Unit 2.5

System of Equations using Numerical Methods

❖ **A System of Linear Equations:**

Definition: Numerical method in which we start with some random (Initial) solution of system of equations and use previous iteration values in rearranged equations to find next values are called **Iterative Methods**.

We will see following Two Iterative Methods:

1. **Gauss Jacobi's Method:**
   In this method, we will use **previous iteration values** to calculate next value.

2. **Gauss Seidel Method:**
   In this method, we will use **latest two values** instead of previous iteration values to calculate next value.

Note:

i. A sufficient condition for method to converge is that the coefficient matrix A of order n should be **strictly or irreducibly diagonally dominant.**
   i.e. $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$, $for\ every\ 1 < i < n$

ii. If the initial value to start the iterations is not provided in the problem then we can assume it to be $x = 0,\ y = 0\ and\ z = 0$

SOME SOLVED EXAMPLES

GAUSS JACOBI'S METHOD

1. Solve the following equations by Gauss-Jacobi's Method

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Rewrite given equations as,

$$x = \frac{1}{20}(17 - y + 2z)$$
$$y = \frac{1}{20}(-18 - 3x + z)$$
$$z = \frac{1}{20}(25 - 2x + 3y)$$

(i) First iteration:

start with $x_0 = 0,\ y_0 = 0\ and\ z_0 = 0$

$$x_1 = \frac{17}{20} = 0.85\ ,\ y_1 = \frac{-18}{20} = -0.9\ ,\ z_1 = \frac{25}{20} = 1.25$$

(ii) Second iteration:

Use $x_1 = 0.85,\ y_1 = -0.9\ and\ z_1 = 1.25$

$$x_2 = \frac{1}{20}\big(17 - (-0.9) + 2(1.25)\big) = 1.02$$
$$y_2 = \frac{1}{20}\big(-18 - 3(0.85) + (1.25)\big) = -0.965$$
$$z_2 = \frac{1}{20}\big(25 - 2(0.85) + 3(-0.9)\big) = 1.03$$

(iii) **Third iteration:**

Use $x_2 = 1.02,\ y_2 = -0.965\ and\ z_2 = 1.03$

$$x_3 = \frac{1}{20}\big(17 - (-0.965) + 2(1.03)\big) = 1.00125$$
$$y_3 = \frac{1}{20}\big(-18 - 3(1.02) + (1.03)\big) = -1.0015$$
$$z_3 = \frac{1}{20}\big(25 - 2(1.02) + 3(-0.965)\big) = 1.00325$$

(iv) **Fourth iteration:**

Use $x_3 = 1.00125,\ y_3 = -1.0015\ and\ z_3 = 1.00325$

$$x_4 = \frac{1}{20}\big(17 - (-1.0015) + 2(1.00325)\big) = 1.0004$$
$$y_4 = \frac{1}{20}\big(-18 - 3(1.00125) + (1.00325)\big) = -1.000025$$
$$z_4 = \frac{1}{20}\big(25 - 2(1.00125) + 3(-1.0015)\big) = 0.99965$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is

$x = 1.0004,\ \ y = -1.0000\ and\ z = 0.9997$

2. Solve the following equations by Gauss-Jacobi's Method (Take three iterations)

$$2x + 20y - 3z = 19$$
$$3x - 6y + 25z = 22$$
$$15x + 2y + z = 18$$

Solution:

First checking the condition of strictly diagonally dominant, we rearrange the system as,

$$15x + 2y + z = 18$$
$$2x + 20y - 3z = 19$$
$$3x - 6y + 25z = 22$$

Rewrite given equations as,

$$x = \frac{1}{15}(18 - 2y - z)$$
$$y = \frac{1}{20}(19 - 2x + 3z)$$
$$z = \frac{1}{25}(22 - 3x + 6y)$$

(i) First iteration:

start with $x_0 = 0,\ y_0 = 0\ and\ z_0 = 0$

$$x_1 = \frac{18}{15} = 1.2\ ,\ y_1 = \frac{19}{20} = 0.95\ ,\ z_1 = \frac{22}{25} = 0.88$$

(ii) Second iteration:

Use $x_1 = 1.2,\ y_1 = 0.95\ and\ z_1 = 0.88$

$$x_2 = \frac{1}{15}\big(18 - 2(0.95) - (0.88)\big) = 1.0147$$
$$y_2 = \frac{1}{20}\big(19 - 2(1.2) + 3(0.88)\big) = 0.962$$
$$z_2 = \frac{1}{25}\big(22 - 3(1.2) + 6(0.95)\big) = 0.964$$

(iii) Third iteration:

Use $x_2 = 1.0147,\ y_2 = 0.962\ and\ z_2 = 0.964$

$$x_3 = \frac{1}{15}\big(18 - 2(0.962) - (0.964)\big) = 1.0075$$
$$y_3 = \frac{1}{20}\big(19 - 2(1.0147) + 3(0.964)\big) = 0.9931$$
$$z_3 = \frac{1}{25}\big(22 - 3(1.0147) + 6(0.962)\big) = 0.9891$$

Hence the final answer (correct up to 4 decimal places) after third iteration is

$x = 1.0075,\ \ y = 0.9931\ and\ z = 0.9891$

3. Solve the following equations by Gauss-Jacobi's Method
$5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$. Start with $(2, 3, 0)$

Solution:
Rewrite given equations as,

$$x = \frac{1}{5}(10 + y - z)$$
$$y = \frac{1}{4}(12 - 2x)$$
$$z = \frac{1}{5}(-1 - x - 5y)$$

(i) First iteration:
start with $x_0 = 2,\ y_0 = 3\ and\ z_0 = 0$

$$x_1 = \frac{1}{5}(10 + 3) = \frac{13}{5} = 2.6$$
$$y_1 = \frac{1}{4}(12 - 4) = \frac{8}{4} = 2$$
$$z_1 = \frac{1}{5}(-1 - 2 - 15) = \frac{-18}{5} = -3.6$$

(ii) Second iteration:
Use $x_1 = 2.6,\ y_1 = 2\ and\ z_1 = -3.6$

$$x_2 = \frac{1}{5}\big((10 + 2 - (-3.6))\big) = 3.12$$
$$y_2 = \frac{1}{4}\big(12 - 2(2.6)\big) = 1.7$$
$$z_2 = \frac{1}{5}\big(-1 - 2.6 - 5(2)\big) = -2.72$$

(iii) **Third iteration:**
Use $x_2 = 3.12,\ y_2 = 1.7\ and\ z_2 = -2.72$

$$x_3 = \frac{1}{5}\big(10 + 1.7 - (-2.72)\big) = 2.884$$
$$y_3 = \frac{1}{4}\big(12 - 2(3.12)\big) = 1.44$$
$$z_3 = \frac{1}{5}\big(-1 - 3.12 - 5(1.7)\big) = -2.524$$

(iv) **Fourth iteration:**
Use $x_3 = 2.884,\ y_3 = 1.44\ and\ z_3 = -2.524$

$$x_4 = \frac{1}{5}\big(10 + 1.44 - (-2.524)\big) = 2.7928$$
$$y_4 = \frac{1}{4}\big(12 - 2(2.884)\big) = 1.558$$
$$z_4 = \frac{1}{5}\big(-1 - 2.884 - 5(1.44)\big) = -2.2168$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is
$x = 2.7928,\ \ y = 1.558\ and\ z = -2.2168$

4. Solve the following equations by Gauss-Jacobi's Method

$15x + y - z = 14\ ,\ x + 20y + z = 23\ ,\ 2x - 3y + 18z = 35$

Solution:

Rewrite given equations as,

$$x = \frac{1}{15}(14 - y + z)$$
$$y = \frac{1}{20}(23 - x - z)$$
$$z = \frac{1}{18}(35 - 2x + 3y)$$

(i) First iteration:

start with $x_0 = 0,\ y_0 = 0\ and\ z_0 = 0$

$$x_1 = \frac{1}{15}(14 - y + z) = \frac{14}{15} = 0.9333$$
$$y_1 = \frac{1}{20}(23 - x - z) = \frac{23}{20} = 1.15$$
$$z_1 = \frac{1}{18}(35 - 2x + 3y) = \frac{35}{18} = 1.9444$$

(ii) Second iteration:

Use $x_1 = 0.9333,\ y_1 = 1.15\ and\ z_1 = 1.9444$

$$x_2 = \frac{1}{15}(14 - 1.15 + 1.9444) = 0.9863$$
$$y_2 = \frac{1}{20}(23 - 0.9333 - 1.9444) = 1.0061$$
$$z_2 = \frac{1}{18}(35 - 2(0.9333) + 3(1.15)) = 2.0324$$

(iii) **Third iteration:**

Use $x_2 = 0.9863,\ y_2 = 1.0061 and\ z_2 = 2.0324$

$$x_3 = \frac{1}{15}(14 - 1.0061 + 2.0324) = 1.0018$$
$$y_3 = \frac{1}{20}(23 - 0.9863 - 2.0324) = 0.9991$$
$$z_3 = \frac{1}{18}(35 - 2(0.9863) + 3(1.0061)) = 2.0025$$

(iv) **Fourth iteration:**

Use $x_3 = 1.0018\ y_3 = 0.9991\ and\ z_3 = 2.0025$

$$x_4 = \frac{1}{15}(14 - 0.9991 + 2.0025) = 1.0002$$
$$y_4 = \frac{1}{20}(23 - 1.0018 - 2.0025) = 0.9995$$
$$z_4 = \frac{1}{18}(35 - 2(1.0018) + 3(0.9991)) = 1.9990$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is

$x = 1.0002,\ \ y = 0.9995\ and\ z = 1.9990$

5. Use Gauss-Seidel method to solve the following equations (Take three iterations)
$3x - 0.1y - 0.2z = 7.85$
$0.1x + 7y - 0.3z = -19.3$
$0.3x - 0.2y + 10z = 71.4$

Solution:
Rewrite given equations as,
$x = \frac{1}{3}(7.85 + 0.1y + 0.2z)$ ……(1)
$y = \frac{1}{7}(-19.3 - 0.1x + 0.3z)$….. (2)
$z = \frac{1}{10}(71.4 - 0.3x + 0.2y)$ …….(3)
(i) **First iteration:**
Start with $y = 0$ *and* $z = 0$
$x = \frac{7.85}{3} = 2.6167$,
We use this value to find y,
i.e. we put $x = 2.6167$ *and* $z = 0$
$y = \frac{1}{7}\big(-19.3 - 0.1(2.6167) + 0.3(0)\big) = -2.7945$,
We use latest two values to find z, i.e.
we put $x = 2.6167$ *and* $y = -2.7945$
$z = \frac{1}{10}\big(71.4 - 0.3(2.6167) + 0.2(-2.7945)\big) = 7.0056$
(ii) **Second iteration:**
We use latest two values to find x, we put $y = -2.7945$ *and* $z = 7.0056$
$x = \frac{1}{3}(7.85 + 0.1(-2.7945) + 0.2(7.0056)) = 2.9906$
We use latest two values to find y, we put $x = 2.9906$ *and* $z = 7.0056$
$y = \frac{1}{7}\big(-19.3 - 0.1(2.9906) + 0.3(7.0056)\big) = -2.4996$
We use latest two values to find z, i.e. we put $x = 2.9906$ *and* $y = -2.4996$
$z = \frac{1}{10}\big(71.4 - 0.3(2.9906) + 0.2(-2.4996)\big) = 7.0003$
(iii) **Third iteration:**
We use latest two values to find x, we put $y = -2.4996$ *and* $z = 7.0003$
$x = \frac{1}{3}(7.85 + 0.1(-2.4996) + 0.2(7.0003)) = 3.0000$
We use latest two values to find y, we put $x = 3$ *and* $z = 7.0003$
$y = \frac{1}{7}\big(-19.3 - 0.1(3) + 0.3(7.0003)\big) = -2.500$
We use latest two values to find z, i.e. we put $x = 3$ *and* $y = -2.5$
$z = \frac{1}{10}\big(71.4 - 0.3(3) + 0.2(-2.5)\big) = 7.000$
Hence the final answer after third iteration is
$x = 3,\ \ y = -2.5$ *and* $z = 7$

6. Solve the following equations by Gauss-Seidel method.

$$28x + 4y - z = 32 \ , \ 2x + 17y + 4z = 35 \ , \ x + 3y + 10z = 24$$

Solution:

Rewrite given equations as,

$$x = \frac{1}{28}(32 - 4y + z) \ \ldots\ldots(1)$$
$$y = \frac{1}{17}(35 - 2x - 4z) \ldots\ldots (2)$$
$$z = \frac{1}{10}(24 - x - 3y) \ \ldots\ldots(3)$$

(i) **First iteration:**

Start with $y = 0 \ and \ z = 0$

$$x = \frac{32}{28} = 1.1429 \ ,$$

We use this value to find y,

i.e. we put $x = 1.1429 \ and \ z = 0$

$$y = \frac{1}{17}\big(35 - 2(1.1429) - 4(0)\big) = 1.9244 \ ,$$

We use latest two values to find z, i.e.

we put $x = 1.1429 \ and \ y = 1.9244$

$$z = \frac{1}{10}\big(24 - 1.1429 - 3(1.9244)\big) = 1.7084$$

(ii) **Second iteration:**

We use latest two values to find x, we put $y = 1.9244 \ and \ z = 1.7084$

$$x = \frac{1}{28}(32 - 4(1.9244) + 1.7084) = 0.9289$$

We use latest two values to find y, we put $x = 0.9289 \ and \ z = 1.7084$

$$y = \frac{1}{17}(35 - 2(0.9289) - 4(1.7084)) = 1.5475$$

We use latest two values to find z, i.e. we put $x = 0.9289 \ and \ y = 1.5475$

$$z = \frac{1}{10}(24 - 0.9289 - 3(1.5475)) = 1.8428$$

(iii) **Third iteration:**

We use latest two values to find x, we put $y = 1.5475 \ and \ z = 1.8428$

$$x = \frac{1}{28}(32 - 4(1.5475) + 1.8428) = 0.9876$$

We use latest two values to find y, we put $x = 0.9876 \ and \ z = 1.8428$

$$y = \frac{1}{17}(35 - 2(0.9876) - 4(1.8428)) = 1.5090$$

We use latest two values to find z, i.e. we put $x = 0.9876 \ and \ y = 1.5090$

$$z = \frac{1}{10}(24 - 0.9876 - 3(1.5090)) = 1.8485$$

(iv) **Fourth iteration:**

We use latest two values to find x, we put $y = 1.5090 \ and \ z = 1.8485$

$$x = \frac{1}{28}(32 - 4(1.5090) + 1.8485) = 0.9933$$

We use latest two values to find y, we put $x = 0.9933 \ and \ z = 1.8485$

$$y = \frac{1}{17}(35 - 2(0.9933) - 4(1.8485)) = 1.5070$$

We use latest two values to find z, i.e. we put $x = 0.9933 \; and \; y = 1.5070$

$$z = \frac{1}{10}(24 - 0.9933 - 3(1.5070)) = 1.8485$$

Hence the final answer after fourth iteration is

$x = 0.9933, \;\; y = 1.5070 \; and \; z = 1.8485$

7. Solve the following equations by Gauss-Seidel method by taking three iterations only.
$10x_1 + x_2 + x_3 = 12 \,,\; 2x_1 + 10x_2 + x_3 = 13 \,,\; 2x_1 + 2x_2 + 10x_3 = 14$

Solution:

Rewrite given equations as,

$$x_1 = \frac{1}{10}(12 - x_2 - x_3)\ldots\ldots(1)$$

$$x_2 = \frac{1}{10}(13 - 2x_1 - x_3)\ldots\ldots(2)$$

$$x_3 = \frac{1}{10}(14 - 2x_1 - 2x_2)\ldots\ldots(3)$$

(i) **First iteration:**

Start with $x_2 = 0 \; and \; x_3 = 0$

$$x_1 = \frac{12}{10} = 1.2 \,,$$

We use this value to find y,

i.e. we put $x_1 = 1.2 \; and \; x_3 = 0$

$$x_2 = \frac{1}{10}(13 - 2(1.2) - 0) = 1.06 \,,$$

We use latest two values to find z, i.e.

We put $x_1 = 1.2 \; and \; x_2 = 1.06$

$$x_3 = \frac{1}{10}(14 - 2(1.2) - 2(1.06)) = 0.948$$

(ii) **Second iteration:**

We use latest two values to find x, we put $x_2 = 1.06 \; and \; x_3 = 0.948$

$$x_1 = \frac{1}{10}(12 - 1.06 - 0.948) = 0.9992$$

We use latest two values to find y, we put $x_1 = 0.9992 \; and \; x_3 = 0.948$

$$x_2 = \frac{1}{10}(13 - 2(0.9992) - 0.948) = 1.00536$$

We use latest two values to find z, i.e. we put $x_1 = 0.9992 \; and \; x_2 = 1.00536$

$$x_3 = \frac{1}{10}(14 - 2(0.9992) - 2(1.00536)) = 0.999088$$

(iii) **Third iteration:**

We use latest two value to find x, we put $x_2 = 1.00536 \; and \; x_3 = 0.999088$

$$x_1 = \frac{1}{10}(12 - 1.00536 - 0.999088) = 0.9996$$

We use latest two values to find y, we put $x_1 = 0.9996 \; and \; x_3 = 0.999088$

$$x_2 = \frac{1}{10}(13 - 2(0.9996) - 0.999088) = 1.00018$$

We use latest two values to find z, i.e. we put $x_1 = 0.9996 \; and \; x_2 = 1.00018$

$$x_3 = \frac{1}{10}(14 - 2(0.9996) - 2(1.00018)) = 1.00052$$

Hence the final answer after third iteration is $x = 0.9996, \;\; y = 1 \; and \; z = 1$

SOME PRACTICE PROBLEMS

JACOBI'S METHOD

I. Solve the following equations by Jacobi's method.

1) $15x + y - z = 14$, $x + 20y + z = 23$, $2x - 3y + 18z = 35$
2) $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$
3) $8x - y + 2z = 13$, $x - 10y + 3z = 17$, $3x + 2y + 12z = 25$
4) $5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$. Start with $(2, 3, 0)$.
5) $5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$
6) $12x + 2y + z = 27$, $2x + 15y - 3z = 16$, $2x - 3y + 25z = 26$
7) $4x + y + 3z = 17$, $x + 5y + z = 14$, $2x - y + 8z = 12$

GAUSS - SEIDEL METHOD

II. Solve the following equations by Gauss-Seidel method.

1) $28x + 4y - z = 32$, $2x + 17y + 4z = 35$, $x + 3y + 10z = 24$
2) $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 27z = 85$
3) $10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$
4) $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$
5) $5x - y = 9$, $-x + 5y - z = 4$, $-y + 5z = -6$
6) $5x + y - z = 10$, $2x + 4y + z = 14$, $x + y + 8z = 20$
7) $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$ by taking three iterations only.
8) $4x - 2y - z = 40$, $x - 6y + 2z = -28$, $x - 2y + 12z = -86$
9) $2x - 4y + 49z = 49$, $43x + 2y + 25z = 23$, $3x + 53y + 3z = 91$
10) $10x_1 - 5x_2 - 2x_3 = 3$, $4x_1 - 10x_2 + 3x_3 = -3$, $x_1 + 6x_2 - 10x_3 = -3$ by taking three iterations only.
11) $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$
12) $25x + 2y - 3z = 48$, $3x + 27y - 2z = 56$, $x + 2y + 32z = 52$. Start with $(1, 1, 0)$.