\cos\theta+i\sin\theta)\right)$$
$$=\log\left(2\cos\theta\cdot e^{i\theta}\right)$$
$$=\log(2\cos\theta)+\log\left(e^{i\theta}\right)$$
$$=\log(2\cos\theta)+i\theta$$

4. Prove that $\log\frac{1}{1-e^{i\theta}}=\log(\frac{1}{2}\text{cosec}\frac{\theta}{2})+i(\frac{\pi}{2}-\frac{\theta}{2})$

Solution: $\log\left(\frac{1}{1-e^{i\theta}}\right)=\log\left(\frac{1}{1-(\cos\theta+i\sin\theta)}\right)$

$$=\log\left(\frac{1}{(1-\cos\theta)-i\sin\theta}\right)$$
$$=\log\left(\frac{1}{2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$
$$=\log\left(\frac{1}{2\sin\frac{\theta}{2}(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2})}\right)$$
$$=\log\left(\frac{1}{2}\text{cosec}\frac{\theta}{2}\cdot e^{i(\frac{\pi}{2}-\frac{\theta}{2})}\right)$$
$$=\log(\frac{1}{2}\text{cosec}\frac{\theta}{2})+i(\frac{\pi}{2}-\frac{\theta}{2})$$

Exercise:

- Prove that $\log\frac{1}{1+e^{i\theta}}=\log(\frac{1}{2}\sec\frac{\theta}{2})-i\frac{\theta}{2}$
- Prove that $\log\left(1+e^{i\theta}\right)=\log(\cos\frac{\theta}{2})+i\frac{\theta}{2}$
- Prove that $\log(1+cos\theta+isin\theta)=\log(\cos\frac{\theta}{2})+i\frac{\theta}{2}$

5. Find the value of $\log\,[\sin(x+i\,y)]$

Solution: We have, $\sin(x+iy)=\sin x\cosh y+icos\,x\,sin\,hy$

$\therefore\ \log\sin(x+iy)=\frac{1}{2}\log(sin^2x\cos h^2y+\ cos^2x\sin h^2y)+i\,tan^{-1}\left(\frac{\cos x\sin hy}{\sin x\cos hy}\right)$

Now, $sin^2x\cos h^2y+\ cos^2x\ sinh^2y=(1-cos^2x)\cos h^2y+cos^2x\,(\cos h^2y-1)$

$$=cosh^2y-cos^2x$$

$$= \left(\frac{1+\cos h\, 2\, y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}(\cos h\, 2\, y - \cos 2x)$$

$$\therefore \log \sin(x+iy) = \frac{1}{2} log\left(\frac{\cos h\, 2\, y - \cos 2x}{2}\right) + i\, tan^{-1}(\cot x \tan hy)$$

6. Prove that $\log\frac{\sin(x+iy)}{\sin(x-iy)} = 2\mathrm{i}\, tan^{-1}(\cot x \tan hy)$

Solution: We have, $\log\frac{\sin(\mathrm{x+iy})}{\sin(\mathrm{x-iy})} = \log\sin(x+iy) - \log\sin(x-iy)$.......1

$$\sin(x+iy) = \sin x \cos hy + icos\, x\, sin\, hy$$

$$\therefore \log\sin(x+iy) = \frac{1}{2}\log(sin^2 x \cos h^2 y + cos^2 x \sin h^2 y) + i\, tan^{-1}\left(\frac{\cos x \sin hy}{\sin x \cos hy}\right)...2$$

$$\log\sin(x-iy) = \frac{1}{2}\log(sin^2 x \cos h^2 y + cos^2 x \sin h^2 y) - i\, tan^{-1}\left(\frac{\cos x \sin hy}{\sin x \cos hy}\right)...3$$

Using (2) & (3) in (1)

We get, $\log\frac{\sin(\mathrm{x+iy})}{\sin(\mathrm{x-iy})} = 2\mathrm{i}\tan^{-1}(\cot \mathrm{x} \tan \mathrm{hy})$

Exercise:

- prove that $\log\frac{\cos(x-iy)}{\cos(x+iy)} = 2\mathrm{i}\, tan^{-1}(\tan x \tan \mathrm{hy})$
- prove that $\log\frac{(x+iy)}{(x-iy)} = 2\mathrm{i}\, tan^{-1}(\mathrm{y/x})$
- prove that $\mathrm{i}\ \log\frac{(x-i)}{(x+i)} = \pi - 2\, tan^{-1}(\mathrm{x})$
- Separate in to real and imaginary parts $tanh^{-1}(\mathrm{x+iy})$

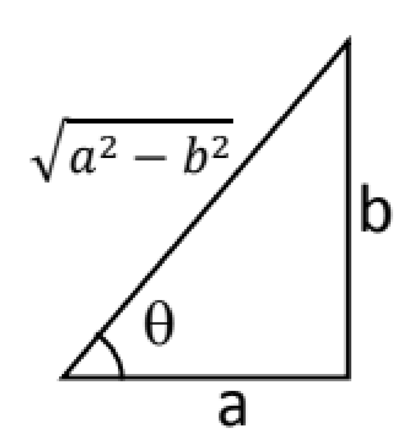
Hint: $\tan h^{-1}(\mathrm{x+iy}) = \frac{1}{2}\log\frac{1+\mathrm{x+iy}}{1-\mathrm{x-iy}}$

7. Show that $\tan\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{2\ \mathrm{ab}}{\mathrm{a^2-b^2}}$

Solution: We have $\log(a-bi) = \frac{1}{2}log(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$$\therefore log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\ \frac{b}{a}$$



$$\therefore i\ log\left(\frac{a-bi}{a+bi}\right) = -2i^2 tan^{-1}\frac{b}{a} = 2tan^{-1}\ \frac{b}{a}$$

$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan\left(2tan^{-1}\frac{b}{a}\right)$$

$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan 2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$$

$$= \frac{2\tan\theta}{1-tan^2\theta} = \frac{2(b/a)}{1-(b^2/a^2)} = \frac{2ab}{a^2-b^2}$$

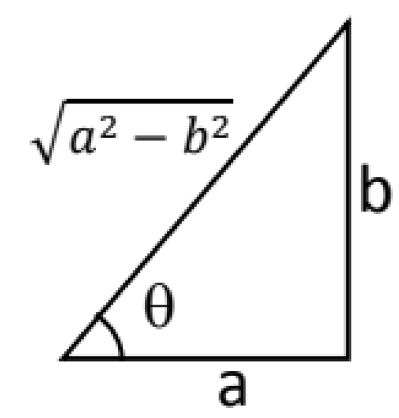
8. Prove that $\cos\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{\mathrm{a^2-b^2}}{\mathrm{a^2+b^2}}$

Solution: We have $\log(a-bi) = \frac{1}{2}log(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$\therefore log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\,\frac{b}{a}$

$\therefore i\,log\left(\frac{a-bi}{a+bi}\right) = -2i^2tan^{-1}\frac{b}{a} = 2tan^{-1}\,\frac{b}{a}$



$cos\left[i\,log\left(\frac{a-ib}{a+ib}\right)\right] = cos\left(2tan^{-1}\frac{b}{a}\right)$

$cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = cos2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$

$$= cos^2\theta - sin^2\theta = \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

Exercise

- Prove that $\sin\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{2\mathrm{ab}}{\mathrm{a^2+b^2}}$

9. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Solution: We have $\sqrt{i} = i^{1/2} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}$

Also $\sqrt{i} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \left(e^{i\,\pi/2}\right)^{1/2} = e^{i\,\pi/4}$

$\therefore \left(\sqrt{i}\right)^{\sqrt{i}} = \{e^{i\,\pi/4}\}^{\left(\frac{1}{\sqrt{2}}+i.\frac{1}{\sqrt{2}}\right)} = e^{i\,\pi/4\sqrt{2} - \pi/4\sqrt{2}} = e^{-\,\pi/4\sqrt{2}}.e^{i\,\pi/4\sqrt{2}}$

$= e^{-\,\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$

$\therefore$ Real Part $= e^{-\,\pi/4\sqrt{2}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$ & Imaginary Part $= e^{-\,\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$

10. Find the principal value of $(1+i)^{1-i}$

Solution: $z = (1+i)^{1-i}$

$\therefore \log z = (1-i)\log(1+i)$

$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + \mathrm{i}\tan^{-1}1\right]$

$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right]$

$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}log2\right) = x + iy\ say$

$\therefore z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

$= e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right]$

$= \sqrt{2}e^{\pi/4}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right] \qquad \because e^{\frac{1}{2}\log 2} = e^{log\sqrt{2}} = \sqrt{2}$

Exercise:

- Separate into real and imaginary parts i^i
- Separate into real and imaginary parts $(1+i)^i$
- Separate into real and imaginary parts $(i)^{(1-i)}$

11. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is

$e^{2m\pi+\alpha}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

Solution: Let $1+i\tan\alpha = r\,e^{i\theta}$

$\therefore r^2 = 1+tan^2\alpha = sec^2\alpha \qquad \therefore r=\sec\alpha$

And $\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \tan^{-1}(\tan\alpha)=\alpha$

Now, $Log\,(1+i\,\tan\alpha) = log\left(r\,e^{i\theta}\right) = \log r + (2m\pi+\theta)i$

$= \log\sec\alpha + (2m\pi+\alpha)i$

$\therefore 1+i\,\tan\alpha = e^{[\log\sec\alpha+(2m\pi+\alpha)i]}$

$\therefore (1+i\,\tan\alpha)^{-i} = e^{-i[\log\sec\alpha+(2m\pi+\alpha)i]}$

$= e^{2m\pi+\alpha}.\,e^{-ilog\sec\alpha}$

$= e^{2m\pi+\alpha}.\,e^{i\,(log\cos\alpha)}$

$= e^{2m\pi+\alpha}.\,[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

12. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Solution: Let $z=(1+i\,tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$\mathrm{Log}\,z = (1+i\tan\beta)log(1+i\,tan\alpha)$

$= (1+i\tan\beta)\left[\frac{1}{2}log(1+tan^2\alpha)+itan^{-1}tan\alpha\right]$

$= (1+i\tan\beta)[\log\sec\alpha + i\alpha]$

$\therefore \log z = (\log\sec\alpha-\alpha\tan\beta)+i(\alpha+tan\beta\log\sec\alpha) = x+iy$ say

Where $x=\log\sec\alpha-\alpha\tan\beta$ and $y=\alpha+tan\beta\log sec\alpha$ ....(i)

Now, $z=e^{x+iy}=e^x.e^{iy}=e^x(cosy+i\sin y)$

Since by data z is real

$\therefore e^x\sin y=0 \qquad \therefore y=0 \qquad \therefore \cos y=1$

$\therefore z=e^x\cos y = e^x = e^{\log sec\alpha-\alpha\tan\beta}$ from (i)

$\therefore z = e^{\log sec\alpha}.\,e^{-\alpha\tan\beta} = sec\alpha.\,e^{-\alpha\tan\beta}$ ....(ii)

But since $y=0$, from (i) $\alpha+tan\beta\log\sec\alpha=0$

$\therefore -\alpha = tan\beta\log sec\alpha$

$\therefore -\alpha\,tan\beta = tan^2\beta.\log sec\,\alpha = log(sec\,\alpha)^{tan^2\beta}$

$\therefore e^{-\alpha\tan\beta} = (sec\,\alpha)^{tan^2\beta}$

$\therefore$from (ii) $z = sec\alpha.\,(sec\,\alpha)^{tan^2\beta} = (sec\,\alpha)^{(1+tan^2\beta)} = (sec\,\alpha)^{sec^2\beta}$

13. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha+i\,\beta$, find α and β

Solution: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha+i\,\beta,$

Taking logarithms of both sides, $log\left(\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}\right) = log(\alpha + i\,\beta)$

$\log(\alpha + i\beta) = (x + iy)\log(a + ib) - (x - iy)\log(a - ib)$

$log(\alpha + i\beta) = (x + iy)\left[\frac{1}{2}log(a^2 + b^2) + i\,tan^{-1}\left(\frac{b}{a}\right)\right] - (x - iy)\left[\frac{1}{2}log(a^2 + b^2) - i\,tan^{-1}\left(\frac{b}{a}\right)\right]$

$log(\alpha + i\beta) = 2i\left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2 + b^2)\right]$

$= 2ik\ say \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2 + b^2)\right]$

$\therefore (\alpha + i\beta) = e^{2ik} = cos2k + isin\,2k$

$\therefore \alpha = cos2k,\ \beta = sin\,2k \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2 + b^2)\right]$

14. If $i^{\alpha+i\beta} = \alpha + i\,\beta$ $\left(\text{or } i^{i^{i\cdots\infty}} = \alpha + i\,\beta\right)$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$ Where n is any positive integer

Solution: Since $i = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

we have $i^{\alpha+i\beta} = \alpha + i\,\beta$

$\left[\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\alpha+i\beta} = \alpha + i\beta$

$\therefore e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta+i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}.e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha\right] = \alpha + i\beta$

Equating real and imaginary parts

$e^{-(4n+1)\frac{\pi}{2}\beta}\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha = \alpha$ and $e^{-(4n+1)\frac{\pi}{2}\beta}\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha = \beta$

Squaring and adding, we get $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

15. Prove that $\log tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = i\,tan^{-1}(\sinh x)$.

Solution: $\log\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right) = log\left\{\frac{1+tan(ix/2)}{1-tan(ix/2)}\right\}$

$= log\left\{\frac{1+i\tan h\,(x/2)}{1-i\tan h(x/2)}\right\}$

$= log[1 + i\tan h\,(x/2)] - log[1 - i\tan h(x/2)\,]$

$= \left[\frac{1}{2}log\left(1 + tanh^2\left(\frac{x}{2}\right)\right) + itan^{-1}tanh\left(\frac{x}{2}\right)\right]$

$-\left[\frac{1}{2}log\left(1 + \tan h^2\left(\frac{x}{2}\right)\right) - i\,tan^{-1}\tan h\left(\frac{x}{2}\right)\right]$

$= 2i\,tan^{-1}\tan h\left(\frac{x}{2}\right) = i.tan^{-1}\left\{\frac{2\tan h(x/2)}{1-tanh^2(x/2)}\right\} = itan^{-1}(\sin hx)$

$\because 2tan^{-1}\alpha = tan^{-1}\left\{\frac{2\alpha}{1-\alpha^2}\right\}$

INVERSE HYPERBOLIC FUNCTIONS:

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.

Similarly we can define $\cosh^{-1}x, \tanh^{-1}x, \coth^{-1}x,\ \mathrm{sech}^{-1}x, \mathrm{cosech}^{-1}x.$

The inverse hyperbolic functions are many valued but we will consider their principal value only.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$ **(ii)** $\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof: (i) $\mathbf{\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)}$

Let $\sinh^{-1} x = y$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$$

$$e^{2y} - 2x\,e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \log\left(x \pm \sqrt{x^2+1}\right)$$

But $x - \sqrt{x^2+1} < 0$ and $\log(-ve)$ is not defined.

$$\therefore y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) $\mathbf{\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)}$

Let $\cosh^{-1}x = y$

$$x = \cosh\ y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$y = \log\left(x \pm \sqrt{x^2-1}\right) \qquad \text{.............................(1)}$$

Consider, $y = \log\left(x - \sqrt{x^2-1}\right)$ ....(2)

$$e^y = x - \sqrt{x^2-1},$$

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1} = x + \sqrt{x^2-1}$$

$$-y = \log\left(x + \sqrt{x^2-1}\right)$$

$$y = -\log\left(x + \sqrt{x^2 - 1}\right) \qquad \text{.....................(3)}$$

Equating equation (2) and (3), we get

$$\log\left(x - \sqrt{x^2 - 1}\right) = -\log\left(x + \sqrt{x^2 - 1}\right) \qquad \text{.......................(4)}$$

From equation (1) and (4), we get

$$y = \pm\log\left(x + \sqrt{x^2 - 1}\right)$$

$$\cosh^{-1}x = \pm\log\left(x + \sqrt{x^2 - 1}\right)$$

$$\therefore\ x = \cosh\{\pm\log\left(x + \sqrt{x^2 - 1}\right)\}$$

$$= \cosh\{\log(x + \sqrt{x^2 - 1})\} \qquad \text{since } \cosh(-z) = \cosh z$$

$$\therefore \cosh^{-1}x = \log\left(x + \sqrt{x^2 - 1}\right)$$

(iii) $\quad \mathbf{\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)}$

Let $\tanh^{-1}x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo-dividendo

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$= \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Solution: Let $\tanh \log\sqrt{x} = \alpha$

$$\log\sqrt{x} = \tanh^{-1}\alpha$$

$$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+\alpha}{1-\alpha}\right)$$

$$x = \frac{1+\alpha}{1-\alpha}$$

$$\frac{x-1}{x+1} = \frac{(1+\alpha)-(1-\alpha)}{(1+\alpha)+(1-\alpha)} = \frac{2\alpha}{2} =$$

$$\therefore \tan h \log\sqrt{x} = \frac{x-1}{x+1}$$

Put $x = 5/3 \ \ and\ x = 7$ and add

$\log h(\log\sqrt{5/3}) + \tan h\,(\log\sqrt{7}) = \frac{(5/3)-1}{(5/3)+1} + \frac{7-1}{7+1} = \frac{2}{8} + \frac{6}{8} = 1$

2. (i) Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1 + x^2 = \cos h^2 y \qquad \therefore x^2 = \cos h^2 y - 1 = \sin h^2 y$

$\therefore x = \sin hy \qquad \therefore y = \sin h^{-1} x \qquad \therefore \cos h^{-1}\sqrt{1+x^2} = \sin h^{-1} x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Solution: Let $\tan h^{-1}x = y \qquad \therefore x = \tan hy$

Now, $\frac{x}{\sqrt{1-x^2}} = \frac{\tan hy}{\sqrt{1-tanh^2 y}} = \frac{\tan hy}{\sqrt{cosh^2 y - \sin h^2 y / cosh^2 y}} \quad = \frac{\sin hy}{\cos hy} \times \frac{\cos hy}{1} = \sin hy$

$\therefore y = \sin h^{-1}\frac{x}{\sqrt{1-x^2}} \qquad \therefore \tan h^{-1}x = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1 + x^2 = \cos h^2 y \qquad \therefore x^2 = \cos h^2 y - 1 = \sin h^2 y \qquad \therefore x = \sin hy$

$\therefore \tan hy = \frac{\sin hy}{\cos hy} = \frac{x}{\sqrt{1+x^2}} \qquad \therefore y = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

Solution: Let $cot\ h^{-1}\left(\frac{x}{a}\right) = y \qquad \therefore \frac{x}{a} = \cot hy \qquad \therefore \tan hy = \frac{1}{\cot hy} = \frac{1}{x/a} = \frac{a}{x}$

$\therefore y = \tan h^{-1}\left(\frac{a}{x}\right) = \frac{1}{2}log\left(\frac{1+(a/x)}{1-(a/x)}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

$\therefore cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

(iii) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

Solution: Let $\sec h^{-1}(\sin\theta) = x \qquad \therefore sin\ \theta = \sec hx \qquad \therefore \sin\theta = \frac{1}{\cos hx} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1}$

$\therefore (\sin\theta)e^{2x} - 2e^x + \sin\theta = 0$ This is a quadratic in e^x

$\therefore e^x = \frac{2 \pm \sqrt{4 - 4sin^2\theta}}{2\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$

$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\,cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$

$\therefore x = \log\cot\left(\frac{\theta}{2}\right) \qquad \therefore \sec h^{-1}(\sin\theta) = \log(\cot\theta/2)$

3. Separate into real and imaginary parts $cos^{-1}e^{i\,\theta}$ ***or*** $cos^{-1}(\cos\theta + i\sin\theta)$

Solution: Let $\cos^{-1} e^{i\theta} = x + iy,\ \ e^{i\theta} = \cos(x + iy)$

$\cos\theta + i\sin\theta = \cos x\cos(iy) - \sin x\sin(iy) = \cos x\cosh y - i\sin x\sinh y$

Equating real and imaginary parts $\cos\theta = \cos x\cosh y$ and $\sin\theta = -\sin x\sinh y$

Since $\cosh^2 y - \sinh^2 y = 1$

$\therefore \left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$

$\therefore \frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{(1-\sin^2\theta)\sin^2 x - \sin^2\theta(1-\sin^2 x)}{(1-\sin^2 x)\sin^2 x} = 1$

$\therefore \sin^2 x - \sin^2 x\sin^2\theta - \sin^2\theta + \sin^2 x\sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore \sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore -\sin^2\theta = -\sin^4 x$

$\therefore \sin^2\theta = \sin^4 x$

$\therefore \sqrt{\sin\theta} = \sin x$ .... (1)

$\therefore x = \sin^{-1}\sqrt{\sin\theta}$

Since $\sin\theta = -\sin x\sinh y$

$\sin\theta = -\sqrt{\sin\theta}\sinh y$ from (1)

$\therefore -\sqrt{\sin\theta} = \sinh y$

$\therefore y = \sinh^{-1}(\sqrt{\sin\theta}) = \log\left(-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$

$\therefore y = \log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

$\therefore \cos^{-1} e^{i\theta} = x + iy = \sin^{-1}\sqrt{\sin\theta} + i\log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

4. Separate into real and imaginary parts $sinh^{-1}(i\,x)$

Solution: Let $\sinh^{-1}(ix) = \alpha + i\beta$

$\therefore ix = \sinh(\alpha + i\beta) = \sinh\alpha\cosh(i\beta) + \cosh\alpha\sinh(i\beta)$

$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

Equating real and imaginary parts $\sinh\alpha\cos\beta = 0$

$\therefore \cos\beta = 0 \quad \therefore \beta = \frac{\pi}{2} \quad \therefore \sin\beta = 1$

Also $\cosh\alpha\sin\beta = x$

$\therefore \cosh\alpha = x \quad \left[\because \sin\frac{\pi}{2} = 1\right]$

$\therefore \alpha = \cosh^{-1} x$

$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1} x + i\frac{\pi}{2}$

5. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}log\left(\frac{1}{5}\right)$

Solution: $\tan z = \frac{i}{2}(1-i)$

$\tan z = \frac{1}{2}(i - i^2) = \frac{1}{2}i + \frac{1}{2}$

$Let\ z = x + iy \ \therefore \tan(x+iy) = \frac{1}{2} + \frac{i}{2}, \quad \tan(x-iy) = \frac{1}{2} - \frac{i}{2}$

$\therefore \tan(2x) = [(x+iy) + (x-iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]+\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1-\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{1}{1-\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{1}{1/2} = 2$$

$\therefore 2\mathrm{x} = \tan^{-1}2 \qquad \therefore x = \frac{1}{2}\tan^{-1}2$

Now, $tan(2iy) = \tan[(x+iy) - (x-iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]-\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1+\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{i}{1+\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{i}{1+(1/2)} = \frac{2}{3}i$$

$\therefore i\tan h\,2y = \frac{2}{3}i \qquad \therefore \tan h\,2y = \frac{2}{3}$

$\therefore 2y = tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}log\left[\frac{1+(2/3)}{1-(2/3)}\right] = \frac{1}{2}\log 5 \qquad \therefore y = \frac{1}{4}\log 5$

$\therefore z = x + iy = \frac{1}{2}tan^{-1}2 + \frac{i}{4}\log 5$

6. Show that $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

Solution: Let $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$\therefore i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{x-a}{x+a} = \frac{e^{-i\theta}-e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \quad [\because i^2 = -1]$

By componendo and dividend $\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta}-e^{i\theta})+(e^{i\theta}+e^{-i\theta})}{(e^{-i\theta}-e^{i\theta})-(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}} = e^{-2i\theta} \qquad \therefore \frac{x}{a} = e^{-2i\theta} \qquad \therefore -2i\theta = \log\frac{x}{a}$

Multiply by i throughout, $2\theta = i\log\frac{x}{a} \qquad \therefore \theta = \frac{i}{2}\log\left(\frac{x}{a}\right)$

$tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

HYPERBOLIC FUNCTIONS

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}, find\ \sinh 2x\ and\ \cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2}$ $\therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$ $\therefore 2\,e^{2x} - 2 = e^{2x} + 1$ $\therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3+(1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$

$\therefore 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$

$\therefore 15e^x - e^{-x} = 2$

$\therefore 15e^{2x} - 2e^x - 1 = 0$ Solving it as a quadratic equation in e^x,

$e^x = \frac{2 \pm \sqrt{4-4(15)(-1)}}{2(15)} = \frac{2\pm 8}{30} = \frac{1}{3}\ or - \frac{1}{5}$

$\therefore x = \log\left(\frac{1}{3}\right)\ or\ x = \log\left(-\frac{1}{5}\right)$

Since x is real, $x = log\left(\frac{1}{3}\right) = -\log 3$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Solution: Let $\sin h^{-1}\ a = \alpha,\ \sin h^{-1} b = \beta\ \ and\ \sin h^{-1}\ x = \gamma$

We are given $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ $\therefore \alpha + \beta = \gamma$

$\therefore \sinh(\alpha + \beta) = \sinh \gamma$

$\therefore \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta = \sinh \gamma$ ....(A)

But $\sinh \alpha$ = a, $\sinh \beta = b$, $\sinh \gamma = x$

$\therefore \cos h\ \alpha = \sqrt{1 + \sin h^2 \alpha} = \sqrt{1 + a^2}$ and $\cos h\ \beta = \sqrt{1 + \sin h^2 \beta} = \sqrt{1 + b^2}$

Putting this values in (A), we get $a\sqrt{1 + a^2} + b\sqrt{1 + b^2} = x$

4. Prove that $16\ sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Solution: LHS $= 16\sinh^5 x$

$= 16\left(\frac{e^x - e^{-x}}{2}\right)^5$

$= \frac{16}{32}(e^{5x} - 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} - 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} - e^{-5x})$

$= \frac{1}{2}(e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x})$

$= \left(\frac{e^{5x} - e^{-5x}}{2}\right) - 5\left(\frac{e^{3x} - e^{-3x}}{2}\right) + 10\left(\frac{e^x - e^{-x}}{2}\right)$

$= \sinh 5x - 5\sinh 3x + 10\sinh x$

$=$ RHS

5. Prove that $16cosh^5x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Solution: $l.h.s = 16cosh^5x$

$= 16\left(\frac{e^x + e^{-x}}{2}\right)^5$ [By Binomial Theorem]

$= \frac{16}{32}[e^{5x} + 5e^{4x}.e^{-x} + 10e^{3x}.e^{-2x} + 10e^{2x}.e^{-3x} + 5e^{x}.e^{-4x} + e^{-5x}]$

$= \frac{(e^{5x} + e^{-5x})}{2} + 5\frac{(e^{3x} + e^{-3x})}{2} + 10\frac{(e^{x} + e^{-x})}{2}$

$= \cos h\, 5\, x + 5\cos h\, 3\, x + 10\cos h\, x = r\,.h.s$

6. Prove that $\frac{1}{1 - \frac{1}{1 - \frac{1}{1 - cosh^2x}}} = cosh^2x$

Solution: $l.h.s = \frac{1}{1 - \frac{1}{1 - \frac{1}{-sinh^2x}}} = \frac{1}{1 - \frac{1}{1 + cosec\, h^2x}} = \frac{1}{1 - \frac{1}{\cot h^2x}} = \frac{1}{1 - \tan h^2x} = \frac{1}{1 - \frac{sinh^2x}{\cos h^2x}} = \frac{\cos h^2x}{\cos h^2x - \sin h^2x} = \cos h^2x$

7. If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Solution: **(i)** $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\therefore e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$

$\therefore e^{-u} = \frac{1 - \tan\theta/2}{1 + \tan\theta/2}$

$\therefore \cosh u = \frac{e^u + e^{-u}}{2}$

$= \frac{1}{2}\left[\frac{(1 + 2\tan\theta/2 + \tan^2\theta/2) + (1 - 2\tan\theta/2 + \tan^2\theta/2)}{1 - \tan^2\theta/2}\right]$

$= \frac{1}{2}\left(\frac{2 + 2\tan^2\theta/2}{1 - \tan^2\theta/2}\right)$

$$= \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\cos\theta} = \sec\theta$$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

(iv) $\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} = \frac{2\sin h(u/2).\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)} = \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta}$ (By (i) and (ii))

$\therefore \tan h\left(\frac{u}{2}\right) = \frac{sin\theta/cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)} = \frac{sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Solution: **(i)** $\cosh x = \sec\theta$

$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$ By definition $\cos hx = \frac{e^x+e^{-x}}{2}$

$\therefore e^x - 2\sec\theta + e^{-x} = 0$

$\therefore (e^x)^2 - 2e^x sec\theta + 1 = 0$

Solving the quadratic in e^x,

$e^x = sec\theta \pm \sqrt{sec^2\theta - 1} = \sec\theta \pm \tan\theta$

$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm log(\sec\theta + \tan\theta)$

(we can prove that $log(\sec\theta - \tan\theta) = -log(\sec\theta + \tan\theta)$)

(ii) Let $tan^{-1}e^{-x} = \alpha$ $\therefore e^{-x} = \tan\alpha$ $\therefore e^x = \cot\alpha$

Now, by data $\sec\theta = \cos hx = \frac{e^x+e^{-x}}{2} = \frac{cot\alpha+tan\alpha}{2}$

$$2\,sec\theta = cot\alpha + tan\alpha = \frac{cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$$

(iii) $\tan h\,\frac{x}{2} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^x-1}{e^x+1} = \frac{sec\theta+tan\theta-1}{sec\theta+tan\theta+1} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{(1-cos\theta)+\sin\theta}{(1+cos\theta)+\sin\theta} = \frac{2sin^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{cos(\theta/2)} = tan\frac{\theta}{2}$$

SEPARATION OF REAL AND IMAGINARY PARTS:

Many a time we are required to separate real and imaginary parts of a given complex function. For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\ \tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

$\therefore 1 - x^2 - y^2 = 2x \cot 2\alpha$ $\qquad \therefore x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$i \tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$\therefore 1 + x^2 + y^2 = 2y \coth 2\beta$ $\qquad$ i.e., $x^2 + y^2 - 2y\coth 2\beta + 1 = 0$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}(e^{i\,\theta})$

Solution: Let $tan^{-1}e^{i\theta} = x + iy$ $\quad \therefore e^{i\theta} = \tan(x + iy)$ $\quad \therefore cos\theta + i\sin\theta = \tan(x + iy)$

Similarly, $cos\theta - i\sin\theta = \tan(x - iy)$

Now, $tan\ 2x = \ tan\ [\ (x + iy) +\ (x - iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)+(cos\theta-i\sin\theta)}{1-(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ cos\theta}{1-(cos^2\theta+sin^2\theta)} = \frac{2\ cos\theta}{1-1} = \frac{2\ cos\theta}{0} = \infty$$

$\therefore 2x = \frac{\pi}{2}$ $\quad \therefore x = \frac{\pi}{4}$

Also $\tan 2\,iy = \tan[(x + iy) - (x - iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)-(cos\theta-i\sin\theta)}{1+(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ isin\theta}{1+(cos^2\theta+sin^2\theta)} = \frac{2isin\theta}{2}$$

$\therefore i\tan h\ 2y = i\sin\theta$ $\qquad \therefore \tan h\ 2y = \sin\theta$

$\therefore 2y = tanh^{-1}\sin\theta$ $\qquad \therefore y = \frac{1}{2}\tan h^{-1}\sin\theta$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Solution: $\sin(\alpha - i\,\beta) = x + i\,y$

$\therefore \sin\alpha\cos h\,\beta + i\cos\alpha\sin h\,\beta = x + iy$

Equating real and imaginary parts, we get, $\sin\alpha\cos h\,\beta = x$ $\ and\ \cos\alpha\sin h\,\beta = y$

$\therefore \frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = sin^2\alpha + cos^2\alpha = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\,, prove\ that$

(i) $\sin\alpha = \pm sin^2 x = \pm \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Solution: $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\cos x\cos(iy) - \sin x\sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we get,

$\cos x\cosh y = \cos\alpha$ and $-\sin x\sinh y = \sin\alpha$

(i) Since $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x\sinh^2 y + \cos^2 x\cosh^2 y = 1$

$\sin^2 x\sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\sin^2 x\sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x\sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x = 1$

$\sinh^2 y - \sin^2 x = 0$

$\therefore \sinh^2 y = \sin^2 x$ ....(i)

$\therefore \sinh y = \pm\sin x$

$\therefore \sin\alpha = -\sin x\sinh y = -\sin x(\pm\sin x) = \pm\sin^2 x$

(ii) $\cos 2x + \cosh 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$ .... from (i)

$= 2$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Solution: We have to separate real part $\pi/6$ and imaginary part α

$\because tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy \qquad \therefore tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$\therefore tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{tan\left(\frac{\pi}{6}+i\alpha\right) + \tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right).\tan\left(\frac{\pi}{6}-i\alpha\right)}$

$\therefore tan\frac{\pi}{3} = \frac{(x+iy)+(x-iy)}{1-(x+iy).(x-iy)}$

$\therefore \sqrt{3} = \frac{2x}{1-x^2-y^2}$

$\therefore 1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$

$\therefore x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1.$

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Solution: We have $x + iy = c\ \cot(u + iv) \qquad \therefore\ x - iy = c\ \cot(u - iv)$

$\therefore 2x = c[\cot(u+iv) + \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\,\frac{[\cos(u+iv)\sin(u-iv)+\sin(u+iv)\cos(u-iv)]}{\sin(u+iv)\sin(u-iv)}$$

$$\therefore 2x = \frac{c\sin[(u-iv)+(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u-iv-u+iv)]/2}$$

$$\therefore x = \frac{c\sin 2u}{-[\cos 2u-\cos 2\,iv]} = \frac{c\sin 2u}{\cos h\;2v-\cos 2u} \quad \text{...........(1)}$$

Now, $2iy = c[\cot(u+iv) - \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} - \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\left[\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$\therefore 2iy = \frac{c\sin[(u-iv)-(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u+iv-u+iv)]/2}$$

$$\therefore\; iy = \frac{c\sin(-2iv)}{-[\cos 2u-\cos 2iv]} = -\frac{i\,c\sin h\;2v}{\cos h\;2v-\cos 2u}$$

$$\therefore\; y = \frac{-\,c\sin h\;2v}{\cos h\;2v-\cos 2u} \quad \text{..........(2)}$$

From (1) & (2) $\dfrac{x}{\sin 2u} = -\dfrac{y}{\sinh 2v} = \dfrac{c}{\cosh 2v-\cos 2u}$

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right), prove\ that\ (u^2+v^2)^2 = 2(u^2-v^2)$

Solution: We have $\dfrac{1}{\sin[(\pi/4)+ix]} = u + iv$

$$\therefore \sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} = \frac{1}{u+iv}\cdot\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore \sin\frac{\pi}{4}\cos i\,x + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}\cos h\,x + i\frac{1}{\sqrt{2}}\sin hx = \frac{u-iv}{u^2+v^2}$$

Equating real and imaginary parts $\cos hx = \sqrt{2}.\left(\frac{u}{u^2+v^2}\right)$; $\sin hx = -\sqrt{2}.\left(\frac{v}{u^2+v^2}\right)$

But $cosh^2x - sinh^2x = 1$

$$\therefore 2\left(\frac{u^2}{(u^2+v^2)^2}\right) - 2\left(\frac{v^2}{(u^2+v^2)^2}\right) = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$

7. If $x + iy = \cos(\alpha + i\,\beta)$ or if $\cos^{-1}(x+iy) = \alpha + i\,\beta$ express x and y in terms of α and β.

Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$

Solution: We have $\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = x + iy$

$\therefore \cos\alpha\cos h\,\beta - i\sin\alpha\sin h\beta = x + iy$

Equating real and imaginary parts $\cos\alpha\cos h\,\beta = x$ *and* $\sin\alpha\sin h\beta = -y$

We know that, in terms of the roots, the quadratic equation is given by

$\lambda^2 - (sum\ of\ the\ roots)\lambda + (product\ of\ the\ roots) = 0$

Hence the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$ is

$\lambda^2 - (cos^2\alpha + cos^2\beta)\lambda + (cos^2\alpha . cos^2\beta) = 0$

This means we have to prove that $x^2 + y^2 + 1 = cos^2\alpha + cos^2\beta$ and $x^2 = cos^2\alpha + cos^2\beta$

Now, $x^2 + y^2 + 1 = cos^2\alpha \cos h^2\beta + sin^2\alpha \sin h^2\beta + 1$

$= cos^2\alpha \cos h^2\beta + (1 - cos^2\alpha)(\cos h^2\beta - 1) + 1$

$= cos^2\alpha \cos h^2\beta + \cos h^2\beta - 1 - cos^2\alpha \cos h^2\beta + cos^2\alpha + 1$

$= cos^2\alpha + \cos h^2\beta = sum\ of\ the\ roots$

And $x^2 = cos^2\alpha \cos h^2\beta$ = Product of the roots

Hence the equation whose roots are $cos^2\alpha$, $\cos h^2\beta$ is $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$

HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $\sinh x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $\cosh x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as $\tan \mathrm{h}x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$\mathrm{cosech}x = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$, $\mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$, and $\coth x = \frac{1}{\tanh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1$, $\tanh(0) = 0$, $\tanh(\infty) = 1$ $\quad \therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x$,
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$

3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\mathrm{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \mathrm{cosec}^2 x$	$\coth^2 x - \mathrm{cosech}^2 x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x\cos y \pm \cos x\, sin\, y$	$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$
15	$\cos(x \pm y) = \cos x\cos y \mp \sin x\, sin\, y$	$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$
16	$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tanh y}$	$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
17	$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \frac{-\coth x \coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$
20	$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
21	$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
22	$2\sin x\cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x\sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

24	$2\cos x\cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x\sin y = \cos\,(x-y) - \cos(x+y)$	$2\sinh x\sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$$\sinh(2\pi i + x) = \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x$$
$$= i\sin\ 2\pi\ \cosh x + \cos\ 2\pi\ \sinh x$$
$$= 0 + \sinh x$$
$$= \sinh x$$

Hence sinh x is a periodic function of period 2πi

Similarly we can prove that cosh x and tanh x are periodic functions of period 2πi and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$, $y = \frac{e^x - e^{-x}}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$, $y = \frac{e^x + e^{-x}}{2}$, $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

If $y = cosh\ x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$, $y = \frac{\sinh x}{\cosh x}$ $\quad \therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\, dx = \sinh x$, $\qquad \int \sinh x\, dx = \cosh x$, $\qquad \int sech^2 x dx = \tanh x$

SOME SOLVED EXAMPLES:

1. If tanh x $= \frac{1}{2}$, $find$ $\sinh 2x$ and $\cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2}$ $\quad \therefore \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2}$ $\quad \therefore 2\,e^{2x} - 2 = e^{2x} + 1$ $\quad \therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - (1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + (1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x+e^{-x}}{2}\right)+8\left(\frac{e^x-e^{-x}}{2}\right)=1$

$\therefore 7e^x+7e^{-x}+8e^x-8e^{-x}=2$

$\therefore 15e^x-e^{-x}=2$

$\therefore 15e^{2x}-2e^x-1=0$ Solving it as a quadratic equation in e^x,

$e^x=\frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)}=\frac{2\pm 8}{30}=\frac{1}{3}\ \ or-\frac{1}{5}$

$\therefore x=\log\left(\frac{1}{3}\right)\ \ or\ x=\log\left(-\frac{1}{5}\right)$

Since x is real, $x=log\left(\frac{1}{3}\right)=-\log 3$

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of $\cos\theta = \cos(2k\pi+\theta)$ and $\sin\theta = \sin(2k\pi+\theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\ \theta + i\sin\ \theta$, we apply De Moivre's theorem

$$z = (cos\ \theta + i\ sin\ \theta)^{\frac{1}{n}} = cos\frac{\theta}{n} + i\ sin\ \frac{\theta}{n}$$

This shows that $\left(cos\frac{\theta}{n} + i\ sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = cos\ \theta + i\ sin\ \theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}^{\frac{1}{n}} = cos\left(\frac{2k\pi+\theta}{n}\right) + i\ sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,…………..,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Solution: Consider $x^3 = 1$ $\quad\therefore x = 1^{1/3}$

$$\therefore x = (\cos 0 + i\sin 0)^{1/3} = (\cos 2\,k\,\pi + i\sin 2\,k\,\pi)^{1/3} = cos\frac{2\,k\,\pi}{3} + i\sin\frac{2\,k\,\pi}{3}$$

Putting $k = 0, 1, 2$, the cube roots of unity are

$x_0 = 1,\ x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$ (say)

And $x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left[cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^2 = \omega^2$

$$\text{Now, } 1+\omega+\omega^2 = 1 + \left(cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - 1 = 0$$

$\therefore 1 + \omega^2 = -\omega$

Now, $(1-\omega)^6 = [(1-\omega)^2]^3 = (1-2\omega+\omega^2)^3$

$$= (-\omega - 2\omega)^3 = (-3\omega)^3 - 27\omega^3 = -27$$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Solution: $\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left((8k+1)\frac{\pi}{4}\right) + i\sin\left((8k+1)\frac{\pi}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) + i\sin\left((8k+1)\frac{\pi}{12}\right)$$

Similarly, $\sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) - i\sin\left((8k+1)\frac{\pi}{12}\right)$

$\therefore \sqrt[3]{\frac{(1+i)}{\sqrt{2}}} + \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = 2\cos\left((8k+1)\frac{\pi}{12}\right)$

Putting $k = 0, 1, 2$ we get the three roots as $2\cos\frac{\pi}{12}$, $2\cos\frac{9\pi}{12}$, $2\cos\frac{17\pi}{12}$

i.e., $2\cos\frac{r\pi}{12}$ where $r = 1, 9, 17$

3. Find the cube roots of $(1 - cos\theta - i\ sin\theta)$.

Solution: $(1 - \cos\theta - i\sin\theta)^{1/3} = \left[2\sin^2\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{1/3}$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) + i\sin\left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{(4k-1)+\theta}{6}\right) + i\sin\left(\frac{(4k-1)+\theta}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get the three roots

4. Find the continued product of all the value of $(-i)^{2/3}$

Solution: $(-i)^{2/3} = \left(0 + i(-1)\right)^{2/3} = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{2/3}$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{2/3}$$

$$= \cos\left((4k+1)\frac{\pi}{3}\right) - i\sin\left((4k+1)\frac{\pi}{3}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right), \left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right), \left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$

∴ Continued product

$$= \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)\left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{8\pi}{3} + \frac{9\pi}{3}\right) - i\sin\left(\frac{\pi}{3} + \frac{8\pi}{3} + \frac{9\pi}{3}\right)$$

$$= \cos 6\pi + i\sin 6\pi$$

$$= 1 - i(0)$$

$$= 1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution: $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left\{\left(cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right\}^{1/4}$

$$= (\cos\pi + i\sin\pi)^{1/4}$$

$$= [cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$$

$$= cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$$

Putting $k = 0,1,2,3$ we get the four roots as,

$\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \left(cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$\therefore \left(cos\frac{r\pi}{4} + i\ sin\frac{r\pi}{4}\right)\ where\ r = 1,3,5,7$

The required product $= cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + \ isin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$

$$= cos4\pi + i\sin 4\pi = 1.$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Solution: $x^7 + x^4 + x^3 + 1 = 0$

$\therefore x^4(x^3 + 1) + (x^3 + 1) = 0$

$\therefore (x^3 + 1)(x^4 + 1) = 0$

$\therefore x^3 = -1, x^4 = -1$

Consider $x^3 = -1$

$\therefore x = (-1 + i0)^{1/3} = (\cos\pi + i\sin\pi)^{1/3} = [\cos(2k+1)\pi - i\sin(2k+1)\pi]^{1/3}$

$$= \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}$$

Putting $k = 0, 1, 2$ we get the three roots

Similarly from $x^4 = -1$ we get the remaining four roots as

$x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$ where $k = 0, 1, 2, 3$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying the given equation by $x - 1$, we get $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

$\therefore$ We have $x^5 - 1 = 0$ $\therefore x^5 = 1 = \cos 0 + i\sin 0$

$\therefore x^5 = cos(2k\pi) + \ i\sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + \ i\sin\frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation.

$x_0 = \cos 0 + i\sin 0 = 1,$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},\quad x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5},\ \ x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

It is clear that 1 is the roots of $x - 1 = 0$

and the remaining roots are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$

i.e., $cos\frac{2\pi}{5} \pm i\ sin\frac{2\pi}{5},\quad cos\frac{4\pi}{5} \pm i\ sin\frac{4\pi}{5}$

8. SOLVE: $x^4 - x^2 + 1 = 0$

Solution: $x^4 - x^2 + 1 = 0$

Multiplying the given equation by $(x^2 + 1)$, we get, $(x^2 + 1)(x^4 - x^2 + 1) = 0$

$\therefore (x^2)^3 + (1)^3 = 0 \qquad \therefore x^6 + 1 = 0 \qquad \therefore x^6 = -1$

$$\therefore x = (-1 + 0i)^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$$
$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/6}$$
$$= \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots of equation

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \qquad x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} \qquad x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i \qquad x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

It is clear that i and $-i$ are the roots of $x^2 + 1 = 0$ and the remaining roots x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$

9. Find the roots common to $x^4 + 1 = 0\ and\ x^6 - i = 0$.

Solution: Consider $x^4 + 1 = 0 \quad \therefore x^4 = -1$

$$x = (-1 + i0)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$$

$$x = \cos\left((2k+1)\frac{\pi}{4}\right) + i\sin\left((2k+1)\frac{\pi}{4}\right)$$

Putting $k = 0, 1, 2, 3$ we get the three roots as

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = 1 \qquad x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ $\quad x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now consider, $x^6 - i = 0$ $\quad \therefore x^6 = i$

$$x = (0+1i)^{1/6} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/6} = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$$

$$= \cos\left((4k+1)\frac{\pi}{12}\right) + i\sin\left((4k+1)\frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots as

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ $\quad x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore$ common roots are $\pm\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

10. If $(1+x)^6 + x^6 = 0$

show that $x = -\frac{1}{2} - \frac{i}{2}cot\frac{\theta}{2}$ *where* $\theta = (2n+1)\,\pi/6\,, n = 0,1,2,3,4,5.$

Solution: $(1+x)^6 + x^6 = 0$ $\quad \therefore \frac{(1+x)^6}{x^6} = -1$

$$\frac{1+x}{x} = (-1)^{1/6} = (\cos\pi + i\sin\pi)^{1/6} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$$

$$= \cos\left((2k+1)\frac{\pi}{6}\right) + i\sin\left((2k+1)\frac{\pi}{6}\right)$$

$$\frac{x+1-x}{x} = \cos\theta + i\sin\theta - 1$$

$$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$$

$$x = \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta} = \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{-2\sin^2(\theta/2) - i2\sin(\theta/2)\cos(\theta/2)}{2(2\sin^2(\theta/2))}$$

$$= -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right) \qquad \text{where } \theta = (2k+1)\frac{\pi}{6}$$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0\ is\ 1 + i$, find all other roots.

Solution: The given equation is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

Since one of the root is $1 + i$

$\therefore$ other root must be $1 - i$ (since roots always occurs as complex conjugate pairs)

$\therefore x = 1 \pm i$ are the two roots

$\therefore x - 1 = \pm i$

$\therefore (x-1)^2 = (\pm i)^2$

$\therefore x^2 - 2x + 1 = -1$

$\therefore x^2 - 2x + 2 = 0$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 4x + 2$ and we obtain

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 4x + 2)(x^2 - 4x + 5)$

$\therefore$ the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$\therefore$ The required remaining roots of given equation are $1 - i,\ 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Solution: We have $x^5 = 1 = \cos O + i \sin O$ $\quad \therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$x_0 = \cos 0 + i \sin 0 = 1, \qquad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$

$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \qquad x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \qquad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$, and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting $x = 1$, we get $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Solution: We have $z^4 = i(z-1)^4$

$$\therefore \left(\frac{z}{z-1}\right)^4 = i = cos\frac{\pi}{2} + i\ sin\frac{\pi}{2} = cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$\therefore \frac{z}{z-1} = \left[cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{1/4}$$

$$= cos(4n+1)\frac{\pi}{8} + i\sin(4n+1)\frac{\pi}{8}$$

$$\therefore \frac{z}{z-1} = \cos\theta + i\sin\theta \qquad \text{where } \theta = (4n+1)\frac{\pi}{8}$$

$$\therefore \frac{z}{z-1-z} = \frac{z}{-1} = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$ Simplifying as in the above example, we get

$$\therefore \frac{z}{-1} = \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$$

$$\therefore -z = -\frac{1}{2} + \frac{i}{2}\ cot\ \frac{\theta}{2}$$

$$\therefore z = \frac{1}{2} - \frac{i}{2}\ cot\ \frac{\theta}{2}, \quad where\ \theta = (4n+1)\frac{\pi}{8}$$

For, $n = 0, 1, 2$, we get three roots, All these roots have the real part $1/2$

14. If ω is a 7th root of unity, prove that

$$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$$

if n is a multiple of 7 and is equal to zero otherwise.

Solution: We have $x = 1^{\frac{1}{7}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{7}}$

$$= \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}, \quad where \;\; n = 0,1,2,3,4,5,6$$

Let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1 \therefore \omega^{7n} = 1^n = 1$

If n is not a multiple of 7, $\therefore \omega^n \neq 1$

Now, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \ldots\ldots\ldots + \omega^{6n} = \frac{1-\omega^{7n}}{1-\omega^n}$ sum of 7 terms of G.P

$$= \frac{1-1}{1-\omega^n} = \frac{0}{1-\omega^n} = 0$$

If n is a multiple of 7, say $n = 7k$

Then, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k}$$

$$= 1+1+1+1+1+1+1 = 7$$

15. Prove that $\sqrt{1+sec(\theta/2)} = \left(1+e^{i\theta}\right)^{-1/2} + \left(1+e^{-i\theta}\right)^{-1/2}$

Solution: We have to show that $\sqrt{1+sec(\theta/2)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

Squaring both sides, we get, $1 + sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

We shall prove this result

Now, $r.h.s = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}}$$

$$= 1 + \frac{2}{\sqrt{2+\left(e^{i\theta}+e^{-i\theta}\right)}} = 1 + \frac{2}{\sqrt{2+2\cos\theta}}$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{4cos^2(\theta/2)}}$$

$$= 1 + \frac{2}{2cos(\theta/2)} = 1 + sec\frac{\theta}{2} = l.h.s$$

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\ cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$are given by $\pm i\ cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha,\alpha^2,\alpha^3,\ldots\alpha^6$ are the roots of $x^7-1=0$, find them and prove that

$(1-\alpha)(1-\alpha^2)\ldots\ldots\ldots\ldots(1-\alpha^6)=7.$

12. Prove that $x^5-1=(x-1)\left(x^2+2x\ cos\frac{\pi}{5}+1\right)\left(x^2+2x\ cos\frac{3\pi}{5}+1\right)=0.$

13. Solve the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a=e^{i\,2\pi/7}$ and $b=a+a^2+a^4,\ c=a^3+a^5+a^6$**.** then prove that b & c are roots of quadratic equation $x^2+x+2=0$.

15. Prove that **(i)** $\sqrt{1-cosce(\theta/2)}=\left(1-e^{i\theta}\right)^{-1/2}-\left(1-e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1-sce(\theta/2)}=\left(1+e^{i\theta}\right)^{-1/2}-\left(1+e^{-i\theta}\right)^{-1/2}$

16. If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$, find all the other roots.