

BAYE'S THEOREM

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If $A_1, A_2, A_3, \dots, A_n$ are n mutually exclusive and exhaustive events from the sample space S , B is any other event from S and if probability of occurrence of A_i 's and probability of occurrence of B given that $A_i, i = 1, 2, 3, \dots, n$ has occurred are known, then probabilities of occurrence of A_i 's given that B has occurred are given by

$$P(A_i/B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^n P(A_i).P(B/A_i)}; i = 1, 2, 3, \dots, n$$

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Note: Three types of probabilities occur in the above theorem $P(A_i), P(B/A_i), P(A_i/B)$.

(i) The probabilities $P(A_i), i = 1, 2, 3, \dots, n$ are called **priori** or simply **prior** probabilities

and $P(A_1) + P(A_2) + P(A_3) + \dots + P(A_n) = 1$

(ii) The probabilities $P(B/A_i)$ is the probability of event B , given each and **every prior** probability.

(iii) The probabilities $P(A_i/B)$ are called **posterior** probabilities.

EXAMPLE-1

The chances of X, Y, Z becoming managers of a certain company are 4:2:3. The probabilities that the bonus scheme will be introduced if X, Y, Z become managers are 0.3, 0.5 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that X is appointed as the manager?

Let event E_1 : Person X becomes manager

event E_2 : Person Y becomes manager

And event E_3 : Person Z becomes manager

$\therefore P(E_1) = \frac{4}{9}$ $P(E_2) = \frac{2}{9}$ $P(E_3) = \frac{3}{9}$ (Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events)

Let event A: Bonus is introduced

$\therefore P(A/E_1) = P(\text{Bonus is introduced under the condition that person X becomes manager}) = 0.3$

$P(A/E_2) = P(\text{Bonus is introduced under the condition that person Y becomes manager}) = 0.5$

$P(A/E_3) = P(\text{Bonus is introduced under the condition that person Z becomes manager}) = 0.8$

\therefore required probability = $P(\text{Person X becomes manager under the condition that bonus Scheme is introduced})$

$$= P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)} = \frac{\left(\frac{2}{15}\right)}{\left(\frac{23}{45}\right)} = \frac{6}{23}$$

EXAMPLE-2

- The members of a consulting firm hire cars from three rental agencies, 60 percent from agency 1, 30 percent from agency 2 and 10 percent from agency 3. 9 percent of the cars from agency 1 need repairs, 20 percent of the car from agency 2 need repairs and 6 percent of the cars from agency 3 need repairs. If a rental car delivered to the consulting firms needs repairs, what is the probability that it came from rental agency 2?

- **Solution:** Let $E_1 \equiv$ events that the car comes from rental agencies 1

- $E_2 \equiv$ events that the car comes from rental agencies 2

- $E_3 \equiv$ events that the car comes from rental agencies 3

- $A \equiv$ event that the car needs repairs

- $P(E_1) = 0.6, P(E_2) = 0.3, P(E_3) = 0.1$

- $P(A/E_1) = 0.09, P(A/E_2) = 0.2$ and $P(A/E_3) = 0.06$

- $$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

- $$= \frac{(0.30)(0.20)}{(0.60)(0.09) + (0.30)(0.20) + (0.10)(0.06)}$$

- $$= \frac{0.060}{0.120} = 0.5$$

EXAMPLE-3

In certain area 25% of women were black, 75% were white. The literacy rates for black women were 48% and for white women was 83%. What proportion of the literate women was black?

Solution: Let $E_1 \equiv$ event that women were black

$E_2 \equiv$ event that women were white

$A \equiv$ event that women were literate

$$\therefore P(E_1) = 25\% = \frac{25}{100} = \frac{1}{4}$$

$$P(E_2) = 75\% = \frac{75}{100} = \frac{3}{4}$$

Since the literacy rates for black women were 48% and for white women were 83%,

$$P(A/E_1) = 48\% = \frac{48}{100} \text{ and } P(A/E_2) = 83\% = \frac{83}{100}$$

By Baye's theorem, the proportion of the literate women who were black is given by

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)} \\ &= \frac{\left(\frac{1}{4}\right)\left(\frac{48}{100}\right)}{\left(\frac{1}{4}\right)\left(\frac{48}{100}\right) + \left(\frac{3}{4}\right)\left(\frac{83}{100}\right)} \\ &= \frac{48}{48 + 249} = \frac{48}{297} = \frac{16}{99} \end{aligned}$$

EXAMPLE-4

If E_1 and E_2 are equally likely, mutually exclusive and exhaustive events and $P(A/E_1) = 0.2, P(A/E_2) = 0.3$. Find $P(E_1/A)$.

Solution: E_1 and E_2 are equally likely, mutually exclusive and exhaustive events

$$\therefore P(E_1) = P(E_2), P(E_1 \cap E_2) = 0 \text{ and } P(E_1 \cup E_2) = 1$$

$$\text{But } P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$\therefore 1 = P(E_1) + P(E_2)$$

$$\therefore P(E_1) + P(E_1) = 1 \quad \dots\dots\dots [\because P(E_1) = P(E_2)]$$

$$\therefore 2 \cdot P(E_1) = 1$$

$$\therefore P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

$$\text{Also, } P(A/E_1) = 0.2 \text{ and } P(A/E_2) = 0.3$$

\therefore by Baye's Theorem

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)} \\ &= \frac{\left(\frac{1}{2}\right)(0.2)}{\left(\frac{1}{2}\right)(0.2) + \left(\frac{1}{2}\right)(0.3)} = \frac{0.2}{0.2+0.3} = \frac{2}{5} = 0.4 \end{aligned}$$

EXAMPLE-5

Suppose that a product is produced in 3 factories X, Y & Z. It is known that factory X produces thrice as many times as factory Y, and that factory Y and Z produce the same number of items. Assume that it is known that 3% of the items produced by each of the factories X and Z are defective while 5% of those manufactured by Y are defective. All the items produced in 3 factories are stocked and an item is selected at random

- (1) What is the probability that this item is defective
- (2) If an item selected at random is found to be defective, what is the probability that it was produced by factory X, Y and Z respectively.

EXAMPLE-5

- **Solution:** Let $E_1 \equiv$ event that the item is produced by factory X
 - $E_2 \equiv$ event that the item is produced by factory Y
 - $E_3 \equiv$ event that the item is produced by factory Z
 - $A \equiv$ event that the item is defective
 - Let the number of items produced by Y and Z each be n . then number of items produced by X = $3n$
 - $P(E_1) = \frac{3n}{3n+n+n} = 0.6, P(E_2) = \frac{n}{5n} = 0.2, P(E_3) = \frac{n}{5n} = 0.2$
 - $P(A/E_1) = P(A/E_2) = 0.03$ and $P(A/E_3) = 0.05$
- (1) $P(\text{item is defective}) = P(A) = \sum_{i=1}^3 P(A \cap E_i)$
$$= \sum_{i=1}^3 P(E_i)P(A/E_i)$$
$$= P(E_1) P(A/E_1) + P(E_2)P(A/E_2) + P(E_3) P(A/E_3)$$
$$= 0.034$$

EXAMPLE-5

Solution:
$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$
$$= \frac{9}{17}$$

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$
$$= \frac{5}{17}$$

$$P(E_3/A) = \frac{P(E_3) \cdot P(A/E_3)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$
$$= \frac{3}{17}$$

EXAMPLE-6

In 2002 there will be 3 candidates for the position of Principal- X, Y, Z, whose chances of getting the appointment are in the proportion 4: 2: 3 respectively. The probability that Mr X is selected who would introduce co-education in the college is 0.3. the probability of Y & Z doing the same are respectively 0.5 & 0.8

- (i) What is the probability that there will be co-education in the college in 2003
- (ii) If there is co-education in the college in 2003, what is the probability that Mr Z is the principal

EXAMPLE-6

Let event E_1 : Mr X becomes Principal

event E_2 : Mr Y becomes Principal

And event E_3 : Mr Z becomes Principal

$$\therefore P(E_1) = \frac{4}{9} \quad P(E_2) = \frac{2}{9} \quad P(E_3) = \frac{3}{9}$$

(Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events)

Let event A: Co-education is introduced

$$\therefore P(A/E_1) = P(\text{co-education is introduced under the condition that Mr X becomes principal}) = 0.3$$

$$P(A/E_2) = P(\text{co-education is introduced under the condition that Mr Y becomes principal}) = 0.5$$

$$P(A/E_3) = P(\text{co-education is introduced under the condition that Mr Z becomes principal}) = 0.8$$

EXAMPLE-6

$$\begin{aligned}(1) \text{ P(Co-education is introduced)} &= P(A) = \sum_{i=1}^3 P(A \cap E_i) \\ &= \sum_{i=1}^3 P(E_i)P(A/E_i) \\ &= P(E_1) P(A/E_1) + P(E_2)P(A/E_2) + P(E_3) P(A/E_3) \\ &= \frac{46}{90}\end{aligned}$$

$$\begin{aligned}(2) P(E_3/A) &= \frac{P(E_3) \cdot P(A/E_3)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)} \\ &= \frac{12}{23}\end{aligned}$$