Discrete Mathematics

Logic

Logic is the discipline that deals with the methods of reasoning.
On an elementary level, logic provides rules and techniques for determining whether a given argument is valid.

Propositions and Logical Operations

Statement of Proposition

- Statement of proposition – a declarative sentence that is either true or false, but not both
- Examples:
  - The earth is round: statement that is true
    2+3=5: statement that is true
  - Do you speak English? This is a question, not a statement

More Examples of Statements of Proposition

- 3-x=5: is a declarative sentence, but not a statement since it is true or false depending on the value of x.
- Take two aspirins: is a command, not a statement.
- The temperature on the surface of the planet Venus is 800ºF: is a declarative statement of whose truth is unknown to us.
- The sun will come out tomorrow: a statement that is either true or false, but not both, although we will have to wait until tomorrow to determine the answer.

Logical Connectives and Compound Statements

- *x*, *y*, *z*, … denote variables that can represent real numbers
- *p*, *q*, *r*,… denote prepositional variables that can be replaced by statements.
  - *p*: The sun is shining today
  - *q*: It is cold
- **Logical connectives** can combine statements or propositional variables to obtain compound statements.

Logical Operations:
Negation

- If p is a statement, the negation of p is the statement *not p*
- Denoted $\sim p$
- If p is true, $\sim p$ is false
- If p is false, $\sim p$ is true
- $\sim p$ is not actually connective, i.e., it doesn't join two of anything
- ***not*** is a unary operation for the collection of statements and $\sim p$ is a statement if p is

- If p: $2+3 > 1$ then If $\sim p$: $2+3 \leq 1$
- If q: It is cold then $\sim q$: It is not the case that it is cold, i.e., It is not cold.

1. Give the negation of each of the following statements.

(a) It will rain tomorrow or it will snow tomorrow.

(b) If you drive, then I will walk

2. In each of the following, form the conjunction and the disjunction of p and q.

(a) p: I will drive my car. q: I will be late.

(b) p: $NUM > 10$ q: $NUM \leq 15$

Solutions:

1. (a) It will not rain tomorrow and it will not snow tomorrow. (b) It is not the case that if you drive, I will walk.

2. (a) I will drive my car and I will be late. I will drive my car or I will be late. (b) $10 < NUM \leq 15$. $NUM > 10$ or $NUM \leq 15$.

Conjunction

- If p and q are statements, then the ***conjunction*** of p and q is the compound statement "**p and q**"
- Denoted $p \wedge q$
- $p \wedge q$ is true only if both p and q are true
- Example:
  - p: : It is snowing.
  - q: I am cold.
  - $p \wedge q$ = ? It is snowing and I am cold.

Disjunction

- If p and q are statements, then the ***disjunction*** of p and q is the compound statement "**p or q**"
- Denoted $p \vee q$
- $p \vee q$ is true if either p or q are true
- Example:
  - Form the disjunction of p and q for each of the following.
  - (a) p: 2 is a positive integer, q: $\sqrt{2}$ is a rational number. $p \vee q$ = ? ….true
  - (b) p: $2 + 3 \neq 5$, q: London is the capital of France.
  - $p \vee q$ = ? ….false

3. Write each of the following in terms of p, q, r, and logical connectives.

(a) Today is Monday and the dish did not run away with the spoon.
(b) Either the grass is wet or today is Monday.
(c) Today is not Monday and the grass is dry.
(d) The dish ran away with the spoon, but the grass is wet.

Exclusive Disjunction

- If p and q are statements, then the ***exclusive disjunction*** is the compound statement, "either p or q may be true, but both are not true at the same time."
- Example:
  - p: It is daytime
  - q: It is night time
  - $p \vee q$ (in the exclusive sense) = ?

Inclusive Disjunction

- If p and q are statements, then the ***inclusive disjunction*** is the compound statement, "either p or q may be true or they may both be true at the same time."
- Example:
  - p: It is cold
  - q: It is night time
  - $p \vee q$ (in the inclusive sense) = ?

Exclusive versus Inclusive

- Depending on the circumstances, some disjunctions are inclusive and some of exclusive.
- Examples of Inclusive
  - “I have a dog” or “I have a cat”
  - “It is warm outside” or “It is raining”
- Examples of Exclusive
  - Today is either Tuesday or it is Thursday
  - Pat is either male or female

Compound Statements

- A ***compound statement*** is a statement made from other statements
- For n individual propositions, there are 2^n possible combinations of truth values
- A truth table contains 2^n rows identifying the truth values for the statement represented by the table.
- Use parenthesis ( ) to denote order of precedence
- $\wedge$ has precedence over $\vee$

Truth Tables are Important Tools for this Material!

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Make a truth table for the statement $(p \wedge q) \vee (\sim p)$.

Compound Statement Example

$(p \wedge q) \vee (\sim p)$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

- 4. Make the Truth Table for the following:
- (a) $(\sim p \land q) \lor p$
- (b) $(p \lor q) \land r$

Quantifiers

- Back in Section 1.1, a set was defined
  $\{x \mid P(x)\}$
- For an element t to be a member of the set, P(t) must evaluate to "true"
- **P(x)** is called a **predicate** or a **propositional function**

If Q(n): n + 3 = 6, then
(a) Q(5) is the statement--------------------- .
(b) Q(m) is the statement -------------------.

Computer Science Functions

- if P(x), then execute certain steps
- while Q(x), do specified actions

The predicates P(x) and Q(x) are called the guards for the block of
programming code. Often the **guard** for a block is a conjunction or disjunction.

1. **IF** $N < 10$ **THEN**
    a. Replace N with $N + 1$
    b. **RETURN**

Here the statement $N < 10$ is the guard.

1. **WHILE** $t \in T$ and $s \in S$
    a. **PRINT** $t + s$
    b. **RETURN**

Here the compound statement $t \in T$ and $s \in S$ is the guard.

Universal quantification of a predicate P(x)

- Universal quantification of predicate P(x) = **For all values of x**, P(x) is true
- Denoted **$\forall x\ P(x)$**
- The symbol $\forall$ is called the universal quantifier
- The order in which multiple quantifications are considered does not affect the truth value (e.g., $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$)

Examples:

- P(x): -(-x) = x
  - This predicate makes sense for all real numbers x.
  - The universal quantification of P(x), $\forall x\, P(x)$, is a true statement, because for all real numbers, -(-x) = x
- Q(x): x+1<4
  - $\forall x\, Q(x)$ is a false statement, because, for example, Q(5) is not true

Existential quantification of a predicate P(x)

- Existential quantification of a predicate P(x) is the statement "There **exists a value of x** for which P(x) is true."
- Denoted **∃x P(x)**
- Existential quantification may be applied to several variables in a predicate
- The order in which multiple quantifications are considered does not affect the truth value

(a) Let $Q(x)$: $x + 1 < 4$. The existential quantification of $Q(x)$, $\exists x\, Q(x)$, is a true statement, because Q(2) is a true statement.

(b) The statement $\exists y \;\; y + 2 = y$ is false. There is no value of y for which the propositional function $y + 2 = y$ produces a true statement. ◆

Applying both universal and existential quantification

- Order of application does matter
- Example: Let A and B be n x n matrices
- The statement $\forall A\ \exists B\ \ A + B = I_n$
- Reads "for every A there is a B such that $A + B = I_n$"

$\exists \mathbf{B}\, \forall \mathbf{A}\ \ \mathbf{A} + \mathbf{B} = \mathbf{A}$ is true. What is the value for $\mathbf{B}$ that makes the statement true?

1. Write an English sentence corresponding to each of the following.

(a) $\forall x\ \exists y\ R(x, y)$ (b) $\exists x\ \forall y\ R(x, y)$

2. Write an English sentence corresponding to each of the following.

(a) $\sim(\exists x\ P(x))$ (b) $\sim(\forall x\ Q(x))$

Answers :

1.

(a) For all x there exists a y such that x + y is even

(b) There exists an x such that, for all y, x + y is even

2.

(a) It is not true that there is an x such that x is even. (b) It is not true that, for all x, x is a prime number.

Assigning Quantification to Proposition

- Let p: $\forall x\, P(x)$
- The negation of p is false when p is true and true when p is false
- For p to be false, there must be at least one value of x for which P(x) is false.
- Thus, p is false if $\exists x\, \sim P(x)$ is true.
- If $\exists x\, \sim P(x)$ is false, then for every x, $\sim P(x)$ is false; that is $\forall x\, P(x)$ is true.

Okay, what exactly did the previous slide say?

- Assume a statement is made that "for all x, P(x) is true."
  - If we can find one case that is not true, then the statement is false.
  - If we cannot find one case that is not true, then the statement is true.
- Example: $\forall$ positive integers, n,
  $P(n) = n^2 + 41n + 41$ is a prime number.
  - This is false because $\exists$ an integer resulting in a non-prime value, i.e., $\exists$ n such that P(n) is false.
  - Why ? What is that value of n?

In Exercises 8 and 9, find the truth value of each proposition if p and r are true and q is false.

8. **(a)** $\sim p \wedge \sim q$ **(b)** $(\sim p \vee q) \wedge r$

(c) $p \vee q \vee r$ **(d)** $\sim(p \vee q) \wedge r$

9. **(a)** $\sim p \wedge (q \vee r)$ **(b)** $p \wedge (\sim(q \vee \sim r))$

(c) $(r \wedge \sim q) \vee (p \vee r)$ **(d)** $(q \wedge r) \wedge (p \vee \sim r)$

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