MATHEMATICS DIGITAL ASSIGNMENT

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Batch : 10 (COMPUTER SCIENCE (CORE))

Slot : D2 + TD2

Subject Code : MAT1011

Q 1) Bacterium population: When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4 t^2 - 10^3 t^2$. Find the growth rates at

a) $t = 0$ hours

b) $t = 5$ hours

c) $t = 10$ hours

Ans: The size of the population at time t (hours) was : $b(t) = 10^6 + 10^4 t - 10^3 t^2$

On differentiating:

$$b'(t) = 10^4 - 2(10^3 t) = 10^3 (10 - 2t)$$

(a) $b'(0) = 10^4 - 2(10^3 \times 0) = 10^4$

$b'(0) = 10^4$ bacteria / hour

(b) $b'(5) = 10^3 (10 - 2 \times 5) = 10^3 (10 - 10) = 0$

$b'(5) = 0$ bacteria / hour

(c) $b'(10) = 10^4 - 2(10^3 \times 10) = 10^4 - 2 \times 10^4 = -10^4$

$b'(10) = -10^4$ bacteria / hour.

Q 2) Use implicit differentiation to find dy/dx of $x\cos(2x+3y) = y\sin x$

Ans : $x\cos(2x+3y) = y\sin x$

On differentiation we get,

$$\Rightarrow -x\sin(2x+3y)(2+3y') + \cos(2x+3y)$$

$$= y\cos x + y'\sin x$$

$$\Rightarrow -2x\sin(2x+3y) - 3xy'\sin(2x+3y) + \cos(2x+3y)$$

$$= y\cos x + y'\sin x$$

$$\Rightarrow \cos(2x+3y) - 2x\sin(2x+3y) - y\cos x =$$
$$(\sin + 3x\sin(2x+3y))y'$$

$$\Rightarrow y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3x\sin(2x+3y)}$$

Q3) Check whether the function

$$f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$$

satisfies the hypotheses of the Mean Value Theorem on the given interval, Give reasons for your answer.

Ans: Yes, the function exist as

At $x = 2$

R.H.S : $f(2) = 6 \times 2 - (2)^2 - 7$
$= 12 - 4 - 7 = 1$

L.H.S : $f(2) = 2 \times 2 - 3 = 1$

$f(2) = 1$

Since R.H.S = L.H.S = $f(2) = 1$

$\therefore$ Continuous at $x = 2$ and every polynomical function is continuous in range $[0, 3]$.

$$f'(x) = \begin{cases} 2, & 0 \le x \le 2 \\ 6 - 2x, & 2 < x \le 3 \end{cases}$$

R.H.S = $f'(2) = 2$

L.H.S = $f'(2) = 6 - 2 \times 2 = 2$

Since R.H.S = L.H.S

$\therefore$ differentiable at $x = 2$ and every polynomial function is differentiable in range $(0, 3)$.

Thus the mean Value Theorem exists for the function $f(x)$ at its every point.

Q 4) (a) Identifying extrema of $f(x) = \frac{3}{2}t^4 - t^6$

~~(a)~~

Ans: $f(t) = \frac{3}{2}t^4 - t^6$

On differention

$f'(t) = \frac{3}{2} \times 4t^3 - 6t^5 = 6t^3 - 6t^5 = 6t^3(1-t)(1+t)$

Critical points at $t = 0, \pm 1$

$f'(t)$ is thus increasing on $(-\infty, -1) \cup (0, 1)$

and $f'(t)$ thus decreasing on $(-1, 0) \cup (1, \infty)$

Thus the local maxima are $f(-1) = \frac{3}{2}(-1)^4 - (-1)^6$

$= \frac{3}{2} - 1 = \frac{1}{2}$

and the local maxima at $f(1) = \frac{3}{2}(1)^4 - (1)^6$

$= \frac{1}{2}$

The local minimum at $f(0) = \frac{3}{2}(0)^4 - (0)^6 = 0$

$\therefore$ The absolute maximum is $\frac{1}{2}$ at $t = 1$ & $t = -1$.

There is no absolute minimum value.

Interval	Condition of the function
$(-\infty, -1) \cup (0, 1)$	Increasing
$(-1, 0) \cup (1, \infty)$	Decreasing

Q 4) (6) $f'(x) = \dfrac{(x-2)(x+4)}{(x+1)(x-3)}$, $x \neq -1, 3$

(a) What are the critical points of f ?

(b) On what intervals is f increasing or decreasing?

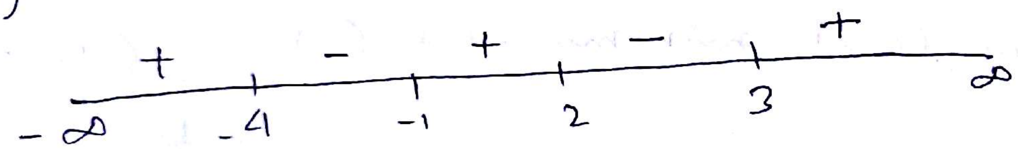
(C) At what points, if any, does f assume local maximum and minimum values ?

Ans : The function's derivative is :-

$$f'(x) = \frac{(x-2)(x+4)}{(x+1)(x-3)}$$

(a) critical points of f are at $x = 2$, $x = -4$, $x = -1$, $x = 3$

(b)



The function is increasing on the interval
$(-\infty, -4) \cup (-1, 2) \cup (3, \infty)$

The function is decreasing on the interval
$(-4, -1) \cup (2, 3)$

(C) Local maximum at $x = -4$ as the function is increasing on its left and decreasing on its right.

Similarly, local maximum at $x = 2$.

Interval	Increasing / decreasing condition
$(-\infty, -4) \cup (-1, 2) \cup (3, \infty)$	Increasing
$(-4, -1) \cup (2, 3)$	Decreasing

Q, 5) Graph the rational function

$$y = \frac{x^3 + x - 2}{x - x^2}$$

Ans : $y = \frac{x^3+x-2}{x-x^2} = \frac{(x-1)\ (x^2+x+2)}{(x-1)\ (-x)}$ [On Simplification]

Since 1 and 0 are roots of the denominator, the domain is $(-\infty, 0) \cup (0,1) \cup (1,\infty)$.

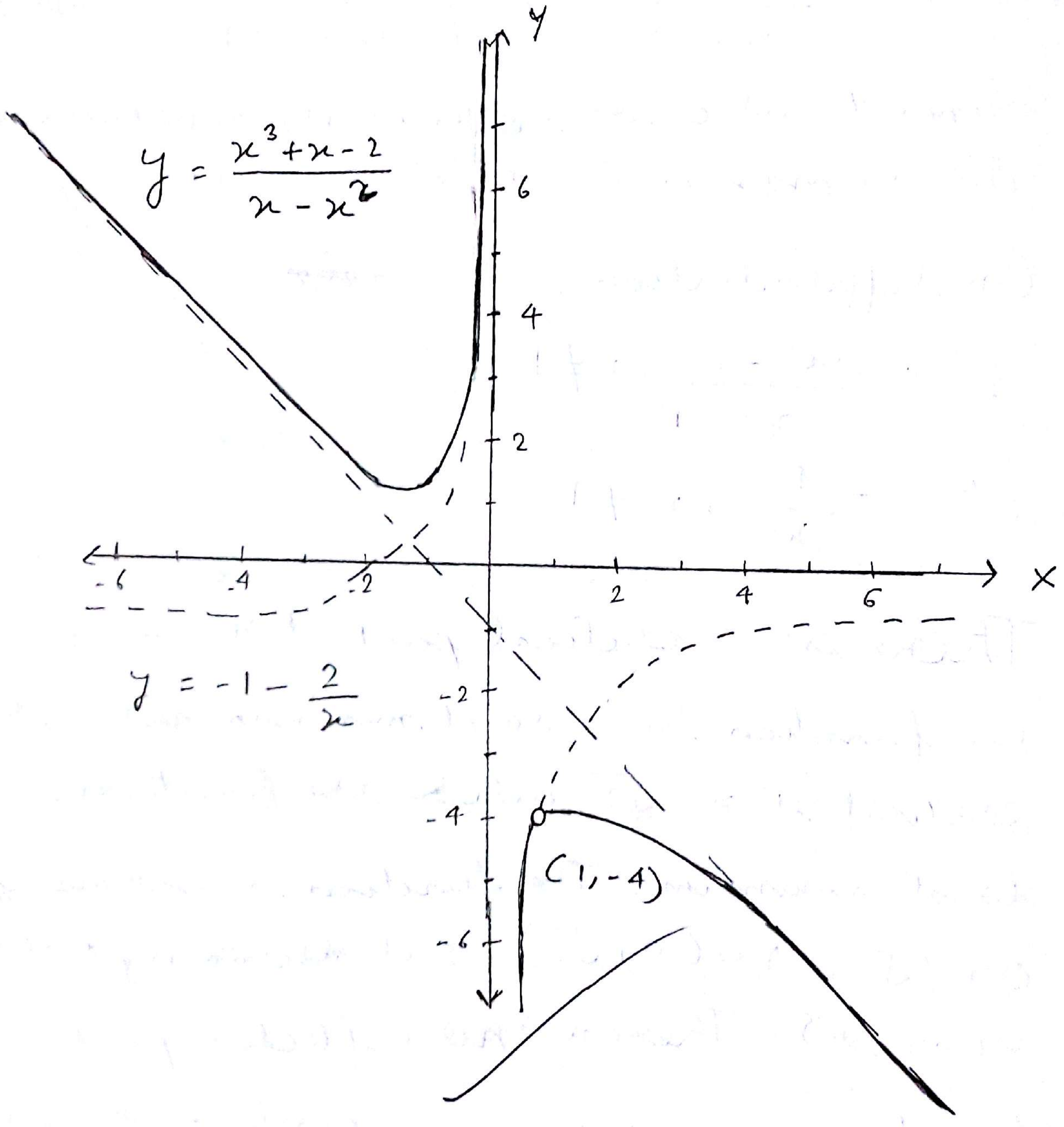
On differentiation, ~~Ang~~

$$y' = \frac{-x^2-2}{x^2}, \quad x \neq 1 \;;$$

$$y'' = -\frac{4}{x^2}, \quad x \neq 1$$

There is a critical point at $x = -\sqrt{2}$ where the function has a local minimum and a point critical at $x = \sqrt{2}$ where the function has a local maximum. The function is increasing on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$ and decreasing on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$. There are no inflection points. The function is concave up on $(-\infty, 0)$ and concave down on $(0,1) \cup (1,\infty)$. The y axis is a vertical asymptote. Dividing numerator by denominator gives $y = -x - 1 - \frac{2}{x}$ which shows that the line $y = -x-1$ is an

Oblique asymptote. The graph has a hole at the point $(1, -4)$.



Q 6) The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$ and $x = 0$. The cross-sections perpendicular to the x-axis are:

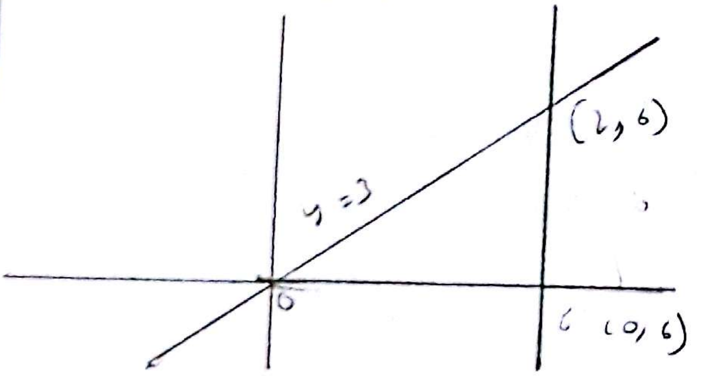
a) rectangles of height 10

b) rectangles of perimeter 20

Calculate volume of each case by slicing method.

Ans : (a) $A(x)$ = area of rectangle

As we know, area = length × height



$= (6 - 3x) \times 10$

$= 60 - 30x$

As we are finding the Volume ($V(x)$) according to x axis, thus $a = 0$ (lower value) & $b = 2$ (upper value).

$$V = \int_a^b A(x)\,dx = \int_0^2 (60 - 30x)\,dx = \left[60x - 15x^2\right]_0^2$$

$= (120 - 60) - 0 = 60$ cube units.

(b) $A(x)$ = area of rectangle

Since Perimeter = 20 units ∴ $20 = 2(\text{length} + \text{height})$

∴ Height $= \dfrac{20 - 2(\text{length})}{2}$

As length $= 6 - 3x$ ∴ Height $= \dfrac{20 - 2(6 - 3x)}{2}$

∴ Area = length × height

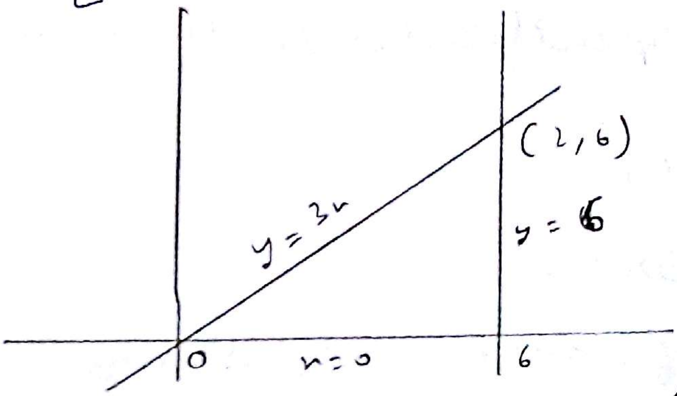
Similarly in this case also, $a = 0$ & $b = 2$

$$V = \int_a^b A(x)\,dx = \int_0^2 (6 - 3x)\left(\frac{20 - 2(6 - 3x)}{2}\right) dx$$

$$= \int_0^2 (6 - 3x)(4 + 3x)\,dx = \int_0^2 (24 + 6x - 9x)\,dx$$

$$= \left[24x + 3x^2 - 3x^3\right]_0^2 = (48 + 12 - 24) = 36$$

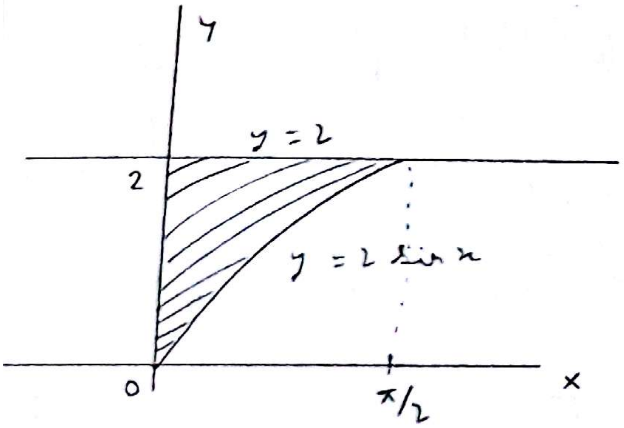
cubic units



Q.7) The region is the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$.

Find the volume of the solid generated by revolving the region about the given line.

Ans: $R(x) = 2 - 2\sin x = 2(1 - \sin x)$



Since we are solving with respect to the line $y = 2$,

$\therefore$ The upper value $(b) = \frac{\pi}{2}$

& the lower value $(a) = 0$.

$$\therefore V = \int_0^{\pi/2} \pi [R(x)]^2 \, dx = \pi \int_0^{\pi/2} 4 (1 - \sin x)^2 \, dx =$$

$$= 4\pi \int_0^{\pi/2} (1 + \sin^2 x - 2\sin x) \, dx$$

$$= 4\pi \int_0^{\pi/2} \left(1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right) dx$$

$$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$$

$$= 4\pi \left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right]$$

$$= \pi(3\pi - 8)$$

Thus the volume is $\pi(3\pi - 8)$ cube. unit

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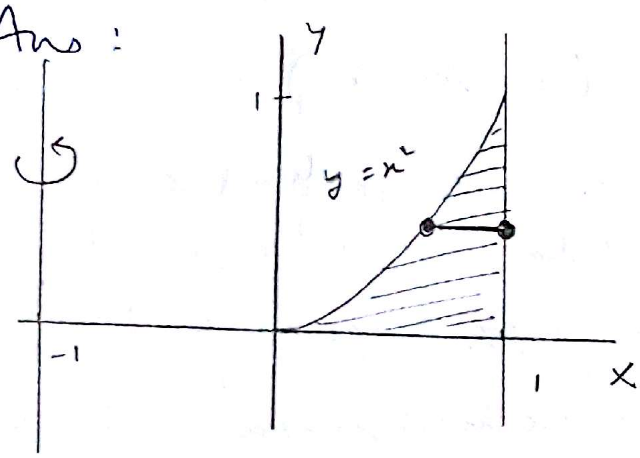
Q 8) The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$.

Find the volume of the solid generated by revolving each region about the given axis.

Ans :



$R(y) = 2$ & $r(y) = 1 + \sqrt{y}$

$$V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$

$$= \pi \int_0^1 \left[4 - (1+\sqrt{y})^2\right] dy$$

$$= \pi \int_0^1 (4 - 1 - 2\sqrt{y} - y)\, dy$$

$$= \pi \int_0^1 (3 - 2\sqrt{y} - y)\, dy = \pi\left[3y - \frac{4}{3} y^{3/2} - \frac{y^2}{2}\right]_0^1$$

$$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18 - 8 - 3}{6}\right) = \frac{7\pi}{6}$$

Thus the volume of the solid generated is $\frac{7\pi}{6}$ cube units.

Q9) Express the following function in terms of unit step function and hence find its Laplace transform.

$$f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$$

Ans:

By applying unit step function method:

$$f(t) = \cos t\,[u(t-0) - u(t-\pi)] + 1\,[u(t-\pi) - u(t-2\pi)] + \sin t\,[u(t-2\pi)]$$

$$= \cos t + (1 - \cos t)\,u(t-\pi) + (\sin t - 1)(u(t-2\pi))$$

Since $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

and $L(\sin at) = \frac{a}{s^2+a^2}$, $L(\cos at) = \frac{s}{s^2+a^2}$, $L(1) = \frac{1}{s}$

$$L\{f(t)\} = L\{\cos t + (1-\cos t)u(t-\pi) + (\sin t - 1)\,u(t-2\pi)\}$$

$$= L\{\cos t\} + L\{(1-\cos t)\,u(t-\pi)\} + L\{(\sin t - 1)\,u(t-2\pi)\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

Thus, the Laplace conversion is:

$$\frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

Q 10) Evaluate $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)\,dt$

Ans :

$$L\left[(1+3t+t^2)H(t-2)\right]$$

$$= e^{-2s} L\left[1+3(t+2)+(t+2)^2\right]$$

$$= e^{-2s} L\left[1+3t+6+t^2+4+4t\right]$$

$$= e^{-2s} L\left[11+7t+t^2\right] = e^{-2s}\left[\frac{11}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]$$

$$\therefore \int_0^\infty e^{-2s}(1+3t+t^2)H(t-2)\,dt$$

$$= e^{-2s}\left(\frac{11}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right)$$

Obtaining the expression with $s=1$:-

$$\int_0^\infty e^{-2s}(1+3t+t^2)H(t-2)\,dt = e^{-2}(11+7+2)$$

$$= 20e^{-2}.$$

21/8/16