# Module 3
# Partial Differentiation and Application

## Unit 3.1
## Partial derivatives of first and higher order

### ❖ Partial Derivatives of the First Order:

Let $z = f(x, y)$ be a function of two independent variables $x$ and $y$.

- The derivative of $z$ with respect to $x$ keeping $y$ constant, if it exists is called the **partial derivative of $z$ with respect to $x$** and it is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x$.

  Thus, $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x\,,\ y)-f(x,y)}{\delta x}$

- The derivative of $z$ with respect to $y$ keeping $x$ constant, if it exists is called the **partial derivative of $z$ with respect to $y$** and it is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y$.

  Thus, $\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x,\ y+\delta y)-f(x,y)}{\delta y}$

### ❖ Partial Derivatives of Higher Order:

The partial derivatives of higher order, if they exist, can be obtained from partial derivatives of the first order by using the above definitions again.

- Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is the second order partial derivative of z w.r.t. x

  and is denoted by $\frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}$

- Similarly, we have $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$
- $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$
- $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$

**Note:**

1. If $u = f(x, y)$ possesses continuous second-order partial derivatives $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$

   then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ this is called commutative property.

2. Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation.

### ❖ Differentiation of a Function of a Function:

Let $z = f(u)$ and $u = \Phi(x, y)$

So that $z$ is function of $u$ and $u$ itself is a function of two independent variables $x$ and $y$.

The two relations define $z$ as a function of $x$ and $y$.

In such cases $z$ may be called a **function of a function** of $x$ and $y$.

**e.g. (i)** $z = \frac{1}{u}$ and $u = \sqrt{x^2 + y^2}$

**(ii)** $z = \tan u$ and $u = x^2 + y^2$

- If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$ possesses first order partial derivatives then,

  $\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{dx}$ i.e. $\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}$

  Similarly,

  $\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$ i.e. $\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y}$

**Examples:**

1. If $z = (ax + by)^n$ then

$$\frac{\partial z}{\partial x} = n(ax + by)^{n-1}. a \quad \text{and} \quad \frac{\partial z}{\partial y} = n(ax + by)^{n-1}. b$$

2. If $z = sin(ax + by)$ then

$$\frac{\partial z}{\partial x} = cos(ax + by) . a \quad \text{and} \quad \frac{\partial z}{\partial y} = cos(ax + by) . b$$

3. If $z = cos(ax + by)$ then

$$\frac{\partial z}{\partial x} = -sin(ax + by) . a \quad \text{and} \quad \frac{\partial z}{\partial y} = -sin(ax + by) . b$$

4. If $z = e^{(ax+by)}$ then

$$\frac{\partial z}{\partial x} = e^{(ax+by)}. a \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{(ax+by)}. b$$

5. If $z = x^y$ then

$$\frac{\partial z}{\partial x} = y. x^{y-1} \quad \text{and} \quad \frac{\partial z}{\partial y} = x^y. logx$$

## SOME SOLVED EXAMPLES

1. If $u = \cos(\sqrt{x}+\sqrt{y})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x}+\sqrt{y})sin(\sqrt{x}+\sqrt{y}) = 0.$$

**Solution:**

We have,

$$\frac{\partial u}{\partial x} = -\sin(\sqrt{x}+\sqrt{y})\frac{1}{2\sqrt{x}}$$

$$\frac{\partial u}{\partial y} = -\sin(\sqrt{x}+\sqrt{y})\frac{1}{2\sqrt{y}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin(\sqrt{x}+\sqrt{y})\cdot\frac{1}{2}(\sqrt{x}+\sqrt{y})$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x}+\sqrt{y})\sin(\sqrt{x}+\sqrt{y}) = 0$$

2. If $z(x+y) = x^2+y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

**Solution:**

Since $z = \frac{x^2+y^2}{x+y}$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2)}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\therefore \text{LHS} = \left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2$$

$$= \left[2\cdot\frac{(x^2-y^2)}{(x+y)^2}\right]^2 = \left[2\cdot\frac{(x-y)}{(x+y)}\right]^2 = 4\frac{(x-y)^2}{(x+y)^2}$$

Putting the values of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\text{RHS} = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = 4\frac{(x-y)^2}{(x+y)^2}$$

$\therefore$ LHS = RHS

3. If $u = \log(tanx + tany)$, prove that

$$sin2x.\frac{\partial u}{\partial x} + sin2y.\frac{\partial u}{\partial y} = 2.$$

**Solution:**

We have, $u = \log(tanx + tany)$

$$\frac{\partial u}{\partial x} = \frac{1}{tanx+tany}.sec^2x$$

$$sin2x.\frac{\partial u}{\partial x} = \frac{2\,sinx\,cosx}{tanx+tany}.\frac{1}{cos^2x} = \frac{2\,tanx}{tanx+tany}$$

Similarly,

$$sin2y.\frac{\partial u}{\partial y} = \frac{2\,tany}{tanx+tany}$$

Hence, $sin2x.\frac{\partial u}{\partial x} + sin2y.\frac{\partial u}{\partial y} = \frac{2\,tanx}{tanx+tany} + \frac{2\,tany}{tanx+tany} = \frac{2(tanx+\,tany)}{tanx+tany} = 2$

4. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = y^2u^3$

**Solution:**

Since, $u = (1 - 2xy + y^2)^{-1/2}$,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = y\left[(1 - 2xy + y^2)^{-1/2}\right]^3 = yu^3$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)\left[(1 - 2xy + y^2)^{-1/2}\right]^3 = (x - y)u^3$$

$$\therefore \text{LHS} = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyu^3 - y(x - y)u^3 = xyu^3 - xyu^3 + y^2u^3 = y^2u^3$$

5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

**Solution:**

$$\text{LHS} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \qquad \ldots\ldots\ldots\ldots \text{(i)}$$

Now, $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2 - 3yz)$, $\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\left(\frac{x^2 + y^2 + z^2 - xy - yz - zx}{x^3 + y^3 + z^3 - 3xyz}\right) = \frac{3}{(x + y + z)}$$

$\{\because (x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) = x^3 + y^3 + z^3 - 3xyz\}$

Hence from (i),

$$\text{LHS} = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{(x+y+z)}$$

$$= 3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$= -\frac{9}{(x+y+z)^2} = \text{RHS}$$

6. If $z = x^y + y^x$, verify that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

**Solution:**

Differentiating $z$ partially w.r.t. $y$ we get,

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating this partially w.r.t. $x$ we get,

$$\frac{\partial^2 z}{\partial x\,\partial y} = yx^{y-1}\cdot\log x + x^y\cdot\frac{1}{x} + 1\cdot y^{x-1} + xy^{x-1}\log y$$

$$= yx^{y-1}\cdot\log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y \qquad \text{.......(i)}$$

Now, differentiating $z$ partially w.r.t. $x$, we get,

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating this again partially w.r.t. $y$, we get,

$$\frac{\partial^2 z}{\partial y\,\partial x} = x^{y-1} + y\cdot x^{y-1}\log x + \frac{y^x}{y} + xy^{x-1}\log y$$

$$= yx^{y-1}\log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y \qquad \text{.......(ii)}$$

From (i) and (ii) $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$

7. If $u = x^3 y + e^{xy^2}$, verify that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

**Solution:**

Differentiating $u$ partially w.r.t. $x$ we get,

$$\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2} y^2$$

Differentiating this partially w.r.t. $y$ we get,

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2 + e^{xy^2}\,2y + y^2 e^{xy^2}\,2xy \qquad \text{.......(i)}$$

Now, differentiating $u$ partially w.r.t. $y$, we get,

$$\frac{\partial u}{\partial y} = x^3 + e^{xy^2}\,2xy$$

Differentiating this again partially w.r.t. $x$, we get,

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2 + e^{xy^2}\,2y + 2xy\,e^{xy^2}\,y^2 \qquad \text{.......(ii)}$$

From (i) and (ii) $\quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$

8. If $u = e^{x^2+y^2+z^2}$, prove that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = 8\,xyzu$.

**Solution:**

$$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$$

$$= 2z \cdot e^{x^2+y^2+z^2} \cdot 2y$$

$$= 4yz \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) = 4yz \cdot e^{x^2+y^2+z^2} \cdot 2x$$

$$= 8xyz \cdot e^{x^2+y^2+z^2}$$

$$= 8xyzu$$

9. If $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)$.

**Solution:**

$$\frac{\partial \theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n e^{-r^2/4t} \cdot \left(\frac{r^2}{4t^2}\right)$$

$$= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta \qquad \text{.......(i)}$$

Also, $\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \cdot \left(-\frac{2r}{4t}\right) = -\frac{r\theta}{2t}$

$$\therefore r^2\frac{\partial \theta}{\partial r} = -\frac{r^3\theta}{2t}$$

$$\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right) = -\frac{1}{2t}\frac{\partial}{\partial r}(r^3\theta)$$

$$= -\frac{1}{2t}\left[r^3\frac{\partial \theta}{\partial r} + 3r^2\theta\right]$$

$$= -\frac{1}{2t}\left[r^3\frac{r\theta}{2t} + 3r^2\theta\right]$$

$$= -\frac{1}{2t}\left[\frac{r^4\theta}{2t} + 3r^2\theta\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{1}{2t}\left[-\frac{r^2\theta}{2t} + 3\theta\right] \qquad \text{.......(ii)}$$

$\therefore$ Equating (i) and (ii), we get, $\quad \frac{n}{t} = -\frac{3}{2t} \qquad \therefore n = -\frac{3}{2}$

10. If $u = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$ and $a^2+b^2+c^2=1$,

Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

**Solution:**

$u = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$

Differentiating $u$ partially w.r.t. $x$, $\frac{\partial u}{\partial x} = 6(ax+by+cz)a - 2x$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. $x$, $\frac{\partial^2 u}{\partial x^2} = 6a\cdot a - 2 = 6a^2 - 2$

Differentiating $u$ partially w.r.t. $y$, $\frac{\partial u}{\partial y} = 6(ax+by+cz)b - 2y$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. $y$, $\frac{\partial^2 u}{\partial y^2} = 6b\cdot b - 2 = 6b^2 - 2$

Differentiating $u$ partially w.r.t. $z$, $\frac{\partial u}{\partial z} = 6(ax+by+cz)c - 2z$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. $z$, $\frac{\partial^2 u}{\partial z^2} = 6c\cdot c - 2 = 6c^2 - 2$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2+b^2+c^2) - 6$

$= 6(1) - 6 \qquad [\because a^2+b^2+c^2=1]$

$= 0$

11. If $u = f(r)$, $r^2 = x^2+y^2+z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$.

**Solution:**

$u = f(r)$

Differentiating $u$ partially w.r.t. $x$,

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(r) = \frac{d}{dr}f(r)\cdot\frac{\partial r}{\partial x}$

$= f'(r)\cdot\frac{\partial r}{\partial x} \qquad \ldots\ldots\ldots\ldots\ldots \text{(i)}$

But $r^2 = x^2+y^2+z^2$

Differentiating $r^2$ partially w.r.t. $x$,

$2r\frac{\partial r}{\partial x} = 2x \qquad \Rightarrow \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$

Substituting in Eq. (1), $\frac{\partial u}{\partial x} = f'(r)\cdot\frac{x}{r}$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. $x$,

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$

$= f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(-\frac{1}{r^2}\right)\cdot\frac{\partial r}{\partial x}$

$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2}f'(r)\cdot\frac{x}{r}$

$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r) \qquad \ldots\ldots\ldots\ldots \text{(ii)}$

Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)$ ………….. (iii)

and $\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)$ ………….. (iv)

Adding Equations (ii), (iii) and (iv),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$$

$$= \frac{f''(r)}{r^2}\cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$$

$$= f''(r) + \frac{2f'(r)}{r}$$

12. If $= cos\theta - rsin\theta,\ y = sin\theta + rcos\theta$ , prove that $\frac{\partial r}{\partial x} = \frac{x}{r}$.

**Solution:**

To find $\frac{\partial r}{\partial x}$, first express $r$ in terms of $x$ and $y$

$$x^2 + y^2 = (cos\theta - rsin\theta)^2 + (sin\theta + rcos\theta\ )^2$$
$$= cos^2\theta - 2r\ cos\theta\ sin\theta + r^2sin^2\theta + sin^2\theta + 2r\ sin\theta cos\theta + r^2cos^2\theta$$
$$= cos^2\theta + sin^2\theta + r^2(cos^2\theta + sin^2\theta)$$
$$= 1 + r^2$$
$$\therefore r^2 = x^2 + y^2 - 1$$

Differentiating partially w.r.t. $x$,

$$2\,r\,\frac{\partial r}{\partial x} = 2x$$
$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

13. If $z = u(x,y)\ e^{ax+by}$ where $u(x,y)$ is such that $\frac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b such that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$.

**Solution:**

We have, from $z = u(x,y)e^{ax+by}$ …………… (i)

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\cdot e^{ax+by} + u\cdot e^{ax+by}\cdot a = e^{ax+by}\left(\frac{\partial u}{\partial x} + au\right) \quad \text{…………… (ii)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\cdot e^{ax+by} + u\cdot e^{ax+by}\cdot b = e^{ax+by}\left(\frac{\partial u}{\partial y} + bu\right) \quad \text{…………… (iii)}$$

Differentiating (iii) partially w.r.t. $x$,

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\cdot a\cdot\left(\frac{\partial u}{\partial y} + bu\right) + e^{ax+by}\left(\frac{\partial^2 u}{\partial x \partial y} + b\cdot\frac{\partial u}{\partial x}\right) \quad \text{…………… (iv)}$$

But since by data $\frac{\partial^2 u}{\partial x \partial y} = 0$, we get,

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left(a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu\right) \qquad \text{............... (v)}$$

Further by data $\quad \dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} + z = 0 \qquad$ ............... (vi)

Putting the values from (v), (ii), (iii) and (i) in (vi), we get,

$$e^{ax+by}\left[a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right] = 0$$

$$\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + au(b-1) - u(b-1)\right] = 0$$

$$\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (b-1) \cdot u(b-1)\right] = 0$$

Since $u \neq 0,\ \dfrac{\partial u}{\partial x} \neq 0$ and $\dfrac{\partial u}{\partial y} \neq 0$

We should have $a - 1 = 0,\ b - 1 = 0$

i.e., $a = 1,\ b = 1$

14. If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$

**Solution:**

$a^2x^2 + b^2y^2 = c^2z^2$

Differentiating partially w.r.t. $x$,

$2a^2x = 2c^2z \cdot \dfrac{\partial z}{\partial x}$

$\dfrac{\partial z}{\partial x} = \dfrac{a^2x}{c^2z}$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. $x$,

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{a^2}{c^2}\left(\dfrac{1}{z} - \dfrac{x}{z^2} \cdot \dfrac{\partial z}{\partial x}\right) = \dfrac{a^2}{c^2 z}\left(1 - \dfrac{x}{z} \cdot \dfrac{a^2x}{c^2z}\right)$

$\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{c^2z}\left(1 - \dfrac{a^2x^2}{c^2z^2}\right)$

Similarly, $\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(1 - \frac{b^2y^2}{c^2z^2}\right)$

Hence, $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(2 - \frac{a^2x^2+b^2y^2}{c^2z^2}\right)$

$$= \frac{1}{c^2z}\left(2 - \frac{c^2z^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}(2 - 1) = \frac{1}{c^2z}$$

## SOME PRACTICE PROBLEMS

1. If $u = (1-2xy+y^2)^{\frac{-1}{2}}$ then prove that
$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$ and $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right] = 0$
2. If $u = \sin(\sqrt{x}+\sqrt{y}+\sqrt{z})$ , prove that
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}(\sqrt{x}+\sqrt{y}+\sqrt{z})cos(\sqrt{x}+\sqrt{y}+\sqrt{z})$
3. If $u = \log(\tan x + \tan y + \tan z)$, then show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$
4. If $u = e^{xyz}$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial x} = (1+3xyz+x^2y^2z^2)e^{xyz}$.
5. If $u = x^3y + e^{xy^2}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.
6. If $u = \log(x^2+y^2)$ , prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.
7. If $u = x^y$ , prove that $\frac{\partial^3 u}{\partial x^2\,\partial y} = \frac{\partial^3 u}{\partial x \partial y\,\partial x}$.
8. If $z = x^y + y^x$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.
9. If $z(x+y) = (x-y)$, Find $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$.
10. $u = \tan^{-1}\left(\frac{y}{x}\right)$ Find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ .
11. If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2-y^2}{x^2+y^2}$.
12. If $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$ $[or\ \frac{1}{u^2} = x^2+y^2+z^2\ or\ u = \frac{1}{r}\ and\ r = \sqrt{x^2+y^2+z^2}]$,
then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ .
13. If $u = f\left(\frac{x^2}{y}\right)$, prove that $x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0$ and $x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$
14. If $z = \log(e^x + e^y)$, prove that $rt - s^2 = 0, where, r = \frac{\partial^2 z}{\partial x^2}$ , $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x\,\partial y}$ .
15. If $u = log(x^3 + y^3 - x^2y - xy^2)$ , prove that
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$ and $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$
16. If $u = f(r)$ & $r^2 = x^2+y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.
17. If $u = f(r^2)$ & $r^2 = x^2+y^2+z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2f''(r^2) + 6f'(r^2)$
18. If $z = \tan(y+ax) + (y-ax)^{\frac{3}{2}}$, then show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$ .
19. If $u = x\log(x+r) - r,\ r^2 = x^2+y^2$, prove that $i)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{x+r}$ $ii)\ \frac{\partial^3 u}{\partial x^3} = -\left(\frac{x}{r^3}\right)$

20. If $u\,(x,t) = a\ e^{-gx}\ \sin\ (nt\ -\ gx)$ where a, g, n are constants, satisfying the equation $\frac{\partial u}{\partial t} = a^2\,\frac{\partial^2 u}{\partial x^2}$, prove that $g = \frac{1}{a}\sqrt{\frac{n}{2}}$.

21. If $v = r^n(3cos^2\theta - 1)$ then, find the value of n so that $\frac{\partial}{\partial r}\left(r^2\ \frac{\partial v}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial v}{\partial \theta}\right) = 0$

22. If $u = e^{xyz}f\left(\frac{xy}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + z\ \frac{\partial u}{\partial z} = 2xyzu,\quad y\frac{\partial u}{\partial y} + z\ \frac{\partial u}{\partial z} = 2xyzu,$

Hence, show that $x\ \frac{\partial^2 u}{\partial z \partial x} = y\frac{\partial^2 u}{\partial z \partial y}$.

23. If $z = c\,t^{-\frac{1}{2}}\ e^{-\frac{x^2}{4a^2 t}}$, prove that $\frac{\partial z}{\partial t} = a^2\ \frac{\partial^2 z}{\partial x^2}$.

24. If $u = \frac{e^{x+y+z}}{e^x + e^y + e^z}$, show that $u_x + u_y + u_z = 2u$.

25. If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$ ,

show that $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} = 0$.

26. If $x = \cos\theta - r\sin\theta, y = \sin\theta + r\cos\theta$, prove that $\frac{\partial r}{\partial x} = \frac{x}{r}$.

27. If $x = r\cos\theta, y = r\sin\theta$ prove that

$i)\ \frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}\quad ii)\ \frac{\partial x}{\partial \theta} = r^2\ \frac{\partial \theta}{\partial x}\quad iii)\ \left(x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r}\right)^2 = x^2 + y^2.$

$iv)\ \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]\quad v)\ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

# Module 3
# Partial Differentiation and Application

## Unit 3.2
## Composite Functions

❖ **$z = f(x, y)$ as composite function of one variable :**

Let $z = f(x, y)$ and $x = \Phi(t)$, $y = \Psi(t)$ so that $z$ is function of $x$ , $y$ and $x$ , $y$ are function of third variable $t$.

The three relations define $z$ as a function of $t$. In such cases $z$ is called a **composite function of $t$**.

**e.g. (i)** $z = x^2 + y^2$ , $x = at^2$ , $y = 2at$

**(ii)** $z = x^2y + xy^2$ , $x = a\cos t$ , $y = b\sin t$

In above examples $z$ is a composite function of one variable $t$.

**Differentiation:** Let $z = f(x, y)$ posses continuous first order partial derivatives and $x = \Phi(t)$, $y = \Psi(t)$ posses continuous first order derivatives then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

❖ **$z = f(x, y)$ as composite function of two variables :**

Let $z = f(x, y)$ and $x = \Phi(u, v)$, $y = \Psi(u, v)$ so that z is function of $x$ , $y$ and $x$ , $y$ are function of $u$ , $v$.

The three relations define $z$ as a function of $u$ , $v$. In such cases $z$ is called a **composite function of $u$ , $v$.**

**e.g. (i)** $z = xy$ , $x = e^u + e^{-v}$ , $y = e^{-u} + e^v$

**(ii)** $z = x^2 - y^2$ , $x = 2u - 3v$ , $y = 3u + 2v$

In above examples $z$ is a composite function of two variables $u$ and $v$

**Differentiation:** Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \Phi(u, v)$, $y = \Psi(u, v)$ possess continuous first order partial derivatives then,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

## SOME SOLVED EXAMPLES

1. If $u = x^2y^3,\ x = \log t\ ,\ y = e^t$, find $\frac{du}{dt}$

**Solution:**

$u = x^2y^3,\ x = \log t\ ,\ y = e^t$

$\therefore u$ is Composite Function of one variable $t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$
$$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$

Substituting $x$ and $y$,

$$\frac{du}{dt} = 2(\log t)e^{3t}\cdot\frac{1}{t} + 3(\log t)^2e^{2t}\cdot e^t$$
$$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2e^{3t}$$

2. If $u = xy + yz + zx$ where $x = \frac{1}{t},\ y = e^t,\ z = e^{-t}$, find $\frac{du}{dt}$

**Solution:**

$u = xy + yz + zx,\ x = \dfrac{1}{t},\ y = e^t,\ z = e^{-t}$

$\therefore u$ is Composite Function of one variable $t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$
$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

Substituting $x,\ y$ and $z$,

$$\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$$
$$= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$

3. If $z = x^2y + y^2x\ ,\ x = at^2\ ,\ y = 2at$ find $\frac{dz}{dt}$

**Solution:**

$z = x^2y + y^2x\ ,\ x = at^2\ ,\ y = 2at$

$\therefore z$ is Composite Function of one variable $t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + y^2)(2at) + (x^2 + 2yx)(2a)$$

Substituting $x$ and $y$,

$$\frac{dz}{dt} = (2at^2.2at + (2at)^2)2at + ((at^2)^2 + 2(2at)at^2)2a$$
$$= (4a^2t^3 + 4a^2t^2)2at + (a^2t^4 + 4a^2t^3)2a$$
$$= 8a^3t^4 + 8a^3t^3 + 2a^3t^4 + 8a^3t^3$$
$$= 10a^3t^4 + 16a^3t^3$$

4. If $z = e^{xy}$, $x = t\cos t$, $y = t\sin t$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$

**Solution:**
$z = e^{xy}$, $x = t\cos t$, $y = t\sin t$
$\therefore z$ is Composite Function of one variable $t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= e^{xy}y(\cos t - t\sin t) + e^{xy}x(\sin t + t\cos t)$$

At $t = \frac{\pi}{2}$, $x = 0$, $y = \frac{\pi}{2}$

Hence, $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^2}{4}$

5. If $z = sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ find $\frac{dz}{dt}$

**Solution:**
$z = sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$

$\therefore z$ is Composite Function of one variable $t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{\sqrt{1-(x-y)^2}}3 + \frac{(-1)}{\sqrt{1-(x-y)^2}}12t^2$$
$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

Substituting $x$ and $y$,

$$\frac{dz}{dt} = \frac{3 - 12t^2}{\sqrt{1 - (3t - 4t^3)^2}}$$

6. If $x^2 = au + bv,\ \ y^2 = au - bv \ and\ z = f(x, y)$,

Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

**Solution:**

$z = f(x, y),\ \ x^2 = au + bv,\ \ y^2 = au - bv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{a}{2x} + \frac{\partial z}{\partial y}\cdot\frac{a}{2y}$$

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} \qquad \text{............ (i)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{b}{2x} + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y} \qquad \text{............ (ii)}$$

Adding Equations (i) and (ii),

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} + \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y}$$

$$= \frac{\partial z}{\partial x}\left(\frac{au+bv}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{au-bv}{2y}\right)$$

$$= \frac{\partial z}{\partial x}\left(\frac{x^2}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{y^2}{2y}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

7. If $z = f(u,\ v)\ and\ u = \log(x^2 + y^2)\ ,\ v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$

**Solution:**

$z = f(u,\ v),\ u = \log(x^2 + y^2)\ ,\ v = \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot\frac{1}{x^2+y^2}\cdot 2x + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v} \qquad \text{................ (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}\cdot\frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \text{................ (2)}$$

Subtracting Eq. (1) from Eq. (2),

Hence, $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1 + v^2)\frac{\partial z}{\partial v}$

8. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

**Solution:**

Let $l = x^2 - y^2$, $m = y^2 - z^2$, $n = z^2 - x^2$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x$$

$$\frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$$

Consider, $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x} = \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-2x)$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n} \quad \text{……….. (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y} = \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\therefore \frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \quad \text{……….. (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z} = \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

$$\therefore \frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \quad \text{……….. (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

9. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $\varphi$ is function of $x, y, z$ then prove that

$$x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z} = u\frac{\partial \varphi}{\partial u} + v\frac{\partial \varphi}{\partial v} + w\frac{\partial \varphi}{\partial w}$$

**Solution:**

$\varphi$ is function of $x, y, z$ and $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$

Hence $\varphi$ is composite function of three variables $u, v, w$

$$\frac{\partial \varphi}{\partial u} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial \varphi}{\partial x}(0) + \frac{\partial \varphi}{\partial y}\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial \varphi}{\partial z}\frac{\sqrt{v}}{2\sqrt{u}}$$

$$u\frac{\partial\varphi}{\partial u} = \frac{u\sqrt{w}}{2\sqrt{u}}\frac{\partial\varphi}{\partial y} + \frac{u\sqrt{v}}{2\sqrt{u}}\frac{\partial\varphi}{\partial z} = \frac{\sqrt{uw}}{2}\frac{\partial\varphi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial\varphi}{\partial z}$$

As, $y = \sqrt{wu},\ z = \sqrt{uv}$

$$u\frac{\partial\varphi}{\partial u} = \frac{1}{2}\left(y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right) \quad \text{..........(i)}$$

Similarly,

$$v\frac{\partial\varphi}{\partial v} = \frac{1}{2}\left(x\frac{\partial\varphi}{\partial x} + z\frac{\partial\varphi}{\partial z}\right) \quad \text{..........(ii)}$$

$$w\frac{\partial\varphi}{\partial u} = \frac{1}{2}\left(x\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}y\right) \quad \text{..........(iii)}$$

Adding (i), (ii) and (iii)

$$u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w} = \frac{1}{2}\left(y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z} + x\frac{\partial\varphi}{\partial x} + z\frac{\partial\varphi}{\partial z} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right)$$

$$= x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}$$

10. If $x = e^u cosec\, v,\ y = e^u \cot v\ and\ z$ is a function of x and y, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

**Solution:**

$z = f(x,\ y),\ x = e^u cosec\, v,\ y = e^u \cot v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u cosec\, v + \frac{\partial z}{\partial y}e^u \cot v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\left(-e^u cosec\, v \cot v\right) + \frac{\partial z}{\partial y}\left(-e^u cosec\, v\right)$$

$$\text{R.H.S} = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^u cosec^2\, v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u} cosec\, v \cot v + \right.$$
$$(-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2\left(e^{2u} cosec^2\, v \cot^2 v\right) + (-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 v +$$
$$\left.(-\sin^2 v)2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u} cosec^3\, v \cot v\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\left(cosec^2 - \cot^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\cot^2 v - cosec^2\, v\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

$= $ L.H.S.

11. If $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$, show that $\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$ where u is a $f(x,y)$ .

**Solution:**

Adding $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$,

$2x=2e^{\theta}(\cos\Phi+i\sin\Phi)$

$\therefore x=e^{\theta}\cdot e^{i\Phi}=e^{\theta+i\Phi}$

Subtracting results, $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$

$2y=2e^{\theta}(\cos\Phi-i\sin\Phi)$

$\therefore y=e^{\theta-i\Phi}$

Now, $u$ is a function of $x$, $y$ and $x$, $y$ are functions of $\theta$ and $\Phi$

$$\therefore \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}=\frac{\partial u}{\partial x}\cdot x+\frac{\partial u}{\partial y}\cdot y \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore \frac{\partial}{\partial\theta}\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$\therefore \frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (i)]}$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (ii)]}$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Also,
$$\frac{\partial u}{\partial\Phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\Phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\Phi}$$

$$=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}\cdot i+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}\cdot(-i)$$

$$=\frac{\partial u}{\partial x}\cdot ix-i\frac{\partial u}{\partial y}\cdot y \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$\therefore \frac{\partial}{\partial\Phi}\equiv i\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right) \quad \ldots\ldots\ldots\ldots \text{(v)}$$

$$\therefore \frac{\partial^2 u}{\partial\Phi^2}=\frac{\partial}{\partial\Phi}\left(\frac{\partial u}{\partial\Phi}\right)=\frac{\partial}{\partial\Phi}\left[i\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (iv)]}$$

$$=i\left[i\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (v)]}$$

$$=-\left[x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

$\therefore$ Adding the two results, (v) and (vi) we get,

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$

## SOME PRACTICE PROBLEMS

1. If $u = x^2 + y^2 + z^2$, where, $x = e^t, y = e^t \sin t, z = e^t \cos t$

   prove that $\frac{du}{dt} = 4e^{2t}$.

2. If $z = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, prove that $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}$.

3. If $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$.

4. If $u = f\,[e^{y-z}, e^{z-x}, e^{x-y}]$, then show that $u_x + u_y + u_z = 0$.

5. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

7. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$,

   prove that $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0$.

8. If $u = f\,(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

9. If $u = f\,(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$

10. If $x = u + v + w, y = uv + vw + wu, z = uvw$, *and* $\phi$ is a function of $x, y$ & $z$,

    then prove that $x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$.

11. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ *and* $\phi$ is a function of $x, y$ & $z$ then prove that,

    $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$.

12. If $z = f\,(x, y)$, $x = r\cos\theta, y = r\sin\theta$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

13. If $z = f\,(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then show that

    $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

14. If $w = \phi\,(u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$, prove that $\frac{\partial w}{\partial v} = 0$ is equivalent

    to $(x + y)\frac{\partial w}{\partial x} + (x - y)\frac{\partial w}{\partial y} = 0$.

15. If $z = f\,(x, y), x = u\cosh v, y = u\sinh v$, prove that, $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$.

16. If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, prove that

(i) $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

17. If $z = f(x, y)$, $x = e^u \sec v$, $y = e^u \tan v$,

prove that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - cos^2 v \left(\frac{\partial z}{\partial v}\right)^2\right]$

18. If $z = f(u, v)$, $u = e^x, v = e^y$, prove that $\frac{\partial^2 z}{\partial x \partial y} = uv \frac{\partial^2 z}{\partial u \partial v}$.

19. If $z = f(u, v)$, $u = lx + my, v = ly - mx$,

prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

20. If $z = f(u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$,

prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4\sqrt{u^2 + v^2}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

# Module 3
# Partial Differentiation and Application

## Unit 3.3

## Maxima and Minima of Function of Two Independent Variables

❖ **Maxima:**

A function of two variable, $f(x, y)$ is said to be maximum at point $(a, b)$ if $f(a, b) > f(a+h, b+k)$ for some $h, k$.

❖ **Maxima:**

A function of two variable, $f(x, y)$ is said to be minimum at point $(a, b)$ if

$f(a, b) < f(a+h, b+k)$ for some $h, k$.

❖ **Method To find maxima or minima of a function of two variables**

1) Given $f(x, y)$, solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for $x$ and $y$ simultaneously.
2) Suppose $(a, b)$ is solution of above equations then it is called as **Stationary Point**.
3) Calculate the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at $(a, b)$.
4) DECISION:

   For point $(a, b)$,

   a. If $rt - s^2 > 0$ and $r < 0$ or $t < 0$, $f(x, y)$ is **maximum** at$(a, b)$.
   b. If $rt - s^2 > 0$ and $r > 0$ or $t > 0$, $f(x, y)$is **minimum** at $(a, b)$.
   c. If $rt - s^2 < 0$ then $f(x, y)$ is **neither maximum nor minimum** at $(a, b)$.
      Such point is known as a saddle point.
   d. If $rt - s^2 = 0$, **test fails**.

## SOME SOLVED EXAMPLES

1. Discuss the maxima and minima $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.

**Solution:**

We have $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.

$$f_x = 3x^2 + y^2 - 24x + 21$$
$$f_y = 2xy - 4y$$

We now solve the equations $f_x = 0, f_y = 0$

$$3x^2 + y^2 - 24x + 21 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

and $2xy - 4y = 0$

$$\Rightarrow 2y(x-2) \Rightarrow x = 2 \text{ or } y = 0.$$

When x = 2, $(i)$gives

$$12 + y^2 - 48 + 21 = 0$$

$$\therefore \quad y^2 - 15 = 0 \qquad \therefore y^2 = 15 \qquad \therefore \quad y = \pm\sqrt{15}.$$

$\therefore$ The possible stationary points are $(2, \sqrt{15})$, $(2, -\sqrt{15})$

When y = 0, (1) gives

$3x^2 - 24x + 21 = 0 \Rightarrow x^2 - 8x + 7 = 0$

$$\therefore \quad (x-7)(x-1) = 0 \quad \therefore \quad x = 7, 1.$$

The other possible stationary points are (7, 0), (1, 0).

Now,

$$r = f_{xx} = 6x - 24,$$
$$s = f_{xy} = 2y,$$
$$t = f_{yy} = 2x - 4$$

Hence, all stationary points of $f(x, y)$ are $(7,0), (1,0), (2, \sqrt{15}), (2, -\sqrt{15})$

We write a tabular form and find Maxima and Minima

| Sr. No. | Point | $r$ | $t$ | $s$ | $rt - s^2$ | Conclusion |
|---|---|---|---|---|---|---|
| 1 | (7,0) | $18 > 0$ | 10 | 0 | $180 > 0$ | (7,0) is minima. |
| 2 | (1,0) | $-18 < 0$ | –2 | 0 | $36 > 0$ | (1,0) is maxima. |
| 3 | $(2, \sqrt{15})$ | $-12 < 0$ | 0 | $2\sqrt{15}$ | $-60 < 0$ | Neither maxima nor minima |
| 4 | $(2, -\sqrt{15})$ | $-12 < 0$ | 0 | $-2\sqrt{15}$ | $-60 < 0$ | Neither maxima nor minima |

Maximum Value at (1,0) is $f(1,0) = 1 + 0 - 12 - 0 + 21 + 10 = 20$

Minimum Value at (7,0) is $f(7,0)$ =343 + 0 – 588 – 0 + 147 + 10 = - 88.

2. Discuss the maxima & minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

**Solution:**

Let $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$f_x = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$f_y = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

We now solve the equations $f_x = 0, f_y = 0$ simultaneously,

$$4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0 \ldots\ldots\ldots.. (i)$$

$$4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y \ldots\ldots\ldots.. (ii)$$

On adding $(i)$ & $(ii)$, we get

$$x^3 + y^3 = 0$$

$$\therefore (x+y)(x^2 - xy + y^2) = 0$$

$$\therefore x = -y$$

Substituting $x = -y$ in eq $(i)$,

$$x^3 - x - x = 0$$

$$\therefore x^3 - 2x = 0$$

$$\therefore x(x^2 - 2) = 0$$

$\therefore x = 0$ or $x = \pm\sqrt{2}$

Since $x = -y$,

For $x = 0 \Rightarrow y = 0$

For $x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

For $x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

$\therefore$ Stationary points are (0,0), $(\sqrt{2}, -\sqrt{2})$&$(-\sqrt{2}, \sqrt{2})$.

$$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

We write a tabular form and find Maxima and Minima

| Sr. No. | Point | $r$ | $t$ | $s$ | $rt - s^2$ | Conclusion |
|---|---|---|---|---|---|---|
| 1 | (0,0) | $-4 < 0$ | $-4$ | 4 | 0 | Test fails. |
| 2 | $(\sqrt{2}, -\sqrt{2})$ | $20 > 0$ | 20 | 4 | $384 > 0$ | $(\sqrt{2}, -\sqrt{2})$is minima. |
| 3 | $(-\sqrt{2}, \sqrt{2})$ | $20 > 0$ | 20 | 4 | $384 > 0$ | $(-\sqrt{2}, \sqrt{2})$is minima. |

Minimum Value at $(\sqrt{2}, -\sqrt{2})$ *and* $(-\sqrt{2}, \sqrt{2})$ is

$$f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = 4 + 4 - 4 - 8 - 4 = -8$$

3. Find the stationary values of $x^3 + y^3 - 3a\,xy,\ a > 0$

**Solution:**

We have $f\,(x,\ y) = x^3 + y^3 - 3a\,xy$

$f_x = 3x^2 - 3ay,$

$f_y = 3y^2 - 3ax$

We now solve, $f_x = 0,\ \&\ f_y = 0.$

$$x^2 - ay = 0 \Rightarrow y = x^2/\,a$$

and $y^2 - ax = 0$

To eliminate y, we put $y = x^2/\,a$ in this equation.

$\therefore \quad x^4 - a^3x = 0 \qquad \therefore \quad x(x^3 - a^3) = 0$

Hence, $x = 0$ or $x = a$.

When $x = 0 \Rightarrow y = 0$ and when $x = a \Rightarrow y = a$.

$\therefore\ (0,\ 0)$ and $(a,\ a)$ are stationary points.

Now,

$r = f_{xx} = 6x,$

$s = f_{xy} = -3a$

$t = f_{yy} = 6y$

We write a tabular form and find Maxima and Minima

| Sr. No. | Point | $r$ | $t$ | $s$ | $rt - s^2$ | Conclusion |
|---|---|---|---|---|---|---|
| 1 | (0,0) | 0 | $-3a$ | 0 | $-9a^2 < 0$ | Neither maxima nor minima |
| 2 | $(a, a)$ | $6a > 0$, as $a > 0$ | $-3a$ | $6a$ | $27a^2 > 0$ | $(a, a)$ is minima. |

Minimum Value at $(a, a)$ is $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

4. Find the stationary values of $sin\,x \cdot siny \cdot sin\,(x + y)$.

**Solution:**

We have $f\,(x,\ y) = sin\,x \cdot \sin y \cdot \sin\,(x + y)$

$f_x = sin\,y\,[cos\,x \cdot sin\,(x + y) + sin\,x \cdot cos\,(x + y)] = \sin y \cdot \sin\,(2x + y)$

Similarly, $f_y = sin\,x \cdot sin\,(x + 2y)$

Now, we solve $f_x = 0$ *and* $f_y = 0$.

$\therefore \quad \sin y \sin(2x+y) = 0 \Rightarrow \frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0 \quad \ldots\ldots\ldots\ldots (i)$

and $\sin x \sin(x+2y) = 0 \Rightarrow \frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0 \quad \ldots\ldots\ldots\ldots (ii)$

Equating $(i)$ & $(ii)$, we get

$[\cos 2x - \cos(2x+2y)] = [\cos 2y - \cos(2x+2y)] \Rightarrow \cos 2x = \cos 2y$

$\Rightarrow x = y$

From $(i)$ we get,

$\frac{1}{2}[\cos 2x - \cos 4x] = 0 \Rightarrow \cos 2x - (2\cos^2 2x - 1) = 0$

$\Rightarrow 2\cos^2 2x - \cos 2x - 1 = 0$

$\Rightarrow \cos 2x = 1$ or $\cos 2x = -\frac{1}{2}$

$\Rightarrow 2x = 0$ or $2x = \frac{2\pi}{3} \Rightarrow x = 0$ or $x = \frac{\pi}{3}$

As $x = y$,

$y = 0$ or $y = \frac{\pi}{3}$

$\therefore$ (0, 0) and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are possible stationary points.

$r = f_{xx} = 2\sin y \cdot \cos(2x+y)$

$s = f_{xy} = \cos y \cdot \sin(2x+y) + \sin y \cdot \cos(2x+y) = \sin(2x+2y)$

$t = f_{yy} = 2\sin x \cdot \cos(x+2y)$

We write a tabular form and find Maxima and Minima

| Sr. No. | Point | $r$ | $t$ | $s$ | $rt - s^2$ | Conclusion |
|---|---|---|---|---|---|---|
| 1 | (0,0) | 0 | 0 | 0 | 0 | Test Fails |
| 2 | $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ | $-\sqrt{3} < 0$ | $-\sqrt{3}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{9}{4} > 0$ | $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is maxima. |

Maximum Value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

5. Divide 90 into three parts such that the sum of their products taken two at a time is maximum.

**Solution**:

Let three parts of the 90 are $x, y$ & $z$.

$\therefore x + y + z = 90 \ldots\ldots\ldots..(i)$

Function to be maximized $f\left(x,\ y\right) = xy + yz + zx$

$$= xy + y\,(90 - x - y) + x\,(90 - x - y) \quad from(i)$$

$$= 90x + 90y - xy - x^2 - y^2$$

$f_x = 90 - y - 2x,$

$f_y = 90 - x - 2y$

Solving $f_x = 0$ & $f_y = 0$

$\therefore \quad 2x + y = 90 \quad \& \quad x + 2y = 90$

Solving above equations we get,

$x = 30 \ \& \ \ y = 30$

Hence (30,30) is stationary point.

$r = f_{xx} = -2,$

$s = f_{xy} = -1,$

$t = f_{yy} = -2$

At (30,30)

$r = -2 < 0,\ S = -1,\ t = -2$

$rt - s^2 = 4 > 0$

Function has maxima at (30,30).

From $(i)$

$z = 90 - x - y = 30$

$\therefore$ Required three parts of the 90 are 30,30 &30.

6. A rectangular box with open top has capacity of 32 cubiccms. Find the dimensions of the box such that the material required is minimum.

**Solution:**

Let the dimensions of the box be $x, y, z$.

$\therefore Volume = V = xyz = 32 \qquad \ldots\ldots\ldots.. (i)$

Minimum material required if surface area is minimum.

Considering that the box is rectangular with open top,

$Surface\ Area = xy + 2yz + 2zx$

$$f(x,\ y) = xy + 2yz + 2zx = xy + \frac{64}{x} + \frac{64}{y} \qquad from(i)$$

$f_x = y - \frac{64}{x^2}$,

$f_y = x - \frac{64}{y^2}$

Solving $f_x = 0$ & $f_y = 0$

$y - \frac{64}{x^2} = 0 \quad \therefore 64 = x^2y \quad \therefore y = \frac{64}{x^2} \quad \ldots\ldots\ldots.. (ii)$

$x - \frac{64}{y^2} = 0 \quad \therefore \quad 64 = xy^2 \quad \ldots\ldots\ldots.. (iii)$

$\therefore \quad 64 = x \cdot \frac{(64)^2}{x^4}$

$\therefore x^3 = 64 \quad \therefore \quad x = 4$

For $x = 4$, $\quad y = \frac{64}{x^2} = 4$

Hence, (4,4) is stationary point

$r = f_{xx} = \frac{64(2)}{x^3}$,

$s = f_{xy} = 1$,

$t = f_{yy} = \frac{64(2)}{y^3}$

At (4,4), $r = 2\ , s = 1\ and\ t = 2$

$rt - s^2 = 4 - 1 = 3 > 0$, $\ \& r = 2 > 0$, So, function has minima at (4,4)

From $(i)$

$z = \frac{32}{xy} = 2$

Hence, surface area is minimum if dimensions of box are $x = 4, y = 4\ and\ z = 2$.

7. Divide 24 into three parts such that the product of the first, square of the second and cube of the third is maximum.

**Solution:**

Let three parts of the 24 are $z, y$ & $x$.

$z + y + x = 24 \quad \ldots\ldots\ldots\ldots (i)$

We want product $zy^2 \cdot x^3$ to be maximum.

Hence,

$f(x,\, y) = zy^2 \cdot x^3 = (24 - y - x)y^2x^3 \quad \ldots\ldots\ldots.. from(i) \;\; z = 24 - y - x$

$= 24y^2x^3 - y^3x^3 - y^2x^4$

$f_x = 72y^2x^2 - 3y^3x^2 - 4y^2x^3$

$f_y = 48x^3y - 3x^3y^2 - 2x^4y$

Solving $f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 72y^2x^2 - 3y^3x^2 - 4y^2x^3 = 0$

$\Rightarrow y^2x^2(72 - 3y - 4x) = 0$

$f_y = 0 \Rightarrow 48x^3y - 3x^3y^2 - 2x^4y = 0$

$\Rightarrow x^3y(48 - 3y - 2x) = 0$

As, we want maximum product $x, y, z \neq 0$

$\Rightarrow (72 - 3y - 4x) = 0 \; and \; (48 - 3y - 2x) = 0$

$\therefore \quad 4x + 3y = 72 \;\; and \;\; 2x + 3y = 48$

$\therefore \;\; x = 12, \; y = 8$

Hence (12,8) is stationary point.

$r = f_{xx} = 144y^2x - 6y^3x - 12y^2x^2$

$s = f_{xy} = 144yx^2 - 9y^2x^2 - 8yx^3$

$t = f_{yy} = 48x^3 - 6x^3y - 2x^4$

At (12,8) ,

$r = 144 \times 64 \times 12 - 6 \times 512 \times 12 - 12 \times 64 \times 144 = -36864$

$s = 144 \times 8 \times 144 - 9 \times 64 \times 144 - 8 \times 8 \times 1728 = -27648$

$t = 48 \times 1728 - 6 \times 1728 \times 8 - 2 \times 20736 = -41472$

$rt - s^2 = 1528823808 - 764411904 > 0, \ \& r < 0$

So, function has minima at (12,8)

From $(i)$

$z = 24 - y - x = 24 - 8 - 12 = 4$

$\therefore \ \ z = 4, y = \ 8, x = \ 12.$

Hence three parts of 24 are 4, 8 and 12 such that the product of the first, square of the second and cube of the third is maximum.

## SOME PRACTICE PROBLEMS

1. Find stationary points of the following functions and discuss the maxima & minima at those points.
    1) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
    2) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$
    3) $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$
    4) $x^3 y^2 (1 - x - y)$
    5) $x^2 y^3 (1 - x - y)$
    6) $xy(3a - x - y)$
    7) $x^2 y - 3x^2 - 2y^2 - 4y + 3$
    8) $y^2 + 4xy + 3x^2 + x^3$
    9) $xy(3 - x - y)$
    10) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
    11) $2(x^2 - y^2) - x^4 + y^4$
    12) $xy + a^3 \left(\frac{1}{x} + \frac{1}{y}\right)$
2. A real number $k, k > 0$ is divided into 3 parts such that the sum of their products taken two at a time is maximum. Find the numbers.
3. A rectangular box, open at top has volume V. Find dimensions of the box requiring least material for its construction.
4. Find the maximum value of $cosA\ cosB\ cosC$, where $A, B, C$ are angles of a triangle.
5. Find the maximum volume of a parallelepiped inscribed in a sphere $x^2 + y^2 + z^2 = a^2$.