A Stroll through Orthogonal Polynomials

Definition : Given a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, it is fully determined by the values $L(x^n)$ $n \geq 0$. These values $L(x^n)$ are called the moments of L and are denoted as μ_n $(n \geq 0)$

Definition : A sequence $\{p_n(x)\}_{n \geq 0}$ of polynomials is called an orthogonal polynomial seq. (OPS) wrt a linear $L : \mathbb{R}[x] \to \mathbb{R}$

(i) $\deg(p_n(x)) = n \quad \forall \quad n \geq 0$

(ii) $L(p_m(x)\, p_n(x)) = \lambda_n \delta_{m,n}$ for some $\lambda_n \neq 0 \quad \forall\, m, n \geq 0$

If $\lambda_n = 1 \;\; \forall\, n \geq 0$, it is called orthonormal

Proposition 1

Let L be a moment functional (another name) and let $\{P_n(x)\}_{n \geq 0}$ be a seq of poly. TFAE

(a) $\{P_n(x)\}$ is OPS wrt L

(b) $L(\pi(x)\, P_n(x)) = 0 \quad \forall \quad \pi \in \mathbb{R}[x]$ with $\deg(\pi) < n$, $\forall n$

and $L(\pi(x)\, p_n(x)) \neq 0$ if $\deg(\pi) = n$

(c) $L(x^m P_n(x)) = K_n \delta_{m,n}$ for some $K_n \neq 0$

proof

$a \Rightarrow b$:

$\{p_n(x)\}$ forms a basis for $\mathbb{R}[x]$ & hence $\pi(x)$ can be expressed in terms of $p_i(x)$. If $\deg(\pi) < n$, all the terms $L(p_i(x)\, p_n(x))$ are 0 & if $\deg(\pi) = n$, then the $L(p_n(x)\, p_n(x))$ term is the only non-zero term

$b \Rightarrow c$:

Trivially true

$c \Rightarrow a$:

Assume for now that p_n has degree n

Then $L(p_n(x)\, p_m(x))$ (WLOG assume $n \le m$)

$$= \sum_{i=1}^{n} a_i \, L(x^i p_m(x)) = \sum_{i=1}^{n} a_i K_i \delta_{im}$$

$$= \begin{cases} 0 & n < m \\ a_n K_n & n = m \end{cases}$$

Since $a_n \neq 0$, we are done.

Now we have to show that each p_n has degree n

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Proposition 2

If $\{P_n(x)\}$ is an OPS for L, it is unique upto scalar multiplication in each polynomial

proof

Let $\{P_n(x)\}$, $\{Q_n(x)\}$ be two OPS wrt L

write $P_n(x) = \sum_{i=0}^{n} c_i \, Q_i(x)$ ($\because$ basis)

Multiply both sides with $Q_j(x)$ and apply L to get $(0 \le j \le n)$

$$L(P_n(x)\, Q_j(x)) = c_j \, L(Q_j(x)\, Q_j(x))$$

$$\therefore \quad c_j = \frac{L(P_n(x)\, Q_j(x))}{L(Q_j(x)\, Q_j(x))}$$

Since $\{P_j\}$ is OPS & $\deg(Q_j) = j$,

$L(P_n(x)\, Q_j(x)) = 0 \quad \forall\ 0 \le j < n$ and is non zero for $j = n$.

Denominator always non zero since $\{Q_j\}$ is OPS

$\therefore$ Only c_n survives and we have our result ■

Remark : We may now assume that an OPS is monic (that is every P_n is monic)

Definition : Given the moments μ_n of some linear L, we define the associated infinite size matrix $\underline{M_L}$ whose $(i,j)^{th}$ entry is μ_{i+j} (index the rows & columns by $\mathbb{N} = \{0, 1, \dots\}$)

Theorem 3

Given a linear functional L, L admits an OPS iff every $n \times n$ determinant hooked at $(1,1)$ is non-zero

Proof

To better understand the condition, we show the $n \times n$ blocks in the matrix

$$\begin{matrix} \boxed{\mu_0} & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

We have shown the first three blocks.
These determinants should be non-zero

Let $P_n(x) = \sum_{k=0}^{n} c_{n,k}\, x^k$

$\{P_n(x)\}$ is an OPS iff $\mathcal{L}(x^m P_n(x)) = K_n \delta_{mn}$ for some $K_n \neq 0$ $\forall$ $m \leq n$

This is equivalent to the system

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix} \begin{bmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ c_{n,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ K_n \end{bmatrix}$$

Suppose an OPS for $\mathcal{L}$ exists, via induction, it is easy to observe that K_n determines the OPS

Thus, our linear system has a unique solution determined by K_n

Hence, by Crammer's rule, the determinants aren't zero

Conversely, if they aren't zero, we can get a unique solution & construct our OPS.

One needs to argue that each P_n has degree n but this follows because $c_{nn} = \frac{K_n \Delta_{n-1}}{\Delta_n} \neq 0$ $(n \geq 1)$

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Theorem 4 (Favard's theorem)

Given a monic seq $\{p_n(x)\}$, it is an OPS for some $\mathcal{L}$ iff $\exists$ sequences $\{b_n\}_{n\geq 0}$ & $\{\lambda_n\}_{n\geq 1}$ s.t. $\lambda_n \neq 0$ $\forall n \geq 1$ and $x\, p_n(x) = p_{n+1}(x) + b_n p_n(x) + \lambda_n p_{n-1}(x)$ with $p_0(x) = 1$ and $p_1(x) = x - b_0$

proof

lemma : Define a seq of poly as in the above reccurence relation & define $L : \mathbb{k}[x] \to \mathbb{k}$ as $L(1) = 1$ and $L(p_n(x)) = 0 \ \forall \ n \geq 1$

Then, $L(x^n \, p_k(x) \, p_\ell(x)) = \lambda_1 \cdots \lambda_\ell \cdot$ WGF (Motzkin paths from $(0,k)$ to (n,ℓ))

proof :

Base case is $n = 0$

We want to show $L(p_k(x) \, p_\ell(x)) = \lambda_1 \cdots \lambda_\ell \, \delta_{k\ell}$

For $k = \ell = 0$ it is trivially true

WLOG $k \geq \ell$

Now $p_\ell(x) = x \, p_{\ell-1}(x) - b_{\ell-1} \, p_{\ell-1}(x) - \lambda_{\ell-1} \, p_{\ell-2}(x)$

$$\therefore L(p_k p_\ell) = L(x \, p_k p_{\ell-1}) - b_{\ell-1} L(p_k p_{\ell-1}) - \lambda_{\ell-1} L(p_k p_{\ell-2})$$

$$= L(x \, p_k p_{\ell-1}) \quad (\because \text{By induc hypo, they are } 0)$$

$$= L(p_{\ell-1}(p_{k+1} + b_k p_k + \lambda_k p_{k-1}))$$

$$= \lambda_k L(p_{\ell-1} p_{k-1}) \quad (\because \text{Induc hypo})$$

$$= \lambda_k \lambda_{k-1} L(p_{\ell-2} p_{k-2})$$

$$= \lambda_k \cdots \lambda_1 \, \delta_{k,\ell}$$

We only need to argue that $L(p_1(x) \, p_1(x)) = \lambda_1$

$$L((x-b_0)^2) = L(x^2) - 2b_0 L(x) - b_0^2 L(1)$$

Now $L(x) = b_0 \quad (\because L(p_1) = 0)$ & $L(1) = 1 \quad (\because \text{given})$

Also, $p_2(x) = x \, p_1(x) - b_1 p_1(x) - \lambda_1 p_0(x)$

$$= x(x - b_0) - b_1(x - b_0) - \lambda_1$$

$$= x^2 - (b_0 + b_1)x + b_0 b_1 - \lambda_1$$

$L(p_2) = 0 \Rightarrow L(x^2) = (b_0 + b_1) b_0 - b_0 b_1 + \lambda_1$

$= b_0^2 + \lambda_1$

$\therefore \; L(x^2) - 2b_0 L(x) - b_0^2 L(1)$

$= b_0^2 + \lambda_1 - 2b_0^2 - b_0^2 = \lambda_1$ & we are done with the base case

Suppose the result is true upto $n-1$

$L(x^n p_k p_\ell) = L(x^{n-1}(x p_k) p_\ell)$

$= L(x^{n-1} p_{k+1} p_\ell) + b_k L(x^{n-1} p_k p_\ell) + \lambda_k L(x^{n-1} p_{k-1} p_\ell)$

Using induction hypothesis,

$L(x^{n-1} p_{k+1} p_\ell) = \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k+1) \to (n-1, \ell)$)

$= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(1, k+1) \to (n, \ell)$)

$= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k) \to (n, \ell)$ with first step being U step)

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$= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k) \to (n, \ell)$ with first step being H step) $\times \frac{1}{b_k}$

$L(x^{n-1} p_{k-1} p_\ell) = \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k-1) \to (n-1, \ell)$)

$= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(1, k-1) \to (n, \ell)$)

$= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k) \to (n, \ell)$ with first step being D) $\times \frac{1}{\lambda_k}$

It now follows that LHS $= \lambda_1 \cdots \lambda_\ell$ WGF (motzkin $(0, k) \to (n, \ell)$)

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Let $\{P_n(x)\}$ be OPS for some L (assume monic)

Write $x P_n(x) = P_{n+1}(x) + \sum_{k=0}^{n} \alpha_{n,k} P_k(x)$

Multiplying by $P_j(x)$ & applying L, we have

$L(x P_j P_n) = L(P_j P_{n+1}) + \sum_{k=0}^{n} \alpha_{n,k} L(P_j P_k)$

For $j = 0, 1, \dots, n-2$, we have

$L(x P_j P_n) = \alpha_{n,j} L(P_j^2)$

Since $j \leq n-2$, $x P_j$ has degree at most $n-1$ & hence

LHS is 0. Thus $\alpha_{n,j} = 0$

We now have

$x P_n(x) = P_{n+1}(x) + \alpha_{n,n-1} P_{n-1}(x) + \alpha_{n,n} P_n(x)$

$L(x P_n P_{n-1}) = \alpha_{n,n-1} L(P_{n-1}^2)$

Since $x P_{n-1}$ has degree n, LHS is non zero implying

that $\alpha_{n,n-1}$ is non zero. This forms our sequence λ_n

with $\lambda_n = \alpha_{n,n-1}$ $(n \geq 1)$

We dont care about $\alpha_{n,n}$. It is our sequence b_n

with $b_n = \alpha_{n,n}$

$p_0(x) = 1$ is forced by monic property

Let $p_1(x) = x + a$

$x p_0(x) = p_1(x) + \alpha_{0,0} p_0(x)$

i.e. $x = x + a + \alpha_{0,0}$

$\therefore \alpha_{0,0} = b_0 = -a$

$\therefore p_1(x) = x - b_0$

This proves one direction of Favard's theorem.

Conversely, let the recurrence be satisfied

Define L as $L(1) = 1$, $L(p_n(x)) = 0$ $\forall n$

By lemma,

$$L(P_k P_\ell) = \lambda_1 \cdots \lambda_\ell \text{ wt of (motzkin } (0,0) \to (0,\ell))$$
$$= \lambda_1 \cdots \lambda_\ell \, \delta_{k\ell}$$
$$= K_\ell \, \delta_{k\ell}$$

And hence $\{P_n\}$ is OPS wrt L

(It is easy to give a degree argument for P_n via induction using recurrence)