EXACT DIFFERENTIAL EQUATIONS:

Definition : A differential equation which is obtained from its primitive differentiation only and without any operation of elimination or reduction is called an **exact differential equation**.

If $u = c$ where u is a function of x and y is primitive then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$ is an exact differential equation. Thus, an exact differential equation is obtained from its promitive by equating its total differential to zero.

For example, If $u = x^2 + y^2 = c$ then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 2xdx + 2ydy$

Equating $du = 0$, we get the equation $x\,dx + y\,dy = 0$ which is exact.

The Necessary and Sufficient Condition for equation $\boldsymbol{Mdx + Ndy = 0}$ to be exact is $\boldsymbol{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

i.e. If the equation $M\,dx + N\,dy = 0$ is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and conversely if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is exact.

Rule for finding the solution:

Rule 1: Integrate M w.r.t x treating y constant and Integrate only those terms in N which are free from x w.r.t. y. Equate the sum to a constant. This is the solution.

In symbols, $\int M\,dx$ (treating y constant) + $\int (Terms$ in N free from $x)\,dy = c$.

Rule 2: Integrate N w.r.t y treating x constant and Integrate only those terms in M which are free from y w.r.t. x. Equate the sum to a constant. This is the solution.

In symbols, $\int N\,dy$ (treating x constant) + $\int (Terms$ in M free from $y)\,dx = c$.

EXAMPLES:

1. $x\,dx + y\,dy = \frac{a(x\,dy - y\,dx)}{x^2+y^2}$

Solution: The equation can be written as $\left\{x + \frac{ay}{x^2+y^2}\right\}dx + \left\{y - \frac{ax}{x^2+y^2}\right\}dy = 0$

$\therefore M = x + \frac{ay}{x^2+y^2}, \quad \therefore \frac{\partial M}{\partial y} = \frac{a}{x^2+y^2} - \frac{2ay^2}{(x^2+y^2)^2} = \frac{ax^2-ay^2}{(x^2+y^2)^2}$

$\therefore N = y - \frac{ax}{x^2+y^2}, \quad \therefore \frac{\partial N}{\partial x} = -\frac{a}{x^2+y^2} + \frac{2ax^2}{(x^2+y^2)^2} = \frac{ax^2-ay^2}{(x^2+y^2)^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact

Now, $\int M\,dx = \int x\,dx + ay\int \frac{dx}{x^2+y^2} = \frac{x^2}{2} + ay\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) = \frac{x^2}{2} + a\cdot\tan^{-1}\left(\frac{x}{y}\right)$

And $\int$(terms in N free from x) $dy = \int y\,dy = \frac{y^2}{2}$

$\therefore$ This solution is $\frac{x^2}{2} + \frac{y^2}{2} + a\tan^{-1}\left(\frac{x}{y}\right) = c$ i.e. $x^2 + y^2 + 2a\tan^{-1}\left(\frac{x}{y}\right) = c$

2. $2(1 + x^2\sqrt{y})ydx + (x^2\sqrt{y} + 2)x\,dy = 0$

Solution: Here, $M = 2y + 2x^2y^{3/2}$; $N = x^3\sqrt{y} + 2x$

$\therefore \frac{\partial M}{\partial y} = 2 + 3x^2y^{1/2}$; $\frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2$ $\qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The equation is exact

Now, $\int M\,dx = \int (2y + 2x^2y^{3/2})\,dx = 2xy + \frac{2}{3}x^3y^{3/2}$

And $\int$(terms in N free from x) $dy = \int 0\cdot dy = 0$

$\therefore$ The solution is $2xy + \frac{2}{3}x^3y^{3/2} = c$

3. $\frac{y}{x^2}cos\left(\frac{y}{x}\right)dx - \frac{1}{x}cos\left(\frac{y}{x}\right)dy + 2xdx = 0$

Solution: We have $\left[2x + \frac{y}{x^2}\cos\left(\frac{y}{x}\right)\right]dx + \left[-\frac{1}{x}\cos\frac{y}{x}\right]dy = 0$

$\therefore M = 2x + \frac{y}{x^2}\cos\frac{y}{x}$ and $N = -\frac{1}{x}\cos\frac{y}{x}$

$\therefore \frac{\partial M}{\partial y} = \frac{1}{x^2}\cos\left(\frac{y}{x}\right) - \frac{y}{x^2}\sin\left(\frac{y}{x}\right)\cdot\frac{1}{x}$; $\frac{\partial N}{\partial x} = \frac{1}{x^2}\cos\left(\frac{y}{x}\right) - \frac{1}{x}\sin\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact

$\therefore \int M\,dx = \int\left(2x + \frac{y}{x^2}\cos\frac{y}{x}\right)dx = \int 2x\,dx + \int\cos\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}dx = x^2 + I_2$

For I_2, put $\frac{y}{x} = t, -\frac{y}{x^2}dx = dt$

$\therefore \int M\,dx = x^2 - \int\cos t\,dt = x^2 - \sin t = x^2 - \sin\left(\frac{y}{x}\right)$

$\int(\text{terms in } N \text{ free from } x)\,dy = 0$

$\therefore$ The solution is $x^2 - \sin\frac{y}{x} = c$

4. $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0, given\ y(0) = 4$

Solution: Here $M = 1 + e^{x/y}, N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$; $\frac{\partial N}{\partial x} = e^{x/y}\cdot\frac{1}{y}\left(1 - \frac{x}{y}\right) - e^{x/y}\left(\frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The equation is exact

$\therefore \int M\,dx = \int\left(1 + e^{x/y}\right)dx = x + ye^{x/y}$

$\int(\text{terms in } N \text{ free from } x)\,dy = \int 0\,dy = 0$

$\therefore$ The solution is $x + ye^{x/y} = c$

By data when $x = 0, y = 4$ $\quad\therefore 4 = c$

The particular solution is $x + ye^{x/y} = 4$

INTEGRATING FACTOR :

Some times a given differential equation is not exact but is rendered exact if it is multiplied by a suitable factor, Such a factor is called an **Integrating Factor**

Standard rules of obtaining integrating factors.

Rule 1 : If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N$ is a function of x only, say f(x) then $e^{\int f(x)dx}$ is an integrating factor.

Rule 2 : If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/M$ is a function of y only say f(y) then $e^{-\int f(y)dy}$ is an integrating factor.

Rule 3 : If the equation is of the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$ and $Mx - Ny \neq 0$ then $1/(Mx - Ny)$ is an integrating factor.

Rule 4 : If the equation $M\, dx + N\, dy = 0$ is homogeneous and $Mx + Ny \neq 0$ then $1/(Mx + Ny)$ is an integrating factor.

EXAMPLES:

1. $(x^2 + y^2 + 1)dx - 2xy\, dy = 0$

Solution: We have, $M = x^2 + y^2 + 1$ and $N = -2xy$ $\therefore \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = -2y$

$$\therefore \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$$

$$\therefore IF = e^{\int -(2/x)dx} = e^{-2\log x} = e^{\log(1/x^2)} = \frac{1}{x^2}$$

Multiplying by the IF, we get, $\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx + \left(-\frac{2y}{x}\right) dy = 0$, which is exact

Now, $\int M\, dx = \int \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = x - \frac{y^2}{x} - \frac{1}{x}$ and

$\int N\, dy = \int (\text{terms in } N \text{ free from } x)\, dy = 0$

$\therefore$ The solution is $x - \frac{y^2}{x} - \frac{1}{x} = c$, i.e. $x^2 - y^2 - 1 = cx$

2. $y(xy + e^x)dx - e^x dy = 0$

Solution: We have, $M = y(xy + e^x)$ and $N = -e^x$ $\therefore \frac{\partial M}{\partial y} = 2xy + e^x$ and $\frac{\partial N}{\partial x} = -e^x$

$$\therefore \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} = \frac{-e^x - 2xy - e^x}{y(xy + e^x)} = \frac{-2(xy + e^x)}{y(xy + e^x)} = \frac{-2}{y} = f(y)$$

$$\therefore IF = e^{\int(-2/y)dy} = e^{-2\log y} = e^{\log(1/y^2)} = \frac{1}{y^2}$$

Multiply the given equation by IF, we get, $\left(x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2} dy = 0$, which is exact

$\therefore \int M\, dx = \int \left(x + \frac{e^x}{y}\right) dx = \frac{x^2}{2} + \frac{e^x}{y}$

$\int (\text{terms in } N \text{ free from } x)\, dy = 0$

$\therefore$ The solution is $\frac{x^2}{2} + \frac{e^x}{y} = c$

3. $\frac{dy}{dx} = -\frac{x^2y^3 + 2y}{2x - 2x^3y^2}$

Solution: The equation can be written as $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$

i.e. $y(2 + x^2y^2)dx + x(2 - 2x^2y^2)dy = 0$

We have, $M = y(2 + x^2y^2)$ and $N = x(2 - 2x^2y^2)$

$\therefore \frac{\partial M}{\partial y} = 2 + 3x^2y^2$ and $\frac{\partial N}{\partial x} = 2 - 6x^2y^2$ The DE is not exact.

$Mx - Ny = 2xy + x^3y^3 - 2xy + 2x^3y^3 = 3x^3y^3 \neq 0$

$\therefore IF = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$

Multiply the given equation by IF, we get, $\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$

Now, $\int M\,dx = \int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx = \frac{1}{3}\log x - \frac{1}{3x^2y^2}$

And $\int(\text{terms in } N \text{ free from } x)\,dy = \int -\frac{2}{3}y\,dy = -\frac{2}{3}\log y$

$\therefore$ The solution is $\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$

$\therefore \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c$

4. $\left[2x\, sin\,h\left(\frac{y}{x}\right) + 3y cos\,h\left(\frac{y}{x}\right)\right]dx - 3x.cos\,h\left(\frac{y}{x}\right).dy = 0$

Solution: Here, $M = 2x\sinh\left(\frac{y}{x}\right) + 3y\cosh\left(\frac{y}{x}\right)$ and $N = -3x\cosh\left(\frac{y}{x}\right)$

$\therefore \frac{\partial M}{\partial y} = 2x\cdot\cosh\left(\frac{y}{x}\right)\cdot\frac{1}{x} + 3\cosh\left(\frac{y}{x}\right) + 3y\sinh\left(\frac{y}{x}\right)\cdot\frac{1}{x} \quad = 5\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sinh\left(\frac{y}{x}\right)$

$\therefore \frac{\partial N}{\partial x} = -3\cosh\left(\frac{y}{x}\right) - 3x\sinh\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) \quad = -3\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sin\left(\frac{y}{x}\right)$

The DE is not exact.

$\therefore \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{8\cosh\left(\frac{y}{x}\right)}{-3x\cosh\left(\frac{y}{x}\right)} = -\frac{8}{3x} = f(x)$

$\therefore IF = e^{\int f(x)dx} = e^{\int -(8/3x)dx} = e^{-(8/3)\log x} = x^{-(8/3)}$

Multiply the given equation by IF, we get,

$\left[2x^{-5/3}\cdot\sinh\left(\frac{y}{x}\right) + 3x^{-8/3}\cdot y\cdot\cosh\left(\frac{y}{x}\right)\right]dx - 3x^{-5/3}\cdot\cosh\left(\frac{y}{x}\right)\cdot dy = 0$

Since $\int M\,dx$ is rather a complex integral

$\therefore \int N\,dy$ treating x constant

$= \int -3x^{-5/3}\cdot\cosh\left(\frac{y}{x}\right)dy = -3x^{-5/3}\cdot\sinh\left(\frac{y}{x}\right)\cdot x = -3x^{-2/3}\cdot\sinh\left(\frac{y}{x}\right)$

$\int(\text{terms in } M \text{ free from } y)\,dx = 0$

$\therefore$ The solution is $x^{-2/3}\cdot\sinh\left(\frac{y}{x}\right) = -\frac{c'}{3} = c$

5. Solve $(x^2 - xy + y^2)dx - xydy = 0$

Solution: The DE is not exact. (show this !!!)

The given differential equation is homogeneous

and $Mx + Ny = x^3 - x^2y + xy^2 - xy^2 = x^2(x-y)$

$\therefore \frac{1}{(Mx+Ny)} = \frac{1}{x^2(x-y)}$ is an integrating factor

Multiply the given equation by IF, we get, $\frac{x^2-xy+y^2}{x^2(x-y)}dx - \frac{xy}{x^2(x-y)}dy = 0$

$\therefore \frac{x^2-xy}{x^2(x-y)}dx + \frac{y^2}{x^2(x-y)}dx - \frac{y}{x(x-y)}dy = 0$

$\therefore \left[\frac{1}{x} + \frac{1}{x-y} - \frac{1}{x} - \frac{y}{x^2}\right]dx + \left[\frac{1}{x} - \frac{1}{x-y}\right]dy = 0$ (By partial fraction)

$\therefore \left[\frac{1}{x-y} - \frac{y}{x^2}\right]dx + \left[\frac{1}{x} - \frac{1}{x-y}\right]dy = 0$

$\therefore \int M\,dx = \int\frac{dx}{x-y} - \int\frac{y}{x^2}dx = \log(x-y) + \frac{y}{x}$

$\int$(terms in N free from x) $dy = 0$

$\therefore$ The solution is $\log(x-y)+\frac{y}{x}=c$

6. $(xy\,sin\,xy\ +\ cos\,xy)\,y\,dx\ +\ (xy\,sin\,xy - cos\,xy)\,x\,dy\ =\ 0$

Solution: We have, $M=(xy\,sin\,xy\ +\ cos\,xy)\,y$ and $N=\ (xy\,sin\,xy - cos\,xy)\,x$

The DE is not exact. (show this !!!)

$Mx-Ny=x^2y^2\sin xy+xy\cos xy-x^2y^2\sin xy+xy\cos xy=2xy\cos xy$

$\therefore IF=\frac{1}{Mx-Ny}=\frac{1}{2xy\cos xy}$

Multiply the given equation by IF, we get, $\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)dx+\frac{1}{2}\left(x\tan xy-\frac{1}{y}\right)dy=0$

$\therefore \left(y\tan xy+\frac{1}{x}\right)dx+\left(x\tan xy-\frac{1}{y}\right)dy=0$, which is exact

$\therefore \int M\,dx=\int\left(y\tan xy+\frac{1}{x}\right)dx=\log\sec xy+\log x$

$\int$(terms in N free from x) $dy=\int-\frac{1}{y}dy=-\log y$

$\therefore$ The solution is $\log\sec xy+\log x=\log y+\log c$ i.e. $x\sec xy=cy$

7. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy\ =\ 0$

Solution : The DE is not exact. (show this !!!)

The given differential equation is homogeneous

and $Mx+Ny=x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2$

Hence, $\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$ is an integrating factor

Multiply the given equation by IF, we get, $\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}-\frac{3}{y}\right)dy=0$ which is exact

Now, $\int M\,dx=\int\left(\frac{1}{y}-\frac{2}{x}\right)dx=\frac{x}{y}-2\log x$

And $\int$(terms in N free from x) $dy=\int\frac{3}{y}dy=3\log y$

$\therefore$ The solution is $\frac{x}{y}-2\log x+3\log y=-\log c$

i.e. $\frac{x}{y}+\log\frac{cy^3}{x^2}=0$ i.e. $\log\frac{cy^3}{x^2}=-\frac{x}{y}$ i.e. $\frac{cy^3}{x^2}=e^{-x/y}$

8. If $f(x)$ a function of x only is an integrating factor of $(x^4e^x-2mxy^2)dx+2mx^2ydy=0$. find $f(x)$ and then solve the equation.

Solution: The given equation is $(x^4e^x-2mxy^2)dx+2mx^2ydy=0$

Multiplying the equation by $f(x)$,

$f(x)\cdot(x^4e^x-2mxy^2)dx+f(x)\cdot 2mx^2ydy=0$

$\therefore M=f(x)\cdot(x^4e^x-2mxy^2),\quad N=f(x)\cdot 2mx^2y$

$\therefore \frac{\partial M}{\partial y}=f(x)(-4m\,xy),\qquad \frac{\partial N}{\partial x}=f(x)4m\,xy+f'(x)2mx^2y$

Since, now equation is exact $\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\therefore f(x)(-4m\,xy)=f(x)4m\,xy+f'(x)2mx^2y$

$\therefore -4mxy\cdot f(x)=f'(x)\cdot mx^2y$

$\therefore f'(x)=-\frac{4}{x}f(x)$ $\therefore \frac{f'(x)}{f(x)}=-\frac{4}{x}$ $\therefore \int\frac{f'(x)}{f(x)}dx=-4\int\frac{dx}{x}$

$\therefore \log f(x)=-4\log x=\log x^{-4}$

$\therefore f(x) = x^{-4}$

Now, multiply the given equation by x^{-4} so that it becomes exact and solve as above.

9. If $(x+y)^k$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$. find k and solve the equation.

Solution: The given equation is $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$.

Multiplying the equation by $(x+y)^k$, we get,

$(x+y)^k(4x^2 + 2xy + 6y)dx + (x+y)^k(2x^2 + 9y + 3x)dy = 0$ …………… (1)

$\therefore M = (x+y)^k(4x^2 + 2xy + 6y), \quad N = (x+y)^k(2x^2 + 9y + 3x)$

$\therefore \frac{\partial M}{\partial y} = k(x+y)^{k-1}(4x^2 + 2xy + 6y) + (x+y)^k(2x+6),$

$\frac{\partial N}{\partial x} = k(x+y)^{k-1}(2x^2 + 9y + 3x) + (x+y)^k(4x+3)$

Since, now equation is exact $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore k(x+y)^{k-1}(4x^2 + 2xy + 6y) + (x+y)^k(2x+6)$

$= k(x+y)^{k-1}(2x^2 + 9y + 3x) + (x+y)^k(4x+3)$

$\therefore k(4x^2 + 2xy + 6y) + (x+y)(2x+6) = k(2x^2 + 9y + 3x) + (x+y)(4x+3)$

$\therefore 4kx^2 + 2kxy + 6ky + 2x^2 + 6x + 2xy + 6y = 2kx^2 + 9ky + 3kx + 4x^2 + 3x + 4xy + 3y$

$\therefore k(2x^2 + 2xy - 3y - 3x) = (2x^2 + 2xy - 3y - 3x) \qquad \therefore k = 1$

Putting $k = 1$ in (1), we get,

$(x+y)(4x^2 + 2xy + 6y)dx + (x+y)(2x^2 + 9y + 3x)dy = 0$ which is exact

$(4x^3 + 2x^2y + 6xy + 4x^2y + 2xy^2 + 6y^2)dx + (2x^3 + 9xy + 3x^2 + 2x^2y + 9y^2 + 3xy)dy = 0$

$\therefore (4x^3 + 6x^2y + 6xy + 2xy^2 + 6y^2)dx + (2x^3 + 12xy + 3x^2 + 2x^2y + 9y^2)dy = 0$

$\therefore \int M\,dx = \int(4x^3 + 6x^2y + 6xy + 2xy^2 + 6y^2)\,dx = x^4 + 2x^3y + 3x^2y + x^2y^2 + 6y^2x$

$\int(\text{Terms in } M \text{ free from } y)\,dy = \int 9y^2\,dy = 3y^3$

$\therefore$ The solution is $x^4 + 2x^3y + 3x^2y + x^2y^2 + 6y^2x + 3y^3 = c$

LINEAR DIFFERENTIAL EQUATIONS

Definition: A differential equation is said to be linear if the dependent variable and its derivatives appear only in the first degree. The form of the linear equation of the first order is

$\frac{dy}{dx} + Py = Q$ Where P and Q are function of x or constants only.

For example, $\frac{dy}{dx} + 3xy = x^2$, $\frac{dy}{dx} + y = e^x$ are linear equations.

Method to solve Linear Differential Equations :

(1) First write the equation with the coefficient of $\frac{dy}{dx}$ unity i.e. in the form $\frac{dy}{dx} + Py = Q$

(2) Find $\int Pdx$ and further $I.F = e^{\int P\,dx}$

(3) Multiply the equation by Integrating factor $e^{\int P\,dx}$ it becomes exact and hence can be solved by mere integration.

(4) The solution is $y.\left(e^{\int P\,dx}\right) = \int \left(\left(e^{\int P\,dx}\right).Q\right)\,dx + c$

ANOTHER FORM OF LINEAR DIFFERENTIAL EQUATION :

A differential equation of the form $\frac{dx}{dy} + p'x = Q'$ Where P' and Q' are functions of y only is also a linear differential equation with x and y having interchanged the positions.

Its solution is, $x.\left(e^{\int P'\,dy}\right) = \int \left(\left(e^{\int P'\,dy}\right).Q'\right)dy + c$

EXAMPLES

1. $\frac{dy}{dx} + \left(\frac{1-2x}{x^2}\right)y = 1$

Solution : This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Now, $\int P\,dx = \int \left(\frac{1-2x}{x^2}\right)dx = \int \frac{dx}{x^2} - 2\int \frac{dx}{x} = -\frac{1}{x} - 2\log x$

$\therefore e^{\int Pdx} = e^{-(1/x)-2\log x} = e^{-1/x} \cdot e^{-2\log x} = e^{-1/x} \cdot \frac{1}{x^2}$

$\therefore$ The solution is $ye^{\int Pdx} = \int e^{\int Pdx} \cdot Qdx + c$

$\therefore ye^{-1/x} \cdot \frac{1}{x^2} = \int e^{-1/x} \cdot \frac{1}{x^2} Qdx + c$

$ye^{-1/x} \cdot \frac{1}{x^2} = \int e^{-1/x} \cdot \frac{1}{x^2} dx + c$ $\because \int e^{f(x)} \cdot f'(x)dx = e^{f(x)} + c$

$\therefore$ The solution is $ye^{-1/x} \cdot \frac{1}{x^2} = e^{-1/x} + c$ $\therefore y = x^2 + ce^{1/x} \cdot x^2$

2. $(1 + x + xy^2)dy + (y + y^3)dx = 0$

Solution: We have, $1 + x(1 + y^2) + y(1 + y^2)\frac{dx}{dy} = 0$

$\therefore \frac{dx}{dy} + \frac{x}{y} = -\frac{1}{y(1+y^2)}$

This is a linear differential equation of the form $\frac{dx}{dy} + P'x = Q'$

Now, $\int P'\,dy = \int \frac{dy}{y} = \log y$ $\therefore e^{\int P'dy} = e^{\log y} = y$

$\therefore$ This solution is $x \cdot e^{\int P'dy} = \int e^{\int P'dy} \cdot Q'dy + c$

$\therefore xy = \int y\left[-\frac{1}{y(1+y^2)}\right]dy + c = -\int \frac{dy}{1+y^2} = -\tan^{-1} y + c$

$\therefore xy + \tan^{-1} y = c$

3. $(1 + y^2)dx = \left(e^{tan^{-1}y} - x\right)dy$

Solution: The equation can be written as $\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{e^{\tan^{-1} y}}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy} + P'x = Q'$

Now, $\int P'\,dy = \int \frac{1}{1+y^2}dy = \tan^{-1} y$ $\therefore e^{\int P'dy} = e^{\tan^{-1} y}$

$\therefore$ The solution is $x \cdot e^{\int P' dy} = \int e^{\int P' dy} \cdot Q' \, dy + c$

$\therefore xe^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2} \cdot dy + c$

put $\tan^{-1} y = t$ $\quad \therefore \frac{1}{1+y^2} \cdot dy = dt$

$\therefore xe^{\tan^{-1} y} = \int e^{2t} \cdot dt + c = \frac{1}{2} e^{2t} + c$ $\qquad \therefore xe^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + c$

EQUATION REDUCIBLE TO LINEAR FORM :

(1) The equation of the type $f'(y)\frac{dy}{dx} + P.f(y) = Q$ Where P and Q are functions of x only can be reduced to linear form as follows.

Let us put $f(y) = v$ then $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ The equation reduces to $\frac{dv}{dx} + Pv = Q$ which is linear.

Its solution is $v.\left(e^{\int P\,dx}\right) = \int \left(\left(e^{\int P\,dx}\right).Q\right) dx + c$

(2) The equation of the type $f'(x)\frac{dx}{dy} + Pf(x) = Q$ Where P and Q are functions of y only can also be reduced to linear form as follows.

Let us put $f(x) = v$ then $f'(x)\frac{dx}{dy} = \frac{dv}{dy}$

$\therefore$ The equation reduces to $\frac{dv}{dy} + Pv = Q$ which is linear.

Its solution is $v.\left(e^{\int P\,dy}\right) = \int \left(\left(e^{\int P\,dy}\right).Q\right) dy + c$

4. $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Solution: Dividing by $\cos^2 y$ the equation can be written as $\sec^2 y \frac{dy}{dx} + \sec^2 y \cdot \sin 2y \cdot x = x^3$ ……..(1)

$\therefore \sec^2 y \frac{dy}{dx} + 2\tan y \cdot x = x^3$

Put $\tan y = v$ and differentiate w.r.t. x, we get $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$

Hence, from (1), we get $\frac{dv}{dx} + 2v \cdot x = x^3$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

$\therefore \int P\,dx = \int 2x\,dx = x^2$ $\qquad \therefore IF = e^{\int P dx} = e^{x^2}$

$\therefore$ The solution is $ye^{\int P dx} = \int e^{\int P dx} \cdot Q dx + c$

$\therefore\ ve^{x^2} = \int e^{x^2} x^3\,dx + c$

To find the integral on R.H.S. put $x^2 = t,\ \therefore x^2 dx = dt \quad \therefore\ xdx = \frac{dt}{2}$

$\therefore \int e^{x^2} x^3\,dx = \int e^t \cdot t \cdot \frac{dt}{2} = \frac{1}{2}[t \cdot e^t - \int e^t \cdot dt] = \frac{1}{2}[te^t - e^t] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$

$\therefore$ The solution is $ve^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

Re sub. $v = tan\,y$

$\therefore \tan y \cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$ $\qquad \therefore \tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$

5. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Solution: The equation can be written as $\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$ i.e. $e^y\frac{dy}{dx} + e^y \cdot e^x = e^{2x}$ ……..(1)

Now, put $e^y = v$ and differnetiate w.r.t. x, $\quad e^y \frac{dy}{dx} = \frac{dv}{dx}$

Hence, from (1), we get $\frac{dv}{dx} + e^x \cdot v = e^{2x}$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Its solution is $ve^{\int P dx} = \int e^{\int P dx} \cdot Q\, dx + c$

$\therefore ve^{\int e^x dx} = \int e^{\int e^x dx} \cdot e^{2x}\, dx + c$

$\therefore ve^{e^x} = \int e^{e^x} \cdot e^{2x} \cdot dx + c$

To find the integral on R.H.S. put $e^x = t \quad \therefore e^x dx = dt$

$\therefore \int e^{e^x} e^x \cdot e^x \cdot dx = \int e^t \cdot t\, dt = e^t(t-1)$

$\therefore$ The solution is $ve^{e^x} = e^{e^x}(e^x - 1) + c$

$\therefore v = (e^x - 1) + ce^{-e^x}$

Re sub. $v = e^y \qquad \therefore e^y = e^x - 1 + ce^{-e^x}$

6. $\frac{dy}{dx} = \frac{y^3}{e^{2x}+y^2}$

Solution: The equation can be written as $e^{2x} + y^2 = y^3 \frac{dx}{dy} \qquad \therefore \frac{dx}{dy} - \frac{1}{y} = e^{2x} \cdot \frac{1}{y^3}$

Dividing by e^{-2x}, $\quad e^{-2x}\frac{dx}{dy} - e^{-2x} \cdot \frac{1}{y} = \frac{1}{y^3}$

Putting $e^{-2x} = v$, $\quad \therefore -2e^{-2x}\frac{dx}{dy} = \frac{dv}{dy}$, we get,

$-\frac{1}{2} \cdot \frac{dv}{dy} - \frac{1}{y} \cdot v = \frac{1}{y^3}$ i.e. $\frac{dv}{dy} + \frac{2}{y} \cdot v = -\frac{2}{y^3}$

This is a linear differential equation of the form $\frac{dv}{dy} + Pv = Q$

$\therefore e^{\int P dy} = e^{\int (2/y) dy} = e^{2\log y} = e^{\log y^2} = y^2$

The solution is $ve^{\int P dy} = \int e^{\int P dy} \cdot Q\, dy + c$

$\therefore v \cdot y^2 = \int y^2 \left(-\frac{2}{y^3}\right) dy + c$

$\therefore vy^2 = \int -\frac{2}{y} dy + c \qquad \therefore vy^2 = -2\log y + c \qquad \therefore e^{-2x}y^2 + 2\log y = c$

APPLICATION OF DIFFERENTIAL EQUATIONS

Q.1 A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Solve the Differential equation to express v in terms of x.

Solution: The given differential equation is $xv\frac{dv}{dx} + v^2 = gx$ ....(1)

put $v^2 = y \quad \therefore 2v\frac{dv}{dx} = \frac{dy}{dx}$

$\therefore$ From (1) $x\left(\frac{1}{2}\frac{dy}{dx}\right) + y = gx$

$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right)y = 2g$

It is linear differential equation with $P = \frac{2}{x}$ and $Q = 2g$

$\therefore$ Its solution is $ye^{\int P dx} = \int Q \cdot e^{\int P dx}\, dx + c$

$ye^{\int \frac{2}{x} dx} = \int 2g \cdot e^{\int \frac{2}{x} dx} dx + c$

$yx^2 = \int 2g \cdot x^2 dx + c$

$yx^2 = \frac{2}{3}gx^3 + c$

$v^2x^2 = \frac{2g}{3}x^3 + c$

Q.2. In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t if at $t = 0, i = 0$ and L, R, E are constants.

Solution: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $i \cdot e^{\int (R/L)dt} = \int \frac{E}{L} \cdot e^{\int (R/L)dt} dt + c$

$\therefore i \cdot e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\, dt + c = \frac{E}{L} \cdot e^{Rt/L} \cdot \frac{L}{R} + c = \frac{E}{R}e^{Rt/L} + c$

When $t = 0, i = 0 \qquad \therefore c = -\frac{E}{R}$

$\therefore i \cdot e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R} = \frac{E}{R}\left(e^{Rt/L} - 1\right)$

$\therefore i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

Q.3. In a circuit containing inductance L resistance R and voltage E, the current i is given by $E = Ri + L\frac{di}{dt}$ .If $L = 640h, R = 250\ \Omega\ and\ E = 500$volts and $i = 0$ when $t = 0$, find the time that elapses before the current reaches 90% of its maximum value.

Solution: As in above example the current is given by $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

Maximum value of i is I, can be obtained when $t \to \infty$

$\therefore I = \frac{E}{R}(1 - 0) = \frac{E}{R}$

When $i = 90\%$ if $I = \frac{9}{10}I = \frac{9E}{10R}$

We get $\frac{9}{10}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

$\frac{9}{10} = 1 - e^{-Rt/L}$

$$e^{-Rt/L} = 1 - \frac{9}{10} = \frac{1}{10}$$
$$e^{Rt/L} = 10$$
$$Rt/L = \log 10$$
$$t = \frac{L}{R}\log 10 \text{ sec}$$
$$\therefore t = \frac{640}{250}\log 10 = 5.89 \text{ sec}$$

Q.4. The charge q on the plate of a condenser of capacity C charged through a resistance R by the steady voltage V satisfies the differential equation $R\frac{dq}{dt} + \frac{q}{C} = V$. If $q = 0 \ at \ t = 0$, show that $q = CV(1 - e^{-t/RC})$. Find also the current flowing into the plate.

Solution: We are given that $\frac{dq}{dt} + \frac{1}{Rc} \cdot q = \frac{V}{R}$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

Its solution is $q \cdot e^{\int (1/Rc)dt} = \int e^{\int (1/Rc)dt} \cdot \frac{V}{R} dt + k$

$\therefore q \cdot e^{t/RC} = \int e^{t/RC} \cdot \frac{V}{R} dt + k = \frac{V}{R} \cdot \frac{e^{t/RC}}{(1/RC)} + k$

By data when $t = 0, q = 0$ $\qquad \therefore k = -CV$

$\therefore \ q \cdot e^{t/RC} = CV \cdot e^{t/RC} - CV = e^{t/RC}\left(CV - CVe^{-t/RC}\right)$

$\therefore \ q = CV\left(1 - e^{-t/RC}\right)$

Further $i = \frac{dq}{dt} = CV \cdot e^{-t/RC} \cdot \frac{1}{RC} = \frac{V}{R} \cdot e^{-t/RC}$

Q.5. An equation in the theory of stability of an aeroplane is $\frac{dv}{dt} = g \cos\alpha - kv$, v being velocity and g, k being constants. It is observed that at time $t = 0$, the velocity $v = 0$. Solve the equation completely.

Higher Order Linear DE

Wednesday, February 17, 2021 12:00 PM

degree = 1. order > 1

HIGHER ORDER DIFFERENTIAL EQUATION

- **Definition:** An equation of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots . + P_n y = X$ ......(1) Where $P_1, P_2, \ldots \ldots \ldots P_n$ are constants and X is a function of x only is called a **linear differential equation with constant coefficients**.

For example $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = \cos x$

$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 7y = \sinh^2 x$

THE OPERATOR D

- Let D be the symbol which denotes differentiation with respect to x, say, of the function which immediately follows it i.e. D stands for $\frac{d}{dx}$.
- Thus, if y is a differentiable function of x then $D(y) = \frac{d}{dx}(y)$ or $Dy = \frac{dy}{dx}$,
- $D(Dy) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ $\frac{d^2y}{dx^2} + 4y = e^x \rightarrow D^2y + 4y = e^x$
- Let us further denote the operation of D repeated twice, thrice, ……. n times by $D^2, D^3, \ldots \ldots \ldots D^n$.
- With this notation, $D^2 y = \frac{d^2y}{dx^2}, D^3 y = \frac{d^3y}{dx^3}, \ldots \ldots . D^n y = \frac{d^n y}{dx^n}$
- From this point of view the symbol D is called as operator and the function y on which it **operates** is called **operand**.
- With this notation the differential equation (1) can be written as
- $D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + .. + P_n y = X$
- i.e. $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots . + P_n)y = X$ i.e. $\boxed{f(D)y = X}$

$\frac{d^5y}{dx^5} - 4\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - 5y = e^x$

$D^5y - 4D^4y + 3D^3y - D^2y + Dy - 5y = e^x$

$(D^5 - 4D^4 + 3D^3 - D^2 + D - 5)y = e^x$

$f(D)y = e^x$

METHOD TO FIND COMPLETE SOLUTION

$y = y_c + y_p$

- Complete Solution = Complementary Function (C.F.) + Particular integral (P.I.).
- (1) Write the given differential equation in the form $f(D)y = X$
- **(2)** Write the associated equation $f(D)y = 0$ $\rightarrow y_c$
- **(3)** Write the **auxiliary equation** by putting $D = m$ in the terms within bracket when the equation is written in the symbolic form as $f(m) = 0$
- $(D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots + P_n)y = 0$
- Auxiliary equation is $m^n + P_1m^{n-1} + P_2m^{n-2} + \cdots + P_n = 0$
  It is an equation of n[th] degree in m having n roots says $m_1, m_2, m_3 \ldots .. m_n$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^x$$

$$D^3y + 3D^2y + 3Dy + y = e^x$$

$$(D^3 + 3D^2 + 3D + 1)y = e^x \rightarrow f(D)y = X$$

Associated eq^n $f(D)y = 0$ ie $(D^3+3D^2+3D+1)y = 0$

Auxillary eq^n is $f(m)=0$ ie $m^3 + 3m^2 + 3m + 1 = 0$

- **(4)** Write the **complementary function (C.F)** as follows:
- **Case (i)** when roots are **real and different** $m_1, m_2, \cdots, m_n$
- The C.F. is $y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \cdots + C_ne^{m_nx}$

$$(D^2 - 3D + 2)y = 0$$

Aux $\rightarrow m^2 - 3m + 2 = 0$ $(m-2)(m-1) = 0$ $\Rightarrow m = 1, 2$

$$C.F. \Rightarrow y_c = c_1e^x + c_2e^{2x}$$

- **Case (ii)** When roots are **real and equal** i.e. repeated
- **(a)** Suppose the auxiliary equation has got two equal roots. Say, each m_1 and Let the other roots be $m_3, m_4 \ldots .. m_n$ then the C.F is
- $y = (C_1 + C_2x)e^{m_1x} + C_3e^{m_3x} + \ldots + C_ne^{m_nx}$

for eg. $m = 1, 1, 2, 3, -6$

$$y_c = (c_1 + c_2x)e^x + c_3e^{2x} + c_4e^{3x} + c_5e^{-6x}$$

- **(b)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and let the other roots be $m_4, m_5 \ldots \ldots \ldots .. m_n$ then the C.F. is

- **(b)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and let the other roots be $m_4, m_5 \ldots\ldots\ldots..m_n$ then the C.F. is
- $y = (C_1 + C_2x + C_3x^2)e^{m_1x} + C_4e^{m_4x} + \ \ldots.+C_ne^{m_nx}$

for eg. $m = 2, 2, 2, -1, -3, 7$

$y_c = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x} + c_5e^{-3x} + c_6e^{7x}$

- **(c)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and next two equal roots say, each m_2, and let the other roots be $m_6, m_7 \ldots\ldots\ldots\ldots..m_n$ then the C.F is

$$y = (C_1 + C_2x + C_3x^2)e^{m_1x} + (C_4 + C_5x)e^{m_2x} + C_6e^{m_6x} + \ \ldots\ldots.+C_ne^{m_nx}$$

for eg. $m = 3, 3, 3, -1, -1, 2, 5$

$y_c = (c_1 + c_2x + c_3x^2)e^{3x} + (c_4 + c_5x)e^{-x} + c_6e^{2x} + c_7e^{5x}$

- **Case (iii)** when roots are **Imaginary and different** (complex)
- Suppose the auxiliary equation has got two roots $(\alpha + i\beta)$ *and* $(\alpha - i\beta)$ then the part of the solution of the equation corresponding to these roots will be $e^{\alpha x}(C_1 \cos\beta x + \ C_2 \sin\beta x)$

for eg. $3+2i, 3-2i, 1+i, 1-i$

$y_c = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

$\quad + e^{x}(c_3 \cos x + c_4 \sin x)$

- **Case (iv)** when roots are **Imaginary and equal** i.e. repeated
- **(a)** Suppose $(\alpha \pm i\beta)$ occurs twice then the part of the solution with reference to these roots will be $e^{\alpha x}[(C_1 + C_2x)cos\beta x + (C_3 + C_4x)sin\beta x]$

for eg. Suppose $1 \pm i$ repeats twice

$y_c = e^{x}[(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x]$

- **(b)** Suppose $(\alpha \pm i\beta)$ occurs thrice then the part of the solution with reference to these roots will be $e^{\alpha x}[(C_1 + C_2x + C_3x^2)\cos\beta x + (C_4 + C_5x + C_6x^2)\sin\beta x]$

Suppose we get roots of Auxilliary eqn as

$m = 1, 1, -2, 3, 5 \pm i, (2 \pm 3i)$ twice

$$y_c = (c_1 + c_2 x)e^{x} + c_3 e^{-2x} + c_4 e^{3x} + e^{5x}(c_5 \cos x + c_6 \sin x)$$
$$+ e^{2x}\left[(c_7 + c_8 x)\cos 3x + (c_9 + c_{10} x)\sin 3x\right]$$

- **(5)** When the r.h.s $X = 0$ then complete solution = complementary function (i.e no need to find particular integral)
- **(6)** When the r.h.s $X \neq 0$ then we find Particular Integral using following rules.
- $P.I = \frac{1}{f(D)} X$
- The method of finding Particular Integral depends upon the nature of the right hand side X.

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$EXAMPLE - 1: \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Soln:- $D^3y - 6D^2y + 11Dy - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)y = 0$

$f(D)\, y = 0$

Auxillary eqn $f(m) = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$m = 1, 2, 3$

$\therefore$ Roots are distinct and real

$\therefore$ The solution $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

EXAMPLE-2: $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$

Soln:- $D^3y - 5D^2y + 8Dy - 4y = 0$

$$(D^3 - 5D^2 + 8D - 4)\, y = 0$$
$$f(D)\, y = 0$$

Auxillary Eq^n (A.E.) is $f(m) = 0$

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$m = 1, 2, 2$$

The roots are real and repeated

Sol^n is $y = c_1 e^x + (c_2 + c_3 x) e^{2x}$

Ex 3:- $\dfrac{d^4y}{dx^4} + k^4 y = 0$

Sol^n:- $D^4 y + k^4 y = 0$

$$(D^4 + k^4)\, y = 0$$

A.E is $m^4 + k^4 = 0$

$$m^4 + 2m^2k^2 + k^4 - 2m^2k^2 = 0$$
$$(m^2 + k^2)^2 - (\sqrt{2}mk)^2 = 0$$
$$(m^2 + k^2 + \sqrt{2}mk)(m^2 + k^2 - \sqrt{2}mk) = 0$$
$$m^2 + \sqrt{2}mk + k^2 = 0$$
$$m = \frac{-\sqrt{2}k \pm \sqrt{2k^2 - 4k^2}}{2}$$
$$m = \frac{k}{\sqrt{2}}(-1 \pm i)$$

Similarly, $m^2 - \sqrt{2}mk + k^2 = 0$

$$m = \frac{k}{\sqrt{2}}(1 \pm i)$$

$$m^4 + k^4 = 0$$
$$m^4 = -k^4$$
$$m = k(-1)^{1/4}$$
$$= k\left[\cos\pi + i\sin\pi\right]^{1/4}$$
$$= k\left[\cos\left(\frac{2k+1}{4}\right)\pi + i\sin\left(\frac{2k+1}{4}\right)\pi\right]$$
$$k = 0, 1, 2, 3$$

we have complex, distinct roots

$$m = \frac{k}{\sqrt{2}}(1 \pm i), \quad \frac{k}{\sqrt{2}}(-1 \pm i)$$

$\therefore$ The soln is

$$y = e^{\frac{k}{\sqrt{2}}x}\left(c_1 \cos\frac{k}{\sqrt{2}}x + c_2 \sin\frac{k}{\sqrt{2}}x\right)$$

$$+ e^{-\frac{k}{\sqrt{2}}x}\left(c_3 \cos\frac{k}{\sqrt{2}}x + c_4 \sin\frac{k}{\sqrt{2}}x\right)$$

Ex 4 :- $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$

Soln :-

$$D^4y + 2D^2y + y = 0$$

$$(D^4 + 2D^2 + 1)y = 0$$

A·E is $m^4 + 2m^2 + 1 = 0$

$$(m^2+1)^2 = 0$$

$m^2 + 1 = 0$ twice

$m^2 = -1$ twice

$m = \pm i$ twice

roots are complex and repeated

$m = 0 \pm i$ twice

$$y = e^{0x}\left[(c_1 + c_2 x)\sin x + (c_3 + c_4 x)\cos x\right]$$

Ex 5 $\{(D^2+1)^3(D^2+D+1)^2\}y = 0$

Soln :- $\{(D^2+1)^3(D^2+D+1)^2\}y = 0$

A·E. $(m^2+1)^3(m^2+m+1)^2 = 0$

$(m^2+1)^3=0 \Rightarrow m^2+1=0$ 3 times

$\Rightarrow m^2=-1$ 3 times

$\Rightarrow m=\pm i$ 3 times

Also $(m^2+m+1)^2=0 \Rightarrow m^2+m+1=0$ 2 times

$\Rightarrow m=\dfrac{-1\pm i\sqrt{3}}{2}$ 2 times

$\therefore$ Roots are complex and repeated

$m=\pm i$ 3 times, $-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$ 2 times.

$\therefore$ soln is

$$y=(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x$$

$$+e^{-\frac{1}{2}x}\left[(c_7+c_8x)\cos\tfrac{\sqrt{3}}{2}x+(c_9+c_{10}x)\sin\tfrac{\sqrt{3}}{2}x\right]$$

Ex:- $\{(D-1)^4(D^2+2D+2)^2\}y=0$

AE is $(m-1)^4(m^2+2m+2)^2=0$

$(m-1)^4=0 \Rightarrow m-1=0$ 4 times

$= m=1$ repeated 4 times

$(m^2+2m+2)^2=0 \Rightarrow m^2+2m+2=0$ 2 times

$\Rightarrow m=-1\pm i$ 2 times

$\therefore$ soln is

$$y=(c_1+c_2x+c_3x^2+c_4x^3)e^{x}$$

$$+e^{-x}\left[(c_5+c_6x)\cos x+(c_7+c_8x)\sin x\right]$$

Ex:- $(D^2-2D-4)y=0$.

A.E. $m^2-2m-4=0$

$$m=\frac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5}$$

$$y=c_1e^{(1+\sqrt{5})x}+c_2e^{(1-\sqrt{5})x}$$

$$=e^{x}\left[A\cosh\sqrt{5}x+B\sinh\sqrt{5}x\right]$$

Ex:- $\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}+8\frac{d^2y}{dx^2}+8\frac{dy}{dx}+4y=0$

Soln:- A.E. $m^4+4m^3+8m^2+8m+4=0$

$m=-1\pm i$ repeated 2 times

$$y=e^{-x}\left[(c_1+c_2x)\cos x+(c_3+c_4x)\sin x\right]$$

PI WHEN RHS= e^{ax}

Monday, February 22, 2021 11:30 AM

METHOD TO FIND PI

$f(D)y = X$ where X is some function of x

complete solution = complementary Function (C.F.) + particular Integral (PI)

$$y = y_c + y_p$$

$D \to \frac{d}{dx}$

$\frac{1}{D} \to \int dx$

To find y_c, we take $f(D)y = 0$

Auxillary eqn $f(m) = 0$

find roots of A.E. and write y_c.

To find y_p, $f(D)y = X \Rightarrow y_p = \frac{1}{f(D)} X$

This depends on X

- **Case (i) When the r.h.s. $X = e^{ax}$.**
- $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$

$$\frac{1}{D^3+D^2+D} e^{2x} = \frac{1}{2^3+2^2+2} e^{2x} = \frac{1}{8+4+2} e^{2x} = \frac{e^{2x}}{14}$$

- When $f(a) = 0$, $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ If $f'(a) \neq 0$

$\frac{1}{D^2-1} e^{x}$ if we put $D=1$, $D^2-1=0$

$$= x \cdot \frac{1}{2D} e^{x} = x \cdot \frac{1}{2} e^{x} = \frac{xe^x}{2}$$

- When $f'(a) = 0$, $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$ If $f''(a) \neq 0$

$\frac{1}{(D+2)^2} e^{-2x}$ if we put $D=-2$, $(D+2)^2 = 0$

$= x \cdot \frac{1}{} e^{-2x}$ if we put $D=-2$, $D+2=0$

$2(D+2)$

$$= x^2 \cdot \frac{1}{2} e^{-2x} = \frac{x^2 e^{-2x}}{2}$$

- When $f''(a) = 0$, $\frac{1}{f(D)} e^{ax} = x^3 \frac{1}{f'''(a)} e^{ax}$ If $f'''(a) \neq 0$ etc

- EXAMPLE -1: $(D^3 - 2D^2 - 5D + 6)\, y = (e^{2x} + 3)^2$

Soln :- Associated eqn is $(D^3 - 2D^2 - 5D + 6)y = 0$

Auxillary eqn is $m^3 - 2m^2 - 5m + 6 = 0$

$m = -2, 1, 3$

$\therefore$ C.F is $y_c = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x}$

Now $y_p = \frac{1}{D^3 - 2D^2 - 5D + 6} (e^{2x} + 3)^2$

$$= \frac{1}{D^3 - 2D^2 - 5D + 6} (e^{4x} + 6e^{2x} + 9)$$

$$= \underset{(D=4)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{4x} + 6 \underset{(D=2)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{2x}$$

$$+ 9 \underset{(D=0)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{0x}$$

$$y_p = \frac{e^{4x}}{18} - \frac{3}{2}e^{2x} + \frac{3}{2}$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x} + \frac{e^{4x}}{18} - \frac{3e^{2x}}{2} + \frac{3}{2}}$$

EXAMPLE-2: $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3x/2} + 2^x$

Soln :- $(6D^2 + 17D + 12)y = e^{-3x/2} + e^{x\log 2}$

A.E is $6m^2 + 17m + 12 = 0$

$$m = -\frac{4}{3}, -\frac{3}{2}$$

$\therefore$ C.F is $y_c = c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x}$

Now P.I. $= y_p = \frac{1}{6D^2+17D+12}\left(e^{-\frac{3}{2}x} + e^{x\log 2}\right)$

$$= \frac{1}{6D^2+17D+12} e^{-\frac{3}{2}x} + \frac{1}{6D^2+17D+12} e^{x\log 2}$$

$(D = -\frac{3}{2})$ $\quad$ $(D = \log 2)$

$f(D) = 0$

$$= x \cdot \frac{1}{12D+17} e^{-\frac{3}{2}x} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} e^{x\log 2}$$

$(D = -\frac{3}{2})$

$$= -xe^{-\frac{3}{2}x} + \frac{2^x}{6(\log 2)^2 + 17(\log 2) + 12}$$

$\therefore$ complete soln is

$y = y_c + y_p$

$$= c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x} - xe^{-\frac{3}{2}x} + \frac{2^x}{6(\log 2)^2 + 17(\log 2) + 12}$$

EXAMPLE-3: $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh^2 2x$

Soln: $(D^3 - 4D)y = 2\left[\frac{e^{2x} + e^{-2x}}{2}\right]^2 = \frac{1}{2}\left[e^{4x} + e^{-4x} + 2\right]$

A.E is $m^3 - 4m = 0$

$m(m^2 - 4) = 0 \qquad m = 0, 2, -2$

$\therefore$ C.F. is $y_c = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$

$= c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$\therefore$ P.I. $= y_p = \frac{1}{D^3 - 4D} \cdot \frac{1}{2}\left(e^{4x} + e^{-4x} + 2\right)$

$$= \frac{1}{2} \cdot \frac{1}{D^3 - 4D} e^{4x} + \frac{1}{2} \frac{1}{D^3 - 4D} e^{-4x} + \frac{1}{D^3 - 4D} \cdot e^{0x}$$

$(D = 4) \qquad (D = -4) \qquad (D = 0)$

$f(D) = 0$

$$= \frac{1}{2} \cdot \frac{e^{4x}}{48} + \frac{1}{2} \cdot \frac{e^{-4x}}{(-48)} + x \cdot \frac{e^{0x}}{3D^2 - 4}$$

$(D = 0)$

$$= e^{4x} \qquad e^{-4x} \quad - xe^{0x}$$

$$= \frac{e^{4x}}{96} - \frac{e^{-4x}}{96} - \frac{x e^{0x}}{4}$$

$$= \frac{1}{48}\left[\frac{e^{4x} - e^{-4x}}{2}\right] - \frac{x}{4}$$

$$y_p = \frac{\sinh 4x}{48} - \frac{x}{4}$$

$\therefore$ The complete soln is

$$y = y_c + y_p = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{\sinh 4x}{48} - \frac{x}{4}$$

Ex4 $(D^2 - D - 6)y = e^x \cosh 2x$

Soln $\therefore$ A.E is $m^2 - m - 6 = 0$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

$\therefore$ C.F is $y_c = c_1 e^{3x} + c_2 e^{-2x}$

Now P.I. $= y_p = \frac{1}{D^2 - D - 6}(e^x \cosh 2x)$

$$= \frac{1}{D^2 - D - 6}\left[e^x \cdot \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - D - 6} e^{3x} + \frac{1}{2} \frac{1}{D^2 - D - 6} e^{-x}$$

$D = 3$ $\qquad\qquad\qquad D = -1$

$f(D) = 0$

$$= \frac{x}{2} \cdot \frac{1}{2D-1} e^{3x} + \frac{1}{2} \cdot \frac{1}{1+1-6} e^{-x}$$

$(D=3)$

$$= \frac{x}{2} \cdot \frac{1}{5} e^{3x} + \frac{1}{2} \cdot \frac{e^{-x}}{-4}$$

$$y_p = \frac{xe^{3x}}{10} - \frac{e^{-x}}{8}$$

$\therefore$ The complete soln is

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-2x} + \frac{xe^{3x}}{10} - \frac{e^{-x}}{8}$$

$D(e^{ax}) = a e^{ax}$

$D(\sin ax) = a\cos ax$

$D^2(\sin ax) = -a^2 \sin ax$

METHOD TO FIND PI WHEN RHS= $\sin \boldsymbol{ax}$, $\cos \boldsymbol{ax}$

- **Case (ii) When the r.h.s.** $X = \boldsymbol{\sin ax},\ \boldsymbol{\cos ax}$
- $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax,\ \ if\ f(-a^2) \neq 0$ And
$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax,\ \ if\ f(-a^2) \neq 0$

put $D^2 = -a^2$

$$\frac{1}{D^4+3D^2+2}\sin 2x = \frac{1}{(-4)^2+3(-4)+2}\sin 2x = \frac{1}{16-12+2}\sin 2x = \frac{\sin 2x}{6}$$

put $D^2 = -2^2 = -4$

- When $f(-a^2) = 0\ then\ \frac{1}{f(D^2)}\sin ax = \frac{x}{f'(-a^2)}\sin ax,\ \ If\ f'(-a^2) \neq 0$ And
- When $f(-a^2) = 0\ then\ \frac{1}{f(D^2)}\cos ax = \frac{x}{f'(-a^2)}\cos ax,\ \ If\ f'(-a^2) \neq 0$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2D}\cos ax = \frac{x}{2}\int \cos ax\, dx = \frac{x\sin ax}{2a}$$

put $D^2 = -a^2$

EXAMPLE $-$ 1: $(D^3 + D^2 + D + 1)y = \sin^2 x$

Soln :- A.E. $m^3+m^2+m+1=0 \rightarrow m^2(m+1)+1(m+1)=0$

$(m+1)(m^2+1)=0$

$m=-1,\ m^2=-1$

$m=-1,\ m=\pm i$

$m = -1, \pm i$

$\therefore$ C.F is

$$y_c = c_1 e^{-x} + e^{0x}\left[c_2\cos x + c_3 \sin x\right]$$

$$y_c = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$$

Now $PI = y_p = \frac{1}{D^3+D^2+D+1}\sin^2 x$

$$= \frac{1}{D^3+D^2+D+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$\frac{}{D^3+D^2+D+1}\left(\frac{}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^3+D^2+D+1}e^{0x} - \frac{1}{2}\frac{1}{D^3+D^2+D+1}\cos 2x$$

(put $D=0$) (put $D^2=-4$)

$$= \frac{1}{2}\cdot\frac{1}{0+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-4D+(-4)+D+1}\cos 2x$$

$$y_p = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{-3(D+1)}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}\cdot\frac{D-1}{D^2-1}\cos 2x$$

(put $D^2=-4$)

$$= \frac{1}{2} + \frac{1}{6}\cdot\frac{(D-1)}{-5}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{30}(D-1)\cos 2x$$

$$= \frac{1}{2} - \frac{1}{30}\left[D(\cos 2x) - \cos 2x\right]$$

$$= \frac{1}{2} - \frac{1}{30}\left[-2\sin 2x - \cos 2x\right]$$

$$y_p = \frac{1}{2} + \frac{2\sin 2x + \cos 2x}{30}$$

$\therefore$ complete solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{2} + \frac{2\sin 2x + \cos 2x}{30}}$$

EXAMPLE-2: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 27y = \cos 3x$

Soln.:- $(D^3 - 3D^2 + 9D - 27)y = \cos 3x$

A.E is $m^3 - 3m^2 + 9m - 27 = 0 \rightarrow m^2(m-3) + 9(m-3) = 0$

$(m-3)(m^2+9) = 0$

$m = 3, \pm 3i$

$m = 3, 3i, -3i$

$\therefore$ C.F is

$$y_c = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x$$

$$PI = y_p = \frac{1}{D^3 - 3D^2 + 9D - 27} \cos 3x$$

put $D^2 = -3^2 = -9$, Denominator $= 0$

$$= \frac{x}{3D^2 - 6D + 9} \cos 3x$$

put $D^2 = -9$

$$= \frac{x}{-6D - 18} \cos 3x = -\frac{x}{6} \cdot \frac{1}{D+3} \cos 3x$$

$$= -\frac{x}{6} \cdot \frac{D-3}{D^2 - 9} \cos 3x$$

put $D^2 = -9$

$$= -\frac{x}{6} \cdot \frac{D-3}{-18} \cos 3x$$

$$= \frac{x}{108} \left[D(\cos 3x) - 3\cos 3x \right]$$

$$= \frac{x}{108} \left[-3\sin 3x - 3\cos 3x \right]$$

$$\therefore y_p = \frac{-x}{36} (\sin 3x + \cos 3x)$$

$$\therefore \boxed{y = y_c + y_p = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36} [\sin 3x}$$

$$\therefore \boxed{y = y_c + y_p = c_1 e^{3x} + c_2\cos 3x + c_3 \sin 3x - \frac{x}{36}\left[\sin 3x + \cos 3x\right]}$$

EXAMPLE-3: $(D^4 - 1)y = e^x + \cos x \cos 3x$

Soln: A.E is $m^4 - 1 = 0$

$(m^2+1)(m^2-1) = 0$

$m = 1, -1, i, -i$

$\therefore$ C F is $y_c = c_1 e^{x} + c_2 e^{-x} + c_3\cos x + c_4 \sin x$

P I is $y_p = \frac{1}{D^4-1}\left[e^x + \frac{1}{2}(\cos 2x + \cos 4x)\right]$

$$= \frac{1}{D^4-1} e^x + \frac{1}{2}\frac{1}{D^4-1}\cos 2x + \frac{1}{2}\frac{1}{D^4-1}\cos 4x$$

put $D=1$ put $D^2=-4$ put $D^2 = -16$

$f(D)=0$

$$= \frac{x}{4D^3} e^x + \frac{1}{2}\cdot\frac{1}{16-1}\cos 2x + \frac{1}{2}\cdot\frac{1}{256-1}\cos 4x$$

$$y_p = \frac{xe^x}{4} + \frac{\cos 2x}{30} + \frac{\cos 4x}{510}$$

$\therefore y = y_c + y_p$

$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{xe^x}{4} + \frac{\cos 2x}{30} + \frac{\cos 4x}{510}$$

Example-4: $\frac{d^2y}{dx^2} + y = \sin x \sin 2x + 2^x$

Soln: $(D^2+1)y = \sin x \sin 2x + 2^x$

AE is $m^2+1=0$ $m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

PI. $y_p = \dfrac{1}{D^2+1}\left[\sin x \sin 2x + 2^x\right]$

$$= \frac{1}{D^2+1}\left[\frac{1}{2}(\cos x - \cos 3x) + e^{x\log 2}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{1}{D^2+1}\cos 3x + \frac{1}{D^2+1}e^{x\log 2}$$

put $D^2=-1$
$D^2+1=0$

put $D^2=-9$

put $D=\log 2$

$$= \frac{1}{2}\cdot\frac{x}{2D}\cos x - \frac{1}{2}\cdot\frac{1}{-8}\cos 3x + \frac{1}{(\log 2)^2+1}e^{x\log 2}$$

$$= \frac{x}{4}\int \cos x\, dx + \frac{1}{16}\cos 3x + \frac{2^x}{(\log 2)^2+1}$$

$$= \frac{x\sin x}{4} + \frac{\cos 3x}{16} + \frac{2^x}{(\log 2)^2+1}$$

$\therefore$ $y = y_c + y_p$

$$= c_1\cos x + c_2 \sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16} + \frac{2^x}{(\log 2)^2+1}$$

PI WHEN RHS = x^m

Wednesday, February 24, 2021 11:30 AM

- **Case (iii) When the r.h.s. $X = x^m$ where m is a positive integer**
- When $X = x^m$, we write f(D) in ascending powers of D by putting it in the form $1 + F(D)$.
- Then, $P.I = \frac{1}{f(D)} x^m = \frac{1}{1+\emptyset(D)} x^m = [1 + \emptyset(D)]^{-1} x^m$
- By expanding the bracket by the formula,
- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots.$ **or**
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$ **or**
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots\ldots.$ **or**
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$

$$\frac{1}{D^2+D+1} x^3 = \frac{1}{1+(D^2+D)} x^3 \qquad [1+\phi(D)]^{-1}$$

$$= [1 + (D^2+D)]^{-1} x^3$$

$$(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \ldots$$

$$= [1 - (D^2+D) + (D^2+D)^2 - (D^2+D)^3 + (D^2+D)^4 \ldots] x^3$$

$$= [1 - D^2 - D + \cancel{D^4} + 2D^3 + D^2 - \cancel{D^6} - 3\cancel{D^5} - 3\cancel{D^4} - D^3] x^3$$

$$= x^3 - 6x - 3x^2 + 12 + 6x - 6 = x^3 - 3x^2 + 6$$

- and by operating each term of the expansion on xm we get the required Particular Integral.
- It is obvious that in the expansion, terms beyond the m^th power of D need not be written
- since the derivatives of x^m of powers higher than m are zero.

- EXAMPLE-1: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2 \rightarrow (D^3 - 2D + 4)y = 3x^2 - 5x + 2$

<u>Solution</u> A.E is $m^3 - 2m + 4 = 0$

$$m = -2,\ 1 \pm i$$

$\therefore$ The C.F is $y_c = c_1 e^{-2x} + e^{x}\left[c_2 \cos x + c_3 \sin x\right]$

The PI is $y_p = \dfrac{1}{D^3 - 2D + 4}\left(3x^2 - 5x + 2\right)$

$$= \frac{1}{4} \cdot \frac{1}{1 + \left(\frac{D^3 - 2D}{4}\right)}\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[1 + \left(\frac{D^3 - 2D}{4}\right)\right]^{-1}\left(3x^2 - 5x + 2\right)$$

(wkt $(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \cdots\cdots$)

$$y_p = \frac{1}{4}\cdot\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3 + \cdots\cdots\right]\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[1 + \frac{2D}{4} + \frac{4D^2}{16}\right]\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[\left(3x^2 - 5x + 2\right) + \frac{1}{2}\left(6x - 5\right) + \frac{1}{4}(6)\right]$$

$$y_p = \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$$

The complete solution is

$$y = y_c + y_p = c_1 e^{-2x} + e^{x}\left[c_2 \cos x + c_3 \sin x\right] + \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$$

EXAMPLE-2: $\left(D^3 - D^2 - 6D\right)y = x^2 + 1$

Soln: A.E is $m^3 - m^2 - 6m = 0$

$$m\left(m^2 - m - 6\right) = 0$$

$$m = 0, -2, 3$$

C.F is $y_c = c_1 + c_2 e^{-2x} + c_3 e^{3x}$

PI is $y_p = \dfrac{1}{D^3 - D^2 - 6D}(x^2+1)$

$$= \frac{-1}{6D} \cdot \frac{1}{1+\left(\frac{D-D^2}{6}\right)}(x^2+1)$$

$$= \frac{-1}{6D} \cdot \left[1+\left(\frac{D-D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$y_p = \frac{-1}{6D}\left[1-\left(\frac{D-D^2}{6}\right)+\left(\frac{D-D^2}{6}\right)^2 \cdots\cdots\right](x^2+1)$$

$$= \frac{-1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\right](x^2+1)$$

$$= \frac{-1}{6D}\left[x^2+1-\frac{1}{6}(2x)+\frac{2}{6}+\frac{2}{36}\right]$$

$$= \frac{-1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)dx$$

$$y_p = -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

$\therefore$ The complete solution is

$$y = y_c + y_p = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25x}{18}\right]$$

EXAMPLE-3: $\frac{d^3y}{dt^3}+\frac{dy}{dt} = \cos t + t^2 + 3 \rightarrow (D^3+D)y = \cos t + t^2 + 3$

Sol^n:- A.E. $m^3 + m = 0$

$m = 0, \pm i$

The C.F is $y_c = c_1 + c_2 \cos t + c_3 \sin t$

$$y_p = \frac{1}{D^3+D}(\cos t + t^2 + 3)$$

$$= \frac{1}{D^3+D}(\cos t) + \frac{1}{D^3+D}(t^2+3)$$

$$\frac{1}{D^3+D}\cos t = \frac{t}{3D^2+1}\cos t = \frac{-t}{2}\cos t$$

$D^2 = -1^2$ deno $= 0$ put $D^2 = -1$

$$\frac{1}{D^3+D}(t^2+3) = \frac{1}{D[1+D^2]}(t^2+3)$$

$$= \frac{1}{D}\cdot[1+D^2]^{-1}(t^2+3)$$

$$= \frac{1}{D}[1 - D^2 + D^4 - D^6 \cdots](t^2+3)$$

$$= \frac{1}{D}(1-D^2)(t^2+3)$$

$$= \frac{1}{D}(t^2+3-2)$$

$$= \frac{1}{D}(t^2+1)$$

$$= \int (t^2+1)\,dt$$

$$= \frac{t^3}{3} + t$$

$$\therefore y_p = -\frac{t}{2}\cos t + \frac{t^3}{3} + t$$

$$\therefore y = y_c + y_p = c_1 + c_2\cos t + c_3 \sin t - \frac{t}{2}\cos t + \frac{t^3}{3} + t$$

<u>Another way</u>

Another way

$$\frac{1}{D^3+D}\cos t = \frac{1}{D[1+D^2]}\cos t = \frac{1}{D}\cdot\frac{1}{1+D^2}\cos t$$

put $D^2=-1$
$D^2+1=0$

$$= \frac{1}{D}\cdot\frac{t}{2D}\cos t$$

$$= \frac{1}{D}\cdot\frac{t}{2}\sin t$$

$$= \frac{1}{2}\int t\sin t$$

$$\frac{1}{D^3+D}\cos t \qquad = \frac{1}{2}\left[-t\cos t + \sin t\right]$$

$$\therefore\ y = c_1 + c_2\cos t + c_3\sin t - \frac{t}{2}\cos t + \frac{1}{2}\sin t + \frac{t^3}{3} + t$$

METHOD TO FIND PI WHEN RHS= $e^{ax}\,V$

- **Case (iv) When the r.h.s. $X = e^{ax}\,V$ where V is a function of x.**
- $\frac{1}{f(D)} e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$

$EXAMPLE-1: \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \left(x^2e^x\right)^2$

Soln:-

$$(D^2-4D+3)y = x^4e^{2x}$$

A.E is $m^2-4m+3=0$

$m = 1, 3$

C.F is $y_c = c_1e^x + c_2e^{3x}$

$$y_p = \frac{1}{D^2-4D+3}\,e^{2x}.x^4$$

$$\frac{1}{f(D)}e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}V$$

$$= e^{2x}\frac{1}{(D+2)^2-4(D+2)+3}x^4$$

$$= e^{2x}.\frac{1}{D^2+4D+4-4D-8+3}x^4$$

$$= e^{2x}.\frac{1}{D^2-1}x^4$$

$$= -e^{2x}.\frac{1}{1-D^2}x^4$$

$$= -e^{2x}\left(1-D^2\right)^{-1} x^4$$

$$\left[\text{wkt } (1-t)^{-1} = 1+t+t^2+t^3+\cdots\cdots\right]$$

$$y_p = -e^{2x}\left[1+D^2+D^4+D^6+\cdots\right]x^4$$

$$= -e^{2x}\left(1+D^2+D^4\right)x^4$$

$$y_p = -e^{2x}\left(x^4+12x^2+24\right)$$

$\therefore$ The complete soln is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{3x} - e^{2x}\left(x^4+12x^2+24\right)$$

EXAMPLE – 2: $(D^3+1)y = e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$

Soln:- A·E is $m^3+1=0$

$$m = -1,\ \frac{1 \pm i\sqrt{3}}{2}$$

$$y_c = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{D^3+1}\, e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{x/2}\cdot\frac{1}{\left(D+\frac{1}{2}\right)^3+1}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{\ [illegible]\ }\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$D^3 + \frac{3}{2}D^2 + \frac{3}{4}D + \frac{9}{8}$$

put $D^2 = -\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{-3}{4}$

denominator becomes zero

$$= e^{\frac{x}{2}} \cdot \frac{x}{3D^2 + 3D + \frac{3}{4}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

put $D^2 = -\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{-3}{4}$

$$\therefore y_p = e^{\frac{x}{2}} \frac{x}{3D - \frac{3}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}} \cdot \frac{x\left(3D + \frac{3}{2}\right)}{9D^2 - \frac{9}{4}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{e^{\frac{x}{2}} \cdot x \left[3\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{3}{2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}{9\left(\frac{-3}{4}\right) - \frac{9}{4}}$$

$$\therefore y_p = -\frac{xe^{\frac{x}{2}}}{6}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$\therefore$ complete solution is

$$y = y_c + y_p$$

$$= c_1e^{-x} + e^{\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] - \frac{xe^{\frac{x}{2}}}{6}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

- $EXAMPLE - 3$: $(D^2 + 2)y = e^x cosx + x^2 e^{3x}$

Sol^n:- A.E is $m^2+2=0$

$$m = \pm\sqrt{2}i$$

$\therefore$ CF is $y_c = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)$

$$y_p = \frac{1}{D^2+2}\left(e^x\cos x + x^2e^{3x}\right)$$

$$= \frac{1}{D^2+2}e^x\cos x + \frac{1}{D^2+2}e^{3x}x^2$$

$$\left[\frac{1}{f(D)}e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}.V\right]$$

$$y_p = e^x.\frac{1}{(D+1)^2+2}\cos x + e^{3x}.\frac{1}{(D+3)^2+2}x^2$$

$$= e^x.\frac{1}{D^2+2D+3}\cos x + e^{3x}.\frac{1}{D^2+6D+11}x^2 \quad \text{--- (1)}$$

(put $D^2=-1$)

$$\frac{1}{D^2+2D+3}\cos x = \frac{1}{-1+2D+3}\cos x = \frac{1}{2D+2}\cos x$$

$$= \frac{1\,(D-1)}{2(D+1)(D-1)}\cos x = \frac{D-1}{2(D^2-1)}\cos x$$

(put $D^2=-1$)

$$= \frac{-1}{4}(D-1)\cos x$$

$$= \frac{-1}{4}(-\sin x - \cos x) = \frac{1}{4}(\sin x + \cos x)$$

Also $\frac{1}{D^2+6D+11}x^2 = \frac{1}{11\left[1+\frac{D^2+6D}{11}\right]}x^2$

$$= \frac{1}{11}\left[1+\frac{D^2+6D}{11}\right]^{-1}x^2$$

$$\left[(1+t)^{-1} = 1-t+t^2-t^3\ldots\ldots\right]$$

$$= \frac{1}{11}\left[1-\left(\frac{D^2+6D}{11}\right)+\left(\frac{D^2+6D}{11}\right)^2+\cdots\right]x^2$$

$$= \frac{1}{11}\left[1-\frac{D^2}{11}-\frac{6D}{11}+\frac{36D^2}{121}\right]x^2$$

$$= \frac{1}{11}\left[x^2-\frac{2}{11}-\frac{12x}{11}+\frac{72}{121}\right]$$

$$= \frac{1}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

Substituting in (1)

$$y_p = e^x\cdot\frac{1}{4}(\sin x+\cos x)+\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

$\therefore$ The complete solution is

$$y = y_c+y_p$$

$$= c_1\cos(\sqrt{2}x)+c_2\sin(\sqrt{2}x)+\frac{e^x}{4}(\sin x+\cos x)$$

$$+\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

EXAMPLE-4: $(D^2-1)y = x^2\sin 3x$

Sol^n

Soln:- A.E is $m^2-1=0$

$$m=\pm 1$$

$\therefore$ C.F is $y_c = c_1 e^{x} + c_2 e^{-x}$

$$y_p = \frac{1}{D^2-1}(x^2 \sin 3x)$$

$\left[e^{i3x} = \cos 3x + i \sin 3x \right]$

$$= \text{I.P of } \frac{1}{D^2-1}\left(x^2 e^{3ix}\right)$$

$$\left[\frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \cdot \frac{1}{f(D+a)} V \right]$$

$$= \text{I.P. of } e^{i3x} \cdot \frac{1}{(D+3i)^2-1} \cdot x^2$$

$$= \text{I.P of } e^{i3x} \cdot \frac{1}{D^2+6iD-10} x^2$$

$$= \text{I.P. of } e^{i3x} \cdot \left(\frac{-1}{10}\right) \frac{1}{\left[1-\left(\frac{D^2+6iD}{10}\right)\right]} x^2$$

$$= \text{I.P of } \left(\frac{e^{i3x}}{-10}\right)\left[1-\left(\frac{D^2+6iD}{10}\right)\right]^{-1} x^2$$

$$\left[(1-t)^{-1} = 1+t+t^2+t^3+\cdots\cdots \right]$$

$$= \text{I.P of } \left(-\frac{e^{i3x}}{10}\right)\left[1+\left(\frac{D^2+6iD}{10}\right)+\left(\frac{D^2+6iD}{10}\right)^2+\cdots\right] x^2$$

$$= I.P \text{ of } \left(\frac{-e^{i3x}}{10}\right)\left[1+\frac{D^2}{10}+\frac{3iD}{5}-\frac{36}{100}D^2\right]x^2$$

$$= I.P \text{ of } \left(\frac{-e^{i3x}}{10}\right)\left[x^2+\frac{2}{10}+\frac{3i(2x)}{5}-\frac{72}{100}\right]$$

$$= I.P. \text{ of } \left(-\frac{e^{i3x}}{10}\right)\left[\left(x^2-\frac{52}{100}\right)+i\left(\frac{6x}{5}\right)\right]$$

$$= I.P \text{ of } \left(\frac{-1}{10}\right)\left(\cos 3x+i\sin 3x\right)\left[\left(x^2-\frac{52}{100}\right)+i\left(\frac{6x}{5}\right)\right]$$

$$y_p = \left(\frac{-1}{10}\right)\left[\frac{6x}{5}\cos 3x+\left(x^2-\frac{13}{25}\right)\sin 3x\right]$$

The complete solution is

$$y = y_c + y_p$$

$$= c_1e^{x}+c_2e^{-x}-\frac{1}{10}\left[x^2\sin 3x+\frac{6x}{5}\cos 3x-\frac{13}{25}\sin 3x\right]$$

Example-5: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = xe^{-x}\cos x$

$$(D^2+2D+1)\,y = xe^{-x}\cos x$$

Sol^n :- A.E is $m^2+2m+1=0$

$$(m+1)^2=0 \qquad m=-1,-1$$

$\therefore$ C.F is $y_c = c_1e^{-x}+c_2xe^{-x} = (c_1+c_2x)e^{-x}$

$$y_p = \frac{1}{(D+1)^2}\,xe^{-x}\cos x$$

$$\frac{1}{f(D)}e^{ax}V = e^{ax}.\frac{1}{f(D+a)}V$$

$$y_p = \frac{1}{(D+1)^2} x e^{-x} \cos x$$

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} V$$

$$= e^{-x} \frac{1}{(D-1+1)^2} x \cos x$$

$$\frac{1}{D} \to \text{Integration}$$

$$= e^{-x} \cdot \frac{1}{D^2} x \cos x$$

$$= e^{-x} \cdot \frac{1}{D} \int x \cos x \, dx$$

$$= e^{-x} \cdot \frac{1}{D} \left[x \sin x - \int (1) \sin x \right]$$

$$= e^{-x} \cdot \frac{1}{D} \left[x \sin x + \cos x \right]$$

$$= e^{-x} \cdot \int (x \sin x + \cos x) \, dx$$

$$= e^{-x} \left\{ x(-\cos x) - \int (1)(-\cos x) \, dx + \sin x \right\}$$

$$= e^{-x} \left\{ -x \cos x + \sin x + \sin x \right\}$$

$$y_p = e^{-x} \left\{ 2 \sin x - x \cos x \right\}$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) e^{-x} + e^{-x} \left\{ 2 \sin x - x \cos x \right\}$$

METHOD OF VARIATION OF PARAMETERS

Sunday, April 24, 2022 11:24 AM

This is one of the methods for finding the Particular Integral (P.I.) of a linear differential equation whose Complimentary function (C.F.) is known.
Though the method is general, we will illustrate it by applying it to a second order and third order differential equation.

(1) Consider the linear equation of second order with constant coefficients. $aD^2y + bDy + cy = X$

i.e. $(aD^2 + bD + c)\, y = X$ $\quad f(D)y = X$

$m = 1, 2$

$C.F = c_1 e^{x} + c_2 e^{2x}$

$y_1 = e^{x}$

$y_2 = e^{2x}$

Let Complementary function $= c_1\, y_1 + c_2 y_2$ then Particular Integral $= uy_1 + vy_2$ where

$u = -\int \frac{y_2 X}{W} dx$ & $V = \int \frac{y_1 X}{W} dx$ & $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow$ wronskin

$\therefore$ General solution = Complementary function + Particular Integral.

① $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore$ C.F $= y_c = c_1 \cos ax + c_2 \sin ax$

$= c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos ax, \quad y_2 = \sin ax$

let $PI = uy_1 + vy_2$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} =$

$= a(\cos^2 ax + \sin^2 ax)$

$W = a$

Now $u = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin ax \cdot \sec ax}{a} dx$

$= \frac{-1}{a}\int \tan ax\, dx = \frac{-1}{a}\left[\frac{\log|\sec ax|}{a}\right]$

$= \frac{1}{a^2} \log|\cos ax|$

$$V = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos ax \cdot \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int dx = \frac{x}{a}$$

$$\therefore\ y_p = PI = u y_1 + V y_2 = \frac{1}{a^2}\log|\cos ax| \cdot \cos ax$$

$$+ \frac{x}{a}\sin ax$$

$\therefore$ Complete solution $= y_c + y_p$

$$= c_1\cos ax + c_2 \sin ax + \frac{1}{a^2}\cos ax \cdot \log|\cos ax|$$

$$+ \frac{x}{a}\sin ax$$

② $(D^2 - 2D + 2)\,y = e^x \tan x$

Soln: A.E. $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

$$\therefore\ CF = y_c = e^x(c_1\cos x + c_2 \sin x)$$

$$= c_1 e^x \cos x + c_2 e^x \sin x$$

$$= c_1 y_1 + c_2 y_2$$

$y_1 = e^x\cos x \qquad y_2 = e^x \sin x$

Let $y_p = PI = u y_1 + V y_2$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$

$$= e^{2x}(\sin x\cos x + \cos^2 x) - e^{2x}(\sin x\cos x - \sin^2 x)$$

$$= e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$$

Now $u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\,dx$

$$\int W \qquad \int \qquad e^{2x}$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int \sec x\,dx + \int \cos x\,dx$$

$$u = -\log|\sec x + \tan x| + \sin x$$

Also, $V = \int \frac{y_1 X}{W} = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx$

$$V = \int \sin x\,dx = -\cos x$$

$\therefore\ y_p = u y_1 + V y_2$

$$= e^x \cos x \left[-\log|\sec x + \tan x| + \sin x\right]$$

$$+ e^x \sin x\,(-\cos x)$$

$$y_p = -e^x \cos x \log|\sec x + \tan x|$$

$\therefore$ complete solⁿ $= y_c + y_p$

$$= c_1 e^x \cos x + c_2 e^x \sin x - e^x \cos x \log|\sec x + \tan x|$$

(3) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

A.E. $\rightarrow$ $m^2 + 3m + 2 = 0$

$$m = -1, -2$$

$$CF = y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$y_1 = e^{-x} \qquad y_2 = e^{-2x}$

$\therefore y_1 = e^{-x} \quad , \quad y_2 = e^{-2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x}$$
$$= -e^{-3x}$$

$$u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{e^{-2x}\, e^{e^x}}{-e^{-3x}}\, dx = \int e^x . e^{e^x}\, dx$$

put $e^x = t \Rightarrow e^x dx = dt$

$$u = \int e^t\, dt = e^t = e^{e^x}$$

$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^{-x} . e^{e^x}}{-e^{-3x}}\, dx = -\int e^{2x} . e^{e^x}\, dx$$

put $e^x = t \Rightarrow e^x dx = dt$

$$v = -\int t e^t\, dt = -\left[t(e^t) - \int (1)\, e^t\, dt \right]$$
$$= -\left[te^t - e^t \right] = -\left[e^x e^{e^x} - e^{e^x} \right]$$

$$v = -e^{e^x}(e^x - 1)$$

$$\therefore y_p = u y_1 + v y_2 = e^{e^x} . e^{-x} - e^{e^x}(e^x - 1)\, e^{-2x}$$
$$= e^{e^x}\left[e^{-x} - e^{-x} + e^{-2x} \right]$$

$$y_p = e^{-2x} . e^{e^x}$$

$$\therefore y = y_c + y_2$$
$$= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} . e^{e^x}$$

(2) Consider the linear equation of third order with constants coefficient $aD^3y + bD^2y + cDy + dy = X$

i.e. $(aD^3 + bD^2 + cD + d)\,y = X$ $\quad f(D)y = X$

Let Complementary function $= c_1 y_1 + c_2 y_2 + c_3 y_3$ then Particular Integral $= uy_1 + vy_2 + wy_3$ where

$u = \int \frac{(y_2y_3' - y_3y_2')X}{W}dx,$ $\quad v = \int \frac{(y_3y_1' - y_1y_3')X}{W}dx,$ $\quad w = \int \frac{(y_1y_2' - y_2y_1')X}{W}dx$

Where $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$\therefore$ General Solution = Complementary function + Particular Integral.

① $(D^3 + 4D)\,y = 4\cot 2x$

Soln: A.E. $m^3 + 4m = 0 \Rightarrow m = 0, \pm 2i$

$\therefore y_c = CF = c_1 e^{0x} + c_2\cos 2x + c_3 \sin 2x$

$= c_1 + c_2 \cos 2x + c_3 \sin 2x$

$= c_1 y_1 + c_2 y_2 + c_3 y_3$

$\therefore y_1 = 1, \quad y_2 = \cos 2x, \quad y_3 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8(\cos^2 2x + \sin^2 2x) = 8$$

$PI = y_p = uy_1 + vy_2 + wy_3$

Now $u = \int \frac{(y_2 y_3' - y_3 y_2')X}{W} dx$

$$= \int \frac{[\cos 2x \cdot (2\cos 2x) - \sin 2x(-2\sin 2x)]\,4\cot 2x}{8} dx$$

$$u = \int \cot 2x \, dx = \frac{1}{2}\log|\sin 2x|$$

$$V = \int \frac{(y_3 y_1' - y_1 y_3')}{W} X \, dx =$$

$$= \int \frac{[\sin 2x (0) - (1)(2\cos 2x)]}{8} \cdot 4\cot 2x \, dx$$

$$= -\int \frac{\cos^2 2x}{\sin 2x} dx = -\int \frac{1 - \sin^2 2x}{\sin 2x} dx$$

$$= -\int \operatorname{cosec} 2x \, dx + \int \sin 2x \, dx$$

$$V = -\frac{1}{2} \log|\operatorname{cosec} 2x - \cot 2x| - \frac{1}{2}\cos 2x$$

$$W = \int \frac{(y_1 y_2' - y_2 y_1')}{W} \cdot X \, dx$$

$$= \int \frac{[(1)(-2\sin 2x) - \cos 2x (0)]}{8} \cdot 4\cot 2x \, dx$$

$$W = -\int \cos 2x \, dx = -\frac{1}{2}\sin 2x$$

$$\therefore y_p = u y_1 + V y_2 + W y_3$$

$$= \frac{1}{2}\log|\sin 2x| + \cos 2x \cdot \left(-\frac{1}{2}\right)\left[\log|\operatorname{cosec} 2x - \cot 2x| + \cos 2x\right]$$

$$-\frac{1}{2}\sin^2 2x$$

$$= \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}(\cos^2 2x + \sin^2 2x)$$

$$y_p = \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}$$

$\therefore$ complete solution

$$y = y_c + y_p$$
$$= c_1 + c_2 \cos 2x + c_3 \sin 2x$$
$$+ \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}$$

$$y = C + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2}\log|\sin 2x|$$
$$- \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$

$$\left[C = c_1 - \frac{1}{2} \right]$$