Vector Algebra

$(\bar{a} \times \bar{b}) \times \bar{c} = [\bar{a}\,\bar{b}\,\bar{c}]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

solve this determinant

$r \rightarrow p \rightarrow q \rightarrow r$

property: $pqr, qrp, rpq = [p\,q\,r]$

i) $(\bar{a} \times \bar{b}) \cdot \bar{c} = (\bar{b} \times \bar{c}) \cdot \bar{a} = (\bar{c} \times \bar{a}) \cdot \bar{b}$

ii) $(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{a} \cdot (\bar{b} \times \bar{c})$

iii) $[\bar{a}\,\bar{b}\,\bar{c}] = -[\bar{b}\,\bar{a}\,\bar{c}]$ This is called box product

iv) $[\bar{a}\,\bar{a}\,\bar{b}] = 0$ .... [2 repeating terms then 0]

v) If $\bar{a}, \bar{b}, \bar{c}$ are coplanar then $[\bar{a}\,\bar{b}\,\bar{c}] = 0$

If they are in same plane $\theta = 0$

vi) Volume of Tetrahedron $= \frac{1}{6}[\bar{a}\,\bar{b}\,\bar{c}]$

where $\bar{a}, \bar{b}, \bar{c}$ are sides of Tetrahedron.

vii) Volume of parallelopiped whose coterminal edges are $\bar{a}, \bar{b}, \bar{c} = [\bar{a}\,\bar{b}\,\bar{c}]$

Q show that $(\bar{p} + \bar{q}) \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})] = 2[\bar{p}\,\bar{q}\,\bar{r}]$

$\bar{p} \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})] + \bar{q} \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})]$

$= \bar{p} \cdot [\bar{q} \times (\bar{r} + \bar{p}) + \underset{0}{\bar{r}\,(\bar{r} + \bar{p})}] + \bar{q} \cdot [\bar{q} \times (\bar{r} + \bar{p}) + \bar{r} \times (\bar{r} + \bar{p})]$

$= \bar{p} \cdot [\bar{q} \times \bar{r} + \bar{q} \times \bar{p} + \bar{r} \times \bar{r} + \bar{r} \times \bar{p}] +$
$\bar{q} \cdot [\bar{q} \times \bar{r} + \bar{q} \times \bar{p} + \underset{0}{\bar{r} \times \bar{r}} + \bar{r} \times \bar{p}]$

$= \cancel{\bar{p}} [\; \bar{p} \cdot [\bar{q} \times \bar{r}] + \bar{p} \cdot [\underset{0}{\bar{q} \times \bar{p}}] + \bar{p} \cdot [\underset{0}{\bar{r} \times \bar{p}}]$

$+\bar{q}\cdot[\bar{q}\times\bar{r}] + \bar{q}\cdot[\bar{q}\times\bar{p}] + \bar{q}\cdot[\bar{r}\times\bar{p}]$

$\quad\uparrow 0 \qquad\qquad \uparrow 0$

$= \bar{p}\cdot[\bar{q}\times\bar{r}] + \bar{q}\cdot[\bar{r}\times\bar{p}]$

$= [\bar{p}\,\bar{q}\,\bar{r}] + [\bar{p}\,\bar{q}\,\bar{r}]$

$= 2[\bar{p}\,\bar{q}\,\bar{r}]$

Vector Triple Product [VTP]

VTP of 3 vectors $\bar{a}, \bar{b}, \bar{c}$ is given by

$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$

or $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

Q If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors then prove $\bar{a}\times\bar{b}$, $\bar{b}\times\bar{c}$, $\bar{c}\times\bar{a}$ are also coplanar

A) $[\bar{a}\,\bar{b}\,\bar{c}] = 0$

To prove: $[\bar{a}\times\bar{b}\;\; \bar{b}\times\bar{c}\;\; \bar{c}\times\bar{a}] = 0$

LHS: $[(\bar{a}\times\bar{b})\times(\bar{b}\times\bar{c})]\cdot(\bar{c}\times\bar{a})$

let $\bar{b}\times\bar{c} = \bar{m}$

$= [(\bar{a}\times\bar{b})\times\bar{m}]\cdot(\bar{c}\times\bar{a})$

$= [(\bar{a}\cdot\bar{m})\bar{b} - (\bar{b}\cdot\bar{m})\bar{a}]\cdot\bar{c}\times\bar{a}$

$= [(\bar{a}\cdot(\bar{b}\times\bar{c}))\bar{b} - (b\cdot(b\times c))\bar{a}]\cdot\bar{c}\times\bar{a}$

$\uparrow$ 0 since coplanar $\qquad \uparrow 0$

$= 0$

Q $[\bar{d} \times (\bar{a} \times \bar{b})] \cdot (\bar{a} \times \bar{c}) = [\bar{a}\,\bar{b}\,\bar{c}]\,(\bar{a} \cdot \bar{d})$

$LHS = [(\bar{d} \cdot \bar{b})\bar{a} - (\bar{d} \cdot a)\bar{b}] \cdot \bar{a} \times \bar{c}$

$= (\bar{d} \cdot \bar{b})\bar{a} \cdot (\bar{a} \times \bar{c}) - (\bar{d} \cdot \bar{a})\bar{b} \cdot (\bar{a} \times \bar{c})$

$\uparrow 0$ $\qquad \hookrightarrow -[a\,b\,c] = -a \cdot [b \times c]$

$= \cancel{-}(\bar{d} \cdot \bar{a})(\cancel{-}\bar{a} \cdot b[\times c])$

$= [\bar{a}\,\bar{b}\,\bar{c}][\bar{a} \cdot \bar{d}]$

Q $(\bar{a} \times \bar{b}) \cdot (\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a}) = [\bar{a} \cdot (\bar{b} \times \bar{c})]^2$

1st cross prod. then dot prod

$LHS = (\bar{a} \times \bar{b}) \cdot (\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a})$

$[\bar{a} \times \bar{b}$

Q vector $\bar{u}, \bar{v}$ and $\bar{w}$ are non-coplanar then $\bar{u}\times\bar{v}, \bar{v}\times\bar{w}, \bar{w}\times\bar{u}$ are also non-coplanar vectors. Hence obtain scalar l, m, n such that $\bar{u} = l(\bar{v}\times w) + m(\bar{v}\times\bar{u}) + n(\bar{u}\times\bar{v})$

$[\bar{u}\ \bar{v}\ \bar{w}] \neq 0$

$[u\times v \quad v\times w \quad w\times u] \neq 0$

$\bar{u}.\bar{u} = l(u.(\bar{v}\times w)) + m(u.(\bar{v}\times\bar{u}) + n(\bar{u}.(\bar{u}\times\bar{v}))$

$\uparrow 0$ $\quad \uparrow 0$

$$\frac{\bar{u}\cdot\bar{u}}{\bar{u}.(\bar{v}\times\bar{w})} = l = \frac{\bar{u}\cdot\bar{u}}{[\bar{u}\ \bar{v}\ \bar{w}]}$$

similarly $\dfrac{\bar{u}\cdot\bar{v}}{[\bar{u}\ \bar{v}\ \bar{w}]} = m, \quad n = \dfrac{\bar{u}\cdot\bar{w}}{[\bar{u}\ \bar{v}\ \bar{w}]}$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

lagrange's Identity

Q Prove that $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) + (\bar{b}\times\bar{c})\cdot(\bar{a}\times\bar{d}) + ((\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{d})) = 0$

$$\begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{b}\cdot\bar{a} & \bar{c}\cdot\bar{a} \\ \bar{b}\cdot\bar{d} & \bar{c}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{c}\cdot\bar{b} & \bar{a}\cdot\bar{b} \\ \bar{c}\cdot d & \bar{a}\cdot\bar{d} \end{vmatrix}$$

$= \cancel{(\bar{b}\cdot\bar{a})\times(\bar{c}\cdot d)} - (\bar{b}\cdot d)\times(\bar{c}\cdot\bar{a})$
$+ \cancel{(\bar{a}\cdot\bar{c})\times(\bar{b}\cdot\bar{d})} - (\bar{b}\cdot d)\times(\bar{a}\cdot\bar{d})$
$+ \bar{c}\cdot\bar{b}\times(\bar{a}\cdot\bar{d}) - (\bar{a}\cdot\bar{b})\times(\bar{c}\cdot\bar{d})$

$= 0$

Vector Product of 4 Vectors

$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$

$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

$= [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

Q $\bar{d} \cdot [\bar{a} \times [\bar{b} \times (\bar{c} \times \bar{d})]]$

$= (\bar{b} \cdot \bar{d})\ [\bar{a}\ \bar{c}\ \bar{d}]$ prove this

LHS $= \bar{d} \cdot [\bar{a} \times [\bar{b} \times (\bar{c} \times \bar{d})]]$

$= \bar{d} \cdot [\bar{a} \times ([\bar{b} \cdot \bar{d}]\bar{c} - [\bar{b} \cdot \bar{c}]\bar{d})]$

.... expanded vector triple product

$= \bar{d} \cdot [(\bar{b} \cdot \bar{d})(\bar{a} \times c) - (\bar{b} \cdot \bar{c})(\bar{a} \times \bar{d})]$

$= (\bar{b} \cdot \bar{d})(\bar{d} \cdot (\bar{a} \times \bar{c})) - (\bar{b} \cdot \bar{c})(\bar{d} \cdot (\bar{a} \times \bar{d}))$

$\uparrow 0$

$\Rightarrow \bar{b} \cdot \bar{d}\ [\bar{d}\ \bar{a}\ \bar{c}]$

$\because [\bar{d}\ \bar{a}\ \bar{c}] = [\bar{a}\ \bar{c}\ \bar{d}]$

$= \bar{b} \cdot \bar{d}\ [\bar{a}\ \bar{c}\ \bar{d}] =$ RHS

Q P.T $[(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{d} = (\bar{a} \cdot \bar{d})[\bar{a}\ \bar{b}\ \bar{c}$

LHS $= [(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{d}$

Vector Differentiation

1) gradient:
gradient of scalar point function (no $\bar{i}, \bar{j}, \bar{k}$) ϕ at point $p(x, y, z)$ is given by

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

2) Directional Derivative (DD)
DD of function f in direction of unit vector $\hat{u}$ at point (a, b) is given by

$$D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (1, 0, 0)$$

$$= \frac{\partial f}{\partial x}$$

Property:
i) ~~$D_{\hat{u}} f(a,b)$~~

$\therefore \frac{\partial f}{\partial x}$ is DD of f in direction of x-axis / $\hat{i}$.

$\frac{\partial f}{\partial y}$ along y-axis $(0,1,0)$, $\frac{\partial f}{\partial z}$ along z-axis $(0,0,1)$

i) $\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

ii) $D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u}$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (u_1, u_2, u_3)$$

$$D_{\hat{u}} f(a,b) = |\nabla f|\,|\hat{u}|\cos\theta$$
$$= |\nabla f| \cos\theta$$

$D_{\hat{u}} f$ is max if $\cos\theta = 1$ ie $\theta = 0$ ie.
$D_{\hat{u}} f$ is max in direction of ∇f.

$$F(z) = \frac{1}{(z-1)(z-2)} \qquad 1<|z|<2$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$$= Az - 2A + Bz - B$$

$$1 = z(A+B) + (-2A-B)$$

$A+B=0$ $\qquad$ $-2A-B=1$

$A=-B$ $\qquad$ $-2(-B)-B=1$

$2B-B=1$

$B=1 \quad A=-1$

$$\frac{1}{z-2} - \frac{1}{z-1}$$

a)

$$F(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$z^{-1}[F(z)] = z^{-1}\left(\frac{1}{z-2}\right) - z^{-1}\left(\frac{1}{z-1}\right)$$

$$f(k) = 4^k$$

$$4^k z^{-k} \qquad \left(\frac{4}{z}\right)^k$$

$$\left(\frac{4}{z}\right)^0 + \left(\frac{4}{z}\right)^1 + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \ldots$$

$1 + t \ldots$

$$\frac{1}{1-\frac{4}{z}} \qquad \frac{z}{z-4}$$

$$F(z) = \frac{1}{z-2} \qquad |z|<2$$

$$z^{-1}\left(\frac{1}{z-2}\right)$$

$$F(z) = \frac{1}{z-2} \quad |z| < 2$$

z

-2^{k-1} $k \le 0$ -1^{k} $k > 0$ $\frac{z}{4-z}$

-2^{k-1} $k \le 0$

$$e^{4z}$$

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properties of gradient

& magnitude of max direction $= |\nabla f|$

$D_{\hat{a}} f$ is minimum if $\cos\theta = -1$ i.e. $\theta = \pi$

i.e. $D_{\hat{a}} f$ is min in direction of $-\nabla f$

& magnitude of minimum direction $= -|\nabla f|$

3) Total differential (df)

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (dx, dy, dz)$$

$$df = \nabla f \cdot d\bar{r}$$

Q Find $\phi(r)$ s.t $\nabla\phi = \frac{-\bar{r}}{r^5}$ & $\phi(1) = 0$

$$\nabla\phi = \frac{-\bar{r}}{r^5}, \quad r = |\bar{r}|$$

$$= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = -\frac{(xi + yj + zk)}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial \phi}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{5/2}} \qquad \frac{\partial \phi}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial \phi}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{5/2}}$$

$$\partial \phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= \frac{-1}{(x^2+y^2+z^2)^{5/2}} [x\,dx + y\,dy + z\,dz]$$

$$x^2+y^2+z^2 = t$$

$$2x\,dx + 2y\,dy + 2z\,d\cancel{z} = 2t\,dt$$

$$x\,dx + y\,dy + z\,dz = t\,dt$$

$$d\phi = -\frac{1}{t^5} t\,dt = -\frac{1}{t^4} dt$$

$$\int d\phi = -\int \frac{1}{t^4} dt = -\frac{t^{-3}}{(-3)} + c$$

$$= \cancel{t}\ \frac{1}{3t^3} + c$$

$$= \frac{1}{3(x^2+y^2+z^2)^{3/2}} + c$$

$$\phi(r) = \frac{1}{3r^{3}} + c$$

but $\phi(1) = 0 \Rightarrow \frac{1}{3} + c = 0 \quad c = -\frac{1}{3}$

Q $\phi(r) = \frac{1}{3r^2} - \frac{1}{3}$

Prove that $\nabla f(r) = f'(r) \frac{\bar{r}}{r}$

Hence find if $\nabla f = 2r^4 \bar{r}$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$f \to r \to x, y, z$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \times \frac{\partial r}{\partial x} = f'(r)$$

but $r = \sqrt{x^2+y^2+z}$

$$r^2 = x^2+y^2+z^2$$

$$2r\,dr = 2x\,dx$$

$$dr = \frac{x}{r} dx$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial f}{\partial x} = f'(r) \frac{x}{r}$$

$$\frac{\partial f}{\partial y} = f'(r) \frac{y}{r}$$

$$\frac{\partial f}{\partial z} = f'(r) \frac{z}{r}$$

$$\nabla f = \frac{f'(r)}{r} [xi + yj + zk]$$

$$= \frac{f'(r)}{r} \bar{r}$$

$$\nabla f = 2r^4 \bar{r} = \frac{f'(r)}{r} \bar{r}$$

$$2r^4 = \frac{f'(r)}{r}$$

$$f'(r) = 2r^5$$

$$f(r) = \frac{2r^6}{6} + c$$

$$= \frac{r^6}{3} + c$$

Q P.T $\nabla \left[\frac{\bar{a} \cdot \bar{r}}{r^n} \right] = \frac{\bar{a}}{r^n} = \frac{n(\bar{a} \cdot \bar{r})\bar{r}}{r^{n+2}}$

let $\phi = \frac{\bar{a} \cdot \bar{r}}{r^n}$

$$= \frac{a_1 x + a_2 y + a_3 z}{r^n}$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \quad (1)$$

$$\frac{\partial \phi}{\partial x} = r^n a_1 - (a_1 x + a_2 y + a_3 z)$$

$r = \sqrt{x^2+y^2+z^2}$ $\quad r \to x, y, z$

$$r^2 = x^2 + y^2 + z^2$$

$$f(r) = r^n$$

$f \to r \to x, y, z$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \times \frac{\partial r}{\partial x} = n r^{n-1} \frac{x}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{r^n a_1 - (a_1 x + a_2 y + a_3 z) n r^{n-1} \, x/r}{r^{2n}}$$

similarly $$\frac{\partial \phi}{\partial y} = \frac{r^n a_2 - (a_1 x + a_2 y + a_3 z) n r^{n-1} \, y/r}{r^{2n}}$$

similarly $$\frac{\partial \phi}{\partial z} = \frac{r^n a_3 - (a_1 x + a_2 y + a_3 z) n r^{n-1} \, z/r}{r^{2n}}$$

Note: $\because a_1 x + a_2 y + a_3 z = \bar{a} \cdot \bar{r}$

$$\frac{n r^{n-1}}{r} = n r^{n-2}$$

$$\nabla \phi = \frac{1}{r^{2n}} \left[r^n (a_1 x + a_2 y + a_3 z) - (\bar{a} \cdot \bar{r}) n r^{n-2} \times (xi + yj + zk) \right]$$

$$= \frac{1}{r^{2n}} \left[r^n \bar{a} - (\bar{a} \cdot \bar{r}) n r^{n-2} \, \bar{r} \right]$$

$$= \frac{\bar{a}}{r^n} - \frac{(\bar{a} \cdot \bar{r}) n}{r^{n+2}} \bar{r}$$

Q $\phi = x^4 + y^4 + z^4$ at $A \equiv (1, -2, 1)$
$B \equiv (2, 6, -1)$. Find DD in dir of AB

A) $\phi = x^4 + y^4 + z^4$

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$= 4x^3 + 4y^3 + 4z^3$$

at $A \equiv (1, -2, 1)$

$$4(1)^3 + 4(-2)^3 + 4(1)^3$$
$$= 4 - 32 + 4 = 24$$

~~at $B \equiv (2, 6, -1)$~~
~~$4(2)^3 + 4(6)^3 + 4(-1)^3$~~

$$\overline{AB} = \bar{B} - \bar{A}$$
$$= i + 8j - 2k$$

$$\hat{u} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{i + 8j - 2k}{\sqrt{1^2 + 8^2 + (-4)^2}} = \frac{i + 8j - 2k}{\sqrt{69}}$$

$$D_{\hat{u}}\phi(A) = \nabla\phi(A) \cdot \hat{u}$$
$$= (4i - 32j + 4k)\left(\frac{i + 8j - 2k}{\sqrt{69}}\right)$$

$$= \frac{4 - 256 - 8}{\sqrt{69}} = \frac{-260}{\sqrt{69}}$$

Q2 Find DD of $\phi = x^2 + y^2 + z^2$ in direction of line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at point $(1, 2, 3)$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi = 2xi + 2yj + 2zk$$

at $\equiv (1, 2, 3)$

$$2(1)i + 2(2)j + 2(3)k$$
$$= 2i + 4j + 6k \cancel{= 12}$$

Note Eq^n of line $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$

then vector parallel to this line is given by
$\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

vector parallel to $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is

$3i + 4j + 5k$

$\hat{u} = \frac{3i+4j+5k}{\sqrt{9+16+25}} \Rightarrow \frac{3i+4j+5k}{\sqrt{50}}$

$D_{\hat{u}}\phi(A) = \nabla\phi(A)\cdot\hat{u}$

$= (2i+4j+6k)\frac{(3i+4j+5k)}{\sqrt{50}}$

$= \frac{6+16+30}{\sqrt{50}} = \frac{52}{\sqrt{50}}$

Q Find DD of $\phi = e^{2x}\cos yz$ at $(0,0,0)$ in direction of tangent to the curve
1 $x = a\sin t,\ y = a\cos t,\ z = at$ at $t = \frac{\pi}{4}$

$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\nabla\phi = 2e^{2x}\cos yz\, i - ze^{2x}\sin yz\, j - ye^{2x}\sin yz\, k$

at $(0,0,0)$

$= \frac{2}{2}i \cancel{- \frac{1}{2}j - \frac{1}{2}k}$

tangent to $\bar{r} = \dfrac{\partial \bar{r}}{dt}$

$= a\cos t\, i - a\sin t\, j + ak$

at $t = \pi/4$

$$\frac{\partial r}{\partial t} = \frac{a}{\sqrt{2}} i - \frac{a}{\sqrt{2}} j + ak$$

$$\hat{i} = \frac{\dfrac{\partial \bar{r}}{\partial t}}{\left|\dfrac{\partial \bar{r}}{\partial t}\right|} = \frac{\dfrac{a}{\sqrt{2}} i - \dfrac{a}{\sqrt{2}} j + ak}{\sqrt{\dfrac{a^2}{2}}}$$

Q Find DD $\phi = xy^2 + yz^3$ at $(2,-1,-1)$ in direction of normal to surface $x\log z - y^2 + 4 = 0$ at $(-1, 2, 1)$

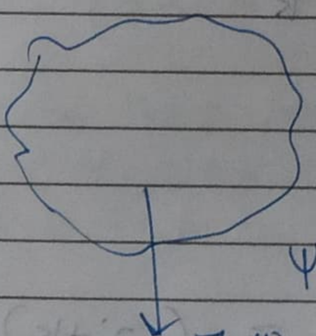
$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$= y^2 i + (2xy + z^3) j + 3z^2 y\, k$

at $A = (2, -1, +1)$

$= (-1)^2 i + (2(2)(-1) + (+1)^3) j + 3(+1)^2(-1) k$

$= i - 3j - 3k$



$\psi(x, y, z) = c$

$\nabla \psi$

$\equiv$ normal to surface

let $\psi = x\log z - y^2 + 4$

normal to ψ is $\nabla\psi = \log z\, i - 2yj + \frac{x}{z} k$

at $B \equiv (-1, 2, 1)$

$\nabla\psi = -4j - k$

$\hat{u} = \frac{\nabla\psi}{|\nabla\psi|} = \frac{-4j - k}{\sqrt{17}}$

$D_{\hat{u}}\phi(A) = \nabla\phi(A) \cdot \hat{u}$

$= (i - 3j - 3k) \cdot \left(\frac{-4j - k}{\sqrt{17}}\right)$

$= \frac{12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$

Q Find angle between surfaces $x\log z + 1 - y^2 = 0$ and $x^2y + z = 2$ at $(1,1,1)$

let $\phi = x\log z + 1 - y^2$

let $\psi = x^2y + z - 2$

$\nabla\phi = \log z\, i - 2yj + \cancel{xk}\ x/z\, k$

$\nabla\psi = 2xyi + x^2j + 1\hat{k}$

at $1,1,1$

$\nabla\phi = 0 - 2j + k$

$\nabla\psi = 2i + j + k$

$\cos\theta = \frac{\nabla\phi \nabla\psi}{|\nabla\phi||\nabla\psi|} = \frac{(-2j+k)\cdot(2i+j+k)}{\sqrt{5}\ \sqrt{6}}$

$$= \frac{-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

$$= \pi - \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$$

Q Find values of a, b, c if directional derivatives of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, -1) has max magnitude 64 in direction parallel to z-axis.

A $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$$= (ay^2 + 3cz^2x^2)i + (2axy + bz)j + (by + 2czx^3)k$$

at (1, 2, -1)

$$\nabla\phi = (4a + 3c)i + (4a - b)j + (2b - 2c)k$$

we know that DD is maximum in direction of $\nabla\phi$ and magnitude of max direction $= |\nabla\phi| = 64$ (1)

$$|\nabla\phi| = \sqrt{(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2} = 64$$ (2)

vector parallel to z axis is $0i + 0j + k$ (3)

From 1 and 3

$\nabla\phi$ and $0i + 0j + k$ are parallel

Note: whenever 2 vectors are parallel their components are proportional.

$$\frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1} = t$$

$4a + 3c = 0$

$4a - b = 0$ (4)

$2b - 2c = t$

Substi (4) in (2)

$$\sqrt{0 + 0 + t^2} = 64$$

$t = 64$

$\therefore 4a + 3c = 0$

$4a - b = 0$

$2b - 2c = 64$

$a = 6 \quad b = 24 \quad c = -8$

Q Find b, c if normal to surface $ax^2 + bxz + z^2y = 0$ at $(-1, 1, 2)$ is parallel to normal to surface.

$\phi = ax^2 + bxz + z^2y = 0$

at $(-1, 1, 2)$

$\psi = x^2 - y^2 + 2z = 2$ at $(1,1,1)$
find $\nabla\phi$, $\nabla\psi$

Divergence & Curl

let $\bar{f} = f_1 i + f_2 j + f_3 k$
then Div $\bar{f} = \nabla\cdot\bar{f}$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(f_1, f_2, f_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

gradient of scalar f^n is vector f^n
divergence of vector f^n is scalar f^n

If $\nabla\cdot\bar{f} = 0$ then $\bar{f}$ is called solenoidal vector field.

$$\text{grad } \phi = \nabla\phi$$
$$= \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\text{curl } \bar{f} = \nabla\times\bar{f}$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

If $\nabla\times\bar{f} = \bar{0}$ then $\bar{f}$ is called irrotational

Formulae:

1) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

2) $\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$

Divergence

3) $\nabla\cdot(\bar{f}\times\bar{g}) = \bar{g}\cdot(\nabla\times\bar{f}) - \bar{f}\cdot(\nabla\times\bar{g})$

4) $\nabla\times(\phi\bar{f}) = \phi(\nabla\times\bar{f}) + \nabla\phi\times\bar{f}$

Curl

Q If $\bar{a}$ is a constant vector such that magnitude of $|\bar{a}| = a$ then prove that $\nabla\cdot[(\bar{a}\cdot\bar{r})\bar{a}] = a^2$

let $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{r} = xi + yj + zk$

$\bar{a}\cdot\bar{r} = \phi \quad \bar{f} = \bar{a}$

$$\nabla\cdot[\phi\bar{f}] = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$$

$$= (\bar{a}\cdot\bar{r})(\nabla\cdot\bar{a}) + \bar{a}\cdot(\nabla(a_1x + a_2y + a_3z))$$

$$= (\bar{a}\cdot\bar{r})\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right)$$

$$+ \bar{a}\cdot(a_1 x + a_2 y + a_3 z)$$

compare ϕ

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$\bar{a}$ is constant $\therefore \frac{\partial a_1}{\partial x} \cdots = 0$

$$(\bar{a}\cdot\bar{r})(0) + \bar{a}\cdot(a_1 i + a_2 j + a_3 k)$$

$$= 0 + \bar{a}\cdot(\bar{a})$$

$$= \cancel{(\bar{a})^2} \downarrow \text{substi values of } \bar{a}$$

$$= a_1^2 + a_2^2 + a_3^2 = |a|^2 = \underline{\underline{a^2}}$$

Q Prove that $\nabla\left[\nabla\cdot\left(\frac{\bar{r}}{r}\right)\right] = -\frac{2\bar{r}}{r^3}$

let $\bar{r} = \bar{f}$, $\phi = \frac{1}{r}$

$$\therefore \nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$$

$$\nabla\cdot\bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial y}{\partial z} = 3$$

$$= \frac{1}{r}(\nabla\cdot\bar{r}) + \bar{r}\left(\nabla\cdot\frac{1}{r}\right)$$

$$= \frac{3}{r} + \bar{r}\cdot\left(-\frac{1}{r^2}\right)\frac{\bar{r}}{r}$$

$$\left[\nabla f(r) = f''(r)\frac{\bar{r}}{r}\right.$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\nabla\times\bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

ϕ is always scalar function

Vector $\cdot$ vector = scalar $\quad \bar{r} \cdot \bar{r} = r^2$

DATE

Q Prove that $\nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right] = \frac{n(n-2)}{r^{n+1}}$

$$\nabla \left(\frac{1}{r^n} \right) = \frac{-n}{r^{n+1}} \frac{\bar{r}}{r} = \frac{-n\bar{r}}{r^{n+2}} \quad \cdots \nabla f(r) = f'(r)\frac{\bar{r}}{r}$$

$$r \nabla \left(\frac{1}{r^n} \right) = \frac{-n\bar{r}}{r^{n+1}}$$

$$\phi = \frac{1}{r^{n+1}} \qquad \bar{f} = \bar{r}$$

$$\nabla \cdot (\phi \bar{f}) = -n \left[\nabla \cdot \frac{\bar{r}}{r^{n+1}} \right] = \frac{1}{r^{n+1}} (\nabla \cdot \bar{r})$$

$$+ \bar{r} \cdot \nabla \left(\frac{1}{r^{n+1}} \right)$$

$$= -n \left[\frac{1}{r^{n+1}} \times 3 + \bar{r} \cdot \left(\frac{-(n+1)}{r^{n+2}} \right) \frac{\bar{r}}{r} \right.$$

$$= -n \left[\frac{3}{r^{n+1}} - \frac{n+1}{r^{n+1}} \right]$$

$$= -n \left[\frac{2-n}{r^{n+1}} \right]$$

$$= \frac{n-2}{r^{n+1}}$$

Q $\nabla \cdot \left[\frac{f(r)}{r} \bar{r} \right] = \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$

Hence prove $\nabla \cdot (r^n \bar{r}) = (n+3) r^n$

let $\phi = \frac{f(r)}{r}$, $\bar{f} = \bar{r}$

LHS

$$\nabla \cdot (\phi \bar{f}) = \phi (\nabla \cdot \bar{f}) + \bar{f} \cdot (\nabla \phi)$$

.. formula

$$= \frac{f(r)}{r} (\underbrace{\nabla \cdot \bar{r}}_{3}) + \bar{r} \cdot \left(\nabla \frac{f(r)}{r} \right) \rightarrow \frac{u}{v} \text{ rule}$$

$$= \frac{3f(r)}{r} + \bar{r} \cdot \underbrace{\left[\frac{r f'(r) - f(r)}{r^2} \right] \frac{\bar{r}}{r}}$$

$$\hookrightarrow \nabla f(r) = f'(r) \frac{\bar{r}}{r}$$

$$= \frac{3f(r)}{r} + \frac{r f'(r) - f(r)}{r}$$

$$= \frac{3f(r)}{r} + f'(r) - \frac{f(r)}{r}$$

$$= \frac{2f(r)}{r} + f'(r)$$

$$RHS = \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right] = \frac{1}{r^2} \left[r^2 f'(r) + f(r) 2r \right]$$

$$= f'(r) + \frac{2f(r)}{r}$$

LHS = RHS

Take $f(r) = r^{n+1}$

$$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$$

$$\nabla \cdot (r^n \bar{r}) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 r^{n+1} \right]$$

Q. Prove That:

$$\overline{F} = \overset{f_1}{(x+2y+az)}i + \overset{f_2}{(bx-3y-z)}j + \overset{f_3}{(4x+cy+2z)}k$$

is solenoidal. Determine constant a, b, c if $\overline{F}$ is irrotational

PT $\nabla \cdot \overline{F} = 0$

$$\nabla \cdot \overline{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1 + (-3) + 2$$

$$= 0$$

given:

$$\nabla \times \overline{F} = \overline{0}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-2y-z & 4x+cy+2k \end{vmatrix}$$

$$= i(c+1) - j(4-a) + k(b-2) = \overline{0}$$

$c+1=0 \qquad c=-1$

$a-4=0 \qquad a=4$

$b-2=0 \qquad b=2$

Q. If $\overline{F} = (y^2 - 2xyz^3)i + (3+2xy-x^2y^3)j + (6z^3 - 3x^2yz^2)k$

find ϕ such that $\overline{F} = \nabla\phi$ and $\phi(1,0,1) = 8$

Ans) $\overline{F} = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3$$

$$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2$$

Now

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\cancel{= -2yz^3 + 2x + 18z^2 - 6x^2yz}$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz$$

$$= (y^2dx + 2xy\,dy) - (2yxz^3dx + x^2z^3dy + 3x^2yz^2dz) + 3dy + 6z^3dz$$

Now

$d(xy^2) = y^2dx + 2xy\,dy$ ... (diff once with x and once with y)

$d(x^2yz^3) = 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$

... (diff once with x, once with y, once with z)

$3dy$ can also be written as $d(3y)$ ∵ deri of $3y = 3$

similarly $6z^3dz$ as $d\left(\frac{6z^4}{4}\right)$ ∵ deri of $\frac{6z^4}{4} = 6z^3$

$$= d(xy^2) - d(x^2yz^3) + d(3y) + d\left(\frac{6y^4}{4}\right)$$

$$\therefore d\phi = d\left(xy^2 - x^2yz^3 + 3y + \frac{6y^4}{4}\right)$$

$\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + c$

but $\phi(1,0,1) = 8$ (given)

$$\frac{3}{2} + c = 8$$

$$c = 8 - \frac{3}{2} = \frac{13}{2}$$

Q $\overline{F} = (y\sin z - \sin x)i + (x\sin z + 2yz)j + (xy\cos z + y^2)k$

$\overline{F}$ is irrotational then there's always a function ϕ such that $\overline{F} = \nabla\phi$.

$$\frac{\partial\phi_1}{\partial x} = y\sin z - \sin x$$

$$\frac{\partial\phi_2}{\partial y} = x\sin z + 2yz$$

$$\frac{\partial\phi_3}{\partial z} = xy\cos xy + y^2$$

$$d\phi = \frac{\partial\phi_1}{\partial x}dx + \frac{\partial\phi_2}{\partial y}dy + \frac{\partial\phi_3}{\partial z}dz$$

$$= (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos(z) + y^2)\,dz$$

$$= (y\sin z\,dx + x\sin z\,dy + xy\cos z\,dz) + (2yz\,dy + y^2\,dz) - \sin x\,dx$$

$$= d(xy\sin z) + d(y^2 z) + d(\cos x)$$

$$\phi = xy\sin z + y^2 z + \cos x + c$$