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Perron - Frobenius Theory

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All matrices will be square of order $n \times n$ and we will only work on $\mathbb{R}$ (eigenvalues may be in $\mathbb{C}$)

Definition : We say a matrix A is non-negative (resp. positive) if all its entries are non-negative (resp. positive) and we write $A \geq 0$ (resp. $A > 0$)

Definition : The spectral radius of a matrix is its maximum eigenvalue by modulus

A standard result on norms is that on a finite dim vec space, all norms are equivalent. Thus, we may use the following norms :

For a vector v, $\|v\|$ = sum of all entries of v

for a matrix A, $\|A\|$ = sum of all entries of A

Definition : If $A \geq 0$, we say A is primitive if $A^k > 0$ for some $k \geq 1$

We first wish to prove Perron's theorem

Theorem 1 (Perron theorem)

Suppose A is a primitive matrix with spectral radius λ. Then we have the following properties

(i) λ is an eigenvalue of A with alg. multiplicity 1

(ii) λ is the unique largest eigenvalue by modulus

(iii) λ has strictly positive eigenvectors

This theorem is very powerful and finds a wide range of application because most matrices encountered in real life are non-negative

We have a lemma before proving Perron's theorem

Lemma 2

Let V be a finite dim$_n$ real vec. space and $T : V \to V$ be linear. Suppose S is a polyhedron containing 0 in its interior and some positive power of T maps S strictly into its interior, then the spectral radius of T is less than 1

proof

Eigenvalues of T and T^k are related through powers. Thus, Spectral radius of $T^k < 1$ iff spectral radius of $T < 1$ holds.

Thus, we may assume WLOG that $T(S) \subsetneq S$

It is clear that the spectral radius ≤ 1. Suppose v is an eigenvector with eigenvalue λ st. $|\lambda| > 1$.

Scale v by α so that αv lies on ∂S (the boundary of S) (Note that $\alpha \in \mathbb{R}$, $\alpha > 0$)

$T(\alpha v) = \alpha \lambda v$

$\therefore \| T(\alpha v) \| = \| \alpha \lambda v \| = |\lambda| \, \|\alpha v\| > \|\alpha v\|$

Now $\alpha v \in \partial S$ and T is mapping this point outside S which is a contradiction.

Now we show that spectral radius is less than 1

$T(S) \subsetneq S \Rightarrow T(T(S)) \subsetneq T(S) \subsetneq S$

Continuing this way we get $T^k(S) \subsetneq S$

Now we know $T(S) \cap \partial S = \varnothing$. Suppose for the sake of contradiction, there is an eigenvalue λ st. $|\lambda| = 1$

Let $\lambda = e^{i\theta}$ & the eigenvector be v_0

Case 1 : λ is a root of unity ie. θ/π is rational

Suppose $\lambda^m = 1$

Then $T v_0 = \lambda v_0$ & $T^m(v_0) = v_0$

Scale v_0 so that αv_0 $(\alpha \in \mathbb{R}, \alpha > 0)$ lies on ∂S.

Then $T^m(\alpha v_0) = \alpha v_0$. $\therefore \alpha v_0 \in T^m(S) \cap \partial S$

which is a contradiction

Case 2 : λ is not a root of unity ie. θ/π is irrational.

Split $v_0 = x + iy$. Write $W = \operatorname{span}_{\mathbb{R}}\{x, y\}$.

Clearly $x \neq 0$ & $y \neq 0$ else it forces $v = 0$.

Also if $x = \beta y$, then $v_0 = (\beta + i) y$

$\therefore \quad Tv_0 = \lambda v_0 \Rightarrow Ty = \lambda y \quad (\because \beta \in \mathbb{R} \Rightarrow \beta + i \neq 0)$

But T is a real operator & $\lambda \in \mathbb{C}$, $y \in \mathbb{R}$. Thus it forces $y = 0$.

Hence W is a 2-dim subspace of V

Now W is a T invariant subspace because T carries both x & y back into W ($\because$ Real part of $T(v) = T(x)$ and imaginary part of $T(v) = T(y)$)

Explicitly, $T(x) = \mathrm{Re}\left(e^{i\theta}(x+iy)\right) = \cos\theta\, x - \sin\theta\, y$

$T(y) = \mathrm{Im}\left(e^{i\theta}(x+iy)\right) = \sin\theta\, x + \cos\theta\, y$

Thus, wrt the basis $\{x, y\}$, $T|_W$ is given by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This is a rotation on W

It is well known that for any point u on a circle of radius $\|u\|$, the set $\{ R_\theta^n(u) \mid n = 1, 2, \ldots \}$ comes arbitrarily close to every point on that circle provided θ/π is irrational.

Now $W \cap \partial S \neq \emptyset$ since 0 lies in the interior of S and also in W (W is a plane through origin & S is a closed set containing the origin)

Let $p \in W \cap \partial S$

Keep rotating p : $T(p), T^2(p), \ldots$

Since S is closed, $T(S)$ is closed and by result on irrational rotations we can find a sequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} T^{n_k}(p) = p$

By closedness of $T(S)$, limit of sequence in $T(S)$ lies in $T(S) \Rightarrow p \in T(S)$

$\therefore p \in T(S) \cap \partial S$. Contradiction.

proof of theorem 1

Let $S = \{ v \in \mathbb{R}^n \mid v \geq 0, \|v\| = 1 \}$

Define $F : S \to S$ as $F(v) = \dfrac{Av}{\|Av\|}$

(Note that $A \geq 0$, $v \geq 0 \Rightarrow Av \geq 0$)

Brower's theorem : Every cont. function from a non-empty convex compact subset K (of a Euclidean space) to K itself has a fixed point.

Applying it here, $\exists\ r \in S$ such that $Ar = \|Ar\|\, r$

Thus $r \geq 0$ is an evec with eval $\|Ar\| = \lambda > 0$

Now $A^k > 0$ for some k

$A^k r = \lambda^k r \Rightarrow r = \lambda^{-k} A^k r > 0$ ($\because B > 0, v \geq 0 \Rightarrow Bv > 0$)

Thus $r > 0$ is an evec with eval $\lambda > 0$

Now let $r = (r_1, r_2, \dots, r_n)$

Let $R = \text{diag}(r_1, \dots, r_n)$

Let $P = \lambda^{-1} R^{-1} A R$ & $\mathbb{1}$ be the all 1's vector

$P\mathbb{1} = \lambda^{-1} R^{-1} A R \mathbb{1} = \lambda^{-1} R^{-1} A r = \lambda^{-1} R^{-1} \lambda r = R^{-1} r = \mathbb{1}$

Thus every row of P sums to 1

Suppose $Pv = \lambda v$ for some $\lambda > 1$

Rows of P are non negative and sum to 1.

Now P is also primitive ($\because P^k = \lambda^{-k} R^{-1} A^k R > 0$)

$P\mathbb{1} = \mathbb{1}$. Scale $\mathbb{1}$ to v so that $Pv = v$ with $v \in S$

Consider $S' = S - v$. Clearly $0 \in S'$

We claim $P^k(S') \subsetneq S'$

Let $x - v \in S'$ ($x \in S$ so that $x_i \geq 0\ \forall i$)

$P^m(x - v) + v = P^m x$

Now $(P^m x)_i = \sum_{t=1}^{n} P^m_{it} x_t > 0$

$\therefore P^k(x - v) \in \text{int}(S')$

By lemma, P^k has spectral radius < 1 (on S')

Note that $v = \frac{1}{n}\mathbb{1}$ & hence S' is given by

$S' = \{ x \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0 \}$

Thus, on $W = \text{span}\{\mathbb{1}\}$, P has eigenvalue 1 and on $W^\perp = S'$, P has eigenvalue < 1.

Thus P has eigenvalue 1 & every other eigenvalue has modulus strictly smaller than 1

Thus $A = \lambda R P R^{-1}$ has eigenvalue λ & every other e.val has modulus strictly smaller than λ

Further the evec r corresponding to λ satisfies $r > 0$ ∎

Apart from these main results, there are several other properties found.

So far, for a primitive matrix A, we have obtained a perron root r (which has alg. mult 1) which dominates every other root (ie. $|\lambda| < r$) and its e.vector, called the perron vector has all components positive.

The following corollary says that the perron vector is the only eigenvector with positive (moreover, non-negative) components

Corollary 3

If w is a non-negative e.vec with eval β, then β must be the spectral radius

proof

Choose k st. $A^k > 0$.

$A^k > 0$, $w \geq 0 \Rightarrow A^k w = \beta^k w > 0$

$A^k > 0 \Rightarrow A^{k+1} > 0$

Case 1 : k = odd

Then $\beta^{k+1} w > 0 \Rightarrow w > 0$ $\quad (\because \beta^{even} \geq 0)$

and $w > 0$, $\beta^k w > 0 \Rightarrow \beta^k > 0 \Rightarrow \beta > 0$

Case 2 : k = even

Then $\beta^k w > 0 \Rightarrow w > 0$ $\quad (\because \beta^{even} \geq 0)$

and $w > 0$, $\beta^{k+1} w > 0 \Rightarrow \beta^{k+1} > 0 \Rightarrow \beta > 0$

Overall, $\beta > 0$ and $w > 0$

Now let v_0, λ be a perron vector & the perron root

$w > 0$ & $v_0 > 0$. WLOG let $v_0 < w$ (scale v_0 down)

Then $\forall\, n > 0$, $\lambda^n v_0 = A^n v_0 \leq A^n w = \beta^n w$

ie. $\lambda^n v_0 \leq \beta^n w \quad \forall\, n > 0$

If $\beta < \lambda$, this is not possible. Hence $\beta \geq \lambda$

But λ is perron root $\Rightarrow \beta = \lambda$

■

Proposition 4 (Collatz - Wielandt inequalities)

Let A be primitive with perron root r. Let $x \geq 0$ with $x \neq 0$

Define $m(x) = \min\limits_{1 \leq i \leq n} \dfrac{(Ax)_i}{x_i}$, $\quad M(x) = \max\limits_{1 \leq i \leq n} \dfrac{(Ax)_i}{x_i}$

Then $m(x) \leq r \leq M(x)$ with $r = \max\limits_{x > 0} m(x) = \min\limits_{x > 0} M(x)$

proof

We first show $r = \max_{x>0} m(x)$ so that $m(x) \leq r$ follows $\forall\ x \geq 0$ st. $x \neq 0$

Let perron vector be v_0

$m(v_0) = \min_{1\leq i\leq n} \frac{(A v_0)_i}{(v_0)_i} = r \Rightarrow \max_{x>0} m(x) \geq r$

Now $m(x)\, x_i \leq (Ax)_i \quad \forall\ i$

$\therefore\ 0 \leq m(x)\, x \leq Ax$

A has perron root r. A^T is also primitive and has same spectrum as A & hence A^T also has perron root r with some perron vector w_0 ie. $A^T w_0 = r\, w_0$

Thus, $m(x) \langle w_0, x \rangle \leq \langle w_0, Ax \rangle$

$= \langle A^T w_0, x \rangle$

$= r \langle w_0, x \rangle$

$\therefore\ m(x) \leq r$

$\therefore\ \max_{x>0} m(x) \leq r$

Thus $\max_{x>0} m(x) = r$

Similarly, one shows $r = \min_{x>0} M(x)$

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Corollary 5

If a & b denote min & max row sums of A, then $a \leq r \leq b$

proof

Apply above proposition to $x = \mathbb{1}$

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We now move on to another class of matrices called irreducible matrices.

Proposition 6

The following are equivalent for $A \geq 0$

(a) For any linear subspace W spanned by some standard basis vectors $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$, $A(W) \not\subseteq W$ (ofc $A(\mathbb{R}^n) \subseteq \mathbb{R}^n$)

(b) There is no perm matrix P st. $PAP^{-1} = \begin{bmatrix} E & F \\ 0 & G \end{bmatrix}$

(c) $\forall\ (i,j) \in [n]^2$, $\exists\ k = k(i,j)$ st. $A^k_{ij} > 0$

Proof

(a) $\Rightarrow$ (b) :

Suppose there is a permutation matrix P st. $PAP^{-1} = \begin{bmatrix} E & F \\ O & G \end{bmatrix}$

Let $I = \{1, 2, \dots, k\}$, $J = \{k+1, \dots, n\}$ where E is $k \times k$

and G is $(n-k) \times (n-k)$

Consider $W = \text{span}\{e_i \mid i \in I\}$

$PAP^{-1} e_i$ = i^{th} column of PAP^{-1}

For $i \in I$, it gives i^{th} column of E appended with 0's.

Clearly $PAP^{-1} e_j \in \text{span}\{e_i \mid i \in I\}$ $\forall j \in I$

$\therefore PAP^{-1}(W) \subseteq W$

$\therefore A(P^{-1}W) = P^{-1}PAP^{-1}(W) \subseteq P^{-1}W$

This is a contradiction to (a)

(b) $\Rightarrow$ (c) :

$A \geq 0 \Rightarrow A^k \geq 0 \quad \forall k$

Suppose $\exists\ i_0, j_0$ st. $A^k_{i_0, j_0} = 0 \quad \forall k = 1, 2, \dots$

Combinatorially, in the digraph on $[n]$ with edge $i \to j$ whenever $a_{ij} > 0$, no directed walks exist from i_0 to j_0

Define $I = \{r \mid \text{there is a directed walk from } r \text{ to } j_0\}$

$J = \{r \mid \text{there is no walk from } r \text{ to } j_0\}$

Observe that if $r \in J$ and $a_{rs} > 0$, then $s \in J$

(if $s \in I$, then we get a walk $r \to s \rightsquigarrow j_0$ contradicting the fact that $r \in J$)

Thus $\forall s \in I, \forall r \in J$ there is no edge $r \to s$

Reorder the labelling of A by a suitable permutation P so that all vertices in I come before J.

Thus $PAP^{-1} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ contradicting (b)

(c) $\Rightarrow$ (a) :

Suppose $I \subsetneq [n]$ st. if $V = \text{span}\{e_i \mid i \in I\}$ then $A(V) \subseteq V$

$\therefore A e_j \in V \quad \forall j \in I$

$\therefore$ Now $Ae_j = j^{th}$ colum of $A \in \text{span}\{e_i \mid i \in I\}$

$\therefore a_{ij} = 0$ whenever $j \in I$, $i \notin I$

We claim that $A^k(i, j) = 0 \quad \forall j \in I, \ i \notin I$

It is trivially true for $k = 1$

Suppose it holds true for powers $1, 2, \dots, k$

$A^{k+1} = A^k A$

$\therefore A^{k+1}(i,j) = \sum_{t=1}^{n} A^k_{it} A_{tj}$

Now $i \notin I$ & $j \in I$

If $t \in I$, $A^k_{it} = 0$ by induction hypothesis

If $t \notin I$, $A_{tj} = a_{tj} = 0$

$\therefore A^{k+1}_{ij} = 0$

This proves the claim which contradicts (c) ∎

Definition: A matrix $A \geq 0$ is called <u>irreducible</u> if it satisfies any one of the equivalent statements above

Proposition 7

If T is irreducible, then $I+T$ is primitive

proof

Obtain k_{ij} for every $(i,j) \in [n]^2$ st. $T^{k_{ij}}_{ij} > 0$

Let $K = \max\{k_{ij} \mid 1 \leq i,j \leq n\}$

$(I+T)^K_{u,v} = \sum_{r=0}^{K} \binom{K}{r} T^r_{u,v}$ (for some $1 \leq u, v \leq n$)

For $r = k_{u,v} \in [0, K]$, $T^r_{u,v} > 0$ and every $T^r_{u,v}$ is anyways ≥ 0

$\therefore (I+T)^K_{u,v} > 0$

This holds $\forall u,v \Rightarrow (I+T)^K > 0$ ∎

Definition: Given $A \geq 0$, we define the <u>period of order i</u> to be the gcd of all powers m such that $A^m_{ii} > 0$

Proposition 8

If A is irreducible, the period is independent of i

Proof

Let $P(i) = \{m \in \mathbb{N}\setminus\{0\} \mid A^m(i,i) > 0\}$

Similarly one defines $P(j)$.

The aim is to show that $\gcd(P(i)) = \gcd(P(j))$

Due to irreducibility, $\exists\ p, q \in \mathbb{N}\setminus\{0\}$ st.

$A^p(i,j) > 0$ and $A^q(j,i) > 0$

Let $m \in P(j)$

$$A^{m+p+q}(i,i) = \sum_{t_1,t_2=1}^{n} A^p(i,t_1)\, A^m(t_1,t_2)\, A^q(t_2,i)$$
$$\geq A^p(i,j)\, A^m(j,j)\, A^q(j,i)$$
$$> 0$$

$\therefore\ m+p+q \in P(i)$

If $\gcd(P(i)) = d_i$, $\gcd(P(j)) = d_j$ then

$d_j \mid m \Rightarrow d_i \mid m+p+q$

Further, $A^{p+q}(i,i) \geq A^p(i,j)\, A^q(j,i) > 0$

$\therefore\ d_i \mid p+q$

$\therefore\ d_i \mid m$

Thus, $d_j \mid m \Rightarrow d_i \mid m$

Similarly $d_i \mid m \Rightarrow d_j \mid m$

Thus $d_i \mid d_j$ & $d_j \mid d_i \Rightarrow d_i = d_j$ ∎

Proposition 9

Every irreducible matrix with period 1 is primitive and conversely, every primitive matrix is irreducible with period 1

Proof

Let A be primitive with $A^k > 0$ for a minimal possible k.

Clearly A is irreducible.

Consider the least m such that $A^m(1,1) > 0$.

Let $P(1) = \{ m \in \mathbb{N}\setminus\{0\} \mid A^m(1,1) > 0 \}$

Then $m \in P(1)$ & $k \in P(1)$

But $k+1, k+2, \dots$ all are part of $P(1)$

$\therefore \gcd(P(1)) = 1$

Conversely, let A be irreducible with period 1

Consider $P(i) = \{ m \in \mathbb{N}\setminus\{0\} \mid A^m(i,i) > 0 \}$

$\gcd(P(1)) = 1$

Let $m_1, m_2 \in P(1)$. Let $\lambda = m_1 m_2 - m_1 - m_2$. This is the Frobenius number of m_1 & m_2

Now $A^{m_1}(i,i) > 0$, $A^{m_2}(i,i) > 0$

and every $N > \lambda$ can be written as $\alpha m_1 + \beta m_2$

and $A^{\alpha m_1 + \beta m_2}(i,i) \geq (A^{m_1}(i,i))^{\alpha} (A^{m_2}(i,i))^{\beta} > 0$

Thus we can obtain $M(i)$ st. $\forall\, n \geq M(i)$, $A^n(i,i) > 0$

By irreducibility of A, obtain $m(i,j)$ st. $A^{m(i,j)}(i,j) > 0$

Thus $\forall\, n \geq M(i)$,

$A^{n+m(i,j)}(i,j) \geq A^n(i,i)\, A^{m(i,j)}(i,j) > 0$

Thus, let $M = \max_{i,j} M(i) + m(i,j)$ so that $\forall\, n \geq M$

$A^n(i,j) > 0$

$\therefore A^n > 0$

We now have the Perron-Frobenius theorem for irreducible matrices of a general period d, which due to the above proposition is a generalisation of Perron's theorem

Theorem 10 (Perron - Frobenius theorem)

Let $T \geq 0$ be irreducible. Then there is a unique positive real number θ_0 with the following properties

(a) $\exists$ real vector $x_0 > 0$ st. $Tx_0 = \theta_0 x_0$

(b) θ_0 has alg mult = geom mult = 1

(c) For every eigenvalue θ of T, $|\theta| \leq \theta_0$. Further, if $|\theta| = \theta_0$, then $\theta = \theta_0 \exp(2\pi i m/d)$ for $m = 0, 1, \dots, d-1$ (that is d of the eigenvalues have modulus as r_0) where d is the period of T

(d) Any non-negative left or right e.vec of T has eval θ_0 :
Let $t \in \mathbb{R}$, $x \geq 0$, $x \neq 0$
If $Tx \leq tx$, then $x > 0$, $t \geq \theta_0$. Further $t = \theta_0$ iff $Tx = tx$
If $Tx \geq tx$, then $t \leq \theta_0$. Further, $t = \theta_0$ iff $Tx = tx$

(e) If $0 \leq S \leq T$ or if S is a principal minor of T, and S has eval σ then $|\sigma| \leq \theta_0$. $|\sigma| = \theta_0$ occurs only if $S = T$

(f) Given a complex matrix S, let $|S|$ denote the matrix obtained by S by taking modulus of every entry of S. If $|S| \leq T$ and S has eval σ, then $|\sigma| \leq \theta_0$ with equality only if the following hold

(i) $|S| = T$

(ii) $\exists$ diagonal matrix E with $|E| = I$ and $\exists\, z$ with $|z| = 1$ so that $S = zETE^{-1}$

Proof

(a) Let $P = (I+T)^k$ for some k s.t. $P > 0$ (using prop 7). Also note $PT = TP$

Let $B = \{x \mid x \geq 0, x \neq 0\}$

For $x \in B$, define a function $\theta : B \to \mathbb{R}$ as

$$\theta(x) = \max\{t \in \mathbb{R} \mid tx \leq Tx\}$$
$$= \max\{t \in \mathbb{R} \mid t_i \leq \frac{(Tx)_i}{x_i}, x_i \neq 0\}$$
$$= \min_{x_i \neq 0} \frac{(Tx)_i}{x_i}$$

claim 1 : $\theta(\alpha x) = \theta(x) \quad \forall \alpha \in \mathbb{R}, \alpha > 0$

proof : $\theta(\alpha x) = \min_{\alpha x_i \neq 0} \frac{(T(\alpha x))_i}{(\alpha x)_i} = \min_{x_i \neq 0} \frac{(Tx)_i}{x_i}$

claim 2 : $\theta(Px) \geq \theta(x)$ with equality iff x is an eigenvector of T

proof : $\theta(x)\, x \leq Tx$

Thus $P(\theta(x)\, x) \leq PTx = TPx$

$\therefore \theta(x)\, Px \leq T\, Px$

Now $\theta(Px) = \min_{(Px)_i \neq 0} \frac{(TPx)_i}{(Px)_i} \geq \min_{(Px)_i \neq 0} \frac{\theta(x)\cancel{(Px)_i}}{\cancel{(Px)_i}}$

The equality holds iff $Tx = \theta(x)\, x$ i.e. x is an eigenvector of T

Now define $C = \{x \in B \mid \|x\| = 1\} \subseteq B$

claim 3 : $\theta(x)$ is continuous on $P(C)$ but not on C

proof : Well for every $x \geq 0$, $Px > 0$ since $P > 0$.

Thus θ is continuous on $P(C)$ but some x_i may be 0 on C and hence we get jumps. Try the example of $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ at $(1,0)$

C is compact $\Rightarrow$ $P(C)$ is compact $\Rightarrow$ θ attains maximum on $P(C)$, at, say x_0

Now $\theta_0 := \sup_{x \in B} \theta(x)$

$$= \sup_{x \in C} \theta(x) \quad (\because \text{by claim 1})$$
$$= \sup_{x \in P(C)} \theta(x) \quad (\because \text{by claim 2})$$
$$= \theta(x_0)$$

$x_0 > 0$ and x_0 is an eigenvector of T with $Tx_0 = \theta_0 x_0$

(b) It suffices to show that the algebraic multiplicity is 1 since $x_0 > 0 \Rightarrow 1 \leq \text{geom mult} \leq \text{alg. mult} = 1$

Unfortunately we need to use (e) to prove alg. mult = 1 and to prove (e) we need geom. mult = 1.
Thus we first show that geom mult = 1 & give the proof that alg. mult = 1 as (b′) after (f)
Claim : If $z \in B$ such that $Tz \geq \theta_0 z$, then z is an eigenvector of T and $z > 0$
proof : $Tz \geq \theta_0 z$
By definition, $\theta(z) \geq \theta_0$ but since θ_0 is the supremum of B, $\theta(z) = \theta_0 \Rightarrow Tz = \theta_0 z$
Now $P > 0$, $z \geq 0 \Rightarrow Pz > 0$
But $Pz = (I+T)^k z = (1+\theta_0)^k z > 0$
Thus $z > 0$. This proves our claim
Suppose $Tx = \theta_0 x$. We show $\lambda x = x_0$ for some $\lambda \neq 0$
Note that $x_0 > 0$ from (a). Choose λ so that $y = x_0 + \lambda x \geq 0$ but $y \not> 0$. We have $y \in B$
Now $Ty = Tx_0 + \lambda Tx = \theta_0 (x_0 + \lambda x) = \theta_0 y$
In particular $Ty \geq \theta_0 y \Rightarrow y > 0$ by claim which is a contradiction. The only possible choice is $y = 0 \Rightarrow -\lambda x = x_0$

(c) Let $Tx = \theta x$. Define $|x| = (|x_1|, \dots, |x_n|) \in \mathbb{R}^n$
$T|x| \geq |Tx| = |\theta x| = |\theta|\,|x|$
By definition $\theta(|x|) \geq |\theta|$
But $|x| \geq 0$ & θ achieves maximum at x_0 & it is θ_0
$\therefore |\theta| \leq \theta(|x|) \leq \theta_0 \Rightarrow |\theta| \leq \theta_0$
Now suppose $Tx = \theta x$ with $|\theta| = \theta_0$
This means $|\theta| = \theta(|x|) = \theta_0$
In particular, equality is achieved in the triangle inequality $T|x| \geq |Tx|$ (i.e. $|\sum_{j=1}^n t_{ij} x_j| \leq \sum_{j=1}^n t_{ij} |x_j|$)
(Note that $t_{ij} \geq 0$)
Thus every $t_{ij} x_j$ has the same argument (i.e. they all lie on the same line in $\mathbb{C}$)
Now T has period d. Thus its underlying directed graph has every cycle length some multiple of d with d being gcd of all cycle lengths
Fix a vertex i_0 in the graph and define the sets
$V_k = \{ j \mid \text{there is a walk from } i_0 \text{ to } j \text{ of length } k \pmod d \}$

Define $L_j = \{ l \in \mathbb{N} \mid$ there is a walk of length l from i_0 to $j \}$

L_j is non-empty since T is irreducible

Let $l_1, l_2 \in L_j$

Suppose we have a walk of length m from j to i_0.

Then we get cycles (at i_0) of lengths $l_1 + m$ & $l_2 + m$.

Thus $d \mid l_1 + m$, $d \mid l_2 + m \Rightarrow l_1 \equiv l_2 \pmod d$

Since $d = \gcd$ (cycle lengths), there is a cycle of length λd (some $\lambda \geq 1$) and hence, we have walks of length $1, 2, \dots, d$ starting from i_0.

This shows that every $n \in \mathbb{N}$ lies in some L_j & $\forall j, L_j \neq \emptyset$

Now observe that $V_k = \{ j \mid L_j \subseteq k + d\mathbb{Z} \}$

Clearly, every index j satisfies $j \in V_k$ for a unique k and every V_k is non-empty since every $n \in \mathbb{N}$ is is some L_j

Now we claim that the V_k are disjoint.

Suppose $j \in V_{k_1} \cap V_{k_2}$

Then $L_j \subseteq k_1 + d\mathbb{Z}$ & $L_j \subseteq k_2 + d\mathbb{Z}$ but for a fixed j, L_j determines the k uniquely (everything else in L_j is a multiple of $k \Rightarrow k_1 = k_2$

Thus $V(G) = V_0 \sqcup V_1 \sqcup \dots \sqcup V_{d-1}$

Suppose there is an edge $i \to j$ with $i \in V_k$ $(0 \leq k < d)$

This means that there is a path of length k from i_0 to i

Thus we get a path of length $k+1$ from $i_0 \to j$ showing that $j \in V_{k+1}$ ($k+1$ is taken mod d)

Thus, if we order the vertices according to $V_0 < \dots < V_{d-1}$, the matrix T now becomes $P^{-1} T P$ (P is some perm. matrix) given by

$$P^{-1} T P = \begin{bmatrix} 0 & T^{(0,1)} & 0 & \cdots & 0 \\ 0 & 0 & T^{(1,2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{(d-1,0)} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\therefore P^{-1} T^d P = \operatorname{diag}(B^0, \dots, B^{d-1})$ for some blocks B_i

[In fact $B_i = T^{(i,i+1)} T^{(i+1,i+2)} \cdots (T^{i-1,i})$ (indices mod d)]

Let $P^{-1} T P = \tilde{T}$. Eigenvalues of T and $\tilde{T}$ are same.

Let x be an eigenvector of $\tilde{T}$. Decompose x as $x = (x^{(0)}, \dots, x^{(d-1)})$ according to column sizes of $T^{(i,i+1)}$

Let the eigenvalue be θ

Then $(x^{(0)}, \zeta x^{(1)}, \zeta^2 x^{(2)}, \dots, \zeta^{d-1} x^{(d-1)})$ is also an eigenvector of $\tilde{T}$ with eigenvalue ζ for some d^{th} root of unity ζ

This shows that the spectrum of T is invariant under d^{th} rotations of $\mathbb{C}$

This gives the existence of at least d θ s.t. $|\theta| = \theta_0$ (these are $\theta_0, \zeta\theta_0, \zeta^2\theta_0, \dots, \zeta^{d-1}\theta_0$ for $\zeta = e^{2\pi i/d}$)

Now we show that these are all.

Recall the blocks B_k of $P^{-1}T^d P$.

$B_k = T^{(k,k+1)} T^{(k+1,k+2)} \dots T^{(k+d-1,k)}$ (indices mod d)

A path of length m from a vertex $i \in V_k$ ends up in V_{k+m} ($k+m$ taken mod d) & hence within the same set V_k, all paths $i \to j$ have length divisible by d ie. in the graph, $\exists$ m so that there is a walk of length md from i to j

This shows that $(B_k)^{md}_{ij} > 0$ by property of edges going from V_i to V_{i+1}

Thus all B_k are primitive

Since all edges $i \to j$ for $i, j \in V_k$ have length d, every cycle in B_k has length divisible by d and hence gcd (all cycle lengths) $= 1$

Hence every block has period 1

This shows that the B_k are primitive.

Now if we have a primitive matrix T with $T^\alpha > 0$, replacing T by T^α in the initial stages gives that all x_j have the same angular part (all $t_{ij} > 0$)

Consequently, $x = e^{i\beta} y$ for some β & some $y \geq 0$

Thus $Ty = \theta y \Rightarrow \theta$ is real

Thus $|\theta| = \theta_0 \Rightarrow \theta = \theta_0$ is unique.

This finishes the $d = 1$ part (primitive case)

(We couldve used theorem 1 itself but we'd rather not!)

Now each B_k is primitive and hence has exactly one eigenvalue of maximum modulus.

The spectrum of a block diagonal matrix is a multiset union of spectrum of each block

But each block has the same spectrum since each block is a permutation of $T^{(0,1)} T^{(1,2)} \cdots T^{(d-1,0)}$ and AB & BA have the same spectrum.

Let Ω be the largest (unique) e.val of a block.

Then this eigenvalue occurs d times in the spectrum of $P^{-1}T^d P$ and every other eigenvalue of $P^{-1}T^d P$ is smaller than Ω by modulus

But T^d has largest eval θ_0^d & hence $\Omega = \theta_0^d$.

Thus T^d & $P^{-1}T^d P$ have d largest eigenvalues given by θ_0^d & hence

Now if θ is an eigenvalue of T such that $|\theta| = \theta_0$ then θ^d is an eval of $P^{-1}T^d P$ ($\& \, T^d$) but $|\theta^d| = \theta_0^d$ and hence there can only be d many θ.

This proves (c).

(d) Repeating items (a), (b), (c) with row vectors, we get real $\eta_0 > 0$ such that $\exists \, y_0 > 0$ with $y_0^t T = \eta_0 y_0^t$, η_0 is the largest by modulus & has alg & geom mult equal to 1

But T & T^t has same spectrum $\Rightarrow \eta_0 = \theta_0$

Thus we get y_0 st $T^t y_0 = \theta_0 y_0$

Assume $Tx \le tx$ for some $x \geq 0$, $x \neq 0$, $t \in \mathbb{R}$

$\therefore \langle y_0, Tx \rangle \le \langle y_0, tx \rangle$ (since $y_0 > 0$)

$\therefore \langle T^t y_0, x \rangle \le t \langle y_0, x \rangle$

$\therefore \theta_0 \langle y_0, x \rangle \le t \langle y_0, x \rangle$

$\therefore \theta_0 \le t$ ($\because \langle y_0, x \rangle > 0$ since $y_0 > 0, x \geq 0, x \neq 0$)

Equality holds iff $\langle y_0, Tx \rangle = \langle y_0, tx \rangle$

ie. $y_0^t (Tx - tx) = 0 \Rightarrow Tx = tx$ ($\because y_0 > 0$)

We are left to show that $x > 0$

Now $0 \le Tx \le tx$, $x \geq 0, x \neq 0 \Rightarrow t \geq 0$

$\therefore 0 < Px < (1+t)^k x \Rightarrow x > 0$

The same is mimicked for the $Tx \geq tx$ case.

(e) Let $0 \le S \le T$

Let $s \neq 0$ be an e.vec of S with eval σ : $Ss = \sigma s$

Consider $s^* = |s|$ (coordinate wise modulus)

$Ts^* \geq S s^* \geq |\sigma| s^*$

$s^* \geq 0$, $s^* \neq 0$, $|\sigma| \in \mathbb{R}$

By part (d), $|\sigma| \le \theta_0$ and equality is achieved iff $Ts^* = |\sigma| s^* \Rightarrow (T-S)s^* = 0$

But $s^* \geq 0 \Rightarrow T - S = 0 \Rightarrow T = S$ (only neccessary)

(f) Let $S_s = \sigma s$ $(s \neq 0)$. Let $s^* = |s|$

$Ts^* \geq |S| s^* \geq |\sigma| s^*$

Again by (d), $|\sigma| \leq \theta_0$ and equality holds iff

$Ts^* = |\sigma| s^* \Rightarrow (T - |S|) s^* = 0 \Rightarrow T = |S|$

Further $|S| s^* = |\sigma| s^* = |\sigma s| = |S s|$

$\therefore \sum |S_{ij}| |s_j| = |\sum S_{ij} s_j|$

Thus all $S_{ij} s_j$ have same argument in $\mathbb{C}$

Let $E_i = \frac{s_i}{|s_i|}$, $c = \frac{\sigma}{|\sigma|}$

Then it is clear that $S_{ij} = c E_{ii} E_{jj}^{-1} |S_{ij}|$

(b') To show that θ_0 has multiplicity 1 in the characteristic polynomial $\det(xI - T)$, we take the derivative of the polynomial and evaluate it at θ_0 to get a non-zero value.

It is a well known result (Jacobi's formula) that

$$\frac{d}{dt}(\det(A(t))) = \text{trace}\left(\text{adj}(A(t)) \frac{d}{dt}(A(t))\right)$$

Here, $(\det(xI - T))' = \text{trace}(\text{adj}(xI - T))$

$= \sum_{j=1}^{n} \det(xI - T^{\{i\}})$

where $T^{\{i\}}$ is T without the j^{th} row & column.

Now, evaluating at θ_0, we get

$\sum_{j=1}^{n} \det(\theta_0 I - T^{\{i\}})$

Obtain matrices $M^{\{i\}}$ as

$$M^{\{i\}}(a,b) = \begin{cases} T^{\{i\}}(a,b) & \text{if } a,b < i \text{ or } a,b > i \\ 0 & a = i \text{ or } b = i \end{cases}$$

(Note that we deleted the i^{th} row & col of T to get $T^{\{i\}}$. Replacing these with 0's instead of deleting, yields $M^{\{i\}}$)

Now $0 \leq M^{\{i\}} \leq T$

By (e), all eigenvalues of $M^{\{i\}}$ have modulus $\leq \theta_0$ and if an eigenvalue has modulus θ_0 then $M^{\{i\}} = T$ which is not true since if T has an entire row and column 0, then it cannot be irreducible.

Now Spectrum of $M^{\{i\}}$ is spectrum of $T^{\{i\}}$ with an extra 0.

Thus, θ_0 is not an eigenvalue of $T^{\{i\}}$.

Hence, $\det(\theta_0 I - T^{\{i\}}) \neq 0$ (no-eval is θ_0)

Further, every eval of $T^{\{i\}}$ is an eval of $M^{\{i\}}$ and hence satisfies $|\lambda| < \theta$

$\therefore$ evals of $\theta_0 I - T^{\{i\}}$ are of the form $\theta_0 - z$ for z such that $|z| < \theta_0$

But the matrix $\theta_0 I - T^{\{i\}}$ is real & hence complex evals occur in pairs.

$(\theta_0 - z)(\overline{\theta_0 - z}) = |\theta_0 - z|^2 > 0$ $(\because |z| < \theta_0)$

and if $z \in \mathbb{R}$, $\theta_0 - z > 0$ holds anyways

Thus, $\det(\theta_0 I - T^{\{i\}}) > 0$

Hence the overall sum is positive & in particular non-zero completing the proof

$\blacksquare$