9.8 FOURIER INTEGRAL THEOREM

If the function $f(x)$ is piecewise continuous in every finite interval and absolutely integrable in $(-\infty, \infty)$ then the Fourier integral is given by

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\,d\omega$$

Proof The complex form of Fourier series of $f(x)$ is given by

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where
$$c_n=\frac{1}{2l}\int_{-l}^{l} f(t)e^{-\frac{in\pi t}{l}}\,dt$$

Putting $\frac{n\pi}{l}=\omega_n$,

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{1}{2l}\int_{-l}^{l} f(t)e^{-i\omega_n(t-x)}\,dt$$

Interchanging summation and integration,

$$f(x)=\frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t)e^{-i\omega_n(t-x)}\cdot\frac{\pi}{l}\right]dt=\frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t)e^{-i\omega_n(t-x)}\,\Delta\omega_n\right]dt \qquad \ldots (9.7)$$

where $\Delta\omega_n=\omega_{n+1}-\omega_n=\frac{(n+1)\pi}{l}-\frac{n\pi}{l}=\frac{\pi}{l}$

As $l\to\infty$, $\Delta\omega_n\to 0$ and the infinite series in Eq. (9.7) becomes an integral from $-\infty$ to ∞.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}\,d\omega\right]dt \qquad [\because l\to\infty, \Delta\omega_n\to d\omega]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}\,d\omega\,dt$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}\left[\cos\omega(t-x)-i\sin\omega(t-x)\right]d\omega\,dt$$

Since cos $\omega(t-x)$ is an even function and sin $\omega(t-x)$ is an odd function of ω in $(-\infty, \infty)$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot 2\int_0^{\infty} \cos\omega(t-x)\,d\omega\,dt$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\,d\omega \qquad \text{... (9.8)}$$

Equation (9.8) is called the Fourier integral of $f(x)$.

Fourier Cosine and Sine Integrals

The Fourier integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)dt\,d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos\omega t\ \cos\omega x + \sin\omega x\ \sin\omega t\right)dt\,d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos\omega x\left(\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt\right) + \frac{1}{\pi}\int_0^{\infty}\sin\omega x\left(\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt\right)d\omega$$

If $f(t)$ is an even function, $f(t)$ cos ωt is an even function of t and $f(t)$ sin ωt is an odd function of t,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\omega x\int_0^{\infty} f(t)\cos\omega t\,dt\,d\omega \qquad \text{... (9.9)}$$

Equation (9.9) is called the *Fourier cosine integral* of $f(x)$, provided $f(x)$ is even.

If $f(t)$ is an odd function, $f(t)$ cosωt is an odd function of t, and $f(t)$ sin ωt is an even function of t,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\omega x\int_0^{\infty} f(t)\sin\omega t\,dt\,d\omega \qquad \text{... (9.10)}$$

Equation (9.10) is called the *Fourier sine integral* of $f(x)$, provided $f(x)$ is odd.

EXAMPLE 9.28 *Find the Fourier integral representation of the function*

$$\begin{aligned} f(x) &= 0, && x<0 \\ &= \frac{1}{2}, && x=0 \\ &= e^{-x}, && x>0 \end{aligned}$$

Solution: The Fourier integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\,d\omega$$

$$= \frac{1}{\pi}\left[\int_0^{\infty}\int_{-\infty}^{0} 0\cdot\cos\omega(t-x)\,dt + \int_0^{\infty}\int_0^{\infty} e^{-t}\cos\omega(t-x)\,dt\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-t}(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)dt\,d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\omega x\int_0^\infty e^{-t}\cos\omega t\,dt + \sin\omega x\int_0^\infty e^{-t}\sin\omega t\,dt\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\omega x\left|\frac{e^{-t}}{1+\omega^2}(-\cos\omega t + \omega\sin\omega t)\right|_0^\infty\right.$$

$$\left. + \sin\omega x\left|\frac{e^{-t}}{1+\omega^2}(-\sin\omega t - \omega\cos\omega t)\right|_0^\infty\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\omega x}{1+\omega^2} + \frac{\omega\sin\omega x}{1+\omega^2}\right]d\omega = \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega$$

EXAMPLE 9.29 *Find the Fourier integral representation of the function*

$$f(x) = 1, \quad |x| < 1$$
$$= 0, \quad |x| > 1$$

Hence, evaluate (i) $\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega$ (ii) $\int_0^\infty \frac{\sin\omega}{\omega}d\omega$

Solution: The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(t)\cos\omega t\,dt\,d\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^1 1\cdot\cos\omega t\,dt\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left|\frac{\sin\omega t}{\omega}\right|_0^1 d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega$$

(i) $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \qquad \text{... (1)}$$

At $|x| = 1$, i.e., $x = \pm 1$, $f(x)$ is discontinuous.

At $x = 1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \frac{1}{2}(1+0) = \frac{1}{2}$$

At $x = -1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right] = \frac{1}{2}(0+1) = \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} \dfrac{\pi}{2}, & |x|<1 \\ \dfrac{\pi}{4}, & |x|=1 \\ 0, & |x|>1 \end{cases}$$

(ii) Putting $x=0$ in Eq. (1),

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}f(0) = \frac{\pi}{2} \qquad [\because f(0)=1]$$

EXAMPLE 9.30 *Find the Fourier cosine integral of the function*

$$\begin{aligned} f(x) &= \cos x, \quad |x|<\frac{\pi}{2} \\ &= 0, \quad |x|>\frac{\pi}{2} \end{aligned}$$

Solution: The Fourier cosine integral of $f(x)$ is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(t)\cos\omega t\,dt\,d\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^{\frac{\pi}{2}}\cos t\cos\omega t\,dt\,d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\,\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\cos(1+\omega)t+\cos(1-\omega)t\right]dt\,d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{\sin(1+\omega)t}{1+\omega}+\frac{\sin(1-\omega)t}{1-\omega}\right]_0^{\frac{\pi}{2}}d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega}+\frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right]d\omega = \frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1+\omega}+\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega}\right]d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\,\frac{2\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}\,d\omega$$

EXAMPLE 9.31 *Express the function*

$$\begin{aligned} f(x) &= 1, \quad 0\le x<\pi \\ &= 0, \quad x>\pi \end{aligned}$$

as a Fourier sine integral and, hence, evaluate $\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega$.

Solution: The Fourier sine integral of $f(x)$ is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(t)\sin\omega t\,dt\,d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\pi 1\cdot\sin\omega t\,dt\,d\omega=\frac{2}{\pi}\int_0^\infty \sin\omega x\left|\frac{-\cos\omega t}{\omega}\right|_0^\pi d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega$$

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega=\frac{\pi}{2}f(x)$$

$$=\begin{cases}\dfrac{\pi}{2}, & 0\le x<\pi\\ 0, & x>\pi\end{cases} \qquad ...(1)$$

At $x=\pi$, $f(x)$ is discontinuous.

$$f(x)=\frac{1}{2}\left[\lim_{x\to\pi^-}f(x)+\lim_{x\to\pi^+}f(x)\right]=\frac{1}{2}(1+0)=\frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega=\begin{cases}\dfrac{\pi}{2}, & 0\le x<\pi\\ \dfrac{\pi}{4}, & x=\pi\\ 0, & x>\pi\end{cases}$$

EXAMPLE 9.32

Express the function $f(x)=\sin x,\quad 0\le x\le\pi$
$\qquad\qquad=0,\quad x>\pi$

as a Fourier sine integral and show that

$$\int_0^\infty \frac{\sin\omega x\sin\pi\omega}{1-\omega^2}d\omega=\frac{\pi}{2}\sin x,\quad 0\le x\le\pi.$$

Solution: The Fourier sine integral of $f(x)$ is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(t)\sin\omega t\,dt\,d\omega=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\pi \sin t\sin\omega t\,dt\,d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\frac{1}{2}\int_0^\pi\left[\cos(\omega-1)t-\cos(\omega+1)t\right]dt\,d\omega=\frac{1}{\pi}\int_0^\infty \sin\omega x\left|\frac{\sin(\omega-1)t}{\omega-1}-\frac{\sin(\omega+1)t}{\omega+1}\right|_0^\pi d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \sin\omega x\left|\frac{\sin(\pi\omega-\pi)}{\omega-1}-\frac{\sin(\pi\omega+\pi)}{\omega+1}\right|d\omega=\frac{1}{\pi}\int_0^\infty \sin\omega x\left|\frac{-\sin\pi\omega}{\omega-1}+\frac{\sin\pi\omega}{\omega+1}\right|d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \sin\omega x\left(\frac{-2\sin\pi\omega}{\omega^2-1}\right)d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x\sin\pi\omega}{1-\omega^2}d\omega$$

Hence,
$$\int_0^\infty \frac{\sin\omega x\sin\pi\omega}{1-\omega^2}d\omega = \frac{\pi}{2}f(x)$$
$$= \frac{\pi}{2}\sin x, \qquad 0 \le x \le \pi$$
$$= 0, \qquad x > \pi$$

EXERCISE 9.7

1. Find the Fourier integral representations of the following functions:

(i) $f(x) = x, \quad |x| < 1$
$\quad = 0, \quad |x| > 1$

(ii) $f(x) = -e^{ax}, \quad x < 0$
$\quad = e^{-ax}, \quad x > 0$

$$\left[\textbf{Ans.: } \text{(i)} \int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{ip\omega^2} e^{i\omega x}\, d\omega \quad \text{(ii)} \frac{2}{\pi}\int_0^\infty \sin\omega x \frac{\omega}{a^2+\omega^2}d\omega\right]$$

2. Find the Fourier sine integral of $f(x) = e^{-ax} - e^{-bx}$.

$$\left[\textbf{Ans.: } \frac{2}{\pi}\int_0^\infty \frac{(b^2-a^2)\omega\sin\omega x}{(a^2+\omega^2)(b^2+\omega^2)}d\omega\right]$$

3. Find the Fourier cosine integral of $f(x) = e^{-ax}$.

$$\left[\textbf{Ans.: } \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}d\omega\right]$$

4. Express the function

$f(x) = \frac{\pi}{2}, \quad 0 < x < \pi$
$\quad = 0, \quad x > \pi$

as a Fourier sine integral and show that
$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega = \frac{\pi}{2}.$$

$$\left[\textbf{Ans.: } \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega\right]$$

9.9 FOURIER TRANSFORM

The Fourier integral of $f(x)$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-x)}dt\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}e^{i\omega x}dt\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\, d\omega$$

where
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$$

Hence, the Fourier transform of $f(x)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx$$

and the inverse Fourier transform is given by

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}\,d\omega$$

Notes

(i) The Fourier transform pair can also be given by

$$F(\omega)=\int_{-\infty}^{\infty}f(x)e^{i\omega x}\,dx$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega x}\,d\omega$$

(ii) The Fourier transform pair can also be given by

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\omega x}\,dx$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega x}\,d\omega$$

Fourier Cosine and Sine Transforms

From the Fourier cosine integral,

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}f(t)\cos\omega x\cos\omega t\,dt\,d\omega=\frac{2}{\pi}\int_0^{\infty}\cos\omega x\left[\int_0^{\infty}f(t)\cos\omega t\,dt\right]d\omega$$
$$=\frac{2}{\pi}\int_0^{\infty}F_c(\omega)\cos\omega x\,d\omega$$

where $F_c(\omega)=\int_0^{\infty}f(t)\cos\omega t\,dt$ and is known as the Fourier cosine transform.

Hence, the Fourier cosine transform of $f(x)$ is given by

$$F_c(\omega)=\int_0^{\infty}f(x)\cos\omega x\,dx$$

and the inverse Fourier cosine transform is given by

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(\omega)\cos\omega x\,d\omega$$

Similarly, from the Fourier sine integral,

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}f(t)\sin\omega x\sin\omega t\,dt\,d\omega=\frac{2}{\pi}\int_0^{\infty}\sin\omega x\left[\int_0^{\infty}f(t)\sin\omega t\,dt\right]d\omega$$
$$=\frac{2}{\pi}\int_0^{\infty}F_s(\omega)\sin\omega x\,d\omega$$

where $F_s(\omega)=\int_0^{\infty}f(t)\sin\omega t\,dt$ and is known as the Fourier sine transform.

Hence, the Fourier sine transform of $f(x)$ is given by

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\, dx$$

and the inverse Fourier sine transform is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\, d\omega$$

Notes

(i) The Fourier cosine transform pair can also be given by

$$F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega)\cos\omega x\, d\omega$$

(ii) The Fourier sine transform pair can also be given by

$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\omega)\sin\omega x\, d\omega$$

9.10 PROPERTIES OF THE FOURIER TRANSFORM

9.10.1 Linearity

If $F\{f_1(x)\} = F_1(\omega)$ and $F\{f_2(x)\} = F_2(\omega)$ then $F\{af_1(x) + bf_2(x)\} = aF_1(\omega) + bF_2(\omega)$, where a and b are any constants.

Proof $F\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx$

$$F\{af_1(x)+bf_2(x)\} = \int_{-\infty}^{\infty}\left[af_1(x)+bf_2(x)\right]e^{-i\omega x}\, dx = a\int_{-\infty}^{\infty} f_1(x)e^{-i\omega x}dx + b\int_{-\infty}^{\infty} f_2(x)e^{-i\omega x}\, dx$$

$$= aF_1(\omega)+bF_2(\omega)$$

9.10.2 Change of Scale

If $F\{f(x)\} = F(\omega)$ then $F\{f(ax)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$.

Proof $F\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\, dx$$

Putting $ax = t$, $x = \frac{t}{a}$, $dx = \frac{dt}{a}$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega\left(\frac{t}{a}\right)} \frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(t) e^{-i\left(\frac{\omega}{a}\right)t} dt = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

9.10.3 Shifting in *x*

If $F\{f(x)\} = F(\omega)$ then $F\{f(x-a)\} = e^{-ia\omega} F(\omega)$.

Proof $F\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x} dx$$

Putting $x - a = t$, $x = a + t$, $dx = dt$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega(a+t)} dt = e^{-ia\omega}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = e^{-ia\omega} F(\omega)$$

9.10.4 Shifting in ω

If $F\{f(x)\} = F(\omega)$ then $F\{f(x) e^{i\omega_0 x}\} = F(\omega - \omega_0)$.

Proof $F\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$F\{f(x)e^{i\omega_0 x}\} = \int_{-\infty}^{\infty} f(x) e^{i\omega_0 x} e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(x) e^{-i(\omega-\omega_0)x} dx = F(\omega - \omega_0)$$

9.10.5 Differentiation

If $F\{f(x)\} = F(\omega)$ then $F\{f'(x)\} = i\omega F(\omega)$, $f(x) \to 0$ as $x \to \pm\infty$.

Proof $F\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx = \left|e^{-i\omega x} f(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega) e^{-i\omega x} f(x)\, dx$$

$$= 0 + i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = i\omega\, F(\omega)$$

9.10.6 Convolution

If $F\{f_1(x)\} = F_1(\omega)$ and $F\{f_2(x)\} = F_2(\omega)$ then $F\{f_1(x) * f_2(x)\} = F_1(\omega)\, F_2(\omega)$.

Proof By definition of convolution,

$$f_1(x)*f_2(x)=\int_{-\infty}^{\infty} f_1(u)f_2(x-u)\,du$$

$$F\{f_1(x)*f_2(x)\}=\int_{-\infty}^{\infty}[f_1(x)*f_2(x)]e^{-i\omega x}dx=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(u)f_2(x-u)\,du\right]e^{-i\omega x}dx$$

Putting $x-u=t$, $x=t+u$, $dx=dt$

$$F\{f_1(x)*f_2(x)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u)f_2(t)e^{-i\omega(t+u)}du\,dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u)f_2(t)e^{-i\omega t}e^{-i\omega u}\,du\,dt$$

$$=\int_{-\infty}^{\infty} f_1(u)e^{-i\omega u}\,du\int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}\,dt=F_1(\omega)F_2(\omega)$$

EXAMPLE 9.33 *Find the Fourier transform of* $f(x)=xe^{-x},\ x>0$
$=0,\quad x<0$

Solution: The Fourier transform of $f(x)$ is given by

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx=\int_0^{\infty} xe^{-x}e^{-i\omega x}\,dx$$

$$=\int_0^{\infty} xe^{-(1+i\omega)x}\,dx=\left|x\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}-\frac{e^{-(1+i\omega)x}}{[-(1+i\omega)]^2}\right|_0^{\infty}=\frac{1}{(1+i\omega)^2}$$

EXAMPLE 9.34 *Find the Fourier transform of* $f(x)=a-|x|,\quad |x|\le a$
$=0,\quad |x|>a$

Hence, show that $\displaystyle\int_0^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}.$

Solution:
$$f(x)=a-(-x)=a+x,\quad -a<x\le 0$$
$$=a-x,\quad 0\le x<a$$
$$=0,\quad |x|>a$$
$$\left[\because |x|=-x\quad x<0;\quad =x\quad x>0\right]$$

Also, $f(-x)=f(x)$
The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx=\int_{-\infty}^{\infty} f(x)(\cos\omega x-i\sin\omega x)\,dx$$

$$=2\int_0^{\infty} f(x)\cos\omega x\,dx\qquad\left[\because \int_{-a}^{a} f(x)dx=2\int_0^{a} f(x)dx,\text{ if } f(-x)=f(x);\quad =0,\ \text{if } f(-x)=-f(x)\right]$$

$$=2\int_0^{a}(a-x)\cos\omega x\,dx=2\left[\left|(a-x)\left(\frac{\sin\omega x}{\omega}\right)-(-1)\left(-\frac{\cos\omega x}{\omega^2}\right)\right|_0^{a}\right]$$

$$=2\left(-\frac{\cos a\omega}{\omega^2}+\frac{1}{\omega^2}\right)=\frac{2(1-\cos a\omega)}{\omega^2}$$

Taking the inverse Fourier transform,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(1-\cos a\omega)}{\omega^2}(\cos\omega x+i\sin\omega x)\,d\omega$$

$$=\frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos a\omega}{\omega^2}\right)\cos\omega x\,d\omega\quad\left[\begin{array}{ll}\because \int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx, & \text{if } f(-x)=f(x)\\ =0, & \text{if } f(-x)=-f(x)\end{array}\right]$$

Putting $x=0$,

$$f(0)=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos a\omega}{\omega^2}d\omega$$

$$a=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos a\omega}{\omega^2}d\omega$$

Putting $a=2$,

$$2=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos 2\omega}{\omega^2}d\omega$$

$$2=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$$

$$\int_0^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega=\frac{\pi}{2}$$

Changing the variable ω to x,

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}$$

EXAMPLE 9.35 *Find the Fourier transform of*

$$\begin{aligned}f(x)&=1+\frac{x}{a}, && -a<x<0\\ &=1-\frac{x}{a}, && 0<x<a\\ &=0, && \textit{otherwise}\end{aligned}$$

Solution:

$$\begin{aligned}f(-x)&=1-\frac{x}{a}, && -a<-x<0 \quad \text{or} \quad 0<x<a\\ &=1+\frac{x}{a}, && 0<-x<a \quad \text{or} \quad -a<x<0\\ &=0, && \text{otherwise}\end{aligned}$$

$$f(-x)=f(x)$$

The function $f(x)$ is an even function. The Fourier transform of $f(x)$ is given by

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx=\int_{-\infty}^{\infty} f(x)(\cos\omega x-i\sin\omega x)\,dx$$

$$=2\int_0^{\infty} f(x)\cos\omega x\,dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x)=f(x) \atop \qquad =0, \qquad \text{if } f(-x)=-f(x)\right]$$

$$=2\int_0^a\left(1-\frac{x}{a}\right)\cos\omega x dx = 2\left[\left(1-\frac{x}{a}\right)\left(\frac{\sin\omega x}{\omega}\right)-\left(-\frac{1}{a}\right)\left(\frac{-\cos\omega x}{\omega^2}\right)\right]_0^a$$

$$=2\left[-\frac{1}{a}\frac{\cos a\omega}{\omega^2}+\frac{1}{a\omega^2}\right]=\frac{2}{a\omega^2}(1-\cos a\omega)=\frac{4}{a\omega^2}\sin^2\frac{a\omega}{2}$$

EXAMPLE 9.36 *Find the Fourier transform of* $f(x)=e^{-ax}, \quad x>0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad =-e^{ax}, \quad x<0$

Solution: $f(-x)=e^{ax}, \quad x<0$
$\qquad\qquad =-e^{-ax}, \quad x>0$
$\qquad f(-x)=-f(x)$

The function $f(x)$ is an odd function. The Fourier transform of $f(x)$ is given by

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx=\int_{-\infty}^{\infty} f(x)(\cos\omega x-i\sin\omega x)\,dx$$

$$=-2i\int_0^{\infty} f(x)\sin\omega x\,dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x)=f(x) \atop \qquad =0, \qquad \text{if } f(-x)=-f(x)\right]$$

$$=-2i\int_0^{\infty} e^{-ax}\sin\omega x dx=-2i\left[\frac{e^{-ax}}{a^2+\omega^2}(-a\sin\omega x-\omega\cos\omega x)\right]_0^{\infty}$$

$$=-\frac{2i\omega}{a^2+\omega^2}$$

EXAMPLE 9.37 *Find the Fourier cosine and sine transforms of*
$f(x)=x, \qquad 0<x<1$
$\quad =2-x, \qquad 1<x<2$
$\quad =0, \qquad x>2$

Solution: The Fourier cosine transform of $f(x)$ is given by

$$F_c(\omega)=\int_0^{\infty} f(x)\cos\omega x dx=\int_0^1 x\cos\omega x+\int_1^2(2-x)\cos\omega x dx+\int_2^{\infty}0\cdot\cos\omega x dx$$

$$=\left|x\left(\frac{\sin\omega x}{\omega}\right)-\left(\frac{-\cos\omega x}{\omega^2}\right)\right|_0^1+\left|(2-x)\left(\frac{\sin\omega x}{\omega}\right)-(-1)\left(\frac{-\cos\omega x}{\omega^2}\right)\right|_1^2$$

$$= \left(\frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2}\right) + \left(\frac{-\cos 2\omega}{\omega^2} - \frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2}\right)$$

$$= \frac{2\cos\omega - 1 - \cos 2\omega}{\omega^2} = \frac{2\cos\omega - (1+\cos 2\omega)}{\omega^2} = \frac{2\cos\omega - 2\cos^2\omega}{\omega^2}$$

$$= \frac{2}{\omega^2}\cos\omega(1-\cos\omega)$$

The Fourier sine transform of $f(x)$ is given by

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx = \int_0^1 x\sin\omega x\,dx + \int_1^2 (2-x)\sin\omega x\,dx + \int_2^\infty 0\cdot\sin\omega x\,dx$$

$$= \left| x\left(\frac{-\cos\omega x}{\omega}\right) - (1)\left(\frac{-\sin\omega x}{\omega^2}\right)\right|_0^1 + \left|(2-x)\left(\frac{-\cos\omega x}{\omega}\right) - (-1)\left(\frac{-\sin\omega x}{\omega^2}\right)\right|_1^2$$

$$= \left(\frac{-\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right) + \left(\frac{-\sin 2\omega}{\omega^2} + \frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right)$$

$$= \frac{2\sin\omega - \sin 2\omega}{\omega^2} = \frac{2\sin\omega - 2\sin\omega\cos\omega}{\omega^2} = \frac{2}{\omega^2}\sin\omega(1-\cos\omega)$$

EXAMPLE 9.38 *Find the Fourier sine and cosine transforms of (i)* x^{m-1} *(ii)* $\frac{1}{\sqrt{x}}$.

Solution: (i) The Fourier sine transform of $f(x)$ is given by

$$F_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx = \int_0^\infty x^{m-1}\sin\omega x\,dx$$

$$= \int_0^\infty x^{m-1}(-\text{Imaginary part of } e^{-i\omega x})\,dx = -\text{Imaginary part of} \int_0^\infty x^{m-1}e^{-i\omega x}dx \quad \ldots (1)$$

Let $$I = \int_0^\infty x^{m-1}e^{-i\omega x}dx = \frac{\Gamma m}{(i\omega)^m} = \frac{\Gamma m}{\omega^m}(-i)^m = \frac{\Gamma m}{\omega^m}\left(e^{-\frac{i\pi}{2}}\right)^m = \frac{\Gamma m}{\omega^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right) \quad \ldots (2)$$

Imaginary part $\int_0^\infty x^{m-1}e^{-i\omega x}dx = -\frac{\Gamma m}{\omega^m}\sin\frac{m\pi}{2}$

Substituting in Eq. (1),

$$F_s(\omega) = \frac{\Gamma m}{\omega^m}\sin\frac{m\pi}{2}$$

The Fourier cosine transform of $f(x)$ is given by

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \int_0^\infty x^{m-1}\cos\omega x\,dx = \int_0^\infty x^{m-1}(\text{Real part of } e^{-i\omega x})\,dx$$

$$= \text{Real part of} \int_0^\infty x^{m-1}e^{-i\omega x}dx = \frac{\Gamma m}{\omega^m}\cos\frac{m\pi}{2} \quad \text{[Using Eq. (2)]}$$

(ii) Putting $m = \frac{1}{2}$,

$$F_s(\omega) = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\omega}} \sin\frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{\omega}}\frac{1}{\sqrt{2}} = \sqrt{\frac{\pi}{2\omega}}$$

$$F_c(\omega) = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\omega}} \cos\frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{\omega}}\frac{1}{\sqrt{2}} = \sqrt{\frac{\pi}{2\omega}}$$

EXAMPLE 9.39 *Find f(x) if its Fourier sine transform is* $\frac{1}{\omega}e^{-a\omega}$. *Hence, deduce* $F_s^{-1}\left(\frac{1}{\omega}\right)$.

Solution: The inverse Fourier sine transform of $F_s(\omega)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega x\, d\omega = \frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega}e^{-a\omega}\sin\omega x\, d\omega$$

Differentiating w.r.t. x using DUIS,

$$f'(x) = \frac{2}{\pi}\int_0^{\infty}\frac{e^{-a\omega}}{\omega}\omega\cos\omega x\, d\omega = \frac{2}{\pi}\int_0^{\infty} e^{-a\omega}\cos\omega x\, d\omega$$

$$= \frac{2}{\pi}\left|\frac{e^{-a\omega}}{a^2+x^2}(-a\cos\omega x + x\sin\omega x)\right|_0^{\infty} = \frac{2}{\pi}\frac{a}{a^2+x^2}$$

Integrating w.r.t. x,

$$f(x) = \frac{2}{\pi}\int\frac{a}{a^2+x^2}dx = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) + c$$

At $\quad x = 0, f(0) = 0$

$$0 = \frac{2}{\pi}\tan^{-1}0 + c$$

$$c = 0$$

Hence, $\quad f(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) = F_s^{-1}\left(\frac{1}{\omega}e^{-a\omega}\right)$

Putting $a = 0$,

$$F_s^{-1}\left(\frac{1}{\omega}\right) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$$

EXAMPLE 9.40 *Solve the integral equation* $\int_0^{\infty} f(x)\sin\omega x\, dx = 1, \quad 0 \le \omega < 1$

$$= 2, \quad 1 \le \omega < 2$$

$$= 0, \quad \omega \ge 2$$

Solution: Solve the integral equation means find $f(x)$. The Fourier sine transform of $f(x)$ is given by

$$F_s(\omega)=\int_0^\infty f(x)\sin\omega x\,dx$$

$$=\begin{cases}1, & 0\le\omega<1\\ 2, & 1\le\omega<2\\ 0, & \omega\ge 2\end{cases}$$

The inverse Fourier sine transform of $F_s(\omega)$ is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,d\omega=\frac{2}{\pi}\left[\int_0^1 1\cdot\sin\omega x\,d\omega+\int_1^2 2\cdot\sin\omega x\,d\omega\right]$$

$$=\frac{2}{\pi}\left[\left|\frac{-\cos\omega x}{x}\right|_0^1+2\left|\frac{-\cos\omega x}{x}\right|_1^2\right]=\frac{2}{\pi x}\left[(1-\cos x)+2(\cos x-\cos 2x)\right]$$

$$=\frac{2}{\pi x}(1+\cos x-2\cos 2x)$$

EXAMPLE 9.41 *Find the Fourier cosine transform of* $f(x)=\frac{1}{1+x^2}$. *Hence, derive the Fourier sine transform of* $\frac{x}{1+x^2}$.

Solution: The Fourier cosine transform of $f(x)$ is given by

$$F_c(\omega)=\int_0^\infty f(x)\cos\omega x\,dx=\int_0^\infty\frac{1}{1+x^2}\cos\omega x\,dx \qquad \text{... (1)}$$

Differentiating w.r.t. ω using DUIS,

$$F_c'(\omega)=\int_0^\infty\frac{-x\sin\omega x}{1+x^2}dx=-\int_0^\infty\frac{x^2\sin\omega x}{x(1+x^2)}dx$$

$$=-\int_0^\infty\frac{\left[(1+x^2)-1\right]\sin\omega x}{x(1+x^2)}dx=-\int_0^\infty\frac{\sin\omega x}{x}dx+\int_0^\infty\frac{\sin\omega x}{x(1+x^2)}dx$$

$$=-\frac{\pi}{2}+\int_0^\infty\frac{\sin\omega x}{x(1+x^2)}dx \qquad \left[\because\int_0^\infty\frac{\sin\omega x}{\omega}d\omega=\frac{\pi}{2}\right] \qquad \text{... (2)}$$

Differentiating again w.r.t. ω using DUIS,

$$F_c''(\omega)=0+\int_0^\infty\frac{x\cos\omega x}{x(1+x^2)}dx=F_c(\omega)$$

$$F_c''(\omega)-F_c(\omega)=0$$

Solving the above differential equation,

$$F_c(\omega)=c_1e^{\omega}+c_2e^{-\omega}$$

$$F_c'(\omega)=c_1e^{\omega}-c_2e^{-\omega}$$

When $\omega = 0$, from Eq. (1),

$$F_c(0) = \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$$c_1 + c_2 = \frac{\pi}{2} \quad \dots (3)$$

When $\omega = 0$, from Eq. (2),

$$F_c'(0) = -\frac{\pi}{2}$$

$$c_1 - c_2 = -\frac{\pi}{2} \quad \dots (4)$$

Solving Eqs (3) and (4),

$$c_1 = 0, \quad c_2 = \frac{\pi}{2}$$

$$F_c(\omega) = \frac{\pi}{2} e^{-\omega}$$

$$F_s(\omega) = \int_0^\infty \frac{x}{1+x^2} \sin \omega x\,dx = -F_c'(\omega) = \frac{\pi}{2} e^{-\omega}$$

EXAMPLE 9.42 *Solve the integral equation* $\int_0^\infty f(x)\cos\omega x\,dx = 1-\omega, \quad 0<\omega<1$

$= 0, \quad \omega > 1$

Hence, show that $\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}$.

Solution: Solve the integral equation means find $f(x)$. The Fourier cosine transform of $f(x)$ is given by

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \begin{cases} 1-\omega, & 0<\omega<1 \\ 0, & \omega>1 \end{cases}$$

The inverse Fourier cosine transform of $F_c(\omega)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega = \frac{2}{\pi}\int_0^1 (1-\omega)\cos\omega x\,d\omega$$

$$= \frac{2}{\pi}\left|(1-\omega)\left(\frac{\sin\omega x}{x}\right) - (-1)\left(\frac{-\cos\omega x}{x^2}\right)\right|_0^1 = \frac{2}{\pi}\left(\frac{-\cos x}{x^2} + \frac{1}{x^2}\right) = \frac{2}{\pi x^2}(1-\cos x)$$

$$= \frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)$$

Substituting $f(x)$ in the given integral equation,

$$\int_0^\infty \frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)\cos\omega x\,dx = 1-\omega, \quad 0<\omega<1$$

$$= 0, \quad \omega>1$$

Putting $\omega = 0$,

$$\frac{4}{\pi}\int_0^{\infty} \frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)dx = 1$$

$$\int_0^{\infty} \frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)dx = \frac{\pi}{4}$$

Putting $\frac{x}{2} = u$, $dx = 2du$

$$\int_0^{\infty} \frac{\sin^2 u}{(2u)^2} 2\,du = \frac{\pi}{4}$$

$$\int_0^{\infty} \frac{\sin^2 u}{u^2} du = \frac{\pi}{2}$$

EXERCISE 9.8

1. Find the Fourier transforms of the following functions:

(i) $f(x) = e^{ix}, \quad a<x<b$
$\quad = 0, \quad x<a,\ x>b$

(ii) $f(x) = \frac{1}{2a}, \quad |x| \le a$
$\quad = 0, \quad |x|>a$

(iii) $f(x) = x^2, \quad |x|<a$
$\quad = 0, \quad |x|>a$

(iv) $f(x) = 1-|x|, \quad |x|<1$
$\quad = 0, \quad |x|>1$

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i)} \dfrac{1}{i(1-\omega)}\left[e^{i(1-\omega)b} - e^{i(1-\omega)a}\right] \\ \text{(ii)} \dfrac{\sin a\omega}{a\omega} \\ \text{(iii)} \dfrac{2}{\omega^3}\left[(a^2\omega^2-2)\sin a\omega + \dfrac{4a}{\omega^2}\cos a\omega\right] \\ \text{(iv)} \dfrac{2}{\omega^2}(1-\cos a\omega)\end{array}\right]$$

2. Find the Fourier cosine transform of e^{-ax}, $a>0$. Hence, find $F_c\{xe^{-ax}\}$ and $F\{|x|e^{-a|x|}\}$.

$$\left[\textbf{Ans.: } \frac{a}{\omega^2+a^2}, \frac{a^2-\omega^2}{(a^2+\omega^2)}, \frac{2(a^2-\omega^2)}{(a^2+\omega^2)^2}\right]$$

3. Find the Fourier sine transform of e^{-ax}, $a>0$. Hence, find $F_s\{xe^{-ax}\}$ and $F\{xe^{-a|x|}\}$.

$$\left[\textbf{Ans.: } \frac{\omega}{\omega^2+a^2}, \frac{2a\omega}{(\omega^2+a^2)^2}, \frac{-4ia\omega}{(\omega^2+a^2)^2}\right]$$

4. Find the Fourier cosine transform of
$f(x) = \cos x, \quad 0<x<a$
$\quad = 0, \quad x>a$

$$\left[\textbf{Ans.: } \frac{1}{2}\left[\frac{\sin a(\omega+1)}{\omega+1} + \frac{\sin a(\omega-1)}{\omega-1}\right]\right]$$

5. Find the Fourier sine transform of $f(x) = \frac{x}{1+x^2}$.

$$\left[\textbf{Ans.: } \frac{\pi}{2}e^{-\omega}\right]$$

6. Find the Fourier transform of $e^{-ax^2}\cos bx$.

$$\left[\text{Ans.}: \sqrt{\frac{\pi}{4a^2}}\left[e^{\frac{-(\omega+b)^2}{4a}}+e^{\frac{-(\omega-b)^2}{4a}}\right]\right]$$

7. Find the inverse Fourier transform of $\dfrac{1}{(1+\omega^2)^2}$.

$$\left[\text{Ans.}: \frac{1}{4}(1+x)e^{-x}\right]$$

8. Find the inverse Fourier transform of

$$F(\omega)=1+\omega^2, \quad |\omega|<1$$
$$=0, \quad |\omega|>1$$

$$\left[\text{Ans.}: \frac{1}{\pi x^3}(x^2\sin x+x\cos x-\sin x)\right]$$

9. Find the inverse Fourier transform of $e^{-\omega^2}$.

$$\left[\text{Ans.}: \frac{1}{2\sqrt{\pi}}e^{-\frac{x^2}{4}}\right]$$

10. Find the Fourier sine transform of

$$f(x)=0, \quad 0<x<a$$
$$=x, \quad a\le x\le b$$
$$=0, \quad x>b$$

$$\left[\text{Ans.}: \frac{1}{\omega}(a\cos a\omega-b\cos b\omega)+\frac{1}{b^2}(\sin b\omega-\sin a\omega)\right]$$

11. Find the Fourier sine transforms of

(i) $\dfrac{x}{1+x^2}$ (ii) $\dfrac{1}{x}$

$$\left[\text{Ans.}: \text{(i) } \frac{\pi}{2}e^{-\omega} \quad \text{(ii) } \frac{\pi}{2}\right]$$

12. Find the Fourier sine and cosine transforms of

$$f(x)=x^3, \quad 0\le x\le 1$$
$$=0, \quad x>0$$

$$\left[\text{Ans.}: \frac{2\sin\omega}{\omega^2}-\frac{\cos\omega}{\omega}+\frac{2(\cos\omega-1)}{\omega^3},\ \frac{\sin\omega}{\omega}+\frac{2\cos\omega}{\omega^2}-\frac{2\sin\omega}{\omega^3}\right]$$

13. Find $f(x)$ satisfying the integral equation:

(i) $\int_0^\infty f(x)\sin\omega x\,dx=\dfrac{\sin\omega}{\omega}$

(ii) $\int_0^\infty f(x)\sin\omega x\,dx=1-\omega, \quad 0\le\omega\le 1$
$=0, \quad \omega>1$

$$\left[\begin{aligned}\text{Ans.: (i) } f(x)&=1, \quad 0<x<1\\ &=0, \quad x\ge 1\\ \text{(ii) } f(x)&=\frac{2}{\pi x^2}(x-\sin x)\end{aligned}\right]$$

9.11 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If $F\{f(x)\}=F(\omega)$ and $F\{g(x)\}=G(\omega)$ then

$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\bar{G}(\omega)\,d\omega$$

where the bar implies the complex conjugate.

Proof $\displaystyle\int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx=\int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{G}(\omega)e^{-i\omega x}d\omega\right\}dx$

[Using the inversion formula for Fourier transform]

9.12 FINITE FOURIER TRANSFORMS

9.12.1 Finite Fourier Cosine Transform

If the function $f(x)$ is piecewise continuous in the interval $(0, l)$ then the finite Fourier cosine transform of $f(x)$ is given by

$$F_c(n)=\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

and the inverse finite Fourier cosine transform of $F_c(n)$ is given by

$$f(x)=\frac{1}{l}F_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\frac{n\pi x}{l}$$

Proof The half-range cosine series of $f(x)$ is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \qquad \text{... (9.11)}$$

where

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Integrating both the sides of Eq. (9.11) w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\,dx=\int_0^l a_0\,dx+\int_0^l\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\,dx=a_0\int_0^l dx+\sum_{n=1}^{\infty}a_n\int_0^l\cos\frac{n\pi x}{l}\,dx=a_0 l+0=a_0 l$$

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}F_c(0) \qquad \text{[From definition]} \qquad \text{... (9.12)}$$

Multiplying both the sides of Eq. (9.11) by $\cos\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=a_0\int_0^l\cos\frac{n\pi x}{l}\,dx+\int_0^l\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx=0+\frac{l}{2}a_n=\frac{l}{2}a_n$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}F_c(n) \qquad \text{[From definition]} \qquad \text{... (9.13)}$$

Substituting Eqs (9.12) and (9.13) in Eq. (9.11),

$$f(x)=\frac{1}{l}F_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\frac{n\pi x}{l}$$

9.12.2 Finite Fourier Sine Transform

If the function $f(x)$ is piecewise continuous in the interval $(0, l)$ then the finite Fourier sine transform of $f(x)$ is given by

$$F_s(n)=\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

and the inverse finite Fourier sine transform of $F_s(n)$ is given by

$$f(x)=\frac{2}{l}\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{l}$$

Proof The half-range sine series of $f(x)$ is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \qquad \text{... (9.14)}$$

where $$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Multiplying both the sides of Eq. (9.14) by $\sin\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(0, l)$,

$$\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx=\int_0^l\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\sin\frac{n\pi x}{l}\,dx=\frac{l}{2}b_n$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}F_s(n) \qquad \text{[From definition] ... (9.15)}$$

Substituting Eq. (9.15) in Eq. (9.14),

$$f(x)=\frac{2}{l}\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{l}$$

EXAMPLE 9.46 *Find the finite Fourier cosine and sine transforms of the function*

$$f(x)=1, \qquad 0<x<\frac{\pi}{2}$$
$$=-1, \qquad \frac{\pi}{2}<x<\pi$$

Solution: The finite Fourier cosine transform of $f(x)$ is given by

$$F_c(n)=\int_0^{\pi}f(x)\cos nx\,dx=\int_0^{\frac{\pi}{2}}1\cdot\cos nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(-1)\cos nx\,dx$$

$$=\int_0^{\frac{\pi}{2}}\cos nx\,dx-\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx=\left|\frac{\sin nx}{n}\right|_0^{\frac{\pi}{2}}-\left|\frac{\sin nx}{n}\right|_{\frac{\pi}{2}}^{\pi}=\frac{2}{n}\sin\frac{n\pi}{2}$$

The finite Fourier sine transform of $f(x)$ is given by

$$F_s(n)=\int_0^{\pi}f(x)\sin nx\,dx=\int_0^{\frac{\pi}{2}}1\cdot\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(-1)\sin nx\,dx$$

$$=\int_0^{\frac{\pi}{2}}\sin nx\,dx-\int_{\frac{\pi}{2}}^{\pi}\sin nx\,dx=\left|\frac{-\cos nx}{n}\right|_0^{\frac{\pi}{2}}-\left|\frac{-\cos nx}{n}\right|_{\frac{\pi}{2}}^{\pi}$$

$$=-\frac{1}{n}\cos\frac{n\pi}{2}+\frac{1}{n}+\frac{1}{n}\cos n\pi-\frac{1}{n}\cos\frac{n\pi}{2}=\frac{1}{n}\left(\cos n\pi-2\cos\frac{n\pi}{2}+1\right)$$

EXAMPLE 9.47 *Find the finite Fourier cosine and sine transforms of the function $f(x) = e^{ax}$, $0 < x < l$.*

Solution: The finite Fourier cosine transform of $f(x)$ is given by

$$F_c(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^l e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \left|\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{1}{a^2+\frac{n^2\pi^2}{l^2}}\left[e^{al}a(-1)^n - a\right] = \frac{al^2}{n^2\pi^2+a^2l^2}\left[(-1)^n e^{al} - 1\right]$$

The finite Fourier sine transform of $f(x)$ is given by

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^l e^{ax}\sin\frac{n\pi x}{l}\,dx = \left|\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{e^{al}}{a^2+\frac{n^2\pi^2}{l^2}}\left(-\frac{n\pi}{l}\right)(-1)^n + \frac{1}{a^2+\frac{n^2\pi^2}{l^2}}\left(\frac{n\pi}{l}\right) = \frac{n\pi l}{n^2\pi^2+a^2l^2}\left[1-(-1)^n e^{al}\right]$$

EXAMPLE 9.48 *Find $f(x)$ if its finite Fourier cosine transform is given by*

$$F_c(n) = \frac{1}{(2n+1)^2}\cos\frac{2n\pi}{3} \text{ in } 0 < x < 1.$$

Solution: The inverse finite Fourier cosine transform of $F_c(n)$ is given by

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\frac{n\pi x}{l} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{2n\pi}{3}\cos n\pi x \qquad [\because l = 1]$$

EXAMPLE 9.49 *Find $f(x)$ if its finite Fourier sine transform is given by*

$$F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2},\quad 0 < x < \pi$$

Solution: The inverse finite Fourier sine transform of $F_s(n)$ is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{l}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2\pi^2}\sin nx \qquad [\because l = \pi]$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin nx$$

EXERCISE 9.10

1. Find the finite Fourier cosine and sine transforms of the following functions:

(i) $f(x) = 1$ in $(0, l)$

(ii) $f(x) = x$ in $(0, \pi)$

(iii) $f(x) = x^2$ in $(0, 1)$

(iv) $f(x) = x^3$ in $(0, 2)$

(v) $f(x) = x(\pi - x)$ in $(0, \pi)$

Ans.:

(i) $F_c(n) = 0, \quad n \neq 0$
$= l, \quad n = 0$
$F_s(n) = \frac{l}{n\pi}\left[1-(-1)^n\right]$

(ii) $F_c(n) = \frac{1}{n^2}\left[(-1)^n - 1\right], \; n \neq 0$
$= \frac{\pi^2}{2}, \quad n = 0$
$F_s(n) = \frac{\pi}{n}(-1)^{n+1}$

(iii) $F_c(n) = \frac{2(-1)^n}{n^2\pi^2}, \quad n \neq 0$
$= \frac{1}{3}, \quad n = 0$
$F_s(n) = \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left[(-1)^n - 1\right]$

(iv) $F_c(n) = \frac{3(-1)^n}{n^2\pi^2} - \frac{6}{n^4\pi^4}, \quad n \neq 0$
$= 4, \quad n = 0$
$F_s(n) = \frac{(-1)^n}{n\pi} + \frac{6(-1)^n}{n^3\pi^3}$

(v) $F_c(n) = -\frac{\pi}{n^2}\left[1+(-1)^n\right], \; n \neq 0$
$= \frac{\pi^3}{6}, \quad n = 0$
$F_s(n) = \frac{2}{n^3}\left[1-(-1)^n\right]$

2. Find the finite Fourier cosine transforms of the following functions:

(i) $f(x) = \sin x, \quad 0 < x < \pi$

(ii) $f(x) = \left(1 - \frac{x}{\pi}\right)^2, \quad 0 < x < \pi$

Ans.:

(i) $F_c(n) = \frac{-\pi}{n^2-1}\left[1+(-1)^n\right], F_c(0) = 2$

(ii) $F_c(n) = \frac{2}{\pi n^2}, \; F_c(0) = \frac{\pi}{3}$

3. Find the finite Fourier sine transforms of the following functions:

(i) $f(x) = \frac{x}{\pi}, \quad 0 < x < \pi$

(ii) $f(x) = \cos ax, \quad 0 < x < \pi$

Ans.:

(i) $F_s(n) = \frac{(-1)^{n+1}}{n}$

(ii) $F_s(n) = \frac{n}{n^2 - a^2}\left[1 - \cos a\pi(-1)^n\right]$

4. Find $f(x)$, if

(i) $F_c(n)=\dfrac{1}{n^2\pi^2+1}\left[e(-1)^n-1\right],\quad 0<x<1$

(ii) $F_s(n)=\dfrac{2l}{n\pi}\sin^2\dfrac{n\pi}{4},\quad 0<x<l$

Ans.:

(i) $f(x)=e-1+2\sum_{n=1}^{\infty}\dfrac{\left[e(-1)^n-1\right]}{n^2\pi^2+1}\cos n\pi x$

(ii) $f(x)=\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin^2 n\pi}{4n}\sin\dfrac{n\pi x}{l}$

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