NORMAL DISTRIBUTION

1. Definition: A continuous random variable X is said to follow normal distribution with parameter m (called mean) and σ^2 (Called variance), if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < x < \infty,\ \ -\infty < m < \infty,\ \ \sigma^2 > 0$$

Remarks:

(i) A continuous random variate X following normal distribution with mean m and standard deviation σ is referred to as $\boldsymbol{X \sim N(m, \sigma)}$

(ii) If X is a normal variate with parameter m, σ then $Z = \frac{X-m}{\sigma}$ is also a normal variate with mean $= 0$ and standard deviation $= 1$. It is called **Standard Normal Variate**. **(S.N.V.)**

(iii) Mean, median and mode of the normal distribution are equal to m

(iv) Mean deviation about mean is $M.D = \frac{4}{5}\sigma$

(v) First quartile $Q_1 = m - \frac{2}{3}\sigma$ and third quartile $Q_3 = m + \frac{2}{3}\sigma$ $\therefore$ Quartile Deviation= $\frac{Q_3 - Q_1}{2} = \frac{2}{3}\sigma$

2. Linear Combination (Additive Property):

(i) Let $X_i, i = 1,2,3, \ldots.. n$ be n independent normal variates with mean m_i and variance σ_i^2.
Let their linear combination be denoted by Y i.e. $Y = a_1X_1 + a_2X_2 + \ \ldots \ldots + a_nx_n$
Then Y is also a normal variate with mean m and variance σ^2
where $m = a_1m_1 + a_2m_2 + \ \ldots a_nm_n$ and $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ \ldots. + a_n^2\sigma_n^2$

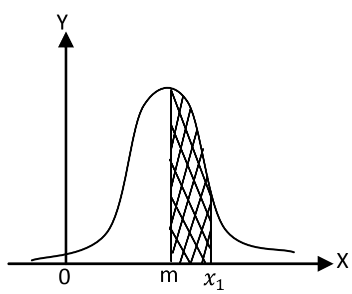
(ii) $Y = X_1 + X_2$ is normal variate with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$

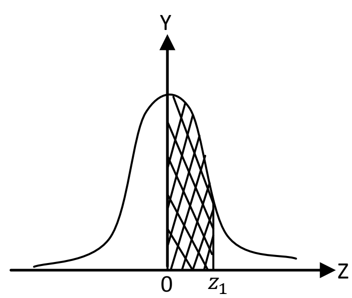
(iii) $Y = X_1 - X_2$ is also a normal variate with mean $m_1 - m_2$ and variance $\sigma_1^2 + \sigma_2^2$

(iv) Comparing Normal Distribution with Poisson Distribution we find that sum of two Normal or Poisson Variates is a Normal or Poisson variate, But although difference of two normal variates is a normal variate, the difference of two Poisson variates is not a Poisson variate.

3. Area Property:

(i) If X is a normal variate with mean m and variance σ^2 and Z is standard normal variate (with mean zero and variance one) then the area under the normal curve of X between $X = m$ and $X = x_1$ is equal to the area under the S.N Curve of Z between $Z = 0$ to $Z = z_1$ (say, corresponding to x_1)





(ii) Since standard normal curve is symmetrical about the y – axis it is enough to find the areas to the right. The areas to the left of y – axis at equal distance will be equal

(iii) The total area under the curve is unity. Hence, because of symmetry the area under S.N.V to the right of the y – axis is 0.5

(iv) To find the probability that X – will be lie between x_1 and x_2 $(x_1 < x_2)$, we find the corresponding values of S.N.V. Z (from $Z = \frac{X-m}{\sigma}$) say z_1 and z_2 and find the area from z_1 and z_2 under the S.N. curve. The required probability is this area.

$P(x_1 \leq X \leq x_2) = P(z_1 \leq Z \leq z_2) =$ area between $Z = z_1$ and $Z = z_2$ under the S.N. Curve

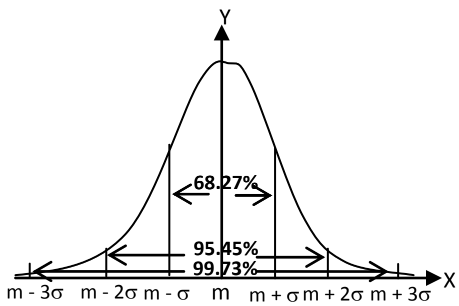
(v) The area under the normal curve is distributed as follows

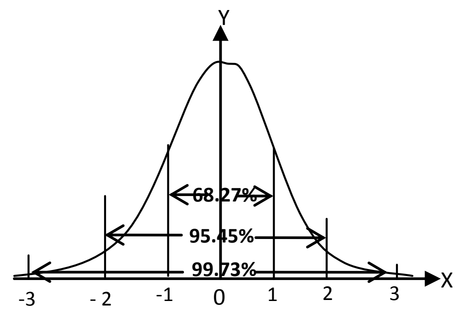
(a) The area between $x = m - \sigma$ and $x = m + \sigma$ is 68.27%

(b) The area between $x = m - 2\sigma$ and $x = m + 2\sigma$ is 95.45%

(c) The area between $x = m - 3\sigma$ and $x = m + 3\sigma$ is 99.73%

These areas under the normal curve and standard normal curve are shown below.





4. Normal approximation to the Binomial Distribution:

It can be proved, that if X is a Binomial variate with parameter n and p (i.e mean = np and $S.D = \sqrt{npq}$ where $= 1 - p$) then $Z = \frac{X-np}{\sqrt{npq}}$ is a standard Normal Variate if $n \to \infty$ (i.e is large) and neither p nor q is small.

Remarks:

(i) Normal distribution can be used in place of binomial distribution when np and nq are both greater than 15.

(ii) Normal Distribution can also be obtained from Poission distribution when the parameter $m \to \infty$.

EXERCISE

1. Find k and the mean and standard deviation of the normal distribution given by $y = k\, e^{-\left(\frac{x^2}{18} - x + \frac{9}{2}\right)}$
2. Write down the equation of the normal curve with mean 10 and variance 36. What is the quartile deviation of the distribution ?
3. For a normal distribution the mean is 50 and the standard deviation is 15. Find (i) Q_1 and Q_3, (ii) mean deviation & the interquartile range

4. If X is normally distribution with mean 10 and standard deviation 2, find $P(-3 \leq X \leq 12)$ and $P(|x| \geq 5)$

5. If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that **(i)** $26 \leq X \leq 40$, **(ii)** $X \geq 45$

6. If Z is a standard normal variate, find c such that **(i)** $P(-c < Z < c) = 0.95$, **(ii)** $P(|Z| > c) = 0.01$
If X is a normal variate with the mean 120 and standard deviation 10, find c such that **(i)** $P(X > c) = 0.02$ **(ii)** $P(X < c) = 0.05$

7. If X is a normal variate with mean 30 and standard deviation 6, find the value of $X = \mathrm{x}_1$ such that $\mathrm{P(X \geq x_1) = 0.05}$.

8. If X is a normal variate with mean 25 and standard deviation 5, find the value of $X = x_1$ such that $P(X \leq x_1) = 0.01$.

9. For a normal distribution the first quartile is 46 and the variance is 144. Find
(i) mode, **(ii)** limits of central 50% items, **(iii)** mean deviation.

10. The mean and the standard deviation of a normal distribution are 70 and 15. Find the quartile deviation and mean deviation.

11. The first and the third quartiles of a normal distribution are 36 and 44. Find the mean, standard deviation and the mean deviation

12. The marks obtained by students in a college are normally distribution with mean 65 and variance 25. If 3 students are selected at random from this college what is the probability that at least one of them would have scored more than 75 marks ?

13. The weights of 4000 students are found to be normally distributed with mean 50 kgs. and standard deviation 5 kgs. Find the probability that a student selected at random will have weight
(i) less than 45 kgs. **(ii)** between 45 and 60 kgs.

14. The sizes of 10, 000 items are normally distribution with mean 20 cms and standard deviation 4 cm. Find the probability that an item selected at random will have size between **(i)** 18cms and 23 cms, **(ii)** above 26 cms.

15. Mean and standard deviation of chest measurements of 1200 soldiers are 85cms and 5cms respectively. How many of them are expected to have their chest measurements exceeding 95cms. assuming the measurements to follow the normal distribution ?

16. The life of army shoes is normally distribution with mean 8 months and standard deviation 2months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months ?

17. The height of 22 year old boys is distributed normally with mean 63'' and standard deviation 2.5'' A boy is eligible if his height is between 62'' and 56''. Find the expected number of boys out of 180 who will be ineligible because of excess height.

18. If the heights of 500 students is normally distributed with mean 68 inches and standard deviation 4 inches, estimate the numbers of students having heights **(i)** greater than 72 inches, **(ii)** less than 62 inches, **(iii)** between 65 and 71 inches.

19. In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with as average of life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for **(i)** more than 2150 hours. **(ii)** less than 1950 hours.

20. The mean I. Q. of a large number of children of age 14 is 100 with S. D. 16. Assuming the distribution of I. Q. to be normal, find the percentage of the children having I. Q. between 70 and 120.

21. The average selling price of house in a city is Rs. 50, 000 with standard deviation of Rs. 10,000. Assuming the distribution of selling price to be normal find **(i)** the percentage of house that sell for more than Rs. 55,000, **(ii)** the percentage of houses selling between Rs. 45,000 and Rs. 60,000.
(Area between t = 0 and t = 1 is 0.3413 and between t = 0 and t = 0.5 is 0.1915).

22. A large number of automobile batteries have average life of 24 months. If 34 percent of them average between 22 and 26 months and 272 of them last longer than 29 months how many were in the group tested ? Assume the distribution to be normal.

23. In a factory a large number of workers have average daily income of Rs. 120. If 38.3% of them have income between Rs. 100 – 140 and 528 of them get more than Rs. 170, how many workers were interrogated ?
(Area for S. N. V. between z = 0 and z =19.15 is 0.5 and that between z = 0 and z = 1.25 is 0.3944).

24. The income distribution of workers in a certain factory was found to be normal with mean of Rs. 500 and standard deviation equal to Rs. 50. There were 228 persons above Rs. 600. How many persons were there in all ? (Area under the S. N. curve between height at 0 and 2 is 0.4772).

25. The arithmetic mean of the weights of a group of boys is 105 lbs with standard deviation of 5 lbs. If there were 456 boys having weights more than 115 lbs, how many students were there in the group ?
(Given : For S. N. V. z area from z = 0 to z = 2 is 0.4772).

26. Monthly salary X in a big organization is normally distributed with mean Rs. 3000 and standard deviation of Rs. 250. What should be the minimum salary of a worker is this organization, so that the probability that he belongs to top 5% workers ?

27. The heights of 1000 cakes baked with certain mix have a normal distribution with a mean of 5.75 cms. and standard deviation of 0.75 cms. Find the numbers of cakes having heights between 5 cms and 6.25 cms. Also find the maximum height of the flattest 200 cakes.

28. Monthly salaries of 1000 workers have a normal distribution with mean of Rs. 575 and a standard deviation of Rs. 75. Find the numbers of workers having salaries between Rs. 500 and Rs. 625 p.m. Also find the minimum salary of the highest paid 200 workers.

29. Assuming that the diameters of 100 brass plugs taken consecutively from a Normal distribution with mean 0.7515 cm. an standard deviation 0.0020cm. how many plugs are likely to be rejected if the approved diameter is 0.752 $\pm$ 0.004 cms ?

30. The diameters of can tops produced by a machine are normally distributed with standard deviation of 0.01 cms. At what mean diameter the machine be set so that not more than 5% of the can tops produced by the machine have diameters exceeding 3 cms ?

31. Find the mean and the standard deviation of a normal distribution of marks in an examination where 58% of the candidates obtained marks below 75, 4% got above 80 and the rest between 75 and 80
(For a S. N. V. the area under the curve between $z = \pm 0.2$ is 0.16 and between $z = \pm 1.8$ is 92)

32. In a distribution exactly normal 7% of items are under 35 and 89% are under 63. What are the mean and standard deviation ?

33. If X_1 and X_2 are two independent random variates with means 30 and 25 variances 16 and 12 and if $Y = 3X_1 - 2X_2$, find $P(60 \leq Y \leq 80)$.

34. Two independent random variates X and Y have normal distributions with means 46 and 45 and standard deviations 2 and 2.5 respectively. Find the probability that a randomly chosen pair of values of X and Y will

(i) differ by 1.5 or more **(ii)** add up to 100 or more

35. If X and Y are two independent random variates $N(3, 4)$ and $N(8, 5)$, find the probability that a point (X, Y) will lie between the lines $5X + 3Y = 8$ and $5X + 3Y = 15$

36. If X and Y are two independent normal random variates such that their mean are 8,12 and standard deviations are 2 and $4\sqrt{3}$ respectively, find the value α such that $P[(2X - Y) \leq 2\alpha] = P[(X + 2Y) \geq 3\alpha]$.

37. If X and Y are independent normal variates with the same mean μ but with variances 4 and 48 such that $P(X + 2Y \geq 4) = P(2X - Y \leq 3)$, find μ.

38. In an examination marks obtained by students in Mathematics, Physics and Chemistry are normally distributed with means 51, 53 and 46 with standard deviation $15, 12, 16$ respectively. Find the probability of securing total marks **(i)** 180 or above, **(ii)** 80 or below.

39. The probability that an electronic component will fail in less 1200 hours of continuous use is 0.25. Using normal approximation to Binomial distribution, find the probability that among 200 such components fewer than 45 will fail in less than 1200 hours of continuous use.

40. Using normal distribution, find the probability of getting 55 heads in the toss of 100 fair coins. Compare the result with that obtained from Binomial distribution.

41. A random variable has a Binomial distribution with $\mathrm{n} = 30$ and $\mathrm{p} = 0.60$. Using normal approximation to Binomial distribution, find the probabilities that it will take **(i)** the value 14, **(ii)** a value less than 12.

42. Find the probability of getting 30 to 35 diamond cards when cards are drawn with replacement from 100 pack of cards which is well – shuffled every time before a card is drawn, using **(i)** Normal distribution, **(ii)** Binomial distribution.

ANSWERS

1.	$k = \frac{1}{3\sqrt{2\pi}}, m = 9, \sigma = 3$					**2.**	4			
3.	**(i)**	$40 , 60$	**(ii)**	$12, 20$		**4.**	$0.8413; \ 0.9938$			
5.	**(i)**	0.7653	**(ii)**	0.0014						
6.	**(i)**	1.96	**(ii)**	2.58		**(iii)**	140.5		**(iv)**	103.6
7.	$z_1 = 1.64, \ x_1 = 39.84$					**8.**	$z_1 = -2.33, \ x_1 = 13.35$			
9.	**(i)**	54,	**(ii)**	46; 62,		**(iii)**	9.6			
10.	**(i)**	10,	**(ii)**	12		**11.**	$40 , 6 , \frac{24}{5}$			
12.	0.07					**13.**	**(i)** 0.1587,		**(ii)**	0.8185
14.	**(i)**	0.4649,	**(ii)**	0.0668		**15.**	27		**16.**	2386
17.	118		**18.**	**(i)** 79,		**(ii)**	33,		**(iii)**	273
19.	**(i)**	67,	**(ii)**	184		**20.**	86.4%			
21.	**(i)**	30.85%,	**(ii)**	53.28%		**22.**	2000		**23.**	5,000
24.	10,000		**25.**	20,000		**26.**	Rs. 3410		**27.**	590, 5.12
28.	Rs. 590, Rs. 638		**29.**	5		**30.**	2.984			
31.	$m = 74.4$ mark. $\sigma = 3.125$					**32.**	$\sigma = 10.33, m = 50.3$			
33.	0.0730		**34.**	**(i)** 0.6541		**(ii)**	0.0025		**35.**	0.061
36.	$\alpha = 6$		**37.**	$\frac{27}{35}$		**38.**	$0.1151, \ 0.0026$		**39.**	0.1841
40.	**B.D.** 0.04847. **N.D.** 0.0484					**41.**	**(i)** 0.0486		**(ii)**	0.0078
42.	**(i)**	N.D : 0.141	**(ii)**	B.D : 0.140						