EXPERIMENT – VI

Anirban Mukherjee

16BIT0200

Lateral Vibration of Hanging Rope

Aim:

To find the power series solution of the Bessel's equation of order zero by the method of Frobenius and visualize it using MATLAB.

Problem Statements:

- Many differential equations arising from physical problems are linear with variable co-efficient.
- A general solution in terms of known function does not exist for these types of equations.
- Such equations can be solved by finding the solution in the form of an infinite convergent series.
- This is what we aim to do using MATLAB.

Matlab Commands:

- coeffs(P, var) – returns coefficients of the polynomial ‘P’ with respect to the variable ‘var’.

- collect(P, var) – rewrites ‘P’ in terms of the powers of the variable ‘var’.

- n=numel(A) – returns the number of elements ‘n’, in array ‘A’ ,equivalent to prod(size(A)).

- simplify(S) – performs an algebraic simplification of S.

- J = besselj(nu,Z) - computes the Bessel function of the first kind, where ‘nu’ is order and ‘Z’ is an argument.

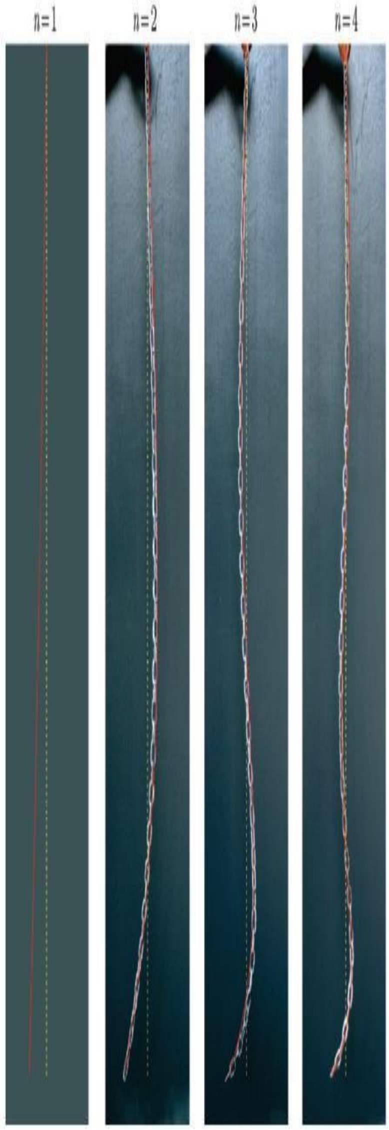
- Y = bessely(nu,Z) - computes Bessel function of the second kind, where **‘nu’ represents order and ‘Z’ is an argument.**

Description of Physical Experiment:

•A flexible uniform chain/rope/cable of length L and constant linear density (mass/unit length) of ρ gm/cm is fixed at the upper end (x = L). The x-axis is vertical, measured up from the equilibrium position of the free end of the chain.

•u(x,t) represents the displacement function for a point x on the chain.

•The displacements are small compared with the length of the chain, so that the displacement can be neglected.



•The tension in the chain is due to the weight below point , w(x) = ρgx, and the difference in the horizontal components of the tension at the ends of a small interval Δx of chain is the accelerating force.

• For any displacement of angle α, the restoring force is

$$F(x) = W \sin\alpha \sim Wu_x$$

•The difference in force between points on the change at x and $x + \Delta x$ is thus $\Delta F = \Delta x\, (Wu_x)_x$.

•From Newton's 2^{nd} law, $f = ma = mu_{tt}$,

$\Delta x\, \rho g\, [x\, u_x]_x = \rho \Delta x\, u_{tt}$ **Or** $u_{tt} = g(u_x + xu_{xx})$

<u>The Vertical Solution</u>

• Separate the variables with the solution function of the form

u(x,t) = F(x)G(t) ----------- (A)

$u_{tt} = F(x)\, G''(t)$, $u_x = F'(x)\, G(t)$, $u_{xx} = F''(x)\, G(t)$

using A, we get:

$$\frac{G''(t)}{G(t)} = \frac{gxF(x) + gF'(x)}{F(x)}$$

•The time function is thus just a cosine function of the form G(t) = cos(ωt + φ).

• The angular frequency 'ω' has units of $time^{-1}$.

• The chain will probably begin its oscillation at maximum displacement (initial velocity = 0), so that the phase angle ‘φ’ will be 0.

•The governing equation is therefore given by,

$$xF''(x) + F'(x) + \frac{\omega^2}{g} F(x) = 0 \quad ----(8)$$

•This ODE represents the Bessel’s equation, which has a typical form:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

Where ‘p’ is the order of the Bessel function (in our case p = 0).

Note: Eq. (8) has equal roots and hence one have to use the case: 2 type of solution whose conditions are discussed later.

<u>**Connection to Mathematics:**</u>

<u>Basic Definition- Singular Point</u>

Consider the Differential Equation of the form,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \text{ ------ } (1)$$

If $P_0(a) \neq 0$, then is called an ordinary point of (1), otherwise singular point.

<u>Basic Definition - Regular Singular Point</u>

A singular point x = a of (1) is called regular if (1) is expressed in the form:

$$\frac{d^2y}{dx^2}+\frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

Where $Q_1(x)$ and $Q_2(x)$possess derivatives of all orders in the neighbourhood of ‘a’.

• A singular point which is not regular is called an irregular singular point.

• When x = a is a regular singular point of (1), then it can be solved using the method of Frobenius.

<u>Frobenius Method</u>

• Let b(x) and c(x) be any functions that are analytic at x=0 (‘x’ is regular singular point). Then the ODE:

$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$ ------ (2)

has at least one solution that can be represented in the form

$y(x) = x^r \sum_{m=0}^{\infty} x^m a_m$ ------ (3)

where $a_0 \neq 0$, the exponent 'r' may be real or complex.

• Bessel's equation is given by,

$y'' + \frac{1}{x} y' + \left(\frac{x^2 - n^2}{x^2}\right) y = 0$ ------ (4)

And eq.(4) is identical to eq. (2) with b(x) = 1 and
c(x) = $x^2 - n^2$ analytic at x = 0.

Steps involved in the Frobenius method for solving (2),

<u>Step : 1</u>

Multiplying equation (2) by x^2,

$x^2 y'' + x b(x) y' + c(x) y = 0$ ------ (4)

Equation (3) can be written as

$y(x) = \sum_{m=0}^{\infty} x^{m+r} a_m$ ------ (5)

<u>Step : 2</u>

Taking derivative of equation (5)

$$y'(x) = a_m \sum_{m=0}^{\infty} (m+r)x^{m+r-1}$$

$$y''(x) = a_m \sum_{m=0}^{\infty} a_m (m+r)(m+r-1)x^{m+r-2}$$

Substituting these values in equation (4) and equating the sum of coefficients of each power x^r, x^{r+1}, x^{r+2}, … to zero, we obtain a system of equations involving the unknown coefficients a_m.

The corresponding equation is,

$[\mathrm{r}(\mathrm{r}-1) + b_0\mathrm{r} + c_0]a_0 = 0$ ------ (6)

since $a_0 \neq 0$,

$\mathrm{r}(\mathrm{r}-1) + b_0\mathrm{r} + c_0 = 0$ ------ (7)

Basis of Solutions

Case : 1

Distinct roots (r_1 and r_2) not differing by an integer. Basis is,

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots)$ and
$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots)$

Case : 2

Double root (r1=r2=r, equal roots). A basis is,

$y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \cdots)$ and
$y_2(x) = y_1(x)\, lnx + x^r(A_1x + A_2x^2 + \cdots)$ (x>0)

Case : 3

Roots differing by an integer. A basis is

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots)$ and
$y_2(x) = ky_1(x)\, lnx + x^{r_2}(A_1x + A_2x^2 + \cdots)$,
where the roots are so denoted that r1 - r2>0 and k may turn out to be zero.

Code:

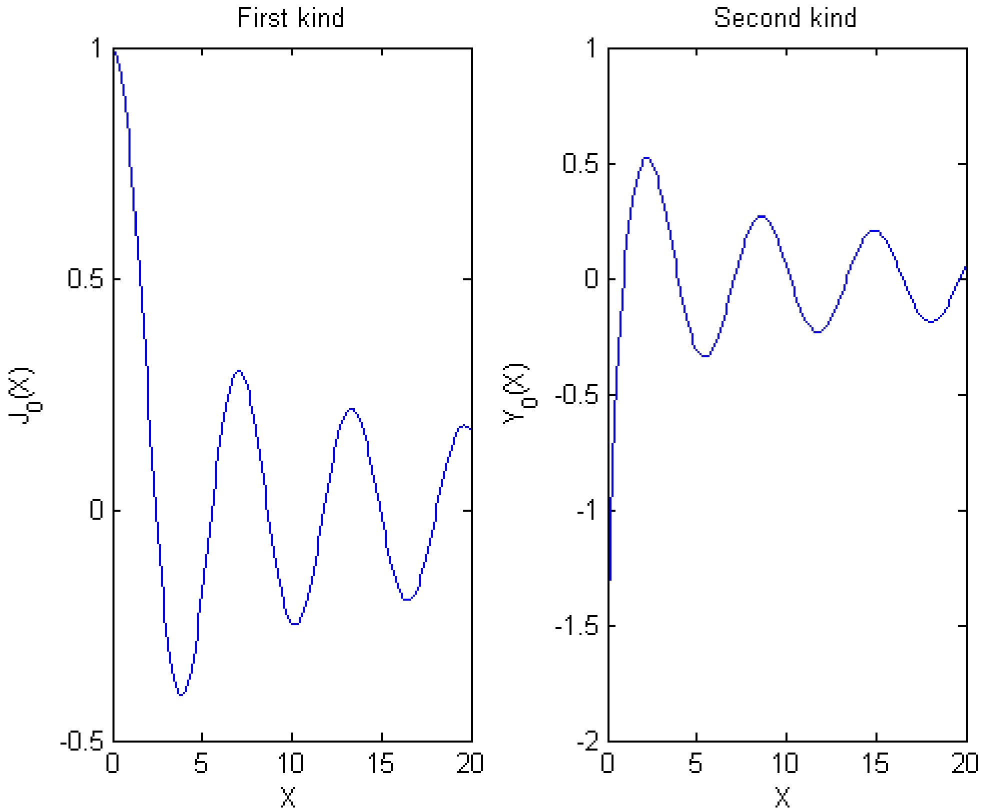
```
clc
clear all
close all
syms x a0 a1 a2 a3 a4 m c1 c2
y=a0*x^m+a1*x^(m+1)+a2*x^(m+2)+a3*x^(m+3)+a4*x^(m+4);
eq=x^2*diff(y,x,2)+x*diff(y,x,1)+x^2*y;
eq1=collect(eq)
eq2=coeffs(simplify(eq1*x^(1-m)),x)
eq3=solve(eq2(1),m)
a1=solve(eq2(2),a1)
a2=solve(eq2(3),a2)
a3=subs(solve(eq2(4),a3))
a4=subs(solve(eq2(5),a4))
ss=a0*x^m+a1*x^(m+1)+a2*x^(m+2)
+a3*x^(m+3)+a4*x^(m+4);
y1=subs(ss,m,eq3(1))
y2=subs(diff(ss,m),m,eq3(1))
gs=c1*y1+c2*y2
X = 0:0.1:20;
Y = zeros(5,numel(X));
J = zeros(5,numel(X));
Y0 = bessely(0,X);
J0=besselj(0,X);
subplot(1,2,1),plot(X,J0)
title('First kind')
xlabel('X')
ylabel('J_0(X)')
subplot(1,2,2),plot(X,Y0)
title('Second kind')
xlabel('X')
ylabel('Y_0(X)')
```

MATLAB output:

```
eq1 =


(a0*x^m + a1*x^(m + 1) + a2*x^(m + 2) + a3*x^(m + 3) + a4*x^(m + 4) +
a0*m*x^(m - 2)*(m - 1) + a1*m*x^(m - 1)*(m + 1) + a2*x^m*(m + 1)*(m + 2) +
a3*x^(m + 1)*(m + 2)*(m + 3) + a4*x^(m + 2)*(m + 3)*(m + 4))*x^2 + (a2*x^(m
+ 1)*(m + 2) + a3*x^(m + 2)*(m + 3) + a4*x^(m + 3)*(m + 4) + a0*m*x^(m - 1)
+ a1*x^m*(m + 1))*x


eq2 =
```

```
[ a0*m^2, a1*m^2 + 2*a1*m + a1, a2*m^2 + 4*a2*m + a0 + 4*a2, a3*m^2 +
6*a3*m + a1 + 9*a3, a4*m^2 + 8*a4*m + a2 + 16*a4, a3, a4]


eq3 =

 0
 0


a1 =

0


a2 =

-a0/(m^2 + 4*m + 4)


a3 =

0


a4 =

a0/((m^2 + 4*m + 4)*(m^2 + 8*m + 16))


ss =

a0*x^m - (a0*x^(m + 2))/(m^2 + 4*m + 4)
```

```
y1 =

a0 - (a0*x^2)/4


y2 =

(a0*x^2)/4 + a0*log(x) - (a0*x^2*log(x))/4


gs =

c1*(a0 - (a0*x^2)/4) + c2*((a0*x^2)/4 + a0*log(x) - (a0*x^2*log(x))/4)
```



My Work:

Using the same code, we can use it in Information Technology by using it in software for traffic operations.

This can be used in bridges to determine the vibration of that bridge by giving the necessary equations and conditions, then we can directly calculate the vibrations in the bridge.

We then develop an error signalling system which would alert the traffic control system if the traffic is too much and the bridge might be in risk.

This can also determine the damage in a bridge over time and would similarly give out warning if the bridge requires repair work. This is extremely crucial for Indian Road Networks.

This can also be fit in building construction lifts where labourers work to paint or wash windows. It would calculate the vibration and determine if the worker is safe or should he employ more caution.