INVERSE HYPERBOLIC FUNCTIONS:

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.

Similarly we can define $\cosh^{-1}x, \tanh^{-1}x, \coth^{-1}x,\ \mathrm{sech}^{-1}x, \mathrm{cosech}^{-1}x$.

The inverse hyperbolic functions are many valued but we will consider their principal value only.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\ (x + \sqrt{x^2+1})$ **(ii)** $\cosh^{-1}x = \log\ (x + \sqrt{x^2-1})$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof: (i) $\mathbf{\sinh^{-1} x = \log\ (x + \sqrt{x^2+1})}$

Let $\sinh^{-1} x = y$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$$

$$e^{2y} - 2x\, e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \log\ (x \pm \sqrt{x^2+1})$$

But $x - \sqrt{x^2+1} < 0$ and $\log(-ve)$ is not defined.

$$\therefore y = \log\ (x + \sqrt{x^2+1})$$

$$\sinh^{-1}x = \log(x + \sqrt{x^2+1})$$

(ii) $\mathbf{\cosh^{-1}x = \log(x + \sqrt{x^2-1})}$

Let $\cosh^{-1}x = y$

$$x = \cosh\ y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$y = \log\ (x \pm \sqrt{x^2-1}) \qquad \text{.............................(1)}$$

Consider, $y = \log\ (x - \sqrt{x^2-1})$ ....(2)

$$e^y = x - \sqrt{x^2-1},$$

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1} = x + \sqrt{x^2-1}$$

$$-y = \log\ (x + \sqrt{x^2-1})$$

$$y = -\log\left(x + \sqrt{x^2 - 1}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equating equation (2) and (3), we get

$$\log\left(x - \sqrt{x^2 - 1}\right) = -\log\left(x + \sqrt{x^2 - 1}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From equation (1) and (4), we get

$$y = \pm\log\left(x + \sqrt{x^2 - 1}\right)$$

$$\cosh^{-1}x = \pm\log\left(x + \sqrt{x^2 - 1}\right)$$

$$\therefore\ x = \cosh\{\pm\log\left(x + \sqrt{x^2 - 1}\right)\}$$

$$= \cosh\{\log(x + \sqrt{x^2 - 1})\} \qquad \text{since } \cosh(-z) = \cosh z$$

$$\therefore \cosh^{-1}x = \log\left(x + \sqrt{x^2 - 1}\right)$$

(iii) $\boldsymbol{\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)}$

Let $\tanh^{-1}x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo-dividendo

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$= \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh \log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Solution: Let $\tanh \log\sqrt{x} = \alpha$

$$\log\sqrt{x} = \tanh^{-1}\alpha$$

$$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+\alpha}{1-\alpha}\right)$$

$$x = \frac{1+\alpha}{1-\alpha}$$

$$\frac{x-1}{x+1} = \frac{(1+\alpha)-(1-\alpha)}{(1+\alpha)+(1-\alpha)} = \frac{2\alpha}{2} =$$

$$\therefore \tan h \log\sqrt{x} = \frac{x-1}{x+1}$$

Put $x = 5/3$ *and* $x = 7$ and add

$$\log h(\log\sqrt{5/3}) + \tan h\,(\log\sqrt{7}) = \frac{(5/3)-1}{(5/3)+1} + \frac{7-1}{7+1} = \frac{2}{8} + \frac{6}{8} = 1$$

2. (i) Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y$ $\quad\therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1+x^2 = \cos h^2 y$ $\quad\therefore x^2 = \cos h^2 y - 1 = \sin h^2 y$

$\therefore x = \sin hy$ $\quad\therefore y = \sin h^{-1}x$ $\quad\therefore \cos h^{-1}\sqrt{1+x^2} = \sin h^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Solution: Let $\tan h^{-1}x = y$ $\quad\therefore x = \tan hy$

Now, $\frac{x}{\sqrt{1-x^2}} = \frac{\tan hy}{\sqrt{1-tanh^2y}} = \frac{\tan hy}{\sqrt{cosh^2y - \sin h^2y/cosh^2y}} \quad = \frac{\sin hy}{\cos hy} \times \frac{\cos hy}{1} = \sin hy$

$\therefore y = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}$ $\quad\therefore \tan h^{-1}x = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}(\sqrt{1+x^2}) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y$ $\quad\therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1+x^2 = \cos h^2y$ $\quad\therefore x^2 = \cos h^2y - 1 = \sin h^2y$ $\quad\therefore x = \sin hy$

$\therefore \tan hy = \frac{\sin hy}{\cos hy} = \frac{x}{\sqrt{1+x^2}}$ $\quad\therefore y = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore cosh^{-1}(\sqrt{1+x^2}) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

Solution: Let $cot\ h^{-1}\left(\frac{x}{a}\right) = y$ $\quad\therefore \frac{x}{a} = \cot hy$ $\quad\therefore \tan hy = \frac{1}{\cot hy} = \frac{1}{x/a} = \frac{a}{x}$

$\therefore y = \tan h^{-1}\left(\frac{a}{x}\right) = \frac{1}{2}log\left(\frac{1+(a/x)}{1-(a/x)}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

$\therefore cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

(iii) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

Solution: Let $\sec h^{-1}(\sin\theta) = x$ $\quad\therefore sin\ \theta = \sec hx$ $\quad\therefore \sin\theta = \frac{1}{\cos hx} = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{e^{2x}+1}$

$\therefore (\sin\theta)e^{2x} - 2e^x + \sin\theta = 0$ This is a quadratic in e^x

$\therefore e^x = \frac{2\pm\sqrt{4-4sin^2\theta}}{2\sin\theta} = \frac{1\pm\cos\theta}{\sin\theta}$

$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\,cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$

$\therefore x = \log\cot\left(\frac{\theta}{2}\right)$ $\quad\therefore \sec h^{-1}(\sin\theta) = \log(\cot\theta/2)$

3. Separate into real and imaginary parts $cos^{-1}e^{i\,\theta}$ ***or*** $cos^{-1}(\cos\theta + i\sin\theta)$

Solution: Let $\cos^{-1} e^{i\theta} = x + iy,\ \ e^{i\theta} = \cos(x+iy)$

$\cos\theta + i\sin\theta = \cos x\cos(iy) - \sin x\sin(iy) = \cos x\cosh y - i\sin x\sinh y$

Equating real and imaginary parts $\cos\theta = \cos x\cosh y$ and $\sin\theta = -\sin x\sinh y$

Since $\cosh^2 y - \sinh^2 y = 1$

$\therefore \left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$

$\therefore \frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{(1-\sin^2\theta)\sin^2 x - \sin^2\theta(1-\sin^2 x)}{(1-\sin^2 x)\sin^2 x} = 1$

$\therefore \sin^2 x - \sin^2 x\sin^2\theta - \sin^2\theta + \sin^2 x\sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore \sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore -\sin^2\theta = -\sin^4 x$

$\therefore \sin^2\theta = \sin^4 x$

$\therefore \sqrt{\sin\theta} = \sin x$ .... (1)

$\therefore x = \sin^{-1}\sqrt{\sin\theta}$

Since $\sin\theta = -\sin x\sinh y$

$\sin\theta = -\sqrt{\sin\theta}\sinh y$ from (1)

$\therefore -\sqrt{\sin\theta} = \sinh y$

$\therefore y = \sinh^{-1}(\sqrt{\sin\theta}) = \log(-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1})$

$\therefore y = \log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$

$\therefore \cos^{-1} e^{i\theta} = x + iy = \sin^{-1}\sqrt{\sin\theta} + i\log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$

4. Separate into real and imaginary parts $sinh^{-1}(i\,x)$

Solution: Let $\sinh^{-1}(ix) = \alpha + i\beta$

$\therefore ix = \sinh(\alpha + i\beta) = \sinh\alpha\cosh(i\beta) + \cosh\alpha\sinh(i\beta)$

$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

Equating real and imaginary parts $\sinh\alpha\cos\beta = 0$

$\therefore \cos\beta = 0 \quad \therefore \beta = \frac{\pi}{2} \quad \therefore \sin\beta = 1$

Also $\cosh\alpha\sin\beta = x$

$\therefore \cosh\alpha = x \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$

$\therefore \alpha = \cosh^{-1} x$

$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1} x + i\frac{\pi}{2}$

5. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}log\left(\frac{1}{5}\right)$

Solution: $\tan z = \frac{i}{2}(1-i)$

$\tan z = \frac{1}{2}(i - i^2) = \frac{1}{2}i + \frac{1}{2}$

$Let\ z = x + iy \ \therefore \tan(x+iy) = \frac{1}{2} + \frac{i}{2}, \quad \tan(x - iy) = \frac{1}{2} - \frac{i}{2}$

$\therefore \tan(2x) = [(x+iy) + (x-iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]+\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1-\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{1}{1-\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{1}{1/2} = 2$$

$\therefore 2x = \tan^{-1}2 \qquad \therefore x = \frac{1}{2}\tan^{-1}2$

Now, $tan(2iy) = \tan[(x+iy) - (x-iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]-\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1+\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{i}{1+\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{i}{1+(1/2)} = \frac{2}{3}i$$

$\therefore i \tan h\, 2y = \frac{2}{3}i \qquad \therefore \tan h\, 2y = \frac{2}{3}$

$\therefore 2y = tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}log\left[\frac{1+(2/3)}{1-(2/3)}\right] = \frac{1}{2}\log 5 \qquad \therefore y = \frac{1}{4}\log 5$

$\therefore z = x + iy = \frac{1}{2}tan^{-1}2 + \frac{i}{4}\log 5$

6. Show that $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

Solution: Let $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$\therefore i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{x-a}{x+a} = \frac{e^{-i\theta}-e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \quad [\because i^2 = -1]$

By componendo and dividend $\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta}-e^{i\theta})+(e^{i\theta}+e^{-i\theta})}{(e^{-i\theta}-e^{i\theta})-(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}} = e^{-2i\theta} \qquad \therefore \frac{x}{a} = e^{-2i\theta} \qquad \therefore -2i\theta = \log\frac{x}{a}$

Multiply by i throughout, $2\theta = i\log\frac{x}{a} \qquad \therefore \theta = \frac{i}{2}\log\left(\frac{x}{a}\right)$

$tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

HYPERBOLIC FUNCTIONS

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}, find \sinh 2x \; and \cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2} \quad \therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2} \quad \therefore 2\, e^{2x} - 2 = e^{2x} + 1 \quad \therefore\; e^{2x} = 3$

Now, $sin\; h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3+(1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\; x + 8sinh\; x = 1$ for real values of x.

Solution: $7cosh\; x + 8sinh\; x = 1$

Putting the values of $coshx$ and $sin\; hx$, we get

$\therefore 7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$

$\therefore 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$

$\therefore 15e^x - e^{-x} = 2$

$\therefore 15e^{2x} - 2e^x - 1 = 0$ Solving it as a quadratic equation in e^x,

$e^x = \frac{2 \pm \sqrt{4-4(15)(-1)}}{2(15)} = \frac{2 \pm 8}{30} = \frac{1}{3} \; or -\frac{1}{5}$

$\therefore x = \log\left(\frac{1}{3}\right) \; or\; x = \log\left(-\frac{1}{5}\right)$

Since x is real, $x = log\left(\frac{1}{3}\right) = -\log 3$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Solution: Let $\sin h^{-1}\; a = \alpha,\; \sin h^{-1} b = \beta \; and \; \sin h^{-1}\; x = \gamma$

We are given $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x \qquad \therefore \alpha + \beta = \gamma$

$\therefore \sinh(\alpha + \beta) = \sinh \gamma$

$\therefore \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta = \sinh\gamma$ ....(A)

But $\sinh\alpha$ = a, $\sinh\beta = b$, $\sinh\gamma = x$

$\therefore \cos h\; \alpha = \sqrt{1 + \sin h^2\alpha} = \sqrt{1+a^2}$ and $\cos h\; \beta = \sqrt{1+\sin h^2\beta} = \sqrt{1+b^2}$

Putting this values in (A), we get $a\sqrt{1+a^2} + b\sqrt{1+b^2} = x$

4. Prove that $16\; sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Solution: LHS $= 16\sinh^5 x$

$= 16\left(\frac{e^x - e^{-x}}{2}\right)^5$

$= \frac{16}{32}(e^{5x} - 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} - 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} - e^{-5x})$

$= \frac{1}{2}(e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x})$

$= \left(\frac{e^{5x} - e^{-5x}}{2}\right) - 5\left(\frac{e^{3x} - e^{-3x}}{2}\right) + 10\left(\frac{e^{x} - e^{-x}}{2}\right)$

$= \sinh 5x - 5\sinh 3x + 10\sinh x$

$=$ RHS

5. Prove that $16cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Solution: $l.h.s = 16cosh^5 x$

$= 16\left(\frac{e^x + e^{-x}}{2}\right)^5$ [By Binomial Theorem]

$= \frac{16}{32}[e^{5x} + 5e^{4x}.e^{-x} + 10e^{3x}.e^{-2x} + 10e^{2x}.e^{-3x} + 5e^{x}.e^{-4x} + e^{-5x}]$

$= \frac{(e^{5x} + e^{-5x})}{2} + 5\frac{(e^{3x} + e^{-3x})}{2} + 10\frac{(e^{x} + e^{-x})}{2}$

$= \cos h\, 5\,x + 5\cos h\, 3\,x + 10\cos h\, x = r\,.h.s$

6. Prove that $\frac{1}{1 - \frac{1}{1 - \frac{1}{1 - cosh^2 x}}} = cosh^2 x$

Solution: $l.h.s = \frac{1}{1 - \frac{1}{1 - \frac{1}{-sinh^2 x}}} = \frac{1}{1 - \frac{1}{1 + cosec\,h^2 x}} = \frac{1}{1 - \frac{1}{\cot h^2 x}} = \frac{1}{1 - \tan h^2 x} = \frac{1}{1 - \frac{sinh^2 x}{\cos h^2 x}} = \frac{\cos h^2 x}{\cos h^2 x - \sin h^2 x} = \cos h^2 x$

7. If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Solution: **(i)** $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\therefore e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$

$\therefore e^{-u} = \frac{1 - \tan\theta/2}{1 + \tan\theta/2}$

$\therefore \cosh u = \frac{e^u + e^{-u}}{2}$

$= \frac{1}{2}\left[\frac{(1 + 2\tan\theta/2 + \tan^2\theta/2) + (1 - 2\tan\theta/2 + \tan^2\theta/2)}{1 - \tan^2\theta/2}\right]$

$= \frac{1}{2}\left(\frac{2 + 2\tan^2\theta/2}{1 - \tan^2\theta/2}\right)$

$$= \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\cos\theta} = \sec\theta$$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

(iv) $\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} = \frac{2\sin h(u/2).\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)} = \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta}$ (By (i) and (ii))

$\therefore \tan h\left(\frac{u}{2}\right) = \frac{sin\theta/cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)} = \frac{sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Solution: **(i)** $\cosh x = \sec\theta$

$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$ By definition $\cos hx = \frac{e^x+e^{-x}}{2}$

$\therefore e^x - 2\sec\theta + e^{-x} = 0$

$\therefore (e^x)^2 - 2e^x sec\theta + 1 = 0$

Solving the quadratic in e^x,

$e^x = sec\theta \pm \sqrt{sec^2\theta - 1} = \sec\theta \pm \tan\theta$

$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm log(\sec\theta + \tan\theta)$

(we can prove that $log(\sec\theta - \tan\theta) = -log(\sec\theta + \tan\theta)$)

(ii) Let $tan^{-1}e^{-x} = \alpha$ $\therefore e^{-x} = \tan\alpha$ $\therefore e^x = \cot\alpha$

Now, by data $\sec\theta = \cos hx = \frac{e^x+e^{-x}}{2} = \frac{cot\alpha+tan\alpha}{2}$

$$2\ sec\theta = cot\alpha + tan\alpha = \frac{cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$$

(iii) $\tan h\ \frac{x}{2} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^x-1}{e^x+1} = \frac{sec\theta+tan\theta-1}{sec\theta+tan\theta+1} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{(1-cos\theta)+\sin\theta}{(1+cos\theta)+\sin\theta} = \frac{2sin^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{cos(\theta/2)} = tan\frac{\theta}{2}$$

SEPARATION OF REAL AND IMAGINARY PARTS:

Many a time we are required to separate real and imaginary parts of a given complex function. For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore\ \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\ \tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

$\therefore 1 - x^2 - y^2 = 2x\cot 2\alpha \qquad \therefore x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$\therefore 1 + x^2 + y^2 = 2y\coth 2\beta \qquad$ i.e., $x^2 + y^2 - 2y\coth 2\beta + 1 = 0$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}(e^{i\,\theta})$

Solution: Let $tan^{-1}e^{i\theta} = x + iy \quad \therefore e^{i\theta} = \tan(x + iy) \quad \therefore cos\theta + i\sin\theta = \tan(x + iy)$

Similarly, $cos\theta - i\sin\theta = \tan(x - iy)$

Now, $tan\ 2x = tan\ [\,(x + iy) + (x - iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)+(cos\theta-i\sin\theta)}{1-(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ cos\theta}{1-(cos^2\theta+sin^2\theta)} = \frac{2\ cos\theta}{1-1} = \frac{2\ cos\theta}{0} = \infty$$

$\therefore 2x = \frac{\pi}{2} \quad \therefore x = \frac{\pi}{4}$

Also $\tan 2\,iy = \tan[(x + iy) - (x - iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)-(cos\theta-i\sin\theta)}{1+(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ isin\theta}{1+(cos^2\theta+sin^2\theta)} = \frac{2isin\theta}{2}$$

$\therefore i\tan h\ 2y = i\sin\theta \qquad \therefore \tan h\ 2y = \sin\theta$

$\therefore 2y = tanh^{-1}\sin\theta \qquad \therefore y = \frac{1}{2}\tan h^{-1}\sin\theta$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Solution: $\sin(\alpha - i\,\beta) = x + i\,y$

$\therefore \sin\alpha\cos h\,\beta + i\cos\alpha\sin h\,\beta = x + iy$

Equating real and imaginary parts, we get, $\sin\alpha\cos h\,\beta = x$ *and* $\cos\alpha\sin h\,\beta = y$

$\therefore \frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = sin^2\alpha + cos^2\alpha = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\,, prove\ that$

(i) $\sin\alpha = \pm sin^2 x = \pm \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Solution: $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\cos x\cos(iy) - \sin x\sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we get,

$\cos x\cosh y = \cos\alpha$ and $-\sin x\sinh y = \sin\alpha$

(i) Since $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x\sinh^2 y + \cos^2 x\cosh^2 y = 1$

$\sin^2 x\sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\sin^2 x\sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x\sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x = 1$

$\sinh^2 y - \sin^2 x = 0$

$\therefore \sinh^2 y = \sin^2 x$ ....(i)

$\therefore \sinh y = \pm\sin x$

$\therefore \sin\alpha = -\sin x\sinh y = -\sin x(\pm\sin x) = \pm\sin^2 x$

(ii) $\cos 2x + \cosh 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$ .... from (i)

$= 2$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Solution: We have to separate real part $\pi/6$ and imaginary part α

$\because tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$ $\quad\therefore tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$\therefore tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{tan\left(\frac{\pi}{6}+i\alpha\right)+\tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right).\tan\left(\frac{\pi}{6}-i\alpha\right)}$

$\therefore tan\frac{\pi}{3} = \frac{(x+iy)+(x-iy)}{1-(x+iy).(x-iy)}$

$\therefore \sqrt{3} = \frac{2x}{1-x^2-y^2}$

$\therefore 1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$

$\therefore x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1.$

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Solution: We have $x + iy = c\,\cot(u + iv)$ $\quad\therefore\ x - iy = c\,\cot(u - iv)$

$\therefore 2x = c[\cot(u+iv) + \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\,\frac{[\cos(u+iv)\sin(u-iv)+\sin(u+iv)\cos(u-iv)]}{\sin(u+iv)\sin(u-iv)}$$

$$\therefore 2x = \frac{c\sin[(u-iv)+(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u-iv-u+iv)]/2}$$

$$\therefore x = \frac{c\sin 2u}{-[\cos 2u-\cos 2\,iv]} = \frac{c\sin 2u}{\cos h\,2v-\cos 2u} \quad \text{............(1)}$$

Now, $2iy = c[\cot(u+iv) - \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} - \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\left[\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$\therefore 2iy = \frac{c\sin[(u-iv)-(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u+iv-u+iv)]/2}$$

$$\therefore\ iy = \frac{c\sin(-2iv)}{-[\cos 2u-\cos 2iv]} = -\frac{i\,c\sin h\,2v}{\cos h\,2v-\cos 2u}$$

$$\therefore\ y = \frac{-\,c\sin h\,2v}{\cos h\,2v-\cos 2u} \quad \text{..........(2)}$$

From (1) & (2) $\dfrac{x}{\sin 2u} = -\dfrac{y}{\sinh 2v} = \dfrac{c}{\cosh 2v-\cos 2u}$

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right), prove\ that\ (u^2+v^2)^2 = 2(u^2-v^2)$

Solution: We have $\dfrac{1}{\sin[(\pi/4)+ix]} = u + iv$

$$\therefore \sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} = \frac{1}{u+iv}\cdot\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore \sin\frac{\pi}{4}\cos i\,x + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}\cos h\,x + i\frac{1}{\sqrt{2}}\sin hx = \frac{u-iv}{u^2+v^2}$$

Equating real and imaginary parts $\cos hx = \sqrt{2}.\left(\frac{u}{u^2+v^2}\right)$; $\sin hx = -\sqrt{2}.\left(\frac{v}{u^2+v^2}\right)$

But $cosh^2x - sinh^2x = 1$

$$\therefore 2\left(\frac{u^2}{(u^2+v^2)^2}\right) - 2\left(\frac{v^2}{(u^2+v^2)^2}\right) = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$

7. If $x + iy = \cos(\alpha + i\,\beta)$ or if $\cos^{-1}(x+iy) = \alpha + i\,\beta$ express x and y in terms of α and β.

Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$

Solution: We have $\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = x + iy$

$\therefore \cos\alpha\cos h\,\beta - i\sin\alpha\sin h\beta = x + iy$

Equating real and imaginary parts $\cos\alpha\cos h\,\beta = x\ \ and\ \ \sin\alpha\sin h\beta = -y$

We know that, in terms of the roots, the quadratic equation is given by

$\lambda^2 - (sum\ of\ the\ roots)\lambda + (product\ of\ the\ roots) = 0$

Hence the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$ is

$\lambda^2 - (cos^2\alpha + cos^2\beta)\lambda + (cos^2\alpha . cos^2\beta) = 0$

This means we have to prove that $x^2 + y^2 + 1 = cos^2\alpha + cos^2\beta$ and $x^2 = cos^2\alpha + cos^2\beta$

Now, $x^2 + y^2 + 1 = cos^2\alpha \cos h^2\beta + sin^2\alpha \sin h^2\beta + 1$

$= cos^2\alpha \cos h^2\beta + (1 - cos^2\alpha)(\cos h^2\beta - 1) + 1$

$= cos^2\alpha \cos h^2\beta + \cos h^2\beta - 1 - cos^2\alpha \cos h^2\beta + cos^2\alpha + 1$

$= cos^2\alpha + \cos h^2\beta = sum\ of\ the\ roots$

And $x^2 = cos^2\alpha \cos h^2\beta$ = Product of the roots

Hence the equation whose roots are $cos^2\alpha$, $\cos h^2\beta$ is $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$

HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\, x$ and is given as, $\sinh x = \frac{e^x - e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\, x$ and is given as, $\cosh x = \frac{e^x + e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as $\tan \mathrm{hx} = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\mathrm{cosechx} = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad \mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$, and $\quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1,\ \tanh(0) = 0,\ \tanh(\infty) = 1$ $\quad \therefore |\tanh x| \leq 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x$,
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$

3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\mathrm{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \mathrm{cosec}^2 x$	$\coth^2 x - \mathrm{cosech}^2 x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x\cos y \pm \cos x\, sin\, y$	$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$
15	$\cos(x \pm y) = \cos x\cos y \mp \sin x\, sin\, y$	$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$
16	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x\ \tanh y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x\ \tanh y}$
17	$\cot(x \pm y) = \dfrac{\cot x\cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \dfrac{-\coth x\coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$
20	$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
21	$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
22	$2\sin x\cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x\sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

24	$2\cos x \cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x \sin y = \cos\,(x-y) - \cos(x+y)$	$2\sinh x \sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$$\begin{aligned}\sinh(2\pi i + x) &= \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x\\ &= i\sin\ 2\pi\ \cosh x + \cos\ 2\pi\ \sinh x\\ &= 0 + \sinh x\\ &= \sinh x\end{aligned}$$

Hence sinh x is a periodic function of period $2\pi i$

Similarly we can prove that cosh x and tanh x are periodic functions of period $2\pi i$ and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$, $y = \frac{e^x - e^{-x}}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{e^x+e^{-x}}{2} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$, $y = \frac{e^x+e^{-x}}{2}$, $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh x$

If $y = cosh\ x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$, $y = \frac{\sinh x}{\cosh x}$ $\quad \therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\, dx = \sinh x$, $\qquad \int \sinh x\, dx = \cosh x$, $\qquad \int sech^2 x dx = \tanh x$

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}, find\ \sinh 2x\ and\ \cosh 2x$ -

Solution: $\tan hx = \frac{e^x-e^{-x}}{e^x+e^x} = \frac{1}{2}$ $\therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$ $\quad \therefore 2\,e^{2x} - 2 = e^{2x} + 1$ $\quad \therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x}-e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x}+e^{-2x}}{2} = \frac{3+(1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x+e^{-x}}{2}\right)+8\left(\frac{e^x-e^{-x}}{2}\right)=1$

$\therefore 7e^x+7e^{-x}+8e^x-8e^{-x}=2$

$\therefore 15e^x-e^{-x}=2$

$\therefore 15e^{2x}-2e^x-1=0$ Solving it as a quadratic equation in e^x,

$e^x=\frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)}=\frac{2\pm 8}{30}=\frac{1}{3}\ \ or -\frac{1}{5}$

$\therefore x=\log\left(\frac{1}{3}\right)\ \ or\ x=\log\left(-\frac{1}{5}\right)$

Since x is real, $x=log\left(\frac{1}{3}\right)=-\log 3$