

Laplace Transform.

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$$1) L[e^{at}] = \frac{1}{s-a} \quad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$2) L[e^{-at}] = \frac{1}{s+a} \quad L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$3) L[\sin at] = \frac{a}{s^2+a^2} \quad L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$$

$$4) L[\cos at] = \frac{s}{s^2+a^2} \quad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$5) L[\sinh at] = \frac{a}{s^2-a^2} \quad L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$$

$$7) L[\cosh at] = \frac{s}{s^2-a^2} \quad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$$

$$8) L[t^n] = \frac{n!}{s^{n+1}} \quad L^{-1}\left[\frac{1}{s^n}\right] = t^{n-1}$$

$$9) L[1] = \frac{1}{s} \quad L^{-1}\left[\frac{1}{s}\right] = 1$$

Prop. i) $L[e^{at} f(t)] = \Phi(s-a)$

$L[e^{-at} f(t)] = \Phi(s+a)$

ii) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \Phi(s)$

iii) $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \Phi(s) ds$

iv) $L[f(at)] = \frac{1}{a} \Phi\left(\frac{as}{a}\right) = \frac{1}{a} \Phi\left(\frac{s}{a}\right)$

v) $L\left[\int_0^t f(t) dt\right] = \Phi(s)$

For finding L^{-1} of $\log L \sin^4 \omega t$ etc.

$$L^{-1}[\phi(s)] = e^{-\int s} \left[\frac{d}{dt} [\phi'(s)] \right]_{t=0} \quad (1)$$

$$\text{Convolution} = \int_0^\infty f_1(t) f_2(t-u) du \quad (2)$$

$$L^{-1}[\phi(s) \phi_2(s)] = \int_0^\infty f_1(u) f_2(t-u) du \quad (3)$$

$$f_1 f_2 = \int_0^\infty f_1(t-u) f_2(u) du \quad (4)$$

$$\text{Special Function} \quad (5)$$

$$\text{Periodic} = \left[\begin{matrix} f_1 \\ f_2 \end{matrix} \right] \int_0^\infty [f_1(t+T) - f_1(t)] dt \quad (6)$$

$$L[f(t)] = \left[\begin{matrix} f_1 \\ f_2 \end{matrix} \right] \int_0^\infty e^{-st} [f_1(t+T) - f_1(t)] dt \quad (7)$$

$$\text{Ans} = \left[\begin{matrix} f_1 \\ f_2 \end{matrix} \right] \int_0^\infty (s-2\pi i)^{-1} e^{-st} [f_1(t+T) - f_1(t)] dt \quad (8)$$

$$\text{Heaviside} = (t+a)^{-1} = (t+1)^{-1} e^{a \cdot 1} \quad (9)$$

$$L^{-1}[e^{-as} \frac{1}{s-a}] = H(t-a) - H(t+a) \quad (10)$$

$$L^{-1}[f(t) H(t-a)] = e^{-as} f(t-a) \quad (11)$$

$$I \int f(t) dt = (f_1, t) \quad \text{for } a \leq t \leq b, \\ (f_2, t) \quad \text{for } b \leq t \leq c \quad (12)$$

$$f(t) = f_1(t) H(t-a) + f_2(t-b) H(t-b) \Rightarrow \\ L_2(t) [H(t-b) - H(t-c)] \quad (13)$$

Diodes (delta) \Rightarrow $\delta(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$

$$\# L[\delta(t-a)] = e^{-as}$$

$$\# L[\delta(t-a)f(t)] = e^{-as}f(a).$$

Solution of diff using Laplace.

$$\bar{y} = \frac{\Phi(s)}{(s+a)} \bar{y}(0) + L[y(t)].$$

$$L[y'] = s\bar{y} - y(0)$$

$$L[y''] = s^2\bar{y} - sy(0) - y'(0).$$

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$$L[y] = \frac{1}{s+a} \Rightarrow L[y''] = s^2 \frac{1}{s+a}$$

$$s^2 \frac{1}{s+a} = s^2 + s^2 a = s^2(1+a)$$

$$s^2 + s^2 a = s^2(1+a)$$

Fourier

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)]$$

$$\downarrow [a, a+2l]$$

$$a_0 = \frac{1}{2l} \int_a^{a+2l} f(u) du.$$

$$a_n = \frac{1}{l} \int_a^{a+2l} f(u) \cos\left(\frac{n\pi u}{l}\right) du.$$

$$b_n = \frac{1}{l} \int_a^{a+2l} f(u) \sin\left(\frac{n\pi u}{l}\right) du.$$

Parseval's identity.

$$\frac{1}{\pi} \int_0^{2\pi} f^2(u) du = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Odd Function $\Rightarrow a_0 = a_n = 0$

Even Function $\Rightarrow b_n = 0$.

Half Ranges → To find a_0

Period $\Rightarrow 2l$

Cosine

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad b_n = 0$$

Sine

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Sine.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad a_0 = a_n = 0$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Parsevals

$$\frac{2}{l} \int_0^l f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Complex Form of Fourier S. 1104

$[a, a+2l]$.

$$\text{and } f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = (b)$$

$$c_n = \frac{1}{2l} \int_a^{a+2l} f(x) e^{-inx} dx = (b)$$

$$c_0 = (a_0)$$

$$c_n = \frac{1}{2} [a_n + i b_n].$$

Fourier Transform.

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx. \quad (b)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(s) ds.$$

$$\text{if } s = t + i\omega \Rightarrow f(t) = (b)$$

Fourier Transform $f(\omega) = \begin{cases} 1 & |\omega| \leq a \\ 0 & |\omega| > a \end{cases}$

$$f(\omega) = \begin{cases} 1 & -a \leq \omega \leq a \\ 0 & \text{o.w.} \end{cases}$$

a] $\int_{-\infty}^{\infty} s \sin s ds$ b] $\int_{-\infty}^{\infty} s \sin s \cos s ds$

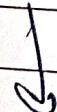
$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\omega} f(s) ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{is\omega} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a [\cos(s\omega) + i \sin(s\omega)] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a 2 \cos(s\omega) ds = 1$$

$$f(s) = \frac{2}{\sqrt{2\pi}} \left[\frac{\sin s}{s} \right]_0^a = \frac{2}{\sqrt{2\pi}} \frac{\sin a}{s}$$



Fourier Transform

$$0 > |k| \quad F(s) = \frac{1}{2} + \int_{-\infty}^{\infty} e^{-ks} f(s) ds - \text{reduced}$$

$$0 < |k| \quad 0 \sqrt{2\pi} \quad -\infty$$

$$0 > |k| \quad R = 1/2 = \infty$$

$$\text{Ansatz: } \frac{1}{2} + \int_{-\infty}^{\infty} e^{-is\alpha} \frac{2}{2} \left(\sin \frac{s\alpha}{2} \right) ds$$

$$= \frac{1}{2 \cdot \pi \cdot i} \int_{-\infty}^{\infty} e^{-is\alpha} \left(\sin \frac{s\alpha}{2} \right) s^{-1} ds = \frac{1}{2 \cdot \pi \cdot i}$$

$$k \cdot \pi / 2 = 0 \quad \Rightarrow \quad 1 =$$

$$\int_0^\infty \frac{ds}{s^{1/2}}$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{s\alpha}{2}}{s} ds$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\sin \frac{s\alpha}{2}}{s} ds$$

$$1 = \frac{1}{\pi} \int_0^\infty \frac{\sin \frac{s\alpha}{2}}{s} ds$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\sin \frac{s\alpha}{2}}{s} ds$$

$$\frac{1}{2} \cdot \frac{\pi}{\sqrt{2}} = \int_0^\infty \frac{\sin \frac{s\alpha}{2}}{s} ds = \frac{1}{2} \cdot \frac{\pi}{\sqrt{2}}$$

→ problem solved

Inverse Fourier Transform.

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b)
$$f(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isn f(s)} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isn(2\pi)} ds = \frac{1}{\sqrt{2\pi}} \left[\sin s \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cos s + i \sin s \right]_0^\infty \sin s$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cos s \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos s ds$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} & |u| \leq a \\ 0 & |u| > a \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a \cos s ds$$

above is 0 because $\sin 0 = 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a 1 ds = \frac{2a}{\sqrt{2\pi}}$$

Fourier Integral

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x) dt dx$$

Cosine integral or even.

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda t \cos \lambda x dt dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t dt dx$$

$$\Rightarrow \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t dt dx$$

Sine integral or odd,

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t dt dx$$

Find the Fourier integral.

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{else } |x| > 1 \end{cases} \quad -1 < x < 1$$

∞ even.

$$F(n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos \lambda x) f(\sin \lambda t) \cos \lambda t dt$$

$$(20) \int_{-\pi}^{\pi} = \int_{-\pi}^{\pi}$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty \cos \lambda t dt dt$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[\frac{\sin \lambda t}{\lambda} \right]_0^\infty$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \sin \lambda x$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \sin \lambda x$$

$$= \frac{1}{2} \int_0^\infty (\cos 2\lambda x + \sin 2\lambda x)$$

$$2 \int_0^\infty \cos \lambda x \sin \lambda x = \frac{\pi}{2} F(n)$$

$$\int_0^\infty \sin(2\lambda x) dx = -\frac{1}{2} \int_0^\infty \cos(2\lambda x) dx$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

$$F(n) = \int_{-\pi}^{\pi} f(x) e^{-inx}$$

$$= \int_{-1}^1 e^{-inx} dx$$

$$= \frac{1}{in} [e^{-inx}]$$

$$= \frac{1}{in} [e^{-inx} - e^{inx}]$$

$$= \frac{1}{in} [2 \sin nx]$$

$$= \frac{1}{in} [2 \sin n(-1) - 2 \sin n(1)]$$

$$= \frac{1}{in} [-2 \sin n - 2 \sin n]$$

Z - Transform.

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$\mathcal{Z}[f(k)] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} f(k) z^{-k}$. (6.1)

Change of Scale.

~~With $\mathcal{Z}[a^k f(k)] = F(az)$~~ (6.2)

$$\mathcal{Z}[a^{-k} f(k)] = F(az)$$

Shifting prop

$$\mathcal{Z}[f(k+n)] = z^n F(z)$$

$-z \frac{d}{dz} \mathcal{Z}[k^n f(k)] = z^n F(z)$.

$\mathcal{Z}[k^n f(k)] = \left(-z \frac{d}{dz} \right)^n F(z)$.

$\mathcal{Z}[f(k)] = - \int_z^\infty q^{-1} F(q) dq$.

I.V.T.

$$F(z) = \mathcal{Z}[f(k)]$$

then $f(0) = \lim_{z \rightarrow \infty} F(z)$

$$f(1) = \lim_{z \rightarrow \infty} z [F(z) - f(0)]$$

$$f(2) = \lim_{z \rightarrow \infty} z^2 [F(z) - f(0) - f(1)z]$$

Imp. Series (P.T. + T.M.T. & S.P.T.)

$$1. \frac{a + ar + ar^2 + \dots}{(1-r)} = \frac{a}{1-r} \quad |r| < 1$$

$$2. \frac{1 - r + r^2 - r^3 + \dots}{(1+r)} = \frac{(1+r)^{-1}}{1-r} \quad |r| < 1$$

$$3. \frac{1 + r + r^2 + r^3 + \dots}{(1-r)} = \frac{(1-r)^{-1}}{1+r} \quad |r| < 1$$

$$4. \frac{1 + nr + n(n-1)r^2 + \dots}{2! \cdot (n-1)!} = \frac{(1+r)^n}{(1-r)^{n+1}}$$

$$5. \frac{1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots}{(1-r)^{n+1}} = e^{\frac{r}{1-r}}$$

$$6. \cancel{\log} = -r - \frac{r^2}{2} - \frac{r^3}{3} - \dots = \log(1-r)$$

$$7. \frac{1}{2} + \frac{r}{3} + \frac{r^2}{4} + \dots = \frac{1}{2} \log(1+r)$$

$$8. \frac{1}{2} + \frac{r}{3} + \frac{r^2}{4} + \dots = (1-r)^{-\frac{1}{2}}$$

Std Form.

$$z(1) = \bar{z} \quad z[\cos k\theta] = \bar{z}^2 - \bar{z} \cos \theta$$

$$z[\sin k\theta] = \frac{1}{2i} [z - \bar{z}] = \frac{\bar{z} \sin \theta}{\bar{z}^2 - 2\bar{z} \cos \theta + 1}$$

$$z(a^n) = \frac{1}{2i} [(1 - \frac{1}{a^n})z - \bar{z}]$$

$$z(a^n) = \frac{1}{2i} \frac{z - \bar{z}}{a^n}$$

$$Z[g(k) * f(k)] = Z[g(k)] * Z[f(k)].$$

$$Z^{-1}[F(z)G(z)] = \sum_{k=0}^{\infty} f(k) * g(k).$$

$$\sum_{k=0}^{\infty} f(k+1) = \sum_{m=0}^{\infty} f(m)g(k-m).$$

$$\sum_{k=0}^{\infty} f(k+1) = \sum_{m=0}^{\infty} f(m)g(k-m).$$

$$(k+1) = \frac{1}{(z-1)(z-3)} \left[(z-1)z + 3z \right] (k+1) = k + 1$$

$$f(k+1) = z^{-1} \left[-z + \frac{g(k)}{(z-1)} \right] = z^{-1} \left[\frac{g}{z-3} \right]$$

$$(k+1) = i - \sum_{k=0}^{\infty} g(k) = 3^k$$

$$(k+1) = \sum_{m=0}^{k-1} -f(m) + g(k-m)$$

$$(k+1) = \sum_{m=0}^{k-1} -f(m) + 3^{k-m} = 1 - \sum_{m=0}^{k-1} 3^{k-m}$$

$$= 1 - 3^k \sum_{m=0}^{k-1} 3^{-m}$$

$$0.1000000000000000 = 1 - 3^k \sum_{m=0}^{k-1} 3^{-m}$$

$$0.1000000000000000 = 3^k \left[\frac{(1/3)^{k+1} - 1}{1 - 1/3} \right] \rightarrow \text{Sum of } n \text{ terms in G.P.}$$

$$= 3^k \frac{[(1/3)^{k+1} - 1]}{-2/3} = \frac{1}{2} [3^{k+1} - 1]$$

Vector and Algebra

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$$[a b c] = a \cdot (b \times c) \quad (\text{Volume}) \quad (1)$$

Volume of parallelepiped = $[a b c]$

Volume of tetrahedron = $\frac{1}{6} [a b c]$.

$$[a b c] = - [b a c].$$

$$\delta \cdot \vec{D} = \vec{n} \cdot \vec{D}$$

Vector product of 3 vectors.

$$(a \times b) \times (c \times d)$$

$$= [a b d] c + [a b c] d - [b c d] a$$

$$= [a b (d+c)] a - [a b c] d.$$

Vector triple product = $a \cdot (b \times c)$

$$(a \times b) \times c = (c \cdot a) b - (c \cdot b) a$$

Scalar product of four vectors.

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}$$

fundamental = $0 = 4 \times 0$

fundamental value

$$\frac{ab}{pb} \vec{a} + \frac{cb}{pb} \vec{a} + \frac{ab}{nb} \vec{b} = \vec{ab}$$

Vector Differentiation

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i) Gradient ($d \times 1$) . $\phi = [\partial \phi / \partial x \quad \partial \phi / \partial y \quad \partial \phi / \partial z]^T$

$$\text{grad } \phi = \nabla \phi = \left[\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right]$$

To find it is perpendicular to surface

ii) directional derivative

$$[\text{Dir. d.}] = [\partial \phi / \partial \vec{a}]$$

$$= \nabla \phi \cdot \vec{a} = \nabla \phi \cdot \hat{a}$$

value is 1st column value

iii) Divergence $(d \times 3) \times (3 \times 1)$

$$[\text{div. f}] = \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} =$$

$$[\partial f_i / \partial x \quad \partial f_i / \partial y \quad \partial f_i / \partial z]^T$$

$\nabla \cdot F = 0$ \rightarrow Solenoidal field

iv) (curl), $\vec{f} (m.s) = m \times (d \times 1)$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{and } \vec{f} \cdot \vec{D} = (F_1 \times \vec{D}_1) \vec{D}_3 = (d \times 1) \vec{D}_3$$

$$\nabla \times \vec{f} = 0 \rightarrow \text{irrotational}$$

v) scalar potential

$$\int \partial \phi = \int \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

Rem $\nabla \frac{1}{\phi} = \left(-\frac{1}{\phi^2} \right) \frac{\vec{\sigma}}{\phi}$

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Imp Lemma 10. $\nabla \cdot (\phi \vec{F}) = \phi \nabla \cdot \vec{F} + \vec{F} \cdot (\nabla \phi)$

$$\nabla \cdot (\phi \vec{F}) = \phi \nabla \cdot \vec{F} + \vec{F} \cdot (\nabla \phi).$$

Proof: $\nabla \cdot (\phi \vec{F}) = \phi \nabla \cdot \vec{F} + \vec{F} \cdot (\nabla \phi)$

- smooth ab

$$\begin{aligned} \nabla \cdot (\phi \vec{F}) &= \nabla \cdot (\phi \vec{F}) + \vec{F} \cdot (\nabla \phi) \\ &\stackrel{(\text{smooth ab})}{=} \phi \nabla \cdot \vec{F} + \vec{F} \cdot (\nabla \phi) \end{aligned}$$

- smooth ab

$$\vec{F} \cdot (\nabla \phi) = \vec{F} \cdot (\nabla \phi)$$

- smooth ab

$$\phi \nabla \cdot \vec{F} + \vec{F} \cdot (\nabla \phi) = \phi \nabla \cdot \vec{F} + \vec{F} \cdot \nabla \phi$$

Vector Integration.

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Line Integral is obtained as

$$(\nabla \times \vec{F}) \cdot d\vec{r} = (\vec{A} \cdot \vec{F}) - (\vec{B} \cdot \vec{F})$$

$$\int d\phi = \int F \cdot dr. \quad \phi \rightarrow \text{scalar potential}$$

Green's theorem.

$$\int p \, dx + q \, dy = \iint \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \, dx \, dy$$

Stoke's theorem.

$$\oint F \cdot dr = \iint \hat{n} \cdot (\nabla \times F) \, ds$$

Gauss - div theorem

$$\iint_S \hat{n} \cdot \vec{F} \, ds = \iiint \nabla \cdot \vec{F} \, dv.$$