Laplace Transform.

Page No.	
Date	

1] $L[e^{at}] = \frac{1}{s-a}$ $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2] $L[e^{-at}] = \frac{1}{s+a}$ $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

3] $L[\sin at] = \frac{a}{s^2+a^2}$ $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

4] $L[\cos at] = \frac{s}{s^2+a^2}$ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6] $L[\sinh at] = \frac{a}{s^2-a^2}$ $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

7] $L[\cosh at] = \frac{s}{s^2-a^2}$ $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

8] $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n}$

9] $L[1] = \frac{1}{s}$ $L^{-1}\left[\frac{1}{s}\right] = 1$

Prop. i] $L[e^{at} f(t)] = \phi(s-a)$
$L[e^{-at} f(t)] = \phi(s+a)$

ii] $L[t^n f(t)] = (-1)^n \frac{d^{(n)}}{ds} \phi(s)$

iii] $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\, ds$

iv] $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

v] $L\left[\int_0^t f(t)\, dt\right] = \frac{\phi(s)}{s}$

$e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

For Finding L^{-1} of log / $\tan^{-1}$ / $\cot^{-1}$ etc.

$$L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)].$$

Convolution

$$L^{-1}[\phi_1(s)\,\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du.$$

Special Function

Periodic

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt.$$

Heaviside

$$L^{-1}\left[\frac{e^{-as}}{s}\right] = H(t-a)$$

$$L[f(t).H(t-a)] = e^{-as} L[f(t+a)]$$

If $f(t) = f_1(t)$ for $a \le t \le b$
$\quad f_2(t)$ for $b \le t \le c$

$$f(t) = f_1(t)[H(t-a) - H(t-b)] + f_2(t)[H(t-b) - H(t-c)].$$

Dirac delta.

$L[\delta(t-a)] = e^{-as}$

$L[\delta(t-a)\, f(t)] = e^{-as} f(a).$

Solution of diff using laplace.

$\bar{y} = \phi(s)$; $\bar{y} = L[y(t)]$.

$L[y'] = s\bar{y} - y(0)$.

$L[y''] = s^2\bar{y} - s\,y(0) - y'(0)$.

Fourier

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$\hookrightarrow [a, a+2l]$.

$$a_0 = \frac{1}{l} \int_0^{a+2l} f(x)\, dx.$$

$$a_n = \frac{1}{l} \int_0^{a+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_0^{a+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) . dx.$$

Parseval's identity.

$$\frac{1}{\pi} \int_0^{2\pi} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Odd Func^n $\Rightarrow$ $a_0 = a_n = 0$

even Function $\rightarrow$ $b_n = 0$.

Page No.
Date

Half Range

Period $\Rightarrow 2l$.

Cosine

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad b_n = 0$$

~~# Sine~~

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$f(x)$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Sine.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad a_0 = a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Parsevals

$$\frac{2}{l}\int_0^l f^2(x) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Complex Form of Fourier.

$[a, a+2l]$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2}[a_n - ib_n] = \frac{1}{2l}\int_{a}^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$c_0 = \frac{a_0}{2}$$

$$c_{-n} = \frac{1}{2}[a_n + ib_n].$$

Fourier Transform.

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} F(s)\, ds.$$

Fourier Transform $f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$

$$= \begin{cases} 1 & -a \le x \le a \\ 0 & \text{o.w.} \end{cases}$$

a] $\int_0^\infty \frac{\sin s}{s}\, ds$ b] $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$.

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [\cos(sx) + i \cancel{\sin(sx)}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_0^a \cos sx\, dx.$$

$$f(s) = \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s}\right]_0^a = \frac{2}{\sqrt{2\pi}} \frac{\sin as}{s}$$

Fourier Transform.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sa}{s}\right) ds.$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \int_{-\infty}^{\infty} e^{-isx} \left(\frac{\sin sa}{s}\right) = \begin{cases} 1 \\ 0 \end{cases}$$

* $x = 0$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s}$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin sa}{s}$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin sa}{s}$$

Inverse Fourier Transform.

Page No.	
Date	

b] $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \cdot \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sa}{s}\right] ds$

$= \frac{\cancel{2}}{\cancel{2}\pi} \int_{-\infty}^{\infty} [\cos sx + i\cancel{\sin sx}] \frac{\sin sa}{s}$

$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \sin sa}{s}\, ds$

$= \begin{cases} \pi & |x| \leq a \\ 0 & |x| \geq a. \end{cases}$

Fourier Integral

$$f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$$

Cosine integral or even.

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda t . \cos\lambda x\, dt\, d\lambda$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \cos\lambda x \int_{0}^{\infty} f(t) \cos\lambda t\, dt\, d\lambda$$

$$\Rightarrow \frac{2}{\pi} \int_{0}^{\infty} \cos\lambda x \int_{0}^{\infty} f(t) \cos\lambda t\, dt\, d\lambda.$$

Sine integral or odd.

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin\lambda x \int_{0}^{\infty} f(t) \sin\lambda t\, dt\, d\lambda$$

Find the Fourier integral

$$f(x) = \begin{cases} 1 & |x| < 1 \quad -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$$

even.

$$f(x) = \frac{1}{\pi} \cdot 2 \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^1 \cos \lambda t \, dt \, d\lambda.$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[\frac{\sin \lambda t}{\lambda} \right]_0^1$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos \lambda x \sin \lambda}{\lambda}$$

$$2 \int_0^\infty \frac{\cos \lambda x \sin \lambda}{\lambda} = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Z - Transform.

$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$

Change of Scale.

$Z[a^k f(k)] = F(z/a)$ &

$Z[a^{-k} f(k)] = F(az)$

Shifting prop

$Z[f(k \pm n)] = z^{\pm n} F(z)$.

$-z \dfrac{d}{dz} Z[k^{n-1}] = Z[k^n]$.

$Z\left[k^n f(k)\right] = \left(-z \dfrac{d}{dz}\right)^n F(z)$.

$Z\left[\dfrac{f(k)}{k}\right] = -\int_z^{\infty} z^{-1} F(z)\, dz$.

IVT.

$F(z) = Z[f(k)]$

then $f(0) = \lim_{z \to \infty} F(z)$

$f(1) = \lim_{z \to \infty} z\,[F(z) - f(0)]$

$f(2) = \lim_{z \to \infty} z^2\,[F(z) - f(0) - f(1)/z]$.

Page No. | Date

Imp. Series.

1. $a + ar + ar^2 + \dots = \frac{a}{1-r} \quad |r| < 1$

2. $1 - x + x^2 - x^3 + \dots = (1+x)^{-1} \quad |x| < 1$

3. $1 + x + x^2 + x^3 + \dots = (1-x)^{-1} \quad |x| < 1$

4. $1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^2 = (1+x)^n$

5. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$

6. $\cancel{\log} \; -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = \log(1-x)$

7. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \log(1+x)$

8. $1 + nx + \frac{n(n+1)}{2!} x^2 + \dots = (1-x)^{-n}$

Std Form.

$Z(1) = \frac{z}{z-1} \qquad Z[\cos k\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$

$Z(k) = \frac{z}{(z-1)^2} \qquad Z[\sin k\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$Z(a^n) = \frac{z}{z-a}$

Page No.
Date

$$Z[g(k) * f(k)] = Z[g(k)] * Z[f(k)].$$

$$Z^{-1}[F(z)\,G(z)] = f(k) * g(k).$$

$$= \sum_{m=0}^{k} f(m)\, g(k-m).$$

e.g. $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right].$

$$f(k) = Z^{-1}\left[\frac{z}{(z-1)}\right] \quad g(k) = Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1 \qquad g(k) = 3^k.$$

$$Z^{-1}[F(z)\,G(z)] = \sum_{m=0}^{k} f(m)\, g(k-m)$$

$$= \sum_{m=0}^{k} 1^k\, 3^{k-m}$$

$$= 1^k \cdot 3^k \sum_{m=0}^{k} 3^{-m}$$

$$= 3^k \left[\frac{(1/3)^{k+1} - 1}{1/3 - 1}\right] \rightarrow \text{sum of n terms in g.p.}$$

$$= 3^k \frac{\left[(1/3)^{k+1} - 1\right]}{-\frac{2}{3}} = \frac{1}{2}\left[3^{k+1} - 1\right].$$

Vector Algebra

$[a\ b\ c] = a \cdot (b \times c)$

Volume of parallelepiped $= [a\ b\ c]$

Volume of tetrahedron $= \frac{1}{6}[a\ b\ c]$.

$[a\ b\ c] = -[b\ a\ c]$.

Vector product of 4 vector.

$(a \times b) \times (c \times d)$

$= \cancel{[a\ b\ d]\ c}\ [a\ c\ d]\ b - [b\ c\ d]\ a$

$= [a\ b\ d]\ c - [a\ b\ c]\ d$.

Vector triple product.

$(a \times b) \times c = (c \cdot a)\, b - (c \cdot b)\, \bar{a}$

Scalar product of four vector.

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}$$

Vector Differentiation.

i] Gradient.

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

ii] directional derivative

$$= \nabla \phi \cdot \frac{\bar{a}}{|a|} = \nabla \phi \cdot \hat{a}$$

iii] Divergence

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\nabla \cdot F = 0 \rightarrow$ Solenoidal

iv] Curl.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$\nabla \times F = 0 \rightarrow$ irrotational

v] Scalar potential

$$\int d\phi = \int \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

Rem $\nabla \frac{1}{r} = \left(-\frac{1}{r^2}\right) \frac{\bar{r}}{r}$

Imp Formula.

$$\nabla \cdot (\phi \bar{f}) = \phi(\nabla \cdot f) + \bar{f} \cdot (\nabla \phi).$$

Vector Integration.

Page No.
Date

Line Integral :-

$$\int \vec{F} \cdot d\vec{r}$$

$$\int d\phi = \int F \cdot dr.$$ $\phi \rightarrow$ Scalar potential

Greens theorem.

$$\int p\, dx + q\, dy = \iint \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\, dx\, dy$$

Stoke's theorem.

$$\oint F \cdot dr = \iint \hat{n} \cdot (\nabla \times F)\, ds$$

Gauss - div theorem

$$\iint_S \hat{n} \cdot \bar{F}\, ds = \iiint \nabla \cdot \bar{F}\, dv.$$