(1) OM ASHISH MISHRA 16BCE0789

U+V & U∩W (7/9/18) (C2)

$u_1 = (1, 3, -2, 2, 3)$ | $w_1 = (1, 3, 0, 2, 1)$

$u_2 = (1, 4, -3, 4, 2)$ | $v_2 = (1, 5, -6, 6, 3)$

$u_3 = (2, 3, -1, -2, 10)$ | $v_3 = (2, 5, 3, 2, 1)$

(2) Find the rank & nullity of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

Ans (1)

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 5 \\ -2 & -3 & -1 & 0 & -6 & 3 \\ 2 & 4 & -2 & 2 & 6 & 2 \\ 3 & 2 & 10 & 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 & 0 & 5 \\ 2 & 4 & -2 & 2 & 6 & 2 \\ 0 & -1 & 4 & -2 & 0 & -5 \end{bmatrix} \begin{matrix} \\ (R_2 - 3R_1) \\ (R_3 + R_4) \\ \\ (R_5 - 3R_1) \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 2 & -4 & 0 & 4 & -2 \\ 0 & -1 & 4 & -2 & 0 & -5 \end{bmatrix} \begin{matrix} \\ \\ (R_3 - R_2) \\ (R_4 - 2R_1) \\ \\ \end{matrix}$$

$(R_1 + R_5)$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 2 & -4 & 0 & 4 & -2 \\ 0 & -1 & 4 & -2 & 0 & -5 \end{bmatrix} \begin{matrix} \\ (R_4 - 2R_2) \\ \\ \\ (2R_5 + R_4) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 6 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 0 & 4 & -4 & 4 & -12 \\ 0 & -1 & 4 & -2 & 0 & -5 \end{bmatrix}$$

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A

$$\begin{bmatrix} 1 & 0 & 6 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 0 & 4 & -4 & 4 & -12 \\ 0 & -1 & 4 & -2 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 0 & 1 & -1 & 1 & -3 \\ 0 & 1 & -4 & -2 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ \\ (R_4/4) \\ (R_5/(-1)) \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 6 \\ 0 & 0 & 1 & -1 & 1 & -3 \\ 0 & 1 & -4 & -2 & 0 & 5 \end{bmatrix} (R_1 + R_4) = \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -3 \\ 0 & 1 & -4 & -2 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ (R_3 + 2R_4) \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -3 \\ 0 & 1 & 0 & -2 & 0 & 5 \end{bmatrix} \begin{matrix} (R_1 - 7R_3) \\ \\ \\ (R_4 - R_3) \\ (R_5 - (-4)R_3) \end{matrix}$$

∴ we get 4 pivot
4 leading 1's.

$\boxed{\therefore \dim(A) = (U+V) = 4}$: $\{u_1, u_2, v_3, v_2\}$

<u>N(A)</u> :- $x_1 = 0$
$x_3 = 0$
$-x_4 + x_5 - 3x_6 = 0$
$x_2 - 2x_4 + 5x_6 = 0$

Let $x_3 = s$
$x_5 = t$

$\therefore x_5 = x_4 + 3x_6$
$x_2 = x_4 + 5x_6$

$$\therefore \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ \end{bmatrix}} s + \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \\ \end{bmatrix}} t$$

∴ Since x_5, x_2 can be represented in the form of other variables. Therefore, the $\boxed{V \wedge w = (2)}$

Ans 2) $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & 2 \end{bmatrix} \begin{matrix} \\ R_1 - R_2 \\ 2R_1 - R_3 \end{matrix}$

$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 3 & -3 & 2 \end{bmatrix} (3R_3 - 3R_2) = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 2 \end{bmatrix} R_2/(-4)$

$= \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 2 \end{bmatrix} (R_1 - R_3) = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 \end{bmatrix} (R_3 - 2R_2)$

$= \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 \end{bmatrix} (R_1 + R_3) = \begin{bmatrix} 3 & 9 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 \end{bmatrix} (3R_1 + R_3)$

$= \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{matrix} (R_1/3) \\ \\ (R_3/3) \end{matrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} R_3/(-1)$

$\boxed{\therefore \dim R(A) = 3 = \text{rank } A.}$

~~$\therefore$ The dim R(A) =~~

N(A)

$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 + 3x_2 = 0$

$x_4 = 0$

$x_2 - x_3 = 0$

Let $x_2 = s$

$\therefore x_1 = -3s \qquad x_3 = s$

$\therefore$ The nullity :- $s = \langle -3, 1, 1, 0 \rangle$

$\therefore$ $\boxed{\text{The Nullity is :- } 1.}$

Uniqueness

(i) For each $b \in R^m$, $Ax = b$ has atmost one solution x in R^n

(ii) The column vectors of A are linearly independent

(iii) $\dim C(A) = \text{rank } A = n$ (hence, $n \leq m$)

(iv) $R(A) = R^n$

(v) $N(A) = \{0\}$

(vi) A has a left inverse [i.e, C such that $CA = I_n$]

(i) → (ii)

if column vectors of A are linearly independent then $Ax = 0$ $\therefore x$ will be having trivial solution ~~They~~ Therefore it can clearly have only one solution

(ii) → (iii) Since all the column vectors are linearly independent then it will be ~~basis~~ basis only if $C(A)$ or $\dim C(A) = n \leq m$.

(iii) $\Leftrightarrow$ (iv) Since $\dim C(A) = \text{rank } A = n$

$\therefore$ this is only possible if $R(A) = R^n$.

(iv) $\Leftrightarrow$ (v) $\dim R(A) + \dim N(A) = n$

& $\dim R(A) = n$ $\therefore \dim N(A) = 0$.

(ii) $\Leftrightarrow$ (iv) Columns of A are linearly independent so that rank $A = n$. Now if we extend. n to m rows for basis of R^m by adding $m-n$ additional independent vectors to them.

$\therefore$ $m \times m$ matrix B is formed. Then the matrix B has rank m, and hence it is invertible.

$\therefore$ Let C be $n \times m$ matrix obtained by B^{-1} by throwing away last $m-n$ rows. Since the first n columns of S constitute. The matrix A we have $CA = I_n$.

$$I_m = B^{-1}B = \begin{bmatrix} C_{n\times m} \\ \hline * * * * \end{bmatrix} \begin{bmatrix} A_{m\times n} & \vdots & * \\ & & * \end{bmatrix} = \begin{bmatrix} I_n & \vdots & 0 \\ \hline 0 & \vdots & I \end{bmatrix}$$

(vi) $\Rightarrow$ (i) :- Let C be a left inverse of A. If $Ax = b$ has no solution, then we are done. Suppose that $Ax = b$ has 2 solutions, say x_1 & x_2. Then,

$$x_1 = C[Ax_1] = Cb = C[Ax_2] = x_2$$

Hence, there can only be one solution (at most).

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Invertibility

(i) For each $b \in R^m$, $Ax = b$ has atleast one solution x in R^n.

(ii) The column vectors of A span R^m, i.e, $C(A) = R^m$.

(iii) rank $A = m$ (hence $m \le n$).

(iv) A has a right inverse [i.e, B such that $AB = I_m$].

Proof

(i) $\leftrightarrow$ (ii) $\overset{\text{we know}}{C(A)} \subseteq R^m$. For any $b \in R^m$ there is a ~~sol~~ solution $x \in R^n$ of $Ax = b$ if and only if b is a linear combination vectors of A. $b \in C(A)$. $\therefore R^m = C(A)$.

(ii) $\to$ (iii) $C(A) = R^m$ iff $\dim C(A) = m \le n$.
But $\dim C(A) = \text{rank } A = \dim R(A) \le \min(m, n)$

(i) $\to$ (iv) Let $e_1, e_2 \cdots e_m$ be the standard basis for R^m. Then for each e_i one can find an $x_i \in R^n$ such that $Ax_i = e_i$. $\therefore$ by hypothesis one solution.
If B is $n \times m$ matrix whose column are then x_i's
$B = [x_1, x_2 \ldots x_m]$ Then by matrix multiplication
$AB = A(x_1, x_2 \ldots x_m] = [e_1, e_2 \ldots e_m] = I_m$

(iv) $\to$ (i) : If B is a right inverse of A, then for ~~only~~ any $b \in \mathbb{R}^m$, $x = Bb$ is a solution of $Ax = b$.

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(1) Verify that T is a linear transformation

$T(x,y) = (x+1, 2y, x+y)$

(2) Show that T is invertible & find a formula for T^{-1}

$T(x,y,z) = (3x, x-y, 2x+y+z)$

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(1) In order to prove the following is linear transformation

The condition to be satisfied is:-

$T(x+ky) = T(x) + k\,T(y)$

Let $x = x_1 + k y_1$ $\qquad y = x_2 + k$

Formula

(i) $T(x, y, z) = (3x, x-y, 2x+y+z)$

Let $T^{-1}(r, s, t) = T^{-1}(3x, x-y, 2x+y+z)$

$\therefore r = 3x \Rightarrow x = \frac{r}{3}$

$x - y = s \Rightarrow y = x - s = \frac{r}{3} - s$

$2x + y + z = t \Rightarrow z = t - \frac{2r}{3} - \frac{r}{3} + s$

$z = t - r + s$

$\therefore T^{-1}(r, s, t) = \left(\frac{r}{3}, \frac{r}{3} - s, t - r + s\right)$

$\therefore T^{-1}(x, y, z) = \left(\frac{x}{3}, \frac{x}{3} - y, z - x + y\right)$

(Ans)

Showing that it is invertible

In order to form T is invertible

$T: V \to W \quad \therefore T(v_i) = w_i$

Let $T(u) = x \quad T(v) = y$

$\therefore u = T^{-1}(x) \quad v = T^{-1}(y)$

Addition

~~u + v =~~

$$T(u+v) = T(u) + T(v)$$

$$= T(T^{-1}(x)) + T(T^{-1}(y))$$

$$= (TT^{-1})x + TT^{-1}(y)$$

$$= Id.x + Id.y$$

$$= x + y \rightarrow (A)$$

Scalar Multiplication

$$T(ku) = T(k\,T^{-1}(x)) = k\,T(T^{-1}(x))$$

$$= (TT^{-1})x = k\,Id\,x$$

$$= kx \rightarrow (B)$$

$\therefore$ From (A) & (B) it is proved that T is invertible and the inverse matrix is linear transformation also.

(T) <u>Addition</u>

$T(x,y) = (x+1, 2y, x+y)$

$T(u_1, v_1) + T(u_2, v_2)$

$= (u_1 + u_2 + 2, \cancel{2(}\, 2u_1 + 2v_2,$

$u_1 + u_2 + v_1 + v_2)$

$T(u_1 + u_2, v_1 + v_2) = (\cancel{(u_1 + u_2) + 2}\ (u_1+1) + (u_2+1),$

$2(u_1 + v_2) + (u_1 + u_2) + (v_1 + v_2))$

$\therefore T(u_1 + u_2, v_1 + v_2) = T(u_1, v_1) + T(u_2 + v_2)$

<u>Scalar multiplication</u>

$\cancel{T(ku_1, v_1)}\ T(ku, kv)$

$= (ku + k, 2kv, ku + kv)$

$= k(u+1, 2v, u+v)$

$= kT(u,v)$

$\therefore T(ku, kv) = kT(u,v)$

(Proved) Linear Transformation.

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OM ASHISH MISHRA

16BCE0789

20/9/18

(1) $T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$

Let $\alpha = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$

$\beta = \{(1,3),\ (2,5)\}$ (i) find associated matrix $[T]_\alpha^\beta$

(ii) verify $[T]_\alpha^\beta \cdot [V]_\alpha = [T(V)]_\beta$

(2) for the vector spaces T, of R & R^2 $\alpha = (1, x)$

for $P_1(R)$ & $\beta = \{e_1, e_2\}$ for R^2.

T is mapping from $P_1(R) \to R^2$ is a linear transformation defined by $T(a+bx) = (a, a+b)$ (i) show that T^{-1} is invertible, (ii) Find $[T]_\beta^\alpha$ & $[T^{-1}]_\alpha^\beta$

(3) Let $T: P_2(R) \to P_2(R)$ be the linear transformation defined by $T(f) = (3+x)f' + 2f$ and let $S: P_2(R) \to R^3$ defined by $S(a+bx+cx^2) = (a-b, a+b, c)$ for the basis $\alpha = \{1, x, x^2\}$ for $P_2(R)$ and standard basis $\beta = \{e_1, e_2, e_3\}$ for R^3. Compute $[S]_\alpha^\beta$, $[T]_\alpha$, $[S \circ T]_\alpha^\beta$

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Ans (1) :-

(i)

~~T(e_1) =~~

$T(e_1) = T(1,1,1) = (3(1) + 2(1) - 4(1)), (1) - 8(1) + 3(1))$

$= (1, -1)$

$T(e_2) = T(1,1,0) = (5, -4)$

$T(e_3) = T(1,0,0) = (3, 1)$

$\therefore T(e_1) = e_1 - e_2$ ~~$T(e_3) = 3e_1 + e_2$~~

$T(e_2) = 5e_1 - 4e_2$

~~$\therefore [T]_\alpha^\beta =$~~

$\therefore [T]_\alpha = \begin{bmatrix} 1 & -1 \\ 5 & -4 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 3 \\ -1 & -4 & 1 \end{bmatrix}$ ~~(Ans)~~

~~$T(e_1) = a_1 w_1 + a_2 w_2$~~ ~~$[T]_\beta = \begin{bmatrix} -1 & -4 & 1 \\ 1 & 5 & 3 \end{bmatrix}$~~

$T(1, -1) = a_1 w_1 + a_2 w_2 = a_1(1,3) + a_2(2,5)$

$= (a_1 + 2a_2, 3a_1 + 5a_2)$

$\left.\begin{aligned} a_1 + 2a_2 &= 1 \\ 3a_1 + 5a_2 &= -1 \end{aligned}\right\}$ $\begin{aligned} a_1 &= -7 \\ a_2 &= 4 \end{aligned}$

$$(5,-4) = b_1 w_1 + b_2 w_2$$
$$= (b_1, 3b_1) + (2b_2 + 5b_2)$$
$$= (b_1 + 2b_2, 3b_1 + 5b_2)$$

$$\therefore \left.\begin{array}{l} b_1 + 2b_2 = 5 \\ 3b_1 + 5b_2 = -4 \end{array}\right\} \quad \begin{array}{l} b_1 = -33 \\ b_2 = 19 \end{array}$$

$$(3,1) = c_1 w_1 + c_2 w_2 = c_1(1,3) + c_2(2,5)$$
$$= (c_1, 3c_1) + (2c_2, 5c_2)$$
$$= (c_1 + 2c_2, 3c_1 + 5c_2)$$

$$\therefore \left.\begin{array}{l} c_1 + 2c_2 = 3 \\ 3c_1 + 5c_2 = 1 \end{array}\right\} \quad \begin{array}{l} c_1 = -13 \\ c_2 = 8 \end{array}$$

$$\therefore T(v_1) = -7w_1 + 4w_2$$
$$T(v_2) = -33w_1 + 19w_2$$
$$T(v_3) = -13w_1 + 8w_2$$

$$[T]_\alpha^\beta = \begin{bmatrix} -7 & 4 \\ -33 & 19 \\ -13 & 8 \end{bmatrix}^T = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

(Ans)

(ii) $[T]_\alpha^\beta = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

In order to prove :-

$\cancel{[v]_\alpha} = [T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$

~~$[T]_\alpha$~~ = Since $T: V \to W$ is a linear transformation from an n-dimensional in this case vector space V to a m-dimensional in this case vector space W. For fixed ordered basis $\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$ for V & $\beta = \{(1,3), (2,5)\}$ for W, there corresponds a unique associated $m \times n$ matrix $[T]_\alpha^\beta$ for T such that any vector $x \in V$ the coordinate vector $[T(v)]_\beta$ of $T(v)$ with respect to β is given as a matrix product of the associated matrix $[T]_\alpha^\beta$ for T and the coordinate vector $[v]_\alpha$ i.e;-

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

<u>The Prove</u> :-

$[T(v)]_\beta$ is a bases.

$\therefore$ on spanning :-

$$[T(v)]_\beta = v_1 T(\vec{x}_1) + v_2 T(\vec{x}_2) + \dots\dots v_n T(\vec{x}_n)$$

$$= [T(\vec{x}_1) \; T(\vec{x}_2) \dots T(\vec{x}_n)] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{(matrix Representation)}$$

$= [T]_{\alpha}^{\beta} [V]_{\alpha}$

$[T]_{\alpha}^{\beta}$ = The linear transformation matrix representation from $\alpha \rightarrow \beta$ in column wise representation,

$[V]_{\alpha}$ = The column wise representation of directors which will help in spanning and resulting in basis formation.

Hence proved.

Ans ② :-

(i) $\alpha = \{1, x\}$ $\beta = \{e_1, e_2\}$

$e_1 = (1, 0)$ $e_2 = (0, 1)$

$T(a \cdot 1 + b x) = (a, a+b)$

↳ As it is clear from the function, that the linear transformation is seen in the matrix due to co-efficients.

$\therefore T(1) = e_1 (1, 0) =$ Checking for co-efficient of a.

$= (1, 1) = e_1 + e_2$

$T(x) = e_2 = (0, 1) =$ Checking for co-efficient of $~~b~~$,

$= (0, 1) = e_2$

$\therefore [T]_\alpha^\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ [Column wise representation]

(Ans)

Now, checking for invertibility. In order to prove invertibility, it is enough to prove isomorphism of $[T]_\alpha^\beta$ & $[T^{-1}]_\beta^\alpha$.

$T: V \to W$

Clearly $\dim V = \dim W$. Let $[T]_\alpha^\beta$ & $[T^{-1}]_\beta^\alpha$ are square and of same size.

Thus

$$[T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [T \circ T^{-1}]_\beta = [id]_\beta \to \text{A}$$

Only if $[T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1}$

Suppose this is true:-

$\therefore$ On Multiplying $\left[[T]_\alpha^\beta\right]^{-1}$ on both sides of A

$$\left[[T]_\alpha^\beta\right]^{-1} [T]_\alpha^\beta [T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1} [id]_\beta$$

$$[id]_\beta [T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1} (id)_\beta$$

$$\therefore [T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1}$$

Hence, the contradiction is correct and they are equal. Therefore T is invertible.

Moreover, $\det\left([T]_\alpha^\beta\right) = 1 - 0 = 1$ is not zero, therefore it is not zero. Therefore inverse can be found. (Hence Proved)

(iii) $\left([T]_\alpha^\beta\right)^{-1} = \left([T^{-1}]_\beta^{\alpha}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (Ans)

Ans ③ :-

$\alpha = \{1, x, x^2\}$ $\quad \beta = \{e_1, e_2, e_3\}$

$\alpha = 1 \Rightarrow e_1 = (1, 0, 0)$
$\alpha = x \Rightarrow e_2 = (0, 1, 0)$
$\alpha = x^2 \Rightarrow e_3 = (0, 0, 1)$

$T(f) = (3 + x) f' + 2f$

$\therefore T(1) = (3 + x)(0) + 2(1) = 2 = (2, 0, 0)$

$T(x) = (3 + x)(1) + 2x = 3 + 3x = (3, 3, 0)$

$T(x^2) = (3 + x)(2x) + 2x^2 = 6x + 4x^2 = (0, 6, 4)$

$\therefore [T]_\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 6 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$ (Ans)

$S(a + bx + cx^2) = (a - b, a + b, c)$

Since we are dealing with coefficients of the terms,

$\therefore S(1) = e_1 = (1, 0, 0) = S(a.1 + b.0 + c.0) = S(a)$

= Collecting coefficient of 'a'.

$= (1, 1, 0) = e_1 + e_2$

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$\therefore\ S(x) = e_2 = (0,1,0) = S(a.0 + b.1 + c.0) = S(b)$

$=$ collecting coefficient of 'b'

$= (-1, 1, 0) = -e_1 + e_2$

$\therefore\ S(x^2) = e_3 = (0,0,1) = S(a.0 + b.0 + c.1) = S(c)$

$=$ collecting coefficient of 'c'

$= (0,0,1) = e_3$

$$\therefore\ [S]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(Ans)}$$

$$\therefore\ [S \circ T]_\alpha^\beta = [S]_\alpha^\beta \cdot [T]_\alpha \quad \text{(Prove in question ①)}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -6 \\ 2 & 6 & 6 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{(Ans)}$$

C2

OM ASHISH MISHRA — 21/9/18

16BCE0789

① Find the basis change matrix from α to β where
$\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$ &
$\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$

Ans①:-

$$\alpha = \{\underset{v_1}{(1,1,1)}, \underset{v_2}{(1,1,0)}, \underset{v_3}{(1,0,0)}\}$$

$$\beta = \{\underset{w_1}{(2,0,3)}, \underset{w_2}{(-1,4,1)}, \underset{w_3}{(3,2,5)}\}$$

$$v_1 = a_1 w_1 + a_2 w_2 + a_3 w_3$$

$$(1,1,1) = a_1(2,0,3) + a_2(-1,4,1) + a_3(3,2,5)$$

$$(1,1,1) = (2a_1 - a_2 + 3a_3,\ 4a_2 + 2a_3,\ 3a_1 + a_2 + 5a_3)$$

$$1 = 2a_1 - a_2 + 3a_3$$

$$1 = 4a_2 + 2a_3 \Rightarrow \boxed{\cancel{a_3 = \frac{1-4a_2}{2}}}$$

$$1 = 3a_1 + a_2 + 5a_3$$

$$\boxed{\cancel{2a_1 = \frac{1 + a_2 + 3\left(1 - \frac{4a_2}{2}\right)}{2}}}$$

$$\cancel{\frac{3}{2} + \frac{3a_2}{2} + \frac{9}{2} - \frac{12a_2}{4}} + a_2 + \cancel{\frac{5 - 20a_2}{2}}\ \frac{5 - 20a_2}{2} = 1$$

$\Rightarrow$

$a_1 = -2 \qquad a_2 = -\frac{1}{2} \qquad a_3 = \frac{3}{2}$

$$(1,1,0) = b_1 w_1 + b_2 w_2 + b_3 w_3$$

$$= b_1(2,0,3) + b_2(-1,4,1) + b_3(3,2,5)$$

$$(1,1,0) = (2b_1 - b_2 + 3b_3,\ 4b_2 + 2b_3,\ 3b_1 + b_2 + 5b_3)$$

$$1 = 2b_1 - b_2 + 3b_3$$

$$1 = 4b_2 + 2b_3$$

$$0 = 3b_1 + b_2 + 5b_3$$

$b_1 = \frac{-13}{3} \qquad b_2 = \frac{-7}{6} \qquad b_3 = \frac{17}{6}$

$$(1,0,0) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

$$(1,0,0) = c_1(2,0,3) + c_2(-1,4,1) + c_3(3,2,5)$$

$$(1,0,0) = (2c_1 - c_2 + 3c_3,\ 4c_2 + 2c_3,\ 3c_1 + c_2 + 5c_3)$$

$$1 = 2c_1 - c_2 + 3c_3$$

$$0 = 4c_2 + 2c_3$$

$$0 = 3c_1 + c_2 + 5c_3$$

$$c_1 = -3 \qquad c_2 = -1 \qquad c_3 = 2$$

$$[id]_\alpha^\beta = \begin{bmatrix} -2 & -13/2 & -3 \\ -1/2 & -7/6 & -1 \\ 3/2 & 17/6 & 2 \end{bmatrix}$$

(Ans)

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② verify if $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ & $\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$ are similar

Ans

~~det~~ $A = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$

$\det(A) = -4 - 0 = -4$ $\det(B) = -10 + 6 = -4$

$\therefore \det(A) = \det(B)$

$tr(A) = 4 - 1 = 3$ $tr(B) = 5 - 2 = 3$

$tr(A) = tr(B)$

$p(t) = t^2 - tr(A)\,t + \det(A)$

$= t^2 - 3t - 4 = (t-4)(t+1)$

$\therefore$ Eigen values $=$ A & B $= (4, -1)$. Hence both A & B are diagonalizable. There exist non singular matrices S & P such that $S^{-1}AS = P^{-1}BP^{-1}$ and hence $= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

$PS^{-1}ASP^{-1} = B$ Putting $U = SP^{-1}$ we have

$U^{-1}AU = B$

Since the product of invertible matrices is invertible, the matrix U is invertible.

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③ Show that $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ cannot be similar to a diagonal matrix.

<u>Ans:-</u>

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ Let the diagonal matrix be $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

In order to be similar we need to have a ~~det(A) value~~. determinant value to prove the matrix can be invertible.

$\therefore |A| = (1 - 0) = 1.$

whereas the determinant value of B is not always non-singular or zero $|B| = ab - 0 = ab$.

Thus it is not always possible to find the similarity.

Even if the trace, rank and determinant match,

$Q = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

$QBQ^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -14 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -4 \\ 4 & 14 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$ ∴ Not similar.

④ Let A & B are similar $n \times n$ matrix

Show that

i) $\det A = \det B$

(ii) $tr(A) = tr(B)$

(iii) $rank(A) = rank(B)$

Ans

(i) $\boxed{\det A = \det B}$

Since A & B are similar $\therefore B = P^{-1} A P$

$$\det(B) = \det(P^{-1} A P)$$

$$= \det(P^{-1}) \det(A) \det(P)$$

$$= \det(P)^{-1} \det(A) \det(P)$$

$$\boxed{\det(B) = \det(A)}$$

(ii) ~~tr(A) = Let A = CD B = DC~~

Let C & D be similar matrix

$$A = CD \qquad B = DC$$

C & D are $n \times n$ matrix

$$tr(C) = \sum_{i=j}^{n} x_{ij}$$

$$tr(D) = \sum_{i=j}^{n} y_{ij}$$

$$tr(A) = \sum_{i=j} tr(CD) = \sum_{i=j}^{n} x_{ij} \sum_{i=j}^{n} y_{ij}$$

$$= \sum_{i=j}^{n} y_{ij} \sum_{i=j}^{n} x_{ij} = \sum_{i=j}^{n} (DC) = tr(B)$$

$$\therefore tr(CD) = tr(DC)$$

$\therefore$ Let A & B be 2 similar matrix $\therefore B = PAP^{-1}$

~~$\therefore A = \sum^{n} x_{ij}$ B =~~

$$tr(B) = tr(PAP^{-1})$$

$$tr(B) = tr(P(AP^{-1}))$$

$$tr(B) = tr((AP^{-1})P) \text{ from above}$$

$tr(B) = tr(A(P^{-1}P))$ $[\because P^{-1}P = I_d]$

$\boxed{tr(B) = tr(A)}$

(iii) Rank (A) = rank (B)

Since it is given that both the matrix are similar therefore, the dimension are A other to be same $n \times n$. dim A = dim B.

~~The ba~~

Therefore the base variables are equal also. Thus the number of free variables are equal.

If no of free variables are same.

Nullity (B) = dim $N(B) = K$ = dim $N(A)$ = Nullity(A)

Thus both will have same number of base variables as pivot elements.

$\therefore$ The Rank $(A) = N -$ Nullity $(A) = N -$ Nullity(B)
= Rank(B).

$\therefore$ $\boxed{\text{Rank}(A) = \text{Rank}(B)}$. $\boxed{\text{Hence Proved}}$

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