Definition: Any rectangular array of mathematical objects (reals, complex numbers, functions, etc.) is called a matrix. $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$ denotes an $m \times n$ matrix whose i^{th} row, j^{th} column entry is a_{ij}

Definition: A row vector is a $1 \times m$ matrix and a column vector is an $m \times 1$ matrix

Definition: Two matrices are said to be equal if their corresponding entries are equal

Definition: A matrix B is said to be the transpose of a matrix A if $b_{ij} = a_{ji}$ $\forall$ $1 \le i \le m$, $1 \le j \le n$.

Definition: A matrix is called a square matrix if the number of columns equals the number of rows in it

THEOREM 1

The transpose of a matrix is unique

THEOREM 2

* $(A^t)^t = A$

Definition: A matrix is symmetric if it equals its transpose and skew symmetric if it equals the negative transpose i.e. $a_{ij} = -a_{ji}$ $\forall$ $1 \le i \le m$, $1 \le j \le n$

i.e. $A = A^t$ and $A = -A^t$

THEOREM 3

All symmetric matrices are square

Definition: If A, B are two $m \times n$ matrices, we define their sum to be C, an $m \times n$ matrix whose entries are given by $C = [c_{ij}] = [a_{ij} + b_{ij}]$

Definition: We define multiplication of a matrix A by a scalar λ as $\lambda A = B$ and the entries of matrix B are given by $b_{ij} = \lambda a_{ij}$, $1 \le i \le m$, $1 \le j \le n$

Definition: We define matrix multiplication product of a matrix $A_{m \times n}$, $B_{n \times p}$ to be a $m \times p$ matrix $C_{m \times p}$ s.t. $c_{ij} = \sum_{t=1}^{n} a_{it} b_{tj}$ $\forall$ $1 \le i \le m$, $1 \le j \le p$

THEOREM 4

Matrix multiplication is associative but not commutative and further, matrix multiplication distributes over matrix sum

THEOREM 5

(i) $(A+B)^t = A^t + B^t$

(ii) $(\lambda A)^t = \lambda A^t$

(iii) $(AB)^t = B^t A^t$

Definition: Given v, w column vectors of the same order $n \times 1$, their dot product is defined to be $v \cdot w = \sum_{j=1}^{n} v_j w_j$

THEOREM 6

$v \cdot w = v^t w$ (note: LHS is dot product, RHS is matrix multiplication)

notation: All elements of $\mathbb{R}^n$ shall be treated as column vectors of dimension $n \times 1$

Definition: A map $f : \mathbb{R}^n \to \mathbb{R}$ is said to be linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ where $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{R}^n$.

THEOREM 7 [Structure theorem for linear maps $\mathbb{R}^n \to \mathbb{R}$]

Any linear map $f : \mathbb{R}^n \to \mathbb{R}$ is of the form

$$f(x) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \quad \text{for } x = [x_1 \cdots x_n]^t$$

Definition: A linear system consisting of m linear equations and n variables is defined as

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + \cdots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n &= b_m \end{aligned}$$

This can be realized as $Ax = b$ where $A = [a_{ij}]$, $x = [x_1, \ldots, x_n]^t$, $b = [b_1, \ldots, b_m]^t$

We write $A^+ = [A \mid b]$ and call it the augmented matrix

THEOREM 8

The following three operations generate a new linear system which is equivalent to the original linear system in terms of solutions

(i) multiplying an equation with a non-zero scalar

(ii) interchanging two equations

(iii) multiplying an equation with any scalar and adding it to another equation

THEOREM 9

Each of the above mentioned operation corresponds to an operation on A^+ as follows :

(i) Applying $M_j(c)$ to A^+ for $c \neq 0$

(ii) Applying P_{ij} to A^+

(iii) Applying $E_{jk}(c)$ to A^+ (multiplying k^{th} row of A^+ by c and adding it to the j^{th} row)

Definition : The above 3 operations are called as elementary row transformations

THEOREM 10

- Operation P_{ij} is redundant since

$$P_{ij} = E_{ij}(1)\, E_{ji}(-1)\, M_i(-1)\, E_{ij}(-1)$$

Definition : Row Echelon Form (REF) of a matrix is the matrix obtained after applying ERT's such that each row starts with more no. of zeroes than prev. row

Definition: The first non zero entry in each row is called a pivot of the REF

THEOREM 11

The number of pivots in any REF of a matrix cannot exceed the number of rows of the matrix

THEOREM 12

REF of a matrix is not unique

Definition: Given REF of a matrix A, we define the reduced REF to be the further reduced form of the REF such that each pivot is 1 and every entry above a pivot is a 0

THEOREM 13

RREF of a given matrix is unique

THEOREM 14

A linear system has a unique solution if the no. of pivots equals the number of rows

THEOREM 15

If in a linear system, the no. of pivots is less than the no. of rows, the system has no solutions if at least one b_k corresponding to a non-pivot row is non zero and the system has infinitely many solutions if every b_k corresponding to non-pivot rows is a 0

Definition: $\mathcal{E}_{ij}^{(m)}$ is defined to be the $m \times m$ matrix whose i, j^{th} entry is 1 and the rest of the entries are 0

Definition: The identity matrix of order m is defined to be $I_m := \sum_{t=1}^{m} \mathcal{E}_{tt}^{(m)}$

THEOREM 16

I_m is the identity for matrix multiplication of square matrices of order $m \times m$

Definition: We define elementary row matrices (ERM's) of order $m \times m$ as

(i) $P_{ij} := \sum_{\substack{t=1 \\ t \neq i, t \neq j}}^{m} \mathcal{E}_{tt}^{(m)} + \mathcal{E}_{ij}^{(m)} + \mathcal{E}_{ji}^{(m)}$

(ii) $E_{ij}(c) := I_m + c\,\mathcal{E}_{ij}^{(m)}$

(iii) $M_j(c) := \sum_{\substack{t=1 \\ t \neq j}}^{m} \mathcal{E}_{tt}^{(m)} + c \cdot \mathcal{E}_{jj}^{(m)}$

note: $i \neq j$ throughout

THEOREM 17

The three matrices defined above, when right multiplied by a matrix A, precisely correspond to the action described in theorems 8 and 9

THEOREM 18

$\forall A, \exists E_1, \ldots, E_n$ (ERM's) s.t. $E_1 E_2 \cdots E_n A$ is the RREF of A

THEOREM 19

Given any square matrix A, $\exists$ elementary row matrices $E_1, \dots, E_n$ so that $E_1 \cdots E_n A$ is either the identity matrix or has its last row 0

THEOREM 20

For a given square matrix A of order $m \times m$, a matrix B of order $m \times m$ is said to be its inverse and denoted A^{-1} if $AB = I_m = BA$

THEOREM 21

(i) The inverse is unique if it exists

(ii) $(AB)^{-1} = B^{-1} A^{-1}$ (if B, A are invertible)

(iii) Each ERM is invertible as $P_{ij}^{-1} = P_{ji}$,

$$M_j(c)^{-1} = M_j\left(\tfrac{1}{c}\right), \quad E_{jk}(c)^{-1} = E_{jk}(-c)$$

THEOREM 22

A square matrix is invertible iff it is a product of ERM's

Definition: A variable involved in the linear system described earlier is said to be simple if it corresponds to a pivot column and otherwise, it is called a free variable

THEOREM 23

All simple variables can be expressed in terms of free variables and free var's can be assigned any value to satisfy the linear system

Definition: $\hat{e}_j := [0,0,\dots,1,0,0..0]^t$ and $\hat{f}_k :=$ $[0,0,\dots 1,0,0,\dots 0]$ are the standard referential vectors in $\mathbb{R}^n$

THEOREM 24

Given any $m\times n$ matrix A, $A\hat{e}_j$ is the j^{th} column of A while $\hat{f}_k A$ is the k^{th} row of A

Definition: A subset V of $\mathbb{R}^n$ is called a vector subspace if

(i) V is non empty

(ii) $x, y \in V \Rightarrow ax+by \in V \quad \forall\, a,b \in \mathbb{R}$

THEOREM 25

The only vector subspaces of $\mathbb{R}^3$ are

(i) singleton set containing origin

(ii) lines through origin

(iii) planes through origin

(iv) $\mathbb{R}^3$ itself entirely

Definition: Given vectors $v_1,\dots,v_k \in \mathbb{R}^n$, we define a linear combination of these to be $a_1v_1+\dots+a_kv_k$ for scalars $a_1,\dots,a_k$

Definition: A linear span of a set of vectors $v_1,\dots v_k \in \mathbb{R}^n$ is the set of all possible linear combinations of these vectors

THEOREM 26

linear span of any set of vectors forms a vector subspace

Definition: A set of vectors $\{v_1, \dots, v_k\}$ is said to be linearly independent if $a_1 v_1 + \dots + a_k v_k = 0 \Rightarrow a_i = 0 \; \forall i$

consequently, the set is linearly dependent if $\exists$ non zero a_i (at least one) so that $a_1 v_1 + \dots + a_k v_k = 0$

Convention: $\emptyset$ is linearly independent, $\text{span}\{\emptyset\} := \{0\}$

THEOREM 27

If $0 \in \{v_1, \dots, v_k\} = S$, then S is not LI

THEOREM 28

Subset of LI set is LI; any set containing LD set is LD

Definition: We say that a set of vectors B is the basis for a vector subspace V if

(i) B is LI

(ii) $\text{span}\{B\} = V$

THEOREM 29

Every vector space has a basis and every basis of a vector space has the same number of elements

THEOREM 30

If V is a vector space spanned by k vectors, any set of $k+1$ vectors in V is not LI

Definition: dimension of a vector space is defined to be the number of elements in a basis of that space

THEOREM 31

Any set of vectors $\{v_1, \dots, v_n\}$ s.t. $v_i \in \mathbb{R}^n$ is LI iff $\det [v_1\ v_2\ v_3 \cdots v_n] \neq 0$

NOTE: This course does not cover what 'det' means

THEOREM 32

Let $\{v_1, \dots, v_n\}$ be a set of given vectors. Define $A = [a_{ij}]$ with $a_{ij} = v_i \cdot v_j = v_i^t v_j$

Det (A) is often known as the gram-determinant

The set is LI iff $\det(A) > 0$

Definition: Regarding columns as vectors, the column space of a matrix is the vector space spanned by the columns of the matrix (similarly row space)

Definition: The dimension of the row space is called as row rank and that of the column space is called as column rank

THEOREM 33

For any S such that span (S) = V, we can find $S_0 \subseteq S$ such that S_0 is the basis

THEOREM 34

Performing row operations does not affect the row rank or the column rank (same goes with column operations)

THEOREM 35

Row rank = Column rank for any matrix

Definition: We define the null space of a matrix $A_{m \times n}$ to be $N(A) = NS(A) = \ker(A) = \{v \in \mathbb{R}^n \mid Av = 0\}$

The nullity is the dimension of the null space

Definition: We define pivotal rank of a matrix to be the no. of pivots in that matrix

THEOREM 36

row rank = pivotal rank

Henceforth, we refer to this common value as rank

THEOREM 37 [Rank-nullity theorem]

rank (A) + nullity (A) = no of columns in A

THEOREM 38

For a given solution v_0 of $Av = b$, the set of all solutions is given by $\{v_0 + n \mid n \in NS(A)\}$

THEOREM 39

$\det(A^t) = \det(A)$

THEOREM 40

Let $A_{n \times n}$ be a matrix. If a $k \times k$ submatrix of A has a non-zero det then rank $(A) \geq k$ and the converse also holds ~~for~~ true

Definition: A matrix is said to have a ~~deter~~ determinantal rank k if $\exists$ $k \times k$ submatrix of non zero det s.t. all $(k+1) \times (k+1)$ matrices have zero determinant

THEOREM 41

determinantal rank = rank (A)

THEOREM 42

(i) Performing P_{ij} changes sign of det

(ii) Performing $E_{ij}(k)$ has no effect

(iii) Performing $M_i(c)$ scales the det by $\frac{1}{c}$

THEOREM 43

Suppose $\dim(V) = k$.

(i) If S is LI having k vectors, S is a basis

(ii) If S is s.t. $\mathrm{span}(S) = V$, $|S| = k$, S is a basis

THEOREM 44

TFAE for a square matrix

(i) A is invertible

(ii) rows of A are LI

(iii) columns of A are LI

(iv) $\det(A) \neq 0$

(v) $\mathrm{rank}(A) = n$

(vi) $\mathrm{nullity}(A) = 0$

(vii) A has n pivots

THEOREM 45 [Kronecker-Capelli]

$Ax = b$ has a solution iff $\text{rank}(A) = \text{rank}(A^+)$

Definition: The determinant of a matrix is a function $\det : \mathbb{R}^n \times \mathbb{R}^n \cdots \times \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\det(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n) = 1$

(ii) $\det(c_1, \ldots, c_i, \ldots, c_j, \ldots c_n) = -\det(c_1, \ldots, c_j, \ldots, c_i, \ldots c_n)$

(iii) $\det(\alpha c_1' + \beta c_1'', c_2, \ldots, c_n)$

$$= \alpha \det(c_1', c_2, \ldots, c_n) + \beta \det(c_1'', c_2, c_3, \ldots, c_n)$$

THEOREM 46

The det function above is unique

THEOREM 47

TFAE for $f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ (a multilinear function)

(1) f is skew symmetric

(ii) $f(c_1, \ldots, c_i, \ldots c_j, \ldots c_n) = 0$ whenever $c_i = c_j$

(iii) $f(c_1, \ldots, c_i, c_{i+1}, \ldots c_n) = 0$ whenever $c_i = c_{i+1}$

Definition:

Given a_{ij} element, the determinant of the $(n-1) \times (n-1)$ matrix left after deleting the i^{th} row and j^{th} column from the $n \times n$ matrix A of which a_{ij} was an element, is denoted M_{ij} and called as the minor of a_{ij}

Definition:

The cofactor of $a_{ij} \in A_{n \times n}$ is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}$$

THEOREM 48

For an $n \times n$ matrix A,

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij}$$

Definition: Given a $n \times n$ matrix A, we define the adjugate matrix. $\operatorname{adj}(A)$ as $[A_{ji}]$ where A_{ij} is the cofactor of a_{ij}

THEOREM 49

For any $n \times n$ matrix A, $A \cdot \operatorname{adj}(A) = \det(A) \cdot I_n$ and in particular if $\det(A) \neq 0$, $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

THEOREM 50

$\det(AB) = \det(A)\det(B)$ provided A, B are square and of same size

THEOREM 51

If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a $2n \times 2n$ matrix where A, B, C, D are themselves $n \times n$ matrices, then,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} M & P \\ N & Q \end{bmatrix} = \begin{bmatrix} AM + BN & AP + BQ \\ CM + DN & CP + DQ \end{bmatrix}$$

THEOREM 52

The determinant of a skew symmetric matrix of odd order is zero ie. if $A_{2n+1 \times 2n+1}$ is such that $A = -A^t$, then $\det(A) = 0$

THEOREM 53

$\det \begin{bmatrix} I & P \\ 0 & Q \end{bmatrix} = \det Q$ where all are $n \times n$ blocks

THEOREM 54

If A is a skew symmetric matrix of odd order,

$\det \begin{bmatrix} I & A \\ -A & I \end{bmatrix} = \det (I + A^2)$ where A is $n \times n$

and I is also $n \times n$ (n is odd)

THEOREM 55

Let $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ and A be a $2n \times 2n$ matrix

s.t. $A^t J A = J$. Then $\det A = \pm 1$. In fact

$\det(A) = +1$ (showing $\det A \neq -1$ is very difficult)

THEOREM 56 (determinants of block matrices)

(i) $\det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det(A) \det(D) = \det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$

(ii) $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)$ assuming

that A is invertible

Definition: For a given vector space V, we define an

inner product $(,)$ on V that maps $(,):$

$V \times V \to \mathbb{R}$ such that

(i) $(x, y) = \overline{(y, x)}$

(ii) $(ax + by, z) = a(x, z) + b(y, z)$

(iii) $(x, x) \geq 0$ with equality iff $x = 0$

Definition: The standard vector inner product is

defined as $((v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)) := \sum_{i=1}^{n} v_i \overline{w_i}$

Definition: $v_1, v_2 \in V$ are orthogonal if $(v_1, v_2) = 0$ and $v \in V$ is a unit vector if $(v, v) = 1$.

Definition: We define a norm on the vector space as $\|\cdot\| : V \to \mathbb{R}$ as $\|v\| := \sqrt{(v,v)}$

THEOREM 57 (Parallelogram law)

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2(\|v_1\|^2 + \|v_2\|^2)$$

THEOREM 58 (Pythagoras theorem)

$$(v_1, v_2) = 0 \Rightarrow \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$$

THEOREM 59 (Cauchy - Schwarz) ☆

$|(v_1, v_2)| \leq \|v_1\| \cdot \|v_2\|$ with equality iff v_1, v_2 are linearly dependent

THEOREM 60

If a set of vectors in mutually orthogonal, they are linearly independent

Definition: Angle between two vectors v, w is defined to be the unique $\theta \in [0, \pi]$ so that $\cos\theta = \left(\frac{v}{\|v\|}, \frac{w}{\|w\|}\right)$

THEOREM 61

Given any vector space & an inner product, we can always find an orthonormal basis for it (orthogonal, unit vectors)

THEOREM 62 (Gram - Schmidt process)

Let $\{v_1, \dots, v_k\}$ be a set of non zero vectors in an inner product space V.

Construct a sequence of vectors $\{w_n\}_{n=1}^{r}$ such that

$$w_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = \frac{v_2 - (v_2, w_1) w_1}{\|v_2 - (v_2, w_1) w_1\|}$$

$$\vdots$$

$$w_r = \frac{v_r - (v_r, w_1) w_1 - \dots - (v_r, w_{r-1}) w_{r-1}}{\|v_r - (v_r, w_1) w_1 - \dots - (v_r, w_{r-1}) w_{r-1}\|}$$

where r is such that $\forall\, y \geq r+1$, $w_y = 0$

Then we have the following claims :

(i) $\operatorname{span}(v_1, \dots, v_t) = \operatorname{span}(w_1, \dots, w_t)$

$\forall\, t = 1, 2, \dots, r$

(ii) $\{w_1, \dots, w_r\}$ is an orthonormal system

(iii) If $r = k$, $\{v_1, \dots, v_k\}$ are LI and if $r < k$, they are LD

Definition: A real symmetric matrix A is said to be non negative if $u^t A u \geq 0 \ \forall\, u \in \mathbb{R}^n$ and is said to be positive definite if $u^t A u > 0 \ \forall\, u \in \mathbb{R}^n \setminus \{0\}$

THEOREM 63

$\langle u, v \rangle = u^t A v$ defines an inner product if A is a positive definite matrix

THEOREM 64

(i) A, B are pos def $\Rightarrow$ $\alpha A + \beta B$ is pos def for $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$ or whatever the field is)

(ii) A is pos def, P is non singular $\Rightarrow$ $P^t A P$ is positive definite

Definition: We define the trace of a $n \times n$ matrix A to be $tr(A) = \sum_{t=1}^{n} a_{tt}$

Definition: The adjoint of a matrix is A^* and is given by $[\overline{a_{ji}}]$ (conjugate transpose)

Definition: A is said to be self-adjoint or hermitian if $A = A^*$ and skew hermitian if $A = -A^*$

THEOREM 65

Any matrix A can be expressed as $A = B + iC$ where B is hermitian and C is skew-hermitian. Precisely,

$B = \frac{1}{2}(A + A^*)$, $C = \frac{1}{2i}(A - A^*)$

Definition: A matrix A is said to be normal if $AA^* = A^*A$, unitary if $AA^* = I = A^*A$

Definition: For a $n \times n$ matrix A, a non zero vector v is said to be its eigen vector if $Av = \lambda v$ for some scalar λ and the scalar λ corresponding to the vector v is called the eigenvalue

THEOREM 66

All eigen values are roots of $\det(A - xI) = 0$. This polynomial is called the characteristic polynomial

THEOREM 67

Eigenvectors corresponding to distinct eigenvalues are linearly independent

Definition: $A_{n \times n}$ is said to be diagonalisable if there is a basis of $\mathbb{R}^n$ consisting of eigen vectors of A

Definition: A $k \times k$ principal minor is a $k \times k$ sub determinant such that diagonal of the submatrix is a part (not necessarily contiguous) of the main diag

THEOREM 68

all eigenvalues distinct $\Rightarrow$ diagonalisable

THEOREM 69

In the equation $\det(xI - A) = 0$, the coefficient of x^k is $(-1)^{n-k} P_{n-k}$ where P_j = sum of e.vals taken j at a time

THEOREM 70

If A has e·vals $\lambda_1, \dots, \lambda_n$ listed with multiplicities and $g(A)$ is a polynomial in A, then,

$g(\lambda_1), g(\lambda_2), \dots, g(\lambda_n)$ are e·vals of $g(A)$ and consequently, $\det(g(A)) = g(\lambda_1) \cdots g(\lambda_n)$

Definition: A, B are similar if $\exists P$ s.t. $P^{-1}AP = B$

THEOREM 71

Similar matrices have same characteristic equation

THEOREM 72

For a diagonalisable matrix A, if $P = [v_1, \dots v_n]$ is an $n \times n$ matrix of the LI eigenvectors of A with e·vals $\lambda_1, \lambda_2, \dots, \lambda_n$, then,

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

Definition: Algebraic multiplicity of an eigenvalue λ is the highest power of $(x-\lambda)$ which divides the characteristic polynomial

Definition: Geometric multiplicity of an eigenvalue λ is the nullity of $A - \lambda I$

THEOREM 73

$GM(\lambda) \leq AM(\lambda)$ with equality iff A is diagonalisable (holds for any e·val λ)

THEOREM 74 (Cayley Hamilton theorem)

Every matrix satisfies its own characteristic polynomial

THEOREM 75 (Spectral theorem)

Hermitian matrices are unitarily diagonalisable

i.e. $\exists\ B$ s.t. $B^* A B = \text{diag}(\lambda_1, \ldots, \lambda_n)$

and B is unitary i.e. $B^* B = I$

Moreover, $\lambda_i \in \mathbb{R} \quad \forall\ i = 1, 2, \ldots, n$

THEOREM 76

~~AB = BA~~ Let A, B be hermitian.

$AB = BA$ iff $\exists$ common unitary matrix U which diagonalises both A, B

corollary: commuting hermitian matrices have a common eigen vector

THEOREM 77

A is normal iff A is diagonalisable

THEOREM 78 (Schur's theorem)

Given any $A_{n\times n}$ $\exists\ U$ (unitary) s.t. $U^* A U$ is upper triangular

THEOREM 79 (Cauchy-Binet)

$A_{k\times n}$, $B_{n\times k}$ are matrices. $\det(AB)$ is the sum of $k\times k$ principal minors of BA $(k \leq n)$

THEOREM 80 (Bessel's ineq, Parseval's formula)

Let $S = \{w_1, \dots, w_k\}$ be an orthonormal set.

$\forall$ vectors v, $\|v\|^2 \geq \sum_{i=1}^{k} |(v, w_i)|^2$ with equality iff $v \in \text{span}(S)$

THEOREM 81 (Least square approximation)

Let W be a strict subspace of $\mathbb{R}^n$ of dimension k $(k < n)$. Let $b \in \mathbb{R}^n$. The point $w \in W$ such that $\|b - w\|$ is the least is given by $w = \sum_{i=1}^{k} (b, v_i) v_i$ where $\{v_i\}$ is an orthonormal basis of W

THEOREM 82 (Gram Schmidt and Grammian)

Recall theorems 32 and 62. Slightly perturbing the notation,

$$v_k^N = \text{component of } v \text{ normal to span}\{v_1, \dots, v_{k-1}\}$$

$$= v_k - (v_k, w_1)w_1 - (v_k, w_2)w_2 - \dots - (v_k, w_{k-1})w_{k-1}$$

$G(v_1, \dots, v_k) = \det(A)$ where $a_{ij} = (v_i, v_j)$

We have, $$\|v_k^N\|^2 = \frac{G(v_1, \dots v_k)}{G(v_1, \dots, v_{k-1})}$$

THEOREM 83 (Hadamard's inequality)

Let A be an $n \times n$ matrix and v_i the vector corresponding to the i^{th} column i.e. $v_i = Ae_i$

Then $(\det(A))^2 \leq \|v_1\|^2 \cdot \|v_2\|^2 \cdot \ldots \cdot \|v_n\|^2$

TUTORIAL 1

1) Show that every square matrix A can be written as a sum of a symmetric and skew symmetric matrix in a unique way

Ans $A = \left(\frac{A + A^t}{2}\right) + \left(\frac{A - A^t}{2}\right)$ where

$S = \frac{A + A^t}{2}$, $T = \frac{A - A^t}{2}$ are the required sym & skew-sym matrices

Further, if $A = S_1 + T_1 = S_2 + T_2$

then $S_1 - S_2 = T_2 - T_1$ — ①

$\therefore S_1^t - S_2^t = T_2^t - T_1^t$

$\therefore S_1 - S_2 = -T_2 + T_1 = T_1 - T_2$ — ②

from ① and ②, $S_1 = S_2$, $T_1 = T_2$

2) Prove for square matrices A, B:

(i) A, B are symm $\Rightarrow \alpha A + \beta B$ is symm

(ii) " skew symm $\Rightarrow$ " skew symm

(iii) " upper Δ $\Rightarrow$ " upper Δ

Ans (i) $A = A^t$, $B = B^t$

$$(\alpha A + \beta B)^t = (\alpha A)^t + (\beta B)^t$$
$$= \alpha A^t + \beta B^t$$
$$= \alpha A + \beta B$$

(ii) $$(\alpha A + \beta B)^t = \alpha A^t + \beta B^t$$
$$= \alpha(-A) + \beta(-B)$$
$$= -\alpha A - \beta B$$
$$= -(\alpha A + \beta B)$$

(iii) A is upper Δ

$\Rightarrow$ $a_{ij} = 0 \quad \forall \; i < j$

$\therefore \alpha a_{ij} = 0 \quad \forall \; i < j$

$\therefore \alpha A$ is upper Δ

Ills βB is upper Δ

$\therefore \alpha a_{ij} + \beta b_{ij} = 0 \quad \forall \; i < j$

$\therefore \alpha A + \beta B$ is upper Δ

3) A, B are symmetric of same size. Prove that AB is symmetric iff $AB = BA$

Ans $(AB)^t = AB$

iff $B^t A^t = AB$

iff $BA = AB$

4) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a map from $\mathbb{R}$ to $\mathbb{R}^2$. Show that its range is a line through 0. Similarly find range of $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^3$

Ans $\begin{bmatrix} 1 \\ -1 \end{bmatrix} [x] = \begin{bmatrix} x \\ -x \end{bmatrix}$.

$\begin{bmatrix} x \\ -x \end{bmatrix}$ ~~[illegible]~~ is a line through 0

with the equation $x = -y$

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u-v \\ -u+2v \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\begin{bmatrix} u-v \\ -u+2v \\ v \end{bmatrix}$ is a plane given by

$$x + y = z \quad \text{(eliminate } u, v\text{)}$$

Another way:

$$\begin{bmatrix} u-v \\ -u+2v \\ v \end{bmatrix} = u\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

The RHS represents the plane generated by vectors $\begin{matrix} 1 \\ -1 \\ 0 \end{matrix}$ and $\begin{matrix} -1 \\ 2 \\ 1 \end{matrix}$

5) Find images of the unit square, unit circle, unit disc under:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Ans (i) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

unit square: $0 \le x \le 1$, $0 \le y \le 1$

$$\left.\begin{matrix} \therefore \; 0 \le x+y \le 2 \\ 0 \le y \le 1 \end{matrix}\right\} \text{parallelogram}$$

(why not square? Think!)

unit circle :

$$x^2+y^2 = 1$$

$$\begin{aligned} X &= x+y \\ Y &= y \end{aligned} \implies \begin{aligned} x &= X - Y \\ y &= Y \end{aligned}$$

$$\therefore \quad X^2 + Y^2 - 2XY + Y^2 = 1$$

(disc is similar)

(ii) $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -x+y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

unit square : $0 \le x \le 1$

$0 \le y \le 1$

$\therefore \quad 0 \le -X \le 1$

$0 \le Y - X \le 1$

$\therefore \quad -1 \le X \le 0$

$X \le Y \le 1+X$

unit circle $\quad x^2 + y^2 = 1$

$\therefore \quad (-X)^2 + (Y-X)^2 = 1$

$\therefore \quad 2X^2 - 2XY + Y^2 = 1$

(disc is similar)

(iii) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$:

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -3y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$x \sim X/2$

$y \sim \frac{Y}{-3}$

~~unit d~~

unit square:

$0 \le x \le 1$

and $0 \le y \le 1$

$\therefore$ $0 \le X \le 2$

$-3 \le Y \le 0$

unit circle : $\left(\frac{X}{2}\right)^2 + \left(\frac{Y}{-3}\right)^2 = 1$

(disc is similar)

(iv) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x + y = X$

$0 = Y$

$\left.\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array}\right\} \Rightarrow 0 \le x+y \le 2$

$\Rightarrow 0 \le X \le 2$

$\therefore$ segment $[0, 2]$ on X-axis

For the unit circle

$x+y$ varies b/w $-\sqrt{2}$ to $\sqrt{2}$

$\therefore$ $[-\sqrt{2}, \sqrt{2}]$ on X-axis

disc is same as above

(v) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$:

$x + y = X$

$x + y = Y$

unit sq : $\min(x+y) = 0$ $\max(x+y) = 1$

$\therefore$ $[0, 1]$ on line $x = y$

unit circle/disc : $\min(x+y) = -\sqrt{2}$ $\max(x+y) = \sqrt{2}$

$\therefore$ $\{-\sqrt{2}, +\sqrt{2}\}$ on $x = y$

(vi) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

unit sq : y varies from 0 to 1

$\therefore [0,1]$ along x axis

unit circle or disc : y varies from -1 to 1

$\therefore [-1,1]$ along x axis

6) Find inverses using ERO's

(i) $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -x & e^x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

Ans (i) $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & -3 \\ 0 & 1 & -4 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \end{bmatrix}$

$\xrightarrow{R_3 = -\frac{1}{7}R_3} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 7 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 = R_2 + 3R_3 \\ R_2 = -R_2]{R_1 = R_1 - 7R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying all of these on I_3

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

$\xrightarrow{R_3 = -\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ +\frac{2}{7} & -\frac{1}{7} & -\frac{1}{7} \end{bmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{bmatrix} 7 & -3 & 0 \\ -2 & 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} & -\frac{1}{7} \end{bmatrix} \xrightarrow[R_2 = R_2 + 3R_3]{R_1 = R_1 - 7R_3} \begin{bmatrix} -\frac{13}{7} & \frac{10}{7} & -\frac{1}{7} \\ 8/7 & -4/7 & 3/7 \\ \frac{2}{7} & -\frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

Others can be done similarly.

$$\begin{bmatrix} 1 & -x & e^x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x & -x^2-e^x \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^{-1} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

7) Find the last row in the inverses of

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix}$

Ans

$$\begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix} \xrightarrow[R_3+7R_1]{R_3=} \begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow[R_2-8R_1]{R_2=} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -8 \\ 0 & 3 & 8 \end{bmatrix}$$

$$\xrightarrow[=R_3-3R_2]{R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -8 \\ 0 & 0 & 32 \end{bmatrix} \xrightarrow[=\frac{1}{32}R_3]{R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply to I_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 31 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 31/32 & -3/32 & 1/32 \end{bmatrix}$$

(argue why this works with yourself)

$$\cancel{\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftarrow \frac{2}{3}R_3} \begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -\frac{5}{3} & 0 & 1 \end{bmatrix} \xrightarrow{R_4+\frac{5}{3}R_3} \begin{bmatrix} 2 & 0 & -1 & 4 \\ 8 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ & & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{stuff}]{\text{usual}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{becomes}} \begin{bmatrix} -1 & 2 & -1 & 1 \end{bmatrix}$$

(from I_4)

(I don't see a Shorter way)

8) A markov matrix is $M_{n \times n}$ s.t. $m_{ij} \geq 0$ and $\sum_{j=1}^{n} m_{ij} = 1 \ \forall i$. Prove that product of markov matrices is a markov matrix

Ans Let $C = AB$ for A, B markov

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} \sum_{t=1}^{n} a_{it} b_{tj}$$

$$= \sum_{t=1}^{n} \sum_{j=1}^{n} a_{it} b_{tj}$$

$$= \left(\sum_{t=1}^{n} a_{it} \left(\sum_{j=1}^{n} b_{tj} \right) \right)$$

$$= \sum_{t=1}^{n} a_{it} \times 1 = 1$$

9) <simple grade 10 matrix multiplication>

10) Let A be a square matrix. Prove that there are ERM's s.t. their product together is E and EA is either identity or has last row zero

Ans If A is invertible, we have all pivots in place and ~~that~~ ~~two~~ EA is identity if EA is not the identity for any possible product of ERM's E, then A is not invertible and hence has lesser than n pivots.

Thus, we can exchange ~~pivots~~ rows to leave the last row pivot free.

Using above pivots, first m entries can be made zero.

Converting to RREF will give last row zero.

11) List all possibilities for RREF having exactly 1 pivot (4×4 matrix)

Ans $\begin{bmatrix} \boxed{1} & * & * & * \\ & & 0 & \end{bmatrix}$ $\begin{bmatrix} 0 & \boxed{1} & * & * \\ & & 0 & \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & \boxed{1} & * \\ & & 0 & \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & \boxed{1} \\ & & 0 & \end{bmatrix}$

hypothesis : $n \times n$ ~~pivot~~ matrix, m pivots has $\binom{n}{m}$ possibilities

(It is true. think why)

12) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Verify that $(A-I)^3 = 0$ and hence the inverse is $A^2 - 3A + 3I$

Ans Check it yourself

$$A^3 - 3A^2 + 3A - I = 0$$

$$\therefore A(A^2 - 3A + 3I) = I$$

13) $X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

(i) Prove $X^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} \quad \forall\ n \geq 1$

(ii) $X^0 := I$. Prove $e^X = e^{\lambda}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(iii) Show that (i) holds $\forall\ n \leq 0$

<u>Ans</u> (i) $X^1 = \begin{bmatrix} \lambda^1 & 1\cdot\lambda^{1-1} \\ 0 & \lambda^1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

which is true (base case ✓)

Assume true for $n = k$

$$X^{k+1} = X^k \cdot X = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{bmatrix}$$

Hence proved

(ii) $e^X = I + X + \frac{X^2}{2!} + \cdots$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \lambda^n & \sum_{n=0}^{\infty} n\lambda^{n-1} \\ 0 & \sum_{n=0}^{\infty} \lambda^n \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix} = e^{\lambda}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(iii) reverse induction OR put $n = -t$

and do normal induction

TUTORIAL 2

1) <Not in syllabus>

2) Find LI or LD

(i) $\{(1,-1,1), (1,1,-1), (0,1,0), (-1,1,1)\}$

(ii) $\{(1,9,9,8), (2,0,0,3), (2,0,0,8)\}$

<u>Ans</u> (i) No! There are 4 vectors in $\mathbb{R}^3$.

Never possible

(ii) $$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 8 & 3 & 8 \end{bmatrix} \xrightarrow{GEM} \begin{bmatrix} \boxed{1} & 2 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix}$$

no of pivots < no of vectors $\Rightarrow$ LD

no of pivots = no of vectors $\Rightarrow$ LI

Here, $3 = 3 \Rightarrow$ LI

3) Find ranks

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$m^2 - n^2 \neq 0$

<u>Ans</u> Applying GEM to each one, we get

$$\begin{bmatrix} \boxed{2} & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{m+n} & m+n \\ 0 & \boxed{m-n} \\ p & p \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{4} & 5 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

rank 1 rank 2 rank 3

(note: $m+n \neq 0$ <u>and</u> $m-n \neq 0$)

4) Solve using GEM

(i)
$$2x_3 - 2x_4 + x_5 = 2$$
$$2x_2 - 8x_3 + 14x_4 - 5x_5 = 2$$
$$x_2 + 3x_3 + x_5 = \alpha$$

(ii)
$$2x_1 - 2x_2 + x_3 + x_4 = 1$$
$$-2x_2 + x_3 + 7x_4 = 0$$
$$3x_1 - x_2 + 4x_3 - 2x_4 = -2$$

Ans

(i)
$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 2 & -8 & 14 & -5 & 2 \\ 0 & 1 & 3 & 0 & 1 & \alpha \end{array}\right]$$

$\downarrow$ GEM (make it RREF)

$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 0 & 3 & -1/2 & \alpha - 3 \\ 0 & 0 & \boxed{1} & -1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 16 - 2\alpha \end{array}\right]$$

$\alpha \neq 8 \Rightarrow$ no solution

$\alpha = 8 \Rightarrow \infty$ solutions

$$x_3 = 1 + x_4 - \frac{x_5}{2}$$

$$x_2 = 5 - 3x_4 + \frac{x_5}{2}$$

x_1, x_4, x_5 arbitrary

(ii) $\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & 7 & 0 \\ 3 & -1 & 4 & -2 & -2 \end{array}\right]$

$\downarrow$ RREF

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -3 & 1/2 \\ 0 & \boxed{1} & 0 & -3 & -1/2 \\ 0 & 0 & \boxed{1} & 1 & -1 \end{array}\right]$$

$\therefore \quad x_3 = -1 - x_4$

$x_2 = -\frac{1}{2} + 3x_4$

$x_1 = \frac{1}{2} + 3x_4$

x_4 is arbitrary

5) (Not in syllabus)

6) Check if solvable using ranks of A, A^+. Write a basis for the solutions and describe the general solution (for homogenous)

(i) $-2x_4 + x_5 = 2$

$2x_2 - 2x_3 + 14x_4 - x_5 = 2$

$2x_2 + 3x_3 + 13x_4 + x_5 = 3$

(ii) $2x_1 - 2x_2 + x_3 + x_4 = 1$

$-2x_2 + x_3 - x_4 = 2$

$x_1 + x_2 + 2x_3 - x_4 = -2$

Ans (i) $\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 2 & 3 & 13 & 1 & 3 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 0 & 0 & 3.3 & 8 \\ 0 & 0 & \boxed{1} & 0 & 0.3 & 0 \\ 0 & 0 & 0 & \boxed{1} & -0.5 & -1 \end{array}\right]$

$\rho(A) = \rho(A^+) = 3 \rightarrow$ solvable

for homogenous, (entire right column 0)

$x_4 = 0.5 x_5$

$x_3 = -0.3 x_5$

$x_2 = -3.3 x_5$

x_1, x_5 arbitrary

$\therefore$ Basis is $\{ e_1, [0, 3.3, 0.3, -0.5, -1] \}$

for $x_1 = 0, x_5 = 0$,

a particular solution is $(0, 8, 0, -1, 0)$

$\therefore$ general $= (0, 8, 0, -1, 0) + \text{span}\{e_1, (0, 3.3, 0.3, -0.5, -1)\}$

(ii) $\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & -2 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 1 & -0.5 \\ 0 & \boxed{1} & 0 & 0 & -1.1 \\ 0 & 0 & \boxed{1} & -1 & -0.2 \end{array}\right]$

$\rho(A) = \rho(A^+) = 3 \Rightarrow$ solvable

for homogenous,

$x_3 = x_4$

$x_2 = 0$

$x_1 = -x_4$

x_4 arbitrary

~~basis is {e...}~~

basis is $\{(-1, 0, 1, 1)\}$

particular solution (at $x_4 = 0$) is $(-0.5, -1.1, -0.2, 0)$

$\therefore$ general is $(-0.5, -1.1, -0.2, 0) + \text{span}\{(-1, 0, 1, 1)\}$

7) Is the given set of vectors a VS

(i) All vectors $[a, b, c]$ s.t.

$3a - 2b + c = 0$

$4a = -5b$

(ii) all $v \in \mathbb{R}^3$ s.t. $\|v\| < 1$

Ans (i) $3a - 2b + c = 0$

$4a + 5b = 0$

$\Rightarrow (a, b, c)$ ~~= ...~~

$= \left(a, -\frac{4a}{5}, -\frac{23a}{5}\right)$

line equation

$\therefore$ Yes

(ii) No!

$0,0,1 \in V$

$1,0,0 \in V$

But $1,0,1 \notin V$

8) For $a < b$, consider

$$x + y + z = 1$$

$$ax + by + 2z = 3$$

$$a^2x + b^2y + 4z = 9$$

Find (a,b) s.t. we have ∞ solutions

<u>Ans</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ a & b & 2 & 3 \\ a^2 & b^2 & 4 & 9 \end{array}\right]$$

$b-a \neq 0$ (since $a < b$)

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{b-a} & 2-a & 3-a \\ 0 & 0 & (a-2)(b-2) & (a-3)(b-3) \end{array}\right]$$

we want solvable & ∞ soln

$\therefore$ 1 or 2 pivots

<u>and</u> $\text{rank}(A) = \text{rank}(A^+)$

$(a-2)(b-2)$ and $(a-3)(b-3)$ can ~~cannot~~

simultaneously be 0

we want rank (A) = rank (A^+)

But rank $(A) = 2$ only

(we saw it is at least 2)

$\therefore (a-2)(b-2) = 0$

and $(a-3)(b-3) = 0$

$\therefore (2,3)$, $(3,2)$ are the solutions but $a < b$

$\Rightarrow (2,3)$ is the only solution

q) Show that row space & column dimension does not change by row operations

Ans row space unchanged :

let rows be identified as vectors

$\{r_1, \ldots, r_n\}$

Then we know that

$$\text{span}\{r_1, \ldots, r_n\}$$
$$= \text{span}\{r_1, \ldots, \alpha r_i, \ldots, r_n\}$$
$$= \text{span}\{r_1, \ldots r_i, \ldots \alpha r_i + r_j, \ldots, r_n\}$$
$$= \text{span}\{r_1, \ldots r_j, \ldots, r_i, \ldots, r_n\}$$

$\therefore$ Row space is unchanged

col rank unchanged :

$c_1, \ldots, c_m$ are colums

$Ec_1, \ldots Ec_m$ are new columns

E is invertible. (ERM product)

Let $c_1 \cdots c_m$ be s.t. the rank

is p

$\therefore c_{i_1}, c_{i_2}, \ldots, c_{i_p}$ are LI

iff $Ec_{i_1}, Ec_{i_2}, \ldots, Ec_{i_p}$ are LI

$\therefore$ $\cancel{\text{rank}}$ dim $(\text{span}(c_1, \ldots, c_m)) = p$

TUTORIAL 3

1) Find rank by determinants

(i) $\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}$

Ans (i) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \xrightarrow{\det} -4 \neq 0$ $\therefore$ rank ≥ 2

$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \xrightarrow{\det} -60$ $\therefore$ rank $= 3$

(ii) $\det\begin{pmatrix} 4 & 3 \\ -8 & -6 \end{pmatrix} = \det\begin{pmatrix} 4 & 3 \\ 16 & 12 \end{pmatrix} = \det\begin{pmatrix} -8 & -6 \\ 16 & 12 \end{pmatrix} = 0$

$\therefore$ rank < 2 $\Rightarrow$ rank $= 1$

(iii) $\det\begin{pmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{pmatrix} = 0$

$\det\begin{pmatrix} -2 & -\sqrt{3} \\ -1 & 0 \end{pmatrix} \neq 0$

$\therefore$ rank $= 2$

2) Find β so that Crammer is applicable. For other values of β, find no. of solutions

$$x + 2y + 3z = 20$$
$$x + 3y + z = 13$$
$$x + 6y + \beta z = \beta$$

Ans

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \beta \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 20 \\ 13 \\ \beta \end{bmatrix}}_{b}$$

$$\det(A) = \beta + 5$$

Crammers rule is applicable for $\beta \neq -5$

and we get

$$x_1 = \frac{\det\begin{pmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ \beta & 6 & \beta \end{pmatrix}}{\beta + 5}$$

$$x_2 = \frac{\det\begin{pmatrix} 1 & 20 & 3 \\ 1 & 13 & 1 \\ 1 & \beta & \beta \end{pmatrix}}{\beta + 5}$$

$$x_3 = \frac{\det\begin{pmatrix} 1 & 2 & 20 \\ 1 & 3 & 13 \\ 1 & 6 & \beta \end{pmatrix}}{\beta + 5}$$

For $\beta = -5$, the REF of the augmented matrix is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 7 & 34 \\ 0 & \boxed{1} & -2 & -7 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right]$$

$\text{rank}(A) = 2 \qquad \text{rank}(A^+) = 3$

$\therefore$ not solvable

3) find if the set is LI

$\{(a,b,c),\ (b,c,a),\ (c,a,b)\}$

Ans
$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \xrightarrow{\text{det}} 3abc - a^3 - b^3 - c^3$$

They will be LI if we have full rank

$$\text{rank} = 3 \Rightarrow 3abc \neq a^3 + b^3 + c^3$$

$$a^3 + b^3 - c^3 - 3abc = \left(\frac{a+b+c}{2}\right)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$$

$\therefore$ LD if $a+b+c = 0$ or $a = b = c$

4) Find λ to apply Crammer. Discuss solvability for other values of λ

$$x + \lambda z = \lambda - 1$$
$$x + \lambda y = \lambda + 1$$
$$\lambda x + y + 3z = 2\lambda - 1$$

Ans $\det A \neq 0$

$$\Rightarrow \det \begin{pmatrix} 1 & 0 & \lambda \\ 1 & \lambda & 0 \\ \lambda & 1 & 3 \end{pmatrix} \neq 0$$

$$\therefore \lambda(\lambda^2 - 4) \neq 0$$

$$\therefore \lambda \neq 0,\quad \lambda \neq 2,\quad \lambda \neq -2$$

$$A^+ = \left[\begin{array}{ccc|c} 1 & 0 & \lambda & \lambda - 1 \\ 1 & \lambda & 0 & \lambda + 1 \\ \lambda & 1 & 3 & 2\lambda - 1 \end{array}\right]$$

for $\lambda = 0$, RREF is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

Not soluable

for $\lambda = 2$, RREF is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 1 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Soluable ans

$y = 1 + z$

$x = 1 - 2z$

z is arbitrary

5) Finding adjugate matrices (grade 11/12 stuff)

6) Finding inverses of the matrices in (5) above

7) Solve using Crammer & verify using GEM

(i)
$$5x - 3y = 37$$
$$-2x + 7y = -38$$

(ii)
$$x + 2y + 3z = 20$$
$$7x + 3y + z = 13$$
$$x + 6y + 2z = 0$$

Ans I will only solve (ii)

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \\ 0 \end{bmatrix}$$

$\det A = 91$

$$\det\begin{pmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix} = 182$$

$$\det\begin{pmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{pmatrix} = -273$$

$$\det\begin{pmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{pmatrix} = 728$$

$\therefore \ x = \dfrac{182}{91}, \quad y = \dfrac{-273}{91}, \quad z = \dfrac{728}{91}$

$\therefore \ x = 2, \quad y = -3, \quad z = 8$

GEM

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right]$$

$$\Big\downarrow \text{RREF}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$\therefore \ x = 2, \quad y = -3, \quad z = 8$

9) Invert $\begin{bmatrix} 1 & 2^{-1} & 3^{-1} \\ 2^{-1} & 3^{-1} & 4^{-1} \\ 3^{-1} & 4^{-1} & 5^{-1} \end{bmatrix}$ $\left(x^{-1} = \dfrac{1}{x} \right)$

Just do it. No shortcuts

Ans

$$A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

10) Find det $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ & generalise for $n \times n$

Ans

$$\det \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Claim:
$$\det \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ & & \vdots & & \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

$$= \prod_{1 \leq j < k \leq n} (x_k - x_j)$$

proof is by induction but a degree argument would suffice at this stage.

11) Prove that if Wronskian of n functions doesn't vanish at some x, all functions are LI

Ans

$$W(f_1, \ldots, f_n) = \det \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ & \vdots & & \\ f_1^{(n-1)} & \cdots & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Consider $\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = 0$

(we want to prove $\alpha_1 = \dots = \alpha_n = 0$)

Differentiating, we get more equations and finally,

$$\underbrace{\begin{bmatrix} b_1 & \cdots & b_n \\ & \vdots & \\ b_1^{(n-1)} & & b_n^{(n-1)} \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

if $\det(A) \neq 0$, A is invertible

and $\alpha_1 = \dots = \alpha_n = 0$ is the only unique solution

(note: functions are LI $\not\Rightarrow$ wronskian is not zero)

TUTORIAL 4

1) Let u, v be vectors $(u \neq 0)$. Let $w = v - \frac{(v,u)}{\|u\|^2} u$.

Show that $v = \frac{(v,u)}{\|u\|^2} u + w$ is the resolution of v into components parallel to u and perpendicular to u

Ans $(w, u) = (v, u) - \frac{(v,u)}{\|u\|^2}(u,u)$

$= (v, u) - (v, u) = 0$

$\therefore$ $w \perp u$ and v is indeed $\frac{(v,u)u}{\|u\|^2} u + w$

2) Verify that $\{(2,-2,1), (1,2,2), (2,1,-2)\}$ is an orthogonal set. Is it a basis? If yes, express $(1,1,1)$ as a linear combination. Verify Bessel's inequality

Ans $(2,-2,1)\cdot(1,2,2) = 2-4+2 = 0$

$(1,2,2)\cdot(2,1,-2) = 2+2-4 = 0$

$(2,1,-2)\cdot(2,-2,1) = 4-2-2 = 0$

$\therefore$ indeed orthogonal.

orthogonal vectors are linearly independent

since if $c_1v_1 + \dots + c_nv_n = 0$,

then $c_1(v_1,v_1) + 0 + 0 + \dots + 0 = 0\cdot v_1 = 0$

$\therefore c_1 = 0 \quad (v_1 \neq 0)$

$\therefore$ They form a basis (since cardinality is 3)

$$\frac{1}{9}(2,-2,1) + \frac{5}{9}(1,2,2) + \frac{1}{9}(2,1,-2)$$

$$= (1,1,1)$$

Bessels:

$\|v\|^2 = \|(1,1,1)\|^2 = 3$

$(v\cdot x_1)^2 = 1/9$

$(v\cdot x_2)^2 = 25/9$

$(v\cdot x_3)^2 = 1/9$

$3 \geq \frac{1}{9} + \frac{25}{9} + \frac{1}{9}$ (✓ verified)

note: Bessel is applied to $\boxed{\text{orthonormal}}$ set

3) use gram schmidt to orthogonalise

$\{1111,\ 11-1-1,\ 1100,\ -1\ 1\ -1\ 1\}$

<u>Ans</u> $w_1 = (1,1,1,1)/\|\ \ "\ \ \| = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$$w_2 = \frac{v_2 - (v_2 \cdot w_1) w_1}{\|\ \ \ "\ \ \ \|}$$

$$= \frac{(1\ 1\ -1\ -1) - 0}{\|\ \ "\ \ \|}$$

$$= (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2})$$

$$w_3 = \frac{(1,1,0,0) - (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) - (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}{\|\ \ \ "\ \ \ \|}$$

$$= 0$$

$r = 2$

$\therefore$ not a basis

but span $\{1111,\ 11-1-1\}$

$= \text{span}\{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2});\ (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})\}$

and R(t) is orthogonal

4) orthogonalise:

$\{\underset{v_1}{(1100)}, \underset{v_2}{(1010)}, \underset{v_3}{(1001)}, \underset{v_4}{(0110)}, \underset{v_5}{(0101)}, \underset{v_6}{(0011)}\}$

<u>Ans</u> $w_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$

$$w_2 = \frac{(1\ 0\ 1\ 0) - \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)}{\|\ \ \ "\ \ \ \|} = (\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0)$$

$$= \sqrt{\tfrac{2}{3}}\,(\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0)$$

$w_3 = \left(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2} \right)$

$w_4 = \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$

$w_5 = 0$

$\therefore$ not LI but we have 4 vectors in our orthonormal set

5) write orthogonal basis for the solution space of

(i)
$$-2x_4 + x_5 = 0$$
$$2x_2 - 2x_3 + 14x_4 - x_5 = 0$$
$$2x_2 + 3x_3 + 13x_4 + x_5 = 0$$

(ii)
$$2x_1 - 2x_2 + x_3 + x_4 = 0$$
$$-2x_2 + x_3 - x_4 = 0$$
$$x_1 + x_2 + 2x_3 - x_4 = 0$$

(iii)
$$2x_3 - 2x_4 + x_5 = 0$$
$$2x_2 - 8x_3 + 14x_4 - 5x_5 = 0$$
$$x_2 + 3x_3 + x_5 = 0$$

(iv)
$$2x_1 - 2x_2 + x_3 + x_4 = 0$$
$$-2x_2 + x_3 + 7x_4 = 0$$
$$3x_1 - x_2 + 4x_3 - 2x_4 = 0$$

Ans I will do only one of them $\rightarrow$ (iii)

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & -2 & 1 & 0 \\ 0 & 2 & -8 & 14 & -5 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 0 & 3 & -1/2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad x_3 = x_4 - x_5/2$

$x_2 = \frac{1}{2} x_5 - 3x_4$

$x_1 \; x_4 \; x_5$ arbitrary

Pick $x_1\, x_4\, x_5$ to be

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$$

(why 2? Because we have $x_5/2$ so it simplifies calculation)

basis is $\{10000, 0\,{-3}\,110, 01\,{-1}\,02\}$

(Dont be an idiot and choose $x_1\, x_4\, x_5$ to be 000 111 222)

6) Find LI or not using Gram Schmidt ...

Ans Just use the stuff we did in Q4. That is, theorem 62

7) Orthonormalise in $\mathbb{C}^5$ using standard hermitian dot product

$\{(1,i,0,0,0), (0,1,i,0,0), (0,0,1,i,0), (0,0,0,1,i)\}$

Ans $\quad \omega_1 = \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0, 0, 0\right)$

$\omega_2 = \left(\frac{1}{\sqrt{6}} i, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\, i, 0, 0\right)$

$$w_3 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} i, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2} i, 0 \right)$$

$$w_4 = \left(\frac{-1}{2\sqrt{5}} i, -\frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}} i, \frac{1}{2\sqrt{5}}, \frac{2}{\sqrt{5}} i \right)$$

Note:

$(v_1, \cdots, v_n) = v$

$(w_1, \cdots, w_n) = w$

$$v \cdot w = \langle v, w \rangle = (v, w)$$

$$:= \sum_{i=1}^{n} v_i \overline{w_i} \quad \text{is the inner prod}$$

8) Let $S = \{v_1, \cdots, v_k\}$, T = set obtained by applying G-S to S. Suppose T has no zero vector and $T = \{w_1, \cdots, w_k\}$, show that spans of both S, T are same and show that w_j is orthogonal to $\{w_1, \cdots, w_{k-1}\}$ (there was a typo)

Ans we use induction for the first part, on j,

$$T_1 = \{w_1\} \Rightarrow \text{span } T_1 = \text{span}\{v_1\}$$

$$T_2 = \{w_1, w_2\} \Rightarrow \text{span } T_2$$

$$= \{c_1 w_1 + c_2 w_2\}$$

$$= \left\{ c_1 \frac{v_1}{\|v_1\|} + c_2 \left(\frac{v_2 - (c_1 \cdot w_1) w_1}{\|v_2 - (c_1 \cdot w_1) w_1\|} \right) \right\}$$

$$= \{ a v_1 + b v_2 - c w_1 \}$$

$$= \{ \alpha v_1 + \beta v_2 \}$$

$$= \text{span}\{T_2\}$$

Now assume for general n that

$\text{span}\, T_n' = \text{span}\, S_n$ (assume)

$$\text{span}\, T_{n+1} = \{ c_1 w_1 + c_2 w_2 + \cdots + c_n w_n + c_{n+1} w_{n+1} \}$$

$$= \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + c_{n+1} w_{n+1} \}$$

(induc hyp.)

But by defn of w_{n+1},

$w_{n+1} \in \text{span}\{ v_1, \ldots, v_{n+1} \}$

$\therefore \text{span}\, T_{n+1} = \text{span}\, S_{n+1}$

$$w_j = \frac{v_j - (v_j, w_1) w_1 - \cdots - (v_j, w_{j-1}) w_{j-1}}{\| \quad \| \cdots \|}$$

$$(w_j, v_t) = \frac{(v_j, v_t) - (v_j, w_1)(w_1, v_t) - \cdots - (v_j, w_{j-1})(w_{j-1}, v_t)}{\| \quad \| \cdots \|}$$

not directly possible

so do by induction

Base case is trivially true

Assume true for upto j-1

~~so that~~ $\cancel{(w_{j-1}, v_t) = 0 \ \forall\, t}$

so that $(w_j, v_t) = (v_j, v_t)$

omit

$\therefore (w_j, w_t) = \dfrac{(v_j, w_t) - \sum_{i=1}^{j-1} (v_j, w_i)(w_i, w_t)}{\| \quad \cdots \quad \|}$
$(t<j)$

If we use strong induction and assume that

$w_k \perp w_1, \dots, w_{k-1}$

$\forall\ k = 1, 2, 3, \dots, j-1$,

we get $(w_j, w_t) = \dfrac{(v_j, w_t) - (v_j, w_t)(w_t, w_t)}{\| \quad \cdots \quad \|}$

$= 0$

i) Let $\{p, q, r\}$ be LI in $\mathbb{R}^3$. Let $\{x, y, z\}$ be obtained after applying G-S. Prove that z must be a multiple of $\vec{p} \times \vec{q}$ (cross product)

Ans WLOG assume p, q, r are unit vectors

Then $x = p$

$$y = \frac{q - (q,p)\,p}{\| q - (q,p)\,p \|}$$

$$z = \frac{r - (r,y)\,y - (r,p)\,p}{\| \quad " \quad \|}$$

Now span $\{x, y\}$ = span $\{p, q\}$

But $\{x, y, z\}$ is an orthonormal set

$\therefore z \perp x,\ z \perp y \Rightarrow z \perp \text{span}\{x,y\} = \text{span}\{p, q\}$

In $\mathbb{R}^3$, $z = \lambda\, \vec{p} \times \vec{q}$

10) If θ = angle b/w $\vec{v}, \vec{w}$, prove

(i) $\theta = 0$ iff $w = \frac{\|w\|}{\|v\|} v$

(ii) $\theta = \pi$ iff $w = -\frac{\|w\|}{\|v\|} v$

(iii) $\|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \cos\theta$

Ans (i) $\theta = \cos^{-1}\left(\frac{v}{\|v\|}, \frac{w}{\|w\|}\right) \quad (\in [0, \pi])$

$\theta = 0$ iff $\left(\frac{v}{\|v\|}, \frac{w}{\|w\|}\right) = 1$

iff $(v, w) = \|v\| \|w\|$

iff $\frac{(v, w)}{\|w\|} = \|v\|$

iff $\frac{(v, w)}{(w, w)} \|w\| = \|v\|$

$\therefore \frac{\|v\|}{\|w\|} = \frac{(v, w)}{(w, w)} = \lambda$

$\therefore \lambda w = v$

OR

$|(v, w)| \leq \|v\| \|w\|$ with equality

iff v, w are LD

$\therefore$ v, w are LD

$\Rightarrow v = \lambda' w$

(ii) $\theta = \pi$ iff $\left(\frac{v}{\|v\|}, \frac{w}{\|w\|}\right) = -1$

again, $v = \lambda' w$

But $\lambda' < 0$ (check directly by working)

~~[illegible]~~ Let $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

$$\Leftarrow) \quad v = \begin{bmatrix} \lambda' w_1 \\ \vdots \\ \lambda' w_n \end{bmatrix}$$

$$\therefore \frac{(v,w)}{\|v\|\,\|w\|} = -1$$

iff $$\frac{\lambda'(w_1^2 + \dots + w_n^2)}{\sqrt{w_1^2+\dots+w_n^2}\,\sqrt{w_1^2+\dots+w_n^2}\,|\lambda'|} = -1$$

$$\therefore \frac{\lambda'}{|\lambda'|} = -1$$

$$\therefore \lambda' < 0$$

(iii) $$\|v+w\|^2 = (v+w, v+w)$$
$$= (v,v) + (w,w) + 2(v,w)$$
$$= \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\|\cos\theta$$

TUTORIAL 5

1) Find eval, evec

(i) $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Ans (i) $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

$\therefore 5x - y = \lambda x$

$x + 3y = \lambda y$

$\therefore \dfrac{5-\lambda}{1} = \dfrac{1}{\lambda - 3} \Rightarrow (\lambda-3)(\lambda-5)+1=0$

$\therefore \lambda = 4, 4$

$\therefore 5x - y = 4x$

$\therefore x = y$

$\therefore$ eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

only one eigenvector

(ii) $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} x = \lambda x$

$\therefore (A - \lambda I)x = 0$

x is non zero but still exists

$\therefore \det(A - \lambda I) = 0$

$\therefore \lambda = 3, -1, 1, 0.5$

(do the determinant of $A - \lambda I$)

e.vecs are $1, 0, 0, 0$

$-\frac{1}{4}, 0, 1, 0$

$-\frac{3}{2}, 1, 1, 0$

$\frac{8}{3}, -2, -\frac{8}{3}, 1$

(iii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det \begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix} = 0$$

$$\Rightarrow x^2 + 1 = 0$$

$$\therefore x = \pm i$$

$$\therefore \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pm ix \\ \pm iy \end{bmatrix}$$

$$\therefore \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} \pm ix \\ \pm iy \end{bmatrix}$$

$$x = \pm iy$$

$\therefore \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\begin{bmatrix} 1 \\ +i \end{bmatrix}$ are evecs

corresponding to $i, -i$

2)
3) Not in syllabus

4) Prove that

(i) similar matrices $\Rightarrow$ same char poly

(ii) similar matrices $\Rightarrow$ same evals with same AM

(iii) similar matrices $\Rightarrow$ same evals with same GM

(iv) A, B are similar. A is diag iff B is diag

Ans (i) A, B are similar

$\therefore PAP^{-1} = B$ for some invertible P

$$f_A(x) = \det(xI - A) = \det P \det(xI - A) \det P^{-1}$$

$$f_B(x) = \det(xI - PAP^{-1})$$

$$= \det(xPP^{-1} - PAP^{-1})$$

$$= \det(P(xI - A)P^{-1}) = f_A(x)$$

(ii) follows from (i)

(iii) Let v be an evec of A with eval λ

Then $PAP^{-1}(Pv)$

$= PAv$

$= P\lambda v$

$= \lambda Pv$

$\therefore Pv$ is an evec of B with same eval

$\therefore E_\lambda^{(A)} = \{ x \in V \mid Ax = \lambda x \}$

$E_\lambda^{(B)} = \{ x \in V \mid Bx = \lambda x \}$

are the same spaces since they are independent of basis

$\therefore \dim (E_\lambda^{(A)}) = \dim (E_\lambda^{(B)})$

$\therefore$ nullity $(A - \lambda I) =$ nullity $(B - \lambda I)$

$\therefore GM_\lambda^A = GM_\lambda^B$

(iv) A is diagonalisable

iff $QAQ^{-1} = \text{diag}(\lambda_1 \cdots \lambda_n)$

iff $QP^{-1}PAP^{-1}PQ^{-1} = \text{diag}(\lambda_1 \cdots \lambda_n)$

iff $QP^{-1}B(QP^{-1})^{-1} = \text{diag}(\lambda_1 \cdots \lambda_n)$

5) Show that $\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$ & $\underset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ have same evals.

Are they similar?

Ans $\det \begin{bmatrix} 1-x & 1 \\ 0 & 1-x \end{bmatrix} = (1-x)^2 = 0 \Rightarrow x = 1, 1$

$\det\begin{bmatrix} 1-x & 0 \\ 0 & 1-x \end{bmatrix} = 0 \Rightarrow x = 1, 1$

$\Rightarrow$ same evals but not similar since the geometric multiplicities are different (checking this will suffice in every case)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore x + y = x$

$y = y$

$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is evec

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore x = x$

$y = y$

Any vec is evec

$E_\lambda^{(A)} = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}$

dim = 1

$E_\lambda^{(B)} = \left\{ \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$

$= \mathbb{R}^2$

dim = 2

6) Show that the following pairs are similar

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & \lambda \\ 0 & 0 \end{bmatrix}$

Ans Note :

1) If GM of a common e.val is different, the matrices are NOT similar

2) If all evals of A, B coincide and have same AM, GM then A, B are similar

(i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad\qquad$ $\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$

$1, -1$ $\qquad\qquad$ $1, -1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad\qquad$ $\downarrow \quad \downarrow$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_1 = \{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \}$ $\qquad\qquad$ $E_1 = \{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \}$

$E_{-1} = \{ \begin{bmatrix} t \\ -t \end{bmatrix} : t \in \mathbb{R} \}$ $\qquad\qquad$ $E_{-1} = \{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \}$

~~AM = GM~~

$\therefore$ similar

(ii) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow x^2 + 1 = 0$

$B = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \longrightarrow x^2 + 1 = 0$

$A \pm iI = \begin{bmatrix} \pm i & 1 \\ -1 & \pm i \end{bmatrix}$

nullity $= 1$

$B \pm iI = \begin{bmatrix} \pm i & -1 \\ 1 & \pm i \end{bmatrix}$

nullity $= 1$

(iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to x(x-2) = 0$

$\begin{bmatrix} 2 & \lambda \\ 0 & 0 \end{bmatrix} \to x(x-2) = 0$

distinct eigenvalues $\Rightarrow$ diagonalisable

$\Rightarrow$ similar

(we have shown 3 different methods to test similarity!)

7) Show that $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, A^2 are similar via a transformation of type $X = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix}$

Ans $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$X^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 0 & 1/c \\ 0 & 1/b & 0 \end{bmatrix}$

$X A X^{-1} = \begin{bmatrix} 0 & a/b & 0 \\ 0 & 0 & b/c \\ c/a & 0 & 0 \end{bmatrix} \overset{?}{=} A^2$

$\therefore$ choose $a = b = c = 1$ to make the equality work

8) Show that $A = [a_{ij}]_{n \times n}$, $B = [(-1)^{i+j} a_{ij}]_{n \times n}$ are similar

Ans Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Then $B = \begin{bmatrix} a_{11} & -a_{12} & a_{13} & -a_{14} & \cdots \\ & & \ddots & & \\ & & & \ddots & \end{bmatrix}$

Basically $\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & & & \vdots & & \end{bmatrix}$

use $C = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \\ & & & & & & \ddots \end{bmatrix}_{n \times n}$ no

first make all even numbered rows entirely negative (i.e. flip sign)

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} + & + & + & + & + & \cdots \\ + & + & + & + & + & \cdots \\ + & + & + & + & + & \cdots \\ & & \vdots & & & \end{bmatrix} = \begin{bmatrix} + & + & + & \cdots \\ - & - & - & \cdots \\ + & + & + & \cdots \\ - & - & - & \cdots \end{bmatrix}$$

Now put the same matrix on the RHS to make all alternate columns flip signs

$$\begin{bmatrix} + & + & \cdots \\ - & - & \cdots \\ + & + & \cdots \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \\ & & & & \ddots \end{bmatrix} = \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ & & & & \cdots & \end{bmatrix}$$

and $\begin{bmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{bmatrix}$!

9) find an e.vec basis & diagonalise

(i) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -2.5 & -3 & 3 \\ -4.5 & -4 & 6 \\ -6 & -6 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 1+i \\ 1-i & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 0 \end{bmatrix}$

Ans Just find $\lambda_1, \lambda_2, \lambda_3 \cdots, v_1\ v_2\ v_3 \cdots$

and $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}_{n\times n} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

(i) $\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} \boxed{0} & 0 \\ 0 & \boxed{3} \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ v_1 & v_2 \end{array}$

(ii) $\begin{pmatrix} \frac{(-1+i)\sqrt{2}}{4} & \frac{1}{2} \\ \frac{(1-i)\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 & 1+i \\ 1-i & 4 \end{pmatrix} \begin{pmatrix} \frac{(-1-i)\sqrt{2}}{2} & \frac{(1+i)\sqrt{2}}{2} \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -\sqrt{2}+4 & 0 \\ 0 & \sqrt{2}+4 \end{pmatrix}$$

(iii) $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} -2.5 & -3 & 3 \\ -4.5 & -4 & 6 \\ -6 & -6 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ \frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 15/4 & -3 & 1 \\ -6 & 3 & 0 \\ 9/4 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 4/9 \\ 0 & 1/3 & 8/9 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

10) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$, find evals of AB, BA

Ans $AB = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ evals $\to$ 7, 3

$BA = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}$; evals $\to$ 7, 3, 0

11) Let $A_{m\times n}$, $B_{n\times m}$ be matrices. Show that AB and BA have the same non zero evals

Ans Let $ABv = \lambda v$ $(\lambda \neq 0)$

$Bv = w$

$\Rightarrow Aw = \lambda v$

$\Rightarrow BAw = \lambda(Bv) = \lambda w$

$\therefore \lambda$ is an eval of BA also

So if $m < n$ and

AB has evals $\lambda_1, \dots, \lambda_m$

Then BA has evals $\lambda_1, \dots \lambda_m, \underbrace{0, 0, 0, \dots 0}_{n-m \text{ times}}$

12) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$. Show that A has a repeated eval whose $GM < AM$. If u is an evec of A, solve $Av = \lambda v + u$ for v. Show that $\{u, v\}$ is a basis which almost diagonalizes A to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Ans $(1-x)(3-x)+1=0$

$\therefore x = 2, 2$

$x - y = 2x$

$\therefore x = -y$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an evec.

nullity of $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = 1$

$\therefore GM = 1, \quad AM = 2$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore -v_1 - v_2 = 1$

$\therefore \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is a solution

$\left\{ \begin{matrix} 1 \\ -1 \end{matrix}, \begin{matrix} -1 \\ 0 \end{matrix} \right\}$ is clearly a basis

if $X = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then

$X^{-1} A X = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Switching order of columns in X

gives $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

13) Same as 12

TUTORIAL 6

1) For a sq. matrix A, prove :

(i) e vals real if hermitian

(ii) evals purely imaginary if skew hermitian

Ans (i) $Av = \lambda v$

$$v^* A v = \lambda v^* v = \lambda \|v\|^2$$

$$(v^* A v)^* = (\lambda \|v\|^2)^*$$

$$\therefore v^* A^* v = \bar{\lambda} \|v\|^2$$

$$\therefore v^* A v = \bar{\lambda} \|v\|^2$$

$$\therefore \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is purely real}$$

(ii) $(v^* A v)^* = \bar{\lambda} \|v\|^2$

$$\therefore v^* A^* v = \bar{\lambda} \|v\|^2$$

$$\therefore -v^* A v = \bar{\lambda} \|v\|^2$$

$$\therefore -\lambda \|v\|^2 = \bar{\lambda} \|v\|^2$$

$$\therefore \lambda + \bar{\lambda} = 0 \Rightarrow \lambda \text{ is purely imaginary}$$

2) Show that $A^t A$ has real non negative e vals for any real $m \times n$ A

Ans $A^t A$ is real symmetric $\Rightarrow$ hermitian

$\therefore$ from above we have real e vals

~~$A^t A v = \lambda v \Rightarrow (Av, Av)$~~

Let $A^t A v = \lambda v$

Now

$$\|Av\|^2 = (Av, Av)$$
$$= (Av)^t Av$$
$$= v^t A^t A v$$
$$= \lambda v^t v$$
$$= \lambda (v, v)$$

$$\therefore \lambda = \frac{\|Av\|^2}{\|v\|^2} > 0$$

3) Prove for a ~~real~~ hermitian matrix that e-vecs corresponding to distinct evals are orthogonal

Ans

$$Av_1 = \lambda_1 v_1$$
$$Av_2 = \lambda_2 v_2$$

$$v_1^* A v_2 = \lambda_2 v_1^* v_2$$

$$\| $$

$$v_1^* A^* v_2 = (A v_1)^* v_2 = \lambda_1 v_1^* v_2$$

$$\therefore v_1^t v_2 = 0 \quad (\because \lambda_1 \neq \lambda_2)$$

note: we get similar statements for unitary and skew hermitian matrices

4) Find all real 2×2 skew symm matrices. Also find orthogonal ones among them. Repeat for 3×3

Ans $\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$ is a 2×2 skew symm real matrix

$$\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} = \begin{bmatrix} c^2 & 0 \\ 0 & c^2 \end{bmatrix} = I$$

$\therefore c = \pm 1$

3×3 case :

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$$AA^T = I$$

$$\therefore \quad -A^2 = I$$

$$\therefore \quad A^2 + I = 0$$

$$\therefore \quad a^2 + b^2 = 1 = a^2 + c^2 = b^2 + c^2$$

$$ab = bc = ac = 0$$

~~Case 1 all zero~~

~~case 2 two of them zero~~

~~case 3 but a~~

$ab = 0 \quad \rightarrow \quad a = 0$ or $b = 0$

suppose $a = 0$ Then $b^2 = 1$, $c^2 = 1$

But $bc = 1 \longrightarrow \leftarrow$

suppose $b = 0$ Then $a^2 = c^2 = 1$ but $ac = 0 \rightarrow \leftarrow$

$\therefore$ such a 3×3 orthogonal, real, skew symmetric matrix doesn't exist

5) Same as above moved to complex case

6) Let $q(x,y) = ax^2 + 2hxy + by^2$

Show that $q(x,y) = 1$ is an ellipse if $ab-h^2>0, a>0$

and show that the area bounded is $\frac{\pi}{\sqrt{ab-h^2}}$

Ans q is associated to $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$.

Let the normalised form be $\alpha u^2 + \beta v^2$

we get an ellipse if $\alpha>0, \beta>0$

$\alpha, \beta > 0 \iff \alpha+\beta>0, \alpha\beta>0$

$\iff a+b>0, ab-h^2>0$

But $a+b>0$ iff $a>0$

because if $a<0$, $b>0$ and $a+b>0$,

then $ab<0$ and $-h^2<0$

$\Rightarrow ab-h^2<0 \rightarrow \leftarrow$

Also, area of ellipse $= \pi a' b'$

But $\alpha u^2 + \beta v^2 = 1$

$\therefore a' = \frac{1}{\sqrt{\alpha}}, \quad b' = \frac{1}{\sqrt{\beta}}$

$\therefore \pi a' b' = \frac{\pi}{\sqrt{\alpha\beta}} = \frac{\pi}{\sqrt{ab-h^2}}$

7) Convert to standard form

(i) $10x^2 + 4xy + 7y^2 = 100$

(ii) $3x^2 + 4xy + y^2 = 4$

(iii) $x^2 + 2\sqrt{3}xz + 4yz - 3z^2 = -25$

(iv) $x^2 - 2y^2 + 6yz + 4z^2 = 1$

Ans (i) $\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix} \Rightarrow$ evals are 11, 6

$\therefore 11x^2 + 6y^2 = 100$ (ellipse)

(ii) $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow$ evals are $2-\sqrt{5}, 2+\sqrt{5}$

$\therefore (2-\sqrt{5})x^2 + (2+\sqrt{5})y^2 = 4$ (hyperbola)

(iii) $\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 0 & 2 \\ \sqrt{3} & 2 & -3 \end{bmatrix} \longrightarrow 2, -\sqrt{6}-2, \sqrt{6}-2$

$\therefore 2x^2 - (2+\sqrt{6})y^2 + (\sqrt{6}-2)z^2 = -25$

all three +ve $\Rightarrow$ ellipsoid

axis is evec of -ve eval $\leftarrow$ two +ve, one -ve $\Rightarrow$ elliptic hyperboloid of one sheet

axis is evec of +ve eval $\leftarrow$ one +ve, two -ve $\Rightarrow$ elliptic hyperboloid of two sheets

all -ve $\Rightarrow$ not possible

Here: hyperboloid of ~~ONE~~ TWO sheets

since RHS is negative and needs to be made positive

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix} \longrightarrow 1,\ 1-3\sqrt{2},\ 1+3\sqrt{2}$

2 +ve, 1 −ve $\Rightarrow$ ONE sheet hyperboloid

8) Find evals, mutually orthogonal evecs

(i) $\begin{bmatrix} 5 & \sqrt{6} \\ \sqrt{6} & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 20 & 16 & -18 & 0 \\ 16 & 29 & 0 & -18 \\ -18 & 0 & 29 & -16 \\ 0 & -18 & -16 & 20 \end{bmatrix}$

Ans (i) 7, 2 ; $\begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} -\sqrt{2} \\ \sqrt{3} \end{bmatrix}$

(ii) 0, 0, 4, 4 ; $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(iii) 0, 0, 49, 49 ; $\begin{bmatrix} 9 \\ 0 \\ 10 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 8 \\ -10 \\ 0 \\ -9 \end{bmatrix}$, $\begin{bmatrix} 10 \\ 8 \\ -9 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -9 \\ -8 \\ 10 \end{bmatrix}$

(Do the calculation yourself)

(might need to orthogonalise for the repeated eval cases)

9) Reduce to normal form

(i) $x^2 + 2xy + 5y^2$

(ii) $x^2 + xy - \frac{y^2}{4}$

(iii) $x^2 - 2xy + y^2$

(iv) $x^2 + 2y^2 + z^2 + xy + yz + zx$

(v) $x^2 + z^2 + xy + yz + zx$

(vi) $xy + yz + zx$

(vii) $2x^2 + 2y^2 + z^2 + 2xz - 2yz$

(viii) $3x^2 + z^2 + 2xy - 2yz - 4zx$

(ix) $4x^2 + 9y^2 + z^2 - 12xy - 6yz + 4zx$

Ans (i) $3 \pm \sqrt{5}$

(ii) $\frac{3 \pm \sqrt{41}}{8}$

(iii) $2, 0$

(iv) $5/2, 1, 0.5$

(v) $0.5, \frac{3 \pm \sqrt{17}}{4}$

(vi) $-1, -1, 2$

(vii) $3, 2, 0$

(viii) $0, 2+\sqrt{7}, 2-\sqrt{7}$

(ix) $0, 0, 14$

TUTORIAL 7

1) check if it is a VS. If yes, find dim, basis

(i) $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ s.t.

(a) $x_4 < 0$

(b) $x_1 \le x_2$

(c) $x_1^2 = x_2^2$

(d) $x_1 = x_2 = x_3 = x_4$

(e) $x_1 x_2 = 0$

(all parts independent)

(ii) set of all real functions of the form $a\cos x + b\sin x + c$ where $a, b, c \in \mathbb{R}$

(iii) set of all $n \times n$ real matrices which are

(a) diag

(b) upper Δ

(c) trace 0

(d) symm

(e) anti-symm

(f) invertible

(g) not invertible

(iv) set of all real polynomials of degree 5 together with zero polynomial

(v) real funcs of the form $(ax+b)e^{cx}$

Ans (i) (a) no

$(1,1,1,-1) \in V$

$(-1,-1,-1,1) \notin V$

(b) no

$(1,2,3,4) \in V$

$(-1,-2,-3,-4) \notin V$

(c) no

$(1,1,1,1) \in V$

$(1,-1,1,1) \in V$

but $(2,0,2,2) \notin V$

(d) yes

$(a,a,a,a) \in V$

$(b,b,b,b) \in V$

$\Rightarrow c_1(a,a,a,a) + c_2(b,b,b,b)$

$= (c_1 a + c_2 b, \text{''}, \text{''}, \text{''}) \in V$

is true

(e) no

$(1,0,1,1) \in V$

$(0,1,1,1) \in V$

but $(1,1,2,2) \notin V$

(ii) yes

$\alpha(a\cos x + b\sin x + c) + \beta(a'\cos x + b'\sin x + c')$

is still of the form $A\cos x + B\sin x + C$

(iii) (a) Yes

~~—~~ LC (linear combination) of diag matrices is diagonal

(b) yes

LC of upper Δ is upper Δ

(c) yes

LC of zero trace ~~—~~ is zero trace

since $\text{trace}(\alpha A + \beta B) = \alpha\, tr(A) + \beta\, tr(B)$

(d) Yes

same stuff

(e) yes

same stuff

(f) NO!

$0 \notin V$

(g) NO!

$\varepsilon_{11} \in V$

$\varepsilon_{22} \in V$

$\vdots$

$\varepsilon_{nn} \in V$

$$\varepsilon_{11} + \varepsilon_{22} + \cdots + \varepsilon_{nn} = I_n \notin V$$

(However answer is yes if $n = 1$

since the only type of matrices are $[a]$)

(iv)

NO! ~~—~~ $x^5 + x^2, -x^5 + x^2 \in V$ but $2x^2 \notin V$

(v) No!

$$(x+1)e^{2x} \in V$$

$$(x+1)e^{3x} \in V$$

but $(x+1)(e^{2x}+e^{3x}) \notin V$

2) Find 3 different basis for $\mathbb{R}^2$

Ans Literally any two vectors not parallel or antiparallel will suffice

3) V, W are v.s. of X i.e. $V \underset{v.s}{\subseteq} X$, $W \underset{v.s}{\subseteq} X$ such that $V \cap W = \{0\}$. Let K, L be LI subsets of V, W resp. Prove that $L \cup K$ is LI subset of X.

Ans Let $K = \{v_1, \dots, v_m\}$

$L = \{w_1, \dots, w_n\}$

Let $a_1 v_1 + a_2 v_2 + \dots + a_m v_m + a_{m+1} w_1 + \dots + a_{m+n} w_n = 0$

$$\therefore \sum_{i=1}^{m} a_i v_i = \sum_{j=1}^{n} (-a_{m+j}) w_j = \lambda \text{ (say)}$$

LHS $\in V$, RHS $\in W$

$$\therefore \text{LHS} = \text{RHS} = \lambda \in V \cap W = \{0\}$$

$$\therefore \lambda = 0$$

$\therefore a_i = 0 = a_{m+j} \quad \forall\ i = 1,2,\dots m$
$\qquad\qquad\qquad\qquad j = 1,2,\dots n$

Q) check which functions are linear

(i) x^2

(ii) $5x+1$

(iii) $|x|$

(iv) xy

(v) (x, x)

(vi) $(x, -y)$

(vii) $(2x, 3y)$

(viii) (y, z, x)

Ans (i) No $\quad (x+y)^2 \neq x^2+y^2$

(ii) No $\quad 5(x+y)+1 \neq (5x+1)+(5y+1)$

(iii) No $\quad |x+y| \neq |x|+|y|$

(iv) Yes (in x) $\quad (x_1+x_2)y = x_1y + x_2y$

Yes (in y) $\quad x(y_1+y_2) = xy_1 + xy_2$

No (overall) $\quad (x_1+x_2)(y_1+y_2) \neq x_1y_1 + x_2y_2$

Note: It is bilinear but not linear

(v) $f: \mathbb{R} \to \mathbb{R}^2$

$f(x+y) = (x+y, x+y)$

$f(x)+f(y) = (x, x) + (y, y)$

$= (x+y, x+y)$

$\therefore$ linear

(vi) $(x_1+x_2, -(y_1+y_2)) = (x_1, -y_1) + (x_2, -y_2)$

$\therefore$ linear

(vii) (viii) same as 6 both linear

5) Let $a \in \mathbb{R}^3 \setminus \{0\}$ be fixed. Show that the following are linear maps. Identify domain & codomain and write down the matrices also

(i) $T(x) = \vec{a} \cdot \vec{x}$

(ii) $T(x) = \vec{a} \times \vec{x}$

(iii) $T(X) = X \vec{a}$

(for appropriate square matrix X)

Ans (i) Domain : $\mathbb{R}^3$

Codomain : $\mathbb{R}$

$$T(x_1 + x_2) = \vec{a} \cdot (\vec{x_1} + \vec{x_2})$$

$$= \vec{a} \cdot \vec{x_1} + \vec{a} \cdot \vec{x_2}$$

$$= T(x_1) + T(x_2)$$

$\therefore$ linear

matrix $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^t = [a_1 \; a_2 \; a_3]_{1 \times 3}$

(ii) Domain $= \mathbb{R}^3$

Codomain $= \mathbb{R}^3$

$\vec{a} \times (x_1 + x_2) = \vec{a} \times x_1 + \vec{a} \times x_2$

$\therefore$ linear

$$T((x_1, x_2, x_3)) = a_1 \, a_2 \, a_3 \times x_1 \, x_2 \, x_3$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{bmatrix} a_2 x_3 - x_2 a_3 \\ a_3 x_1 - a_1 x_3 \\ a_1 x_2 - x_1 a_2 \end{bmatrix}$$

$\therefore$ Matrix is

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

(iii) $T(X) = X\vec{a}$

Domain = $M_3(\mathbb{R})$

Codomain = $\mathbb{R}^3$

(since a is 3×1, X has to 3×3)

$T(X+Y) = (X+Y)\vec{a} = T(X) + T(Y)$

$\therefore$ linear

Now we can interpret this as a map from $\mathbb{R}^9 \to \mathbb{R}^3$

$$T\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \\ a_1x_4 + a_2x_5 + a_3x_6 \\ a_1x_7 + a_2x_8 + a_3x_9 \end{bmatrix}$$

$\therefore$ matrix is

$$\begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 \end{bmatrix}$$

6) Let V be set of polynomials of degree at most 3 and W is same but at most 2. Define $D : V \to W$ as $Dp = \frac{d}{dx}(p)$. Define $I : W \to V$ as $Iq = \int_0^x q(t)\,dt$. Write matrices of $D, I, D \circ I, I \circ D$ after fixing a basis of V, W

__Ans__ basis of $V \to \{1, x, x^2, x^3\}$

$W \to \{1, x, x^2\}$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Why ?

$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$

$I(a + bx + cx^2) = ax + \frac{bx^2}{2} + \frac{cx^3}{2}$

$D \circ I(f) = f, \quad I \circ D(f) = f - f(0)$

$\therefore D \circ I$ is just $I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $I \circ D$ is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (multiply out I and D)

(OR) for $I \circ D$:

$I \circ D : V \to V$ as

$I \circ D(a + bx + cx^2 + dx^3)$

$I(b + 2cx + 3dx^2)$

$= bx + cx^2 + dx^3$ (note: a is lost)

7) Let S be a subset of V

Prove that if $S \subseteq V' \subseteq V$ and $V' \underset{\text{subspace}}{\subseteq} V$,

then $\text{span}(S) \subseteq V'$

Ans Let $S = \{s_1, s_2, \ldots, s_m\}$

$S \subseteq V'$

$\Rightarrow s_1, s_2, \ldots, s_m \in V'$

V' is a subspace

$\therefore \text{Span}\{s_1, \ldots, s_m\} \in V'$

$\therefore \text{Span}(S) \subseteq V'$

Note We have shown that span(S) is the smallest subspace of V containing S.

8) $\{v_1, \ldots, v_n\}$ is LI, Prove that

$\{v_1, \ldots, v_{i-1}, v_i + \alpha v_j, v_{i+1}, \ldots, v_n\}$ is LI

Ans ~~$\sum_{j=1, j \neq i}^{n} \beta_j v_j + \beta_i (v_i + \alpha v_j)$~~

Ans $a_1v_1 + a_2v_2 + \cdots + a_i(v_i + \alpha v_j) + \cdots + a_nv_n = 0$

$$\therefore \sum_{t=1}^{n} a_t v_t + \alpha a_i v_j = 0$$

Since $\{v_1 \dots v_n\}$ is LI,

$$a_1 = a_2 = \cdots = a_{i-1} = a_i = \cdots = a_j + a_i\alpha = \cdots = a_n = 0$$

$\therefore a_j = 0$

$\therefore a_m = 0 \ \forall m$

$\therefore$ LI ✓

9) check if VS or not

(i) all poly of deg 3

(ii) all poly of deg ≤ 2

(iii) all $m \times n$ real mat of rank 3

(iv) all $m \times n$ real mat of rank < 3

(v) all solutions of $(y')^2 + 2y + 1 = 0$

(vi) all solutions of $xy' + y = ax^2$

(vii) set of all cont funct on $[0,1]$

(viii) set of all func of $[0,1]$ with finitely many discont

Ans (i) No $0 \notin V$

(ii) yes (check)

(iii) No $\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix} \in V$

but $\begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 0 \end{bmatrix}$ has rank 4

(iv) NO if max possible rank is > 3

else, we just have any ~~set~~ set of $m \times m$ matrices ($m \leq 3$) which is a vector space

(v) NO $0 \notin V$

(vi) NO $0 \notin V$

(vii) NO $x+2 \in V$

$-(x+2) \notin V$

(viii) Yes (sum of two such function is again that type of function & same with scalar multiple)

10) Check if LI or LD on functions from $\mathbb{R} \to \mathbb{R}$. Also find dimension of span of each set

(i) $\{1+t, 1+t^2+2t\}$

(ii) $\{x, |x|\}$

(iii) $\{1, t, 1+t^2, (1+t)^2\}$

Ans (i) Yes it is LI

$a + at + b + bt^2 + 2bt = 0 \quad \forall t$

$\therefore a = b = 0$

$\therefore \dim = 2$

(ii) Yes. LI again $\Rightarrow \dim = 2$

(iii) $\{1, t, 1+t^2, (1+t)^2\}$

$$(1+t)^2 = (1+t^2)\cdot(1) + (t)(2)$$

$$\therefore 1+t^2 \in \text{span}\{t, 1+t^2\}$$

$\therefore$ Not LI

but

$\{1, t, 1+t^2\}$ is LI

$\therefore \dim = 3$

11) Check LI or not

(i) $\{(a,b), (c,d)\}$ s.t. $ad \neq bc$

(ii) $\{(1+i, 2i, 2), (1, 1+i, 1-i)\}$

(iii) $\{v_i\}_{i=1}^{k}$ where $v_i = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1})$ and all α_i are distinct

(iv) $\{e^{\alpha_i x}\}_{i=1}^{n}$ for distinct α_i's

Ans (i) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\det \neq 0 \Rightarrow$ LI

(ii) $(1, 1+i, 1-i) \times (1+i) = (1+i, 2i, 2)$

$\therefore$ LD

(iii) k vectors in $\mathbb{R}^k$. Do the determinant test

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & & \alpha_k \\ \vdots & \vdots & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & & \alpha_k^{k-1} \end{bmatrix} = \prod_{k \geq i > j \geq 1} (\alpha_i - \alpha_j) \neq 0$$

$\therefore$ LI

(iv) Wronskian $= \det \begin{bmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} & \cdots & e^{\alpha_n x} \\ \alpha_1 e^{\alpha_1 x} & - & - & \\ \alpha_1^2 e^{\alpha_1 x} & - & - & \\ & & - & \end{bmatrix}$

$$= e^{(\alpha_1 + \cdots + \alpha_n)x} \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & & - \\ \alpha_1^{n-1} & \cdots & & & \alpha_n^{n-1} \end{bmatrix}$$

$$= e^{(\alpha_1 \cdots + \alpha_n)x} \prod_{n \geq i > j \geq 1} (\alpha_i - \alpha_j)$$

$\neq 0$

i.e. LI

12) For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, prove that (x_1, x_2, x_3, x_4) forms a subspace where $x_4 = \sum_{i=1}^{3} \alpha_i x_i$. Find basis

Ans $x_1, x_2, x_3, x_4 \in V$

$\therefore x_4 = \sum_{i=1}^{3} \alpha_i x_i$

$y_1, y_2, y_3, y_4 \in V$

$\therefore y_4 = \sum_{i=1}^{3} \alpha_i y_i$

Clearly $ax + by \in V$

since $ax_4 + bx_4 = a \sum_{i=1}^{3} \alpha_i x_i + b \sum_{i=1}^{3} \alpha_i y_i$

$= \sum_{i=1}^{3} \alpha_i (ax_i + by_i)$

$x_4 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

Chose $x_1\ x_2\ x_3$ to be $100 \quad 010 \quad 001$

to get $\{(1\,0\,0\,\alpha_1)\ (0\,1\,0\,\alpha_2)\ (0\,0\,1\,\alpha_3)\}$ as basis

dimension $= 3$

13) Find all possible subspaces of $\mathbb{R}^3$

Ans $\dim(\mathbb{R}^3) = 3$

3 dim subspace is obviously $\mathbb{R}^3$ itself

0 dim subspace is $\{0\}$

1 dimensional subspaces are lines

but $0 \in V$

$\Rightarrow$ lines through origin

2 dim $\Rightarrow$ planes through origin

14) - Repeated -

15) $f: \mathbb{R}^3 \to \mathbb{R}$ is linear with

$f(100) = f(010) = f(001) = f(111)$

Find $f(123)$

Ans $f(123) = f((100) + 2(010) + 3f(001))$

$= \lambda + 2\lambda + 3\lambda = 6\lambda$

But $f(1,1,1) = \lambda + \lambda + \lambda = 3\lambda$

But $\lambda = f(1,1,1)$

$\therefore \lambda = 3\lambda$

$\therefore \lambda = 0$

$\therefore f(1,2,3) = 0$

16) Out of scope

17) ''

18) ''

19) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and be viewed as $f = (f_1, \cdots, f_m)^T$ for $f_i \in \mathbb{R}^n \to \mathbb{R}$.

Prove f is linear iff each f_i is linear

Ans $f(ax+by) = af(x) + bf(y)$

iff $\begin{bmatrix} f_1(ax+by) \\ \vdots \\ f_m(ax+by) \end{bmatrix} = a \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} + b \begin{bmatrix} f_1(y) \\ \vdots \\ f_m(y) \end{bmatrix}$

iff $f_i(ax+by) = af_i(x) + bf_i(y) \quad \forall i$

iff f_i is linear $\forall i$

20) Check linear or not. Write matrix if linear

(i) $T(a,b,c,d) = (a,b)$

(ii) $T(a,b,c,d) = (a,b,0,0)$

(iii) $T(a,b) = (3a+b, 0, 0)$

(iv) $T(a,b,c) = (\alpha a, \beta b, \gamma c)$ for $\alpha, \beta, \gamma \in \mathbb{R}$

Ans Easy to check all are linear

(i) $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(ii) $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(iii) $T = \begin{bmatrix} 3 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(iv) $T = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

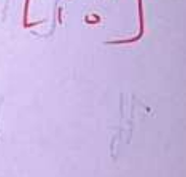
21) Describe geometric meaning:

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

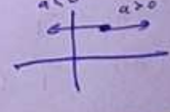
Ans (i) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

(projection on X axis)

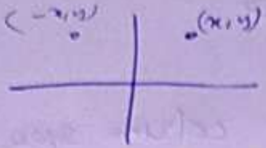


(ii) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$

(bluff x coordinate by a)



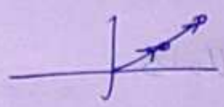
(iii) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$



reflect in Y axis

(iv) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} x \\ y \end{bmatrix}$

scale vector by a



(v) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

reflect in x=y line.

22) repetetive questions

23) give A, B square such that $r(A) = r(B)$ but $r(A^2) \neq r(B^2)$

Ans $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = A \qquad B^2 = 0$

24) Show $\text{rank}(AB) \leq \min\{\text{rank } A, \text{rank } B\}$

~~Ans rank (AB) = dim (col space (AB))~~

Ans Let $v \in$ Null space B

Then $Bv = 0$

$\therefore ABv \Rightarrow NS(B) \subseteq NS(AB)$

$\therefore$ nullity $B \leq$ nullity AB

$\therefore$ rank $(AB) \leq$ rank B

Column space of $AB \subseteq$ column space of A

(Think why?)

$\therefore$ rank $AB \leq$ rank A

$\therefore$ It follows

rank $AB \leq \min\{\text{rank}(A), \text{rank } B\}$

25) check if (f, g) is an inner product on $C^1[a,b]$

(continuously differentiable f on $[a,b]$)

(i) $\int_a^b f'(t)\, g'(t)\, dt$

(ii) $\int_a^b f' g' + \int_a^b f g$

(iii) $f(b)\, g(b)$

(iv) $\int_a^b f \int_a^b g$

Ans (i) No! $(1,1) = \int_a^b 0 \cdot 0 = 0$

(ii) Yes

(a) $(f,g) = (g,f)$ ✓

(b) $(af_1 + bf_2, g) = \int_a^b (af_1 + bf_2) g + (af_1' + bf_2') g'$

$= a \int_a^b (f_1 g + f_1' g') + b \int_a^b f_2 g + f_2' g'$

✓

(c) $$(f,f) = \int_a^b f^2(x) + (f'(x))^2 dx$$

$$= \int_a^b ((f(x))^2 + (f'(x))^2)\, dx$$

$$\geq 0$$

equal to 0 iff $f(x) = 0$, $f'(x) = 0$ on $[a,b]$

i.e. $f = 0$ ✓

(iii) No

$(f,f) = (f(b))^2 \geq 0$

but $(f,f) = 0$

$\Leftrightarrow f(b) = 0$

doesn't necessarily force $f = 0$

(iv) No!

$$(f,f) = \left(\int_a^b f(x)\, dx\right)^2$$

but $(f,f) = 0$

doesn't imply $f = 0$ necessarily

2b) Gram Schmidt (DIY)

use $\langle f,g \rangle = \int_a^b f(t)\, g(t)\, dt$ as inner product on $C^1[a,b]$ and orthogonalise the following

(i) $\{1, \cos x, \cos^2 x\}$ on $[0, \pi]$

(ii) $\{1, \sin x, \cos x, \sin^2 x\}$ on $[-\pi, \pi]$

(iii) $\{1, x, x^2, x^3\}$ over $[-1, 1]$

Ans (i) $\left\{1, \cos x, \frac{\cos 2x}{2}\right\}$

(ii) $\left\{1, \sin x, \cos x, -\frac{1}{2}\cos 2x\right\}$

(iii) $\left\{1, x, \frac{x^2 - 1}{3}, \frac{x^3 - 3x}{5}\right\}$

(Haven't checked them yet! Pls DIY)

27)
28) } Out of scope
29)