Mathematical Foundations for Data Science

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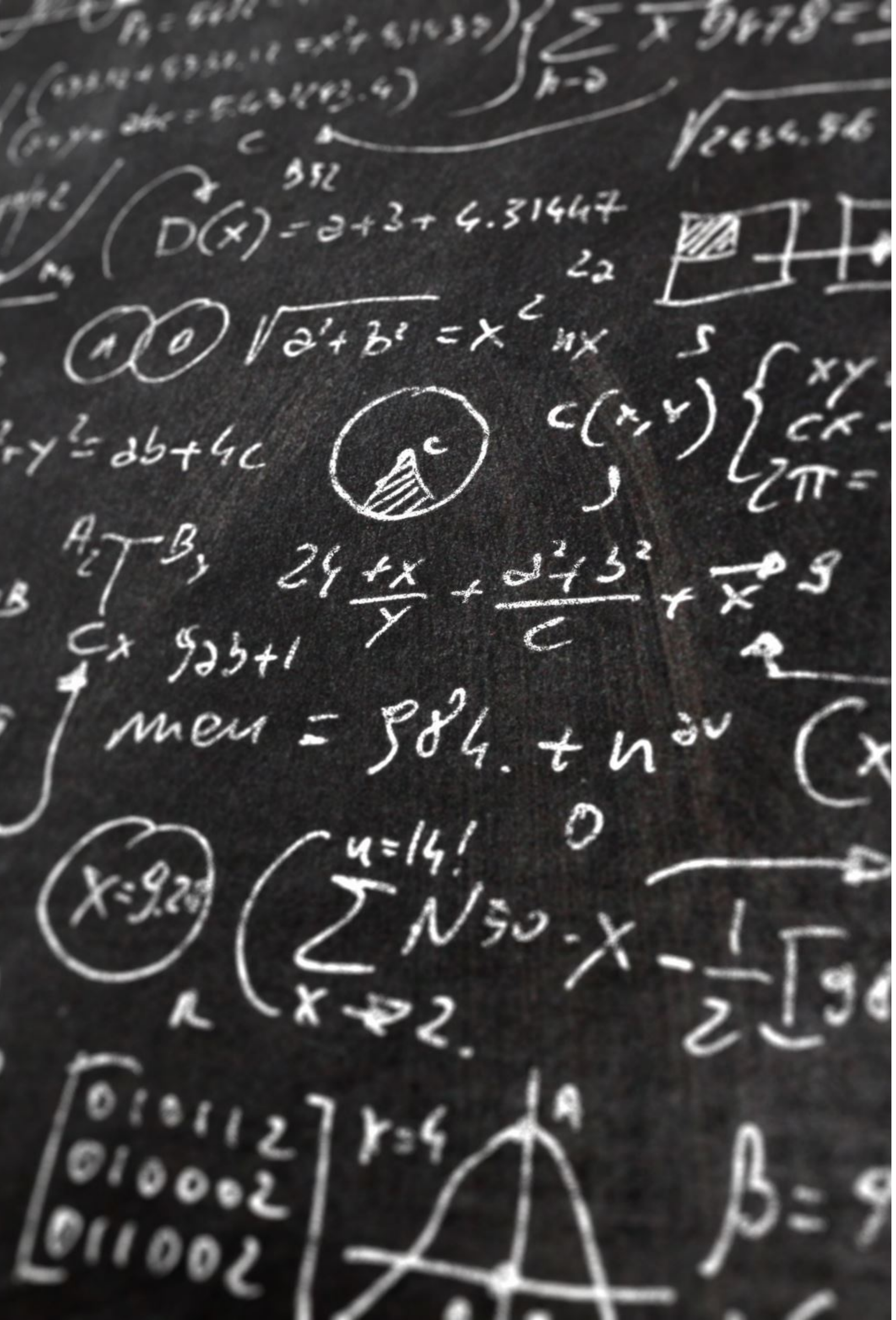


Content

2.1 Linear Algebra: Vectors and matrices. Matrix operations and transformations.

2.2 Probability Distributions: Gaussian distribution, Binomial distribution, Poisson distribution. Properties of Probability Distributions: Mean, variance, and standard deviation of a distribution. Moments and percentiles. Skewness and kurtosis.

2.3 Statistical analysis: Measures of central tendency (mean, median, mode). Measures of dispersion (range, variance, standard deviation). Quartiles, percentiles. Stem and leaf plots, Box plots

Hypothesis testing

What does a matrix look like?

Matrices are everywhere. If you have used a spreadsheet such as Excel or Lotus or written a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program.

Look at the matrix below about the sale of tires in a store – given by quarter and make of tires.

$$\begin{array}{ll} & \begin{array}{cccc} \text{Q1} & \text{Q2} & \text{Q3} & \text{Q4} \end{array} \\ \begin{array}{l} \textbf{Tirestone} \\ \textbf{Michigan} \\ \textbf{Copper} \end{array} & \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \end{array}$$

If one wants to know how many *Copper* tires were sold in *Quarter 4*, we go along the row *Copper* and column *Q4* and find that it is 27.

What is a matrix?

A *matrix* is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix $[A]$ is denoted by

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

Row i of $[A]$ has n elements and is $\begin{bmatrix} a_{i1} & a_{i2} \ldots a_{in} \end{bmatrix}$

and column j of $[A]$ has m elements and is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

The matrix $[A]$ may also be denoted by $[A]_{mxn}$ to show that $[A]$ is a matrix with m rows and n columns.

Each entry in the matrix is called the entry or element of the matrix and is denoted by a_{ij} where I is the row number and j is the column number of the element.

What is a matrix?

The matrix for the tire sales example could be denoted by the matrix $[A]$ as

$$[A]=\begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

There are 3 rows and 4 columns, so the size of the matrix is 3×4. In the above $[A]$ matrix, a_{34} $=27$.

Special Types of Matrices

- Row Vector
- Column Vector
- Submatrix
- Square Matrix
- Upper Triangular Matrix
- Lower Triangular Matrix

- Diagonal Matrix
- Identity Matrix
- Zero Matrix
- Tri-diagonal Matrices
- Diagonally Dominant Matrix

What Is a Vector?

What is a vector?

A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row Vector:

If a matrix [*B*] has one row, it is called a row vector

$$[B] = [b_1 \ b_2 \ \cdots \ \cdots \ b_n]$$

and *n* is the dimension of the row vector.

$$[B] = [25 \ \ 20 \ \ 3 \ \ 2 \ \ 0]$$

Column vector:

If a matrix [*C*] has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

and *m* is the dimension of the vector.

Submatrix

A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, a few submatrices of A are

$[1], [2], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1\ 5], \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, A.$

But the matrices $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not submatrices of A.

Square Matrix

If the number of rows m of a matrix is equal to the number of columns n of a matrix $[A]$, $(m=n)$, then $[A]$ is called a square matrix. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the *diagonal elements* of a square matrix. Sometimes the diagonal of the matrix is also called the *principal or main of the matrix*.

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix as it has the same number of rows and columns, that is, 3. The diagonal elements of $[A]$ are (25,10,7) .

Upper Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0, i > j$ is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

Example 5

Give an example of an upper triangular matrix.

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix}$$

is an upper triangular matrix.

Lower Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0,\ j > i$ is called an lower triangular matrix. That is, all the elements above the diagonal entries are zero.

Example 6

Give an example of a lower triangular matrix.

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.6 & 2.5 & 1 \end{bmatrix}$$

is a lower triangular matrix.

Diagonal Matrix

A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, $(a_{ij} = 0,\ i \neq j)$.

An example of a diagonal matrix.

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any or all the diagonal entries of a diagonal matrix can be zero.

Identity Matrix

A diagonal matrix with all diagonal elements equal to one is called an identity matrix, ($a_{ij} = 0$, $i \neq j$ and $a_{ii} = 1$ for all i).

An example of an identity matrix is,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix

A matrix whose all entries are zero is called a zero matrix, ($a_{ij} = 0$ for all i and j).

Some examples of zero matrices are,

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\mathrm{B}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Tridiagonal Matrix

Tridiagonal matrices are the matrices which are having non-zero elements on the diagonal, super diagonal and subdiagonal. All the rest of the elements are zeros.

An example of a tridiagonal matrix is,

$$\begin{pmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Non-square Matrix

Do non-square matrices have diagonal entries?

Yes, for a $m \times n$ matrix $[A]$, the diagonal entries are $a_{11}, a_{22} ..., a_{k-1,k-1}, a_{kk}$ where k=min$\{m,n\}$.

What are the diagonal entries of

$$[A] = \begin{bmatrix} 3.2 & 5 \\ 6 & 7 \\ 2.9 & 3.2 \\ 5.6 & 7.8 \end{bmatrix}$$

The diagonal elements of $[A]$ are $a_{11} = 3.2$ and $a_{22} = 7$.

Diagonally Dominant Matrix

A $n \times n$ square matrix $[A]$ is a diagonally dominant matrix if

$$\left|a_{ii}\right| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |\, a_{ij} \,| \quad \text{for all} \qquad i = 1,2,\ldots\ldots, n \ \text{ and}$$

$$\left|a_{ii}\right| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |\, a_{ij} \,| \quad \text{for at least one } i,$$

that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

Diagonally Dominant Matrix

Example:

$$[A]=\begin{bmatrix}15 & 6 & 7\\ 2 & -4 & -2\\ 3 & 2 & 6\end{bmatrix}$$

is a diagonally dominant matrix as

$$|a_{11}| = |15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |7| = 13$$

$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |-2| = 4$$

$$|a_{33}| = |6| = 6 \geq |a_{31}| + |a_{32}| = |3| + |2| = 5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

Diagonally Dominant Matrix

$$[B]=\begin{bmatrix}-15 & 6 & 9\\ 2 & -4 & 2\\ 3 & -2 & 5.001\end{bmatrix}$$

is a diagonally dominant matrix as

$$|b_{11}|=|-15|=15\geq|b_{12}|+|b_{13}|=|6|+|9|=15$$
$$|b_{22}|=|-4|=4\geq|b_{21}|+|b_{23}|=|2|+|2|=4$$
$$|b_{33}|=|5.001|=5.001\geq|b_{31}|+|b_{32}|=|3|+|-2|=5$$

The inequalities are satisfied for all rows and it is satisfied strictly greater than for at least one row (in this case it is Row 3).

Diagonally Dominant Matrix

$$[C]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is not diagonally dominant as

$$|c_{22}| = |8| = 8 \le |c_{21}| + |c_{23}| = |64| + |1| = 65$$

Equal Matrices

When are two matrices considered to be equal?

Two matrices [A] and [B] is the same (number of rows and columns are same for [A] and [B]) and $a_{ij}=b_{ij}$ for all i and j.

What would make

$$[A] = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$$

to be equal to

$$[B] = \begin{bmatrix} b_{11} & 3 \\ 6 & b_{22} \end{bmatrix}$$

The two matrices [A] and [B] would be equal if $b_{11}=2$ and $b_{22}=7$.

Matrix Addition and Subtraction

Only possible for matrices of same dimension

Add/subtract matrices element-by-element

Addition example: **C** = **A**+**B**

$$A + B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 4 & 2 & 3 \\ -1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 \\ 7 & 3 & 5 \\ 3 & 8 & 2 \end{bmatrix}$$

Subtraction example: **C** = **A**-**B**

$$\begin{bmatrix} 4 & 2 & -1 \\ 5 & 3 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 2 \\ -4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -3 \\ 9 & 0 & -6 \end{bmatrix}$$

Scalar multiplication

- Scalar x matrix = scalar multiplication

$$\lambda \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \end{pmatrix}$$

Matrix Multiplication

*"When **A** is a **mxn** matrix & **B** is a **kxl** matrix, **AB** is only possible if **n=k**. The result will be an **mxl** matrix"*

$$
\begin{array}{cc} & n \\ m & \begin{matrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \\ A_{10} & A_{11} & A_{12} \end{matrix} \end{array}
\quad \text{x} \quad
\begin{array}{cc} l & \\ \begin{matrix} B_{13} & B_{14} \\ B_{15} & B_{16} \\ B_{17} & B_{18} \end{matrix} & k \end{array}
\quad = \mathbf{m} \text{ x } \mathbf{l} \text{ matrix}
$$

Number of columns in A = Number of rows in B

Matrix multiplication

Sum over product of respective <u>rows</u> and <u>columns</u>

$$\underset{\mathbf{A}}{\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}} \times \underset{\mathbf{B}}{\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}} = \begin{pmatrix} \mathbf{c}_{11} & \mathbf{c}_{12} \\ \mathbf{c}_{21} & \mathbf{c}_{22} \end{pmatrix}$$

Define output matrix

$$= \begin{bmatrix} (1\times 2)+(0\times 3) & (1\times 1)+(0\times 1) \\ (2\times 2)+(3\times 3) & (2\times 1)+(3\times 1) \end{bmatrix}$$

$$= \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{13} & \mathbf{5} \end{pmatrix}$$

Matrix multiplication

- Matrix multiplication is NOT commutative
- $AB \neq BA$
- Matrix multiplication IS associative
- $A(BC)=(AB)C$
- Matrix multiplication IS distributive
- $A(B+C)=AB+AC$
- $(A+B)C=AC+BC$

Transpose of Matrix

- The transpose of an m × n matrix A, we mean a matrix of order n × m having the rows of A as its columns and the columns of A as its rows.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$.

Thus, the transpose of a row vector is a column vector and vice-versa.

- For any matrix A, we have

$$(A^t)^t = A.$$

Vector Products

Two vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Inner product = scalar

Inner product X^TY is a scalar
(1xn) (nx1)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 = \sum_{i=1}^{3} x_i y_i$$

Outer product = matrix

$$\mathrm{xy}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{bmatrix}$$

Outer product XY^T is a matrix
(nx1) (1xn)

Some More Special Matrices

A matrix A over $\mathbb{R}$ is called symmetric if $A^t = A$ and skew-symmetric if $A^t = -A$.

A matrix A is said to be orthogonal if $AA^t = A^tA = I$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$. Then A is a symmetric matrix and B is a skew-symmetric matrix.

Let $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$. Then A is an orthogonal matrix.

Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = \begin{cases} 1 & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$. Then $A^n = \mathbf{0}$ and $A^\ell \neq \mathbf{0}$ for $1 \leq \ell \leq n - 1$. The matrices A for which a positive integer k exists such that $A^k = \mathbf{0}$ are called NILPOTENT matrices. The least positive integer k for which $A^k = \mathbf{0}$ is called the ORDER OF NILPOTENCY.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = A$. The matrices that satisfy the condition that $A^2 = A$ are called IDEMPOTENT matrices.

Matrices over Complex Numbers

(Conjugate Transpose of a Matrix) 1. Let A be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate of A, denoted by $\overline{A}$, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ij}}$.

For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then

$$\overline{A} = \begin{bmatrix} 1 & 4-3i & -i \\ 0 & 1 & -i-2 \end{bmatrix}.$$

2. Let A be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate Transpose of A, denoted by A^*, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ji}}$.

For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then

$$A^* = \begin{bmatrix} 1 & 0 \\ 4-3i & 1 \\ -i & -i-2 \end{bmatrix}.$$

3. A square matrix A over $\mathbb{C}$ is called Hermitian if $A^* = A$.
4. A square matrix A over $\mathbb{C}$ is called skew-Hermitian if $A^* = -A$.
5. A square matrix A over $\mathbb{C}$ is called unitary if $A^*A = AA^* = I$.
6. A square matrix A over $\mathbb{C}$ is called Normal if $AA^* = A^*A$.

Determinants

- Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations (i.e. GLMs).

- The **determinant** is a function that associates a scalar det(*A*) to every square matrix *A*.

  – Input is nxn matrix
  – Output is a single number (real or complex) called the determinant

$$\det(M) = \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{vmatrix} = \sum_{x} \operatorname{sgn}(x) M_{1x_1} M_{2x_2} \cdots M_{nx_n}$$

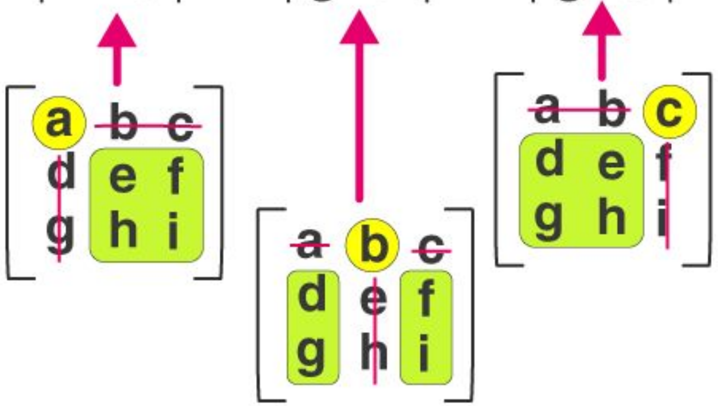
Determinants

- Determinants can only be found for square matrices.
- For a 2x2 matrix A, **det(A) = ad-bc**. Lets have at closer look at that:

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Determinant of a 3 x 3 Matrix Formula

$$|C| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



Determinants

- **Calculate the determinant of the 3 x 3 matrix.** $\begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Use the 3 x 3 determinant formula:

$$\det(A) = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} = 1 \bullet \det\begin{bmatrix} -1 & -3 \\ 3 & 1 \end{bmatrix} - (3) \bullet \det\begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} + 2 \bullet \det\begin{bmatrix} -3 & -1 \\ 2 & 3 \end{bmatrix}$$

= 1[-1 – (-9)] – 3 [-3 – (-6)] + 2 [-9 – (-2)]

= 1 (-1+9) -3 (-3 +6) + 2(-9 + 2)

= 1(8) -3(3) +2(-7)

= 8 -9-14

= -15

$$Therefore,\ the\ determinant\ of\ \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} = -15$$

Cofactor Matrix

- Co-factor matrix is a matrix having the co-factors as the elements of the matrix.
- Co-factor of an element within the matrix is obtained when the minor Mij of the element is multiplied with $(-1)^{i+j}$. $C_{ij} = (-1)^{i+j})M_{ij}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Step-I $\quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Step-II

$$M_{11} = a_{22}.a_{33} - a_{23}.a_{32}$$

Step-III

$$C_{11} = (-1)^{1+1}(a_{22}.a_{33} - a_{23}.a_{32})$$

Step-IV

$$\text{Cofactor Matrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Co-factor of } a_{11} = C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = +(a_{22}.a_{33} - a_{23}.a_{32})$$

$$\text{Co-factor of } a_{23} = C_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}.a_{32} - a_{12}.a_{31})$$

$$\text{Co-factor of } a_{32} = C_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -(a_{11}.a_{23} - a_{13}.a_{21})$$

Similarly we can find the co-factor of each element of the matrix A. Further we can form the co-factor matrix of A by writing the co-factor of each element in the matrix array.

$$\text{Co-factor Matrix of } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Adjoint of the matrix

- The adjoint of a 3 x 3 matrix can be obtained by following two simple steps.

1. Find the co-factor matrix of the given matrix.
2. Transpose of this co-factor matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the co-factor matrix A

$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Inverse of Matrix

- **Definition.** A matrix ***A*** is called **nonsingular** or **invertible** if there exists a matrix ***B*** such that:

$$A\,B = B\,A = I_n$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 + 1/3 & -1/3 + 1/3 \\ -2/3 + 2/3 & \frac{1}{3} + \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Notation.** A common notation for the inverse of a matrix ***A*** is A^{-1}. So: $A\,A^{-1} = A^{-1}\,A = I_n$.

- The inverse matrix is unique when it exists. So if ***A*** is invertible, then A^{-1} is also invertible and then $(A^T)^{-1} = (A^{-1})^T$

•Matrix division: A/B= $A*B^{-1}$

Inverse of Matrix

- The inverse of matrix A can be computed using the inverse of matrix formula, by dividing the adjoint of a matrix by the determinant of the matrix. The inverse of a matrix can be calculated by following the given steps:
- **Step 1:** Calculate the minors of all elements of A.
- **Step 2:** Then compute the cofactors of all elements and write the cofactor matrix by replacing the elements of A by their corresponding cofactors.
- **Step 3:** Find the adjoint of A (written as adj A) by taking the tranpose of cofactor matrix of A.
- **Step 4:** Multiply adj A by reciprocal of determinant.

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

Inverse of Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\boxed{\underbrace{A^{-1}}_{\text{Inverse of A}} = \frac{1}{\underbrace{ad - bc}_{\text{Determinant of A}}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{Adjoint of A}}}$$

Using the inverse matrix to solve equations

- 1. Writing simultaneous equations in matrix form

Consider the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Provided you understand how matrices are multiplied together you will realise that these can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

we have

$$AX = B$$

This is the **matrix form** of the simultaneous equations. Here the unknown is the matrix X, since A and B are already known. A is called the **matrix of coefficients**.

Using the inverse matrix to solve equations

- 2. Solving the simultaneous equations

Given

$$AX = B$$

we can multiply both sides by the inverse of A, provided this exists, to give

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, the identity matrix. Furthermore, $IX = X$, because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered. So

$$X = A^{-1}B$$

$$\boxed{\text{if} \quad AX = B, \qquad \text{then} \qquad X = A^{-1}B}$$

Example

Solve the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Solution

We have already seen these equations in matrix form:

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$.

$$\begin{aligned} A^{-1} &= \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \end{aligned}$$

Then X is given by

$$\begin{aligned} X = A^{-1}B &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -22 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Hence $x = 2$, $y = 1$ is the solution of the simultaneous equations.

Exercises

1. Solve the following sets of simultaneous equations using the inverse matrix method.

a) $\begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned}$ b) $\begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned}$

Answers

1. a) $x = 3, y = -2,$ b) $x = -2, y = 2$.

Eigen vectors

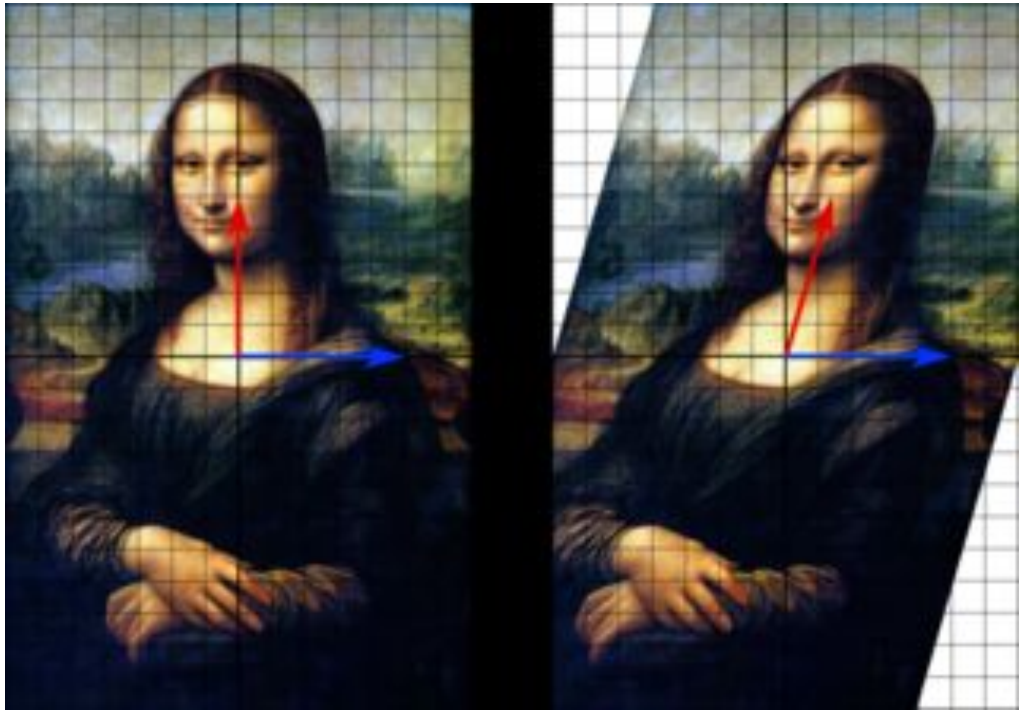
- $M = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$ The resulting vector is not integer multiple of the original vector
- $N = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ The resulting vector is exactly 4 times the original vector. This is called eigenvector.

- *We can transform and change matrices into new vectors by multiplying a matrix with a vector. The multiplication of the matrix by a vector computes a new vector. This is the transformed vector.*

Eigenvector

- If the new transformed vector is just a scaled form of the original vector then the original vector is known to be an **eigenvector** of the original matrix
- An eigenvector is a vector that does not change when a transformation is applied to it, except that it becomes a scaled version of the original vector.
- **Eigenvalue**— The scalar that is used to transform (stretch) an Eigenvector.

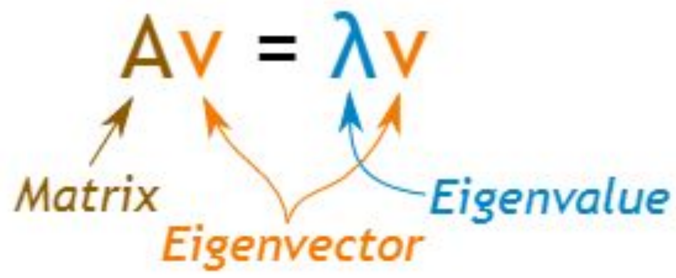
Eigenvector and Eigenvalue



- An eigenvector **does not change direction** in a transformation:
- In this shear mapping the red arrow changes direction, but the blue arrow does not. The blue arrow is an eigenvector of this shear mapping because it does not change direction, and since its length is unchanged, its eigenvalue is 1.
- **1** means no change,
- **2** means doubling in length,
- **−1** means pointing backwards along the eigenvalue's direction

The Mathematics

- For a square matrix **A**, an Eigenvector and Eigenvalue make this equation true:



Av gives us:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6\times1+3\times4 \\ 4\times1+5\times4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

λv gives us :

$$6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

Yes they are equal! So $Av = \lambda v$ as promised.

How do we find these eigen things?

We start by finding the **eigenvalue**:

we know this equation must be true: $Av = \lambda v$

Now let us put in an identity matrix so we are dealing with matrix-vs-matrix:

$$Av = \lambda Iv$$

Bring all to left hand side:

$$Av - \lambda Iv = 0$$

If **v** is non-zero then we can solve for λ using just the determinant:

$$| A - \lambda I | = 0$$

Let's try that equation on our previous example:

Eigenvector and Eigenvalue

Example: Solve for λ:

Start with $| A - \lambda I | = 0$

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Which is:

$$\begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

Calculating that determinant gets:

$$(-6-\lambda)(5-\lambda) - 3\times4 = 0$$

Which then gets us this

$$\lambda^2 + \lambda - 42 = 0$$

$$\lambda = -7 \text{ or } 6$$

Example (continued): Find the Eigenvector for the Eigenvalue $\boldsymbol{\lambda} = \mathbf{6}$:

Start with:

$$Av = \lambda v$$

Put in the values we know:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

After multiplying we get these two equations:

$$\begin{aligned} -6x + 3y &= 6x \\ 4x + 5y &= 6y \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}$$

Either equation reveals that **y** = **4x**, so the **eigenvector** is any non-zero multiple of this:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

And we get the solution shown at the top of the page:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6\times1+3\times4 \\ 4\times1+5\times4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

... and also ...

$$6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

So $Av = \lambda v$

×

Example 1

find the eigenvalues of the matrix

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

Solution

$$[A] - \lambda[I] = \begin{bmatrix} 3-\lambda & -1.5 \\ -0.75 & 0.75-\lambda \end{bmatrix}$$

$$\det([A] - \lambda[I]) = (3-\lambda)(0.75-\lambda) - (-0.75)(-1.5) = 0$$

$$2.25 - 0.75\lambda - 3\lambda + \lambda^2 - 1.125 = 0$$

Example 1 (cont.)

$$\lambda^2 - 3.75\lambda + 1.125 = 0$$

$$\lambda = \frac{-(-3.75) \pm \sqrt{(-3.75)^2 - 4(1)(1.125)}}{2(1)}$$

$$= \frac{3.75 \pm 3.092}{2}$$

$$= 3.421, 0.3288$$

So the eigenvalues are 3.421 and 0.3288.

Example 2

Find the eigenvectors of

$$A = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

Solution

The eigenvalues have already been found in Example 1 as

$$\lambda_1 = 3.421, \lambda_2 = 0.3288$$

Let

$$[X] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the eigenvector corresponding to

$$\lambda_1 = 3.421$$

Example 2 (cont.)

Hence

$$([A]-\lambda_1[I])[X]=0$$

$$\left\{\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} - 3.421\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.421 & -1.5 \\ -0.75 & -2.671 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If

$$x_1 = s$$

then

$$-0.421s - 1.5x_2 = 0$$

$$x_2 = -0.2808s$$

Example 2 (cont.)

The eigenvector corresponding to $\lambda_1 = 3.421$ then is,

$$[X] = \begin{bmatrix} s \\ -0.2808s \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}$$

The eigenvector corresponding to $\lambda_1 = 3.421$ is

$$\begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}$$

Similarly, the eigenvector corresponding to $\lambda_2 = 0.3288$ is

$$\begin{bmatrix} 1 \\ 1.781 \end{bmatrix}$$

Eigen vector and Eigen value

Naïve Gaussian Elimination

Gaussian elimination is a method of solving a linear system Ax = b (consisting of m equations in n unknowns) by bringing the augmented matrix

$$[A \ \ \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

to an upper triangular form

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{mn} & d_m \end{array}\right].$$

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{array}{cc} . & . \\ . & . \\ . & . \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$
\begin{array}{rl}
 & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
- & a_{21}x_1 + \dfrac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \dfrac{a_{21}}{a_{11}}a_{1n}x_n = \dfrac{a_{21}}{a_{11}}b_1 \\
\hline
\end{array}
$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad \vdots \qquad \vdots & \\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad \vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\
0 & 0 & a''_{33} & \cdots & a''_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & a_{nn}^{(n-1)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)}
\end{bmatrix}
$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{n}x_n &= b''_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1\ \ )}
\end{aligned}
$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time, $t\ (\mathrm{s})$	**Velocity,** $v\ (\mathrm{m/s})$
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le \mathrm{t} \le 12.$$

Find the velocity at t=6 seconds .

Example 1 Cont.

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3 \ , \quad 5 \le t \le 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Table1

Time,	**Velocity,**
5	106.8
8	177.2
12	279.2

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right]$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$(n-1)=(3-1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and

multiply it by 64, $\frac{64}{25} = 2.56$

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & | & 106.8 \\ 0 & -4.8 & -1.56 & | & -96.208 \\ 144 & 12 & 1 & | & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$

$$\begin{bmatrix} 25 & 5 & 1 & | & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & | & 615.168 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 & | & 279.2 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 & | & 615.168 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 & | & -335.968 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & | & 106.8 \\ 0 & -4.8 & -1.56 & | & -96.208 \\ 0 & -16.8 & -4.76 & | & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by −4.8

and multiply it by −16.8, $\frac{-16.8}{-4.8}. = 3.5$

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & 335.968 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$\begin{aligned} 25a_1 + 5a_2 + a_3 &= 106.8 \\ a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25} \\ &= 0.290472 \end{aligned}$$

Naïve Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Example 1 Cont.

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472 t^2 + 19.6905 t + 1.08571, \quad 5 \le t \le 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

- However, we could also find all the needed values of velocity at t = 6,7.5,9,11 seconds using matrix multiplication.

$$v(t) = [0.290472 \;\; 19.6905 \;\; 1.08571] \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

- So, if we want to find v(6),v(7.5),v(9),v(11),v(6),v(7.5),v(9),v(11), it is given by

$$\begin{aligned}[v(6)\, v(7.5)\, v(9)\, v(11)] &= [0.290472 \;\; 19.6905 \;\; 1.08571] \begin{bmatrix} 6^2 & 7.5^2 & 9^2 & 11^2 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= [0.290472 \;\; 19.6905 \;\; 1.08571] \begin{bmatrix} 36 & 56.25 & 81 & 121 \\ 6 & 7.5 & 9 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= [129.686 \;\; 165.104 \;\; 201.828 \;\; 252.828]\end{aligned}$$

$$v(6) = 129.686\, m/s$$

$$v(7.5) = 165.1\,04\, m/s$$

$$v(9) = 201.828\, m/s$$

$$v(11) = 252.828\, m/s$$

Can we use Naive Gauss elimination methods to find the determinant of a square matrix?

Theorem 6.1

Let $[A]$ be a $n \times n$ matrix. Then, if $[B]$ is a $n \times n$ matrix that results from adding or subtracting a multiple of one row to another row, then $det(A) = det(B)$ (The same is true for column operations also).

Theorem 6.2

Let $[A]$ be a $n \times n$ matrix that is upper triangular, lower triangular or diagonal, then

$$\begin{aligned}\det(A) &= a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn} \\ &= \prod_{i=1}^{n} a_{ii}\end{aligned}$$

This implies that if we apply the forward elimination steps of the Naive Gauss elimination method, the determinant of the matrix stays the same according to Theorem 6.1. Then since at the end of the forward elimination steps, the resulting matrix is upper triangular, the determinant will be given by Theorem 6.2.

- Find the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Remember in Example 1, we conducted the steps of forward elimination of unknowns using the Naive Gauss elimination method on [A] to give

$$[B] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

According to Theorem 2

$$\begin{aligned} det(A) &= det(B) \\ &= 25 \times (-4.8) \times 0.7 \\ &= -84.00 \end{aligned}$$

LU Decomposition Method

1. *LU Decomposition is another method to solve a set of simultaneous linear equations*

2. *Which is better, Gauss Elimination or LU Decomposition?*

LU Decomposition Method

An LU decomposition of a matrix A is the product of a lower triangular matrix and an upper triangular matrix that is equal to A.

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear eq

If $[A] = [L][U$

Mult

Whic

Remember $[L]^{-1}[L] = [I]$ which l

Now, if $[I][U] = [$

N

Which en

$[A][X] = [C]$

$[L][U][X] = [C]$

$[L]^{-1}$

$[L]^{-1}[L][U][X] = [L]^{-1}[C]$

LU Decomposition using Crout's method of Matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU \text{ where } L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}.$$

Multiplying out LU and setting the answer equal to A gives

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

$$\boxed{U_{11} = 1}, \qquad \boxed{U_{12} = 2}, \qquad \boxed{U_{13} = 4}.$$

Now consider the second row

$$L_{21}U_{11} = 3 \quad \therefore L_{21} \times 1 = 3 \quad \therefore \boxed{L_{21} = 3},$$

$$L_{21}U_{12} + U_{22} = 8 \quad \therefore 3 \times 2 + U_{22} = 8 \quad \therefore \boxed{U_{22} = 2},$$

$$L_{21}U_{13} + U_{23} = 14 \quad \therefore 3 \times 4 + U_{23} = 14 \quad \therefore \boxed{U_{23} = 2}.$$

$$L_{31}U_{11} = 2 \quad \therefore L_{31} \times 1 = 2 \quad \therefore \boxed{L_{31} = 2},$$

$$L_{31}U_{12} + L_{32}U_{22} = 6 \quad \therefore 2 \times 2 + L_{32} \times 2 = 6 \quad \therefore \boxed{L_{32} = 1},$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \quad \therefore (2 \times 4) + (1 \times 2) + U_{33} = 13 \quad \therefore \boxed{U_{33} = 3}$$

We have shown that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

and this is an LU decomposition of A.

Find an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$

Answer
Let

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}$$

then, comparing the left and right hand sides row by row implies that $U_{11} = 3$, $U_{12} = 1$, $L_{21}U_{11} = -6$ which implies $L_{21} = -2$ and $L_{21}U_{12} + U_{22} = -4$ which implies that $U_{22} = -2$. Hence

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$

is an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$

Answer

Using material from the worked example in the notes we set

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

and comparing elements row by row we see that

$$\begin{array}{lll} U_{11} = 3, & U_{12} = 1, & U_{13} = 6, \\ L_{21} = -2, & U_{22} = 2, & U_{23} = -4 \\ L_{31} = 0 & L_{32} = 4 & U_{33} = -1 \end{array}$$

and it follows that

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

Using LU decomposition to solve systems of equations

Once a matrix A has been decomposed into lower and upper triangular parts it is possible to obtain the solution to $AX = B$ in a direct way. The procedure can be summarized as follows

- Given A, find L and U so that $A = LU$. Hence $LUX = B$.
- Let $Y = UX$ so that $LY = B$. Solve this triangular system for Y .
- Finally solve the triangular system $UX = Y$ for X.

The benefit of this approach is that we only ever need to solve triangular systems. The cost is that we have to solve two of them.

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

Solution:

- The first step is to calculate the LU decomposition of the coefficient matrix on the left-hand side. In this case that job has already been done since this is the matrix we considered earlier. We found that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- The next step is to solve $LY = B$ for the vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. That is we consider

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = B$$

which can be solved by **forward substitution**. From the top equation we see that $y_1 = 3$. The middle equation states that $3y_1 + y_2 = 13$ and hence $y_2 = 4$. Finally the bottom line says that $2y_1 + y_2 + y_3 = 4$ from which we see that $y_3 = -6$.

Solution (contd.)

- Now that we have found Y we finish the procedure by solving $UX = Y$ for X. That is we solve

$$UX = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = Y$$

by using **back substitution**. Starting with the bottom equation we see that $3x_3 = -6$ so clearly $x_3 = -2$. The middle equation implies that $2x_2 + 2x_3 = 4$ and it follows that $x_2 = 4$. The top equation states that $x_1 + 2x_2 + 4x_3 = 3$ and consequently $x_1 = 3$.

Therefore we have found that the solution to the system of simultaneous equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \quad \text{is} \quad X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

Solve $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}$

Answer

We found earlier that the coefficient matrix is equal to $LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

First we solve $LY = B$ for Y, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

The top line implies that $y_1 = 0$. The middle line states that $-2y_1 + y_2 = 4$ and therefore $y_2 = 4$. The last line tells us that $4y_2 + y_3 = 17$ and therefore $y_3 = 1$.

Finally we solve $UX = Y$ for X, we have

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

The bottom line shows that $x_3 = -1$. The middle line then shows that $x_2 = 0$, and then the top line gives us that $x_1 = 2$. The required solution is $X = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

Do matrices always have an LU decomposition?

- No. Sometimes it is impossible to write a matrix in the form "lower triangular"×"upper triangular".

Why not?

- An invertible matrix A has an LU decomposition provided that all its leading submatrices have non-zero determinants. The k^{th} leading submatrix of A is denoted A^k and is the k ×k matrix found by looking only at the top k rows and leftmost k columns. For example if

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

then the leading submatrices are

$$A_1 = 1, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$|A_1| = 1,$$
$$|A_2| = (1 \times 8) - (2 \times 3) = 2,$$
$$|A_3| = \begin{vmatrix} 8 & 14 \\ 6 & 13 \end{vmatrix} - 2\begin{vmatrix} 3 & 14 \\ 2 & 13 \end{vmatrix} + 4\begin{vmatrix} 3 & 8 \\ 2 & 6 \end{vmatrix} = 20 - (2 \times 11) + (4 \times 2) = 6$$

The fact that this matrix A has an LU decomposition can be guaranteed in advance because none of these determinants is zero.

Do matrices always have an LU decomposition?

Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ does not have an LU decomposition.

Solution

The second leading submatrix has determinant equal to

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 2) = 0$$

which means that an LU decomposition is not possible in this case.

Which, if any, of these matrices have an LU decomposition?

(a) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$

(a) **Answer**

$|A_1| = 3$ and $|A_2| = |A| = 3$. Neither of these is zero, so A **does** have an LU decomposition.

(b) **Answer**

$|A_1| = 0$ so A **does not** have an LU decomposition.

Can we get around this problem?

Yes. It is always possible to re-order the rows of an invertible matrix so that all of the submatrices have non-zero determinants.

Reorder the rows of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Solution

Swapping the first and second rows does not help us since the second leading submatrix will still have a zero determinant. Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B_3 = B.$$

Now $|B_1| = 1$, $|B_2| = 3 \times 1 - 2 \times 1 = 1$ and (expanding along the first row)

$$|B_3| = 1(15 - 16) - 2(5 - 8) + 3(4 - 6) = -1 + 6 - 6 = -1.$$

All three of these determinants are non-zero and we conclude that B does have an LU decomposition.

Reorder the rows of $A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Answer

Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 3 & -2 \\ -2 & 6 & 1 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}, \quad B_3 = B$$

which have determinants 1, 3 and 45 respectively. All of these are non-zero and we conclude that B does indeed have an LU decomposition.

LU decomposition using Gauss Elimination method

Find LU Decomposition using Gauss Elimination method of Matrix ...

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Solution:

Here $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Using Gaussian Elimination method

$R_2 \leftarrow R_2 - \left(\frac{2}{3}\right) \times R_1 \left[\therefore L_{2,1} = \frac{2}{3} \right]$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & -\frac{2}{3} \\ 4 & 2 & 3 \end{bmatrix}$$

$R_3 \leftarrow R_3 - \left(\frac{4}{3}\right) \times R_1 \left[\therefore L_{3,1} = \frac{4}{3} \right]$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{7}{3} \end{bmatrix}$$

$R_3 \leftarrow R_3 - \left(\frac{1}{2}\right) \times R_2 \left[\therefore L_{3,2} = \frac{1}{2} \right]$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -2 \end{bmatrix}$$

L is just made up of the multipliers we used in Gaussian elimination with 1s on the diagonal.

$\therefore$ LU decomposition for A is

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{4}{3} & \frac{1}{2} & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -2 \end{bmatrix} = LU$$

LU decomposition using Gauss Elimination method

Find LU Decomposition using Gauss Elimination method of Matrix ...

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}$$

Solution:

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}$

Using Gaussian Elimination method

$R_2 \leftarrow R_2 - (-1) \times R_1 \left[\therefore L_{2,1} = -1 \right]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 2 & 4 & 4 \end{bmatrix}$$

$R_3 \leftarrow R_3 - (2) \times R_1 \left[\therefore L_{3,1} = 2 \right]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 2 \end{bmatrix}$$

$R_3 \leftarrow R_3 - (-1) \times R_2 \left[\therefore L_{3,2} = -1 \right]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

L is just made up of the multipliers we used in Gaussian elimination with 1s on the diagonal.

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$\therefore$ LU decomposition for A is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = LU$$

LU decomposition using Gauss Elimination method

Find LU Decomposition using Gauss Elimination method of Matrix ..

$$\begin{bmatrix} 2 & 3 \\ 4 & 10 \end{bmatrix}$$

Solution:

Using Gaussian Elimination method

$R_2 \leftarrow R_2 - (2) \times R_1 \left[\therefore L_{2,1} = 2 \right]$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

L is just made up of the multipliers we used in Gaussian elimination with 1s on the diagonal.

$$\therefore L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\therefore$ LU decomposition for A is

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = LU$$

Find LU Decomposition using Gauss Elimination method of Matrix .

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:?????

Solve the following system of equations using LU Decomposition (LU decomosition using GE) method:

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

Solution: Here, we have

A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ $, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ such that A X = C.

Now, we first consider $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ and convert it to row echelon form using Gauss Elimination Method.

So, by doing

$$R_2 \to R_2 - 4R_1 \qquad (1)$$

$$R_3 \to R_3 - 3R_1 \qquad (2)$$

we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$$

Now, by doing

$$R_3 \to R_3 - (-2)R_2 \qquad (3)$$

we get

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

Hence, we get L = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

(notice that in L matrix, $l_{21} = 4$ is from (1), $l_{31} = 3$ is from (2) and $l_{32} = -2$ is from (3))

Now, we assume Z $= \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ and solve L Z = C.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

So, we have $z_1 = 1,\ 4z_1 + z_2 = 6,\ 3z_1 - 2z_2 + z_3 = 4.$

Solving, we get $z_1 = 1$, $z_2 = 2$ and $z_3 = 5$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Therefore, we get $x_1 + x_2 + x_3 = 1,\ -x_2 - 5x_3 = 2$, $-10x_3 = 5.$

Thus, the solution to the given system of linear equations is $x_1 = 1$, $x_2 = 0.5$, $x_3 = -0.5$ and hence the matrix X = $\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$

Finding the inverse of a square matrix

The inverse $[B]$ of a square matrix $[A]$ is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix using LU Decomposition

How can LU Decomposition be used to find the inverse?

Assume the first column of $[B]$ to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of $[B]$

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of $[B]$

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in $[B]$ can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix $[A]$

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Given $[A][X] = [C]$

1.Decompose $[A]$ into $[L]$ and $[U]$

[L][U][X]=[C]

1.Solve $[L][Z] = [C]$ for $[Z]$ ([Z]=[U][X])

2.Solve $[U][X] = [Z]$ for $[X]$

Using the decomposition procedure, the $[L]$ and $[U]$ matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix (cont.)

Solving for the each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]
2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

This generates the equations:

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$

Example: Inverse of a Matrix (cont.)

Solving for $[Z]$

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56 z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}
$$

Example: Inverse of a Matrix (cont.)

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$\begin{aligned} 25b_{11} + 5b_{21} + b_{31} &= 1 \\ -4.8b_{21} - 1.56b_{31} &= -2.56 \\ 0.7b_{31} &= 3.2 \end{aligned}$$

Example: Inverse of a Matrix (cont.)

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of $[A]$ is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix (cont.)

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix (cont.)

The inverse of $[A]$ is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Inverse of Matrix using CROUT'S METHOD

Two-step procedure to find the inverse of a matrix A:

Step 1. Find the LU decomposition A = LU

Step 2. Find the inverse of $A^{-1} = U^{-1}L^{-1}$ by inverting the matrices U and L.

- **EXAMPLE FOR FINDING INVERSE OF A MATRIX BY CROUT'S METHOD**.
- Find the inverse of the matrix A by using crout's method where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & -2 \end{bmatrix}, U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

To find L^{-1} , LL^{-1} = I. Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_{11} = 1, x_{22} = -1, x_{33} = -1/2, x_{21} = 0, x_{11} + x_{21} + x_{31} = 0, x_{22} + x_{32} = 0,$$
$$x_{31} = -1, x_{32} = 1.$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1/2 \end{bmatrix}$$

To find U^{-1} ,we have $U U^{-1}$ =I.Therefore,

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{32} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating elements,

$$c_{12} - 2 = 0, c_{13} - 2c_{23} + 3 = 0, c_{23} - 4 = 0, c_{12} = 2, c_{23} = 4, c_{13} = 5.$$

$$U^{-1} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A=LU,$A^{-1}=(LU)^{-1}=U^{-1}L^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1/2 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -5/2 \\ -4 & 3 & -2 \\ -1 & 1 & -1/2 \end{bmatrix}$$

Inversion Of A Matrix -Gauss Elimination Method

- Consider a non singular matrix A of order 3.If X is the inverse of A then AX=I, where I is the unit matrix of order 3.Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} and X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Therefore AX=I reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{31} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This equation is equivalent to the nine equations given below:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{...(1)}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{...(2)}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From equations (1),(2)and (3), we can solve for the vectors

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} and \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

using Gauss elimination method. The solution set of each systems (1),(2),(3) will be the corresponding column of the inverse matrix X.

Now the system (1) is equivalent to AX=B, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, X = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} and B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now our aim is to reduce the augmented matrix (A:B) to an upper triangular matrix

$$(A:B)=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right]$$

Now we multiply the first row by $\frac{-a_{i1}}{a_{11}}$ and add to the i^{th} row of

(A:B) where i=2,3.By this all elements in the first column of (A:B) except a_{11} are made zero. Now (A:B) takes the form

$$(A:B)=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ 0 & b_{22} & b_{23} & c \\ 0 & b_{32} & b_{33} & d \end{array}\right]$$

Now take the pivot b_{22} . Now considering b_{22} as pivot we will make b_{32} in the second column of (A:B) as zero. Now multiply the second row by $\frac{-b_{32}}{b_{22}}$ and add to the corresponding element of the

3rd row. Now b_{32} is reduced to zero. Now (A:B) takes the form

$$(A:B)=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ 0 & b_{22} & b_{23} & c \\ 0 & 0 & c_{33} & e \end{array}\right]$$

Now we have the following three equations.

$$a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31}=1$$

$$b_{22}x_{21}+b_{23}x_{3}1=c$$

$$c_{32}x_{31}=e$$

By back substitution method, we get the values of x_{11},x_{21},x_{31}. }
Similarly we can solve the system of equations (2)&(3) and get the values of $x_{12},x_{22},x_{32},x_{13},x_{23},x_{33}$ .By this way we find the inverse of a given square matrix A by Gauss elimination method.

EXAMPLE

QN: By Gaussian elimination find the inverse of $A=\begin{bmatrix} 0 & 1 & 1 \\ 5 & 2 & 0 \\ 3 & -1 & -4 \end{bmatrix}$

$SOLUTION : Now (A:I) is$

$$(A:I)=\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

Since the element a_{11} is zero we will interchange the first and the second row. The reduced system is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

By performing $R_3+(-3)R_1$,we get

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right]$$

By performing R_3+7R_2 , we get

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right]$$

$$\left.\begin{array}{r} x_{11}+2x_{21}=0 \\ x_{21}+x_{31}=1 \\ 3x_{31}=7 \end{array}\right\} \Rightarrow \begin{array}{l} x_{31}=\frac{7}{3} \\ x_{21}=\frac{-4}{3} \\ x_{11}=\frac{8}{3} \end{array}$$

$$\left.\begin{array}{r} x_{12}+2x_{22}=1 \\ x_{22}+x_{32}=0 \\ 3x_{32}=-3 \end{array}\right\} \Rightarrow \begin{array}{l} x_{32}=-1 \\ x_{22}=1 \\ x_{12}=-1 \end{array}$$

$$\left.\begin{array}{r} x_{13}+2x_{23}=0 \\ x_{23}+x_{33}=0 \\ 3x_{33}=1 \end{array}\right\} \Rightarrow \begin{array}{l} x_{33}=\frac{1}{3} \\ x_{23}=\frac{-1}{3} \\ x_{13}=\frac{2}{3} \end{array}$$

Hence $A^{-1}=\begin{bmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ \frac{-4}{3} & 1 & \frac{-1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{bmatrix}$

$$S = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}.$$

Find inverse using guass elimination method

$$\left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 4 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right)$$

LU vs GE

- If you are solving a set of simultaneous linear equations, LU Decomposition method (involving forward elimination, forward substitution and back substitution) would use more computational time than Gaussian elimination (involving forward elimination and back substitution, but NO forward substitution).

Why use LU Decomposition?

- Because, LU Decomposition is computationally more efficient than Gaussian elimination when we are solving several sets of equations with the same coefficient matrix but different right hand sides.
- When you are finding the inverse of a matrix [A]. If one is trying to find the inverse of nxn matrix, then it implies that one needs to solve n sets of simultaneous linear equations of [A][X]=[C] form with the n right hand sides [C] being the n columns of the nxn identity matrix, while the coefficient matrix [A] stays the same.

Why use LU Decomposition?

- The computational time taken for solving a single set of *n* simultaneous linear equations is as follows:
  - **Forward elimination**: Proportional to $\frac{n^3}{3}$
  - **Back substitution**: Proportional to $\frac{n^2}{2}$
  - **Forward substitution**: Proportional to $\frac{n^2}{2}$
- So for LU decomposition method used to find the inverse of a matrix, the computational time is proportional to $\frac{n^3}{3} + n(\frac{n^2}{2} + \frac{n^2}{2}) = \frac{4n^3}{3}$

The forward elimination only needs to be done only once on [A] to generate the L and U matrices for the LU decomposition method. However the forward and back substitution need to be done *n* times.

Why use LU Decomposition?

- Now for Gaussian Elimination used to find the inverse of a matrix, the computational time is proportional to $n\frac{n^3}{3}+n\frac{n^2}{2}=\frac{n^4}{3}+\frac{n^3}{2}$
- Remember that both the forward elimination and back substitution need to be done *n* times.
- Hence for large *n*, for LU Decomposition, the computational time is proportional to $\frac{4n^3}{3}$, while for Gaussian Elimination, the computational time is proportional to $\frac{n^4}{3}$. So for large *n*, the ratio of the computational time for Gaussian elimination to computational for LU Decomposition is $\frac{n^4}{3}/\frac{4n^3}{3}=\frac{n}{4}$
- As an example, to find the inverse of a 2000×2000 coefficient matrix by Gaussian Elimination would take n/4=2000/4=500 times the time it would take to find the inverse by LU Decomposition.

Cramer’s Rule

- Cramer’s rule is one of the important methods applied to solve a system of equations. In this method, the values of the variables in the system are to be calculated using the determinants of matrices. Thus, Cramer’s rule is also known as the determinant method.

Cramer's Rule Formula

Consider a system of linear equations with n variables x_1, x_2, x_3, …, xn written in the matrix form AX = B.

Here,

A = Coefficient matrix (must be a square matrix)

X = Column matrix with variables

B = Column matrix with the constants (which are on the right side of the equations)

Now, we have to find the determinants as:

D = |A|, Dx_1, Dx_2, Dx_3,…, Dx_n

Here, Dx_i for i = 1, 2, 3,…, n is the same determinant as D such that the column is replaced with B.

Thus,

$x_1 = Dx_1/D$; $x_2 = Dx_2/D$; $x_3 = Dx_3/D$; ….; $x_n = Dx_n/D$ {where D is not equal to 0}

Cramer's Rule 2×2

Cramer's rule for the 2×2 matrix is applied to solve the system of equations in two variables.

Let us consider two linear equations in two variables.

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Let us write these two equations in the form of AX = B.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here,

$$Coefficient\ matrix = A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$Variable\ matrix = X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Constant\ matrix = B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

And

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

Therefore,

$\mathbf{x = D_x/D}$

$\mathbf{y = D_y/D}$

Cramer's Rule Example

Solve the following system of equations using Cramer's rule:
2x – y = 5
x + y = 4
Solution:
Given, 2x – y = 5, x + y = 4
Let us write these equations in the form AX = B.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Now,

D = |A|

$$= \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

= 2(1) – (-1)1

= 2 + 1

= 3 ≠ 0

So, the given system of equations has a unique solution.

$$D_x = \begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}$$

= 5(1) − (-1)(4)

= 5 + 4

= 9

$$D_y = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$$

= 2(4) − 5(1)

= 8 − 5

= 3

Therefore,

$x = D_x/D = 9/3 = 3$

$y = D_y/D = 3/3 = 1$

Cramer’s Rule 3×3

To find the Cramer’s rule formula for a 3×3 matrix, we need to consider the system of 3 equations with three variables.

Consider:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let us write these equations in the form AX = B.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now,

$$D = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Therefore, $x = D_x/D$, $y = D_y/D$, $z = D_z/D$; $D \neq 0$

Cramer’s Rule Example – 3×3

Solve the following system of equations using Cramer’s rule:
x + y + z =6
y + 3z = 11
x + z =2y or x – 2y + z = 0

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Now,

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(1+6) - 1(0-3) + 1(0-1) = 7 + 3 - 1 = 9$$

D ≠ 0 so the given system of equations has a unique solution.

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 11 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 6(1+6) - 1(11-0) + 1(-22-0) = 42 - 11 - 22 = 9$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 0 & 11 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 1(11-0) - 6(0-3) + 1(0-11) = 11 + 18 - 11 = 18$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 11 \\ 1 & -2 & 0 \end{vmatrix} = 1(0+22) - 1(0-11) + 6(0-1) = 22 + 11 - 6 = 27$$

Thus,

$x = D_x/D = 9/9 = 1$

$y = D_y/D = 18/9 = 2$

$z = D_z/D = 27/9 = 3$

Cramer's Rule for a System with Two Variables

- Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

Cramer's Rule for a System with Two Variables

- For this system, we have:

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

Cramer’s Rule for a System with Two Variables

- The solution is:

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

Cramer's Rule for a System of Three Variables

- Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \phantom{{}-3y} + 6z = 0 \\ 3x - 2y \phantom{{}+4z} = 5 \end{cases}$$

  - First, we evaluate the determinants that appear in Cramer's Rule.

Cramer's Rule for a System of Three Variables

Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

$\lvert D \rvert = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38$	$\lvert D_x \rvert = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$
$\lvert D_y \rvert = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22$	$\lvert D_z \rvert = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$

Cramer’s Rule for a System of Three Variables

- Now, we use Cramer’s Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19}$$

$$y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

Cramer’s Rule

- Solving the system in previous example using Gaussian elimination would involve matrices whose elements are fractions with fairly large denominators.

  – Thus, in cases like previous two exmples , Cramer’s Rule gives us an efficient way to solve systems of linear equations.

Limitations of Cramer's Rule

- However, in systems with more than three equations, evaluating the various determinants involved is usually a long and tedious task.
  - This is unless you are using a graphing calculator.

Limitations of Cramer's Rule

- Moreover, the rule doesn't apply if $|\, D \,| = 0$ or if D is not a square matrix.
  - So, Cramer's Rule is a useful alternative to Gaussian elimination—but only in some situations.

Cramer’s Rule Questions

- Solve the following system of equations by Cramer’s rule: $2x – 3y + 5z = 113x + 2y – 4z = – 5x + y – 2z = – 3$
- Solve the following system of linear equations using Cramer’s rule: $5x + 7y = -24x + 6y = -3$
- The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat, and 3 kg rice is Rs 70. Find the cost of each item per kg by Cramer’s rule.

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