LOGIC-04

2.1 Propositions and logical operations, Truth tables
2.2 Equivalence, Implications
2.3 Laws of logic, Normal Forms
2.4 Predicates and Quantifiers
2.5 Mathematical Induction

LOGIC

- Study of the logic <u>relationships</u> between <u>objects</u>

and

- Basis of all mathematical reasoning and all automated reasoning

Introduction: PL?

- In Propositional Logic, the objects are called <u>propositions</u>
- **Definition**: **A proposition is a <u>statement</u> that is either <u>true</u> or <u>false</u>, but not both**
- We usually denote a proposition by a letter: *p*, *q*, *r*, *s*, …

Introduction: Proposition

- **Definition**: The value of a proposition is called its truth value; denoted by
  - ***T* or 1 if it is true** or
  - ***F* or 0 if it is false**
- Opinions, interrogative, and imperative are **not propositions**
- **Truth table**

p
0
1

Propositions: Examples

- The following are propositions
  - Today is Monday *M*
  - The grass is wet *W*
  - It is raining *R*
- **The following are not propositions**
  - C++ is the best language *Opinion*
  - When is the pretest? *Interrogative*
  - Do your homework *Imperative*

Logical operations

- Connectives are used to create a compound proposition from two or more propositions
- **Negation** (e.g., $\neg$ a, **~** a, or ā)
- **AND** or logical **Conjunction** (denoted $\wedge$)
- **OR** or logical **Disjunction** (denoted $\vee$)
- **XOR** or exclusive or (denoted $\oplus$)
- **Imply** on (denoted $\Rightarrow$ or $\rightarrow$)
- Biconditional (denoted $\Leftrightarrow$ or $\leftrightarrow$) IFF

We define the meaning (semantics) of the logical connectives using <u>truth tables</u>

Precedence of Logical Operators

- As in arithmetic, an ordering is imposed on the use of logical operators in compound propositions
- However, it is preferable to use parentheses to disambiguate operators and facilitate readability

$$\neg p \vee q \wedge \neg r \;\equiv\; (\neg p) \vee (q \wedge (\neg r))$$

- To avoid unnecessary parenthesis, the following precedence hold:
  1. Negation ($\neg$)
  2. Conjunction ($\wedge$)
  3. Disjunction ($\vee$)
  4. Implication ($\rightarrow$)
  5. Biconditional ($\leftrightarrow$)

Logical Connective: **Negation**

- $\neg p$, the negation of a proposition p, is also a proposition
- Examples:
  - Today is not Monday
  - It is not the case that today is Monday, etc.
- **Truth table**

p	$\neg p$
0	1
1	0

Logical Connective: **Logical AND**

- The logical connective And is true only when both of the propositions are true. It is also called a conjunction
- Examples
  - It is raining and it is warm
  - (2+3=5) and (1<2)
- Truth table

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Logical Connective: **Logical OR**

- The logical <u>disjunction</u>, or logical OR, is true if one or both of the propositions are true.
- Examples
  - It is raining or it is the second lecture
  - $(2+2=5) \vee (1<2)$
  - You may have cake or ice cream
- **Truth table**

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Logical Connective: **Exclusive Or**

- The exclusive OR, or XOR, of two propositions is true when exactly one of the propositions is true and the other one is false
- Example
  - The circuit is either ON or OFF but not both
- Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Connective: **Biconditional** (1)

- **Definition:** The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values. It is false otherwise.
- **Truth table**

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Connective: Biconditional (2)

- The biconditional $p \leftrightarrow q$ can be equivalently read as
  - *p* if and only if *q*
  - *p* is a necessary and sufficient condition for *q*
  - if *p* then *q*, and conversely
  - *p* iff *q* (Note typo in textbook, page 9, line 3)
- Examples
  - $x>0$ **if and only** if x^2 is positive
  - The alarm goes off **iff** a burglar breaks in
  - You may have pudding **iff** you eat your meat

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Examples :

- (i) An integer is even if and only if it is divisible by 2.
- (ii) A right angled triangle is isosceles if and only if the other two angles are each equal to forty-five degrees.
- (iii) Two lines are parallel if and only if they have the same slope

Logical Connective: **Implication** (1)

- **Definition:** Let *p* and *q* be two propositions.

The implication ***p→q*** is **a relationship between two propositions in which the second is a logical consequence of the first**

- *p* is called the hypothesis(If I give you 1 million $)
- *q* is called the conclusion, consequence(then you will become a millionaire)

Note that this is logically equivalent to ¬p V q

Logical Connective: Implication

- The implication of $p \rightarrow q$ can be also read as
  - If p then q
  - p implies q
  - If p, q
  - p only if q
  - q if p
  - q when p
  - q whenever p
  - q follows from p
  - p is a sufficient condition for q (p is sufficient for q)
  - q is a necessary condition for p (q is necessary for p)

Logical Connective: Implication

- Examples
  - If you buy you air ticket in advance, it is cheaper.
  - If x is an integer, then $x^2 \geq 0$.
  - If it rains, then grass gets wet.
  - If 2+2=5, then all unicorns are pink.

Logical Connective: Implication Truth Table

p: It is raining, q: I am wet

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Let p denote :Peter is rich,
q denote :Peter is happy.

Write symbolic form for the following

1. Peter is poor but happy
2. **$\sim p \wedge q$**
3. Peter is neither rich nor happy
4. **$\sim p \wedge \sim q$**
5. Peter is either rich or unhappy
6. **$p \vee \sim q$**

Write the following statements in symbolic form :

p :I will study discrete structures.
q :I will go to a movie.
r : I am in a good mood.

1. If I am not in a good mood, then I will go to a movie.

2. I will not go to a movie and I will study discrete structures.

3. I will go to a movie only if I will not study discrete structures.

4. I will not study discrete structures, then I am not in a good mood.

p :I will study discrete structures.
q :I will go to a movie.
r : I am in a good mood.

Write the following statements in symbolic form :

- If I am not in a good mood, then I will go to a movie.
- $\sim r \rightarrow q$
- I will not go to a movie and I will study discrete structures.
- $\sim q \land p$
- I will go to a movie only if I will not study discrete structures.
- $\sim p \rightarrow q$
- I will not study discrete structures, then I am not in a good mood.
- $\sim p \rightarrow \sim r$

Write logical/conditional propositions

1. There is an error in the program or the data is wrong.
2. If Peter works hard then he will pass the exam.
3. Farmers will face hardship if the dry spell continues.
4. Unless I reach the station on time , I will miss the train.

Converse, Inverse, Contrapositive

- Consider the proposition $p \rightarrow q$ (Conditional If….. then)
  - Its **converse** is the proposition $\mathbf{q \rightarrow p}$
  - Its **contrapositive** is the proposition $\mathbf{\neg q \rightarrow \neg p}$
  - Its **inverse** is the proposition $\mathbf{\neg p \rightarrow \neg q}$

- State the converse, inverse and contrapositive of the following.
- (i) If it is cold then he wears hat.
- Let p: It is cold , q: He wears hat.
- **Converse** $(q \to p)$ **:** If he wears hat then it is cold.
- **Contrapositive** $(\sim q \to \sim p)$ **:** If he does not wear hat, then it is not cold.
- **Inverse** $(\sim p \to \sim q)$ **:** If it is not cold then he does not wear hat.
- (ii) If integer is multiple of 2, then it is even.
- **Converse** $(q \to p)$ **:** If integer is even, then it is multiple of 2.
- **Inverse** $(\sim p \to \sim q)$: If integer is not multiple of 2 then it is not even.
- **Contrapositive** $(\sim q \to \sim p)$ **:** If integer is not even, then it is not multiple of 2.

1. Consider

P:You stay in Mumbai;

Q: You stay in Taj

Determine Converse, Contrapositive and Inverse for

"If you stay in Mumbai , you stay in Taj"

2. Write down the English sentences for converse and contrapositive of : **"If 250 is divisible by 4 then 250 is an even number "**

Let '**a**' be the proposition **'high speed driving is dangerous**' and '**b**' be the proposition **'Rajesh was a wise man.'**

Write down the meaning of the following proposition.

- 1. $a \wedge b$
- 2. $\sim a \wedge b$
- 3. $(a \wedge b) \vee (\sim a \wedge \sim b)$
- **Soln. :**
- **1.** $\mathbf{a \wedge b}$ **:** High speed driving is dangerous and Rajesh was a wise man.
- **2.** $\mathbf{\sim a \wedge b}$ **:** High speed driving is not dangerous and Rajesh is a wise man.
- **3.** $\mathbf{(a \wedge b) \vee (\sim a \wedge \sim b)}$ **:** High speed driving is dangerous and Rajesh was a wise man or neither high speed driving is dangerous nor Rajesh is a wise man.

- Express the proposition 'Either my program runs and it contains no bugs, or my program contains bugs' in symbolic form.
- **Soln. :**
- Let p : My program runs.
- q : My program contains bugs.
- The proposition can be written in symbolic form as,

$$(p \wedge \sim q) \vee q$$

How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

- **Soln: Let q : You can ride the roller coaster.**
- **r : You are under 4 feet tall.**
- **s : You are older than 16 years old.**
- **The sentence can be translated into:**
- $(r \land \neg s) \rightarrow \neg q$

Usefulness of Logic

- Logic is more precise than natural language
  - You may have cake or ice cream.
    - Can I have both?
  - If you buy your air ticket in advance, it is cheaper.
    - Are there or not cheap last-minute tickets?
- For this reason, logic is used for hardware and software <u>specification</u>
  - Given a set of logic statements,
  - One can decide whether or not they are <u>satisfiable</u> (i.e., consistent), although this is a costly process…

Terminology:
Tautology, Contradictions, Contingencies

- Definitions
  - A compound proposition that is always **true**, no matter what the truth values of the propositions that occur in it is called a **TAUTOLOGY**
  - A compound proposition that is always **false** is called a **CONTRADICTION**
  - A proposition that is neither a tautology nor a contradiction is a **CONTINGENCY**
- Examples
  - A simple tautology is $p \vee \neg p$
  - A simple contradiction is $p \wedge \neg p$

Truth Tables

- Truth tables are used to show/define the relationships between the truth values of
  - the individual propositions and
  - the compound propositions based on them

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Logical Equivalences: Example 1

- **Are the propositions ($p \rightarrow q$) and ($\neg p \vee q$) logically equivalent?**
- To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0			
0	1			
1	0			
1	1			

- The two columns in the truth table are identical, thus we conclude that

$$(p \rightarrow q) \equiv (\neg p \vee q)$$

Logical Equivalences: Example 1

- **Are the propositions ($p \rightarrow q$) and ($\neg p \vee q$) logically equivalent?**
- To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

- The two columns in the truth table are identical, thus we conclude that

$$(p \rightarrow q) \equiv (\neg p \vee q)$$

Constructing Truth Tables

- Construct the truth table for the following compound proposition $\sim P \wedge (P \rightarrow Q)$

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T			
T	F			
F	T			
F	F			

Constructing Truth Tables

- Construct the truth table for the following compound proposition ~ P ∧ (P →Q)

$\sim P \wedge (P \rightarrow Q)$

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

. Show that $(P \to Q) \vee (Q \to P)$ is a tautology.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T			
T	F			
F	T			
F	F			

. Show that $(P \to Q) \vee (Q \to P)$ is a tautology.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Prove $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ is a contradiction

Prove $(A \vee B) \wedge (\neg A)$ a contingency

- Construct a truth table for $(P \to Q) \wedge (Q \to R)$.

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

- Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Logical Equivalences: Exercise 25 from Rosen

- Show that $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \vee (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$

LAWS OF LOGIC

Commutative laws

$p \wedge q \Leftrightarrow q \wedge p$

$p \vee q \Leftrightarrow q \vee p$

Associative laws

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

Inverse laws

$p \wedge \neg p \Leftrightarrow F$

$p \vee \neg p \Leftrightarrow T$

Laws of Logic

Distributive laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Idempotent laws

$p \wedge p \Leftrightarrow p$

$p \vee p \Leftrightarrow p$

Identity laws

$p \wedge T \Leftrightarrow p$

$p \vee F \Leftrightarrow p$

Laws of Logic

Domination laws

$p \land F \Leftrightarrow F$

$p \lor T \Leftrightarrow T$

Absorption law

$p \land (p \lor q) \Leftrightarrow p$

$p \lor (p \land q) \Leftrightarrow p$

De Morgan Law:

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Table of Logical Equivalences

Commutative	$p \wedge q \iff q \wedge p$	$p \vee q \iff q$
Associative	$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	$(p \vee q) \vee r \iff p$
Distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee$
Identity	$p \wedge T \iff p$	$p \vee F \iff$
Negation	$p \vee \sim p \iff T$	$p \wedge \sim p \iff$
Double Negative	$\sim (\sim p) \iff p$	
Idempotent	$p \wedge p \iff p$	$p \vee p \iff$
Universal Bound	$p \vee T \iff T$	$p \wedge F \iff$
De Morgan's	$\sim (p \wedge q) \iff (\sim p) \vee (\sim q)$	$\sim (p \vee q) \iff (\sim$
Absorption	$p \vee (p \wedge q) \iff p$	$p \wedge (p \vee q)$
Conditional	$(p \implies q) \iff (\sim p \vee q)$	$\sim (p \implies q) \iff$

Quantifiers

- **Predicates**- Quantifier is **used to quantify the variable of predicates**
- **An assertion that contains one or more variable is called a predicate**
- X+3= 5, x+y >=10…..etc,
- these statements are not propositions since they do not have truth value, but if values assigned can become true or false

Eg : x is a city in India ---------------------------- P(x)

x is the father of y -----------------------------P(x , y)

x + y > = z is denoted by ----------------------P(x, y, z)

– There are two types of quantifier in predicate logic –

– **Universal Quantifier and Existential Quantifier**

Universal **quantifier** “**for all, for every, for each** " is **"∀"**,

Existential **quantifier**“ **there exists** " is "**∃**”

$\forall\ a\ \exists\ b\ P(a, b)$

where $P(a, b)$ denotes $a + b = 0$

$\forall\ a\ \forall\ b\ \forall\ c\ P(a, b, c)$

where $P(a, b, c)$ denotes $a + (b + c) = (a + b) + c$

Note – $\forall\ a\ \exists\ b\ P(x, y) \neq \exists\ a\ \forall\ b\ P(x, y)$

Quantifier

If $M(x)$ is "x is man"

$C(x)$ is "x is clever"

Translate the following statements into English.

(i) $\exists x\ (M(x) \rightarrow C(x))$

(ii) $\forall x\ (M(x) \wedge C(x))$

- **Soln. :**
- (i) There exists a man who is clever.
- (ii) For all men x is man and x is clever.

Write the following two propositions in symbols.

Let $p(x,y)$ denote the predicate '$y = x + 1$'.

(i)'For every number x there is a number y such that $y = x + 1$.'

(ii)'There is a number y such that, for every number x, $y = x + 1$.'

Write the following two propositions in symbols.

Let p(x,y) denote the predicate 'y = x + 1'.

(i)'For every number x there is a number y such that y = x + 1.'

$$\forall x \, \exists y \, P(x, y)$$

(ii)'There is a number y such that, for every number x, y = x + 1.'

$$\exists y \, \forall x \, P(x,y)$$

Write English sentences for the following

1. $\forall x \exists y R(x, y)$ For every even number x there is a prime number y such that x+y is even
2. $\exists x \forall y R(x, y)$ There exists an even number x for all prime numbers y such that x + y is even
3. $\forall x (\sim Q(x))$ For every number x there is a number which is not prime
4. $\exists x (\sim P(x))$ There exists a number x such that x is not even
5. $\forall x P(x)$ where

P(x) : x is even

Q(x) : x is prime nos

R(x, y): x + y is even

NORMAL FORMS(complex form of variables)

- **Conjunctive Normal Form- CNF**
  - Expression $(x_1 \vee x_2 \vee x_3) \wedge (\sim x_1 \vee \sim x_2 \vee \sim x_3)$

  Eg: $(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)$

- **Disjunctive Normal Form- DNF**
  - Expression $(x_1 \wedge x_2 \wedge x_3) \vee (\sim x_1 \wedge \sim x_2 \wedge \sim x_3)$

  Eg: $(A \wedge B) \vee (A \wedge C) \vee (B \wedge C \wedge D)$

Disjunctive Normal Form

- A disjunction of conjunctions where every variable or its negation is represented once in each conjunction (a *minterm*)
  - each minterm appears only once

Example: DNF of $p \oplus q$ is

$(p \wedge \neg q) \vee (\neg p \wedge q)$.

Truth Table

p	q	$p \oplus q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F
T	*F*	*T*	*T*
F	*T*	*T*	*T*
F	F	F	F

Method to construct DNF

1. Construct a truth table for the proposition.
2. Use the rows of the truth table where the proposition is **True to construct minterms.**
   - If the variable is true, use the propositional variable in the minterm
   - If a variable is false, use the negation of the variable in the minterm

1. Connect the minterms with $\vee$'s.

How to find the DNF of $(p \vee q) \to \neg r$

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \to \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$$(p \vee q) \to \neg r \iff (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

How to find the DNF of $(p \vee q) \rightarrow \neg r$

$$(p \vee q) \rightarrow \neg r \iff (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Q2. Find Conjunctive Normal Form

$$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conjunctive Normal Form$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

Truth table:

p	q	r	$p \Leftrightarrow q$	$\neg p \wedge r$	$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	0
1	1	1	1	0	0

Solution: (full) conjunctive normal form (FCNF) of the formula is
$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Find DNF of $(\sim p \rightarrow r) \wedge (p \Leftrightarrow q)$

Find CNF of $(p \Rightarrow q) \Rightarrow r$

$(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive
$\Leftrightarrow (q \vee p) \wedge T$	Negation
$\Leftrightarrow q \vee p$	Identity
$\Leftrightarrow p \vee q$	Commutative

Prove: $p \rightarrow p \vee q$ is a tautology

$p \rightarrow p \vee q$	
$\Leftrightarrow \neg p \vee (p \vee q)$	Implication Equivalence
$\Leftrightarrow (\neg p \vee p) \vee q$	Associative
$\Leftrightarrow (p \vee \neg p) \vee q$	Commutative
$\Leftrightarrow T \vee q$	Negation
$\Leftrightarrow q \vee T$	Commutative
$\Leftrightarrow T$	Domination

Use Logical Equivalences to prove that

$[(p \land \neg(\neg p \lor q)) \lor (p \land q)] \rightarrow p$ is a tautology.

Proof: $[(p \land \neg(\neg p \lor q)) \lor (p \land q)] \rightarrow p$

$\equiv [(p \land (\neg(\neg p) \land \neg q)) \lor (p \land q)] \rightarrow p$	De Morgan's law
$\equiv [(p \land (p \land \neg q)) \lor (p \land q)] \rightarrow p$	Double Negation law
$\equiv [((p \land p) \land \neg q) \lor (p \land q)] \rightarrow p$	Associative law
$\equiv [(p \land \neg q) \lor (p \land q)] \rightarrow p$	Idempotent law
$\equiv [p \land (\neg q \lor q)] \rightarrow p$	Distributive law
$\equiv [p \land (q \lor \neg q)] \rightarrow p$	Commutative law
$\equiv [p \land T] \rightarrow p$	Negation law
$\equiv p \rightarrow p$	Identity law
$\equiv \neg p \lor p$	Equivalence of Implication
$\equiv p \lor \neg p$	Commutative law
$\equiv T$	Negation law

Table of Logical Equivalences

Commutative	$p \wedge q \iff q \wedge p$	$p \vee q \iff q \vee p$
Associative	$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	$(p \vee q) \vee r \iff p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
Negation	$p \vee \sim p \iff T$	$p \wedge \sim p \iff F$
Double Negative	$\sim (\sim p) \iff p$	
Idempotent	$p \wedge p \iff p$	$p \vee p \iff p$
Universal Bound	$p \vee T \iff T$	$p \wedge F \iff F$
De Morgan's	$\sim (p \wedge q) \iff (\sim p) \vee (\sim q)$	$\sim (p \vee q) \iff (\sim p) \wedge (\sim q)$
Absorption	$p \vee (p \wedge q) \iff p$	$p \wedge (p \vee q) \iff p$
Conditional	$(p \implies q) \iff (\sim p \vee q)$	$\sim (p \implies q) \iff (p \wedge \sim q)$

Example 8 :

$p \rightarrow q$ and $\sim p \vee q$ are logically equivalent. (**Implication**).

Solution :

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Similarly, we have the next example, in which the biconditional can also be eliminated.

$$P \leftrightarrow Q = (P \longrightarrow Q) \wedge (Q \longrightarrow P)$$

Example

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent (**De Morgans laws**)

Solution :

Consider the truth tables.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last columns in both the tables are identical. Hence $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

The DeMorgan's Law can be expressed as : **If it is not the case that p and q are both true, then that is the same as saying that at least one of p or q is false.**

De Morgan's $\quad \sim(p \wedge q) \iff (\sim p) \vee (\sim q) \qquad \sim(p \vee q) \iff (\sim p) \wedge (\sim q)$

Example 14 :

Use the laws of logic to show that

$[(p \to q) \wedge \sim q] \to \sim p$ is a tautology.

Solution :

We have,

$[(p \to q) \wedge \sim q] \to \sim p$

$\equiv \sim[(\sim p \vee q) \wedge \sim q] \vee \sim p$ - Implication law

$\equiv \sim[\sim q \wedge (\sim p \vee q)] \vee \sim p$ - Commutative law

$\equiv \sim[(\sim q \wedge \sim p) \vee (\sim q \wedge q)] \vee \sim p$ - Distributive law

$\equiv \sim[(\sim q \wedge \sim p) \vee (q \wedge \sim q)] \vee \sim p$ - Commutative law

$\equiv \sim[(\sim q \wedge \sim p) \vee F] \vee \sim p$ - Complement law

$\equiv \sim(\sim q \wedge \sim p) \vee \sim p$ - Identity law

$\equiv (\sim\sim q \vee \sim\sim p) \vee \sim p$ - First demorgan's law

$\equiv (q \vee p) \vee \sim p$ - Complement law

$\equiv q \vee (p \vee \sim p)$ - Associative law

$\equiv q \vee T$ - Inverse law

$\equiv T$ - Identity law

$\therefore [(p \to q) \wedge \sim q] \to \sim p$ is a tautology.

... determining which law to apply at each step.

Example 1 :

Given the truth value of P and Q as T and those of R and S as F, find the truth value of the following.

$$[P \wedge (Q \wedge R)] \vee \sim [(P \vee Q) \vee (R \vee S)]$$ (May 99, May 2000)

Solution :

We substitute the given truth values in the expression and it reduces to

$$[T \wedge (T \wedge F)] \vee \sim [(T \vee T) \vee (F \vee F)]$$

$$\equiv [T \wedge F] \vee \sim [T \vee F]$$

$$\equiv F \vee \sim (T)$$

$$\equiv F \vee F$$

$$\equiv F$$

Hence the truth value of the above expression is false.

Obtain CNF $(p \wedge q) \vee (p \wedge \neg q)$

$(p \wedge q) \vee (p \wedge \neg q)$

$\equiv ((p \wedge q) \vee p) \wedge ((p \wedge q) \vee \neg q)$

…Distributive Law

$\equiv ((p \vee p) \wedge (q \vee p)) \wedge ((p \vee \neg q) \wedge (q \vee \neg q))$

Obtain DNF $\sim(p \rightarrow (q \wedge r))$

Solution : - $(p \wedge \sim q) \vee (p \wedge \sim r)$

Mathematical Induction

Statement of the principle of mathematical induction :

Let $P(n)$ be a statement involving a natural number n.

1. If $P(n)$ is true for $n = n_0$ and
2. Assuming $P(k)$ is true, $(k \geq n_0)$ we prove $P(k + 1)$ is also true,

then $P(n)$ is true for all natural numbers $n \geq n_0$

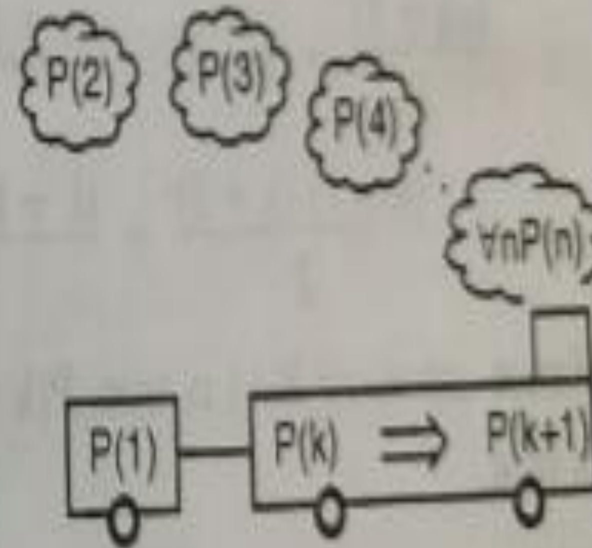

Fig. 2.1 : The principle of induction

Step (1) is called as the **Basis of induction.**

Step (2) is called as the **Induction step.**

The assumption that $P(n)$ is true for $n = k$ is called as the **Induction hypothesis.**

Use **MATHEMATICAL INDUCTION** to prove that

$1 + 2 + 3 + \ldots + n = n(n + 1)/2$ for all positive integers n.

Let the statement P (n) be **$1 + 2 + 3 + \ldots + n = n(n + 1)/2$……………………….(A)**

STEP 1. Basis: We first show that p (1) is true.

Left Side = 1

Right Side = 1 (1 + 1) / 2 = 1

Both sides of the statement are equal hence p (1) is true.

STEP 2-Inductive step : We now assume that p (k) is true

$1 + 2 + 3 + \ldots + k = k(k + 1)/2$ ………………………….(1)

STEP 3:Inductive hypothesis , Show that p (k + 1) is true by adding k + 1 to both sides of the above statement

L.H.S.= $1 + 2 + 3 + \ldots + k + (k + 1) = [k(k + 1)/2] + [(k + 1)]$………..from (1)

$= (k + 1)(k/2 + 1)$ ………………………….(2)

RHS = $(k + 1)(k + 2)/2$: **LHS=RHS**

The last statement may be written as

$1 + 2 + 3 + \ldots + k + (k + 1) = (k + 1)(k + 2)/2$ Which is the statement p(k + 1).

Exercises- Prove each of the following by Mathematical Induction

For n positive integers n, solve the following

→ $1 + 5 + 9 + \ldots\ldots + (4n - 3) = n(2n - 1)$

→ $1^2 + 2^2 + \ldots + n^2 = (n)(n + 1)(2n + 1) / 6$

→ $\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4}$

Prove that for any positive integer number n , $\mathbf{n^3 + 2n}$ is divisible by 3

Statement P (n) is defined by $n^3 + 2n$ is divisible by 3

STEP 1: We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$k^3 + 2k$ is divisible by 3

is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms **(Note: (k+1)^3= k^3+3k^2+3k+1)**

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

- Prove that $5^{n} - 1$ is divisble by 4 for all $n >= 1$

- Prove that $5^{n} - 1$ is divisble by 4 for all n>=1

$5^{1} - 1 = 4$; P(1) is true

$5^{k} - 1$ assume to be true

$= 5^{k+1} - 1$

$= 5^{k} * 5 - 5 + 4$

$5(5^{k} - 1) + 4$

Mathematical Induction

Statement of the principle of mathematical induction :

Let P(n) be a statement involving a natural number n.

1. If P(n) is true for $n = n_0$ and
2. Assuming P(k) is true, $(k \geq n_0)$ we prove P(k + 1) is also true,

then P(n) is true for all natural numbers $n \geq n_0$

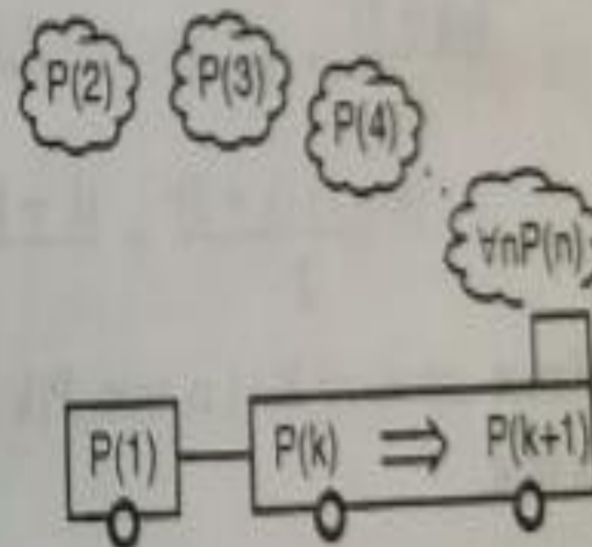

Fig. 2.1 : The principle of induction

Step (1) is called as the **Basis of induction.**

Step (2) is called as the **Induction step.**

The assumption that P(n) is true for n = k is called as the **Induction hypothesis.**

Mathematical Induction

Ex. 1 : Prove by induction :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$ for all natural number values of n.

Soln. : Let P(n) be the statement :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(i) Basis of induction :

for $n = 1$,

$$P(1) : 1 = \frac{1(2)}{2} = 1$$

Hence P(1) is true.

(ii) Induction step :

Assume P(k) is true,

Mathematical Induction

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n + 1)}{2}$$

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k\,(k + 1)}{2} \qquad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P(k + 1) is also true.

$$P(k + 1) : 1 + 2 + 3 + \ldots + k + (k + 1)$$

$$= \frac{(k + 1)\,[\,(k + 1) + 1]}{2} \qquad \ldots(ii)$$

$$= \frac{(k + 1)\,(k + 2)}{2}$$

Using equation (i)

$$\frac{k\,(k + 1)}{2} + (k + 1) = \frac{(k + 1)\,(k + 2)}{2}$$

$$\frac{(k + 1)\,(k + 2)}{2} = \frac{(k + 1)\,(k + 2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also ture. Therefore P(n) is true for all natural number values of n.

Mathematical Induction

Prove that

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

Mathematical Induction

Example 2 :

Prove that

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

Solution :

Let P (*n*) be the statement :

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

(*i*) **Basis of induction :**

For n = 1 P(1) : $\frac{1}{1.4}=\frac{1}{4}$

Hence P(1) is true.

Mathematical Induction

(ii) Induction step :

Assume P(k) is true, and prove P(k + 1) is also true.

$$P(k) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$$

$$P(k+1) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Using induction hypotheis (i),

$$\text{L.H.S. of Equation (ii)} = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

$$\because \left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \right]$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$$

$$= \frac{k+1}{3(k+1)+1}$$

$\therefore$ L.H.S = R.H.S. of Equation (ii)

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Show that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$

Show that $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$

Soln. :

Let $P(n)$: $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$

(i) Basis of induction :

For $n = 1$ $(1)^3 = (1)^2$ which is true

$\therefore$ $P(1)$ is true.

(ii) Induction step :

Assume $P(k)$ is true i.e.,

$$1^3 + 2^3 + 3^3 + \dots + k^3 = (1 + 2 + \dots + k)^2 \quad \dots(i)$$

To prove that $P(k + 1)$ is true i.e.;

$$1^3 + 2^3 + 3^3 + \dots + (k + 1)^3 = [1 + 2 + \dots + (k + 1)]^2$$

$$\begin{aligned}
\text{L.H.S.} &= 1^3 + 2^3 + 3^3 + \dots k^3 + (k + 1)^3 \\
&= [1^3 + 2^3 + 3^3 + \dots k^3] + (k + 1)^3 \\
&= (1 + 2 + \dots + k)^2 + (k + 1)^3 \quad \text{...by induction hypothesis, From Equation (i)} \\
&= \left[\frac{k(k + 1)}{2}\right]^2 + (k + 1)^3 \quad \dots\left[\text{Recall that } 1 + 2 + \dots + k = \frac{k(k + 1)}{2}\right] \\
&= \frac{k^2 (k + 1)^2}{4} + (k + 1)^3 \\
&= (k + 1)^2 \left[\frac{k^2}{4} + (k + 1)\right] \\
&= (k + 1)^2 \left[\frac{k^2 + 4k + 4}{4}\right] \\
&= \frac{(k + 1)^2 (k + 2)^2}{4} = \left[\frac{(k + 1)(k + 2)}{2}\right]^2 \\
&= [1 + 2 + \dots + (k + 1)]^2
\end{aligned}$$

Hence, L.H.S = R.H.S.

Therefore $P(n)$ is true.

Mathematical Induction

Example

Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.

Solution :

(i) **Basis of Induction :** Since it is given that, for all $n \geq 1$, Let $n = 1$.

$$(1)^3 + 2(1) = 1 + 2 = 3$$

3 is divisible by 3.

Mathematical Induction

(ii) **Induction step :**

Assuming that for $n = k$, it is true i.e. $k^3 + 2k$ is divisible by 3.

We will prove it for $n = k + 1$.

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2(1) + 3k(1)^2 + 1^3 + 2k + 2$$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= k^3 + 3k^2 + 5k + 3$$

$$= k^3 + 2k + 3k + 3k^2 + 3$$

$$= k^3 + 2k + 3k^2 + 3k + 3$$

$$= k^3 + 2k + 3(k^2 + k + 1)$$

$$= (k^3 + 2k) + 3(k^2 + k + 1)$$

Note that, since $(k^3 + 2k)$ is divisible by 3 by induction hypothesis (i), and $3(k^2 + k + 1)$ is divisible by 3. Each term is divisible by 3, we can say $n^3 + 2n$ is divisible by 3.

Example 13 :

Prove by mathematical

Mathematical Induction

Example 14 :

Prove by induction that $n^2 + n$ is an even number, for every natural number n.

Solution :

Let P(n) : $n^2 + n$ is an even number

(i) **Basis of induction :**

For $n = 1$

$P(1) : (1)^2 + (1)$ i.e. 2 is an even number

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e. $(k^2 + k)$ is an even number, and prove $P(k + 1)$ is also true.

$$\begin{aligned} P(k+1) \ : \ & (k+1)^2 + (k+1) \\ & = k^2 + 2k + 1 + k + 1 \\ & = (k^2 + k) + 2(k+1) \end{aligned}$$

Now $(k^2 + k)$ is even, by the induction hypothesis, and $2(k + 1)$ is even.

$\therefore$ The sum $(k^2 + k) + 2\,(k + 1)$ is even

$\therefore$ By induction, $k^2 + k$ is an even number, for every natural number n.

Mathematical Induction

Example 15 :

Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction $n \geq 1$. (Dec 96)

Solution :

Let $P(n) : 8^n - 3^n$ is a multiple of 5

(i) **Basis of induction :**

For $n = 1$

$P(1) : 8^1 - 3^1 = 5$ which is divisible by 5

$\therefore$ P(1) is true

(ii) **Induction step :**

Assume P(k) is true i.e. $(8^k - 3^k)$ is a multiple of 5, and prove P(k + 1) is also true.

$$\begin{aligned} P(k+1) &: 8^{k+1} - 3^{k+1} \\ &= 8^k \cdot 8 - 3^k \cdot 3 \\ &= 8^k \cdot 8 - 3^k \cdot 8 + 3^k \cdot 5 \\ &= 8\,(8^k - 3^k) + 3^k \cdot 5 \end{aligned}$$

Now $8(8^k - 3^k)$ is a multiple of 5 by induction hypothesis and $3^k \cdot 5$ is already a multiple of 5.

Hence $8^n - 3^n$ is a multiple of 5, for $n \geq 1$.

Mathematical Induction

Examples

Example 1 :

Obtain the DNF of the form

$(p \rightarrow q) \wedge (\sim p \wedge q)$

Solution : $p \rightarrow q \equiv \sim p \vee q$ (Elimination of biconditional)

Hence $(p \rightarrow q) \wedge (\sim p \wedge q)$

$$\equiv (\sim p \vee q) \wedge (\sim p \wedge q)$$

$$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$$

$$\equiv (\sim p \wedge q) \vee (q \wedge \sim p)$$

(by using the distributive laws, idempotence laws and commutative laws)

Mathematical Induction

Example 2 :

Obtain the DNF of $(p \wedge (p \to q)) \to q$

Solution : $(p \wedge (p \to q)) \to q$

$$\equiv \sim q \vee (p \wedge (\sim p \vee q))$$

$$\equiv \sim q \vee ((p \wedge \sim p) \vee (p \wedge q))$$

$$\equiv \sim q \vee F \vee (p \wedge q)$$

$$\equiv \sim q \vee (p \wedge q) \ldots (p \wedge \sim p \equiv \text{contradiction})$$

Mathematical Induction

Example 3 :

Obtain the DNF of the form

$$p \wedge (p \rightarrow q)$$

Solution :

$$\begin{aligned} p \wedge (p \rightarrow q) &\equiv p \wedge (\sim p \vee q) \\ &\equiv (p \wedge \sim p) \vee (p \wedge q) \\ &\equiv F \vee (p \wedge q) \\ &\equiv (p \wedge q) \end{aligned}$$

3.13.2 Conjunctive Normal Form (CNF) :

An expression of 'n' variables $x_1, x_2, \ldots, x_n$ is said to be a 'maxterm,' if it is of the form $x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_n$

An expression is said to be in **conjunctive normal form** if it is a meet of maxterms.

Example : $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$

Note that a CNF is a tautology if and only if every fundmental disjunction contained in it is a tautology.

Example 1 :

Obtain the CNF of the form

$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

Solution : $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

$$\equiv (\sim p \rightarrow r) \wedge (p \leftrightarrow q))$$

$$\equiv (\sim p \rightarrow r) \wedge ((p \rightarrow q) \wedge (q \rightarrow p))$$

$$\equiv (\sim(\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p))$$

$$\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$$