Discrete Mathematics

Logic - Conditional Statements

Conditional Statement/Implication

- "if *p* then *q*"
- Denoted $p \Rightarrow q$
  - p is called the ***antecedent*** or ***hypothesis***
  - q is called the ***consequent*** or ***conclusion***
- Example:
  - *p*: I am hungry
    q: I will eat
  - *p*: It is snowing
    q: 3+5 = 8

Conditional Statement/Implication (continued)

- In English, we would assume a cause-and-effect relationship, i.e., the fact that p is true would force q to be true.
- If “it is snowing,” then “3+5=8” is meaningless in this regard since *p* has no effect at all on *q*
- At this point it may be easiest to view the operator “$\Rightarrow$” as a logic operation similar to AND or OR (conjunction or disjunction).

Truth Table Representing Implication

- If viewed as a logic operation, p $\Rightarrow$ q can only be evaluated as false if ***p is true and q is false***
- This does not say that *p* causes *q*
- "A false hypothesis implies any conclusion"

- Truth table

Hypothesis p	Conclusion q	$p \Rightarrow q$
T	T	T
T	**F**	**F**
F	T	T
F	F	T

The logical implication can be expressed in various ways, which are described as follows:

- If p then q
- If p, q
- q when p
- q only if P
- q unless ~p
- q whenever p
- p is a sufficient condition for q
- q follow p
- p implies q
- A necessary condition for p is q
- q if p
- q is necessary for p
- p is a necessary condition for q

Examples where $p \Rightarrow q$ is viewed as a logic operation

- If p is false, then any q supports $p \Rightarrow q$ is true.
  - False $\Rightarrow$ True = True
  - False $\Rightarrow$ False = True
- If "2+2=5" then "I am the king of England" is true

Converse and contrapositive

- The converse of $p \Rightarrow q$ is the implication that $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is the implication that $\sim q \Rightarrow \sim p$

Example: What is the converse and contrapositive of *p*: "it is raining" and *q*: I get wet?

- Implication: If it is raining, then I get wet.
- Converse: If I get wet, then it is raining.
- Contrapositive: If I do not get wet, then it is not raining.

Equivalence or biconditional

- If *p* and *q* are statements, the compound statement ***p if and only if q*** is called an ***equivalence*** or ***biconditional***
- Denoted $p \Leftrightarrow q$
  - The only time that the expression can be evaluated as true is, if both statements, p and q, are true or when both statements, p and q are false

p	Q	$p \Leftrightarrow q$
T	**T**	**T**
T	F	F
F	T	F
F	**F**	**T**

Proof of the Contrapositive

Compute the truth table of the statement
$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Tautology and Contradiction

- A statement that is true for all of its propositional variables is called a ***tautology***. (The previous truth table was a tautology.)
- A statement that is false for all of its propositional variables is called a ***contradiction*** or an ***absurdity***

Contingency

- A statement that can be either true or false depending on its propositional variables is called a ***contingency***
- Examples
  - $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology
  - $p \wedge \sim p$ is an absurdity
  - $(p \Rightarrow q) \wedge \sim p$ is a contingency since some cases evaluate to true and some to false.

Contingency Example

The statement $(p \Rightarrow q) \wedge (p \vee q)$ is a contingency

p	q	$p \Rightarrow q$	$p \vee q$	$(p \Rightarrow q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Logically equivalent

- Two propositions are ***logically equivalent*** or ***simply equivalent*** if p $\Leftrightarrow$ q is a tautology.
- Denoted p ≡ q

Example of Logical Equivalence

Prove: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Example of Logical Equivalence

Prove: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Columns 5 and 8 are equivalent, and therefore, p “if and only if” q

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Additional Properties
$(p \Rightarrow q) \equiv ((\sim p) \vee q)$

p	q	$(p \Rightarrow q)$	$\sim p$	$((\sim p) \vee q)$	$(p \Rightarrow q) \Leftrightarrow ((\sim p) \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Additional Properties
$(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$

p	q	$(p \Rightarrow q)$	$\sim q$	$\sim p$	$(\sim q \Rightarrow \sim p)$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Laws Of Logic

- Idempotence: $p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$
- Commutative: $p \wedge q \Leftrightarrow q \wedge p, \quad p \vee q \Leftrightarrow q \vee p$
- Associative: $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r), \quad (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- Distributative: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- Double Negation: $\neg(\neg p) \Leftrightarrow p$
- DeMorgan's Laws: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q, \quad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- Absorbtion: $p \wedge \mathbf{1} \Leftrightarrow p, \quad p \wedge \mathbf{0} \Leftrightarrow \mathbf{0}$
- Dominance: $p \vee \mathbf{1} \Leftrightarrow \mathbf{1}, \quad p \vee \mathbf{0} \Leftrightarrow p$

The following are some other useful logical equivalences.

- $p \rightarrow q \Leftrightarrow \neg p \vee q$
- $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

Two statements s1 and s2 are logically equivalent if s1 ↔ s2 is a tautology.

We use the notation s1 ⇔ s2 to denote the fact (theorem) that s1 ↔ s2 is a tautology, that is, that s1 and s2 are logically equivalent.
Notice that s1 ↔ s2 is a statement and can in general be true or false, and s1 ⇔ s2 indicates the (higher level) fact that it is always true.

Prove the following is a tautology:

$\neg q \vee (p \rightarrow q)$	$\Leftrightarrow \neg q \vee (\neg p \vee q)$	Known L.E.
	$\Leftrightarrow \neg q \vee (q \vee \neg p)$	Commutative
	$\Leftrightarrow (\neg q \vee q) \vee \neg p$	Associative
	$\Leftrightarrow 1 \vee \neg p$	Known tautology
	$\Leftrightarrow 1$	Dominance

$\neg(\neg p \rightarrow q) \vee (p \wedge \neg q) \quad \Leftrightarrow \neg(\neg\neg p \vee q) \vee (p \wedge \neg q)$ Known L.E.

$\Leftrightarrow \neg(p \vee q) \vee (p \wedge \neg q)$ Double Negation

$\Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge \neg q)$ DeMorgan

$\Leftrightarrow (p \wedge \neg p) \vee \neg q$ Dist've (from right to left)

$\Leftrightarrow 0 \vee \neg q$ Known contradiction

$\Leftrightarrow \neg q$ Absorbtion

show that $(p \wedge q) \wedge [(q \wedge \neg r) \vee (p \wedge r)] \Leftrightarrow \neg(p \rightarrow \neg q)$.

LHS

$\Leftrightarrow [(p \wedge q) \wedge (q \wedge \neg r)] \vee (p \wedge q) \wedge (p \wedge r)]$ Distributive

$\Leftrightarrow [((p \wedge q) \wedge q) \wedge \neg r) \vee [((p \wedge q) \wedge p) \wedge r)]$ Associative

$\Leftrightarrow [(p \wedge (q \wedge q)) \wedge \neg r) \vee [((p \wedge p) \wedge q) \wedge r)]$ Commutative, Associative

$\Leftrightarrow [(p \wedge q) \wedge \neg r) \vee [(p \wedge q) \wedge r)]$ Idempotence

$\Leftrightarrow (p \wedge q) \wedge (\neg r \vee r)$ Distributive

$\Leftrightarrow (p \wedge q) \wedge 1$ Known tautology

$\Leftrightarrow (p \wedge q)$ Absorbtion

$\Leftrightarrow \neg\neg(p \wedge q)$ Double Negation

$\Leftrightarrow \neg(\neg p \vee \neg q)$ DeMorgan

$\Leftrightarrow \neg(p \rightarrow q)$ Known L.E.

Using Only And, Or, and Not

p	q	s
0	0	1
0	1	1
1	0	0
1	1	1

$$s \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q).$$

The expression associated with each row of the truth table – a conjunction of variables or their negations – is called a **minterm.**

The compound statement derived using the process consists of the disjunction of a collection of minterms (that is, they are all joined together using “or”).

It is called the **disjunctive normal form** of the statement s.

Try Exercise:

1. Express in words "$\neg\exists a, b, \frac{a}{b} = \sqrt{2}$"

2. Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(a) $(\exists x \in A)(x + 3 = 10)$ (c) $(\exists x \in A)(x + 3 < 5)$

(b) $(\forall x \in A)(x + 3 < 10)$ (d) $(\forall x \in A)(x + 3 \le 7)$

3. Negate each of the following statements:

$$(a)\ \exists x\ \forall y,\ p(x, y); \quad (b)\ \exists x\ \forall y,\ p(x, y); \quad (c)\ \exists y\ \exists x\ \forall z,\ p(x, y, z).$$

Use $\neg\forall x\ p(x) \equiv \exists x \neg p(x)$ and $\neg\exists x\ p(x) \equiv \forall x \neg p(x)$:

4. Find the truth tables for. (a) $p \vee \neg q$; (b) $\neg p \wedge \neg q$.

Verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

5. Negate each of the following statements:

(a) If the teacher is absent, then some students do not complete their homework.

(b) All the students completed their homework and the teacher is present.

(c) Some of the students did not complete their homework or the teacher is absent.

6.

Answers:

1.

For example "$\neg\exists a, b, \frac{a}{b} = \sqrt{2}$" says "it is not the case that there exists (integers) a and b such that $\frac{a}{b} = \sqrt{2}$", or in other words "for all (integers) a and b, $\frac{a}{b} \neq \sqrt{2}$", that is, "$\sqrt{2}$ is irrational".

2. (a) False. For no number in A is a solution to x + 3 = 10. (b) True. For every number in A satisfies x + 3 < 10. (c) True. For if x0 = 1, then x0 + 3 < 5, i.e., 1 is a solution. (d) False. For if x0 = 5, then x0 + 3 is not less than or equal 7. In other words, 5 is not a solution to the given condition.

3.

$$(a)\quad \neg(\exists x \forall y,\ p(x,y)) \equiv \forall x \exists y \neg p(x,y)$$

$$(b)\quad \neg(\forall x \forall y,\ p(x,y)) \equiv \exists x \exists y \neg p(x,y)$$

$$(c)\quad \neg(\exists y\ \exists x\ \forall z,\ p(x,y,z)) \equiv \forall y\ \forall x\ \exists z \neg p(x,y,z)$$

5. (a) The teacher is absent and all the students completed their homework. (b) Some of the students did not complete their homework or the teacher is absent. (c) All the students completed their homework and the teacher is present.

ARGUMENTS

An argument is an assertion that a given set of propositions P1, P2,...,Pn, called premises, yields (has a consequence) another proposition Q, called the conclusion. Such an argument is denoted by $P_1, P_2, \ldots, P_n \vdash Q$

The notion of a "logical argument" or "valid argument" is formalized as follows:

An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be valid if Q is true whenever all the premises P1, P2,...,Pn are true. An argument which is not valid is called fallacy.

The following argument is valid:

$p, p \rightarrow q \vdash q$ (***Law of Detachment***) $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ (***Law of Syllogism***)

The following argument is a fallacy:

$$p \rightarrow q, q \vdash p$$

Consider the following argument:

S_1 : If a man is a bachelor, he is unhappy.
S_2 : If a man is unhappy, he dies young.

S : Bachelors die young

$$p \to q,\ q \to r \vdash p \to r$$

Test the validity of the following argument:

If two sides of a triangle are equal, then the opposite angles are equal.
Two sides of a triangle are not equal.

The opposite angles are not equal.

First translate the argument into the symbolic form $p \to q, \neg p \vdash \neg q$, where p is "Two sides of a triangle are equal" and q is "The opposite angles are equal." ~~By Problem 4.10~~, this argument is a fallacy.

Normal Forms in propositional Logic

1. Conjunctive normal form (CNF):

e.g. $(P \vee Q \vee R) \wedge (P \vee Q) \wedge (P \vee R) \wedge P$

It is conjunction ($\wedge$) of disjunctions ($\vee$)

Where disjunctions are:

1. $(P \vee Q \vee R)$
2. $(P \vee Q)$
3. $(P \vee R)$
4. P

clauses

2. Disjunctive normal form (DNF):

e.g. $(P \wedge Q \wedge R) \vee (P \wedge Q) \vee (P \wedge R) \vee P$

It is disjunction ($\vee$) of conjunctions ($\wedge$)

Procedure to convert a Statement to CNF

1. Eliminate implications and biconditionals using formulas:

- $(P \Leftrightarrow Q) \Longrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
- $P \rightarrow Q \Rightarrow \neg P \vee Q$

2. Apply De-Morgan's Law and reduce NOT symbols so as to bring negations before the atoms. Use:

- $\neg(P \vee Q) \Longrightarrow \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \Longrightarrow \neg P \vee \neg Q$

3. Use distributive and other laws & equivalent formulas to obtain Normal forms.

Conversion to CNF example

Q. Convert into CNF : $((P \rightarrow Q) \rightarrow R)$

Solution:

Step 1: $((P \rightarrow Q) \rightarrow R)$ ==> $((\neg P \vee Q) \rightarrow R)$

==> $\neg(\neg P \vee Q) \vee R$

Step 2: $\neg(\neg P \vee Q) \vee R$ ==> $(P \wedge \neg Q) \vee R$

Step 3: $(P \wedge \neg Q) \vee R$ ==> $\underbrace{(P \vee R) \wedge (\neg Q \vee R)}_{\text{CNF}}$

Convert to CNF

2.$(p \rightarrow q) \wedge (q \vee (p \wedge r))$

$\equiv(\sim p \vee q) \wedge (q \vee (p \wedge r))$

$\equiv (\sim p \vee q) \wedge (q \vee p) \wedge (q \vee r)$

Convert to DNF

1. $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$

2. $(p \rightarrow q) \wedge (\sim p \wedge q) \equiv (\sim p \vee q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \wedge(\sim p \wedge q)) \vee (q \wedge(\sim p \wedge q))$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge q)$

Conversion to CNF example

Q. Convert into CNF : $((P \rightarrow Q) \rightarrow R)$

Solution:

Step 1: $((P \rightarrow Q) \rightarrow R)$ ==> $((\neg P \lor Q) \rightarrow R)$

==> $\neg(\neg P \lor Q) \lor R$

Step 2: $\neg(\neg P \lor Q) \lor R$ ==> $(P \land \neg Q) \lor R$

Step 3: $(P \land \neg Q) \lor R$ ==> $\underbrace{(P \lor R) \land (\neg Q \lor R)}_{\text{CNF}}$