Unit 2.5

System of Equations using Numerical Methods

❖ A System of Linear Equations:

Definition: Numerical method in which we start with some random (Initial) solution of system of equations and use previous iteration values in rearranged equations to find next values are called **Iterative Methods**.

We will see following Two Iterative Methods:

1. **Gauss Jacobi's Method:**
   In this method, we will use **previous iteration values** to calculate next value.

2. **Gauss Seidel Method:**
   In this method, we will use **latest two values** instead of previous iteration values to calculate next value.

Note:

i. A sufficient condition for method to converge is that the coefficient matrix A of order n should be **strictly or irreducibly diagonally dominant.**
   i.e. $a_{ii} \geq \sum_{j\neq i}|a_{ij}|$, $for\ every\ 1 < i < n$

ii. If the initial value to start the iterations is not provided in the problem then we can assume it to be $x = 0,\ y = 0\ and\ z = 0$

SOME SOLVED EXAMPLES

GAUSS JACOBI'S METHOD

1. Solve the following equations by Gauss-Jacobi's Method

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Rewrite given equations as,

$$x = \frac{1}{20}(17 - y + 2z)$$
$$y = \frac{1}{20}(-18 - 3x + z)$$
$$z = \frac{1}{20}(25 - 2x + 3y)$$

(i) First iteration:

start with $x_0 = 0,\ y_0 = 0\ and\ z_0 = 0$

$$x_1 = \frac{17}{20} = 0.85\ ,\ y_1 = \frac{-18}{20} = -0.9\ ,\ z_1 = \frac{25}{20} = 1.25$$

(ii) Second iteration:

Use $x_1 = 0.85,\ y_1 = -0.9\ and\ z_1 = 1.25$

$$x_2 = \frac{1}{20}\big(17 - (-0.9) + 2(1.25)\big) = 1.02$$
$$y_2 = \frac{1}{20}\big(-18 - 3(0.85) + (1.25)\big) = -0.965$$
$$z_2 = \frac{1}{20}\big(25 - 2(0.85) + 3(-0.9)\big) = 1.03$$

(iii) **Third iteration:**

Use $x_2 = 1.02,\ y_2 = -0.965\ and\ z_2 = 1.03$

$$x_3 = \frac{1}{20}\big(17 - (-0.965) + 2(1.03)\big) = 1.00125$$
$$y_3 = \frac{1}{20}\big(-18 - 3(1.02) + (1.03)\big) = -1.0015$$
$$z_3 = \frac{1}{20}\big(25 - 2(1.02) + 3(-0.965)\big) = 1.00325$$

(iv) **Fourth iteration:**

Use $x_3 = 1.00125,\ y_3 = -1.0015\ and\ z_3 = 1.00325$

$$x_4 = \frac{1}{20}\big(17 - (-1.0015) + 2(1.00325)\big) = 1.0004$$
$$y_4 = \frac{1}{20}\big(-18 - 3(1.00125) + (1.00325)\big) = -1.000025$$
$$z_4 = \frac{1}{20}\big(25 - 2(1.00125) + 3(-1.0015)\big) = 0.99965$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is

$x = 1.0004,\ \ y = -1.0000\ and\ z = 0.9997$

2. Solve the following equations by Gauss-Jacobi's Method (Take three iterations)

$$2x + 20y - 3z = 19$$
$$3x - 6y + 25z = 22$$
$$15x + 2y + z = 18$$

Solution:

First checking the condition of strictly diagonally dominant, we rearrange the system as,

$$15x + 2y + z = 18$$
$$2x + 20y - 3z = 19$$
$$3x - 6y + 25z = 22$$

Rewrite given equations as,

$$x = \frac{1}{15}(18 - 2y - z)$$
$$y = \frac{1}{20}(19 - 2x + 3z)$$
$$z = \frac{1}{25}(22 - 3x + 6y)$$

(i) First iteration:

start with $x_0 = 0,\ y_0 = 0\ and\ z_0 = 0$

$$x_1 = \frac{18}{15} = 1.2\ ,\ y_1 = \frac{19}{20} = 0.95\ ,\ z_1 = \frac{22}{25} = 0.88$$

(ii) Second iteration:

Use $x_1 = 1.2,\ y_1 = 0.95\ and\ z_1 = 0.88$

$$x_2 = \frac{1}{15}\big(18 - 2(0.95) - (0.88)\big) = 1.0147$$
$$y_2 = \frac{1}{20}\big(19 - 2(1.2) + 3(0.88)\big) = 0.962$$
$$\mathrm{z}_2 = \frac{1}{25}\big(22 - 3(1.2) + 6(0.95)\big) = 0.964$$

(iii) Third iteration:

Use $x_2 = 1.0147,\ y_2 = 0.962\ and\ z_2 = 0.964$

$$x_3 = \frac{1}{15}\big(18 - 2(0.962) - (0.964)\big) = 1.0075$$
$$y_3 = \frac{1}{20}\big(19 - 2(1.0147) + 3(0.964)\big) = 0.9931$$
$$z_3 = \frac{1}{25}\big(22 - 3(1.0147) + 6(0.962)\big) = 0.9891$$

Hence the final answer (correct up to 4 decimal places) after third iteration is
$x = 1.0075,\ \ y = 0.9931\ and\ z = 0.9891$

3. Solve the following equations by Gauss-Jacobi's Method
$5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$. Start with $(2,3,0)$

Solution:
Rewrite given equations as,

$$x = \frac{1}{5}(10 + y - z)$$
$$y = \frac{1}{4}(12 - 2x)$$
$$z = \frac{1}{5}(-1 - x - 5y)$$

(i) First iteration:
start with $x_0 = 2,\ y_0 = 3\ and\ z_0 = 0$

$$x_1 = \frac{1}{5}(10 + 3) = \frac{13}{5} = 2.6$$
$$y_1 = \frac{1}{4}(12 - 4) = \frac{8}{4} = 2$$
$$z_1 = \frac{1}{5}(-1 - 2 - 15) = \frac{-18}{5} = -3.6$$

(ii) Second iteration:
Use $x_1 = 2.6,\ y_1 = 2\ and\ z_1 = -3.6$

$$x_2 = \frac{1}{5}\big((10 + 2 - (-3.6))\big) = 3.12$$
$$y_2 = \frac{1}{4}\big(12 - 2(2.6)\big) = 1.7$$
$$z_2 = \frac{1}{5}\big(-1 - 2.6 - 5(2)\big) = -2.72$$

(iii) **Third iteration:**
Use $x_2 = 3.12,\ y_2 = 1.7\ and\ z_2 = -2.72$

$$x_3 = \frac{1}{5}\big(10 + 1.7 - (-2.72)\big) = 2.884$$
$$y_3 = \frac{1}{4}\big(12 - 2(3.12)\big) = 1.44$$
$$z_3 = \frac{1}{5}\big(-1 - 3.12 - 5(1.7)\big) = -2.524$$

(iv) **Fourth iteration:**
Use $x_3 = 2.884,\ y_3 = 1.44\ and\ z_3 = -2.524$

$$x_4 = \frac{1}{5}\big(10 + 1.44 - (-2.524)\big) = 2.7928$$
$$y_4 = \frac{1}{4}\big(12 - 2(2.884)\big) = 1.558$$
$$z_4 = \frac{1}{5}\big(-1 - 2.884 - 5(1.44)\big) = -2.2168$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is
$x = 2.7928,\ \ y = 1.558\ and\ z = -2.2168$

4. Solve the following equations by Gauss-Jacobi's Method

$15x + y - z = 14 \ , \ x + 20y + z = 23 \ , \ 2x - 3y + 18z = 35$

Solution:

Rewrite given equations as,

$$x = \frac{1}{15}(14 - y + z)$$
$$y = \frac{1}{20}(23 - x - z)$$
$$z = \frac{1}{18}(35 - 2x + 3y)$$

(i) First iteration:

start with $x_0 = 0, \ y_0 = 0 \ and \ z_0 = 0$

$$x_1 = \frac{1}{15}(14 - y + z) = \frac{14}{15} = 0.9333$$
$$y_1 = \frac{1}{20}(23 - x - z) = \frac{23}{20} = 1.15$$
$$z_1 = \frac{1}{18}(35 - 2x + 3y) = \frac{35}{18} = 1.9444$$

(ii) Second iteration:

Use $x_1 = 0.9333, \ y_1 = 1.15 \ and \ z_1 = 1.9444$

$$x_2 = \frac{1}{15}(14 - 1.15 + 1.9444) = 0.9863$$
$$y_2 = \frac{1}{20}(23 - 0.9333 - 1.9444) = 1.0061$$
$$z_2 = \frac{1}{18}(35 - 2(0.9333) + 3(1.15)) = 2.0324$$

(iii) **Third iteration:**

Use $x_2 = 0.9863, \ y_2 = 1.0061 and \ z_2 = 2.0324$

$$x_3 = \frac{1}{15}(14 - 1.0061 + 2.0324) = 1.0018$$
$$y_3 = \frac{1}{20}(23 - 0.9863 - 2.0324) = 0.9991$$
$$z_3 = \frac{1}{18}(35 - 2(0.9863) + 3(1.0061)) = 2.0025$$

(iv) **Fourth iteration:**

Use $x_3 = 1.0018 \ y_3 = 0.9991 \ and \ z_3 = 2.0025$

$$x_4 = \frac{1}{15}(14 - 0.9991 + 2.0025) = 1.0002$$
$$y_4 = \frac{1}{20}(23 - 1.0018 - 2.0025) = 0.9995$$
$$z_4 = \frac{1}{18}(35 - 2(1.0018) + 3(0.9991)) = 1.9990$$

Hence the final answer (correct up to 4 decimal places) after fourth iteration is

$x = 1.0002, \ \ y = 0.9995 \ and \ z = 1.9990$

5. Use Gauss-Seidel method to solve the following equations (Take three iterations)
$3x - 0.1y - 0.2z = 7.85$
$0.1x + 7y - 0.3z = -19.3$
$0.3x - 0.2y + 10z = 71.4$

Solution:
Rewrite given equations as,
$x = \frac{1}{3}(7.85 + 0.1y + 0.2z)$ ......(1)
$y = \frac{1}{7}(-19.3 - 0.1x + 0.3z)$..... (2)
$z = \frac{1}{10}(71.4 - 0.3x + 0.2y)$ ....(3)
(i) **First iteration:**
Start with $y = 0$ and $z = 0$
$x = \frac{7.85}{3} = 2.6167$,
We use this value to find y,
i.e. we put $x = 2.6167$ and $z = 0$
$y = \frac{1}{7}\left(-19.3 - 0.1(2.6167) + 0.3(0)\right) = -2.7945$,
We use latest two values to find z, i.e.
we put $x = 2.6167$ and $y = -2.7945$
$z = \frac{1}{10}\left(71.4 - 0.3(2.6167) + 0.2(-2.7945)\right) = 7.0056$
(ii) **Second iteration:**
We use latest two values to find x, we put $y = -2.7945$ and $z = 7.0056$
$x = \frac{1}{3}(7.85 + 0.1(-2.7945) + 0.2(7.0056)) = 2.9906$
We use latest two values to find y, we put $x = 2.9906$ and $z = 7.0056$
$y = \frac{1}{7}\left(-19.3 - 0.1(2.9906) + 0.3(7.0056)\right) = -2.4996$
We use latest two values to find z, i.e. we put $x = 2.9906$ and $y = -2.4996$
$z = \frac{1}{10}\left(71.4 - 0.3(2.9906) + 0.2(-2.4996)\right) = 7.0003$
(iii) **Third iteration:**
We use latest two values to find x, we put $y = -2.4996$ and $z = 7.0003$
$x = \frac{1}{3}(7.85 + 0.1(-2.4996) + 0.2(7.0003)) = 3.0000$
We use latest two values to find y, we put $x = 3$ and $z = 7.0003$
$y = \frac{1}{7}\left(-19.3 - 0.1(3) + 0.3(7.0003)\right) = -2.500$
We use latest two values to find z, i.e. we put $x = 3$ and $y = -2.5$
$z = \frac{1}{10}\left(71.4 - 0.3(3) + 0.2(-2.5)\right) = 7.000$
Hence the final answer after third iteration is
$x = 3,\ y = -2.5$ and $z = 7$

6. Solve the following equations by Gauss-Seidel method.

$$28x + 4y - z = 32 \ , \ 2x + 17y + 4z = 35 \ , \ x + 3y + 10z = 24$$

Solution:

Rewrite given equations as,

$$x = \frac{1}{28}(32 - 4y + z) \ ……(1)$$
$$y = \frac{1}{17}(35 - 2x - 4z)….. \ (2)$$
$$z = \frac{1}{10}(24 - x - 3y) \ …….(3)$$

(i) **First iteration:**

Start with $y = 0 \ and \ z = 0$

$$x = \frac{32}{28} = 1.1429 \ ,$$

We use this value to find y,

i.e. we put $x = 1.1429 \ and \ z = 0$

$$y = \frac{1}{17}\big(35 - 2(1.1429) - 4(0)\big) \ = 1.9244 \ ,$$

We use latest two values to find z, i.e.

we put $x = 1.1429 \ and \ y = 1.9244$

$$z = \frac{1}{10}\big(24 - 1.1429 - 3(1.9244)\big) = 1.7084$$

(ii) **Second iteration:**

We use latest two values to find x, we put $y = 1.9244 \ and \ z = 1.7084$

$$x = \frac{1}{28}(32 - 4(1.9244) + 1.7084) = 0.9289$$

We use latest two values to find y, we put $x = 0.9289 \ and \ z = 1.7084$

$$y = \frac{1}{17}(35 - 2(0.9289) - 4(1.7084)) = 1.5475$$

We use latest two values to find z, i.e. we put $x = 0.9289 \ and \ y = 1.5475$

$$z = \frac{1}{10}(24 - 0.9289 - 3(1.5475)) = 1.8428$$

(iii) **Third iteration:**

We use latest two values to find x, we put $y = 1.5475 \ and \ z = 1.8428$

$$x = \frac{1}{28}(32 - 4(1.5475) + 1.8428) = 0.9876$$

We use latest two values to find y, we put $x = 0.9876 \ and \ z = 1.8428$

$$y = \frac{1}{17}(35 - 2(0.9876) - 4(1.8428)) = 1.5090$$

We use latest two values to find z, i.e. we put $x = 0.9876 \ and \ y = 1.5090$

$$z = \frac{1}{10}(24 - 0.9876 - 3(1.5090)) = 1.8485$$

(iv) **Fourth iteration:**

We use latest two values to find x, we put $y = 1.5090 \ and \ z = 1.8485$

$$x = \frac{1}{28}(32 - 4(1.5090) + 1.8485) = 0.9933$$

We use latest two values to find y, we put $x = 0.9933 \ and \ z = 1.8485$

$y = \frac{1}{17}(35 - 2(0.9933) - 4(1.8485)) = 1.5070$

We use latest two values to find z, i.e. we put $x = 0.9933$ *and* $y = 1.5070$

$z = \frac{1}{10}(24 - 0.9933 - 3(1.5070)) = 1.8485$

Hence the final answer after fourth iteration is

$x = 0.9933,\ y = 1.5070$ *and* $z = 1.8485$

7. Solve the following equations by Gauss-Seidel method by taking three iterations only.
$10x_1 + x_2 + x_3 = 12,\ 2x_1 + 10x_2 + x_3 = 13,\ 2x_1 + 2x_2 + 10x_3 = 14$

Solution:

Rewrite given equations as,

$x_1 = \frac{1}{10}(12 - x_2 - x_3)$……(1)

$x_2 = \frac{1}{10}(13 - 2x_1 - x_3)$….. (2)

$x_3 = \frac{1}{10}(14 - 2x_1 - 2x_2)$…….(3)

(i) **First iteration:**

Start with $x_2 = 0$ *and* $x_3 = 0$

$x_1 = \frac{12}{10} = 1.2$,

We use this value to find y,

i.e. we put $x_1 = 1.2$ *and* $x_3 = 0$

$x_2 = \frac{1}{10}(13 - 2(1.2) - 0) = 1.06$,

We use latest two values to find z, i.e.

We put $x_1 = 1.2$ *and* $x_2 = 1.06$

$x_3 = \frac{1}{10}(14 - 2(1.2) - 2(1.06)) = 0.948$

(ii) **Second iteration:**

We use latest two values to find x, we put $x_2 = 1.06$ *and* $x_3 = 0.948$

$x_1 = \frac{1}{10}(12 - 1.06 - 0.948) = 0.9992$

We use latest two values to find y, we put $x_1 = 0.9992$ *and* $x_3 = 0.948$

$x_2 = \frac{1}{10}(13 - 2(0.9992) - 0.948) = 1.00536$

We use latest two values to find z, i.e. we put $x_1 = 0.9992$ *and* $x_2 = 1.00536$

$x_3 = \frac{1}{10}(14 - 2(0.9992) - 2(1.00536)) = 0.999088$

(iii) **Third iteration:**

We use latest two value to find x, we put $x_2 = 1.00536$ *and* $x_3 = 0.999088$

$x_1 = \frac{1}{10}(12 - 1.00536 - 0.999088) = 0.9996$

We use latest two values to find y, we put $x_1 = 0.9996$ *and* $x_3 = 0.999088$

$x_2 = \frac{1}{10}(13 - 2(0.9996) - 0.999088) = 1.00018$

We use latest two values to find z, i.e. we put $x_1 = 0.9996$ *and* $x_2 = 1.00018$

$x_3 = \frac{1}{10}(14 - 2(0.9996) - 2(1.00018)) = 1.00052$

Hence the final answer after third iteration is $x = 0.9996,\ y = 1$ *and* $z = 1$

SOME PRACTICE PROBLEMS

JACOBI'S METHOD

I. Solve the following equations by Jacobi's method.

1) $15x + y - z = 14$, $x + 20y + z = 23$, $2x - 3y + 18z = 35$
2) $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$
3) $8x - y + 2z = 13$, $x - 10y + 3z = 17$, $3x + 2y + 12z = 25$
4) $5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$. Start with $(2, 3, 0)$.
5) $5x - y + z = 10$, $2x + 4y = 12$, $x + 5y + 5z = -1$
6) $12x + 2y + z = 27$, $2x + 15y - 3z = 16$, $2x - 3y + 25z = 26$
7) $4x + y + 3z = 17$, $x + 5y + z = 14$, $2x - y + 8z = 12$

GAUSS - SEIDEL METHOD

II. Solve the following equations by Gauss-Seidel method.

1) $28x + 4y - z = 32$, $2x + 17y + 4z = 35$, $x + 3y + 10z = 24$
2) $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 27z = 85$
3) $10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$
4) $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$
5) $5x - y = 9$, $-x + 5y - z = 4$, $-y + 5z = -6$
6) $5x + y - z = 10$, $2x + 4y + z = 14$, $x + y + 8z = 20$
7) $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$ by taking three iterations only.
8) $4x - 2y - z = 40$, $x - 6y + 2z = -28$, $x - 2y + 12z = -86$
9) $2x - 4y + 49z = 49$, $43x + 2y + 25z = 23$, $3x + 53y + 3z = 91$
10) $10x_1 - 5x_2 - 2x_3 = 3$, $4x_1 - 10x_2 + 3x_3 = -3$, $x_1 + 6x_2 - 10x_3 = -3$ by taking three iterations only.
11) $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$
12) $25x + 2y - 3z = 48$, $3x + 27y - 2z = 56$, $x + 2y + 32z = 52$. Start with $(1, 1, 0)$.

Module 2

Rank of Matrix and System of Equations

Self-Learning Topic

❖ **Properties of Symmetric, Skew Symmetric, Hermitian and Skew Hermitian Matrices**

- Symmetric Matrices with real entries are Hermitian.
- Skew-Symmetric Matrices with real entries are skew-Hermitian.
- If A is Hermitian then $\bar{A} = A^T$ (equivalent condition)

$$A^{\theta} = A \ Take\ transpose\ \ (\bar{A}^T)^T = A^T \ \Rightarrow\ \ \bar{A} = A^T$$

- If A is Skew-Hermitian then $\bar{A} = -A^T$ (equivalent condition)
- If A is Hermitian then iA *is* Skew-Hermitian.

$$(iA)^{\theta} = i^{\theta} A^{\theta} = -iA$$

- If A is Skew-Hermitian then iA *is* Hermitian**.**
- If A is Hermitian then $\bar{A}$ is also Hermitian.

$$(\bar{A})^{\theta} = \left(\bar{\bar{A}}\right)^T = A^T = \bar{A}$$

- If A is skew-Hermitian then $\bar{A}$ is also skew-Hermitian.
- If A is skew symmetric and X is a column matrix then $X^T AX$ is null matrix.

  Proof: Let $X^T AX = B$, taking transpose $(X^T AX)^T = B^T$

  $\therefore\ X^T A^T X = B^T$ But $A^T = -A\ and$

  $$X^T{}_{1\,X\,n} A_{n\,X\,n} X_{n\,X\,1} = B_{1\,X\,1} \quad \therefore\ B^T = B$$

  $$\therefore\ \ -X^T AX = B \ \ \therefore X^T AX = -B \ \ \therefore B = -B \ \ \therefore B = 0$$

- If A is any square matrix then
  $A + A^T$ is symmetric
  $A - A^T$ is skew-symmetric
- If A is any square matrix then
  $A + A^{\theta}$ is Hermitian
  $A + A^{\theta}$ is skew-Hermitian.
- If A and B are symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA

$$Since\ A^T = A\ and\ B^T = B,\ consider\ \ (AB)^T = B^T A^T = BA = AB$$

- If A and B are skew-symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA
- If A is skew symmetric of order n then $|A| = 0$, if n is odd.

$$Since\ A^T = -A,\ \ take\ \det both\ side\ \ |A^T| = |-A|$$

$$\therefore |A| = (-1)^n |A| \therefore |A| = 0\ if\ n\ is\ odd$$

- **Sum of Matrices:**

1. Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew - symmetric matrix.

Proof:

Let A be any square matrix.

$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q,$

Where, $P = \frac{1}{2}(A + A')$

$Q = \frac{1}{2}(A - A')$

To Prove : P is symmetric

$P' = \left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}(A' + (A')') = \frac{1}{2}(A' + A) = P$

To Prove : Q is skew-Hermitian

$Q' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Q$

Uniqueness:

Suppose there is another representation of A = R + S, where R is Symmetric ($R' = R$) and S is skew symmetric ($S' = -S$)

Now, $A' = (R + S)' = R' + S' = R - S$

For uniqueness, we need to prove P=R & Q=S

$$P = \frac{1}{2}(A + A') = \frac{1}{2}(R + S + R - S) = R$$

$$Q = \frac{1}{2}(A - A') = \frac{1}{2}(R + S - (R - S)) = S$$

Hence A = R + S =P+Q . So given representation is unique.

2. Show that every square matrix can be uniquely expressed as sum of Hermitian and skew Hermitian matrices.

Proof:

Let A be any square matrix.

Consider $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta) = P + Q,$

Where, $P = \frac{1}{2}(A + A^\theta)$ $\quad Q = \frac{1}{2}(A - A^\theta)$

To Prove : P is Hermitian

$P^\theta = \left[\frac{1}{2}(A + A^\theta)\right]^\theta = \frac{1}{2}\left(A^\theta + (A^\theta)^\theta\right) = \frac{1}{2}(A^\theta + A) = P$

To Prove : Q is skew-Hermitian

$Q^\theta = \left[\frac{1}{2}(A - A^\theta)\right]^\theta = \frac{1}{2}(A^\theta - A) = -\frac{1}{2}(A - A^\theta) = -Q$

Uniqueness:

Suppose there is another representation of A = R + S, where R is Hermitian ($R^\theta = R$) and S is skew Hermitian ($S^\theta = -S$)

Now, $A^\theta = (R + S)^\theta = R^\theta + S^\theta = R - S$

For uniqueness, we need to prove P = R & Q = S

$P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}(R + S + R - S) = R$

$Q = \frac{1}{2}(A - A^\theta) = \frac{1}{2}(R + S - (R - S)) = S$

Hence A = R + S = P+Q . So given representation is unique.

3. Show that every square matrix can be uniquely expressed as P + iQ, where P and Q both are Hermitian matrices.

Proof:

Let A be any square matrix. Consider $A = \frac{1}{2}(A + A^\theta) + i\left[\frac{1}{2i}(A - A^\theta)\right]$ = P + iQ

Where, $P = \frac{1}{2}(A + A^\theta)$ *and* $Q = \left[\frac{1}{2i}(A - A^\theta)\right]$

To Prove : P is Hermitian

$$P^\theta = \left[\frac{1}{2}(A + A^\theta)\right]^\theta = \frac{1}{2}\left(A^\theta + (A^\theta)^\theta\right) = \frac{1}{2}(A^\theta + A) = P$$

To Prove : Q is Hermitian

$$Q^\theta = \left[\frac{1}{2i}(A - A^\theta)\right]^\theta = \left(\frac{1}{2i}\right)^\theta (A^\theta - A) = -\frac{1}{2i}(A^\theta - A) = \frac{1}{2i}(A - A^\theta) = Q$$

<u>Uniqueness:</u>

Consider another representation, say $A = R + iS$, where R and S are Hermitian.

$(R^\theta = R,\ \ S^\theta = S)$

Then $A^\theta = (R + iS)^\theta = R^\theta + i^\theta S^\theta = R - iS$

Now consider, $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}(R + iS + R - iS) = R$

and $Q = \frac{1}{2i}(A - A^\theta) = \frac{1}{2i}(R + iS - (R - iS)) = S$

Thus we establish R is same as P and S is same as Q. Hence given representation is unique.

4. Show that every Hermitian matrix can be uniquely expressed as P + iQ, where P is real symmetric and Q is real skew symmetric matrix.

Proof:

Let A be any Hermitian matrix.

Consider $A = \frac{1}{2}\left(A + \bar{A}\right) + i\left[\frac{1}{2i}\left(A - \bar{A}\right)\right]$ = P + iQ

Where, $P = \frac{1}{2}\left(A + \bar{A}\right)$ *and* $Q = \left[\frac{1}{2i}\left(A - \bar{A}\right)\right]$

To Prove: P is real symmetric, we show $\bar{P} = P$ *and* $P^T = P$

$\bar{P} = \overline{\frac{1}{2}\left(A + \bar{A}\right)} = \frac{1}{2}\overline{\left(A + \bar{A}\right)} = \frac{1}{2}\left(\bar{A} + A\right) = P$ Hence, P is real

$P^T = \left[\frac{1}{2}\left(A + \bar{A}\right)\right]^T = \frac{1}{2}\left(A^T + \left(\bar{A}\right)^T\right) = \frac{1}{2}(A^T + A^\theta) = \frac{1}{2}\left(\bar{A} + A\right) = P$

Since A is Hermitian, $A^\theta = A$ *and* $A^T = \bar{A}$

Hence P is symmetric.

To Prove : Q is real Skew-symmetric we show $\bar{Q} = Q$ *and* $Q^T = -Q$

$\bar{Q} = \overline{\frac{1}{2\iota}\left(A - \bar{A}\right)} = -\frac{1}{2i}\overline{\left(A - \bar{A}\right)} = -\frac{1}{2i}\left(\bar{A} - A\right) = \frac{1}{2i}\left(A - \bar{A}\right) = Q$

Hence, Q is real.

$Q^T = \left[\frac{1}{2i}\left(A - \bar{A}\right)\right]^T = \frac{1}{2i}\left(A^T - \left(\bar{A}\right)^T\right) = \frac{1}{2i}\left(A^T - A^\theta\right) = \frac{1}{2i}\left(\bar{A} - A\right) = -Q$

Since A is Hermitian, $A^\theta = A \, and \, A^T = \bar{A}$

Uniqueness:

Consider another representation, say $A = R + iS$, where R is real symmetric and S is real skew symmetric.

Then $\bar{A} = \overline{R + \iota S} = \bar{R} + \bar{\iota}\bar{S} = R - \iota S$ (since $\bar{R} = R\,,\ \bar{S} = S$)

Now consider, $P = \frac{1}{2}(A + \bar{A}) = \frac{1}{2}(R + iS + R - iS) = R$

and $Q = \frac{1}{2i}(A - \bar{A}) = \frac{1}{2i}(R + iS - (R - iS)) = S$

Thus, we establish R is same as P and S is same as Q. Hence given representation is unique.

5. Show that every skew Hermitian matrix can be uniquely expressed as P + iQ, where P is real skew symmetric and Q is real symmetric matrix. (Try yourself).

SOME PRACTICE PROBLEMS

1. If A is a skew – symmetric matrix and X is a column matrix, then prove that X^TAX is a null matrix.
2. If A and B are skew – symmetric matrices of order n, then show that AB is symmetric if and only if A and B commute.
3. If $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$ show that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric.

BEYOND SYLLABUS

❖ Reduction of a Matrix to Normal Form PAQ

Theorem: If A is a matrix of rank r, then there exist non – singular matrices P and Q such that PAQ is in the normal form i.e $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

To obtain the matrices P and Q use the following procedure:

1. If A an $m \times n$ matrix, write $A = I_m A I_n$
2. Apply row transformations of A on I.h.s. and the same row transformations on the pre-factor I_m.
3. Apply column transformations on A on I.h.s. and the same column transformations on the post-factor I_n.

   So that, A on the I.h.s is reduced to normal form.

Remark: (i) No transformations are applied on A on the r.h.s.

(ii) The matrices P and Q thus obtained are not unique. They depend upon the transformations used.

SOME SOLVED EXAMPLES

4. Find non – singular matrices P and Q such that PAQ is in normal form, Hence obtain rank of A where A is $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

Solution:

Since A is the matrix of order 3×4, we write $A = I_3 . A_{3\times4} . I_4$

Thus $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

To find non – singular matrices P and Q, we reduce the matrix A on the left hand side to normal form by applying suitable elementary transformations. Every row operation will also be applied to the pre – factor of the product on the right hand side and every column operation to the post factor.

Applying $R_2 - 2R_1,\ R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 - 2C_1,\ C_3 - 3C_1,\ C_4 + 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{C_2}{-6},\ \frac{C_3}{-5},\ \frac{C_4}{7}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 3/5 & 2/7 \\ 0 & -1/6 & 0 & 0 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

Applying $C_3 - C_2,\ C_4 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

Thus, $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

Where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ $and\ Q = \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$

Hence rank of A is 2.

5. Find non – singular matrices P and Q such that PAQ is in normal form. Hence find **(i)** rank of A, **(ii)** A^{-1} , where A is $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

Solution:

Since A is a square matrix of order 4, we write $A = I_4 . A . I_4$

i.e. $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying C_{34}, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, we have $[I_4] = PAQ$ is the required normal form.

Where $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 0 & 0 & 1/3 \end{bmatrix}$ $and\ Q = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Hence rank of A is 4. Since $|A| \neq 0$, therefore A^{-1} exists

To find A^{-1}, we have $PAQ = I$

$\therefore (PAQ)^{-1} = I^{-1}$ $\therefore Q^{-1}A^{-1}P^{-1} = I$ $\{\because I^{-1} = I\}$

$\therefore QQ^{-1}A^{-1}P^{-1} = QI$ $\therefore I\,A^{-1}P^{-1} = Q$ $\therefore A^{-1}P^{-1}P = QP$

$\therefore A^{-1}I = QP$

$$\therefore A^{-1} = QP = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{20}{3} & -5 & -3 & \frac{2}{3} \\ -2 & 1 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

6. Find non singular matrices P and Q such that PAQ is in normal form, where

$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and hence find rank A. Also find A^{-1} if it exists.

Solution: Consider $A = I_3 A I_3$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + R_1$,

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 2C_1, C_3 + 2C_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + 2R_2$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore I_3 = PAQ$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Rank of $A = 3$

Rank A = Order of matrix

$\therefore$ A^{-1} exists

Now, $PAQ = I \qquad \therefore (PAQ)^{-1} = I^{-1}$

$\therefore Q^{-1}A^{-1}P^{-1} = I$

$\therefore \left(QQ^{-1}\right) A^{-1} \left(P^{-1}P\right) = QIP$

$\therefore A^{-1} = QP$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

7. Find non-singular matrices P and Q such that PAQ is in normal form, where $A = \begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$. Also find rank A.

Solution: $A = \begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$

Consider $A = I_3 A I_4$

$$\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_1 \longleftrightarrow C_3$

$$\begin{bmatrix} 1 & 3 & 4 & 6 \\ 2 & 4 & 2 & 2 \\ 5 & 14 & 12 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1, R_3 - 5R_1,$

$$\begin{bmatrix} 1 & 3 & 4 & 6 \\ 0 & -2 & -6 & -10 \\ 0 & -1 & -8 & -14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 3C_1, C_3 - 4C_1, C_4 - 6C_1,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -6 & -10 \\ 0 & -1 & -8 & -14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -3 & -4 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $(-1)R_2 \longleftrightarrow (-1)R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 14 \\ 0 & 2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -3 & -4 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - 2R_2,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 14 \\ 0 & 0 & -10 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ -8 & -1 & 2 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -3 & -4 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_3 - 8C_2, C_4 - 14C_2,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ -8 & -1 & 2 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -8 & -14 \\ 1 & -3 & 20 & 36 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{10}\right) C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ -8 & -1 & 2 \end{bmatrix} A \begin{bmatrix} 0 & 0 & -1/10 & 0 \\ 0 & 1 & 8/10 & -14 \\ 1 & -3 & -2 & 36 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_4 + 18C_3$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ -8 & -1 & 2 \end{bmatrix} A \begin{bmatrix} 0 & 0 & -1/10 & -9/5 \\ 0 & 1 & 4/5 & 2/5 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ -8 & -1 & 2 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 & -1/10 & -9/5 \\ 0 & 1 & 4/5 & 2/5 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Rank of $A = 3$

8. Find non singular matrices P and Q such that PAQ is in normal form, where $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. Also find rank A.

Solution: Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Consider $A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1, R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + R_2$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_3 - C_2, C_4 - 3C_2$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Rank of $A = 2$

SOME PRACTICE PROBLEMS

1. Find non singular matrices P and Q such that PAQ is in normal form, where

$$A=\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ and hence find rank A. Also find A^{-1} if it exists.

2. Find non singular matrices P and Q such that PAQ is in normal form. Also find rank A.

(i) $$A=\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}$$

(iii) $$A=\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$$

(ii) $$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

(iv) $$A=\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{bmatrix}$$