# Discrete

## Unit 4 Combination / Counting

→ **The Pigeonhole Principle:**

- **Theorem 1**: If small K is positive integer K+1 or more objects are placed into K boxes then there is atleast one box containing two or more of the object.

**Proof**: We prove the pigeonhole principle using a prove by contraposition. Suppose ~~of~~ that none of the K boxes contain more than one object then the total no of object would atmost K. This is a contradiction because there are atleast K+1 object.

The pigeonhole principle also called Dirichlet drawer principle.

A function F(~~x~~) from a set ~~which~~ with K+1 or more element to set with K elements is not one to one.

Eg 1: Among any group 367 people, there were atleast two with same birthday because there are only 366 possible birthday.

Sol$^n$: If we apply pigeonhole principle, then we ~~let~~ assume days (366) as a hole and peoples birthday (367) as a ~~object~~ pigeon. So, on atleast one day, two people's birthday possible.

Eg 2: In anygroup, 27 English word there must be atleast two that begin with same letter because their 26 alphabet in english alphabets

Sol$^n$ holes - alphabet
~~object~~ pigeon - English word

Eg 3: How many student must be in a class to ~~guarantee~~ gurantee that atleast 2 students ~~of~~ to score same marks if the exam is of 0-100 marks

Sol$^n$ There are 101 students, the pigeonhole principle show that among any 102 students there must be atleast 2 students with the same score.

$Kn+1 = ?$
$K+1 = 101+1 = 102$

$(1 \times 101) + 1$

Eg 4 Show that for any integers n there is a multiple of n that has only zero's or 0's and 1's in the decimal Expansion.

Sol$^n$

Let n be a positive integer, consider the n+1 integer 1, 11, 111, .., 11....1 (where last integer in this list is the integer with n+1 ~~ones~~ 1s in its decimal Expansion.

Note that there are n possible remainders when an integer is divided by n because there are n+1 integer in the list. So, Using the pigeonhole principle, there must of 2 with same remainder when divided by n. The larger of these integer less than smaller one is the multiple of n which has a decimal Expansion consisting enterely of 0s & 1s.

Unit 6

## → Advanced counting techniques

• Application of Recursion

Suppose no. of bacteria in a ~~colony~~ colony double the every hr if colony begins with 5 bacteria then how many bacteria in n hour.

Let $a_n$ be the no. of bacteria at the end of n hours because the no. of bacteria double every hour, the relationship is $a_n = 2a_{n-1}$, relationship is holds for positive integer n. This recurrence relation ~~ship~~ together with ~~positive~~ intial condition $a_0 = 5$, uniquely determine $a_n$ for all non-negative integer n.

## → Moduling with Recurrence relation

• Rabbits & fibonacci number

Consider the problem, ~~post by~~ which was originally post by Leonardo Pisano also know as fibonacci.

Problem A young pair of Rabbit (both male & female) is placed on an island. A pair

Rabbit does not breed until they are 2 month old. After they are two month old each pair of rabbit produce another pair each month.

Sol^n Denote by $f_n$ the no. of pair of rabbit after n month. we will show that $f_n$, ~~where~~ n = 1, ... and so on are the term of the fibonacci sequence. The Rabbit population can be modelled using a recurrence relation. At the end of the first month, the no. of pair of rabbit on the island is $f_1 = 1$ because this pair does not breed during the second month. $f_2 = 1$ also. To find the pair after n month, ~~and~~ add the no. of pair previous month $f_{n-1}$ and the no. of new born pair is $f_{n-2}$ because each new born pair comes from a pair atleast two month old. Consequently the sequence $\{f_n\}$ satisfy the recurrence relation $\boxed{f_n = f_{n-1} + f_{n-2}}$ for $n \geq 3$, together with

intial condition $f_1 = 1$ & $f_2 = 1$.

Because that Recurrence relation & intial condition uniquely determine the sequence. the no of pair of rabbits on the island after n month is given of nth Fibonacci number

Q ② The Tower of Hanoi puzzle :-

Popular puzzle of late 19 century invented by french mathematician Edouard Lucas called the Tower of Hanoi. Consist of three pegs mounted on a base together with disc of ~~diff~~ different sizes. Initially these disc are placed on the first pegs in order of size

with larger on the bottom.
The rules of puzzle allowed disk to be move one at a time from one peg to another as long as a disk is never placed on a top of a smaller disk.
The goal of the puzzle is to have all the disk on the second peg in order of size with largest on the bottom. Let $H_n$ denote the no. of moves needed to solve the tower of hanoi puzzle with disc. Set up a recurrence relation for the $\{H_n\}$ → sequence of $H_n$.

Sol^n

Begin with n disc on peg 1, we can transfer the top (n-1) disk by following the rules of puzzle to pack 3 using $H_{n-1}$ moves.
We keep the largest disk fixed during these moves. Then we use 1 move to transfer the largest disc to second peg, finally we transfer the (n-1) disc on peg 3 to peg 2 using $H_{n-1}$ moves placing them on disc fixed at peg 2.
∴ The recurrence relation for tower of Hanoi is :- $\boxed{H_n = 2H_{n-1} + 1}$ initially, $H_1 = 1$

We can use the iterative approach to solve the recurrence relation:

$\boxed{H_n = 2^n - 1}$

Assign Q 531 pg → Ex: 4, 5, 3
532

→ Linear homogenous Recurrence Relation of degree 2: Let $c_1$ & $c_2$ be real no. Suppose that $r^2 - c_1 r - c_2 = 0$, has 2 distinct root $r_1$ & $r_2$. Then the sequence $\{a_n\}$ is an solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. If & only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2 \ldots$ where $\alpha_1$ & $\alpha_2$ are constants.

Q What is the soln of recurrence relation with $a_0 = 2$ and $a_1 = 7$

$a_n = a_{n-1} + 2a_{n-2}$

$a_0 = a_1 + a_2$

$a_0 = 2, \; a_1 = 7$

$a_2 = \cancel{a_0 + 2a_1 = 2 + 2(7) = 16}$ $\quad a_2 = a_1 + 2a_0$

$\cancel{= 7 + 9 = 9}$ $\quad = 7 + 2(2)$

$\quad = 11$

$\cancel{a_3 = a_2 + 2a_1}$ $\quad a_3 = a_2 + 2a_1$

$= 11 + 2(7) = 25$

$a_4 = a_3 + 2a_2$

$= 25$

Theorem 1 can be used the solve the problem the characteristic $Eq^n$ $r^2 - r - 2 = 0$. its root are $r = 2$, $r = -1$. Hence the Sequence $a_n$ is the solution to the recurrence relation if & only if

$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 \times 2 + \alpha_2(-1)$

$\alpha_1 + \alpha_2 = 2$

$2\alpha_1 - \alpha_2 = 7$

$3\alpha_1 = 9$

$\boxed{\alpha_1 = 3}$ & $\boxed{\alpha_2 = -1}$

$\boxed{a_n = 3 \times 2^n - (-1)^n}$

Assign

Q Find the Explicit formula for fibonacci no.

Theorem 2 Let $c_1$ & $c_2$ be real no's with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root $r_0$. A sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if & only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2 \ldots$, where $\alpha_1$ and $\alpha_2$ are constant.

Q What is the solution of recurrence relation. if initial condition $a_0 = 1$ and $a_1 = 6$. ?

$a_n = 6a_{n-1} - 9a_{n-2}$

$r^2 - c_1 r - c_2 = 0$

$r^2 - 6r + 9 = 0$

$r^2 - (3+3)r + 9 = 0$

$r^2 - 3r - 3r + 9 = 0$

$r(r-3) - 3(r-3) = 0$

$(r-3)(r-3) = 0$

$\boxed{r = 3, 3}$

$$\begin{array}{c|c} 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

hence the $sol^n$ of the ~~real~~ recurrence relation

$a_n = \alpha_1 3^n + \alpha_2 \times n \times 3^n$

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \times 3 + \alpha_2 \times 3$

$\alpha_1 = 1, \alpha_2 = 1$

$\boxed{a_n = 3^n + n.3^n}$

Assign (545 Pg)

Q) find $sol^n$ to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial $a_0 = 2, a_1 = 5, a_2 = 15$.

Q) On whatapp (from theorem next to · Generalized)

→ Linear Non-homogeneous Recurrence Relations with constant coefficients.

The recurrence relation with constant coefficient,

$\boxed{a_n = 3a_{n-1} + 2n}$

is an Example. That is recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$

where $c_1$ & $c_2$ ... $c_k$ are real no. & $F(n)$ is a function not identically zero depending only on $n$. The recurrence relation. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ is called associated homogenous recurrence relation

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