CHAPTER 3

Z - Transforms

1. Introduction

We have already studied Laplace Transforms and Fourier Transforms. Both these transforms are continuous functions. These transforms are not useful for studying discrete systems. Linear systems in which the input signals are in the form of discrete pulses of short duration are called 'Linear Time Invariant' (LTI) systems. For the analysis of such systems we need Z-transforms. In this chapter we shall first get acquainted with sequences, then study Z-transforms and then inverse Z-transforms. After studying Laplace Transforms and Z-transforms, you will find that Z-transform is the discrete analogue of Laplace Transform. For every operational rule and application of Laplace transform there corresponds an operational rule or application of Z-transform. For example, you will find Linearity Property, Shifting Theorem, Convolution Theorem etc. in both Laplace Transforms and Z-transforms.

2. Sequences

If objects are arranged according to a certain rule, this arrangement is called a sequence. We are particularly interested in sequences whose members are real or complex numbers. So we define a sequence as follows.

Definition : An ordered set of real or complex numbers is called a **sequence**.

We shall denote a sequence by $\{ f(k) \}$ and k-th term of the sequence by $f(k)$. For example, we have a sequence

$$\{ 2^0, 2^1, 2^2, 2^3, \ldots\ldots 2^k, \ldots\ldots \}$$

For $k = 0$, $f(k) = 2^0$; for $k = 1$, $f(k) = 2^1$ ..... Thus, in a sequence we have to take into account, the order of a term k, and the term of k-th order $f(k)$. The set of all such ordered terms $\{ f(0), f(1), \ldots\ldots, f(k), \ldots\ldots \}$ is called a sequence.

1. The most elementary way to denote a sequence is to list all the members of the sequence. For example,

$$\{f(k)\} = \{12, 10, \underset{\uparrow}{8}, 5, 3, 6, 9\}$$

The arrow $\uparrow$ indicates the element corresponding to $k = 0$. The elements on the left of the arrow correspond to $k = -1, -2, -3, \ldots\ldots$ and those on the right correspond to $k = 1, 2, 3, \ldots\ldots$

2. Another way of denoting a sequence is to give the general term in terms of k which varies from $-\infty$ to ∞ taking integral values.

For example $\{f(k)\} = 2^k$ (where k is an integer). This sequence is

$$\{ \ldots\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^2, 2^3, \ldots\ldots \}$$

As illustrations we can have the following sequences and can have many more,

$\{1, 1, 1, 1, \ldots\ldots\}$

$\{1, 2, 3, \ldots\ldots, k, \ldots\ldots\}$

$\{1^2, 2^2, 3^2, \ldots\ldots, k^2, \ldots\ldots\}$

$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots\}$

$\{1 \cdot 2^1, 2 \cdot 2^2, 3 \cdot 2^3, \ldots\ldots, k \cdot 2^k, \ldots\ldots\}$

$\{1 \cdot \alpha^1, 2 \cdot \alpha^2, 3 \cdot \alpha^3, \ldots\ldots, k \cdot \alpha^k, \ldots\ldots\}$

$\left\{\frac{\alpha^1}{1}, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots\ldots, \frac{\alpha^k}{k}, \ldots\ldots\right\}$

$\left\{\frac{5^1}{1!}, \frac{5^2}{2!}, \frac{5^3}{3!}, \ldots\ldots, \frac{5^k}{k!}, \ldots\ldots\right\}$

$\{e^{1 \cdot \alpha}, e^{2 \cdot \alpha}, e^{3 \cdot \alpha}, \ldots\ldots, e^{k \cdot \alpha}, \ldots\ldots\}$

3. Basic Operations On Sequences

We shall see below some properties of sequences through examples.

1. **Addition** : The sum (or difference) of two sequences is obtained by adding (or subtracting) the corresponding terms of the two sequences.

For example, if $\{f(k)\} = 1^3, 2^3, 3^3, 4^3, \ldots\ldots$

$\{g(k)\} = 1^2, 2^2, 3^2, 4^2, \ldots\ldots$

then $\{f(k)\} + \{g(k)\} = \{(1^3 + 1^2), (2^3 + 2^2), (3^3 + 3^2), \ldots\ldots, (k^3 + k^2) \ldots\ldots\}$

$= \{1^2 \cdot 2, 2^2 \cdot 3, 3^2 \cdot 4, \ldots\ldots, k^2 (k+1) \ldots\ldots\}$

$\{f(k)\} - \{g(k)\} = \{(1^3 - 1^2), (2^3 - 2^2), (3^3 - 3^2), \ldots\ldots, (k^3 - k^2), \ldots\}$

$= \{1^2 \cdot 0, 2^2 \cdot 1, 3^2 \cdot 2, \ldots\ldots, k^2 \cdot (k-1), \ldots\ldots\}$

2. **Scalar Multiplication** : If α is a scalar then from a given sequence $\{f(k)\}$ we can obtain another sequence $\alpha \{f(k)\}$ by multiplying each term $f(k)$ of the sequence $\{f(k)\}$ by α.

For example, if $\{f(k)\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots,$ then

$3 \cdot \{f(k)\} = 3, \frac{3}{2}, \frac{3}{3}, \ldots\ldots, \frac{3}{k}, \ldots\ldots$

If $\{f(k)\} = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\ldots, \sqrt{k}, \ldots\ldots,$ then

$2 \cdot \{f(k)\} = 2\sqrt{1}, 2\sqrt{2}, 2\sqrt{3}, \ldots\ldots, 2\sqrt{k}, \ldots\ldots$

3. **Linearity** : If α and β are two scalars then from two sequences $\{f(k)\}$ and $\{g(k)\}$, we can obtain another sequence by multiplying the terms of the two sequences by α and β as above and adding the corresponding terms.

i.e. $\alpha \{f(k)\} + \beta \{g(k)\} = \{\alpha \cdot f(k) + \beta \cdot g(k)\}$

For example, if $\{f(k)\} = 1, 2, 3, \ldots\ldots, k, \ldots\ldots$

and $\{g(k)\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$

then, $2 \cdot \{f(k)\} + 3 \cdot \{g(k)\} = \{2 \cdot f(k) + 3 \cdot g(k)\}$

$$= 2(1) + 3\left(\frac{1}{1}\right),\ 2 \cdot 2 + 3\left(\frac{1}{2}\right),\ 2(3) + 3\left(\frac{1}{3}\right), \ldots\ldots, 2 \cdot k + 3\left(\frac{1}{k}\right) + \ldots\ldots$$

$$= 2 + \frac{3}{1},\ 4 + \frac{3}{2},\ 6 + \frac{3}{3}, \ldots\ldots, 2k + \frac{3}{k}, \ldots\ldots$$

4. Convergence And Divergence : Consider the following sequence

$$\frac{1+1}{1}, \frac{2+1}{2}, \frac{3+1}{3}, \frac{4+1}{4}, \ldots\ldots, \frac{k+1}{k}, \ldots\ldots$$

i.e. $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots, 1 + \frac{1}{k}, \ldots\ldots$ *i.e.*, $2,\ 1{\cdot}5,\ 1{\cdot}33,\ 1{\cdot}25, \ldots\ldots$

It is easy to see that as the number of terms become infinite the sequence goes on decreasing and ultimately takes the value 1. Such a sequence $\{f(k)\}$ is called a convergent sequence.

Definition : If $\{f(k)\}$ is a given sequence and if $f(k)$ tends to a (finite) real number L as k tends to infinity then $\{f(k)\}$ is called a **convergent sequence**.

The following sequences are convergent.

(i) $a, a, a, \ldots\ldots, a, \ldots\ldots, a, \ldots\ldots$ converges to a

(ii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$ converges to 0

(iii) $1 + \frac{1}{2^0}, 1 + \frac{1}{2^1}, 1 + \frac{1}{2^2}, \ldots\ldots, 1 + \frac{1}{2^k}, \ldots\ldots$ converges to 1.

Definition : A sequence which is not convergent *i.e.* which does not tend to a (finite) real number is called a **divergent** sequence.

The following are divergent sequences.

(i) $1, 2, 3, \ldots\ldots, k, \ldots\ldots$ diverges to ∞

(ii) $-1, -2, -3, \ldots\ldots, -k, \ldots\ldots$ diverges to $-\infty$

(iii) $1, 2, 1, 2, 1, 2, \ldots\ldots$ oscillates between 1 and 2

(iv) $0, 1, 0, 1, 0, 1, \ldots\ldots$ oscillates between 0 and 1.

EXERCISE - I

1. Write down the term corresponding to $k = 3$ of the following sequence

$\{-6, -3, -1, \underset{\uparrow}{0}, 2, 4, 6, 8, 10\}$ **[Ans. : 8]**

2. Write down the term corresponding to $k = -3$ of the following sequence.

$\{-12, -10, -9, \underset{\uparrow}{-7}, -5, -3, 1, 4, 6, 10\}$ **[Ans. : – 10]**

3. Write down the sequence if k-th term is 3^k for $-2 \le k \le 4$.

[Ans. : $\frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81$ **]**

4. Write down the sequence whose k-th term is 2^k for $-\infty < k < \infty$.

[Ans. : $\left\{\ldots\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\ldots\right\}$ **]**

5. Write down the sequence whose k-th term $= \begin{cases} 4^k, & k < 0 \\ 3^k, & k \geq 0 \end{cases}$

$$\left[\text{Ans. : } \left\{\ldots\ldots, \frac{1}{64}, \frac{1}{16}, \frac{1}{4}, 1, 3, 9, 27, \ldots\ldots\right\}\right]$$

6. Write down the sequence whose k-th term $= \begin{cases} a^k, & k < 0 \\ b^k, & k \geq 0 \end{cases}$

$$\left[\text{Ans. : } \left\{\ldots\ldots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, b, b^2, b^3, \ldots\ldots\right\}\right]$$

4. Z - transforms

We shall now define Z-transform of a sequence.

Definition : Let $\{f(k)\} = \{\ldots\ldots f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \ldots\ldots\}$ be a sequence of terms where k varies from $-\infty$ to ∞.

Let $z = x + iy$ be a complex number then

$$Z\{f(k)\} = \ldots + f(-3)\, z^3 + f(-2)\, z^2 + f(-1)\, z^{-1} + f(0)\, z^0 + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots$$

$$= \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

is called the **Z-transform of the sequence $\{f(k)\}$**.

In words, the sum of the product of k-th term of the sequence $f(k)$ with z^{-k} taken from $-\infty$ to ∞ is called the **Z-transform of the sequence $\{f(k)\}$**.

Thus,
$$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}}$$

Notation : Unfortunately there is no unanimity in the notations used in the case of Z-transform. Some use x_k, y_k, ... to denote sequences and $x_0, x_1, \ldots\ldots, y_0, y_1, \ldots\ldots$ to denote the terms of the sequences. We shall denote the sequences by $\{f(k)\}$, $\{g(k)\}$, ... the terms of the sequences by $f(1), f(2), \ldots\ldots, g(1), g(2), \ldots$. and Z-transforms by $Z\{f(k)\}$, $Z\{g(k)\}$, ... or by $F(z)$, $G(z)$, ... etc.

Notes ....

1. It is necessary to know which is the zeroth term, first term, second term .... minus first term, minus second term ..... *i.e.* we must know the order of each term. To obtain Z-transform of a sequence we multiply each term by **negative power of z of the order of that term** and take the sum.
2. $Z\{f(k)\}$ is a function of a complex variable z and is defined only if the sum is finite *i.e.* if the **infinite series $\sum f(k)\, z^{-k}$ is absolutely convergent.** We shall denote the Z-transform of the sequence $\{f(k)\}$ by $Z\{f(k)\}$ or by $F(z)$.
3. Wherever necessary we shall denote the sequences by $\{f(k)\}$, $\{g(k)\}$ etc.
4. If Z-transform of $\{f(k)\}$ is $F(z)$ we call **$\{f(k)\}$ the inverse Z transform of $F(z)$ and denote** it by $Z^{-1}[F(z)]$.

Example 1 : If $\{f(k)\} = \{-6, -3, \underset{\uparrow}{0}, 2, 4\}$,

find $Z\{f(k)\}$ where $\uparrow$ denotes the element corresponding to $k = 0$.

Sol. : $Z\{f(k)\} = \Sigma f(k) z^{-k}$

To obtain $Z\{f(k)\}$, we multiply each term $f(k)$ of the sequence by z^{-k}, and take the sum. Multiply the 0^{th} term 0 by z^0, the first term 2 by z^{-1}, the second term 4 by z^{-2}, $(-1)^{st}$ term (-3) by $(z^{-1})^{-1}$ i.e., z, $(-2)^{nd}$ term (-6) by $(z^{-1})^{-2}$ i.e., z^2 and take the sum of these products.

$$\therefore \quad Z\{f(k)\} = f(-2)(z^{-1})^{-2} + f(-1)(z^{-1})^{-1} + f(0)z^0 + f(1)z^{-1} + f(2)z^{-2}$$

where, $f(-2) = -6$, $f(-1) = -3$, $f(0) = 0$, $f(1) = 2$, $f(2) = 4$.

$$\therefore \quad Z\{f(k)\} = \sum_{k=-2}^{2} f(k) z^{-k} = (-6)z^2 + (-3)z^1 + 0 \cdot z^0 + 2z^{-1} + 4z^{-2}$$

$$= -6z^2 + 3z + 0 + \frac{2}{z} + \frac{4}{z^2}$$

Example 2 : If $\{f(k)\} = \{9, 6, \underset{\uparrow}{3}, 0, -3, -6, -9\}$, find $Z\{f(k)\}$.

Sol. : Since 3 is the term corresponding to $k = 0$. We have

$$f(-2) = 9,\ f(-1) = 6,\ f(0) = 3,\ f(1) = 0,\ f(2) = -3,\ f(3) = -6,\ f(4) = -9.$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-2}^{4} f(k) z^{-k} = 9z^2 + 6z^1 + 3z^0 + 0z^{-1} - 3z^{-2} - 6z^{-3} - 9z^{-4}$$

$$= 9z^2 + 6z + 3 + 0 - \frac{3}{z^2} - \frac{6}{z^3} - \frac{9}{z^4}.$$

Example 3 : If $\{f(k)\} = \{2^0, 2^1, 2^2, 2^3, ...\}$ find $Z\{f(k)\}$.

Sol. : Z-transform of the sequence is

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{k=\infty} f(k) z^{-k} = 2^0 z^0 + 2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + ...$$

$$= 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...$$

$$= \frac{1}{1-(2/z)} = \frac{z}{z-2} \quad \text{if } \left|\frac{2}{z}\right| < 1 \qquad \left(\because S_\infty = a + ar + ar^2 + ... = \frac{a}{1-r}, \text{ if } |r| < 1\right)$$

Example 4 : If $\{f(k)\} = \begin{cases} 4^k, & \text{for } k < 0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$, find $Z\{f(k)\}$.

Sol. : The sequence is $\{f(k)\} = \{....., 4^{-4}, 4^{-3}, 4^{-2}, 4^{-1}, 3^0, 3^1, 3^2, 3^3, ...\}$

And Z-transform of $f(k)$ is

$$\therefore \quad Z\{f(k)\} = \{....., 4^{-4}z^4 + 4^{-3}z^3 + 4^{-2}z^2 + 4^{-1}z + 3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + ...\}$$

We write positive powers of z in reverse order.

$$\therefore \quad Z\{f(k)\} = \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + ...\right]$$

$$= \frac{z}{4}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + ...\right] + \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + ...\right]$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(3/z)} \quad \text{if} \left|\frac{z}{4}\right| < 1, \left|\frac{3}{z}\right| < 1$$

$$= \frac{z}{4} \cdot \frac{4}{4-z} + \frac{z}{z-3} = \frac{z}{4-z} + \frac{z}{z-3}$$

$$= \frac{z}{(4-z)(z-3)} \quad \text{if } 3 < |z| < 4.$$

Note ....

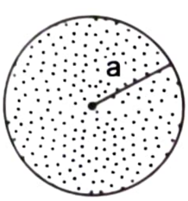
We shall require the following results in finding Z-transforms.

1. $1 + r + r^2 + r^3 + \ldots. = \dfrac{1}{1-r}$ if $|r| < 1$.
2. $1 - x + x^2 - x^3 + \ldots.. = (1+x)^{-1}$ if $|x| < 1$.
3. $1 + x + x^2 + x^3 + \ldots.. = \sum\limits_{k=0}^{\infty} x^k = (1-x)^{-1}$ if $|x| < 1$.
4. $1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(x-2)}{3!}x^3 + \ldots.. = (1+x)^n$
5. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots.. = e^x$
6. If $z = x + iy$, then $|z| = \sqrt{x^2+y^2}$.
7. **The set | z | < a.** Since $z = x + iy$, $|z| = \sqrt{x^2+y^2}$.

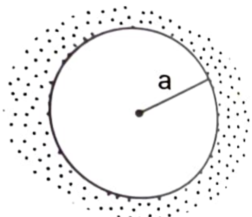
$\therefore$ $|z| < a$ means $\sqrt{x^2+y^2} = a$

i.e. $x^2 + y^2 < a^2$.

This means, $|z| < a$ is the set of points inside the circle of radius a and centre at the origin. By the same reasoning $|z| > a$ is the set of points outside the circle with radius a and centre at the origin.



| z | < a



| z | > a

EXERCISE - II

1. Write down the Z-transforms of the following sequences

(i) $\{f(k)\} = \{8, 6, 4, 2, 0, 1, 3, 5, 7\}$ (ii) $\{f(k)\} = \{-6, -4, -2, 1, 2, 4, 6\}$
$\uparrow$ $\uparrow$

[**Ans.** : (i) $8z^6 + 6z^5 + 4z^4 + 2z^3 + 0 + 1z + 3 + \dfrac{5}{z} + \dfrac{7}{z^2}$

(ii) $-6z^3 - 4z^2 - 2z + 1 + \dfrac{2}{z} + \dfrac{4}{z^2} + \dfrac{6}{z^3}$]

2. Write down the Z-transform of the following sequences

(i) $\{f(k)\} = 3^k$, $k \geq 0$ (ii) $\{f(k)\} = 5^k$, $k \geq 0$

[**Ans.** : (i) $\dfrac{z}{z-3}$, $\left|\dfrac{3}{z}\right| < 1$, (ii) $\dfrac{z}{z-5}$, $\left|\dfrac{5}{z}\right| < 1$.]

Inverse Z-Transform

Definition : If $F(z)$ is the Z-transform of the sequence $\{f(k)\}$ then the sequence $\{f(k)\}$ is called the inverse Z-transform of $F(z)$ and is denoted as

$$\boxed{\{f(k)\} = Z^{-1}[F(z)]}$$

Thus, we have if $Z\{f(k)\} = F(z)$, then $\{f(k)\} = Z^{-1}[f(z)]$ and vice versa.

5. Region of Convergence (ROC)

We shall try to understand this important concept in relation to Z-transforms through two examples.

Example 1 : Consider the sequence $f(k) = \begin{cases} 0 & \text{for } k < 0 \\ 4^k & \text{for } k \geq 0 \end{cases}$

i.e., the sequence $\{f(k)\} = \{4^0, 4^1, 4^2, 4^3, \ldots\ldots, 4^k, \ldots\ldots\}$.

Its Z-transform by definition is

$$Z\{f(k)\} = \Sigma f(k) z^{-k} = \sum_{k=0}^{\infty} 4^k z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + 4^3 z^{-3} + \ldots$$

$$= 1 + \frac{4}{z} + \frac{16}{z^2} + \frac{64}{z^3} + \ldots\ldots$$

Notice that $Z\{f(k)\}$ is a Geometric Progression with common ratio $4/z$. We know that the sum S of infinite terms of a G.P. with first term 1 and common ratio r is given by $S = \frac{1}{1-r}$ if $|r| < 1$.

The sum of the above series *i.e.* of the Z-transform is

$$Z\{f(k)\} = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \text{ if } \left|\frac{4}{z}\right| < 1$$

i.e. $4 < |z|$ *i.e.* $|z| > 4$.

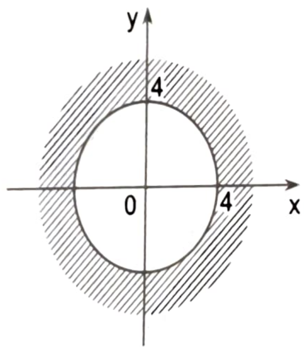
Since, for the Z-transform to exist the corresponding series must be convergent. The above Z-transform is defined only if $|z| > 4$.

Note ....

Note that a G.P. $a, ar, ar^2, \ldots\ldots ar^n \ldots\ldots$ is convergent if $|r| < 1$ and its sum

$$S = \frac{a}{1-r} \text{ where } |r| < 1.$$

But $|z| = 4$ is a circle with centre at the origin and radius $= 4$. Hence, the above Z-transform is defined if $z > 4$ *i.e.* if z is on the exterior of the circle $|z| = 4$.



In this case, we say that the region of convergence of this Z-transform is the exterior of the circle $|z| = 4$.

The region for which $\Sigma f(k) z^{-k}$ is convergent is called the region of convergence denoted in short by R.O.C.

Example 2 : Find the Z-transform and the region of convergence of $f(k) = \begin{cases} 5^k & \text{for } k < 0 \\ 3^k & \text{for } k \geq 0 \end{cases}$.

Sol. : By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

Putting $k = -n$ in the first series, we get

$$Z\{f(k)\} = \sum_{n=1}^{\infty} 5^{-n} z^n + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$Z\{f(k)\} = \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

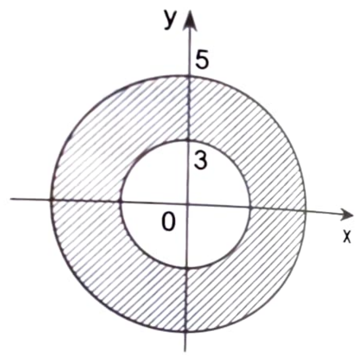
$$= \frac{z}{5}\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{5} \cdot \frac{1}{1-(z/5)} + \frac{1}{1-(3/z)} = \frac{z}{5-z} + \frac{z}{z-3}$$

$$= \frac{2z}{(5-z)(z-3)}$$

Now, $Z\{f(k)\}$ is the sum of two Geometric Progressions with the common ratios $(z/5)$ and $(3/z)$ respectively. The series will be convergent if $|z/5| < 1$ and $|3/z| < 1$. *i.e.* $|z| < 5$ and $3 < |z|$ *i.e.* $3 < |z| < 5$.

But $|z| = 3$ is a circle with centre at the origin and radius 3 and $|z| = 5$ is a circle with centre at the origin and radius 5. Hence, $Z\{f(k)\}$ is convergent if z lies between the annulus as shown in the figure. This is the region of convergence of $Z\{f(k)\}$ which is shown by shaded area.



$\therefore$ ROC is $3 < |z| < 5$.

6. Z-Transforms of Some Standard Functions

Example 1 : Find the Z-transform of *Unit Impulse function*

$$\delta(k) = 1 \quad \text{for } k = 0$$
$$= 0 \quad \text{otherwise}$$

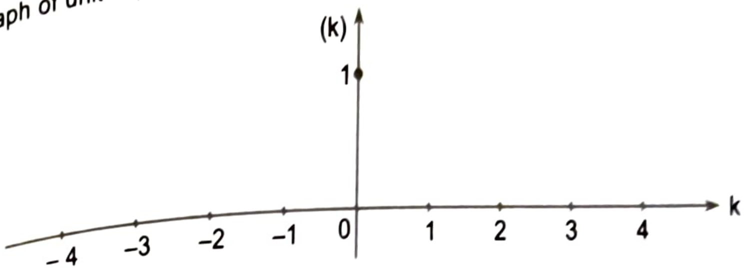
Sol. :
$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$
$$= \{\ldots 0 + 0 + 0 + 1 \cdot z^0 + 0 + 0 + 0 \ldots\}$$
$$= 1 \text{ for all } z$$

$$\therefore \quad \boxed{Z\{\delta(k)\} = 1}$$

This is convergent for all z.

$\therefore$ ROC is whole of z-plane.

The graph of unit impulse function is,



Example 2 : Find the Z-transform of *Discrete Unit Step function*

$U(k) = 1 \quad \text{for } k \geq 0$

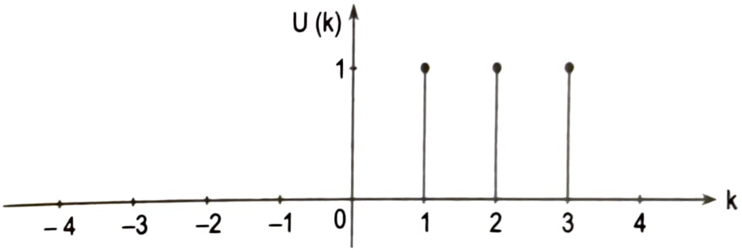
$\quad\;\; = 0 \quad \text{for } k < 0$

Sol. :
$$Z\{U(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$$= \left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\right] = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

$$\therefore \quad \boxed{Z\{U(k)\} = \frac{z}{z-1},} \qquad k \geq 0$$

The graph of discrete unit step function is,



This is convergent if $|1/z| < 1$ *i.e.* $1 < |z|$ *i.e.* $|z| > 1$

$\therefore$ ROC is $|z| > 1$.

Since, $Z\{U(k)\} = \dfrac{z}{z-1}$, $\quad Z^{-1}\left[\dfrac{z}{z-1}\right] = \{U(k)\}$ where Z^{-1} denotes inverse Z-transform.

Example 3 : Find the z-transform of $f(k) = \alpha^k$, $\alpha > 0$, $k \geq 1$.

Sol. : We have $Z\{f(k)\} = \sum_{k=1}^{\infty} \alpha^k z^{-k} = \dfrac{\alpha}{z} + \dfrac{\alpha^2}{z^2} + \dfrac{\alpha^3}{z^3} + \ldots\ldots$

$$= \frac{\alpha}{z}\left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \ldots\ldots\right) \qquad \text{[G.P.]}$$

$$= \frac{\alpha}{z} \cdot \frac{1}{1-(\alpha/z)} = \frac{\alpha}{z-\alpha}, \; |\alpha| < |z|.$$

$$\therefore \quad \boxed{Z\{\alpha^k\} = \frac{\alpha}{z-\alpha},} \qquad k \geq 1$$

Cor. : Putting $\alpha = 1$, $\quad \boxed{Z\{1\} = \frac{1}{z-1},} \qquad k \geq 1$

Example 4 : Find the Z-transform of $f(k) = k\,\alpha^k$, $k \ge 0$.

Sol. : Assuming $f(k) = 0$ for $k < 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^k + \sum_{k=0}^{\infty} k\,\alpha^k z^{-k}$$

$$= 0 + 1 \cdot \frac{\alpha}{z} + 2 \cdot \frac{\alpha^2}{z^2} + 3 \cdot \frac{\alpha^3}{z^3} + \ldots..$$

$$= \frac{\alpha}{z}\left(1 + 2\frac{\alpha}{z} + 3\frac{\alpha^2}{z^2} + \ldots..\right) \qquad \ldots\ldots (1)$$

$$= \frac{\alpha}{z}\left(1 - \frac{\alpha}{z}\right)^{-2} = \frac{\alpha}{z} \cdot \frac{1}{[1 - (\alpha / z)]^2} = \frac{\alpha z}{(z - \alpha)^2}$$

$$\therefore \qquad \boxed{Z\{k\,\alpha^k\} = \frac{\alpha z}{(z - \alpha)^2}}$$

Cor. : Putting $\alpha = 1$, $\boxed{Z\{k\} = \frac{z}{(z-1)^2}}$

Applying D'Alembert's radio test to (1), we find that the series is convergent if $|\alpha / z| < 1$ *i.e.*, $|\alpha| < |z|$ *i.e.*, $|z| > |\alpha|$.

$\therefore$ ROC is $|z| > |\alpha|$

Particular Cases : (i) Find the Z-transform of $f(k) = k2^k$, $k \ge 0$.

(ii) Find the z-transform of $f(k) = k\,2^k + k\,3^k$.

Sol. : Put (i) $\alpha = 2$ and (ii) $\alpha = 2$, $\alpha = 3$, in the above example.

[**Ans. : (i)** $\dfrac{2z}{(z-2)^2}$, $|z| > 2$, **(ii)** $\dfrac{2z}{(z-2)^2} + \dfrac{3z}{(z-3)^2}$, $|z| > 3$]

Example 5 : Find the Z-transform of $f(k) = \dfrac{\alpha^k}{k}$, $k \ge 1$.

Sol. : Assuming $f(k) = 0$ for $k \le 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} z^{-k}$$

$$= \frac{\alpha}{z} + \frac{\alpha^2}{2z^2} + \frac{\alpha^3}{3z^3} + \frac{\alpha^4}{4z^4} + \ldots.. \qquad \ldots\ldots (1)$$

$$Z\{f(k)\} = -\log\left(1 - \frac{\alpha}{z}\right)$$

$$\therefore \qquad \boxed{Z\left(\frac{\alpha^k}{k}\right) = -\log\left(1 - \frac{\alpha}{z}\right)}$$

Applying D'Alembert's Ratio Test to (1), we find that the series is convergent if $|\alpha / z| < 1$ *i.e.*, $|\alpha| < |z|$ *i.e.*, $|z| > |\alpha|$.

$\therefore$ ROC is $|z| > |\alpha|$.

Particular cases : (i) Find Z-transform of $f(k) = \frac{1}{k}$, $k \geq 1$.

(ii) Find the Z-transform of $f(k) = \frac{2^k}{k}$, $k \geq 1$.

Sol. : Put (i) $\alpha = 1$, and (ii) $\alpha = 2$ in the above example.

[**Ans. : (i)** $-\log\left(1-\frac{1}{z}\right)$, $|z| > 1$; **(ii)** $-\log\left(1-\frac{2}{z}\right)$, $|z| > 2$.]

Example 6 : Find the Z-transform of $f(k) = a^k$, $k \geq 0$. **(M.U. 2009, 16)**

Sol. : Assuming that $f(k) = 0$ when $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} a^k z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots.. = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$$\therefore \quad \boxed{Z(a^k) = \frac{z}{z-a}}$$

The series being G.P. is convergent if $1 > |a/z|$ *i.e.* $|z| > |a|$.

$\therefore$ ROC is $|z| > |a|$.

Since, $Z\{a^k\} = \frac{z}{z-a}$, $Z^{-1}\left[\frac{z}{z-a}\right] = a^k$, $k \geq 0$.

Example 7 : Find the Z-transform of $f(k) = b^k$, $k < 0$. **(M.U. 2008, 09)**

Sol. : Assuming that $f(k) = 0$ when $k \geq 0$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{n=1}^{\infty} b^{-n} z^n \text{ where } n = -k$$

(Note the substitution $n = -k$)

$$= \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \ldots. = \frac{z}{b}\left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \ldots..\right)$$

$$= \frac{z}{b} \frac{1}{1-(z/b)} = \frac{z}{b-z}$$

$$\therefore \quad \boxed{Z(b^k) = \frac{z}{b-z}}$$

The series being G.P. is convergent if $1 > |z/b|$ *i.e.* $|b| > |z|$.

$\therefore$ ROC is $|z| < |b|$.

Since, $Z\{b^k\} = \frac{z}{b-z}$, $Z^{-1}\left[\frac{z}{b-z}\right] = b^k$, $k < 0$.

Example 8 : Find Z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k, & k \geq 0 \end{cases}$ **(M.U. 2017, 19)**

Sol. : By Examples 7 and 6, we get

$$Z\{f(k)\} = \frac{z}{b-z} + \frac{z}{z-a} = \frac{z^2 - az + bz - z^2}{(z-a)(b-z)}$$

$$\therefore \quad Z\{f(k)\} = \frac{-z(a-b)}{-(z-a)(z-b)} = \frac{(a-b)z}{(z-a)(z-b)}$$

if $|z| > a$ and $|z| < b$ i.e., $a < |z| < b$.

Note ....

In general the Z-transform of $\{f(k)\} = b^k$, for $k < 0$ and $\{f(k)\} = a^k$ for $k \geq 0$ is $\dfrac{(a-b)z}{(z-a)(z-b)}$ and $a < |z| < b$. This Z-transform exists only if $a < b$.

Example 9 : Find the Z-transform of $f(k) = {}^nC_k,\ 0 \leq k \leq n$.

Sol. : $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^nC_k z^{-k} = \sum_{k=0}^{n} {}^nC_k z^{-k}$

$$= {}^nC_0 + {}^nC_1 \frac{1}{z} + {}^nC_2 \frac{1}{z^2} + \ldots\ldots + {}^nC_n \frac{1}{z^n} = \left(1 + \frac{1}{z}\right)^n$$

The series being finite is obviously convergent if $z \neq 0$.

$$\therefore \quad \boxed{Z({}^nC_k) = \left(1 + \frac{1}{z}\right)^n}$$

$\therefore$ ROC is all of z-plane except the origin.

Example 10 : Find the Z-transform of $f(k) = {}^kC_n,\ k \geq n$.

Sol. : $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^kC_n z^{-k} = \sum_{k=n}^{\infty} {}^kC_n z^{-k}$

To find the sum we change the dummy index k by $k = n + r$.

$$\therefore \quad Z\{f(k)\} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-(n+r)} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-n} z^{-r}$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r z^{-r} \quad \left[\because\ {}^nC_r = {}^nC_{n-r},\ \text{we get } {}^{n+r}C_n = {}^{n+r}C_{n+r-n} = {}^{n+r}C_r\right]$$

$$= z^{-n}\left[1 + {}^{n+1}C_1 z^{-1} + {}^{n+1}C_2 z^{-2} + \ldots\ldots\right] = z^{-n}\left(1 - \frac{1}{z}\right)^{-(n+1)}$$

$$\therefore \quad \boxed{Z({}^kC_n) = z^{-n}\left(1 - \frac{1}{z}\right)^{-(n+1)}}$$

$\therefore$ ROC is $|z| > 1$.

Example 11 : Find the Z-transform of $f(k) = {}^{k+n}C_n$.

Sol. : $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^{k+n}C_n z^{-k}$

But ${}^{k+n}C_n = 0$ if $k + n < n$ i.e. if $k < 0$.

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{\infty} {}^{k+n}C_n z^{-k} = \sum_{k=0}^{\infty} {}^{k+n}C_k z^{-k} \quad \text{(as in the previous example)}$$

$$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + \ldots\ldots$$

$$\therefore \quad Z\{f(k)\} = \left(1 - \frac{1}{z}\right)^{-(n+1)}$$

$$\therefore \quad \boxed{Z\left({}^{k+n}C_n\right) = \left(1 - \frac{1}{z}\right)^{-(n+1)}}$$

As above ROC is $|z| > 1$.

Example 12 : Find the Z-transform of $f(k) = \dfrac{a^k}{k!}$, $k \geq 0$. **(M.U. 2009)**

Sol. :
$$Z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{(a/z)^k}{k!}$$
$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \ldots..$$
$$= e^{a/z}$$ ROC is all of z-plane.

Remark ....

You are advised to memorise the Z-transforms of these standard functions.

EXERCISE - III

Find the Z-transform and its ROC of each of the following sequences $\{f(k)\}$ where $f(k)$ is given by

1. $f(k) = 3^k$, $k \geq 0$
2. $f(k) = 4^k$, $k \geq 0$
3. $f(k) = (1/6)^k$, $k \geq 0$
4. $f(k) = 2$, $k \geq 0$
5. $f(k) = 4$, $k \geq 0$
6. $f(k) = 5$, $k \geq 0$
7. $f(k) = 2^k$, $k < 0$
8. $f(k) = 4^k$, $k < 0$
9. $f(k) = (1/3)^k$, $k < 0$
10. $f(k) = 3^k$, $k < 0$; $= 2^k$, $k \geq 0$
11. $f(k) = 4^k$, $k < 0$; $= 3^k$, $k \geq 0$
12. $f(k) = a^k$, $k < 0$; $= b^k$, $k \geq 0$ $(a, b > 0, a > b)$
13. $f(k) = k3^k$, $k \geq 0$
14. $f(k) = k\,5^k$, $k \geq 0$
15. $f(k) = ka^k$, $k \geq 0$ $(a > 0)$
16. $f(k) = \dfrac{3^k}{k}$, $k > 1$
17. $f(k) = \dfrac{2^k}{k}$, $k > 1$
18. $f(k) = \dfrac{a^k}{k}$, $k > 1$, $a > 0$
19. $f(k) = (1/2)^{|k|}$, for all k
20. $f(k) = (1/4)^{|k|}$, for all k
21. $f(k) = a^k$ for all k $(0 < a < 1)$
22. $f(k) = (3^k / k!)$, $k \geq 0$
23. $f(k) = (5^k / k!)$, $k \geq 0$
24. $f(k) = e^{k\alpha}$, $k \geq 0$
25. $f(k) = \begin{cases} 2^k, & k \leq -1\ldots.. \\ (1/2)^k, & k = 0, 2, 4\ldots.. \\ (1/3)^k, & k = 1, 3, 5\ldots.. \end{cases}$

[**Ans. :** **(1)** $\dfrac{1}{1-(3/z)}$; $|z| > 3$, **(2)** $\dfrac{1}{1-(4/z)}$; $|z| > 4$, **(3)** $\dfrac{1}{1-(1/6z)}$; $|z| > \dfrac{1}{6}$,

(4) $2 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$, **(5)** $4 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$, **(6)** $5 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$,

(7) $\dfrac{z}{2} \cdot \dfrac{1}{1-(z/2)}$; $|z| < 2$, **(8)** $\dfrac{z}{4} \cdot \dfrac{1}{1-(z/4)}$; $|z| < 4$, **(9)** $3z\dfrac{1}{1-3z}$; $|z| < \dfrac{1}{3}$,

(10) $\dfrac{2z}{(3-z)(z-2)}$; $2<|z|<3$, (11) $\dfrac{3z}{(4-z)(z-3)}$; $3<|z|<4$,

(12) $\dfrac{(a-b)z}{(a-z)(z-b)}$; $b<|z|<a$, (13) $\dfrac{3z}{(z-3)^2}$; $|z|>3$,

(14) $\dfrac{5z}{(z-5)^2}$; $|z|>5$, (15) $\dfrac{az}{(z-a)^2}$; $|z|>a$, (16) $-\log\left(1-\dfrac{3}{z}\right)$; $|z|>3$,

(17) $-\log\left(1-\dfrac{2}{z}\right)$; $|z|>2$, (18) $-\log\left(1-\dfrac{a}{z}\right)$; $|z|>a$,

(19) $\dfrac{1}{2}\cdot\dfrac{z}{1-(z/2)}+\dfrac{1}{1-2z}$; $\dfrac{1}{2}<|z|<2$, (20) $\dfrac{1}{4}\cdot\dfrac{z}{1-(z/4)}+\dfrac{1}{1-(1/4z)}$; $\dfrac{1}{4}<|z|<4$,

(21) $\dfrac{az}{1-az}+\dfrac{1}{1-(a/z)}$; $|a|<|z|<\dfrac{1}{|a|}$, (22) $e^{3/z}$, ROC z plane,

(23) $e^{5/z}$, ROC z plane, (24) $\left(1-\dfrac{e^{\alpha}}{z}\right)^{-1}$; $|z|>|e^{\alpha}|$,

(25) $$Z\{f(k)\} = \sum_{-1}^{-\infty} 2^k z^{-k} + \sum_{k=0}^{2n}\left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=0}^{2n-1}\left(\frac{1}{3}\right)^k z^{-k} \text{ as } n\to\infty$$

$$= \sum_{1}^{\infty}\left(\frac{z}{2}\right)^k + \sum_{k=0}^{2n}\left(\frac{1}{2z}\right)^k + \sum_{k=1}^{2n-1}\left(\frac{1}{3z}\right)^k$$

$$= \frac{z}{2-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1} \quad ; \quad \frac{1}{2}<|z|<2\ ]$$

7. Properties of Z-transforms

As in the Laplace transforms we have the following properties of Z-transforms. We shall prove below these properties and use them to solve some problems.

(1) Linearity

If a and b are constants and $\{f(k)\}$ and $\{g(k)\}$ are two sequences which can be added then,

$$\boxed{Z\{a\,f(k)+b\,g(k)\} = a\,Z\{f(k)\}+b\,Z\{g(k)\}}$$

Proof : We have by definition,

$$Z\{a f(k)+b g(k)\} = \sum_{k=-\infty}^{\infty}[a f(k)+b g(k)]z^{-k}$$

$$= \Sigma[a f(k)z^{-k}+b g(k)z^{-k}] = a\,\Sigma f(k)z^{-k}+b\,\Sigma g(k)z^{-k}$$

$$= a\,Z\{f(k)\}+b\,Z\{g(k)\}$$

Corollary : If $a=b=1$, we have $Z\{f(k)+g(k)\} = Z\{f(k)\}+Z\{g(k)\}$ where, $\{f(k)\}$ and $\{g(k)\}$ can be added.

In words, this means, the Z-transform of the sum (or difference when $b=-1$) of two sequences which can be added (or subtracted) is equal to the sum (or difference) of the Z-transforms of the two sequences.

We shall now use this property to solve some problems.

Example 1 : Find $Z\{a^{|k|}\}$. **(M.U. 2008, 13, 18)**

Sol. : We have

$$Z\{a^{|k|}\} = \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= [\ldots + a^3z^3 + a^2z^2 + az] + [1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots]$$

$$= az(1 + az + az^2 + a^3z^3 + \ldots) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots\right)$$

$$= az \cdot \frac{1}{1-az} + 1 \cdot \frac{1}{1-(a/z)} = \frac{az}{1-az} + \frac{z}{z-a}$$

$$= \frac{az^2 - a^2z + z - az^2}{(1-az)(z-a)} = \frac{z(1-a^2)}{(1-az)(z-a)}.$$

$$\boxed{Z\{a^{|k|}\} = \frac{z(1-a^2)}{(1-az)(z-a)}}$$

The series in G.P. are convergent if $1 > |az|$ and $|z| > a$ *i.e.* $\frac{1}{a} > |z|$ and $|z| > a$.

$\therefore$ The ROC is $(1/a) > |z| > a$.

Example 2 : Find the Z-transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$. **(M.U. 2014)**

Sol. : We have

$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \Sigma\left(\frac{1}{3}\right)^{|k|} \cdot z^{-k} = \sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=\infty}^{\infty}\left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[\ldots + \left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right) z\right] + \left[1 + \frac{1}{3}\cdot z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} + \ldots\right]$$

$$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \ldots\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots\right]$$

$$= \frac{z}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \ldots\right]\left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^2 + \ldots\right]$$

$$= \frac{z}{3} \cdot \frac{1}{1-(z/3)} + 1 \cdot \frac{1}{1-[1/(3z)]}, \quad \left|\frac{z}{3}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

$$= \frac{z}{3} \cdot \frac{3}{3-z} + \frac{3z}{3z-1} = \frac{z}{3-z} + \frac{3z}{3z-1}, \quad |z| < 3, \frac{1}{3} < |z|$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3} < |z| < 3.$$

Remark ....

The above Ex. 2 is a particular case of Ex. 1 where $a = 1/3$.

Example 3 : Find the Z-transform $\{3^{|k|}\}$.

Sol. : Putting $a = 3$ in Ex. 1 or proceeding independently, we get

$$Z\{3^{|k|}\} = \frac{-8z}{(1-3z)(z-3)}$$

Example 4 : Find the Z-transform of $f(k) = c^k \cos \alpha k$, $k \geq 0$, where α is real.

Sol. : Assuming $f(k) = 0$ for $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos \alpha k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right] z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \cdot \frac{e^{i\alpha k}}{2} z^{-k} + \sum_{k=0}^{\infty} c^k \cdot e^{-i\alpha k} z^{-k}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{c\,e^{i\alpha}}{z}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{c\,e^{-i\alpha}}{z}\right)^k$$

$$= \frac{1}{2} \cdot \left[\frac{1}{1-(ce^{i\alpha}/z)}\right] + \frac{1}{2}\left[\frac{1}{1-(ce^{-i\alpha}/z)}\right] \qquad \text{.................... (1)}$$

[See note (3), page 3-6]

$$= \frac{1}{2}\left[\frac{z}{z-c\,e^{i\alpha}} + \frac{z}{z-c\,e^{-i\alpha}}\right] = \frac{z}{2}\left[\frac{z - c\,e^{-i\alpha} + z - ce^{i\alpha}}{z^2 - zc\,(e^{i\alpha} + e^{-i\alpha}) + c^2}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2c\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right)}{z^2 - 2zc\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right) + c^2}\right] = \frac{z}{2} \cdot 2\left[\frac{z - c\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right)}{z^2 - 2zc\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right) + c^2}\right]$$

$$= \frac{z\,(z - c\cos\alpha)}{z^2 - 2zc\cos\alpha + c^2} \qquad \text{.................... (2)}$$

$$\therefore \quad \boxed{\mathbf{Z\,(c^k \cos \alpha k) = \frac{z\,(z - c\cos\alpha)}{z^2 - 2zc\cos\alpha + c^2}}}$$

From (1), we find that the series being in G.P. are convergent if $|z| > |ce^{i\alpha}|$ and $|z| > |ce^{-i\alpha}|$ i.e. if $|z| > |c\,(\cos\alpha \pm i\sin\alpha)|$ i.e. if $|z| > |c|$.

Example 5 : Find the Z-transform of

(i) $f(k) = \cos \alpha k$, $k > 0$ where α is real. **(ii)** $f(k) = \cos \dfrac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$\boxed{\mathbf{Z\,\{\cos \alpha\, k\} = \frac{z\,(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1};\ |z| > 1}}$$

Putting $\alpha = \frac{\pi}{3}$, $Z\left\{\cos\frac{k\pi}{3}\right\} = \frac{z\,(z - 1/2)}{z^2 - z + 1}$; $|z| > 1$

Example 6 : Find the Z-transform of $f(k) = c^k \sin \alpha k,\ k \geq 0.$

Sol. : Following the above lines we find that

$$\boxed{Z\{c^k \sin \alpha k\} = \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2},\ |z| > |c|}$$

Example 7 : Find the Z-transform of

(i) $f(k) = \sin \alpha k,\ k \geq 0$ where α is real. **(ii)** $f(k) = \sin \frac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$\boxed{Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1};\ |z| > 1}$$

Putting $\alpha = \frac{\pi}{3}$, $Z\left\{\sin \frac{k\pi}{3}\right\} = \frac{\sqrt{3}\, z/2}{z^2 - z + 1}$.

Example 8 : Find the Z-transform of $f(k) = c^k \cos h\, \alpha\, k,\ k \geq 0.$

Sol. : By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos h\, \alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left(\frac{e^{\alpha k} + e^{-\alpha k}}{2}\right) \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{c^k e^{\alpha k}}{2} \cdot z^{-k} + \sum_{k=0}^{\infty} \frac{c^k e^{-\alpha k}}{2} \cdot z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{\alpha}}{z}\right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{-\alpha}}{z}\right)^k$$

$$= \frac{1}{2}\left[\frac{1}{1 - (ce^{\alpha}/z)}\right] + \frac{1}{2}\left[\frac{1}{1 - (ce^{-\alpha}/z)}\right] \qquad \text{[By note (3), page 3-6]}$$

$$= \frac{1}{2}\left[\frac{z}{z - ce^{\alpha}} + \frac{z}{z - ce^{-\alpha}}\right] = \frac{z}{2}\left[\frac{1}{z - ce^{\alpha}} + \frac{1}{z - ce^{-\alpha}}\right]$$

$$= \frac{z}{2}\left[\frac{z - ce^{-\alpha} + z - ce^{\alpha}}{z^2 - cz(e^{\alpha} + e^{-\alpha}) + c^2}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2c\left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right)}{z^2 - 2cz\left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right) + c^2}\right] = \frac{z}{2} \cdot 2\left[\frac{z - c\left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right)}{z^2 - 2cz\left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right) + c^2}\right]$$

$$= \frac{z(z - c \cos h\, \alpha)}{z^2 - 2cz \cos h\alpha + c^2} \qquad |z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|)$$

Corollary : Putting $c = 1$, we get if $f(k) = \cos h\, \alpha\, k$, then

$$Z\{f(k)\} = \frac{z(z - \cos h\, \alpha)}{z^2 - 2z \cos h\alpha + c^2} \qquad |z| > \max(|e^{\alpha}|, |e^{-\alpha}|)$$

Example 9 : Find the Z-transform of $f(k) = c^k \sin h\, \alpha k,\ k \geq 0$.

Sol. : Following the above lines, we find that,

$$Z\{f(k)\} = \frac{cz \sin h\alpha}{z^2 - 2cz \cos h\alpha + c^2} \qquad |z| > \max(|c e^{\alpha}|, |c e^{-\alpha}|)$$

Corollary : Putting $c = 1$, we get, if $f(k) = \sin h\, \alpha k$, then

$$Z\{f(k)\} = \frac{z \sin h\alpha}{z^2 - 2z \cos h\alpha + 1} \qquad |z| > \max(|e^{\alpha}|, |e^{-\alpha}|)$$

Example 10 : Find the Z-transform of $\left\{\sin\left(\frac{k\pi}{3} + \alpha\right)\right\},\ k \geq 0$.

Sol. : We have

$$Z\{f(k)\} = Z\left\{\sin\left(\frac{k\pi}{3} + \alpha\right)\right\} = Z\left\{\sin\frac{k\pi}{3}\cos\alpha + \cos\frac{k\pi}{3}\sin\alpha\right\}$$

$$= \cos\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\} + \sin\alpha \cdot Z\left\{\cos\frac{k\pi}{3}\right\}$$

$$= \cos\alpha \cdot \frac{z\sin(\pi/3)}{z^2 - 2z\cos(\pi/3) + 1} + \sin\alpha \cdot \frac{z^2 - z\cos(\pi/3)}{z^2 - 2z\cos(\pi/3) + 1}$$

[By Ex. 7 and 5 above]

$$= \frac{z\{z\sin\alpha - \sin\alpha\cos(\pi/3) + \cos\alpha\sin(\pi/3)\}}{z^2 - 2z\cos(\pi/3) + 1}$$

$$= \frac{z\{z\sin\alpha + \sin[(\pi/3) - \alpha]\}}{z^2 - 2z\cos(\pi/3) + 1} = \frac{z[z\sin\alpha + \sin(\pi/3 - \alpha)]}{z^2 - z + 1}.$$

Example 11 : Find the Z-transform of $\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\},\ k \geq 0$.

(M.U. 2009)

Sol. : We have

$$Z\{f(k)\} = Z\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\} = Z\left\{\cos\frac{k\pi}{3} \cdot \cos\alpha - \sin\frac{k\pi}{3} \cdot \sin\alpha\right\}$$

$$= \cos\alpha \cdot Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\}$$

$$\therefore\ F(z) = \cos\alpha \cdot \frac{z^2 - z\cos(\pi/3)}{z^2 - 2z\cdot\cos(\pi/3) + 1} - \sin\alpha \cdot \frac{z\sin(\pi/3)}{z^2 - 2z\cdot\cos(\pi/3) + 1}$$

$$= \frac{z\{z\cos\alpha - [\cos(\pi/3)\cdot\cos\alpha + \sin(\pi/3)\sin\alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1}$$

$$= \frac{z\{z\cos\alpha - \cos[(\pi/3) - \alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1} = \frac{z[z\cos\alpha - \cos(\pi/3 - \alpha)]}{z^2 - z + 1}$$

Example 12 : Find the Z-transform of $\{\sin(ak + b)\},\ k \geq 0$.

(M.U. 2008)

Sol. : We have $\sin(ak + b) = \sin ak \cos b + \cos ak \sin b$

$$\therefore\ Z\{\sin(ak + b)\} = \cos b \cdot Z\{\sin ak\} + \sin b \cdot Z\{\cos ak\}$$

$$= \cos b \cdot \frac{z\sin a}{z^2 - 2z\cos a + 1} + \sin b \cdot \frac{z(z - \cos a)}{z^2 - 2z\cos a + 1}$$

$$\therefore \quad Z\{\sin(ak+b)\} = \frac{z[\sin a \cos b - \cos a \sin b + z \sin b]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[\sin(a-b) + z\sin b]}{z^2 - 2z\cos a + 1}.$$

Example 13 : Find the Z-transform of $\{\cos(ak+b)\}$, $k \geq 0$.

Sol. : We have $\cos(ak+b) = \cos ak \cos b - \sin ak \sin b$

$$\therefore \quad Z\{\cos(ak+b)\} = \cos b \cdot Z\{\cos ak\} - \sin b \cdot Z\{\sin ak\}$$

$$= \cos b \cdot \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1} - \sin b \cdot \frac{z\sin a}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[z\cos b - (\cos a \cos b + \sin a \sin b)]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[z\cos b - \cos(a-b)]}{z^2 - 2z\cos a + 1}.$$

EXERCISE - IV

Find the Z-transforms of $\{f(k)\}$ where $f(k)$ is given by $(k \geq 0)$.

1. $2^{|k|}$ 2. $\left(\frac{1}{2}\right)^k$ 3. $\cos k$ 4. $\cos 2k$ 5. $\sin k$

6. $\sin 2k$ 7. $\cos hk$ 8. $\cos h\, 2k$ 9. $\sin h\, k$ 10. $\sin h\, 2k$

11. $\sin(k+1)$ 12. $2^k \cos k$ 13. $\sin(3k+2)$ 14. $\cos(3k+2)$

15. $\sin\left(\alpha k + \frac{\pi}{2}\right)$ 16. $\cos\left(\alpha k + \frac{\pi}{2}\right)$ 17. $\sin\left(\frac{k\pi}{4} + a\right)$ **(M.U. 2008)**

[**Ans. :** (1) $-\frac{3z}{(1-2z)(z-2)}$, (2) $\frac{3z}{(2-z)(2z-1)}$, (3) $\frac{z(z-\cos 1)}{z^2 - 2z\cos 1 + 1}$, (4) $\frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}$,

(5) $\frac{z\sin 1}{z^2 - 2\cos 1 + 1}$, (6) $\frac{z\sin 2}{z^2 - 2\cos 2 + 1}$, (7) $\frac{z(z-\cos h 1)}{z^2 - 2z\cos h 1 + 1}$, (8) $\frac{z(z-\cos h 2)}{z^2 - 2z\cos h 2 + 1}$,

(9) $\frac{\sin h 1}{z^2 - 2z\cos h 1 + 1}$, (10) $\frac{\sin h 2}{z^2 - 2z\cos h 2 + 1}$, (11) $\frac{z^2\sin 1}{z^2 - 2z\cos 1 + 1}$,

(12) $\frac{z^2 - 2\cos 1 \cdot z}{z^2 - 4\cos 1 \cdot z + 4}$, (13) $\frac{z(\sin 1 + z\sin 2)}{z^2 - 2z\cos 3 + 1}$, (14) $\frac{z(z\cos 2 - \cos 1)}{z^2 - 2z\cos 3 + 1}$,

(15) $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$, (16) $\frac{-z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$, (17) $\frac{z\sin[(\pi/4) - a] + z^2\sin a}{z^2 - \sqrt{2}\cdot z + 1}$.]

(2) Change of Scale

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F(z/a)$ and if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ is $|a| R_1 < |z| < |a| R_2$.

Proof : By definition, $F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

Replacing z by z/a, we get, $F\left(\frac{z}{a}\right) = \sum f(k)\left(\frac{z}{a}\right)^{-k}$ .... (1)

But by definition again, $Z\{a^k f(k)\} = \sum_{k=-\infty}^{\infty} a^k f(k) z^{-k} = \sum f(z)\left(\frac{z}{a}\right)^{-k}$

From (1) and (2), we get $Z\left\{a^k f(k)\right\} = F\left(\frac{z}{a}\right)$ ..... (2)

Further, if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ from (2) will be $R_1 < |z/a| < R_2$ i.e. $|a| R_1 < |z| < |a| R_2$.

Example 1 : Obtain $Z\{1\}$ and hence deduce $Z\{a^k\}$, $k \geq 0$.

Sol. : By definition

$$Z\{1\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots$$

$$= \frac{1}{1-(1/z)} = \frac{z}{z-1}.$$

Now, $a^k = a^k \cdot 1$. Hence, by change of scale property,

$$Z\{a^k\} = Z\{a^k \cdot 1\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}.$$

Example 2 : Find $Z\{c^k \sin \alpha k\}$ from $Z\{\sin \alpha k\}$

Sol. : We know that, $Z\{\sin \alpha k\} = \dfrac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}$

By change of scale property,

$$Z\{c^k \sin \alpha k\} = \frac{(z/c)\sin\alpha}{(z/c)^2 - 2(z/c)\cos\alpha + 1} = \frac{cz \sin\alpha}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 3 : Find $Z\{c^k \cos \alpha k\}$ from $Z\{\cos \alpha k\}$.

Sol. : We know that $Z\{\cos \alpha k\} = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$

By change of scale property,

$$Z\{c^k \cos \alpha k\} = \frac{\frac{z}{c}\left\{\frac{z}{c} - \cos\alpha\right\}}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

(3) Shifting Property

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{f(k+n)\} = z^n F(z)$ and $Z\{f(k-n)\} = z^{-n} F(z)$.

Proof : We have $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$

$$\therefore \quad Z\{f(k+n)\} = \sum f(k+n) z^{-k} = \sum f(k+n) z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)}$$

If we put $k + n = m$ when $k = -\infty$, $m = -\infty$ and when $k = +\infty$, $m = +\infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^n F(z)$$

Changing the sign of n or proceeding as above $Z\{f(k-n)\} = z^{-n} F(z)$.

Example : Find the Z-transform of $\frac{1}{k+1}$, $k \geq 1$. Indicate the region of convergence.

(M.U. 2009)

Sol. : By Ex. 5, page 3-10, putting $\alpha = 1$, we have

$$Z\left(\frac{1}{k}\right) = -\log\left(1 - \frac{1}{z}\right), \quad |z| > 1$$

By Shifting property, $Z\{f(k-n)\} = z^{-n} F(z)$.

Hence, $$Z\left[\frac{1}{k+1}\right] = z \cdot \left\{-\log\left(1 - \frac{1}{z}\right)\right\} = -z\log\left(1 - \frac{1}{z}\right), \quad |z| > 1. \qquad [\text{Put } n = -1]$$

R.O.C. is $|z| > 1$.

Unilateral or one sided or causal sequence : If a sequence $\{f(k)\}$ is defined for the right side only *i.e.* if $\{f(k)\}$ extends to infinity on the right only *i.e.* if k varies from 0 to $+\infty$, the sequence is called **unilateral** or **one sided** or **causal sequence**.

For example, $f(k) = \begin{cases} 0, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$ is a causal sequences.

Theorem : If $\{f(k)\}$ is one sided and if $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$,

then $$Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$$

and $$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r) z^{-n+r}$$

Proof : Since $\{f(k)\}$ is one sided sequence, by data,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} \qquad \text{.................... (1)}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=0}^{\infty} f(k+n) z^{-k} = \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)}$$

Put $k + n = m$. When $k = 0$, $n = m$; when $k = \infty$, $m = \infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=n}^{\infty} f(m) z^{-m}$$

The interval from n to ∞ can be changed to the interval from 0 to ∞ by subtracting from it, the terms in the interval 0 to $n - 1$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=0}^{\infty} f(m) z^{-m} - z^n \sum_{m=0}^{n-1} f(m) z^{-m}$$

$$= z^n \cdot F(z) - z^n \sum_{m=0}^{n-1} f(m) z^{-m}$$

Taking z^n inside the summation,

$$\therefore \quad Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}.$$

Further, from (1) again,

$$Z\{f(k-n)\} = \sum_{k=0}^{\infty} f(k-n)z^{-k} = \sum_{k=0}^{\infty} f(k-n)z^{-(k-n)} \cdot z^{-n}$$

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{k=0}^{\infty} f(k-n)z^{-(k-n)}$$

Put $k-n=m$. When $k=0$, $m=-n$ and when $k=\infty$, $m=\infty$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{k=-n}^{\infty} f(m)z^{-m}$$

As before we split the interval from $-n$ to ∞ into two intervals, $-n$ to -1 and 0 to ∞.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{m=-n}^{-1} f(m)z^{-m} + z^{-n} \sum_{m=0}^{\infty} f(m)z^{-m}$$

$$= z^{-n} F(z) + z^{-n} \sum_{m=-n}^{-1} f(m)z^{-m}$$

Taking z^{-n} inside the summation,

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z) + \sum_{m=-n}^{-1} f(m)z^{-(m+n)}$$

Putting $m=-r$, when $m=-1$, $r=1$ and when $m=-n$, $r=n$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r)z^{-n+r}.$$

Corollary 1 : For one sided sequence ($k \geq 0$), we have

$$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r)z^{-n+r}$$

Since for causal sequence $f(-1) = f(-2) = \ldots\ldots f(-n) = 0$, the second term is zero.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z).$$

Corollary 2 : Since for one sided sequence ($k \geq 0$), we have

$$Z\{f(k+n)\} = z^{n} F(z) - \sum_{m=0}^{n-1} f(m)z^{n-m}$$

Putting $n=1$,

$$Z\{f(k+1)\} = z^{1} F(z) - \sum_{m=0}^{0} f(0)z^{1-0} = zF(z) - zf(0)$$

Putting $n=2$, ... (A)

$$Z\{f(k+2)\} = z^{2} F(z) - \sum_{m=0}^{2-1} f(m)z^{2-m} = z^{2} F(z) - \left[f(0)z^{2-0} + f(1)z^{2-1}\right]$$

$$= z^{2} F(z) - z^{2} f(0) - zf(1).$$

(4) Multiplication by k

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{k f(k)\} = -z \dfrac{d}{dz} F(z)$.

Proof : We have, by definition,

$$Z\{k\,f(k)\} = \sum_{k=-\infty}^{\infty} k\,f(k)\,z^{-k} = \sum k\,f(k)\,z^{-k-1}\cdot z$$

$$= -z\sum -k\,f(k)\,z^{-k-1} = -z\sum f(k)\frac{d}{dz}(z^{-k})$$

$$\therefore \quad Z\{k\,f(k)\} = -z\cdot\frac{d}{dz}\sum f(k)\,z^{-k} = -z\frac{d}{dz}F(z)$$

In general, $Z\{k^n f(k)\} = \left(-z\dfrac{d}{dz}\right)^n F(z)$ ...... (B)

Note that $\left(-z\dfrac{d}{dz}\right)^2 \neq z^2\dfrac{d^2}{dz^2}$, but it is equal to repeated operations $\left(-z\dfrac{d}{dz}\right)\left(-z\dfrac{d}{dz}\right)$.

Corollary 1 : $Z\{k\} = \dfrac{z}{(z-1)^2}$, $|z| > 1$

Proof : By definition,

$$Z\{1\} = \sum_{k=0}^{\infty} 1\cdot z^{-k} = 1 + z^{-1} + z^{-2} + \ldots..$$

$$\therefore \quad Z\{1\} = \frac{1}{1-z^{-1}} = \frac{1}{1-(1/z)},\quad |z^{-1}| < 1$$

$$= \frac{z}{z-1},\quad |z| > 1$$

Now, by the above theorem,

$$Z\{k\} = Z\{k\cdot 1\} = \left(-z\frac{d}{dz}\right)[Z\{1\}]$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = -z\left[\frac{(z-1)1 - z\cdot 1}{(z-1)^2}\right]$$

$$= -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2},\quad |z| > 1.$$

Corollary 2 : $Z\{k^2\} = \dfrac{z(z+1)}{(z-1)^3}$.

Proof : We have proved above that $Z\{1\} = \dfrac{z}{z-1}$.

By the above theorem,

$$Z\{k^2\} = Z\{k^2\cdot 1\} = \left(-z\frac{d}{dz}\right)^2 Z\{1\}$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = \left(-z\frac{d}{dz}\right)\left[\frac{z}{(z-1)^2}\right] \qquad \text{[As above]}$$

$$= -z\left[\frac{(z-1)^2\cdot 1 - z\cdot 2(z-1)\cdot 1}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1)-2z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}.$$

Example : Find Z-transform of $k^2 e^{-2k}$. [See also Ex. 10, page 3-31]

Sol. : We have, $Z\{e^{-2k}\} = Z\{(e^{-2})^k\}$

$$= \frac{z}{z-e^{-2}}$$ [See Ex. 6, page 3-11]

$$= \frac{e^2 z}{e^2 z - 1} \quad \text{for } |z| > e^{-2}.$$

Hence, by the above result (B),

$$Z\{k^2 e^{-2k}\} = (-1)^2 \left\{z\frac{d}{dz}\right\}^2 \left[\frac{e^2 z}{e^2 z - 1}\right] = \left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right)\left(\frac{e^2 z}{e^2 z - 1}\right)$$

$$= \left(z\frac{d}{dz}\right)\left[(e^2 z)\frac{d}{dz}\left(\frac{z}{e^2 z - 1}\right)\right] = z\frac{d}{dz}\left[e^2 z\left\{\frac{(e^2 z - 1)\cdot 1 - z\cdot e^2}{(e^2 z - 1)^2}\right\}\right]$$

$$= -ze^2\frac{d}{dz}\left[\frac{z}{(e^2 z - 1)^2}\right] = -ze^2\left[\frac{(e^2 z - 1)^2\cdot 1 - z\cdot 2(e^2 z - 1)e^2}{(e^2 z - 1)^4}\right]$$

$$= -ze^2\left[\frac{(e^2 z - 1) - z\cdot 2e^2}{(e^2 z - 1)^3}\right] = -ze^2\left[-\frac{e^2 z + 1}{(e^2 z - 1)^3}\right]$$

$$= -ze^2\cdot\frac{e^2 z + 1}{(e^2 z - 1)^3}$$

(5) Division by k

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\left\{\frac{f(k)}{k}\right\} = -\int z^{-1} F(z)\,dz$.

Proof : We have by definition,

$$Z\left\{\frac{f(k)}{k}\right\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k}\cdot z^{-k} = \sum_{k=-\infty}^{\infty} f(z)\cdot\left(\frac{z^{-k}}{k}\right)$$

$$= -\sum_{k=-\infty}^{\infty} f(k)\cdot\int z^{-k-1}\,dz = -\int \Sigma\, f(k) z^{-k-1}\,dz$$

$$= -\int z^{-1}\sum_{k=-\infty}^{\infty} f(k) z^{-k}\,dz = -\int z^{-1} F(z)\,dz$$

(6) Convolution of Two One Sided or Causal Sequences {f (k)} and {g (k)}

Definition : Convolution of two sequences $\{f(k)\}$ and $\{g(k)\}$ denoted by $\{f(k)\} * \{g(k)\}$ is given by

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\, g(k-m) = f_0 g_k + f_1 g_{k-1} + \ldots + f_k g_0$$

We shall denote their convolution by $\{h(k)\}$, *i.e.*, $\{h(k)\} = \{f(k)\} * \{g(k)\}$.

Convolution Theorem : Let $\{f(k)\}$ and $\{g(k)\}$ be any two one sided or causal sequences. Let Z-transform of $\{f(k)\}$, *i.e.*, $Z\{f(k)\} = F(z)$ exist in the region $|z| > (1/R_1)$ and Z-transform of $\{g(k)\}$, *i.e.*, $Z\{g(k)\} = G(z)$ exist in the region $|z| > (1/R_2)$, then Z-transform of their convolution, *i.e.*, $Z\{h(k)\}$ where $\{h(k)\} = \{f(k)\} * \{g(k)\}$ is given by

$$Z\{h(k)\} = Z\{f(k)\}\, Z\{g(k)\}$$

$$H(z) = F(z)\,G(z)$$

i.e., $Z[\{f(k)\} * \{g(k)\}] = F(z)\,G(z)$

i.e., valid in the region $|z| > (1/R)$ where $(1/R) = \max[(1/R_1) \text{ and } (1/R_2)]$.

Proof : $F(z)\,G(z) = Z\{f(k)\}\,Z\{g(k)\}$

$$= \left[\sum_{k=0}^{\infty} f(k) z^{-k}\right]\left[\sum_{k=0}^{\infty} g(k) z^{-k}\right]$$

$$= [f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots + f_k z^{-k} + \ldots]\,[g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots + g_k z^{-k} + \ldots]$$

$$= f_0 g_0 + (f_0 g_1 + f_1 g_0) z^{-1} + (f_0 g_2 + f_1 g_1 + f_2 g_0) z^{-2} + \ldots + (f_0 g_k + f_1 g_{k-1} + \ldots + f_k g_0) z^{-k} + \ldots$$

$$= \Sigma\,(\text{coefficient of } z^{-k})\, z^{-k}$$

$$= \sum_{k=0}^{\infty} (f_0 g_k + f_1 g_{k-1} + \ldots + f_k g_0) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left[\sum_{m=0}^{k} f(m)\, g(k-m)\right] z^{-k}$$

$$= \sum_{k=0}^{\infty} [\{f(k)\} * \{g(k)\}]\, z^{-k} = Z\{h(k)\}$$

$\therefore \quad F(z)\,G(z) = H(z)$

Example 1 : Using the definition of convolution of two sequences, prove that

$$\left\{\frac{1}{k!}\right\} * \left\{\frac{1}{k!}\right\} = \frac{2^k}{k!}.$$

Sol. : By the definition of convolution of two sequences,

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{\infty} f(m)\, g(k-m)$$

$$\therefore \quad \left\{\frac{1}{k!}\right\} * \left\{\frac{1}{k!}\right\} = \sum_{m=0}^{k} \frac{1}{m!} \cdot \frac{1}{(k-m)!}$$

$$= \frac{1}{k!} + \frac{1}{1!\,(k-1)!} + \frac{1}{2!\,(k-2)!} + \ldots + \frac{1}{k!}$$

$$= \frac{1}{k!} + \frac{k}{k!} \cdot \frac{k(k-1)}{2!\,k!} + \ldots + \frac{1}{k!}$$

$$= \frac{1}{k!}\left[1 + k + \frac{k(k-1)}{2!} + \ldots + 1\right]$$

$$= \frac{1}{k!} \cdot (1+1)^k = \frac{2^k}{k!}$$

Example 2 : Verify convolution theorem for $\{f(k)\} = \{k\}$ and $\{g(k)\} = \{k\}$.

Sol. : We have, $Z\{f(k)\} = Z\{k\} = \dfrac{z}{(z-1)^2}$ [By Cor. (1), page 3-23]

Also, $Z\{g(k)\} = Z\{k\} = \dfrac{z}{(z-1)^2}$

$$\therefore \quad F(z)\,G(z) = Z\{f(k)\}\,Z\{g(k)\} = \frac{z}{(z-1)^2} \cdot \frac{z}{(z-1)^2} = \frac{z^2}{(z-1)^4} \qquad \dots\dots (1)$$

Now, $$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\,g(k-m) = \sum_{m=0}^{k} m\,(k-m)$$

$$= \sum_{m=0}^{k} (mk - m^2) = \sum_{m=0}^{k} mk - \sum_{m=0}^{k} m^2$$

$$= k \sum_{m=0}^{k} m - \sum_{m=0}^{k} m^2$$

$$\therefore \quad \{f(k)\} * \{g(k)\} = k\left[\frac{k(k+1)}{2}\right] - \left[\frac{k(k+1)(2k+1)}{6}\right]$$

$$\left[\because\ 1+2+3+\dots+n = \frac{n(n+1)}{2} \text{ and } 1^2+2^2+3^2+\dots+n^2 = \frac{n(n-1)(2n+1)}{6}\right]$$

$$\{f(k)\} * \{g(k)\} = \frac{k(k^2+k)}{2} - \frac{k(2k^2+3k+1)}{6}$$

$$= \frac{k^3+2k^2}{2} - \frac{2k^3+3k^2+k}{6}$$

$$\therefore \quad \{f(k)\} * \{g(k)\} = \frac{3k^3+3k^2-2k^3-3k^2-k}{6} = \frac{k^3-k}{6}$$

$$\therefore \quad Z[\{f(k)\} * \{g(k)\}] = Z\left[\frac{k^3-k}{6}\right] = \frac{1}{6}[Z\{k^3\} - Z\{k\}]$$

$$= \frac{1}{6}\left[\frac{z^3+4z^2+z}{(z-1)^4} - \frac{z}{(z-1)^2}\right] \qquad \text{[See (A), page 3-27]}$$

$$= \frac{1}{6} \cdot \frac{1}{(z-1)^4}[z^3+4z^2+z-z(z-1)^2]$$

$$= \frac{1}{6} \cdot \frac{1}{(z-1)^4}[z^3+4z^2+z-z(z^2-2z+1)]$$

$$= \frac{1}{6} \cdot \frac{1}{(z-1)^4}[z^3+4z^2+z-z^3+2z^2-z]$$

$$\therefore \quad Z[\{f(k)\} * \{g(k)\}] = \frac{1}{6} \cdot \frac{1}{(z-1)^4}(6z^2) = \frac{1}{6} \cdot \frac{z^2}{(z-1)^4} \qquad \dots\dots (2)$$

From (1) and (2), we have

$$F(z)\,G(z) = Z[\{f(k)\} * \{g(k)\}] = Z\{h(k)\} = H(z).$$

Example 3 : Verify convolution theorem for $f(k) = k$ and $g(k) = k^2$.

Sol. : By definition, convolution of $\{f(k)\}$ and $\{g(k)\}$ is

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\,g(k-m) = \sum_{m=0}^{k} g(m)\,f(k-m)$$

$$= \sum_{m=0}^{k} m^2(k-m) = \sum_{m=0}^{k} (m^2 \cdot k) - \sum_{m=0}^{k} m^3$$

$$= k \sum_{m=0}^{k} m^2 - \sum_{m=0}^{k} m^3$$

$$= k\left[\frac{k(k+1)(2k+1)}{6}\right] - \frac{k^2(k+1)^2}{4}$$

$$= k^2(k+1)\left[\frac{2k+1}{6} - \frac{k+1}{4}\right]$$

$$= k^2(k+1)\left[\frac{1}{2}\left\{\frac{2k+1}{3} - \frac{k+1}{2}\right\}\right]$$

$$= k^2(k+1)\left[\frac{1}{2}\left(\frac{4k+2-3k-3}{6}\right)\right]$$

$$= \frac{k^2(k+1)}{12}(k-1) = \frac{k^2}{12}(k^2-1)$$

$$\therefore \quad \{f(k)\} * \{g(k)\} = \frac{k^4 - k^2}{12} \qquad \text{.............. (1)}$$

We shall now find $Z\{k^4\}$.

Now, $Z\{k^2\} = \dfrac{z(z+1)}{(z-1)^3}$ [By Cor. (2), page 3-23]

$\therefore$ By multiplication theorem, page 3-22,

$$Z\{k^3\} = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right] = -z\left[\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)^2[(z-1)(2z+1) - 3(z^2+z)]}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1) - 3(z^2+z)}{(z-1)^4}\right]$$

$$= -z\left[\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right] = -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$\therefore \quad Z\{k^3\} = \frac{z(z^2+4z+1)}{(z-1)^4} = \frac{z^3+4z^2+z}{(z-1)^4} \qquad \text{.............. (A)}$$

Now, $Z\{k^4\} = -z\dfrac{d}{dz}Z\{k^3\} = -z\dfrac{d}{dz}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right]$ [By multiplication theorem]

$$= -z\left[\frac{(z-1)^4(3z^2+8z+1) - (z^3+4z^2+z)4(z-1)^3}{(z-1)^8}\right]$$

$$= -z(z-1)^3\left[\frac{(z-1)(3z^2+8z+1) - 4(z^3+4z^2+z)}{(z-1)^8}\right]$$

$$= \frac{-z(3z^3+8z^2+z-3z^2-8z-1-4z^3-16z^2-4z)}{(z-1)^5}$$

$$\therefore \quad Z\{k^4\} = \frac{-z(-z^3-11z^2-11z-1)}{(z-1)^5} = \frac{z(z^3+11z^2+11z+1)}{(z-1)^5}$$

Hence, from (1), we get,

$$\therefore \quad Z\{f(k) * g(k)\} = Z\left\{\frac{k^4 - k^2}{12}\right\} = \frac{1}{12}[Z\{k^4\} - Z\{k^2\}]$$

$$= \frac{1}{12}\left[\frac{z^4 + 11z^3 + 11z^2 + z}{(z-1)^5} - \frac{z^2 + z}{(z-1)^3}\right]$$

$$= \frac{1}{12} \cdot \frac{1}{(z-1)^5}[z^4 + 11z^3 + 11z^2 + z - (z^2 + z)(z-1)^2]$$

$$= \frac{1}{12} \cdot \frac{1}{(z-1)^5}[z^4 + 11z^3 + 11z^2 + z - (z^2 + z)(z^2 - 2z + 1)]$$

$$= \frac{1}{12} \cdot \frac{1}{(z-1)^5}[z^4 + 11z^3 + 11z^2 + z - (z^4 - 2z^3 + z^2 + z^3 - 2z^2 + z)]$$

$$= \frac{1}{12} \cdot \frac{1}{(z-1)^5}[12z^3 + 12z^2] = \frac{1}{12} \cdot \frac{1}{(z-1)^5} \cdot 12(z^3 + z^2)$$

$$\therefore \quad Z\{f(k) * g(k)\} = \frac{z^3 + z^2}{(z-1)^5} \qquad \ldots\ldots (2)$$

Now, $Z\{f(k)\} = Z\{k\} = \dfrac{z}{(z-1)^2}$ [By Cor. (1), page 3-23]

and $Z\{g(k)\} = Z\{k^2\} = \dfrac{z(z+1)}{(z-1)^3}$ [By Cor. (2), page 3-23]

$$\therefore \quad Z\{f(k)\}\, Z\{g(k)\} = \frac{z}{(z-1)^2} \cdot \frac{z(z+1)}{(z-1)^3} = \frac{z^3 + z^2}{(z-1)^5} \qquad \ldots\ldots (3)$$

Hence, from (2) and (3), we get

$$Z\{f(k)\}\, Z\{g(k)\} = Z\{f(k) * g(k)\} = Z\{h(k)\}$$

$$\therefore \quad F(z)\, G(z) = H(z)$$

Thus, convolution theorem is verified.

Example 4 : If $f(k) = U(k)$ and $g(k) = 2^k\, U(k)$, find Z-transform of $\{f(k) * g(k)\}$. **(M.U. 2008)**

Sol. : We know that $\{f(k)\} = U(k) = \{1, 1, 1, 1, \ldots\}$

$$\therefore \quad Z\{f(k)\} = \sum 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$ [See Ex. 2, page 3-9]

$$\therefore \quad F(z) = \frac{1}{1 - (1/z)} = \frac{z}{z-1}, \quad \left|\frac{1}{z}\right| < 1.$$

By the change of scale property (page 3-19),

$$Z\{g(k)\} = Z\{2^k U(k)\} = \frac{z/2}{(z/2) - 1}$$

$$G(z) = \frac{z}{z-2}, \quad \left|\frac{2}{z}\right| < 1$$

By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z)\, G(z) = \frac{z}{z-1} \cdot \frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)}, \quad |z| > 2.$$

Example 5 : If $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$, then find the Z-transform of $\{f(k) * g(k)\}$.

(M.U. 2009, 14)

Sol. : As above, $\{f(k)\} = \{4^0, 4^1, 4^2, ...\}$

$$\therefore \quad Z\{f(k)\} = \Sigma f(k) z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + ...$$

$$\therefore \quad Z\{f(k)\} = 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + ... = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \quad \left|\frac{4}{z}\right| < 1$$

$$\therefore \quad \{g(k)\} = \{5^0, 5^1, 5^2, ...\}$$

$$\therefore \quad Z\{g(k)\} = \Sigma g(k) z^{-k} = 5^0 z^0 + 5z^{-1} + 5^2 z^{-2} + ...$$

$$= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + ... = \frac{1}{1-(5/z)} = \frac{z}{z-5}, \quad \left|\frac{5}{z}\right| < 1.$$

$\therefore$ By convolution theorem

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{(z-4)} \cdot \frac{z}{(z-5)} = \frac{z^2}{(z-4)(z-5)}, \quad |z| > 5.$$

Alternatively : If $\{h(k)\} = U(k) = \{1, 1, 1, ...\}$, then

$$Z\{h(k)\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + ... = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

$$\therefore \quad F(z) = \frac{z}{z-1}, \quad \left|\frac{1}{z}\right| < 1.$$

By change of scale property,

$$Z\{f(k)\} = Z\{4^k U(k)\} \qquad \therefore \quad F(z) = \frac{z/4}{(z/4)-1} = \frac{z}{z-4}$$

$$Z\{g(k)\} = Z\{5^k U(k)\} \qquad \therefore \quad G(z) = \frac{z/5}{(z/5)-1} = \frac{z}{z-5}$$

$\therefore$ By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{z-4} \cdot \frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}.$$

Example 6 : Find $Z\{f(k)\}$ where $f(k) = \frac{1}{2^k} * \frac{1}{3^k}$.

Sol. :

$$Z\left\{\frac{1}{2^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(2z)^k} = 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + ...$$

$$= \frac{1}{1-[1/(2z)]} = \frac{2z}{2z-1}, \quad |2z| > 1 \text{ i.e. } |z| > \frac{1}{2}.$$

Similarly, $Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}, \quad |z| > \frac{1}{3}.$

By convolution theorem,

$$Z\{f(k)\} = \left(\frac{2z}{2z-1}\right)\left(\frac{3z}{3z-1}\right), \quad |z| > \frac{1}{2}.$$

Example 7 : Find $Z\{f(k) * g(k)\}$ if $f(k) = \dfrac{1}{3^k}$ and $g(k) = \dfrac{1}{5^k}$. **(M.U. 2015)**

Sol. : We have $Z\left\{\dfrac{1}{3^k}\right\} = \sum\limits_{k=0}^{\infty} \dfrac{1}{3^k} \cdot z^{-k} = \sum\limits_{k=0}^{\infty} \dfrac{1}{(3z)^k}$

$$= \frac{1}{1} + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots$$

$$= \frac{1}{1-(1/3z)} \qquad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |3z| > 1\right]$$

$$\therefore \quad Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1} \qquad \left[|z| > \frac{1}{3}\right] \qquad \left[\text{Or by Ex. 6, page 3-11, } Z\left\{\frac{1}{3^k}\right\} = \frac{z}{z-(1/3)}\right]$$

Similarly, $Z\left\{\dfrac{1}{5^k}\right\} = \dfrac{5z}{5z-1} \qquad \left[|5z| > 1 \text{ i.e., } |z| > \dfrac{1}{5}\right]$

$\therefore$ By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \left(\frac{3z}{3z-1}\right)\left(\frac{5z}{5z-1}\right), \qquad |z| > \frac{1}{3}.$$

Example 8 : Find $Z\{f(k) * g(k)\}$ if $f(k) = \dfrac{1}{5^k}$ and $g(k) = \dfrac{1}{7^k}$.

Sol. : We have $Z\left\{\dfrac{1}{5^k}\right\} = \sum\limits_{k=0}^{\infty} \dfrac{1}{5^k} \cdot z^{-k} = \sum\limits_{k=0}^{\infty} \dfrac{1}{(5z)^k}$

$$= \frac{1}{1} + \frac{1}{5z} + \frac{1}{(5z)^2} + \frac{1}{(5z)^3} + \ldots$$

$$= \frac{1}{1-(1/5z)} \qquad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |5z| > 1\right]$$

$$\therefore \quad Z\left\{\frac{1}{5^k}\right\} = \frac{5z}{5z-1} \qquad \left[|z| > \frac{1}{5}\right] \qquad \left[\text{Or by Ex. 6, page 3-11, } Z\left\{\frac{1}{5^k}\right\} = \frac{z}{z-(1/5)}\right]$$

Similarly, $Z\left\{\dfrac{1}{7^k}\right\} = \dfrac{7z}{7z-1} \qquad \left[|7z| > 1 \text{ i.e., } |z| > \dfrac{1}{7}\right]$

$\therefore$ By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \left(\frac{5z}{5z-1}\right)\left(\frac{7z}{7z-1}\right), \qquad |z| > \frac{1}{5}.$$

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{e^{-ak} f(k)\} = F(e^a z)$.

Proof : By definition,

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\{e^{-ak} f(k)\} = \sum e^{-ak} f(k) z^{-k} = \sum f(k)(e^a z)^{-k} = F(e^a z).$$

Example 9 : Find the Z-transform of $\{k e^{-ak}\}$, $k \geq 0$.

Sol. : We know that if $U(k) = 1$, for $k > 0$, then $Z\{U(k)\} = \dfrac{z}{z-1}$.

By the above theorem, $Z[e^{-ak}U(k)] = Z(e^{-ak}) = \dfrac{e^a z}{e^a z - 1}$.

Now, by (4) (property of multiplication by k), page 3-22

$$Z[k e^{-ak}] = -z\frac{d}{dz}\left(\frac{e^a z}{e^a z - 1}\right)$$

$$= -z e^a \cdot \left[\frac{(e^a \cdot z - 1)1 - z(e^a)}{(e^a \cdot z - 1)^2}\right]$$

$$= -z \cdot e^a \cdot \frac{(-1)}{(e^a \cdot z - 1)^2} = \frac{e^a \cdot z}{(e^a z - 1)^2}.$$

Example 10 : Find the Z transform of $\{k^2 e^{-ak}\}$, $k \geq 0$. **(M.U. 2009)**

Sol. : As in Ex. 1 above $\quad Z[k e^{-ak}] = \dfrac{e^a \cdot z}{(e^a z - 1)^2}$

Now, by (4) (property of multiplication by k), page 3-22

$$Z[k^2 e^{-ak}] = \left(-z\frac{d}{dz}\right)F(z)$$

$$= -z\frac{d}{dz}\left[\frac{e^a z}{(e^a z - 1)^2}\right] = -z \cdot e^a \cdot \frac{d}{dz}\left[\frac{z}{(e^a \cdot z - 1)^2}\right]$$

$$\therefore \quad Z[k^2 e^{-ak}] = -z e^a \cdot \left[\frac{(e^a \cdot z - 1)^2 \cdot 1 - z \cdot 2(e^a \cdot z - 1) \cdot e^a}{(e^a \cdot z - 1)^4}\right]$$

$$= -z \cdot e^a \left[\frac{e^a \cdot z - 1 - 2 \cdot z \cdot e^a}{(e^a \cdot z - 1)^3}\right] = z \cdot e^a \frac{(z \cdot e^a + 1)}{(z \cdot e^a - 1)^3}.$$

Example 11 : Find $Z\{e^{-ak} \cos bk\}$.

Sol. : We have already obtained (Ex. 5, page 3-16)

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1}$$

Now, by the above result,

$$Z\{e^{-ak}\cos bk\} = \frac{e^a z(e^a z - \cos b)}{(e^a \cdot z)^2 - 2(e^a z)\cos b + 1}$$

Multiply in the numerator and denominator by e^{-2a}.

$$Z\{e^{-ak}\cos bk\} = \frac{z(z - e^{-a}\cos b)}{z^2 - 2e^{-a} z\cos b + e^{-2a}}.$$

Example 12 : Find $Z\{e^{-ak} \sin bk\}$.

Sol. : We have already proved that (Ex. 7, page 3-17)

$$Z\{\sin bk\} = \frac{z\sin b}{z^2 - 2z\cos b + 1}$$

$\therefore$ By the above property,

$$Z\{e^{-ak}\sin bk\} = \frac{(e^a z)\sin b}{(e^a z)^2 - 2(e^a z)\cos b + 1}$$

$$= \frac{e^{-a} z \cdot \sin b}{z^2 - 2e^{-a} \cdot z\cos b + e^{-2a}}.$$

We give below the list of Z-transforms obtained above.

Table of Z-transforms

1. $Z[\delta(k)] = 1$,	for all z	2. $Z\{U(k)\} = \dfrac{z}{z-1}$,	$\lvert z \rvert > 1$
3. $Z\{1\} = \dfrac{z}{z-1}$,	$\lvert z \rvert > 1$	4. $Z\{k\} = \dfrac{z}{(z-1)^2}$,	$\lvert z \rvert > 1$
5. $Z\{a^k\} = \dfrac{z}{z-a}$,	$k \geq 0$, $\lvert z \rvert > \lvert a \rvert$.	6. $Z\{a^k\} = \dfrac{z}{a-z}$,	$k < 0$, $\lvert z \rvert < \lvert a \rvert$.
7. $Z\{k\,a^k\} = \dfrac{az}{(z-a)^2}$,	$k \geq 0$, $\lvert z \rvert > \lvert a \rvert$.	8. $Z\{{}^nC_k\} = \left(1+\dfrac{1}{z}\right)^n$,	$0 \leq k \leq n$, $\lvert z \rvert > 0$
9. $Z\{{}^kC_n\} = z^{-n}\left(1-\dfrac{1}{z}\right)^{-(n+1)}$,	$\lvert z \rvert > 1$	10. $Z\left\{\dfrac{a^k}{k!}\right\} = e^{a/z}$,	$k \geq 0$ for all z.
11. $Z\{a^{\lvert k \rvert}\} = \dfrac{az}{1-az} + \dfrac{z}{z-a}$,		$\lvert a \rvert < \lvert z \rvert < \dfrac{1}{\lvert a \rvert}$	
12. $Z\{c^k \cos \alpha k\} = \dfrac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$,		$\lvert z \rvert > \lvert c \rvert$	(M.U. 2008)
13. $Z\{c^k \sin \alpha k\} = \dfrac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$,		$\lvert z \rvert > \lvert c \rvert$.	
14. $Z\{c^k \cosh \alpha k\} = \dfrac{z(z - c\cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2}$,		$k \geq 0$	$\lvert z \rvert > \max(\lvert ce^{\alpha} \rvert, \lvert ce^{-\alpha} \rvert)$
15. $Z\{c^k \sinh \alpha k\} = \dfrac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2}$,		$k \geq 0$	$\lvert z \rvert > \max(\lvert ce^{\alpha} \rvert, \lvert ce^{-\alpha} \rvert)$.

Miscellaneous Examples

Example 1 : Find Z-transform of $\{a\cos k\alpha + b\sin k\alpha\}$, $k \geq 0$.

Sol. : $Z\{a\cos k\alpha + b\sin k\alpha\} = aZ\{\cos k\alpha\} + b\,Z\{\sin k\alpha\}$ [By linearity property]

$$= a \cdot \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1} + b \cdot \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}, \quad \lvert z \rvert > 1$$

$$= \frac{az^2 + z(b\sin\alpha - a\cos\alpha)}{z^2 - 2z\cos\alpha + 1}.$$

Example 2 : Find $Z\left\{\sin\left(\dfrac{k\pi}{4} + a\right)\right\}$, $k \geq 0$. (M.U. 2008)

Sol. : Since $\sin\left(\dfrac{k\pi}{4} + a\right) = \sin\dfrac{k\pi}{4}\cos a + \cos\dfrac{k\pi}{4}\sin a$

$$Z\left\{\sin\left(\frac{k\pi}{4}+a\right)\right\} = Z\left\{\sin\frac{k\pi}{4}\cos a+\cos\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a\, Z\cdot\left\{\sin\left(\frac{\pi}{4}k\right)\right\}+\sin a\, Z\cdot\left\{\cos\left(\frac{\pi}{4}\cdot k\right)\right\}$$

$$= \cos a\,\frac{z\sin(\pi/4)}{z^2-2z\cos(\pi/4)+1}+\sin a\,\frac{z[z-\cos(\pi/4)]}{z^2-2z\cos(\pi/4)+1}$$

$$= \frac{\cos a\cdot(z/\sqrt{2})}{z^2-(2z/\sqrt{2})+1}+\frac{\sin a\, z\cdot[z-(1/\sqrt{2})]}{z^2-(2z/\sqrt{2})+1}$$

$$= \frac{z}{\sqrt{2}}\cdot\frac{[\cos a+\sin a\cdot(\sqrt{2}\,z-1)]}{z^2-\sqrt{2}\cdot z+1}.$$

Example 3 : Find $Z\{f(k)\}$ where $f(k)=\cos\left(\frac{k\pi}{4}+a\right),\ k\ge 0.$

Sol. : We have $\cos\left(\frac{k\pi}{4}+a\right)=\cos\frac{k\pi}{4}\cos a-\sin\frac{k\pi}{4}\sin a$

$$\therefore\quad Z\left\{\cos\left(\frac{k\pi}{4}+a\right)\right\} = Z\left\{\cos\frac{k\pi}{4}\cos a-\sin\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a\cdot Z\left\{\cos\frac{k\pi}{4}\right\}-\sin a\cdot Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$= \cos a\cdot\frac{z[z-\cos(\pi/4)]}{z^2-2z\cos(\pi/4)+1}+\frac{\sin a\cdot z\sin(\pi/4)}{z^2-2z\cos(\pi/4)+1}$$

$$= \frac{\cos a\cdot z[z-(1/\sqrt{2})]}{z^2-\sqrt{2}\cdot z+1}-\frac{\sin a\cdot z\cdot 1\sqrt{2}}{z^2-\sqrt{2}\cdot z+1}$$

$$= \frac{z}{\sqrt{2}}\cdot\left[\frac{\cos a(\sqrt{2}z-1)-\sin a}{z^2-\sqrt{2}\cdot z+1}\right].$$

Example 4 : Find $Z\{2^k\cos(3k+2)\},\ k\ge 0.$ **(M.U. 2015)**

Sol. : We have $\cos(3k+2)=\cos 3k\ \cos 2-\sin 3k\sin 2$

$$\therefore\quad Z\{\cos(3k+2)\} = \cos 2\cdot Z\{\cos 3k\}-\sin 2\cdot Z\{\sin 3k\}$$

$$= \cos 2\cdot\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}-\frac{\sin 2\cdot z\sin 3}{z^2-2z\cos 3+1}$$

$$= \frac{z[z\cos 2-(\cos 3\cos 2+\sin 3\sin 2)]}{z^2-2z\cos 3+1}$$

$$= \frac{z\cdot[z\cos 2-\cos 1]}{z^2-2z\cos 3+1}$$

By change of scale property,

If $Z\{f(k)\}=F(z)$, then $Z\{a^k f(k)\}=F\left(\frac{z}{a}\right)$.

$$\therefore\quad Z\{2^k\cos(3k+2)\} = \frac{\frac{z}{2}\left[\frac{z}{2}\cdot\cos 2-\cos 1\right]}{\left(\frac{z}{2}\right)^2-2\left(\frac{z}{2}\right)\cos 3+1}=\frac{z[z\cos 2-2\cos 1]}{z^2-4z\cos 3+4}$$

Example 5 : Find $Z\{2^k \sin(3k+2)\}$, $k \geq 0$. (M.U. 2009)

Sol. : We have $\sin(3k+2) = \sin 3k \cos 2 + \cos 3k \sin 2$

$$\therefore \quad Z\{\sin(3k+2)\} = \cos 2 \cdot Z\{\sin 3k\} + \sin 2 \cdot Z\{\cos 3k\}$$

$$= \cos 2 \cdot \frac{z \sin 3}{z^2 - 2z\cos 3 + 1} + \frac{\sin 2 \cdot z(z - \cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin 3 \cos 2 - \cos 3 \sin 2 + z \sin 2]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin(3-2) + z\sin 2]}{z^2 - 2z\cos 3 + 1} = \frac{z[\sin 1 + z \sin 2]}{z^2 - 2z\cos 3 + 1}.$$

Now, by change of scale property as above,

$$\therefore \quad Z\{2^k \sin(3k+2)\} = \frac{\frac{z}{2}\left[\sin 1 + \frac{z}{2}\sin 2\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1} = \frac{z[2\sin 1 + z\sin 2]}{z^2 - 4z\cos 3 + 4}.$$

Example 6 : Find $Z\{3^k \sin h\,\alpha\, k\}$, $k \geq 0$.

Sol. : We have $Z\{\sin h\,\alpha\, k\} = \dfrac{z \sin h\,\alpha}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$Z\{3^k \sin h\,\alpha\} = \frac{\frac{z}{3}\sin h\,\alpha}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1} = \frac{3z \sin h\,\alpha}{z^2 - 6z\cos h\,\alpha + 9}.$$

Example 7 : Find $Z\{3^k \cos h\,\alpha\, k\}$, $k \geq 0$.

Sol. : We have $Z\{\cos h\,\alpha\, k\} = \dfrac{z(z - \cos h\,\alpha)}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$\therefore \quad Z\{3^k \cos h\,\alpha\, k\} = \frac{\frac{z}{3}\left(\frac{z}{3} - \cos h\,\alpha\right)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1} = \frac{z(z - 3\cos h\,\alpha)}{z^2 - 6z\cos h\,\alpha + 9}.$$

Example 8 : Find $Z\{(k+1)\,a^k\}$, $k \geq 0$.

Sol. : We have $Z\{(k+1)a^k\} = Z\{k\,a^k\} + Z\{a^k\}$

But we know that $Z\{a^k\} = \dfrac{z}{z-a}$.

Now, by (4) (property of multiplication by k), page 3-22

$$\therefore \quad Z\{k\,a^k\} = -z\frac{d}{dz}\left\{\frac{z}{z-a}\right\} = -z\left[\frac{(z-a)1 - z\cdot 1}{(z-a)^2}\right] = -z \cdot \frac{(-a)}{(z-a)^2} = \frac{az}{(z-a)^2}$$

$$\therefore \quad Z\{(k+1)a^k\} = \frac{az}{(z-a)^2} + \frac{z}{(z-a)} = \frac{az + z^2 - az}{(z-a)^2} = \frac{z^2}{(z-a)^2}.$$

Example 9 : Find Z-transform of $\{k^2 a^{k-1}\}$, $k \ge 0$.

Sol. : We know that, $Z\{f(k-n)\} = z^{-n} \cdot F(z)$

$$\therefore \quad Z\{a^{k-1}\} = z^{-1} F(z) \quad \text{where} \quad F(z) = Z\{a^k\} = \frac{z}{z-a}$$

$$\therefore \quad Z\{a^{k-1}\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$$

By the property of multiplication by k,

$$\therefore \quad Z\{k \cdot a^{k-1}\} = -z\frac{d}{dz} \cdot \left(\frac{1}{z-a}\right) = -z \cdot \frac{(-1)}{(z-a)^2} = \frac{z}{(z-a)^2}.$$

By the property of multiplication by k,

$$\therefore \quad Z\{k^2 \cdot a^{k-1}\} = Z\{k \cdot (ka^{k-1})\} = -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right] = -z\left[\frac{(z-a)^2 \cdot 1 - z \cdot 2(z-a) \cdot 1}{(z-a)^4}\right]$$

$$= -z\left[\frac{(z-a) - 2z}{(z-a)^3}\right] = \frac{z(z+a)}{(z-a)^3}, \quad |z| > |a|.$$

Example 11 : Find $Z\{k^2 a^{k-1} U(k-1)\}$. **(M.U. 2016)**

Sol. : We know that, $Z\{U(k)\} = \dfrac{z}{z-1}$.

By change of scale property,

$$Z\{a^k U(k)\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}$$

$$\because \quad Z\{f(k-n)\} = z^{-n} \cdot Z\{f(k)\}$$

$$Z\{a^k U(k-1)\} = z^{-1} \cdot Z\{a^k U(k)\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}.$$

$$\therefore \quad Z\{k^2 \cdot a^2 U(k-1)\} = \left(-z\frac{d}{dz}\right)^2 \left(\frac{1}{z-a}\right) = \frac{z(z+a)}{(z-a)^3} \text{ as above.}$$

EXERCISE - V

Find the Z-transforms of the following sequences.

1. $\{3^k + 5^k\}$, $k < 0$
2. $\{\alpha^k + \beta^k\}$, $k < 0$
3. $\{3^k + 5^k\}$, $k \ge 0$
4. $\{\alpha^k + \beta^k\}$, $k \ge 0$
5. $\{k2^k + k3^k\}$, $k \ge 0$
6. $\{k\alpha^k + k\beta^k\}$, $k \ge 0$
7. $\left\{\dfrac{2^k}{k} + \dfrac{3^k}{k}\right\}$, $k \ge 1$
8. $\left\{\dfrac{\alpha^k}{k} + \dfrac{\beta^k}{k}\right\}$, $k \ge 1$
9. $\left\{3^k + \dfrac{1}{3^k}\right\}$, $k \ge 0$
10. $\left\{\alpha^k + \dfrac{1}{\alpha^k}\right\}$, $k \ge 0$
11. Find $Z\{f(k) * g(k)\}$ where $f(k) = 4^k$ and $g(k) = 7^k$.
12. Find $Z\{f(k) * g(k)\}$ where $f(k) = 5^k$ and $g(k) = 7^k$.
13. Find $Z\{f(k) * g(k)\}$ where $f(k) = \dfrac{1}{4^k}$ and $g(k) = \dfrac{1}{6^k}$.
14. Find $Z\{f(k) * g(k)\}$ where $f(k) = \dfrac{1}{3^k}$ and $g(k) = \dfrac{1}{7^k}$.

15. $\{\sin 5k\},\ k \geq 0$

16. $\{\cos 5k\},\ k \geq 0$

17. $\left\{\sin\left(\frac{k\pi}{2} + a\right)\right\},\ k \geq 0$

18. $\left\{\cos\left(\frac{k\pi}{2} + a\right)\right\},\ k \geq 0$

19. $\left\{\sin\left(\frac{k\pi}{3} + a\right)\right\},\ k \geq 0$

20. $\left\{\cos\left(\frac{k\pi}{3} + a\right)\right\},\ k \geq 0$

21. $\left\{\sin\left(\frac{k\pi}{6} + a\right)\right\},\ k \geq 0$

22. $\left\{\cos\left(\frac{k\pi}{6} + a\right)\right\},\ k \geq 0$

23. $\left\{3^k \cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\},\ k \geq 0$

24. $\left\{3^k \sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\},\ k \geq 0$

25. $\{e^{-2k} \cos 3k\},\ k \geq 0$

26. $\{e^{-3k} \sin 2k\},\ k \geq 0$

[**Ans.** : (1) $\frac{8z - 2z^2}{(3 - z)(5 - z)},\ |z| < 3$

(2) $\frac{(\alpha + \beta) z - 2z^2}{(\alpha - z)(\beta - z)},\ |z| <$ min. of $|\alpha|, |\beta|$

(3) $\frac{2z^2 - 8z}{(z - 3)(z - 5)},\ |z| > 5$

(4) $\frac{2z^2 - (\alpha + \beta) z}{(z - \alpha)(z - \beta)},\ |z| >$ max. of $|\alpha|, |\beta|$

(5) $\frac{2z}{(z - 2)^2} + \frac{3z}{(z - 3)^2},\ |z| > 3$

(6) $\frac{\alpha z}{(z - \alpha)^2} + \frac{\beta z}{(z - \beta)^2},\ |z| >$ max. of $|\alpha|, |\beta|$

(7) $-\log\left[\frac{(z - 2)(z - 3)}{z^2}\right],\ |z| > 3$

(8) $-\log\left[\frac{(z - \alpha)(z - \beta)}{z^2}\right],\ |z| >$ max. of $|\alpha|, |\beta|$

(9) $\frac{z}{z - 3} + \frac{3z}{3z - 1},\ |z| > 3$

(10) $\frac{z}{z - \alpha} + \frac{\alpha z}{\alpha z - 1},\ |z| > |\alpha|$

(11) $\left(\frac{z}{z - 4}\right)\left(\frac{z}{z - 7}\right),\ |z| > 7$

(12) $\left(\frac{z}{z - 5}\right)\left(\frac{z}{z - 7}\right),\ |z| > 7$

(13) $\left(\frac{4z}{4z - 1}\right)\left(\frac{6z}{6z - 1}\right),\ |z| > \frac{1}{4}$

(14) $\left(\frac{3z}{3z - 1}\right)\left(\frac{7z}{7z - 1}\right),\ |z| > \frac{1}{3}$

(15) $\frac{z \sin 5}{z^2 - 2z \cos 5 + 1}$

(16) $\frac{z(z - \cos 5)}{z^2 - 2z \cos 5 + 1}$

(17) $\frac{z^2 \sin a + z \cos a}{z^2 + 1},\ |z| > 1$

(18) $\frac{z^2 \cos a - z \sin a}{z^2 + 1},\ |z| > 1$

(19) $\frac{z}{2}\left[\frac{\cos a \cdot \sqrt{3} + \sin a (2z - 1)}{z^2 - z + 1}\right]$

(20) $\frac{z}{2}\left[\frac{\cos a (2z - 1) - \sin a \cdot \sqrt{3}}{z^2 - z + 1}\right]$

(21) $\frac{z}{2}\left[\frac{\cos a \cdot 1 + \sin a (2z - \sqrt{3})}{z^2 - \sqrt{3} \cdot z + 1}\right]$

(22) $\frac{z}{2}\left[\frac{\cos a (2z - \sqrt{3}) - \sin a \cdot 1}{z^2 - \sqrt{3} \cdot z + 1}\right]$

(23) $\frac{1}{\sqrt{2}} \cdot \frac{z^2 - 3z}{z^2 + 9},\ |z| > 3$

(24) $\frac{1}{\sqrt{2}} \cdot \frac{z^2 + 3z}{z^2 + 9},\ |z| > 3$

(25) $\frac{z(z - e^{-2} \cos 3)}{z^2 - 2e^{-2} z \cos 3 + e^{-4}}$

(26) $\frac{e^{-3} z \sin 2}{z^2 - 2e^{-3} z \cos 2 + e^{-6}}$.]

8. Inverse Z-transforms

We shall now consider the reverse problem *i.e.* given the Z-transform $Z\{f(k)\} = F(z)$ of a sequence to find the original sequence denoted by $\{f(k)\}$ or $Z^{-1}[F(z)]$. We shall consider Z-transforms which are rational functions of z *i.e.* of the form $F(z) = \dfrac{P(z)}{Q(z)}$ and $P(z)$ and $Q(z)$ are algebraic polynomials in z. It should be noted that to find the inverse Z-transform **we should know its region of convergence *i.e.* ROC**. We shall consider the following methods.

(a) Inverse by Partial Fraction method

(b) Inverse by Convolulation Theorem

(a) Method of Partial Fractions

If $F(z)$ can be factorised into partial fractions, linear, quadratic or repeated we express $F(z) = \dfrac{P(z)}{Q(z)}$ as the sum such factors, find the constants and then use the method of Binomial Expansion. This is illustrated in the following problems. If the degree of $P(z)$ is greater than or equal to that of $Q(z)$ we write $\dfrac{F(z)}{z} = \dfrac{P(z)}{Q(z)}$ as in Ex. 3, page 3-41. We now discuss the three cases separately.

(i) *Linear non-repeated factors*

Let the linear non-repeated factor be $\dfrac{z}{z-a}$. Then

$$Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{1}{1-(a/z)}\right] = Z^{-1}\left[1-\left(\frac{a}{z}\right)\right]^{-1}, \quad \text{if } |z| > |a|$$

$$= Z^{-1}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\ldots..\right]$$

$$= Z^{-1}\left[1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots..+a^k2^{-k}+\ldots..\right]$$

$$= \{a^k\}$$

Also, $$Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{z/a}{(z/a)-1}\right] = -Z^{-1}\left[\frac{z/a}{1-(z/a)}\right], \quad \text{if } |a| > |z|$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)\left(1-\frac{z}{a}\right)^{-1}\right] = -Z^{-1}\left[\frac{z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\ldots..\right)\right]$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+\ldots..\right] = -Z^{-1}\left\{\frac{z^k}{a^k}\right\}, \quad k \ge 0$$

$$= -Z^{-1}\left\{a^{-k}z^k\right\}, \quad k \ge 0$$

$$= -Z^{-1}\left\{a^k z^{-k}\right\}, \quad k \le 0$$

$$\therefore \quad Z^{-1}\left(\frac{z}{z-a}\right) = -\left\{a^k\right\}, \quad k \le 0$$

Example 1 : Find inverse Z-transform of $F(z) = \dfrac{z}{(z-1)(z-2)}, |z| > 2$.

(M.U. 2016)

Sol. : We have $F(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{2}{z-2} - \dfrac{1}{z-1}$

Since $|z| > 2$ clearly $|z| > 1$.

$\therefore$ $|z/2| > 1$ and $|z| > 1$.

$\therefore$ $|2/z| < 1$ and $|1/z| < 1$. $\therefore$ We take z common.

$$\therefore \quad F(z) = \frac{2}{z[1-(2/z)]} - \frac{1}{z[1-(1/z)]} = \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^{k-1}}+\ldots\right)$$

$$= \left(\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\ldots+\frac{2^k}{z^k}+\ldots\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^k}+\ldots\right)$$

Coefficient of $z^{-k} = 2^k - 1, k \geq 1$

$\therefore$ $Z^{-1}[F(z)] = \{2^k - 1\}, k \geq 1$.

Example 2 : Find the inverse Z-transform of

$$F(z) = \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}, \quad 3 < z < 4.$$

(M.U. 2018)

Sol. : We have (by partial fractions) $F(z) = \dfrac{1}{z-2} + \dfrac{1}{z-3} + \dfrac{1}{z-4}$

Since $z > 3$, $z > 2$, hence we take z out from the first two terms. Since $4 > z$, we take out 4 from the last bracket.

$$\therefore \quad F(z) = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{z}{4}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right) + \frac{1}{z}\left(1+\frac{3}{z}+\ldots+\frac{3^{k-1}}{z^{k-1}}+\ldots\right)$$

$$- \frac{1}{4}\left(1+\frac{z}{4}+\ldots+\frac{z^k}{4^k}+\ldots\right)$$

From the first series the coefficient of $z^{-k} = 2^{k-1}, k \geq 0$.

From the second series the coefficient of $z^{-k} = 3^{k-1}, k \geq 0$.

From the third series the coefficient of $z^k = -\dfrac{1}{4^{k+1}}, k \geq 0$.

$\therefore$ The coefficient of $z^{-k} = -4^{k-1}, k \leq 0$

$$\therefore \quad Z^{-1}[F(z)] = \{2^{k-1} + 3^{k-1}\}, \quad k \geq 0$$

$$= \{-4^{k+1}\}, \quad k \leq 0$$

Example 3 : Find the inverse Z-transform of $F(z)=\dfrac{1}{(z-3)(z-2)}$ if ROC is (i) $|z|<2$, (ii) $2<|z|<3$, (iii) $|z|>3$. **(M.U. 2008, 09, 13, 17, 18)**

Sol. : We have $F(z)=\dfrac{1}{(z-3)(z-2)}=\dfrac{1}{z-3}-\dfrac{1}{z-2}$

If $|z|<2$, clearly $|z|<3$ $\therefore \left|\dfrac{z}{2}\right|<1$ and $\left|\dfrac{z}{3}\right|<1$

Hence, we take out 3 and 2 from the fractions and write.

$$F(z)=\frac{1}{3[(z/3)-1]}-\frac{1}{2[(z/2)-1]}=-\frac{1}{3[1-(z/3)]}+\frac{1}{2[1-(z/2)]}$$

$$=-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots+\frac{z^k}{3^k}+\ldots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\ldots+\frac{z^k}{2^k}+\ldots\right)$$

$$=-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots+\frac{z^k}{3^{k+1}}+\ldots\right)+\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\ldots+\frac{z^k}{2^{k+1}}+\ldots\right)$$

$$=-(3^{-1}+3^{-2}z+3^{-3}z^2+\ldots..+3^{-k-1}z^k+\ldots..)$$
$$+(2^{-1}+2^{-2}z+2^{-3}z^2+\ldots..+2^{-k-1}z^k+\ldots..)$$

From the first series we find that the coefficient of $z^k=-3^{-k-1}$, $k\geq 0$.

$\therefore$ The coefficient of $z^{-k}=-3^{k-1}$, $k\leq 0$.

From the second series, we find that the coefficient of $z^k=2^{-k-1}$, $k\geq 0$.

$\therefore$ The coefficient of $z^{-k}=2^{k-1}$, $k\leq 0$

$\therefore$ $Z^{-1}[F(z)]=\{-3^{k-1}+2^{k-1}\}$, $k\leq 0$

(ii) If $2<|z|<3$ i.e. $2<|z|$ $\therefore |2/z|<1$ and $|z|<3$ i.e. $|z/3|<1$.

Hence, we take out 3 from the first fraction and z from the second fraction.

$$\therefore \quad F(z)=\frac{1}{3[(z/3)-1]}-\frac{1}{z[1-(2/z)]}=-\frac{1}{3}\cdot\frac{1}{[1-(z/3)]}-\frac{1}{z}\frac{1}{[1-(2/z)]}$$

$$=-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots+\frac{z^k}{3^k}+\ldots\right)-\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots.+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$

$$=-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots+\frac{z^k}{3^{k+1}}+\ldots\right)-\left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right)$$

$$=-[3^{-1}+3^{-2}z+3^{-3}z^2+\ldots..+3^{-k-1}z^k+\ldots..]-\left[\frac{1}{z}+\frac{2}{z^2}+\ldots..+\frac{2^{k-1}}{z^k}+\ldots..\right]$$

From the first series we find that the coefficient of $z^k=-3^{-k-1}$, $k\geq 0$.

$\therefore$ The coefficient of $z^{-k} = -3^{k-1}, k \le 0.$

For the second series the coefficient of $z^{-k} = -2^{k-1}, k \ge 1.$

$$\therefore \quad Z^{-1}[F(z)] = \{-3^{k-1}\},\ k \le 0$$
$$= \{-2^{k-1}\},\ k \ge 1$$

(iii) If $|z| > 3$, clearly $|z| > 2$ *i.e.* $|z/3| > 1$ and $|z/2| > 1$ *i.e.* $|3/z| < 1$ and $|2/z| < 1$. Hence, we take out z from both fractions.

$$\therefore \quad F(z) = \frac{1}{z[1-(3/z)]} - \frac{1}{z[1-(2/z)]} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$
$$= \frac{1}{z}\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\ldots+\frac{3^{k-1}}{z^{k-1}}+\ldots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$
$$= \left(\frac{1}{z}+\frac{3}{z^2}+\frac{3^{k-1}}{z^k}+\ldots\right) - \left(\frac{1}{z}+\frac{2}{z^2}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right)$$

$\therefore$ Coefficient of $z^{-k} = 3^{k-1} - 2^{k-1}, k \ge 1.$

$\therefore$ $Z^{-1}[F(z)] = \{3^{k-1} - 2^{k-1}\}, k \ge 1$

(ii) *Linear repeated factors*

When the linear factors are repeated, we use the above technique and expand $\frac{1}{(z-a)^2}$, $\frac{1}{(z-a)^3}$ by Binomial Theorem as illustrated in the following examples.

Example 1 : Find the inverse Z-transform of $F(z) = \dfrac{z+2}{z^2-2z+1}$, $|z| > 1$. **(M.U. 2008, 14, 15)**

Sol. : We have $F(z) = \dfrac{z+2}{z^2-2z+1} = \dfrac{z+2}{(z-1)^2} = \dfrac{3}{(z-1)^2} + \dfrac{1}{z-1}$

Since, $|z| > 1$, $\dfrac{1}{|z|} < 1$. $\quad \therefore$ We take out z.

$$\therefore \quad F(z) = \frac{3}{z^2[1-(1/z)]^2} + \frac{1}{z[1-(1/z)]} = \frac{3}{z^2}\left(1-\frac{1}{z}\right)^{-2} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$= \frac{3}{z^2}\left(1-(-2)\cdot\frac{1}{z}+\frac{(-2)(-3)}{2!}\cdot\frac{1}{z^2}-\frac{(-2)(-3)(-4)}{3!}\cdot\frac{1}{z^3}+\ldots\right)$$
$$+\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$
$$= \frac{3}{z^2}\left(1+\frac{2}{z}+\frac{3}{z^2}+\frac{4}{z^3}+\ldots+\frac{k-1}{z^{k-2}}+\ldots\right) + \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^{k-1}}+\ldots\right)$$
$$= 3\left(\frac{1}{z^2}+\frac{2}{z^3}+\ldots+\frac{k-1}{z^k}+\ldots\right) + \left(\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^k}\right)$$
$$= \frac{1}{z}+\frac{3+1}{z^2}+\ldots+\frac{3k-3+1}{z^k}+\ldots$$

$$F(z) = \frac{1}{z} + \frac{4}{z^2} + \frac{7}{z^3} + \ldots + \frac{3k-2}{z^k} + \ldots$$

$\therefore$ Coefficient of $z^{-k} = 3k - 2,\ k \geq 1.$

$\therefore$ $Z^{-1}[F(z)] = \{3k - 2\},\ k \geq 1.$

Example 2 : Find the inverse Z-transform of $\dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$, $2 < |z| < 3$.

Sol. : We have $F(z) = \dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \dfrac{1}{z-2} + \dfrac{1}{z-3} + \dfrac{1}{(z-3)^2}$

Since, $2 < |z|$, $|2/z| < 1$ and since $|z| < 3$, $|z/3| < 1$.

$\therefore$ We take out z from the first bracket and 3 out from the last two bracket.

$$\therefore \quad F(z) = \frac{1}{z} \cdot \frac{1}{[1 - (2/z)]} + \frac{1}{3} \cdot \frac{1}{[(z/3) - 1]} + \frac{1}{9} \cdot \frac{1}{[(z/3) - 1]^2}$$

$$F(z) = \frac{1}{z} \cdot \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z} \cdot \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \ldots + \frac{2^{k-1}}{z^{k-1}} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \ldots + \frac{z^k}{3^k} + \ldots\right)$$

$$+ \frac{1}{9}\left(1 + 2 \cdot \frac{z}{3} + 3 \cdot \frac{z^2}{3^2} + \ldots + (k+1)\frac{z^k}{3^k} + \ldots\right)$$

$\therefore$ From the first series we find that the coefficient of z^{-k} is 2^{k-1}, from the second series, we find that the coefficient of z^k is $-\dfrac{1}{3^{k+1}}$ and from the third series, we find that the coefficient of z^k is $\dfrac{k+1}{3^{k+2}}$.

$\therefore$ From the first series, coefficient of $z^{-k} = 2^{k-1},\ k \geq 1.$

From second and third series,

$$\text{Coefficient of } z^k = \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}} = \frac{k+1}{3^{k+2}} - \frac{3}{3^{k+2}} = \frac{k-2}{3^{k+2}},\ k \geq 0$$

$$\therefore \quad \text{Coefficient of } z^{-k} = \frac{-k-2}{3^{-k+2}},\ k \leq 0.$$

Hence, $Z^{-1}[F(z)] = \{2^{k-1}\},\quad k \geq 1$

$$= \left\{\frac{-k-2}{3^{-k+2}}\right\},\ k \leq 0.$$

Example 3 : Find the inverse Z-transform of $F(z) = \dfrac{z^2}{[z - (1/4)][z - (1/5)]}$

(i) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$, **(ii)** $|z| < \dfrac{1}{5}$.

Sol. : (i) Since, the degree of the numerator is equal to the degree of the numerator, we write,

$$\frac{F(z)}{z} = \frac{z}{[z-(1/4)][z-(1/5)]} = \frac{5}{z-(1/4)} - \frac{4}{z-(1/5)}$$

Now $\frac{1}{5} < |z|$ $\quad \frac{1}{|5z|} < 1$ $\quad |z| < \frac{1}{4}$ $\quad |4z| < 1$

$$\frac{F(z)}{z} = \frac{5 \cdot 4}{4z-1} - \frac{4}{z[1-(1/5z)]} = -\frac{20}{1-4z} - \frac{4}{z[1-(1/5z)]}$$

$$= -20(1-4z)^{-1} - \frac{4}{z}\left(1-\frac{1}{5z}\right)^{-1}$$

$$= -20(1+4z+(4z)^2+\ldots) - \frac{4}{z}\left(1+\frac{1}{5z}+\frac{1}{(5z)^2}+\ldots\right)$$

$$= -5(4+4^2z+\ldots+4^k z^{k-1}+\ldots) - \frac{4}{z}\left(1+\frac{1}{5z}+\ldots+\frac{1}{5^k}\cdot\frac{1}{z^k}+\ldots\right)$$

$$F(z) = -5(4z+4^2z^2+\ldots+4^k z^k+\ldots) - 4\left(1+\frac{1}{5z}+\ldots+\frac{1}{5^k}\cdot\frac{1}{z^k}+\ldots\right)$$

In the first series the coefficient of $z^k = -5 \cdot 4^k$, $k \geq 1$.

The coefficient of $z^{-k} = -5 \cdot 4^{-k}$, $\quad k \leq -1$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0.$$

In the second series the coefficient of $z^{-k} = -4 \cdot \left(\frac{1}{5}\right)^k$, $\quad k \geq 0$.

$$Z^{-1}[F(z)] = \left\{-5 \cdot \left(\frac{1}{4}\right)^k \quad -4\left(\frac{1}{5}\right)^k\right\}$$

$$k < 0 \qquad k \geq 0$$

(ii) Since $|z| < 1/5$, clearly $|z| < 1/4$

$|5z| < 1$ and $|4z| < 1$

$$\frac{F(z)}{z} = \frac{5}{z-(1/4)} - \frac{4}{z-(1/5)} = \frac{20}{4z-1} - \frac{20}{5z-1}$$

$$= -\frac{20}{1-4z} + \frac{20}{1-5z} = 20\left[\frac{1}{1-5z} - \frac{1}{1-4z}\right]$$

$$= 20[(1-5z)^{-1} - (1-4z)^{-1}]$$

$$= 20[(1+5z+\ldots+5^{k-1}z^{k-1}+\ldots) - (1+4z+\ldots+4^{k-1}z^{k-1}+\ldots)]$$

$$= 4 \cdot (5+5^2 \cdot z+\ldots+5^k z^{k-1}+\ldots) - 5(4+4^2 \cdot z+\ldots+4^k \cdot z^{k-1}+\ldots)$$

$$F(z) = 4(5z+5^2z^2+\ldots+5^k z^k+\ldots) - 5(4z+4^2z^2+\ldots+4^k z^k+\ldots)$$

In the first series the coefficient of $z^k = 4 \cdot 5^k$, $k \geq 1$.

The coefficient of $z^{-k} = 4 \cdot 5^{-k}$, $\quad k \leq -1$

$$= 4\left(\frac{1}{5}\right)^k \quad k < 0$$

In the second series the coefficient of $z^k = -5 \cdot 4^k$, $\quad k \geq 1$

The coefficient of $z^{-k} = -5 \cdot 4^{-k}$, $\quad k \le 1$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0$$

$$Z^{-1}[F(z)] = \left\{ \underset{k<0}{4\left(\frac{1}{5}\right)^k} - \underset{k<0}{5\left(\frac{1}{4}\right)^k} \right\}$$

(b) Inverse by Convolution Theorem

We have proved in (6), on page 3-24, that

$$H(z) = F(z)\, G(z)$$

When

$$H(z) = Z\{h(k)\} = Z[\{f(k)\} * \{g(k)\}]$$

$$\therefore \quad F(z)\, G(z) = H(z) = Z[\{f(k)\} * \{g(k)\}]$$

Taking inverse of both sides.

$$Z^{-1}[F(z)\, G(z)] = Z^{-1}[\{f(k)\} * \{g(k)\}] = [\{f(k)\} * \{g(k)\}]$$

$$= \sum_{m=0}^{k} f(m)\, g(k-m)$$

Example 1 : Using convolution property, find $Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}$.

Sol. : We have $\dfrac{z^2}{(z-1)(z-3)} = \dfrac{z}{z-1} \cdot \dfrac{z}{z-3}$

Let $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{z}{z-3}$ $\quad \therefore \quad H(z) = \dfrac{z^2}{(z-1)(z-3)}$

Hence, $H(z) = F(z)\, G(z)$.

On page 3-25, we have proved that $H(z) = F(z)\, G(z)$.

Taking inverse Z-transform of both sides.

$$Z^{-1}[H(z)] = Z^{-1}[F(z) \cdot G(z)]$$

$$\therefore \quad Z^{-1}[F(z)\, G(z)] = Z^{-1}[H(z)] = Z^{-1}\{Z\{h(k)\}\}$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\} \quad \text{(1)}$$

Now, if $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{z}{z-3}$.

$$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z}{z-1}\right] = \{1^k\},\ k \ge 0 \quad \text{[By Ex. 2, page 3-72]}$$

Similarly, $\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\dfrac{z}{z-3}\right] = \{3^k\},\ k \ge 0$

Now, $Z^{-1}[F(z)\, G(z)] = \{f(k)\} * \{g(k)\}$ $\quad$ [By (1)]

$$Z^{-1}[F(z)\, G(z)] = \sum_{m=0}^{k} f(m)\, g(k-m) = \sum_{m=0}^{k} (1)^m (3)^{k-m}$$

$$= 3^k \sum_{m=0}^{k} (1)^m (3)^{-m} = 3^k \sum_{m=0}^{k} 1 \cdot 3^{-m} = 3^k \sum_{m=0}^{k} \left(\frac{1}{3}\right)^m$$

$$Z^{-1}[F(z)\,G(z)] = 3^k\left[\frac{(1/3)^{k+1}-1}{(1/3)-1}\right] \quad \text{(G.P.)}$$

$$= 3^k\left[\frac{(1^{k+1}-3^{k+1})/3^{k+1}}{(1-3)/3}\right] = 3^k\cdot\frac{1-3^{k+1}}{3^{k+1}}\cdot\frac{3}{-2}$$

$$= \frac{3^{k+1}(1-3^{k+1})}{3^{k+1}(-2)} = \frac{1}{2}(3^{k+1}-1)$$

Example 2 : Find the inverse Z-transform of $\frac{z^2}{(z-1)(z-2)}$ using convolution theorem.

Sol. : We have $\frac{z^2}{(z-1)(z-2)} = \frac{z}{z-1}\cdot\frac{z}{z-2}$

Let $F(z) = \frac{z}{z-1}$ and $G(z) = \frac{z}{z-2}$. $\quad\therefore\ H(z) = \frac{z^2}{(z-1)(z-2)}$.

$\therefore\ H(z) = F(z)\,G(z) \quad \therefore\ F(z)\,G(z) = H(z)$

Taking inverse Z-transform of both sides,

$$\therefore\ Z^{-1}[F(z)\,G(z)] = Z^{-1}[H(z)] = Z^{-1}\{Z\{h(k)\}\}$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\} \quad \text{.............. (1)}$$

Now, if $F(z) = \frac{z}{z-1}$ and $G(z) = \frac{z}{z-2}$,

$$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z}{z-1}\right] = \{1^k\},\ k \ge 0 \quad \text{[By Ex. 2, page S-72]}$$

Similarly, $\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\frac{z}{z-2}\right] = \{2^k\},\ k \ge 0$

Now, $Z^{-1}[F(z)\,G(z)] = \{f(k)\} * \{g(k)\}$ [By (1)]

$$\therefore\ Z^{-1}[F(z)\,G(z)] = \sum_{m=0}^{k} f(m)\,g(k-m) = \sum_{m=0}^{k}(1)^m(2)^{k-m}$$

$$= 2^k\sum_{m=0}^{k}(1)^m(2)^{-m} = 2^k\sum_{m=0}^{k}1\cdot 2^{-m} = 2^k\sum_{m=0}^{k}\left(\frac{1}{2}\right)^m$$

$$= 2^k\left[\frac{(1/2)^{k+1}-1}{(1/2)-1}\right] \quad \text{(G.P.)}$$

$$= 2^k\cdot\frac{(1-2^{k+1})/2^{k+1}}{(1-2)/2} = 2^k\cdot\frac{1-2^{k+1}}{2^{k+1}}\cdot\frac{2}{-1}$$

$$= \frac{2^{k+1}}{2^{k+1}}\cdot\frac{1-2^{k+1}}{-1} = 2^{k+1}-1$$

Example 3 : Using convolution theorem find $Z^{-1}\left[\frac{z}{z-a}\right]^3$.

Sol. : We shall first find $Z^{-1}\left[\frac{z}{z-a}\right]^2$. $\quad\therefore\ \left(\frac{z}{z-a}\right)^2 = \frac{z}{z-a}\cdot\frac{z}{z-a}$

Let $F(z) = \dfrac{z}{z-a}$ and $G(z) = \dfrac{z}{z-a}$. $\therefore\ H(z) = \dfrac{z^2}{(z-a)^2}$.

On page 3-25, we have proved that

$$H(z) = F(z)\, G(z) \qquad \therefore\ F(z)\, G(z) = H(z)$$

Taking inverse Z-transform of both sides,

$$Z^{-1}[H(z)] = Z^{-1}[F(z)\, G(z)]$$

$$\therefore\ Z^{-1}[F(z)\, G(z)] = Z^{-1}[H(z)] = Z^{-1}\{Z\{h(k)\}\}$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\} \qquad \text{.......... (1)}$$

Now, if $F(z) = \dfrac{z}{z-a}$ and $G(z) = \dfrac{z}{z-a}$,

$$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z}{z-a}\right] = \{a^k\},\ k \ge 0 \qquad \text{[By Ex. 2, page S-72]}$$

Similarly, $$\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\frac{z}{z-a}\right] = \{a^k\},\ k \ge 0$$

Now, $$Z^{-1}[F(z)\, G(z)] = \{f(k)\} * \{g(k)\} \qquad \text{[By (1)]}$$

$$\therefore\ Z^{-1}[F(z)\, G(z)] = \sum_{m=0}^{k} f(m)\, g(k-m) = \sum_{m=0}^{k} \{a\}^m \{a\}^{k-m}$$

$$\therefore\ Z^{-1}[F(z)\, G(z)] = \sum_{m=0}^{k} \{a^{m+k-m}\} = \sum_{m=0}^{k} a^k = a^k \sum_{m=0}^{k} 1$$

$$= a^k [1^0 + 1^1 + 1^2 + \ldots + 1^k]$$

$$= a^k [1 + 1 + \ldots + 1]\ (k+1) \text{ times}$$

$$= a^k (k+1)$$

$$\therefore\ Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = a^k (k+1)$$

Now, $$Z^{-1}\left[\frac{z^3}{(z-a)^3}\right] = Z^{-1}\left[\frac{z^2}{(z-a)^2} \cdot \frac{z}{z-a}\right]$$

Let $F(z) = \dfrac{z^2}{(z-a)^2}$ and $G(z) = \dfrac{z}{z-a}$. $\therefore\ H(z) = \dfrac{z^3}{(z-a)^3}$.

$$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = a^k (k+1)$$

$$\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\frac{z}{z-a}\right] = a^k \qquad \text{.... [By Ex. 2, page S-72]}$$

$$\therefore\ Z^{-1}[F(z)\, G(z)] = Z^{-1}[H(z)] = Z^{-1}\{Z\{h(k)\}\}$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\}$$

$$= \sum_{m=0}^{k} f(m)\, g(k-m) = \sum_{m=0}^{k} (m+1)\, a^m \cdot a^{k-m}$$

$$= \sum_{m=0}^{k} (m+1)\, a^k = a^k \sum_{m=0}^{k} (m+1)$$

$$= a^k [\, 1 + 2 + 3 + \ldots + (k+1)\,]$$

$$= a^k \cdot \frac{1}{2}\,(k+1)(k+2)$$

$$\therefore \quad Z^{-1}\left[\left(\frac{z}{z-a}\right)^3\right] = \frac{1}{2}\cdot a^k (k+1)(k+2).$$

Example 4 : Using convolution property, find $Z^{-1}\left\{\dfrac{z}{(z-1)(z-2)}\right\}$.

Sol. : We have, $\dfrac{z}{(z-1)(z-2)} = \dfrac{z}{z-1}\cdot\dfrac{1}{z-2}$

Let $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{1}{z-2}$. $\quad \therefore \; H(z) = \dfrac{z}{z-1}\cdot\dfrac{1}{z-2}$.

$\therefore \; H(z) = F(z)\, G(z)$

Taking inverse Z-transform of both sides,

$$\therefore \quad Z^{-1}[H(z)] = Z^{-1}[F(z)\,G(z)]$$

$$\therefore \quad Z^{-1}[F(z)\,G(z)] = Z^{-1}[H(z)] = Z^{-1}[Z\{h(k)\}]$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\}$$

Now, if $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{1}{z-2}$,

$$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z}{z-1}\right] = \{1^k\}$$ [By Ex. 2, page S-72]

Similarly, $\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\dfrac{1}{z-2}\right] = \{2^{k-1}\},\; k' > 1$ [By Ex. 1, page S-72]

$\therefore \; k' - 1 > 0. \quad$ Putting $k' - 1 = k,\; k > 0.$

$\therefore \; \{g(k)\} = \{2^k\},\; k > 0$

$$\therefore \quad Z^{-1}[F(z)\,G(z)] = \sum_{m=0}^{k} f(m)\, g(k-m) = \sum_{m=0}^{k} \{1^m\}\{2^{k-m}\}$$

$$= \sum_{m=0}^{k} \{2^{k-m}\} = 2^k \sum_{m=0}^{k} 2^{-m} = 2^k \sum_{m=0}^{k} \left(\frac{1}{2}\right)^m$$

$$= 2^k \left[\frac{(1/2)^{k+1} - 1}{(1/2) - 1}\right] \quad \text{(G.P.)}$$

$$= 2^k \left[\frac{(1-2^{k+1})/2^{k+1}}{(-1)/2}\right]$$

$$= 2^k \left[\frac{(1-2^{k+1})}{2^{k+1}}\cdot\frac{2}{-1}\right]$$

$$\therefore \quad Z^{-1}[F(z)\,G(z)] = \frac{2^{k+1}}{2^{k+1}}\cdot\frac{1-2^{k+1}}{-1} = 2^{k+1} - 1$$

But $k = k' - 1 \qquad \therefore \; k + 1 = k'$

$$\therefore \quad Z^{-1}[F(z)\,G(z)] = 2^k + 1,\; k > 1.$$

EXERCISE - VI

Find the inverse Z-transforms of the following :

1. $\dfrac{1}{z-1}$, $|z|<1$, $|z|>1$

2. $\dfrac{1}{z-3}$, $|z|<3$, $|z|>3$

3. $\dfrac{z}{z-1}$, $|z|<1$, $|z|>1$

4. $\dfrac{z}{z-a}$, $|z|<a$, $|z|>a$, $a>0$

5. $\dfrac{1}{(z-1)^2}$, $|z|<1$, $|z|>1$ **(M.U. 2019)**

6. $\dfrac{1}{(z-5)^2}$, $|z|<5$, $|z|>5$

7. $\dfrac{1}{(z-3)^3}$, $|z|<3$, $|z|>3$

8. $\dfrac{1}{(z-1)^3}$, $|z|<1$, $|z|>1$

9. $\dfrac{1}{z^2-3z+2}$, $|z|>2$

10. $\dfrac{z}{[z-(1/4)][z-(1/5)]}$, $\dfrac{1}{5}<|z|<\dfrac{1}{4}$

11. $\dfrac{z}{(z-2)(z-3)}$, $|z|<2$, $2<|z|<3$, $|z|>3$

12. $\dfrac{1}{[z-(1/2)][z-(1/3)]}$ (i) $\dfrac{1}{3}<|z|<\dfrac{1}{2}$, (ii) $\dfrac{1}{2}<|z|$ **(M.U. 2015)**

13. $\dfrac{3z^2+2z}{z^2-3z+2}$, $1<|z|<2$

14. $\dfrac{z^3}{(z-1)(z-2)^2}$, $|z|>2$

15. $\dfrac{z^3}{(z-3)(z-2)^2}$, $|z|>3$ **(M.U. 2014)**

16. $\dfrac{z^2}{(z-1)[z-(1/2)]}$; $|z|<\dfrac{1}{2}$, $\dfrac{1}{2}<|z|<1$, $|z|>1$

[Ans. : **(1)** (a) -1, $k\le 0$; (b) 1, $k\ge 1$; **(2)** (a) -3^{k-1}, $k\le 0$; (b) 3^{k-1}, $k\ge 1$;

(3) (a) -1, $k<0$; (b) 1, $k\ge 0$; **(4)** (a) $-a^k$, $k<0$; (b) a^k, $k\ge 0$;

(5) (a) $-k+1$, $k\le 0$; (b) $k-1$, $k\ge 2$,

(6) (a) $\dfrac{-k+1}{5^{-k+2}}$, $k\le 0$; (b) $(k-1)\,5^{k-2}$, $k\ge 2$,

(7) (a) $-\dfrac{(-k+1)(-k+2)}{2}\cdot\dfrac{1}{3^{-k+3}}$, $k\le 0$; (b) $\dfrac{(k-2)(k-1)}{2}\cdot 3^{k-3}$, $k\ge 3$,

(8) (a) $-\dfrac{(-k+1)(-k+2)}{2}$, $k\le 0$; (b) $\dfrac{(k-2)(k-1)}{2}$, $k\ge 3$

(9) $2^{k-1}-1$, $k\ge 1$,

(10) $5\left(\dfrac{1}{4}\right)^k+4\left(\dfrac{1}{5}\right)^k$,

$(k\le 0)$, $(k\ge 0)$

(11) (i) 2^k-3^k, $k\le 0$, (ii) -2^k, $(k>0)$, $-3^k\,(k\le 0)$, (iii) 3^k-2^k, $k\ge 0$,