AOD Digital Assignment-1

Name: Om Ashish Mishra

Registration Number: 16BCE0789

---

1. Diagonalize $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and hence find A^4.

Ans:

A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The Eigen Values are given by:

$|A-\lambda I| = 0$

From the above equation:

λ=0,3,15 are the eigen values.

Eigen vectors are given by:-

At $\lambda = 0$, we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

At $\lambda = 3$, we get $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

At $\lambda = 0$, we get $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Let $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ be the modal matrix.

$B^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ is the inverse of B.

$B^{-1}AB = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

$= D$

To find A^4 :-

- ⇨ $B^{-1}AB = D$
- ⇨ $(B^{-1}AB)^4 = D^4$
- ⇨ $A^4 = B^{-1}D^4B$
- ⇨ $A^4 = \begin{bmatrix} 22536 & -22482 & 11214 \\ -22482 & 22509 & -11268 \\ 11214 & -11268 & 5661 \end{bmatrix}$

The Answers:

$A^4 = \begin{bmatrix} 22536 & -22482 & 11214 \\ -22482 & 22509 & -11268 \\ 11214 & -11268 & 5661 \end{bmatrix}$ (Ans)

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ (Ans)

2. Identify the nature, index and signature of the given quadratic form (Q). Also write the canonical form of the same. $Q=-3x_1^2-3x_2^2-3x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$.

Ans:

The matrix of the quadratic form is:-

$$Q = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

The Eigen Values are given by:

$|Q-\lambda I| = 0$

From the above equation:

λ= -1, -4, -4 are the eigen values.

Eigen vectors are given by:-

At $\lambda = -1$, we get $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

At $\lambda = -4$, we get $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

At $\lambda = -4$, we get $X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$; solved by comparing orthogonal matrix's form with X_2.

Let $M = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ be the modal matrix.

Normalized Model Matrix N $= \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$

$$N^TAN = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = D$$

So the canonical form,

$$Y^TAY = [y_1\ y_2\ y_3] \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = -y_1^2 - 4y_2^2 - 4y_3^2 \text{ (Ans)}$$

Nature: Negative definite

Index: 0

Signature: 0 – 3 = -3

---

3. A periodic function $f(t)$ of period 2 is defined by $f(t) = \begin{cases} 3t, & 0 < t < 1, \\ 3, & 1 < t < 2 \end{cases}$. Determine a Fourier series expansion for the function.

Ans:

Since f(t) ={ $\begin{matrix} 3t, 0 < t < 1 \\ 3,1 < t < 2 \end{matrix}$

According Fourier series :-

f(t) = $\frac{a_0}{2}$ + $\sum_{n=1}^{\infty} a_n \cos\frac{n\Pi t}{l} + \sum_{n=1}^{\infty} \mathrm{b}_n \sin\frac{n\Pi t}{l}$

Then $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)dx = \frac{2}{2}\int_0^2 f(t)dx = \int_1^2 3dx + \int_0^1 3tdx = \frac{9}{2}$

Then $a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\Pi t}{l}\ \ dx = \frac{2}{2}\int_0^2 f(t)\cos n\Pi t\ \ dx =$
$\int_1^2 3\cos n\Pi t\ \ dx + \int_0^1 3t\cos n\Pi t\ \ dx = 3\left[\frac{(-1)^n - 1}{n^2\Pi^2}\right]$

Therefore : a_n ={ $\begin{matrix} \frac{-6}{n2\Pi 2}, n = odd \\ 0, n = even \end{matrix}$

Then $b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\Pi t}{l}\ \ dx = \frac{2}{2}\int_0^2 f(t)\sin n\Pi t\ \ dx =$
$\int_1^2 3\sin n\Pi t\ \ dx + \int_0^1 3t\sin n\Pi t\ \ dx = 3\left[\frac{-3-6(-1)^n}{n\Pi}\right]$

Therefore : b_n ={ $\begin{matrix} \frac{3}{n\Pi}, n = odd \\ \frac{-9}{n\Pi}, n = even \end{matrix}$

Substituting a_0 , a_n and b_n in f(x), we get:

f(x) =
$\frac{9}{2} + \sum_{n=1,3,5..}^{\infty} \frac{-6}{n^2\,\Pi^2}\cos nx + \sum_{n=1,3,5..}^{\infty} \frac{3}{n\Pi}\sin nx + \sum_{n=2,4,6}^{\infty} \frac{-9}{n\Pi}\sin nx$
(Ans)

---

4. Find the Fourier series to represent $|\cos x|$ in the interval $(-\pi, \pi)$.

Ans:

As f(-x)=f(x), |cos x| is an even fuction. Therefore $b_n = 0$.

Since f(x) =|cos x|, let

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi} \int_0^{\pi} |\cos x| \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos x \, dx = \frac{4}{\pi}$

Then $a_n = \frac{2}{\pi} \int_0^{\pi} |\cos x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos nx \, dx +$
$\frac{2}{\pi} \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx = \frac{-4 \cos n(\frac{\pi}{2})}{\pi(n^2-1)}$ $(n \neq 1)$

Then $a_1 = \frac{2}{\pi} \int_0^{\pi/2} \cos^2 x \, dx - \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos^2 x \, dx = 0$

Substituting a_0 and a_n in f(x), we get:

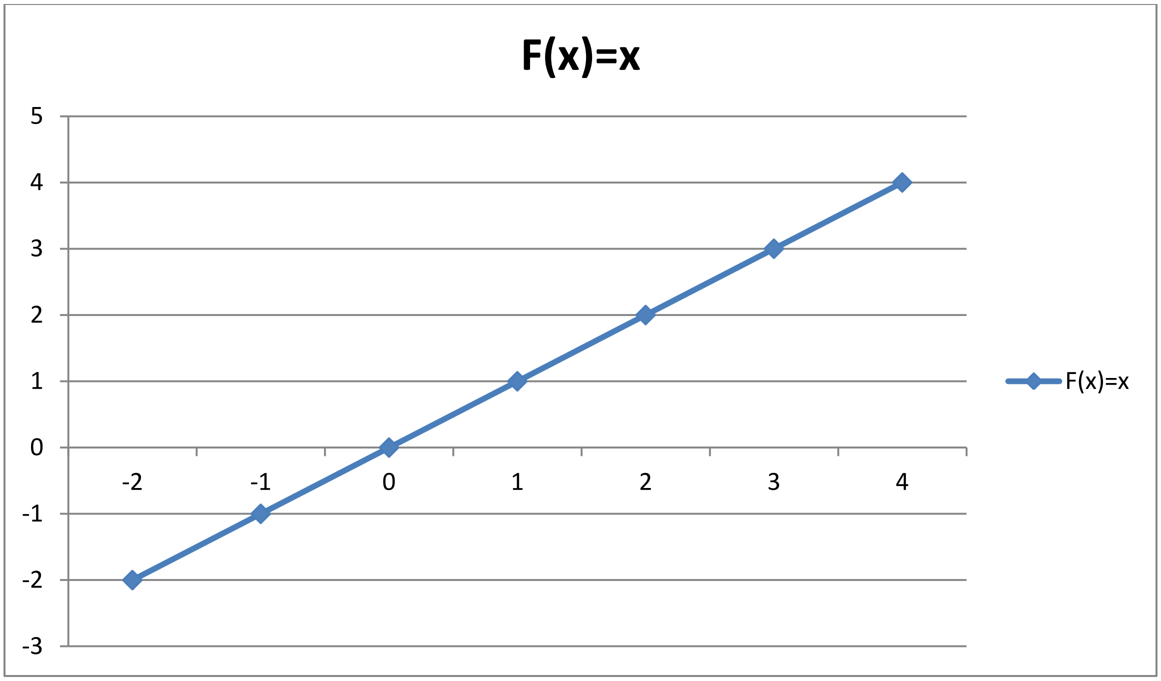
f(x) = $\frac{2}{\pi} + \frac{4}{\pi}\{\frac{1}{3} \cos 2x - \frac{1}{15} \cos 4x + \cdots\}$ (Ans)

---

5. Express $f(x) = x$ as a half range sine series in $0 < x < 2$.

Ans:

The graph f(x)=x is



Since half range sine series is asked in 0<x<2.

Let f(x) = $\sum_{n=1}^{\infty} b_n \sin nx$

Where,

$b_n = \frac{2}{l}\int_{-l}^{l} f(x)\sin\frac{n\Pi x}{2}\ dx = \frac{2}{2}\int_{0}^{2} f(x)\sin\frac{n\Pi x}{2}\ dx = \int_{0}^{2} x\sin\frac{n\Pi x}{2}\ dx = \left[\frac{-4(-1)^n}{n\Pi}\right]$

After substituting b_n in f(x) and hence sine seires for f(x) over the half-range(0,2) is

f(x) = $\frac{4}{\pi}\ (\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + \cdots)$ (Ans)

---

6. Obtain the half range cosine series for $f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l-x), & l/2 \le x \le l \end{cases}$. Hence deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..$

Ans:

Since f(t) ={ $\begin{matrix} kx, 0 \le t \le l/2 \\ k(l-x), \frac{l}{2} \le t \le l \end{matrix}$

According Fourier series for half-range cosine series be

f(t) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\Pi t}{l}$

Then $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^{l/2} kxdx + \frac{2}{l}\int_{l/2}^{l} k(l-x)dx = \frac{kl}{2}$

Then $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\Pi x}{l}\, dx = \frac{2}{l}\int_0^2 f(x)\cos\frac{n\Pi x}{l}\, dx =$
$\frac{2}{l}\int_0^{l/2} kx\cos\frac{n\Pi x}{l}\, dx + \frac{2}{l}\int_{l/2}^{l} k(l-x)\cos\frac{n\Pi x}{l}\, dx = \frac{2kl}{n^2\Pi^2}\left[2\cos\frac{2\Pi x}{l} - (-1)^n - 1\right]$

Substituting a_0 and a_n in f(x), we get:

f(x) = $\frac{kl}{4} - \frac{8kl}{\Pi^2}\left[\frac{1}{4}\cos\frac{2\Pi x}{l} + \frac{1}{36}\cos\frac{6\Pi x}{l} + \frac{1}{100}\cos\frac{10\Pi x}{l} + \cdots \infty\right]$ (Ans)

Putting x=l, we get:-

0 = $\frac{kl}{4} - \frac{8kl}{\Pi^2}\left[\frac{1}{4} + \frac{1}{36} + \frac{1}{100} + \cdots\right]$

Thus, $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots \infty = \frac{\Pi^2}{8}$ (Ans)
