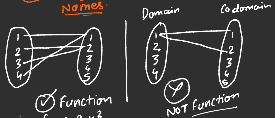
"FUNCTIONS"

THEORY

Definition:
Let X and Y be two non-empty sets.
A rule in which every element of X is assigned to unique element of Y then this rule is called function
It is denoted as $f: X \to Y$
"functions" / "Mappings" / "Transformations"

Names.



Domain : $\{1,2,3,4\}$
Codomain: $\{1,2,3,4,5\}$
Range : $\{1,2\}$

Domain & Codomain
If a function is from A to B then A is the domain of function & B is the codomain.
↳ if $f(a) = b$
$a =$ (preimage of b) OR $b =$ (image of a)

Range
If a function(f) is from A to B, then the set of all those elements of B (codomain) which are related with A (domain) is called as the range of function.

Types of FUNCTIONS

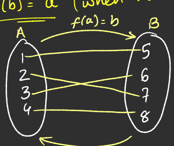
① One-one functions / Injective f^n
A function f is said to be one-one, if and only if $f(a) = f(b)$ then (for implies that)
$\to a = b$

② Onto function / Surjective f^n
A function f from A to B is called as Onto if Range of function = Codomain
i.e.
if and only if for every element $b \in B$ there exists preimage $a \in A$ with $f(a) = b$.

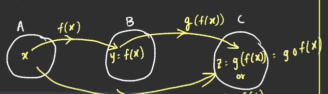
A function is called as "Bijective" if it is both One-one & Onto.

Inverse Functions & Composite f^{ns}

Inverse Functions.
let f be a Bijective function from A to B then the function that assigns $b\ (\in B)$ a unique element $a\ (\in A)$ such that $f(a) = b$
The inverse function of f is shown as
$f^{-1}(b) = a$ (when $f(a) = b$)



Composite function
let f be f^n from set A to set B
g be f^n from set B to set C
Composition of functions f and g denoted (for all $x \in A$) by $g\circ f$ is f^n from A to C
$g\circ f(x) = g(f(x))$



QUESTIONS & Egs.

Q.1 ONE-ONE f^n

Example : Let $f : R \to R$ s.t. $f(x) = ax + b$, $a \neq 0$ is one-one function

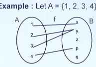
let $f(x) = f(y)$ is true & prove that $x = y$ is coming (if yes, then one-one)
$\therefore \cancel{a}x + \cancel{b} = \cancel{a}y + \cancel{b}$
$\therefore x = y$
$(f(x) = f(y)) \Rightarrow (x = y)$
$\therefore$ The function is one-one function

Q.2
Q If $f(x) = x^2$ one-one? Solve

$f(x) = x^2$
let $f(x) = f(y)$
$x^2 = y^2$
$x^2 - y^2 = 0$
$(x+y)(x-y) = 0$
$\therefore x + y = 0$ or $x - y = 0$
$\therefore x = -y$ or $x = y$
we got this unnecessary condition too! This condition was satisfied but
$\therefore$ NOT an one-one function.

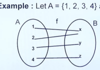
ONTO f^n Q3

Example : Let A = {1, 2, 3, 4} and B = {x, y, z, p, q}



B: codomain = $\{x, y, z, p, q\}$
Here Range f = {x, y, p} which is not onto. as $f \neq B$

Example : Let A = {1, 2, 3, 4} and B = {x, y, z}



Here Range f = {x, y, z} = B which is onto.

Note : If f is not onto then it is called into.

BIJECTIVE f^n

Q.4 **Example :** The function $f : R \to R$, $f(x) = 2x - 3$ is bijective.

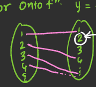
Proving one-one
Let, $f(x) = f(y)$
$\cancel{2}x - \cancel{3} = \cancel{2}y - \cancel{3}$
$x = y$
Hence, one-one proved//

Proving Onto.
$y = f(x)$
$y = 2x - 3$, y is Real No.
writing x in terms of y
$x = \frac{y+3}{2}$
x is Real No.
Given was a relation which goes from
Real No $\to$ Real No//
i.e $R \to R$//
Hence Onto f^n proved//
$\because$ Onto + One-one $\Rightarrow$ BIJECTIVE //

Q.5
1 Show that the mapping $f : Z^+ \to Z^+$ defined by $f(x) = x^2$ for all $x \in Z^+$ is one-one but not onto.

for one-one
let $f(x) = f(y)$
$x^2 = y^2$
$x^2 - y^2 = 0$
$(x+y)(x-y) = 0$
$x = -y$ Disregarding as relation is $Z^+ \to Z^+$ (positive integer)
$x = y$ proved,
Hence one-one f^n

for Onto f^n. $y = f(x) = x^2$



$\nexists x$ such that $f(x) = 2$
Hence proved NOT ONTO f^n

Q.6
2 Find the domain of function
(i) $f(x) = \frac{x}{x^2+1}$ (ii) $f(x) = \sqrt{x-4}$

(i) $f(x) = \frac{x}{x^2+1}$
we know $x^2 + 1 \neq 0$ for any Real values of x
so
Domain of $f(x) = \frac{x}{x^2+1}$ is R ← set of Real nos.

for finding Range of $f(x)$
let $y = \frac{x}{x^2+1}$
$yx^2 - x + y = 0$
→ Apply Quadratic formula,
$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$
x will be real when $y \neq 0$ & $1 - 4y^2 \geq 0$
$-4y^2 \geq -1$
$4y^2 \leq 1$
$|y| \leq \frac{1}{2}$
$\Downarrow$
$-\frac{1}{2} \leq y \leq \frac{1}{2}$

Range: $\left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$

ii) $f(x) = \sqrt{x-4}$
if $x - 4 < 0$, then $f(x)$ is not defined (# imaginary)
so
$x - 4 \geq 0$
$x \geq 4$
so,
Domain of $f(x) = \sqrt{x-4}$ is $[4, \infty)$

Range of $f(x)$
let $y = \sqrt{x-4}$
$y^2 = x - 4$
$x = \sqrt{y^2+4}$
x is real for
$y^2 + 4 \geq 0$
y^2 always ≥ 0 so this condition holds true for every real value y
so Range: R

Q.7
3 Find the range of the function
(a) $y = \frac{x}{1-x}$ (b) $y = \sqrt{9-x^2}$

(a)
writing x in terms of y
$y - yx = x$
$y = x + yx$
$\left(\frac{y}{y+1}\right) = x$
$x = \left(\frac{y}{y+1}\right)$
RHS is real for all Real y except -1
$\therefore$ Range: $R - \{-1\}$

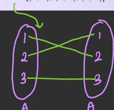
(b)
$y^2 = 9 - x^2$
$x = \sqrt{9 - y^2}$
RHS is real for
$9 - y^2 \geq 0$
$y^2 \leq 9$
$|y| \leq 3$
i.e $-3 \leq y \leq 3$
$\therefore$ Range: $[-3, 3]$

Q.8 To prove this its Inverse exists.
contd..
$\because$ The function is Bijective,
so its inverse $f^{-1}(y)$ exists.
where $f^{-1}(y) = x = \frac{y+3}{2}$

Q.9
Example : If $f : R \to R$ s.t. $f(x) = x^2 + 1$ and $g : R \to R$ s.t. $g(x) = 2x + 1$ are two function then find $g\circ f : R \to R$?

$f(x) = x^2 + 1$
$g(x) = 2x + 1$
$g\circ f(x) = g(f(x))$
$= g(x^2 + 1)$
$= 2(x^2 + 1) + 1$
$g\circ f(x) = 2x^2 + 3$

Q.10
1 Let A = {1, 2, 3}. Define $f : A \to A$ such that $f = \{(1, 2), (2, 1), (3, 3)\}$. Find f^{-1}, f^2 and f^3.



f: $1 \to 2$, $2 \to 1$, $3 \to 3$
f^{-1}: $1 \to 2$, $2 \to 1$, $3 \to 3$
$f^{-1} = \{(1,2), (2,1), (3,3)\}$//

f^2: $f\circ f = f(f(x))$
$f^2(1) = f(f(1)) = f(2) = 1$
$f^2(2) = f(f(2)) = f(1) = 2$
$f^2(3) = f(f(3)) = f(3) = 3$
$f\circ f = \{(1,1), (2,2), (3,3)\}$

$f^3 = f(f(f(x)))$
$f^3(1) = f(f(f(1)))$
$= f(f(2))$
$= f(1)$
$= 2$
similarly
$f^3(2) = 1$
$f^3(3) = 3$
$f^3 = \{(1,2), (2,1), (3,3)\}$

Q.11
2 If $f : A \to B$ and $g : B \to C$ be one-one onto function, then prove that $g\circ f$ is also one-one onto and $(g\circ f)^{-1} = f^{-1} \circ g^{-1}$

Proving one-one $g\circ f$ i.e $g(f(x))$
let $g(f(x_1)) = g(f(x_2))$ (we took previously x, y here x_1, x_2 as u wish)
g is onto f^n,
i.e $g(x) = g(y)$
$\Rightarrow x = y$
$\Rightarrow f(x_1) = f(x_2)$
f is also one-one
i.e $f(x) = f(y)$
$\Rightarrow x = y$
$x_1 = x_2$
we considered $g\circ f(x_2) = g\circ f(x_1)$ & proved $x_1 = x_2$
so $g\circ f$ is one-one proved//

for Onto.

as f, g are onto
$f(x) = y$
$g(y) = z$
$g\circ f(x) = g(f(x))$
$= g(y)$
$g\circ f(x) = z$
Looking like def^n of onto,
so $g\circ f(x)$ is Onto// proved.

Proof for $(g\circ f)^{-1} = f^{-1} \circ g^{-1}$
we have
$g\circ f(x) = z$
so, $g\circ f^{-1}(z) = x$ ..(Taking inverse)
Consider RHS,
$f^{-1} \circ g^{-1}(z) = f^{-1}(g^{-1}(z))$
$= f^{-1}(y)$
$f^{-1} \circ g^{-1}(z) = x$

Hence proved
$(g\circ f)^{-1} = f^{-1} \circ g^{-1}$//

Q.12
3 If $f : X \to Y$ and A, B are two subset of X, prove that (i) $f(A \cup B) = f(A) \cup f(B)$
(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

Sets wali approach.....
Let's consider an element which belongs to $f(A \cup B)$
$x \in f(A \cup B)$
$\Rightarrow f^{-1}(x) \in A \cup B$
$\Rightarrow f^{-1}(x) \in A \cup f^{-1}(x) \in B$
$\Rightarrow x \in f(A) \cup x \in f(B)$
$f(A \cup B) \subseteq f(A) \cup f(B)$
Let's take ulta now (to prove "=")
• $x \in f(A) \cup x \in f(B)$
$\Rightarrow f(x)^{-1} \in A \cup f^{-1}(x) \in B$
$\Rightarrow f(x)^{-1} \in (A \cup B)$
$\Rightarrow x \in f(A \cup B)$
$f(A) \cup f(B) \in f(A \cup B)$
$f(A \cup B) = f(A) \cup f(B)$ Hence proved.

(ii) let $x \in f(A \cap B)$
$f^{-1}(x) \in A \cap f^{-1}(x) \in B$
$x \in f(A) \cap x \in f(B)$
$f(A \cap B) \subseteq f(A) \cap f(B)$
Hence proved

Q.13
4 Let $f : R \to R$ s.t. $f(x) = x^2$ and $g : R \to R$ s.t. $g(x) = x + 3$ then show that $f\circ g \neq g\circ f$

$f(x) = x^2$
$g(x) = x + 3$
$g\circ f = g(f(x)) = g(x^2)$
$g\circ f = x^2 + 3$
$f\circ g = f(g(x)) = f(x+3)$
$= (x+3)^2$
$f\circ g = x^2 + 9 + 6x$
$g\circ f \neq f\circ g$
Hence proved,

Q.14
Let $f : R \to R$, such that $f(x) = 2x + 3$ and $g : R \to R$, such that $g(x) = x^2$, then $g\circ f(x)$ is
(A) $(2x + 3)^2$ (B) $2x^2 + 3$ (C) $2x + 3$ (D) None of these

$g(f(x)) = g(2x+3)$
$= (2x+3)^2$