MODULE - 3

~~Q. Find Fourier Transform of $e^{-x^2/2}$~~

Z- TRANSFORM

Z- Transform of a sequence $\{f(k)\}$ denoted by $Z\{f(k)\}$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

z = Complex no.

LINEARITY PROPERTY

$$Z\{a f(k) + b g(k)\} = a Z[f(k)] + b Z[g(k)]$$

CHANGE OF SCALE PROPERTY

If $Z[f(k)] = F(z)$ then,

$$Z[a^k \cdot f(k)] = \cancel{F\left(\frac{a}{z}\right)}\ F\left(\frac{z}{a}\right)$$

and $Z[a^{-k} \cdot f(k)] = F(a \cdot z)$

SHIFTING PROPERTY

If $Z[f(k)] = F(z)$

then $Z[f(k \pm n)] = z^{\pm n} F(z)$

$$z(1) = \frac{3}{3-1}$$

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Q. Find Z-Transform of e^{ak} for $k \geq 0$

Soln: $Z[e^{ak}] = \sum_{k=0}^{\infty} e^{ak} z^{-k}$

$= \sum_{k=0}^{\infty} \left(\frac{e^a}{z}\right)^k$

$= 1 + \frac{e^a}{z} + \left(\frac{e^a}{z}\right)^2 + \left(\frac{e^a}{z}\right)^3 + \ldots.$

Sum $= \frac{1}{1 - e^a/z}$ if $r < 1$ where $r = |e^a/z|$

ie $\left(e^a/|z|\right) < 1$

$\Rightarrow e^a < |z|$

$\therefore Z[e^{ak}] = \frac{z}{z - e^a}$ for $|z| > e^a$

Q. Find Z-transform of $\sin \alpha k$ and hence find Z-transform of $[e^k \sin \alpha k]$. $(k \geq 0)$

Ans: $Z[\sin \alpha k] = \sum_{k=0}^{\infty} \sin \alpha k \, z^{-k}$

$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} - e^{-i\alpha k}}{2i}\right) z^{-k}$

$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} \left(\frac{e^{i\alpha}}{z}\right)^k - \sum_{k=0}^{\infty} \left(\frac{e^{-i\alpha}}{z}\right)^k\right]$

$= \frac{1}{2i}\left[\left(1 + \frac{e^{i\alpha}}{z} + \left(\frac{e^{i\alpha}}{z}\right)^2 + \ldots.\right) (-) \left(1 + \frac{e^{-i\alpha}}{z} + \left(\frac{e^{-i\alpha}}{z}\right)^2 + \ldots\right)\right]$

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$$= \frac{1}{2i}\left[\frac{1}{1-e^{i\alpha}/z} - \frac{1}{1-e^{-i\alpha}/z}\right]$$

for $\cancel{|z|>0}$
$|e^{i\alpha}/z| < 1$
$|e^{-i\alpha}/z| < 1$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right]$$ for $|e^{i\alpha}| < |z|$
$|e^{-i\alpha}| < |z|$
$|e^{i\alpha}| = |e^{-i\alpha}| = 1$
ie $|z| > 1$

$$= \frac{1}{2i}\left[\frac{z^2 - ze^{-i\alpha} - z^2 + ze^{i\alpha}}{z^2 - ze^{-i\alpha} - ze^{i\alpha} + 1}\right]$$

$$= \frac{1}{2i}\left[\frac{z(e^{i\alpha} - e^{-i\alpha})}{z^2 - 2z\cos\alpha + 1}\right]$$

$$= \frac{2z\sin\alpha}{z^2 - 2z\cos\alpha + 1} = F(z)$$

$$\therefore e^k \sin\alpha k = F\left(\frac{z}{e}\right) = \frac{2z/e \sin\alpha}{\frac{z^2}{e^2} - \frac{2z}{e}\cos\alpha + 1}$$

$$= \frac{2z/e \sin\alpha}{\frac{z^2 - 2ze\cos\alpha + e^2}{e^2}}$$

$$= \frac{2ze\sin\alpha}{z^2 - 2ez\cos\alpha + e^2}$$

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Q. 1) Find $Z[a^{|k|}]$

2) $Z[k^n] = -z \frac{d}{dz} Z(k^{n-1})$

Soln: 1) $Z[a^{|k|}] = \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k}$

$$= \sum_{k=-\infty}^{-1} a^{-k} \cdot z^{-k} + \sum_{k=0}^{\infty} a^k \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{-1} (a \cdot z)^{-k} + \sum_{k=0}^{\infty} a^k \cdot z^{-k}$$

$$= [az + (az)^2 + (az)^3 + \dots] + [1 + \tfrac{a}{z} + (\tfrac{a}{z})^2 + \dots]$$

$$= az[1 + az + (az)^2 + \dots] + [1 + a/z + (a/z)^2 + \dots]$$

$$= az\left[\frac{1}{1-az}\right] + \left[\frac{1}{1-a/z}\right]$$ for $|az| < 1$ and $|a/z| < 1$

$$= \frac{az}{1-az} + \frac{z}{z-a}$$ for $|z| < 1/a$ $|z| > a$

iff $0 < a < 1$

2) $Z[k^n] = -z \frac{d}{dz} Z(k^{n-1})$

$$Z[k^{n-1}] = \sum_{k=-\infty}^{\infty} k^{n-1} z^{-k}$$

$$\frac{d}{dz}[Z(k^{n-1})] = \sum_{k=0}^{\infty} k^{n-1} \frac{d}{dz}(z^{-k})$$

$$= \sum_{k=0}^{\infty} k^{n-1} (-k) z^{-k-1}$$

$$-z \frac{d}{dz} Z(k^{n-1}) = \sum_{k=0}^{\infty} k^n z^{-k} = Z[k^n]$$ //

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Find

(1) $Z(k)$ (2) $Z(k^2)$ (3) $Z(k^2 a^k)$

(4) $Z(a^k \sin k\theta)$ (5) $Z(k^2 e^{ak})$ (6) $Z\left[\frac{1}{k!}\right]$

(7) $Z\left(\frac{a^k}{k!}\right)$ (8) $Z\left(\frac{1}{k}\right)$ (9) $Z\left(\frac{1}{k(k+1)}\right)$

Ans: (1) $Z(k) = \cancel{\sum_{k=\infty}^{\infty} k \cdot k^{n-1}}$ $\cancel{\sum_{k=}^{\infty} k^n}$

$$-z \frac{d}{dz} z(1)$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left[\frac{z-1-z}{(z-1)^2}\right]$$

$$= \frac{z}{(z-1)^2}$$

(2) $Z(k^2) = -z \frac{d}{dz} Z(k)$

$$= -z \frac{d}{dz} \frac{z}{(z-1)^2}$$

(3) $Z(k^2 a^k) = F\left(\frac{z}{a}\right)$

$$= \frac{z/a\,(z/a+1)}{(z/a-1)^2}$$

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5 ~~4~~ $z[k^2 e^{ak}] = z[k^2 (e^a)^k]$

$$= \frac{z/e^a (z/e^a + 1)}{(z/e^a - 1)^3}$$

6 $$Z\left[\frac{1}{k!}\right] = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$$

$$= 1 + \frac{1}{1!\,z^1} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots$$

$$= e^{1/z}$$

7 $$Z\left[\frac{a^k}{k!}\right] = \text{(use change of scale property)}$$

$$= e^{a/z}$$

8 $$Z\left[\frac{1}{k}\right] = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \dots$$

$$-\log(1-z) = +\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \dots\right) \longrightarrow (*)$$

$$\therefore Z\left[\frac{1}{k}\right] = -\log\left(1 - \frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right)$$

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(9) $$Z\left(\frac{1}{k(k+1)}\right) = Z\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= Z\left(\frac{1}{k}\right) - Z\left(\frac{1}{k+1}\right)$$

*

Wkt $Z[f(k+n)] = z^n Z[f(k)]$

$\therefore Z[f(k+1)] = z\, Z[f(k)]$

If $f(k) = \frac{1}{k}$ $\quad f(k+1) = \frac{1}{k+1}$

$$\therefore Z\left[\frac{1}{k(k+1)}\right] = \log\left(\frac{z}{z-1}\right) - z\left[\log\left(\frac{z}{z-1}\right)\right]$$

$$= (1-z)\log\left(\frac{z}{z-1}\right)$$

$Z[k^n] = \cancel{\left(-z\frac{d}{dz}\right)^n F(z)}$ $-z\frac{d}{dz}\left[Z(k^{n-1})\right]$

If $Z[f(k)] = F(z)$
then $Z[k f(k)] = \left(-z\frac{d}{dz}\right) F(z)$

Q. Find $Z(k^2 e^{ak})$

Ans: $Z(e^{ak}) = \frac{z}{z-e^a}$

$$Z(k^2 e^{ak}) = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-e^a}\right)$$

$$= -z\frac{d}{dz}\left[-z\frac{[\cancel{z}-e^a-\cancel{z}]}{(z-e^a)^2}\right]$$

$$= -z\frac{d}{dz}\left[\frac{ze^a}{(z-e^a)^2}\right]$$

$$= -e^a z\frac{d}{dz}\left[\frac{z}{(z-e^a)^2}\right]$$

$$= -e^a z\left[\frac{(z-e^a)^2 - 2z(z-e^a)}{(z-e^a)^4}\right]$$

$$= -e^a z\left[\frac{-z-e^a}{(z-e^a)^3}\right]$$

$$= \frac{z(z+e^a)e^a}{(z-e^a)^3}$$

INITIAL VALUE ~~PROBLEM~~ THEOREM

If $F(z) = Z[f(k)]$

then ~~$F(0) = \lim_{z\to a} F(z)$~~ $f(0) = \lim_{z\to\infty} F(z)$

$$f(1) = \lim_{z\to\infty} z[F(z) - f(0)]$$

$$f(2) = \lim_{z\to\infty} z^2\left[F(z) - f(0) - \frac{f(1)}{z}\right]$$

$$f(3) = \lim_{z\to\infty} z^3\left[F(z) - f(0) - f(1)/z - \frac{f(2)}{z^2}\right]$$

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8. If $F(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$

find $f(0)$, $f(1)$, $f(2)$, $f(3)$

Ans: $F(z) = \dfrac{\cancel{z^2}\left(2 + \dfrac{5}{z} + \dfrac{14}{z^2}\right)}{z^{\cancel{4}2}\left(1 - \dfrac{1}{z}\right)^4}$

$= \dfrac{2 + \dfrac{5}{z} + \dfrac{14}{z^2}}{z^2\left(1 - \dfrac{1}{z}\right)^4}$

$\therefore f(0) = \lim_{z \to \infty} F(z) = 0$

$f(1) = \lim_{z \to \infty} \cancel{z}\left[\dfrac{2 + 15/z + 14/z^2}{z^{\cancel{2}}(1 - 1/z)^4}\right]$

$= \lim_{z \to \infty}\left[\dfrac{2 + 15/z + 14/z^2}{z\,(1 - 1/z)^4}\right]$

$= 0$

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CONVOLUTION :

If $\{f(k)\}$ and $\{g(k)\}$ are two sequences then, their convolution defined as

$$\{f(k)\} * \{g(k)\} = \{h(k)\}$$

$$" \quad = \sum_{m=-\infty}^{+\infty} f(m)\, g(k-m)$$

$$" \quad = \sum_{m=-\infty}^{\infty} g(m) \cdot f(k-m)$$

$$" \quad = \{g(k)\} * \{f(k)\}$$

THEOREM :

If $\{h(k)\}$ is convolution of two sequences $\{f(k)\} * \{g(k)\}$

then $Z\{f(k)\} = Z\{f(k)\} \cdot Z\{g(k)\}$

eg: If $f(k) = 4^k\, U(k)$ $\quad g(k) = 5^k\, U(k)$
find $Z[f*g]$

Soln: $U(k)$ = Unit Sequence.

$$f(k) = 4^k \{1, 1, 1, \ldots\}$$

$$= \{4^0, 4^1, 4^2, \ldots\}$$

$$g(k) = 5^k \{1, 1, 1. \ldots\}$$

$$= \{5^0, 5^1, 5^2, \ldots\}$$

$$\therefore Z[f(k)] = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= 4^0 z^0 + 4^1 z^{-1} + 4^2 z^{-2} + \ldots$$

$$= 1 + \frac{4}{z} + \frac{4^2}{z^2} + \frac{4^3}{z^3} + \dots$$

$$= \frac{1}{1 - 4/z} \quad \text{for } |4/z| < 1$$

$$= \frac{z}{z-4} \quad \text{for } |z| > 4$$

Similarly, $Z[g(k)] = \frac{z}{z-5}$ for $|z| > 5$

$$\therefore Z[f * g] = Z[f] \cdot Z[g]$$

$$= \frac{z^2}{(z-4)(z-5)} \quad \text{for } |z| > 5$$

Q. If $f(k) = \frac{1}{2^k}$ $\quad g(k) = \frac{1}{3^k}$

find $Z[f * g]$

Soln: $Z[f(k)] = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{-k}$

$$= 1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \frac{1}{2^3 z^3} + \dots$$

$$= \frac{1}{1 - 1/2z} \quad \text{for } |1/2z| < 1$$

$$= \frac{2z}{2z-1} \quad \text{for } |2z| > 1$$

$$\Rightarrow |z| > 1/2$$

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Similarly $Z[g(k)] = \dfrac{3z}{3z-1}$ for $|z| > 1/3$

$$Z[f * g] = Z[f] \times Z[g]$$

$$= \frac{6z^2}{(2z-1)(3z-1)} \quad \text{for } |z| > 1/2$$

INVERSE Z- TRANSFORM

I) Inverse by Binomial Expansion

Q. Find $Z^{-1}\left[\dfrac{4z}{z-a}\right]$ for:

~~i) $|z| > |a|$~~ i) $|z| > |a|$ ii) $|z| < |a|$

Ans: i) $\dfrac{4z}{z-a} = \dfrac{4z}{z\left(1-\frac{a}{z}\right)}$

$|z| > |a|$
$|z/a| > 1$
$|a/z| < 1$

$$= 4\left(1-\frac{a}{z}\right)^{-1}$$

$$= 4\left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right]$$

$$= 4\sum_{k=0}^{\infty} a^k z^{-k}$$

$$\Rightarrow Z^{-1}\left[\frac{4z}{z-a}\right] = \{4a^k\} \quad \text{for } |z| > |a|$$

ii) $|z| < |a|$

$|z/a| < 1$

$$\frac{4z}{z-a} = \frac{4z}{-a\left(1-\frac{z}{a}\right)}$$

$$= \frac{-4z}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$= \frac{-4z}{a}\left[1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\dots\right]$$

$$= -4\left[\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\dots\right]$$

$$= -4\sum_{k=1}^{\infty}\left(\frac{z}{a}\right)^k$$

put $k = -m$

$$= -4\sum_{m=-\infty}^{-1}\frac{1}{a^{-m}}z^{-m}$$

$$\therefore Z^{-1}\left[\frac{4z}{z-a}\right] = \left\{-4a^m\right\}_{m=-\infty}^{-1} \quad \text{for } |z|<|a|$$

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II) Inverse By Partial Fraction

Q. Find $Z^{-1}\left[\dfrac{4z^2-2z}{z^3-5z^2+8z-4}\right]$

Ans: $\dfrac{4z^2-2z}{z^3-5z^2+8z-4} = \dfrac{4z^2-2z}{(z-1)(z-2)^2} = \dfrac{2z(z-1)}{(z-1)(z-2)^2} = F(z)$

$$\frac{2z-1}{z^3-5z^2+8z-4} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} = \frac{4z^2-2z}{(z-1)(z-2)^2}$$

~~$= Az - 2A +$~~

$= A(z-2)^2 + B(z-1)(z-2) + C(z-1)$

put ~~z~~ $z=1$

$\therefore A = 1$

put $z = 2$

$\therefore C = 3$

put $z = 0$

$-1 = 4A + 2B - C$

$\therefore B = -1$

$$\therefore F(z) = 2z\left[\frac{1}{z-1} - \frac{1}{z-2} + \frac{3}{(z-2)^2}\right]$$

$$= 2Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{z-2}\right] + 6Z^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$Z^{-1}\left[\frac{z}{z-a}\right] = \{a^n\} \qquad Z^{-1}\left[\frac{z}{(z-a)^2}\right] = \frac{na^n}{a} = na^{n-1}$$

$$\text{but } Z^{-1}\left[\frac{z}{z-1}\right] = 1^n = 1$$

$$\therefore Z^{-1}[F(z)] = 2\times 1^n - 2\times 2^n + 6n2^{n-1}$$

$$= \{2 - 2^{n+1} + 6n2^{n-1}\}$$

Q) Find Inverse Z - Transform of :

$$F(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} \quad \text{for} \quad 2<|z|<3$$

Ans: $F(z) = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-3)^2}$

$$2(z^2-5z+6.5) = A(z-3)^2 + B(z-3)(z-2) + C(z-2)$$

Put $z=2 \Rightarrow A = 1$

$z=3 \Rightarrow C = 1$

$z=0 \Rightarrow B = 1$

$$2(6.5) = 9A + 6B - 2C$$
$$13 = 9 + 6B - 2$$
$$6 = 6B$$
$$B = 1$$

$(1-z)^{-1} = 1+z+z^2+z^3+\dots$

$(1-z)^{-2} = 1+2z+3z^2+4z^3+\dots$

$$\therefore F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$|z| > 2$

$|z/2| > 1$

$|2/z| < 1$

$$\therefore F(z) = \frac{1}{z(1-2/z)} - \frac{1}{3(1-z/3)} + \frac{1}{9\left(1-\frac{z}{3}\right)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right) - \frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}\dots\right)$$

$$+ \frac{1}{9}\left(1+\frac{2z}{3}+\frac{3z^2}{9}+\frac{4z^3}{27}+\dots\right)$$

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

$$Z^{-1}[F(z)] = \{f(k)\}$$

$$= \cancel{\sum_{k=1}^{\infty} \frac{2^k}{k}} \quad \sum_{k=1}^{\infty} \frac{2^{k-1}}{z^k} - \frac{1}{3}\sum_{k=0}^{\infty} \frac{z^k}{3^{k+1}} + \frac{1}{9}\sum_{k=0}^{\infty} \frac{(k+1)z^k}{3^k}$$

Replace $k = -m$

$$= \sum_{k=1}^{\infty} 2^{k-1} z^{-k} - \frac{1}{3}\sum_{m=-\infty}^{0} \frac{1}{3^{-m}} z^{-m} + \frac{1}{9}\sum_{m=-\infty}^{0} \frac{(-m+1)}{3^{-m}} z^{-m}$$

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$$\therefore \quad Z^{-1}\left[F(z)\right] = \begin{cases} 2^{k-1} & k \geq 1 \\ \frac{-1}{3}\left(\frac{1}{3^{-m}}\right) + \frac{1}{9}\left(\frac{1-m}{3^{-m}}\right) & \text{for } -\infty \leq m \leq 0 \end{cases}$$

$$= \begin{cases} 2^{m-1} & \text{for } m \geq 1 \\ \frac{-1}{3^{1-m}} + \frac{(1-m)}{3^{2-m}} & \text{for } -\infty < m \leq 0 \end{cases}$$