Analysis of Algorithms

SY Computer

Introduction

Algorithm:

An Algorithm is a finite sequence of instructions, each of which has a clear meaning and can be performed with a finite amount of effort in a finite length of time.

We represent algorithm using a pseudo language that is a combination of the constructs of a programming language together with informal English statements.

Every algorithm must satisfy the following criteria:

- **Input:** there are zero or more quantities, which are externally supplied;
- **Output:** at least one quantity is produced
- **Definiteness:** each instruction must be clear and unambiguous;
- **Finiteness:** if we trace out the instructions of an algorithm, then for all cases the algorithm will terminate after a finite number of steps;
- **Effectiveness:** every instruction must be sufficiently basic that it can in principle be carried out by a person using only pencil and paper. It is not enough that each operation be definite, but it must also be feasible.

Performance Analysis

- The performance of a program is the amount of computer memory and time needed to run a program.

1. Time Complexity
2. Space Complexity

- How to compare Algorithms?

1. Execution time
2. Number of statements executed
3. Running time Analysis

Time Complexity

The time needed by an algorithm expressed as a function of the size of a problem is called the time complexity of the algorithm.

The time complexity of a program is the amount of computer time it needs to run to completion.

Time Complexity is mainly of 3 Types:

1. Best Case
2. Worst Case
3. Average Case

Space Complexity

- The space complexity of a program is the amount of memory it needs to run to completion. The space need by a program has the following components:
- Instruction space: Instruction space is the space needed to store the compiled version of the program instructions.
- Data space: Data space is the space needed to store all constant and variable values.
- Environment stack space: used to save information needed to resume execution of partially completed functions.
- The space requirement S(P) of any algorithm P may therefore be written as,

  $S(P) = c + S_p(\text{Instance characteristics})$

  where "c" is a constant.

Complexity of Algorithms

- The complexity of an algorithm M is the function f(n) which gives the running time and/or storage space requirement of the algorithm in terms of the size “n” of the input data.

- Approaches to calculate Time/Space Complexity:
  1. Frequency count/Step count Method
  2. Asymptotic Notations – (Order of)

Frequency count/Step count Method

Rules:

1. For comments, declaration

   count = 0

2. return and assignment statement

   count = 1

3. Ignore lower order exponents when higher order exponents are present

   Ex. Complexity of following algo is as follows:

   $f(n) = 6n^3 + 10n^2 + 15n + 3 \Longrightarrow 6n^3$

4. Ignore constant multipliers

   $6n^3 \Longrightarrow n^3$

   $f(n) = O(n^3)$

Example 1: sum of n values of an array

Algorithm sum (int a[], int n	Time Complexity	Space Complexity
`s = 0;`	1	a[] = n words
`for(i=0; i<n; i++)`	n+1	n= 1 word
`{`		s= 1 word
`s=s + a[i];`	n	i= 1 word
`}`		n+3
`return s;`	1	
	2n+3	
	f(n) = O(n)	Space complexity = O(n)

Example 2: Addition of two square Matrices of dimension $n \times n$

```
Algorithm addMat (int a[][], int b[][])
{ int c[][];
   for(i=0; i<n; i++) {
      for(j=0; j<n; j++) {
       c[i][j] = a[i][j] + b[i][j]
    }
  }
}
```

Time Complexity
$n+1$
$n \times (n+1)$
$n \times n$
$n+1+n^2+n+n^2$ $2n^2+2n+1$ $f(n) = O(n^2)$

Space Complexity
a[][] = n^2 words
b[][] = n^2 words
c[][] = n^2 words
i= 1 word
j= 1 word
n= 1 word
$3n^2+3$ Space complexity = $O(n^2)$

Example 3: Multiplication of two Matrices of dimension $n\times n$

```
Algorithm matMul (int a[][], int b[][])
{ int c[][];
  for(i=0; i<n; i++) {
     for(j=0; j<n; j++) {
      c[i][j] = 0;
      for(k=0; k<n; k++){
           c[i][j] += a[i][k] * b[k][j]
     }
   }
  }
}
```

Time Complexity
□ $n+1$
□ $n\times(n+1)$
□ $n\times n$
□ $n\times n\times(n+1)$
$n\times n\times n$
□ $n+1+n^2+n+n^2+n^3+1+n^3+n^2$ □ $2n^3+3n^2+2n+1$ $f(n) = O(n^3)$

Space Complexity
a[][] = n^2 words
b[][] = n^2 words
c[][] = n^2 words
i= 1 word
j= 1 word
k=1 word
n= 1 word
□ $3n^2+4$ Space complexity = $O(n^2)$

Example: loops

1.

```
for(i=0; i<n; i++) {
        statements;
}
```

Time Complexity
? n+1
? n
$f(n) = 2n+1$ $f(n) = O(n)$

2.

```
for(i=n; i>0; i--) {
        statements;
}
```

Time Complexity
? n+1
? n
$f(n) = 2n+1$ $f(n) = O(n)$

Example: loops

3.

```
for(i=1; i<n; i=i+2) {
      statements;
}
```

Time Complexity
⍰ $n+1$
⍰ $n/2$
$f(n) = 3n/2+1$ $f(n) = O(n)$

4.

```
for(i=0; i<n; i++) {
      for(j=0; j<n; j++) {
            statements;
      }
}
```

Time Complexity
⍰ $n+1$
$n(n+1)$
⍰$n \times n$
$f(n) = 2n^2+2n+1$ $f(n) = O(n^2)$

Example: loops (By tracing)

5.

```
for(i=0; i<n; i++) {
    for(j=0; j<i; j++) {
        statements;
    }
}
```

$$1 + 2 + 3 + 4 + \cdots + n = n(n+1)/2$$

$$T(n) = 1 + 2 + 3 + 4 + \cdots + n - 1 = \frac{(n-1)(n)}{2} = O(n^2)$$

Time Complexity		
i	**j**	**statements**
0	0	0
1	0	1
	1	
2	0	2
	1	
	2	
3	0	3
	1	
	2	
	3	
…	…	…
N	0 to n-1	n

Example 5: loops (By tracing)

6.

```
p=0 ;
  for(i=1; p<=n; i++) {
        p=p+i;
   }
}
```

Time Complexity		
i	**p**	**statements**
1	0+1	1
2	1+2	1
3	1+2+3	1
4	1+2+3+4	1
5	1+2+3+4+5	1
6	1+2+3+4+5+6	1
k	1+2+3+4+…+k	???

$=1+2+3+4+...+k>n$

$= \; k(k+1)/2 > n$

$= \; k^2 + k/2 \; > n$

$\cong \; k^2 > n$

$$\boldsymbol{k} = \sqrt{\boldsymbol{n}} \; = \; \boldsymbol{O}(\boldsymbol{n})$$

Example: loops (By tracing)

6.

```
for(i=1; i<n; i=i*2) {
    statements;
 }
}
```

$$i >= n$$
$$i = 2^k$$
$$2^k >= n$$
$$2^k = n$$

$$\boldsymbol{k = log_2 n \;=\; O(log_2 n)}$$

Time Complexity	
i	**statements**
$1*2^0$	1
$1*2$	1
$1*2*2$	1
$1*2*2*2$	1
…	…
…	…
2^k	1

Example: loops (By tracing)

7.

```
for(i=n; i>=1; i=i/2) {
    statements;
  }
}
```

$$i<1$$
$$n/2^k = 1$$
$$n=2^k$$
$$\boldsymbol{k = log_2 n = O(log_2 n)}$$

Time Complexity
i
n
$n/2$
$n/2^2$
$n/2^3$
$n/2^4$
…
$n/2^k$

Example: loops (By tracing)

8.

```
for(i=0; i*i<n; i++) {
    statements;
  }
}
```

$$k * k >= n$$

$$k^2 = n$$

$$k=\sqrt{n}$$

$$\boldsymbol{k = \sqrt{n} = O(\sqrt{n})}$$

Time Complexity	
i	**statements**
1	1
2	2^2
3	3^2
4	4^2
5	5^2
…	…
k	k^2

Example: loops (By tracing)

9.

```
for(i=1; i<n; i=i*2) {
        p++;
 }
for(j=1;j<p;j=j*2){
    statements;
}
```

$$p = \log_2 n$$

$$T(n) = \log_2 p$$

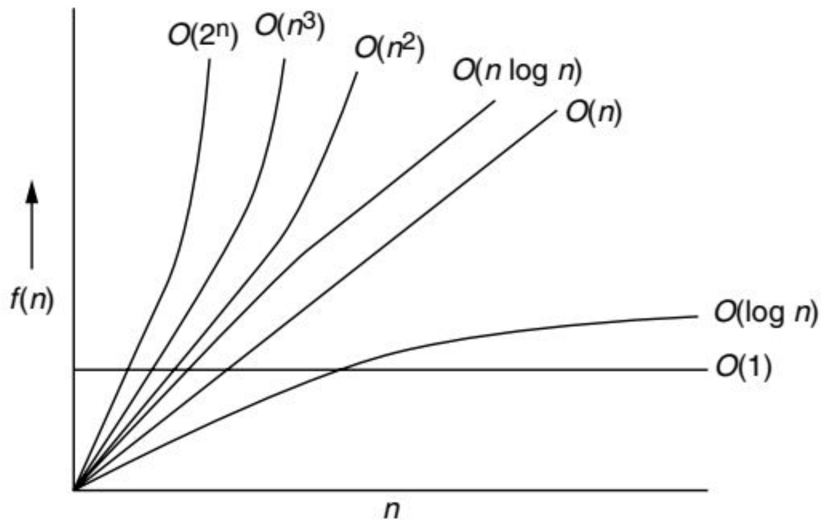
$$T(n) = \log_2 \log_2 n$$

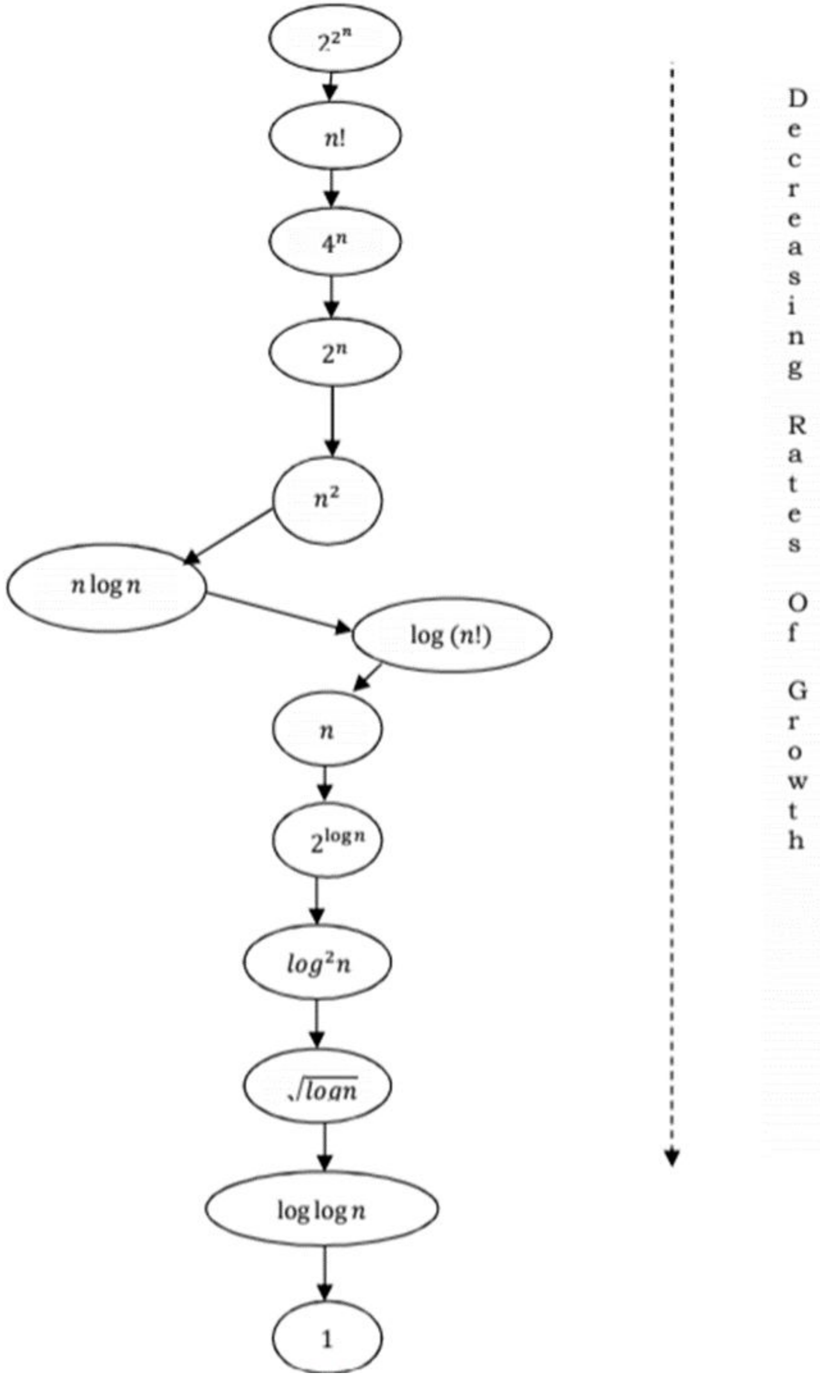
Rate of Growth

- Rate at which the running time increases as a function of input is called Rate of Growth.
- Example:

$$n^4 + 2n^2 + 100n + 500 \approx n^4$$

n^4 , $2n^2$, $100n$ *and* 500 *are individual cost of some functions and approximate to* n^4 *since* n^4 *is highest rate of growth.*





Numerical Comparison of Different Algorithms

n	log2 n	n*log2n	n^2	n^3	2^n
1	0	0	1	1	2
2	1	2	4	8	4
4	2	8	16	64	16
8	3	24	64	512	256
16	4	64	256	4096	65,536
32	5	160	1024	32,768	4,294,967,296
64	6	384	4096	2,62,144	Note 1
128	7	896	16,384	2,097,152	Note 2
256	8	2048	65,536	1,677,216	????????

Asymptotic Notations:

- Asymptotic notations have been developed for analysis of algorithms.
- By the word asymptotic means “for large values of n”
- The following notations are commonly use notations in performance analysis and used to characterize the complexity of an algorithm:

1. Big–OH(O)
2. Big–OMEGA(Ω),
3. Big–THETA (Θ)

Big O notation:

- This notation gives the tight upper bound of the given function
- Represented as:

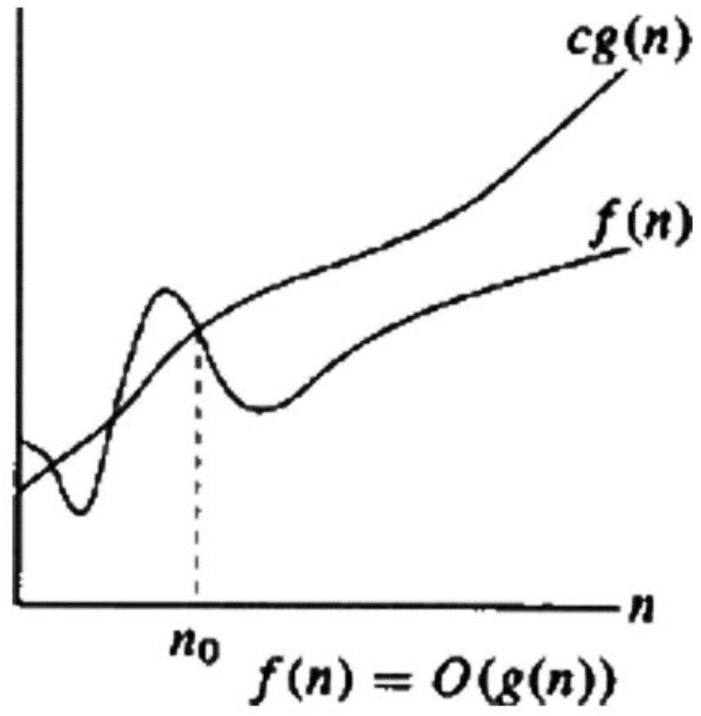
$$f(n) = O(g(n))$$

that means, at larger values of n, upper bound of $f(n)$ is $g(n)$.

Definition:

Big O notation defined as $O(g(n)) =$ $\{f(n)$: there exist positive constants c and no such that

$$0 <= f(n) <= c.g(n) for\ all\ n > n_0\}$$

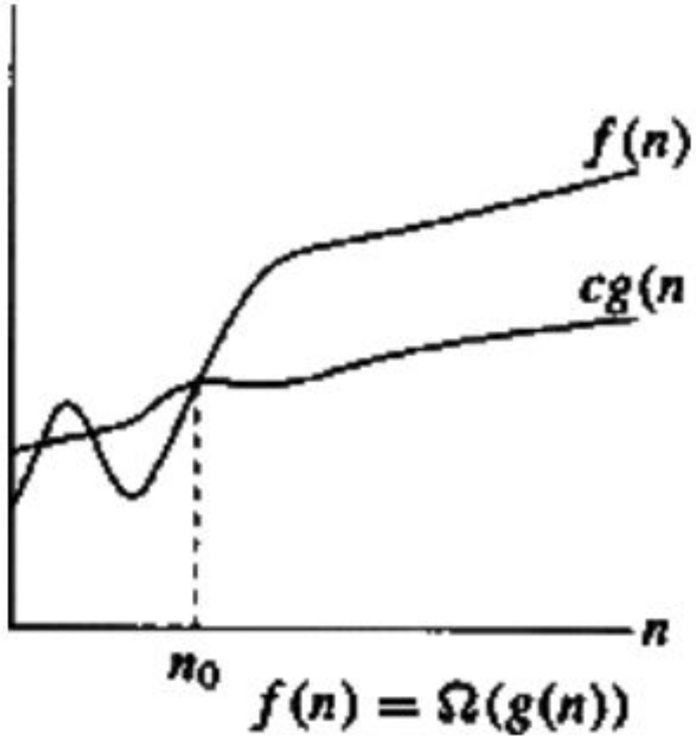


Big Omega (Ω) notation:

- This notation gives the tight lower bound of the given function
- Represented as:

$$f(n) \; = \Omega(g(n))$$

  that means, at larger values of n, lower bound of $f(n)$ is $g(n)$.



Definition:

Big Ω notation defined as $\Omega(g(n))$ = $\{f(n)$: there exist positive constants c and no such that

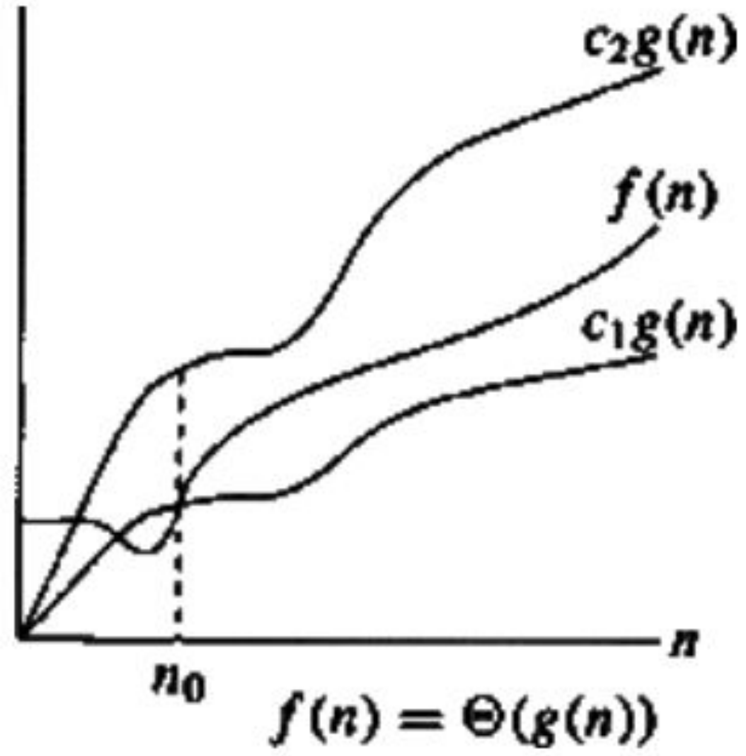
$0 <= c.g(n) \leq= f(n) \quad for\, all\, n > n_0\}$

Big Theta (θ) Notation:

- Average running time of an algorithm is always between lower bound and upper Bound
- Represented as:

$$f(n) \ = \theta(g(n))$$

  that means, at larger values of n, lower bound of $f(n)$ is $g(n)$.



Definition:

Bigθ notation defined as $\theta(g(n))$ = $\{f(n)$: there exist positive constants c_1 and c_2 and n_o such that

$$0 \leq c_1. g(n) \leq f(n) \leq c_1. g(n)$$
$$for \ all \ n > n_0\}$$

Properties of Asymptotic Notations:

1. Transitivity:

   $f(n) = \theta(g(n))$ & $g(n) = \theta(h(n))$

   ➔ $f(n) = \theta(h(n))$

   Valid for O and Ω as well.

2. Reflexivity:

   $f(n) = \theta(f(n))$

   Valid for O and Ω as well.

3. Symmetry:

   $f(n) = \theta(g(n))$, iff $g(n) = \theta(f(n))$

4. Transpose Symmetry:

   $f(n) = \theta\big(g(n)\big)$ iff $g(n) = \Omega(f(n))$

Examples:

----

1. f(n) = n & g(n) = n^2 & h(n)=n^3

   n = O(n^2) ; n2=O(n^3),
   then n = O(n^3)

----

2. f(n) = n^3 = O(n^3) = θ(n^3) = Ω(n^3)

----

3. f(n) = n^2 & g(n) = n^2
   then, f(n) = θ(n^2)

----

4. f(n) = n & g(n) = θ(n^2)
   then n = O(n^2) & n^2= Ω(n)

Properties of Asymptotic Notations:

Observations:

1. If $f(n) = O(kg(n))$ for any constant $k>0$, then $f(n) = O(g(n))$

2. If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $(f_1 + f_2)(n) = O(max(g_1(n), g_2(n))$

3. If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n)\, f_2(n) = O(g_1(n).\, g_2(n))$

4. If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then $f(n) = \theta(g(n))$

Recursion:

- Recursion is an ability of an algorithm to repeatedly call itself until a certain condition is met.
- Such condition is called the base condition.
- The algorithm which calls itself is called a recursive algorithm.
- The recursive algorithms must satisfy the following two conditions:
  1. It must have the base case: The value of which algorithm does not call itself and can be evaluated without recursion.
  2. Each recursive call must be to a case that eventually leads toward a base case.

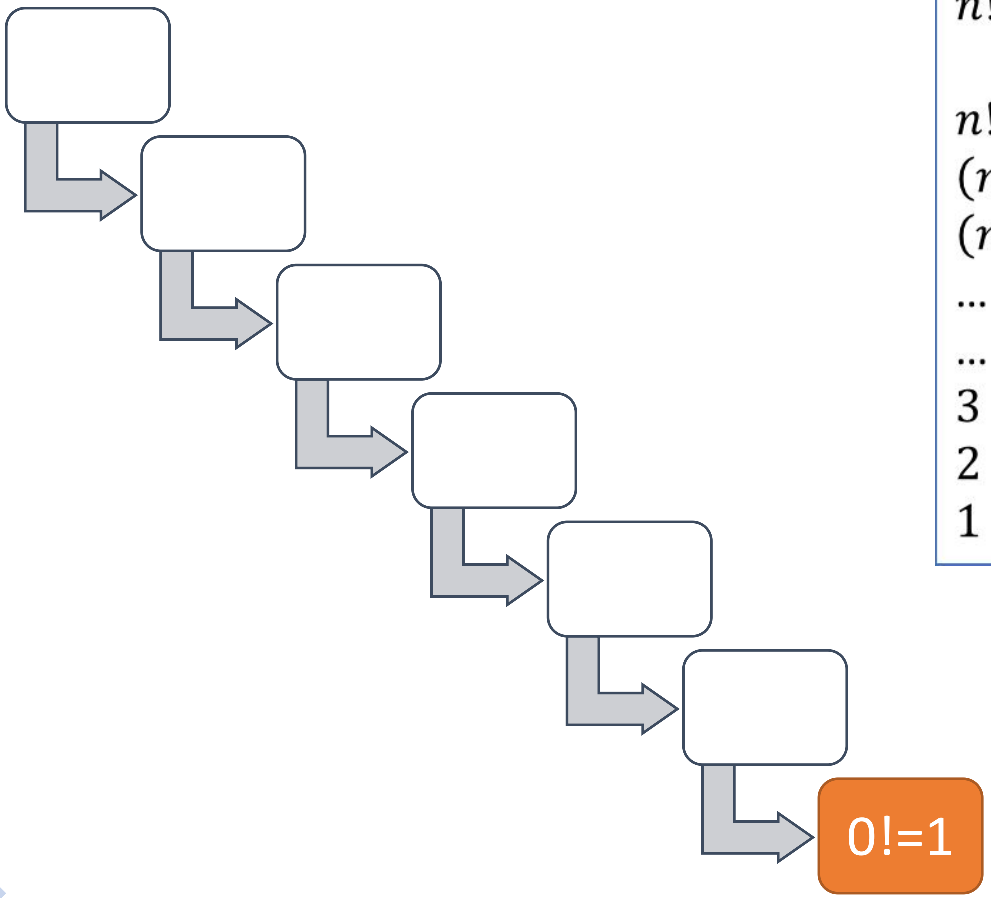
Recursion:

Recurrence Relation:

- An algorithm is said to be recursive if it can be defined in terms of itself.
- The running time of recursive algorithm is expressed by means of recurrence relations.
- A recurrence relation is an equation of inequality that describes a function in terms of its value on smaller inputs.
- It is generally denoted by $\boldsymbol{T(n)}$ where $\boldsymbol{n}$ is the size of the input data of the problem.
- The recurrence relation satisfies both the conditions of recursion, that is, it has both the base case as well as the recursive case.
  - The portion of the recurrence relation that does not contain T is called the base case of the recurrence relation and
  - The portion of the recurrence relation that contains T is called the recursive case of the recurrence relation.

$$\boldsymbol{T(n)} = \begin{cases} \boldsymbol{d} & ;\boldsymbol{n = 1} \\ \boldsymbol{T(n-1) + c} & ;\boldsymbol{n > 1} \end{cases}$$

Example: Factorial



$$n! \; = \; n * (n-1) * (n-2) * (n-3) * \cdots 3 * 2 * 1$$

$$n! \; = \; n * (n-1)$$
$$(n-1)! = (n-1) * (n-2)\,!$$
$$(n-2)! = (n-2) * (n-3)\,!$$
…

…
$$3\,! \; = \; 3 * 2\,!$$
$$2\,! \; = \; 2 * 1\,!$$
$$1\,! \; = \; 1 * 0\,! \; = \; 1 * 1 \; = \; 1$$

Factorial Algorithm:

Recursive Method:

```
Algorithm fact(n)
If (n<0) then return("error");
    else
If (n<2) then return(1);
    else
Return (n*fact(n-1);
End if
End if
End fact
```

$$T(n) = \begin{cases} d & ; n = 1 \\ T(n-1) + c & ; n > 1 \end{cases}$$

Iterative Method:

```
Algorithm fact(n)
If (n<0) then return("error");
    else
If (n<2) then return(1);
    else
prod=1;
End if
End if
For(i=n down to 0)
    do prod=prod*i
    end for
Return prod
End fact
```

Recursion:

Recurrence Relation:

There are various methods to solve recurrence:

1. Iterative Method / Substitution Method
2. Recurrence Tree
3. Master Method/ Master's Theorem

Recursion:

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Recursion:

Recurrence Relation:

There are various methods to solve recurrence:

1. Iterative Method / Substitution Method
2. Recurrence Tree
3. Master Method/ Master’s Theorem

Recursion:

Master's Theorem:

- Dividing Functions
- Decreasing Functions

2. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

$$T(n) = \begin{cases} aT\left(\frac{n}{b}\right) + f(n) & n > 1 \\ \theta(1) & n = 1 \end{cases}$$

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

$$T(n) = \begin{cases} aT\left(\frac{n}{b}\right) + f(n) & n > 1 \\ \theta(1) & n = 1 \end{cases}$$

Where,

$$f(n) = \Theta\left(n^k \log^p n\right)$$

and

$$a \geq 1 \quad ; \quad b > 1 \quad ; \quad k \geq 0$$

and p is a real number

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

$$T(n) = \begin{cases} aT\left(\frac{n}{b}\right) + f(n) & n > 1 \\ \theta(1) & n = 1 \end{cases}$$

Case 1: If $a > b^k$ or $\log_b a > k$

then, $T(n) = \Theta\left(n^{\log_b a}\right)$

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

Case 2: If $a = b^k$ or $\log_b a = k$

then,

A.] If $p > -1$, then

$$T(n) = \Theta\left(n^{\log_b a} \log^{p+1} n\right) \Rightarrow \theta\left(n^k \log^{p+1} n\right)$$

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

Case 2: If $a = b^k$ or $\log_b a = k$

then,

B.]. If $p = -1,$ then

$$T(n) = \Theta\left(n^{\log_b a} \log\log n\right) \Rightarrow \theta\left(n^k \log\log n\right)$$

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

Case 2: If $a = b^k$ or $\log_b a = k$

then,

C.] If $p < -1$, then

$$T(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow \theta\left(n^k\right)$$

Recursion:

1. Dividing functions:

Master's method (for Dividing Functions) provides general method for solving recurrences of the form:

Case 3: If $a < b^k$ or $\log_b a < k$

A.] If $p \geq 0$ then

$$T(n) = \Theta\left(n^{\log_b a} \log^p n\right) \Rightarrow \theta\left(n^k \log^p n\right)$$

B.] If $p < 0$ then

$$T(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow \theta\left(n^k\right)$$

Master's Theorem:

1. $T(n) = 2T(\sqrt{n}) + \log n$

Exercise:

1. $T(n) = 2T\left(\frac{n}{2}\right) + n^3 \log n$

2. $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n^3}{log^2 n}$

Master's Theorem (for Decreasing Function)

Let T(n) be a function defined on positive n

$$T(n) = \begin{cases} c & if\ n \le 1 \\ aT(n-b) + f(n) & if\ n > 1 \end{cases}$$

for some constants c, a>0, b>0, k

where, $f(n) = O(n^k)$ then

$$\begin{aligned} T(n) &= O(n^k) && if\, a < 1 \\ &= O(n^{k+1}) && if\, a = 1 \\ &= O\left(n^k \cdot a^{\frac{n}{b}}\right) && if\, a > 1 \end{aligned}$$

Recursion:

Recurrence Relation:

There are various methods to solve recurrence:

1. Iterative Method / Substitution Method
2. Master Method/ Master's Theorem
3. Recurrence Tree Method

Recursion:

Recurrence Tree Method:

- Recurrence is converted into a tree
- Sum of cost of various nodes at each level is calculated, called pre-level cost then sum of pre-level cost is obtained to determine the total cost of all levels of recursion tree.
- If recurrence relation is:

No. of subproblems

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Cost incurred for dividing and combining

Size of subproblem

Then, $f(n)$ is the root of tree, and each node should have 'a' children and Size of each child node is $\frac{1}{b}$ of parent node.