Unit 2.3
System of Equations

❖ A System of Linear Equations:

Consider a system of **m linear equations in n unknowns**, say $x_1, x_2, x_3, \ldots . x_n$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$
$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3$
……………………………………………………..
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

- This system can be written compactly in matrix notation as

$$\boldsymbol{AX = B}$$

Where,

$\boldsymbol{A = [a_{ij}]_{m\,X\,n}}$ is matrix of coefficients

$\boldsymbol{B = [b_1\ \ b_2\ \ b_3 \ldots b_m]^T}$ is the column vector of order $(m \times 1)$

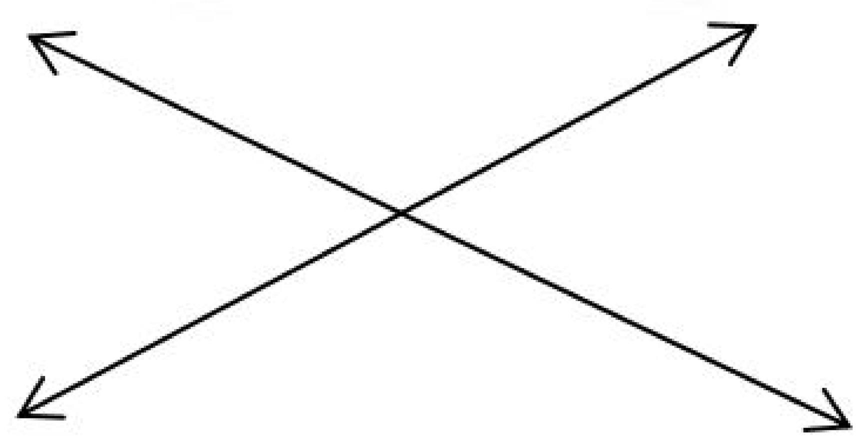
$\boldsymbol{X = [x_1\ x_2\ x_3 \ldots x_n]^T}$ is the column vector of order $(n \times 1)$

- Any vector U satisfying AU = B is said to be a **Solution** of AX = B.
- The matrix [A, B] i.e., the matrix formed by the coefficients and the constants is called the **Augmented Matrix.**
- A system $AX = B$ is
  (i) Homogeneous if $B = O$ and
  (ii) Non - Homogeneous if $B \neq O$

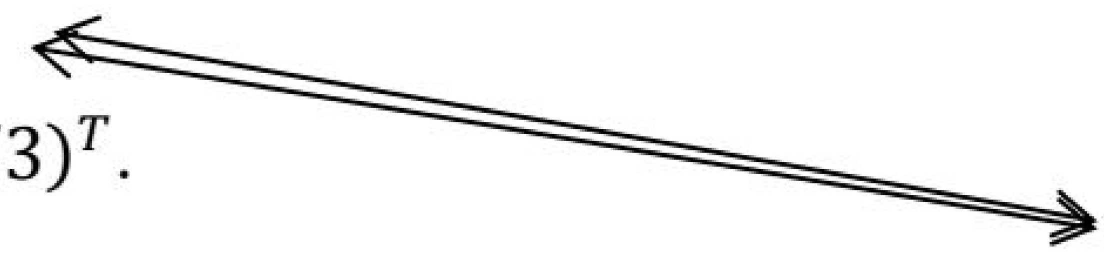
❖ Non-Homogeneous System of Linear Equations:

Different Cases for Solution ($\boldsymbol{AX = B, B \neq 0}$):

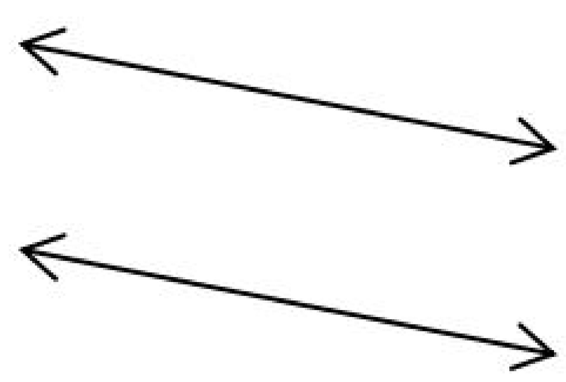
- **Consistent(at least one solution):**
  - **i. Unique Solution:**
    Ex: $4x_1 + 3x_2 = 11$,
    $4x_1 - 3x_2 = 5$
    Solving we get,
    $x_1 = 2$ and $x_2 = 1$



  - **ii. Infinitely Solution:**
    Ex: $4x_1 + 3x_2 = 11$,
    $8x_1 + 6x_2 = 22$
    has more solutions, say $(2,1)^T$ or $(0,11/3)^T$.
    $[k, \frac{11-4k}{3}]^T$ is a general solution for all k



- **Inconsistent (No Solution):**
  Ex: $4x_1 + 3x_2 = 11$,
  $8x_1 + 6x_2 = 20$
  has no solution at all.



❖ Consistency of System of Non-Homogeneous Equations:

➢ m Equations in n Unknowns:

Procedure:

- Write the given system in the matrix form $AX = B$.
- Apply row transformations on A as well as on the column matrix B i.e. on the augmented matrix [A: B] till you get a row echelon form.
- As rank of a matrix in echelon form is equal to the number of non-zero rows.
- Determine the rank of A and the rank of the augmented matrix [A: B] and find the solution by using following cases.

<u>**Case1:**</u>

The **system of equations is consistent** if and only if the coefficient matrix A and the Augmented matrix [A: B] are of the same rank.

i.e. $\boldsymbol{Rank\ of\ (A) = Rank\ of\ [A{:}\ B]}$ $(\boldsymbol{\rho(A) = \rho[A{:}\ B]})$

There are two subcases:

I: If $\boldsymbol{\rho(A) = \rho[A{:}\ B] = r = n}$, no. of unknowns, then system has **unique solution**.

II: If $\boldsymbol{\rho(A) = \rho[A{:}\ B] = r < n}$, no. of unknowns, then system has **infinitely many solutions**.

To write such solution, (n – r)parameters are used.

<u>**Case2:**</u>

The system is said to be **inconsistent** (has **no solution**) when

$\boldsymbol{Rank\ of\ (A) \neq Rank\ of\ [A{:}\ B]\ \ i.e.\ \rho(A) \neq \rho[A{:}\ B]}$

➢ Another way to solve **n equations in n Unknowns(if $|A| \neq 0$):**

- Write the given system in the matrix form $AX = B$
  Where, A is n × n matrix,
  X is n×1 matrix and B is n×1 matrix
- Check that $|A| \neq 0$ and find $A^{-1}$by any suitable method.
- Solution is given by $X = A^{-1}B$.

<u>**NOTE:**</u>

If A is singular matrix ($|A| = 0$), then this inverse method fails. In that case the system may have infinitely many solutions or none.

SOME SOLVED EXAMPLES

1. Test the consistency of the system of equations and solve if consistent.
$x+y+z=6\ ,\ x-y+2z=5\ ,\ 3x+y+z=8\ ,\ 2x-2y+3z=7$

Solution:

Let's write given system of equations in matrix form as AX = B

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 3 & 1 & 1\\ 2 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 5\\ 8\\ 7\end{bmatrix}$$

$By R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 2R_1,$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 3 & -2 & -2\\ 2 & -4 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ -1\\ -10\\ -5\end{bmatrix}$$

$By R_3 \to R_3 - R_2, R_4 \to R_4 - 2R_1,$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 0 & -0 & -3\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ -1\\ -9\\ -3\end{bmatrix}$$

$By R_4 \to 3R_4 - R_3,$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ -1\\ -9\\ 0\end{bmatrix}$$

$\therefore Rank A = Rank[A:B] = 3 = No. of Unknowns$

$\therefore$ System has unique solution.

$$\therefore x+y+z=6$$
$$-2y+z=-1$$
$$-3z=-9$$

On solving $x=1, y=2, z=3.$

2. Test the consistency of the system of equations and solve if consistent.
$2x-y+z=9\ ,\ 3x-y+z=6\ ,\ 4x-y+2z=7\ ,-x+y-z=4$

Solution:

Let's write given system of equations in matrix form as AX = B

$$\begin{bmatrix}2 & -1 & 1\\ 3 & -1 & 1\\ 4 & -1 & 2\\ -1 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 6\\ 7\\ 4\end{bmatrix}$$

$By R_1 \leftrightarrow -R_4$

$$\begin{bmatrix}1 & -1 & 1\\ 3 & -1 & 1\\ 4 & -1 & 2\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-4\\ 6\\ 7\\ 9\end{bmatrix}$$

$$By R_2 \to R_2 - 3R_1, R_3 \to R_3 - 4R_1, R_4 \to R_4 - 2R_1,$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 18 \\ 23 \\ 17 \end{bmatrix}$$

$$By R_4 \to 2R_4 - R_2,$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 18 \\ 23 \\ 16 \end{bmatrix}$$

$\therefore$ Rank A = 3

Rank[A:B] = 4

Rank A $\neq$ Rank[A:B]

$\therefore$ System is inconsistent i.e. system has no solution.

3. Test the consistency of the system of equations and solve if consistent.
   $x_1 - 2x_2 + x_3 - x_4 = 2$, $x_1 + 2x_2 + 2x_4 = 1$, $4x_2 - x_3 + 3x_4 = -1$

   Solution:
   Let's write given system of equations in matrix form as AX = B

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$By R_2 \to R_2 - R_1,$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$By R_3 \to R_3 - R_2,$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\therefore Rank A = Rank[A:B] = 2 < No. of Unknowns = 4$$

$\therefore$ System has infinitely many solutions.

No. of parameters $= 4 - 2 = 2$

$$\therefore x_1 - 2x_2 + x_3 + x_4 = 2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$4x_2 - x_3 + 3x_4 = -1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let $x_2 = s, x_4 = t$

Equation (2) $\Rightarrow x_3 = 1 + 4x_2 + 3x_4 = 1 + 4t + 3s$

Equation (1) $\Rightarrow x_1 = 2 + 2x_2 - x_3 - x_4 = 2 + 2t - (1 + 4t + 3s) - s = 1 - 2t - 3s$

$\therefore$ Solution set is $\{x_1 = 1 - 2t - 3s, x_2 = s, x_4 = t\}$

4. Test the consistency of the equations $x+y+z=3,\ x+2y+3z=4,\ x+4y+9z=6$.

Solution:

The system of equations can be written as $AX=B$, i.e. $\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\4\\6\end{bmatrix}$

$\therefore$ the augmented matrix can be written as $[A:B]=\begin{bmatrix}1&1&1&3\\1&2&3&4\\1&4&9&6\end{bmatrix}$

Applying R_2-R_1 and R_3-R_1 we get $[A:B]\sim\begin{bmatrix}1&1&1&3\\0&1&2&1\\0&3&8&3\end{bmatrix}$

Applying R_3-3R_2, we get $[A:B]\sim\begin{bmatrix}1&1&1&3\\0&1&2&1\\0&0&2&0\end{bmatrix}$

$\therefore$ Rank of $[A:B]$ = Rank of $A = 3$.
$\therefore$ The system has unique solution.
The reduced form of equations can be written as
$x+y+z=3,\quad y+2z=1,\quad 2z=0$
$\therefore$ (iii) $\Rightarrow z=0$ substituting this value in (ii) $y=1$
$\therefore$ (i) $\Rightarrow x+1+0=3\Rightarrow x=2$

Hence the solution set can be written as $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix}$.

5. Solve
$$\begin{aligned}x_1+2x_2+x_3&=2\\2x_1+4x_2+3x_3&=3\\3x_1+6x_2+5x_3&=4\end{aligned}$$

Solution:
The system of equations can be represented by the matrix equation as,
$\begin{bmatrix}1&2&1\\2&4&3\\3&6&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}$

Where the augmented matrix is $[A:B]=\begin{bmatrix}1&2&1&2\\2&4&3&3\\3&6&5&4\end{bmatrix}$

Applying elementary row transformations, the matrix $[A:B]$ can be reduced to Echelon form.

Applying R_2-2R_1 and R_3-3R_1 we get $[A:B]\sim\begin{bmatrix}1&2&1&2\\0&0&1&-1\\0&0&2&-2\end{bmatrix}$

Applying $R_3 - 2R_2$, we get $\quad [A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence $\rho(A) = \rho[A:B]$, therefore the system is consistent.
Further rank $r = 2 < 3$ (number of variables), therefore the system has infinite solutions.
$\therefore (n - r) = 3 - 2 = 1$ (free variable)

The reduced form of the linear equations can be written as,
$x_1 + 2x_2 + x_3 = 2 \quad , x_3 = -1$
Let, $x_2 = k$, an arbitrary constant.
$\therefore x_1 = 3 - 2k$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-2k \\ k \\ -1 \end{bmatrix}$ infinite solutions as k varies.

6. Are the following equations consistent?
$2x + y + z = 4$
$x + y + z = 2$
$5x + 3y + 3z = 6$

Solution:
The system of linear equations can be written in the matrix form $AX = B$
i.e $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$

$\therefore$ the augmented matrix can be written as $[A:B] = \begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 \\ 5 & 3 & 3 & 6 \end{bmatrix}$

Applying R_{12}, we get $\quad [A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 5 & 3 & 3 & 6 \end{bmatrix}$

Applying $R_2 - 2R_1$ *and* $R_3 - 5R_1$, we get $\quad [A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & -4 \end{bmatrix}$

Applying $R_3 - 2R_1$, we get $\quad [A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

$\Rightarrow rank\ of\ A = 2$ and rank of $[A:B] = 3$.
i.e $\rho(A) \neq \rho[A:B]$.
Hence the given system of linear equation is inconsistent and therefore has no solution.

7. Solve the system of equations.

$$x_1 - x_2 + 2x_3 + x_4 = 2$$
$$3x_1 + 2x_2 + x_4 = 1$$
$$4x_1 + x_2 + 2x_3 + 2x_4 = 3$$

Solution:

The system of equations can be represented by the matrix equation as,

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

where the augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{bmatrix}$$

Applying $R_3 - (R_1 + R_2)$, we get $[A:B] \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_2 - 3R_1$, we get $[A:B] \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Hence $\rho(A) = \rho[A:B] = 2$, the system is consistent

Rank $r = 2 < 4$ (Number of variables), therefore the system has infinite solutions.
$\therefore (n - r) = 4 - 2 = 2$ (free variables)
The reduced form of the linear equations is
$x_1 - x_2 + 2x_3 + x_4 = 2$,
$5x_2 - 6x_3 - 2x_4 = -5$
Let $x_3 = p$ and $x_4 = q$, an arbitrary constant for free variables

Substituting back, we get
$x_2 = \frac{1}{5}(-5 + 6p + 2q)$ and $x_1 = \frac{1}{5}(5 - 4p - 3q)$

Hence the infinite values of solution are given by,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(5 - 4p - 3q) \\ \frac{1}{5}(-5 + 6p + 2q) \\ p \\ q \end{bmatrix}$$ as p and q varies.

8. Investigate for what values of λ and μ the equations

$x+2y+z=8, \quad 2x+2y+2z=13, \quad 3x+4y+\lambda z=\mu$

have (i) no solution (ii) unique solution (iii) infinitely many solutions.

Solution:

Let's write given system of equations in matrix form as AX = B

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

Augmented matrix $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 2 & 2 & 13 \\ 3 & 4 & \lambda & \mu \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

$By R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1,$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ \mu-24 \end{bmatrix}$$

$By\, R_3 \to R_3 - R_2,$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ \mu-21 \end{bmatrix}$$

$\therefore\ [A:B] = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & \lambda-3 & \mu-21 \end{bmatrix}$

(i) If $\lambda = 3, \mu \neq 21$, then $\rho(A) \neq \rho[A:B]$

Hence, system is inconsistent and has **no solution**.

(ii) If $\lambda \neq 3, \mu$ may have any value, then $\rho(A) = \rho[A:B] = 3$ (No. of unknowns)

Hence, system is consistent and has **unique solution**.

(iii) If $\lambda = 3, \mu = 21$, then $\rho(A) = \rho[A:B] = 2 <$ No. of Unknowns $= 3$

Hence, system is consistent and has **infinite solutions**.

9. Investigate for what values of λ and μ the equations

$x+y+z=6, \quad x+2y+3z=10, \quad x+2y+\lambda z=\mu$ have

(i) no solution (ii) unique solution (iii) infinitely many solutions.

Solution:

Let's write given system of equations in matrix form as AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$$

$By R_3 \to R_3 - R_2, R_2 \to R_2 - R_1,$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu-10\end{bmatrix}$$

$$\therefore [A:B]=\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 10\\ 0 & 0 & \lambda-3 & \mu-10\end{bmatrix}$$

(i) If $\lambda = 3$, $\mu \neq 10$, then $\rho(A) \neq \rho[A:B]$

Hence, system is inconsistent and has **no solution.**

(ii) If $\lambda \neq 3$, μ may have any value, then $\rho(A) = \rho[A:B] = 3$ (No. of unknowns)

Hence, system is consistent and has **unique solution.**

(iii) If $\lambda = 3, \mu = 10$, then $\rho(A) = \rho[A:B] = 2 <$ No. of unknowns $= 3$

Hence, system is consistent and has **infinite solutions.**

10. Investigate for what values of a and b the following linear equations $x + 2y + 3z = 4,\ x + 3y + 4z = 5,\ x + 3y + az = b$, have (i) no solution, (ii) a unique solution, (iii) An infinite number of solutions.

Solution:

The system of linear equations can be written in the matrix form as $AX = B$

$$\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 4\\ 1 & 3 & a\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 5\\ b\end{bmatrix}$$

where the augmented matrix is $[A:B]=\begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 5\\ 1 & 3 & a & b\end{bmatrix}$

Applying $R_2 - R_1$ and $R_3 - R_1$, we get $[A:B]\sim\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 1 & 1\\ 0 & 1 & a-3 & b-4\end{bmatrix}$

Applying $R_3 - R_2$, $[A:B]\sim\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 1 & 1\\ 0 & 0 & a-4 & b-5\end{bmatrix}$

(i) For no solution:

If $a = 4$ and $b \neq 5$,

Then we have $\rho(A) = 2$ and $\rho[A:B] = 3$.

In this case $(A) \neq \rho[A:B]$.

(ii) For unique solution:
If $a \neq 4$, we have $\rho(A) = \rho[A:B] = 3$
In this case the system is consistent. Further, Since $\rho(A) =$ number of unknowns, therefore the system possesses unique solution if $a \neq 4$ and for any value of b.

(iii) For infinite number of solutions:
If $a = 4$ and $b = 5$, we get $\rho[A:B] = \rho(A) = 2 < 3$, the number of unknowns,
$\therefore$ System of equations is consistent and possesses an infinite number of solutions.

11. For which values of λ following set of equations is consistent? Find and solve equations for those values $x + 2y + z = 3,\ x + y + z = \lambda,\ 3x + y + 3z = \lambda^2$

Solution:
The system of linear equations can be written in the matrix form as $AX = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

where the augmented matrix is $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{bmatrix}$

Applying $R_2 - R_1$ *and* $R_3 - 3R_1$, we get $[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & -5 & 0 & \lambda^2 - 9 \end{bmatrix}$

Applying $R_3 - 5R_2$, we get $[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$

For consistency of the equation, the rank of A and rank of $[A:B]$ must be the same.

From above reduced form of $[A:B]$ it is clear that the rank of $A = 2$.
To have the rank of $[A:B] = 2$,
Consider $\lambda^2 - 5\lambda + 6 = 0$ i.e., $(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2$ *and* $\lambda = 3$

(i) Solution for $\lambda = 2$

The reduced Echelon form of $[A:B]$ is $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$

Substituting $\lambda = 2$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The reduced form of linear equations is
$x + 2y + z = 3$ *and* $y = 1$
Let $z = k$, an arbitrary constant, $\therefore x = 1 - k$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-k \\ 1 \\ k \end{bmatrix}$ has infinite values as k varies

(ii) Solution for $\lambda = 3$:
The reduced Echelon form of $[A: B]$

By substituting $\lambda = 3$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The reduced form of linear equations is
$x + 2y + z = 3$ *and* $y = 0$
Let $z = c$, an arbitrary constant. $\therefore x = 3 - c$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3-c \\ 0 \\ c \end{bmatrix}$ has infinite solutions as c varies.

12. For what values of λ the equations
$3x - 2y + \lambda z = 1,\ 2x + y + z = 2,\ x + 2y - \lambda z = -1$, will have no unique solution? Will the equations have any solutions for this value of λ.

Solution:

(Taking the equations in reverse order)

The system of linear equations can be written in the matrix form as $AX = B$

$$\begin{bmatrix} 1 & 2 & -\lambda \\ 2 & 1 & 1 \\ 3 & -2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Applying $R_2 - 2R_1$ *and* $R_3 - 3R_1$, we get $\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -3 & 1+2\lambda \\ 0 & -8 & 4\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$.............(1)

The equations have unique solutions if the coefficient matrix is non – singular.

$\therefore -12\lambda + 8 + 16\lambda \neq 0,\ 4\lambda \neq -8 \qquad \therefore \lambda \neq -2$

$\therefore$ The equations have unique solutions if $\lambda \neq -2$

and they have no unique solutions if $\lambda = -2$

Further, if $\lambda = -2$, we have from (1) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$

Applying $R_3 - \frac{8}{3}R_2$, we get $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -20/3 \end{bmatrix}$

$\therefore 0x + 0y + 0z = -20/3$ which is absurd

Also the rank of A $= 2 < the\,rank\,of\,[A,\,B] = 3$

$\therefore$ The equations are inconsistent, For $\lambda = -2$ there is no solution.

❖ **Homogeneous System of Linear Equations:**

Solutions of $AX = 0$

- The null column matrix is obviously a solution of $AX = 0$. The solution $X = 0$ is called the **trivial solution** or the **zero – solution**.
  Note: 1. These equations do not have constant term or intercept. Hence geometrically all these equations are lines passing through origin.
  2. Since, they have at least one point of intersection. We will **not** have case of **No solution** here.
- If we could find a non-zero solution $X \neq 0$, then it is called **non – trivial solution.**

➢ **Homogeneous system of m equations in n unknowns**
Procedure:

- Write the given system in the matrix form $AX = 0$..
- Apply row transformations on A till you get a row echelon form.
- As rank of a matrix in echelon form is equal to the number of non-zero rows.
- Determine the rank of A and find the solution by using following cases.

Case I:
If $\boldsymbol{\rho(A) = r = n}$, no. of unknowns then we get **unique solution**.
We get, $x_1 = x_2 \ldots = x_n = 0$ i.e., only possible solution is **zero – solution** or **trivial solution**.
Case II:
If $(\boldsymbol{A}) = \boldsymbol{r} < \boldsymbol{n}$, no. of unknowns then the system has Infinitely many **non – trivial solutions**.
The no of independent solutions i.e parameters is equal to $(n - r)$.

SOME SOLVED EXAMPLES

1. Solve the following system of equations.

$x+2y+3z=0, 2x+3y+z=0, 4x+5y+4z=0, x+2y-2z=0$

Solution:

Let's write given system of equations in matrix form as AX = 0

$$\begin{bmatrix}1&2&3\\2&3&1\\4&5&4\\1&2&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$By\ R_2 \to R_2-2R_1, R_3 \to R_3-4R_1, R_4 \to R_4-R_1,$

$$\begin{bmatrix}1&2&3\\0&-1&-5\\0&-3&-8\\0&0&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$By\ R_3 \to R_3-3R_2,$

$$\begin{bmatrix}1&2&3\\0&-1&-5\\0&0&7\\0&0&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$By\ R_4 \to 7R_4+5R_1,$

$$\begin{bmatrix}1&2&3\\0&-1&-5\\0&0&7\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$Since\ Rank(A)=\rho(A)=3=$ No. of unknowns,

Hence, the system has a trivial solution i. e. $x=0, y=0, z=0$.

2. Find the solution of the system given by
$$x_1-x_2+x_3=0$$
$$x_1+2x_2+x_3=0$$
$$2x_1+x_2+3x_3=0$$

Solution:

The system can be written as $\begin{bmatrix}1&-1&1\\1&2&1\\2&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Applying elementary row transformations on A we will obtain the Echelon form.

Applying R_2-R_1 and R_3-2R_1, we have $\begin{bmatrix}1&-1&1\\0&3&0\\0&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Since $\rho(A) = 3 = no. of\ unknowns$
Hence we get unique solution.
Therefore there exists a trivial solution $x_1 = x_2 = x_3 = 0$

3. Solve the following system of linear equations $\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 0 \\ 2x_1 + x_2 + 5x_3 + 8x_4 &= 0 \\ x_1 + 4x_2 + 6x_3 - 3x_4 &= 0 \end{aligned}$

Solution:

The system can be written as $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 2R_1\ and\ R_3 - R_1$, we have $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 + \frac{2}{3}R_2$, we have $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Hence $\boldsymbol{\rho(A) = 2 < 4}$ (Number of unknowns).
So the system has infinitely non – trivial solution.
$(n - r) = (4 - 2) = 2$ free variables

Then reduced from of system of equations is
$x_1 + 2x_2 + 4x_3 + x_4 = 0$ ……………..(i)
$-3x_2 - 3x_3 + 6x_4 = 0$ …………….(ii)

(ii) $\Rightarrow x_2 + x_3 - 2x_4 = 0 \quad \Rightarrow \boldsymbol{x_2 = -x_3 + 2x_4}$
Substituting in (i), we get $x_1 + 2(-x_3 + 2x_4) + 4x_3 + x_4 = 0$
$\Rightarrow \boldsymbol{x_1 = -2x_3 - 5x_4}$

Let $x_3 = k_1$ and $x_4 = k_2$, where k_1 and k_2 are some parameters.
∴ We have $x_1 = -2k_1 - 5k_2\ and\ x_2 = -k_1 + 2k_2$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2k_1 - 5k_2 \\ -k_1 + 2k_2 \\ k_1 \\ k_2 \end{bmatrix}$ has an infinite solution as $k_1\ and\ k_2$ vary.

4. Solve $\begin{aligned} 3x_1 + 4x_2 - x_3 - 9x_4 &= 0 \\ 2x_1 + 3x_2 + 2x_3 - 3x_4 &= 0 \\ 2x_1 + x_2 - 14x_3 - 12x_4 &= 0 \\ x_1 + 3x_2 + 13x_3 + 3x_4 &= 0 \end{aligned}$

Solution:

The system can be written as $AX = 0$ i.e., $\begin{bmatrix} 3 & 4 & -1 & -9 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -12 \\ 1 & 3 & 13 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying R_{14}, we get $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -12 \\ 3 & 4 & -1 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 2R_1,\ R_3 - 2R_1,\ R_4 - 3R_1$, we get $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -18 \\ 0 & -5 & -40 & -18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_4 - R_3$, we get $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $\frac{R_2}{-3}$, we get $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & -5 & -40 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 + 5R_2$, we get $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Hence $\rho(A) = 3 < 4$ (number of variables).
$\therefore$ the system has infinite non trivial solutions.

The reduced form of system of equations can be written as
$x_1 + 3x_2 + 13x_3 + 3x_4 = 0$ ....(i)
$x_2 + 8x_3 + 3x_4 = 0$ ....(ii)
$-3x_4 = 0$ ....(iii)
Now, since $n = 4,\ r = 3$,
$(n - r) = (4 - 3) = 1$ free variable
(iii) $\Rightarrow x_4 = 0,\quad Let\ x_3 = k$ (arbitrary)
$\therefore$ (ii) $\Rightarrow x_2 + 8k + 0 = 0 \ \Rightarrow\ x_2 = -8k$
And (i) $\Rightarrow x_1 - 24k + 13k + 0 = 0 \Rightarrow x_1 = 11k$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11k \\ -8k \\ k \\ 0 \end{bmatrix}$ will have infinite many solutions as k varies.

5. Determine the values of λ for which the following system of equations possess a non – trivial solution and obtain these solutions for each value of λ.

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

Solution:

The system can be written as $\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

System will possess non trivial solution if rank of coefficient matrix is less than number of variables

i.e., $r < 3$ if $|A| = 0$

$\therefore \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} = 0$

$\therefore 3(-2\lambda + 12) - (4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$

$\therefore \lambda^2 + 8\lambda - 9 = 0 \qquad \therefore (\lambda + 9)(\lambda - 1) = 0$

$\therefore \lambda = -9$ *and* $\lambda = 1$ for which the system possesses a non – trivial solution.

Case (i) For $\lambda = -9$

the system can be written as $\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - \frac{4}{3}R_1,\ R_3 + 6R_1$, we have $\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 10 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 + 3R_2$, we have $\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ The reduced form of system of equations is

$3x_1 + x_2 + 9x_3 = 0$ and $-\left(\frac{10}{3}\right)x_2 - 15x_3 = 0$

$\therefore x_2 = -(9/2)x_3$

$\Rightarrow 3x_1 = (9/2)x_3 - 9x_3 = -(9/2)x_3$

$\therefore x_1 = -(3/2)x_3$

Let $x_3 = a$ (arbitrary)

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(3/2)a \\ -(9/2)a \\ a \end{bmatrix}$ has infinite solutions as 'a' varies

Case (i) For $\lambda = 1$

System can be written as $\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_1 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 4R_1,\ R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 10 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∴ The reduced form of system of equations is

$$x_1 - 3x_2 - 2x_3 = 0\ , \quad 10x_2 + 5x_3 = 0,$$

$$\therefore x_2 = -(1/2)x_3$$

$$\therefore x_1 - 3x_2 - 2x_3 = 0$$

$$\Rightarrow x_1 = -(3/2)x_3 + 2x_3 = (1/2)x_3$$

$$\therefore x_1 = (1/2)x_3$$

Let $x_3 = b$ (arbitrary)

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/2)b \\ -(1/2)b \\ b \end{bmatrix}$ has an infinite solution as 'b' varies.

6. Discuss for all values of k, the following system of equations possesses trivial and non – trivial solutions

$$2x + 3ky + (3k+4)z = 0$$

$$x + (k+4)y + (4k+2)z = 0$$

$$x + 2(k+1)y + (3k+4)z = 0$$

Solution:

The given system of equations can be written as $AX = O$

i.e., $\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying R_{12}, we have $\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 2R_1,\ R_3 - R_1$, we have

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\ldots..(1)$$

If the given system of equations is to posses non – trivial Solutions then the coefficient matrix A must be of rank less than 3. i.e., $|A| = 0$

i.e $(k-8)(-k+2) + 5k(k-2) = 0$

i.e $-k^2 + 10k - 16 + 5k^2 - 10k = 0$

i.e $4k^2 - 16 = 0$ i.e $k^2 = 4$ i.e $k = \pm 2$

Now three cases arise:
Case (i): If $k \neq \pm 2$,
then given system of equations possesses only trivial solution
i.e. $x = y = z = 0$ is the only solution.

Case (ii): If $k = 2$,

then (1) $\Rightarrow \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
The reduced form of the equations is
$x + 6y + 10z = 0\ ,\ 6y + 10z = 0$
Let $z = c$ (arbitrary)
Then we have $y = \frac{-5}{3} c \ and \ x = 0$
Hence the general solution of the given system can be written as $x = 0,\ y = \frac{-5}{3} c,\ z = c$
Or $x = 0,\ y = -5c',\ z = 3c'\ ,\ c' = \frac{c}{3}$ is parameter

Case (iii): If $k = -2$

then (1) $\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $\frac{R_2}{10}$ and $\frac{R_3}{4}$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - R_2$, $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
The reduced form of the equations is
$x + 2y - 6z = 0\ ,\ \ -y + z = 0$
$\Rightarrow y = z \ and \ x = 4z,$
Let $z = b$,
then $x = 4b,\ y = b,\ z = b$ is the general solution.

Unit 2.4
Linearly Dependent and Independent vectors

❖ Vectors:

- **Definition:** An ordered set of n elements x_i is called n – dimensional vector or a **vector of order n** denoted by X.
  $X = [x_1 \ \ x_2 \ \ x_3 \ldots..x_n]$
  The elements $x_1, \ x_2, x_3, \ldots., x_n$ are called **components** of X.
  X is denoted by row matrix or column matrix.
  It is more convenient to denote it as column matrix
  $$X = [x_1 \ \ x_2 \ \ x_3 \ldots\ldots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$$
- The vector, all of whose components are zero, is called a **zero or null vector** and is denoted by 0.

❖ Linear Dependence:

The set of vectors $X_1, \ X_2, \ X_3, \ldots.X_m$ is said to be **Linearly Dependent** if there exist m scalars $k_1, \ k_2, \ k_3 \ldots.k_m$, not all zero, such that $k_1X_1 + k_2X_2 + k_3X_3 + \cdots.+k_mX_m = 0$

NOTE:
If the set of vectors $X_1, X_2, \ \ldots\ldots X_m$ is linearly dependent then any one vectors can be expressed as the linear combination of other vectors.
As $X_1, X_2, \ \ldots\ldots X_m$ are linearly dependent vectors
i.e. if $k_1X_1 + k_2X_2 + k_3X_3 + \cdots..+k_mX_m = 0$ then there is at least one $k_i \neq 0$.
Say $k_1 \neq 0$,
then $-k_1X_1 = k_2X_2 + k_3X_3 + \cdots + k_mX_m$
$\therefore X_1 = \mu_2X_2 + \mu_3X_3 + \cdots..+\mu_mX_m$
Where, $\mu_i = -\frac{k_i}{k_1}; \quad i = 2,3,4, \ldots., m$
Hence, X_1 is expressed as linear combination of $X_2, \ \ldots.X_m$

❖ Linear Independence:

The set of vectors $X_1, \ X_2, \ X_3, \ldots.X_m$ is said to be **Linearly Independent** if there exist m scalars $k_1, \ k_2, \ k_3 \ldots.k_m$ such that
if $k_1X_1 + k_2X_2 + k_3X_3 + \cdots..+k_mX_m = 0$
$\Rightarrow k_i = 0$ for all $i = 1,2, \ldots., m$
Then $X_1, X_2, X_3 \ldots\ldots X_m$ are said to be Linearly Independent.

❖ Procedure to Check Linear Dependence and Independence:

Given set of vectors $X_1, X_2, \ \ldots \ldots X_m$

1. $k_1X_1 + k_2X_2 + k_3X_3 + \cdots .. + k_mX_m = 0$ will form a homogeneous system of equation, say AX = 0. Here unknowns are $k_1,\ k_2,\ \ldots . k_m$ and coefficient matrix is made up of vectors $X_1, X_2, \ \ldots \ldots X_m$ as columns.
2. Apply **row transformations only** on A and reduce the coefficient matrix A to **row echelon form**. Then find the rank(A) i.e. $\rho(A) = r$

   Case(i): when $\rho(A) = r = number\ of\ unknowns$ then system has only trivial solution and vectors are **Independent**.

   Case(ii): when rank A (r) is less than number of variables then system has infinite non-trivial solutions and they can be obtained by assigning $n - r$ variables as parameter and vectors are **Dependent**.

SOME SOLVED EXAMPLES

1. Are the vectors $X_1 = [1\ \ 3\ \ 4\ \ 2],\ \ X_2 = [3\ \ -5\ \ 2\ \ 6],\ X_3 = [2\ \ -1\ \ 3\ \ 4]$ linearly dependent? If so, express X_1 as a linear combination of the others.

Solution:

Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ …………… (i)

$$k_1[1\ 3\ 4\ 2] + k_2[3\ \ -5\ 2\ 6] + k_3[2\ \ -1\ 3\ 4] = [0\ 0\ 0\ 0]$$

$$\therefore k_1 + 3k_2 + 2k_3 = 0,\ 3k_1 - 5k_2 - k_3 = 0,$$

$$4k_1 + 2k_2 + 3k_3 = 0,\ 2k_1 + 6k_2 + 4k_3 = 0$$

which can be written in matrix form as $\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 3R_1,\ R_3 - 4R_1,\ R_4 - 2R_1$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $(-1/7)R_2,\ (-1/5)R_3$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$k_1 + 3k_2 + 2k_3 = 0,\ \ \ 2k_2 + k_3 = 0$

if we put $k_3 = -2t$, we get $k_2 = t,\ k_1 = t$

Now, from (i), we get, $tX_1 + tX_2 - 2tX_3 = 0$ $\quad\quad \therefore X_1 + X_2 - 2X_3 = 0$

Since $k_1,\ k_2,\ k_3$ are not all zero, the vectors are linearly dependent and $X_1 = -X_2 + 2X_3$

2. Examine whether the vectors
$X_1 = [3 \quad 1 \quad 1]$, $X_2 = [2 \quad 0 \quad -1]$, $X_3 = [4 \quad 2 \quad 1]$ are linearly dependent or independent.

Solution:

Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$……………(i)

$\therefore 3k_1 + 2k_2 + 4k_3 = 0$, $\quad k_1 + 0k_2 + 2k_3 = 0, \quad k_1 - k_2 + k_3 = 0$

This is a homogeneous system of equations $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_{13,}$ *we get* $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1$, $R_3 - 3R_1$, we get $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - 5R_2$, we get $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow k_1 - k_2 + k_3 = 0$, $\quad k_2 + k_3 = 0, \quad -4k_3 = 0$

$\Rightarrow k_3 = 0$ and hence $k_2 = 0$ and $k_1 = 0$

Thus non zero values of k_1, k_2, k_3 do not exist which can satisfy equation (i).
Hence by definition the given system of vectors is linearly independent.

3. Show that the vectors X_1, X_2, X_3 are linearly independent and vector X_4 depends upon them, where, $X_1 = [1\ 2\ 4]$, $X_2 = [2 - 1\ 3]$, $X_3 = [0\ 1\ \ 2]$, $X_4 = [-3\ 7\ 2]$

Solution:

Consider $\quad k_1X_1 + k_2X_2 + k_3X_3 = 0$

$\therefore k_1[1\ \ 2\ \ 4] + k_2[2\ -1\ \ 3] + k_3[0\ \ 1\ \ 2] = [0\ \ 0\ \ 0]$

$\therefore k_1 + 2k_3 + 0k_3 = 0, \quad 2k_1 - k_2 + k_3 = 0, \quad 4k_1 + 3k_2 + 2k_3 = 0,$

$\therefore \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - 2R_1$, $R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - R_2$ we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore k_1 + 2k_2 = 0, \quad -5k_2 + k_3 = 0, \quad k_3 = 0$

$\therefore k_3 = 0, \quad k_2 = 0, \quad k_1 = 0.$

Since, the rank = 3 = the number of unknowns,
$\therefore$ only trivial solution is possible.
$\therefore X_1, X_2, X_3$ are linearly independent.

Now, consider the equation

$k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = 0$ ………….(i)

$\therefore k_1[1\ 2\ 4] + k_2[2\ -1\ 3] + k_3[0\ 1\ 2] + k_4[-3\ \ 7\ \ 2] = 0$

$k_1 + 2k_2 + 0k_3 - 3k_4 = 0,$

$2k_1 - k_2 + k_3 + 7k_4 = 0,$

$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$

$$\therefore \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 - 2R_1,\ R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - R_2$ we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore k_1 + 2k_2 - 3k_4 = 0, \quad -5k_2 + k_3 + 13k_4 = 0\ , \quad k_3 + k_4 = 0$

Let $k_4 = t \quad \therefore k_3 = -t$

$\therefore -5k_2 - t + 13t = 0 \qquad \therefore k_2 = \frac{12}{5}t$

$\therefore k_1 + \frac{24}{5}t - 3t = 0 \qquad \therefore k_1 = -\frac{9}{5}t$

Putting the values of $k_1,\ k_2,\ k_3, k_4$ in (i) we get,

$$-\frac{9}{5}tX_1 + \frac{12}{5}tX_2 - tX_3 + tX_4 = 0$$

$$\therefore 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$$

Hence, $X_1,\ X_2, X_3,\ X_4$ are linearly dependent and $X_4 = \frac{9}{5}X_1 - \frac{12}{5}X_2 + X_3$

4. Examine linearly dependence or independence of the following vectors. Find relation between them if they are dependent.
   $[1,1,-1]\ ,\ [2,3,-5]\ ,\ [2,-1,4]$

   Solution:

   Let $X_1 = [1,1,-1]\ , X_2 = \ [2,3,-5]\ , X_3 = \ [2,-1,4]$

   Consider $k_1X_1 + k_2X_2 + k_3X_3 = 0$

   $$\therefore k_1[1,1,-1]\ + k_2\ [2,3,-5]\ + k_3\ [2,-1,4] = [0,0,0]$$

   So we get homogeneous system of linear equations,

   $$k_1 + 2k_2 + 2k_3 = 0$$
   $$k_1 + 3k_2 - k_3 = 0$$
   $$-k_1 - 5k_2 + 4k_3 = 0$$

   Let's write it in matrix form as AX = 0,

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & -5 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$By R_3 \to R_2 - R_1, R_3 \to R_3 + R_1,$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$By R_3 \to R_3 + 3R_2,$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Rank (A) $= 3 =$ No. of Unknowns $= 3$

$\therefore$ System has a trivial solution.

$\therefore k_1 = 0 \ , k_2 = 0 \ , k_3 = 0$

Thus, the vectors are linearly independent.

5. Examine linearly dependence or independence of the following vectors. Find relation between them if they are dependent.
   $[1,2,-1,0]$, $[1,3,1,2]$, $[4,2,1,0]$, $[6,1,0,1]$
   Solution:
   Let $X_1 = [1,2,-1,0]$, $X_2 = [1,3,1,2]$, $X_3 = [4,2,1,0]$, $X_4 = [6,1,0,1]$

   Consider $k_1X_1 + k_2X_2 + k_3X_3 + +k_4X_4 = 0$

   $\therefore k_1 [1,2,-1,0] + k_2[1,3,1,2] + k_3 [4,2,1,0] + k_4[6,1,0,1] = [0,0,0,0]$

   So we get homogeneous system of linear equations,

$$k_1 + k_2 + 4k_3 + 6k_4 = 0$$
$$2k_1 + 3k_2 + 2k_3 + k_4 = 0$$
$$-k_1 + k_2 + k_3 = 0$$
$$2k_2 + k_4 = 0$$

   Let's write it in matrix form as AX = 0

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$By \ \ R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1,$ $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$By\ R_3 \to R_3 - 2R_2, R_4 \to R_4 - R_3,$ $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$By R_4 \to 17R_4 + 5R_3,$ $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & 55 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$Rank(A) = 4 =$ No. of unknowns
$\therefore$ System has a trivial solution.
$\therefore k_1 = 0\ , k_2 = 0\,, k_3 = 0, k_4 = 0$

Thus, the vectors are linearly independent.

6. Show that the rows of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 6 & 2 & -4 \\ -6 & 0 & -3 & -4 \end{bmatrix}$ are linearly dependent and express any row as a linear combination of other rows.

Solution:

Let's represent the rows of given matrix in vector form.

Let $X_1 = [1,0,2,1]\,, X_2 = [3,1,2,1]\ , X_3 = [4,6,2,-4]\,, X_4 = [-6,0,-3,-4]$

Consider $k_1X_1 + k_2X_2 + k_3X_3 + +k_4X_4 = 0$

$\therefore k_1\,[1,0,2,1] + k_2[3,1,2,1] + k_3\,[4,6,2,-4] + k_4[-6,0,-3,-4] = [0,0,0,0]$

So we get homogeneous system of linear equations,

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$k_2 + 6k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 - 3k_4 = 0$$
$$k_1 + k_2 - 4k_3 - 4k_4 = 0$$

Let's write it in matrix form as AX = 0

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$ByR_4 \to R_4 - R_1, R_3 \to R_3 - 2R_1,$ $\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & -8 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$By R_3 \to R_3 + 4R_2, R_4 \to R_4 + 2R_2, \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$By R_3 \to {R_3}/{9}, R_4 \to {R_4}/{2}, \quad \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$By R_4 \to R_4 - R_3, \quad \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Rank(A) $= 3 <$ No. of Unknowns $= 4$

$\therefore$ System has non trivial solution.

No. of Parameters $= 4 - 3 = 1$.

Writing matrix form again in equation form as

$$\mathrm{k}_1 + 3k_2 + 4k_3 - 6k_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$k_2 + 6k_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$2k_3 + k_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let $k_3 = t$, equation (3) $\Rightarrow k_4 = -2t$

equation (2) $\Rightarrow k_2 = -6t$

equation (1) $\Rightarrow k_1 = -3k_2 - 4k_3 + 6k_4 = 2t$

Thus, the vectors are linearly dependent.

We have $k_1X_1 + k_2X_2 + k_3X_3 + +k_4X_4 = 0$

Substituting above values we get,

$$2tX_1 - 6tX_2 + tX_3 - 2tX_4 = 0$$

$$\therefore 2X_1 - 6X_2 + X_3 - 2X_4 = 0$$

$$\therefore X_3 = -2X_1 + 6X_2 + 2X_4$$

SOME PRACTICE PROBLEMS

Non-Homogeneous Equation

1. Test for consistency the following set of equations and obtain the solution if consistent.

(i) $3x+3y+2z=1,\ x+2y=4,\ 10y+3z=-2,\ 2x-3y-z=5$

(ii) $2x-y-z=2,\ x+2y+z=2,\ 4x-7y-5z=2$

(iii) $2x_1+2x_2=-11,\ 6x_1+20x_2-6x_3=-3,\ 6x_2-18x_3=-1$

(iv) $x-2y+3t=0,\ 2x+y+z+t=-4,\ 4x-3y+z+7t=8$

(v) $x_1+x_2+x_3=4,\ 2x_1+5x_2-2x_3=3,\ x_1+7x_2-7x_3=5$

(vi) $5x_1-3x_2-7x_3+x_4=10, -x_1+2x_2+6x_3-3x_4=-3,$
$x_1+x_2+4x_3-5x_4=0$

(vii) $-x_2+x_3=4,\ 3x_1-x_2+x_3=6,\ 4x_1-x_2+2x_3=7, -x_1+x_2-x_3=9$

(viii) $x+2y=1,\ -3x+2y=-2,\ -x+6y=0$

(ix) $2x-y+3z=9,\ x+y+z=6,\ x-y+z=2$

(x) $x+y+4z=6,\ 3x+2y-2z=9,\ 5x+y+2z=13$

(xi) $x_1+x_2+x_3=4,\ 2x_1+5x_2-2x_3=3$

(xii) $x+y+z=6\ ,\ x-y+2z=5\ ,\ 3x+y+z=8\ ,\ 2x-2y+3z=7$

(xiii) $2x-y+z=9\ ,\ 3x-y+z=6\ ,\ 4x-y+2z=7\ , -x+y-z=4$

(xiv) $x_1-2x_2+x_3-x_4=2\ ,\ x_1+2x_2+2x_4=1\ ,\ 4x_2-x_3+3x_4=-1$

(xv) $2x-y+z=8\ ,\ 3x-y+z=6\ ,\ 4x-y+2z=7\ , -x+y-z=4$

(xvi) $5x+3y+7z=4\ ,\ 3x+26y+2z=9\ ,\ 7x+2y+10z=5$

(xvii) $x-2y+3t=2\ ,\ 2x+y+z+t=-4\ ,\ 4x-3y+z+7t=8$

2. Show that the system $\begin{aligned} 2x_1-3x_2+7x_3&=5\\ 3x_1+x_2-3x_3&=13\\ 2x_1+19x_2-47x_3&=32\end{aligned}$ is inconsistent.

3. Investigate for what values of a and b the simultaneous equations $\begin{aligned} 2x-y+3z&=2\\ x+y+2z&=2\\ 5x-y+az&=b\end{aligned}$

will have **(i)** no solution: **(ii)** a unique solution: **(iii)** an infinite number of solutions.

4. Determine the values of a & b such that system

$3x-2y+z=b\ ,\ 5x-8y+9z=3,\ 2x+y+az=-1$ has

(i) no solution (ii) unique solution (iii) infinite number of solutions.

5. Investigate for what values of λ and μ the simultaneous equations $\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu\end{aligned}$

will have **(i)** no solution: **(ii)** a unique solution: **(iii)** an infinite number of solutions.

6. Investigate for what values of λ and μ the equations
$2x+3y+5z=9\,,\ 7x+3y-2z=8\,,\ 2x+3y+\lambda z=\mu$ has
(i) no solution (ii) unique solution (iii) infinite number of solutions.

7. Find the values of λ for which the system of equations
$$\begin{aligned} x+y+4z&=1\\ x+2y-2z&=1\\ \lambda x+y+z&=1 \end{aligned}$$
will have **(i)** a unique solutions **(ii)** no solution

8. Find values of λ for which the set of equations
$$\begin{aligned} x_1+2x_2+x_3&=3\\ x_1+x_2+x_3&=\lambda\\ 3x_1+x_2+3x_3&=\lambda^2 \end{aligned}$$
are consistent and solve equations for those values.

9. For what value of λ the equations

$x\ +\ y\ +\ z=1,\ x\ +\ 2y+\ 4z=\lambda,\ x\ +\ 4y+10z=\ \lambda^2$

have a solution and solve them completely in each case.

10. Show that the system of equation
$$\begin{aligned} -2x+y+z&=a\\ x-2y+z&=b\\ x+y-2z&=c \end{aligned}$$
have no solution unless $a+b+c=0$, in which case they have infinitely many solutions. Find these solutions when $a=1, b=1, c=\ -2$.

Homogeneous Equation

11. Find (trivial or non-trivial) solutions of the following linear equations.

(i) $x_1-\ x_2+2x_3=0,\ x_1+\ \ 2x_2+\ \ x_3=0, 2x_1+x_2+3x_3=0$

(ii) $x_1+2x_2+3x_3+x_4=0,\ x_1+x_2-x_3-x_4=0, 3x_1-x_2+2x_3+3x_4=0$

(iii) $x_1-2x_2+x_3=0, x_1-2x_2-x_3=0, 2x_1-4x_2-5x_3=0$

(iv) $2x_1+3x_2-x_3+x_4=0, 3x_1+2x_2-2x_3+2x_4=0, 5x_1-4x_3+4x_4=0$

(v) $x_1+2x_2+3x_3=0\,,\ 2x_1+3x_2+x_3=0$

(vi) $4x_1+5x_2+4x_3=0\,,\ x_1+2x_2-2x_3=0$

(vii) $x_1+x_2-x_3+x_4=0\,,\ x_1-x_2+2x_3-x_4=0\,,\ 3x_1+x_2+x_4=0$

(viii) $2x-2y-5z=0,\ \ 4x-y+z=0,\ \ 3x-2y+3z=0,\ \ x-3y+7z=0$

12. Find the solution of the system given by
$$\begin{aligned} x_1-2x_2+x_3&=0\\ x_1-2x_2-x_3&=0\\ 2x_1-4x_2-5x_3&=0 \end{aligned}$$
Also find the relation between column vectors of coefficient matrix.

13. Solve the following system of linear equation
$$\begin{aligned} x_1-2x_2-x_3&=0\\ -2x_1+4x_2+2x_3&=0\\ -3x_1-x_2+7x_3&=0\\ 4x_1+3x_2+6x_3&=0 \end{aligned}$$

14. Find k if the system $\begin{matrix} 2x-3y+4z=0 \\ 3x+4y+6z=0 \\ 4x+5y+kz=0 \end{matrix}$ has non trivial solution

15. For what values of λ the following system of equations possesses a non-trivial solution? Obtain the general solution in each case.

$2x-2y+z=\lambda x$, $2x-3y+2z=\lambda y$, $-x+2y=\lambda z$

16. If the following system has non – trivial solutions, prove that $a+b+c=0$ or $a=b=c$, Where $ax+by+cz=0, bx+cy+az=0, cx+ay+bz=0$. Find the non – trivial solution when the condition is satisfied.

Linear Dependence & Independence Of Vectors

17. Are the following vectors linearly dependent? If so find the relation between them.

(i) $X_1=[1\ 1\ 1\ 3], X_2=[1\ 2\ 3\ 4], X_3=[2\ 3\ 4\ 9]$
(ii) $X_1=[2\ 3\ 4-2], X_2=[-1-2-2\ 1], X_3=[1\ 1\ 2-1]$
(iii) $X_1=[1\ 2\ 1\ ], X_2=[2\ 1\ 4], X_3=[4\ 5\ 6], X_4=[1\ 8-3]$
(iv) $X_1=[1-1\ 1\ ], X_2=[2\ 1\ 1], X_3=[3\ 0\ 2]$
(v) $X_1=[1\ 2\ 3\ ], X_2=[2-2\ 6]$
(vi) $X_1=[3\ 1-4\ ], X_2=[2\ \ 2-3], X_3=[0-4\ \ 1]$
(vii) $X_1=[1\ 1\ 1\ 3], X_2=[1\ 2\ 3\ 4], X_3=[2\ 3\ 4\ 7]$
(viii) $X_1=[1\ 1-1\ 1], X_2=[1-1\ 2-1], X_3=[3\ 1\ 0\ 1]$
(ix) $X_1=[1-1\ 2\ 0], X_2=[2\ 1\ 1\ 1], X_3=[3-1\ 2-1], X_4=[3\ 0\ 3\ 1]$
(x) $[1,2,-1,0]$, $[1,3,1,2]$, $[4,2,1,0]$, $[6,1,0,1]$
(xi) $[2,-1,3,2]$, $[1,3,4,2]$, $[3,-5,2,2]$
(xii) $[1,2,-1,0]$, $[1,3,1,3]$, $[4,2,1,-1]$, $[6,1,0,-5]$
(xiii) $[1,\ 3,\ 4,\ 2]$, $[3,\ -5,\ 2,\ 6]$, $[2,-1,\ 3,\ 4]$
(xiv) $[3,\ 1,\ 1]$, $[2,\ 0,-1]$, $[4,\ 2,\ 1]$

18. Show that the rows of the following matrices are linearly dependent and express any row as a linear combination of other rows.

(i) $A=\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ **(ii)** $A=\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 6 & 2 & -4 \\ -6 & 0 & -3 & -4 \end{bmatrix}$