Definition: An equation which contains the differential operator $\frac{d}{dt}$ and is dependent only on a single independent variable is called an ODE

Definition: A differentiable function $\phi : I \to \mathbb{R}$ is a solution of the ODE if it satisfies it. Here, I is some interval

THEOREM 1 (general simple ODE)

Let $\frac{dx}{dt} = f(t)$ for some $f : I \to \mathbb{R}$, a continuous function on $I \subset \mathbb{R}$.

ϕ is a solution iff $\phi(t) = \int_{t_0}^{t} f(s)\,ds + C$ for some $C \in \mathbb{R}$, $t_0 \in I$

THEOREM 2 (constant coeff linear)

$\frac{dx}{dt} = a\,x(t)$ for some $a \in \mathbb{R}$.

ϕ is a solution iff $\phi(t) = C e^{at}$ for some $C \in \mathbb{R}$

THEOREM 3 (first order linear)

$\frac{dx}{dt} = \alpha(t)\,x(t) + \beta(t)$; $\alpha, \beta : I \to \mathbb{R}$

are continuous. ϕ is a solution iff $\varphi(t) = \int_{t_0}^{t} a(s)\,ds$

and $\phi(t) = e^{\varphi(t)} \left(\int_{t_0}^{t} e^{-\varphi(s)} \beta(s)\,ds + C \right)$

THEOREM 4 (separable)

$\frac{dx}{dt} = f(t)\,g(t)$ for f, g continuous real valued functions on $I_1, I_2 \subset \mathbb{R}$. Suppose $|f(m)| > 0 \;\forall\; m \in I_1$.

ϕ is a solution iff $\varphi(\phi(t)) = \int_{t_0}^{t} g(s)\,ds + C$

for $\varphi(u) = \int_{u_0}^{u} \frac{1}{f(y)}\,dy$ and some $C \in \mathbb{R}$,

$t_0 \in I_2$, $u_0 \in I_1$

If $f(m) = 0$ for some $m \in I_1$, $\phi(t) = m$ $\forall\, t$ is a solution for the ODE

Definition: $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$ is called an initial value problem. It is an ODE with value of solution at some point specified. Here $t_0 \in$ domain of solution

THEOREM 5 (special forms)

1) $\frac{dx}{dt} = f(at + bx + c)$ is solved by using $u = at + bx + c$

$(b \neq 0, a \neq 0)$

2) $\frac{dx}{dt} = f\left(\frac{x}{t}\right)$ is solved using $u = \frac{x}{t}$

3) $\frac{dx}{dt} = f\left(\frac{at+bx+c}{ct+gx+h}\right)$; $\frac{at+bx+c}{ct+gx+h} \notin \mathbb{Z}$

Case 1 : $ag = bc$

use $u = at + bc$

Case 2 : $ag \neq bc$

suppose (t_0, x_0) solves $\begin{cases} at+bx+c=0 \\ ct+gx+h=0 \end{cases}$

then we use $y = x - x_0$
$s = t - t_0$

4) $\frac{dx}{dt} = x(b - cx)$ $b, c > 0$

(logistic equation)

stationary solutions : $\phi(t) = 0$
$\phi(t) = \frac{b}{c}$

general solution: $\phi(t) = \frac{b}{c} \frac{1}{1+Ke^{-bt}}$; $K \in \mathbb{R}$, $K \neq 0$

5) $\frac{dx}{dt} = f(t)\, x + g(t)\, x^n$; $n \neq 0, 1$

(Bernoulli equation)

we use $y = x^{1-n}$

6) $\frac{dx}{dt} = f(t)\, x + g(t)\, x^2 + h(t)$

and $\phi(t)$ is a known solution.

(Ricatti equation)

we use $y(t) = \frac{1}{x(t) - \phi(t)}$

Definition: $P(x,y) \frac{dy}{dx} + Q(x,y) = 0$, for P, Q real valued on some $\Omega \subseteq \mathbb{R}^2$, is called an exact differential equation in Ω if $\exists\ F(x,y) \in C^2(\Omega)$ such that $\partial_y F(x,y) = P(x,y)$ and $\partial_x F(x,y) = Q(x,y)$ $\forall\ x, y \in \Omega$ (Ω is open, connected in $\mathbb{R}^n$)

THEOREM 6 (Testing exactness)

In the above content, if P, Q, P_x, Q_y are continuous in some bounded region, then the above ODE is exact iff $P_x = Q_y$ over the bounded region

Definition: $\mu(x,y)$ is called an integrating factor of $P(x,y) \frac{dy}{dx} + Q(x,y) = 0$ if

$\mu(x,y)\, P(x,y) \frac{dy}{dx} + \mu(x,y)\, Q(x,y) = 0$ is exact

THEOREM 7 (finding the IF)

suppose $\frac{Q_y - P_x}{P}$ is a function of only x,

then $\mu(x,y) = \mu(x)$ with $\mu'(x) = \mu(x) \times \left(\frac{Q_y - P_x}{P}\right)$

Suppose $\frac{Q_y - P_x}{Q}$ is a function of only y,

then $\mu(x,y) = \mu(y)$ with $\mu'(y) = \mu(y) \left(\frac{P_x - Q_y}{Q}\right)$

Definitions: A sketch of the parameteric solution of an ODE system in $(x(t), y(t))$ is called the phase portrait on the phase plane which is the XY plane

THEOREM 8 (Prey-Predator / Lotka-Volterra)

$$\begin{cases} \frac{dx}{dt} = x(A - By) \\ \frac{dy}{dt} = y(-C + Dx) \end{cases} \quad ; \; A, B, C, D > 0 \text{ are constants}$$

The stationary solutions are $(0,0)$ and $\left(\frac{C}{D}, \frac{A}{B}\right)$

and $C \log x - Dx + A \log y - By + K = 0$ is the phase portrait

THEOREM 9

All trajectories of the prey predator model form closed Jordan curves that surround $\left(\frac{C}{D}, \frac{A}{B}\right)$

Definition: A banach space of functions refers to the vector space of real valued functions equipped with a norm $\|\cdot\|_\infty : X \to \mathbb{R}$ defined as $\|f\|_\infty := \max\{f(x) \mid x \in \text{Domain}(f)\}$

THEOREM 10 (Banach fixed point theorem)

A contraction is a map $T : C \to X$ where C is a subset of a banach space X such that,

$\|Tx - Ty\|_X \leq \theta \|x-y\|_X$ for some $\theta < 1$ holds for every $x, y \in C$.

The BFPT states that every $T : C \to X$ which is a contraction, has a fixed point in C

THEOREM 11 (Contraction mapping theorem)

Let C be a closed non empty set of Banach space X and let $F : C \to C$ be a contraction. Then F has a

~~unique fixed point~~

unique fixed point $a \in C$ such that

$$\| F^m(x) - a \| \leq \frac{\theta^m}{1-\theta} \| F(x) - x \| \quad \forall x \in C$$

where F^m denotes $(F \circ F \circ \cdots \circ F)$ m times

THEOREM 12 (Existence- Uniqueness)

consider $\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ where $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$

is continuous and U is open with $(t_0, x_0) \in U$

If f is locally lipschitz in the t variable uniformly with respect to the x variable, then there exists a unique local solution $\phi : [t_0 - T_0, t_0 + T_0] \to U$ of the IVP for some $T_0 > 0$

note: locally lipschitz in t wrt x $\Rightarrow$

$$\sup_{\substack{(t,x) \neq (t,y) \\ \text{in } V}} \left| \frac{f(t,x) - f(t,y)}{x - y} \right| = K_V < +\infty$$

holds $\forall$ $V \subseteq U$

note: f is C^1 $\Rightarrow$ f is lipschitz in all variables

note: f is globally lipschitz if K_V doesnt depend on V

i.e. $|f(x_1) - f(x_2)| \leq K |x_1 - x_2| \quad \forall x_1, x_2$

THEOREM 13 (Gronwall's inequality)

(i) Suppose $\varphi(t)$ satisfies $\varphi(t) \le A + \int_0^t (B\varphi(s) + C)\, ds$

$\forall\ t \in [0, T)$ for $B, T > 0$ & $A, C \in \mathbb{R}$, then

$$\varphi(t) \le A e^{Bt} + \frac{C}{B}(e^{Bt} - 1) \quad \forall\ t \in [0, T]$$

(ii) suppose $\varphi'(t) \le \alpha(t)\varphi(t) + \beta(t)$ in $[0, T]$

then $\varphi(t) \le \exp\left(\int_0^t \alpha(s) ds\right)\left[\varphi(0) + \int_0^t \beta(s) \exp\left(-\int_0^s \alpha(r) dr\right) ds\right]$

THEOREM 14

Suppose $f, g \in C(U, \mathbb{R}^n)$ are locally lipschitz in x variable and $x(t), y(t)$ solve the IVP's

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(t, y) \\ y(t_0) = y_0 \end{cases},$$

then $|x(t) - y(t)| \le |x_0 - y_0| e^{K|t - t_0|}$

$$+ \frac{M}{K}\left(e^{K|t - t_0|} - 1\right)$$

where K is lipschitz const of f, $M = \sup_{\substack{(t,x) \in V \\ (t,y) \in V}} |f(t,x) - g(t,y)|$

THEOREM 15 (continuous dependence on data)

Let $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$ be continuous with $f = f(t, x)$. Let f be locally Lipschitz in x. Then,

$\forall\ (t_0, x_0) \in U$, $\exists\ T_0 > 0$, $r_0 > 0$ such that

$\left[t_0 - \frac{T_0}{2}, t_0 + \frac{T_0}{2}\right] \times \underset{\text{ball}}{B}\left(x_0, \frac{r_0}{2}\right) \subset U$ and $\exists$

$\Phi(t, s, y) \in \mathcal{C}\left(\left[t_0 - \frac{T_0}{2}, t_0 + \frac{T_0}{2}\right] \times B\left(x_0, \frac{r_0}{2}\right)\right)$.

Moreover $\Phi(t, t_0, x_0)$ is lipschitz continuous

and we have

$$|\Phi(t, t_0, x_0) - \Phi(s, s_0, y_0)| \le M|t-s| + |x_0 - y_0| e^{K|t-t_0|} + M|s_0 - t_0| e^{K|t-s_0|}$$

where $M = \max\limits_{[t_0 - \frac{T_0}{2}, t_0 + \frac{T_0}{2}] \times B(x_0, r_0)} |f(t,x)|$

K = lipschitz const of f

THEOREM 16 (Maximal interval)

The IVP $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ has a unique maximal

solution defined on a maximal interval $(T_-, T_+) \ni t_0$

THEOREM 17 (extension lemma)

Let $\phi(t)$ be a solution of $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ defined on

(t_-, t_+). (f is locally lipschitz), Then there is an

extension of ϕ to the interval $(t_-, t_+ + \varepsilon)$

for some $\varepsilon > 0$ iff $\exists \{t_m\} \subseteq (t_-, t_+)$ s.t.

$\lim_{m\to\infty} t_m = t_+$ and $(t_+, \lim_{m\to\infty} \phi(t_m)) \in U$.

Analogously, there may be an extension to the left side

THEOREM 18

(i) Suppose ϕ solves $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ on $[t_0, t_+)$ and there is a compact set C such that $\phi(t_m) \in C$ for every sequence $\{t_m\} \to t_+$, then the extension is unique

(ii) Suppose $\forall \tilde{t} > t_0$, $\exists$ C compact s.t. $\phi(t_m) \in C$ for some seq in $(t_0, \tilde{t})$ converging to $\tilde{t}$, then ϕ can be extended to $[t_0, +\infty)$

(iii) If (T_-, T_+) is the maximal interval of existence and $T_+ < +\infty$, then the soln is not contained in any compact set C with $[t_0, t_+] \times C \subset U$

(iv) $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ with f being lipschitz. Suppose (T_-, T_+) is the maximal interval & $T_+ < +\infty$ or $T_- > -\infty$, then

$\lim_{\substack{t\to T^+ \\ t < T_+}} |x(t)| = +\infty$ or $\lim_{\substack{t\to T^- \\ t > T_-}} |x(t)| = +\infty$

THEOREM 19 (Global solution)

consider $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$ with $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ being locally lipschitz and suppose we have that

$|f(t,x)| \leq M_T + K_T |x| \quad \forall\, T$ s.t. (t,x) is in $[-T, T] \times \mathbb{R}^2$, then, any solution of the

- IVP is a global solution

Definition: we express the system of linear ODE's

$$\left\{ x_i'(t) = \sum_{j=1}^{n} a_{ij}(t)\, x_j(t) + b_i(t) \quad \forall\, i = 1, 2, \dots, n \right.$$

as $X'(t) = A(t) X(t) + B(t)$

for $x(t) = [x_1(t) \; \dots \; x_n(t)]^T$

$[A(t)]_{ij} = a_{ij}(t)$

$B(t) = [b_1(t) \; \dots \; b_n(t)]^T$

THEOREM 20

suppose $X'(t) = A\, x(t)$ and $A = \mathrm{diag}(\lambda_1 \dots \lambda_n)$

then, $X(t) = [\, x_1(0) e^{\lambda_1 t} \;\; x_2(0) e^{\lambda_2 t} \; \dots \; x_n(0) e^{\lambda_n t} \,]$

is a solution to the system given.

we can also express it as

$X(t) = \exp(tA)\, X(0)$ (notation in next definition)

Definition: we define $\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ (convergence in theorem 21)

THEOREM 21

Suppose we use the matrix norm

$$\|A\| := \sup_{\substack{x \in \mathbb{C}^n \\ \|x\| = 1}} \|Ax\| ,$$

we see results

(i) $\|AB\| \leq \|A\| \, \|B\|$

(ii) $\{ A \in \mathbb{C}^{n \times n} \}$ forms a banach space with this norm

(iii) $\| \exp(A) \| \leq e^{\|A\|}$

THEOREM 22

- For a diagonal matrix $D = [\lambda_1 \dots \lambda_n]^{diag}$,
  $$\exp(tD) = [e^{t\lambda_1} \dots e^{t\lambda_n}]^{diag}$$
- $\exp(UDU^{-1}) = U \exp D \, U^{-1}$ for a diagonal matrix D
- $\exp(A+B) = \exp(A)\exp(B)$ if $AB = BA$

THEOREM 23

$x(t) = e^{\lambda t} v$ solves $x' = Ax$ where λ is an eigenvector of the constant matrix A corresponding to the e.vec v

THEOREM 24

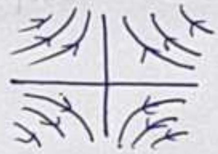
Let $A_{2\times 2}$ have real e.vals $\lambda_1 \neq \lambda_2$ and e.vecs v_1, v_2 respectively. The general solution to $X' = AX$ is then given by

$$x(t) = \alpha e^{\lambda_1 t} v_1 + \beta e^{\lambda_2 t} v_2 \; ; \; \alpha, \beta \in \mathbb{R}$$

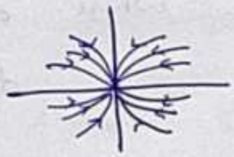
Definitions:

$X' = AX$, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

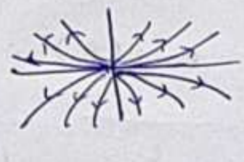
Case 1 : $\lambda_1 < 0 < \lambda_2 \implies$ saddle



Case 2 : $\lambda_1 < \lambda_2 < 0 \implies$ sink

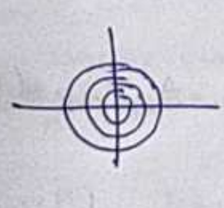


case 3 : $\lambda_2 > \lambda_1 > 0 \implies$ source



Now, $X' = AX$, $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, $\beta \neq 0$

circles



$\beta > 0 \implies$ clockwise arrows

$\beta < 0 \implies$ anticlockwise

$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $\alpha, \beta \neq 0$

$\alpha < 0$ towards origin

$\alpha > 0$ away from origin

spiral

THEOREM 25 (Jordan forms)

For any $A \in \mathbb{C}^{n\times n}$, $\exists\ U$ s.t.

$$U^{-1} A U = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{bmatrix} \quad \text{where each } B_i$$

is a jordan block $\begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} = \lambda_i I + N$

(i) $N_{k\times k}$ is nilpotent of order k

(ii) $$\exp(B_i) = e^{\lambda_i}\begin{bmatrix} 1 & 1 & \cdots & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}_{k\times k} \quad d_1, d_2, \ldots, d_k$$

where diagonal d_i has all entries $\frac{1}{(i-1)!}$

(iii) $$\exp(A) = U \begin{bmatrix} \exp B_1 & & 0 \\ & \ddots & \\ 0 & & \exp B_m \end{bmatrix} U^{-1}$$

(iv) $$\det(\exp(tA)) = e^{t(\mathrm{tr}(A))}$$

(v) $$\frac{d}{dt}\left(\exp(tA)\right) = A\exp(tA) = \exp(tA)A$$

~~THEOREM 26~~ (vi) $$\left.\frac{d}{dt}\left(\det(A+tB)\right)\right|_{t=0} = \det(A)\,\mathrm{tr}(A^{-1}B)$$

THEOREM 26 (Existence uniqueness - autonomous)

Given any $x_0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, there is a unique solution of

$$\begin{cases} x'(t) = A\,x(t) \\ x(0) = x_0 \end{cases}$$

Moreover, $x(t) = \exp(tA)\, x_0$ is the solution.

THEOREM 27

Set of all solutions to $x' = Ax$ forms a vector space of dimension n.

THEOREM 28

A solution of $\{ x'(t) = Ax,\ x(0) = x_0 \}$ converges to $\vec{0}$ at $t \to +\infty$ iff x_0 lies in the subspace spanned by the generalised eigenspaces corresponding to eigen values with negative real part.

Moreover, the solution remains bounded as $t \to +\infty$ iff x_0 lies in the subspace spanned by generalised eigenspaces corresponding to eigenvalues with negative real part $\oplus$ subspace spanned by generalised eigenspaces corresponding to e-vals with vanishing real part.

Definition: A linear system is said to be stable if all solutions remain bounded as $t \to +\infty$ and it is said to be asymptotically stable if all solutions tend to 0 as $t \to +\infty$

THEOREM 29 (Duhamel's formula)

Let $A \in \mathbb{R}^{n\times n}$, $f : I \to \mathbb{R}^n$ be continuous, Then, there exists a unique solution of the IVP

$$\begin{cases} x'(t) = Ax(t) + f(t) \\ x(0) = x_0 \end{cases}$$

Moreover, it is given by

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A) f(s)\, ds$$

THEOREM 30

$x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = 0$ is encoded as

$$x'(t) = Ax(t) \quad \text{for} \quad x(t) = [x \;\; x' \cdots x^{(n-1)}]^T$$

and
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ & & & & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix}_{n\times n}$$

. Now,

suppose the characteristic equation of A is

$$(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$$

then $x_{ij}(t) = t^j \exp(t\lambda_i)$ are n linearly independent solutions of the system.

note: if $\lambda_i = \alpha + \beta i$
$\tilde{\lambda}_i = \alpha - \beta i$,

we use $x_{ij}(t) = t^j \cdot e^{t\alpha_i} \cos(\beta_i t)$
$\tilde{x}_{ij}(t) = t^j e^{t\alpha_i} \sin(\beta_i t)$

THEOREM 31

$x^{(n)} + \cdots + a_0 x = f(t)$ has solution

$$x(t) = x_n(t) + \int_0^t u(t-s) f(s)\, ds$$

where $x_n(t)$ is a solution of the homogenous equation and $u(t)$ is the solution of the homogeneous solution corresponding to

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u^{(n-1)}(0) = 1$$

THEOREM 32

All solutions of $x'(t) = A(t)x(t)$ form an n-dim vec space (note that A isn't const. anymore)

THEOREM 33

The solution of $\begin{cases} x'(t) = A(t)\,x(t) \\ x(t_0) = x_0 \end{cases}$ is given by

$$x(t) = M(t, t_0)\, x_0$$

where $M(t, t_0) = [\, \phi(t, t_0, e_1) \;\cdots\; \phi(t, t_0, e_n) \,]$

where $\{e_i\}$, forms standard basis of $\mathbb{R}^n$ and

$\phi(t, t_0, e_i)$ is the unique solution of

$$\begin{cases} x'(t) = A(t)\,x(t) \\ x(t_0) = e_i \end{cases}$$

Definition: $M(t, t_0)$ is called the principal matrix solution at t_0 and it solves

$$\begin{cases} M'(t, t_0) = A(t)\, M(t, t_0) \\ M(t_0, t_0) = I_{n\times n} \end{cases}$$

THEOREM 34

$M(t, t_0)$ satisfies :

(i) $M(t, t_1)\, M(t_1, t_0) = M(t, t_0)$

(ii) $(M(t, t_0))^{-1} = M(t_0, t)$

note : $(A(t)^{-1})' = -(A(t))^{-1}\, A'(t)\, (A(t))^{-1}$

Definition : $x'(t) = A(t)\,x(t)$ has solutions, say,

$\phi_1(t), \dots, \phi_n(t)$ (all LI).

Construct $U = [\,\phi_1(t) \;\cdots\; \phi_n(t)\,]_{n\times n}$

we define Wronskian as $W(t) = \det(U)$

THEOREM 35

for U, V fundamental matrix solutions,

$$V(t) = U(t)\,U^{-1}(t_0)\,V(t_0)$$

where $U(t)$ is a fundamental matrix solution if the wronskian is non zero for that U

THEOREM 36 (Variation of parameters)

$$\begin{cases} x(t) = A(t)x(t) + g(t) \\ x(t_0) = x_0 \end{cases}$$ has a solution given by

$$x(t) = M(t,t_0)\,x_0 + \int_{t_0}^{t} M(t,s)\,g(s)\,ds$$ for

$A \in C(I, \mathbb{R}^{n\times n})$, $g \in C(I, \mathbb{R}^n)$

$\to$ const coeff matrix

THEOREM 37 (Abels identity / Liouville's formula)

$$M'(t) = A(t)\,M(t) \Rightarrow \det(M(t)) = \det(M(t_0)) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\,ds\right)$$

Definition: Consider the solution $\gamma(t, t_0, x_0)$ of

$$\begin{cases} x' = F(t,x) \\ x(0) = x_0 \end{cases}$$ where F is locally lipschitz in t.

we say;

$\gamma(t, x_0)$ is stable in Lyapunov sense if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. for any $\gamma(t, y_0)$ solution to

$$\begin{cases} x' = F(t,x) \\ x(0) = y_0 \end{cases}$$ with $|y_0 - x_0| < \delta$, we

have $|\gamma(t,y_0) - \gamma(t,x_0)| < \varepsilon \quad \forall t \geq 0$

we say,

γ is assymptotically stable if it is stable and

$\exists r_0$ s.t. $|y_0 - x_0| < r_0 \Rightarrow$ all solution $\gamma(t, y_0)$

satisfy $|\gamma(t,y_0) - \gamma(t,x_0)| \to 0$ as $t \to +\infty$

γ is unstable if it is not stable

γ is exponentially stable if $\exists r_0 > 0$ such that

if $|y_0 - x_0| < r_0$ then any $\gamma(t, y_0)$ satisfies

$$|\gamma(t_0, y_0) - \gamma(t_0, x_0)| \leq C e^{-\alpha t} |y_0 - x_0|$$

for $C > 0$, $\alpha > 0$

Consider $\begin{cases} x' = F(x) \\ x(0) = \xi_0 \end{cases}$ where $F \in C(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^n$

and f is locally lipschitz.

If $F(\xi_0)=0$, $x(t)=\xi_0$ is an equilibrium / steady state solution and ξ is called a fixed point of the given system

The notions of stability transfer from the solution to ξ_0 and ξ_0 can be classified into stable, unstable, etc.

THEOREM 38

$X' = AX$ for A being constant matrix.

Let $r_{max} = \max\{ Re(\lambda) : \lambda \text{ is eval of } A \}$

The trivial solution $x(t)=0$ is

(i) asymptotically stable if $r_{max} < 0$

(ii) unstable if $r_{max} > 0$

(iii) stable if $r_{max} \leq 0$ with corresponding alg, geom multiplicity being equal for every eval with $Re(\lambda)=0$

Definition: Suppose $X' = F(X)$ for $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ being a C^1 function. Let $F(a)=0$ for some $a \in U$

The linearised system is defined as $Y' = (DF(a))\,Y$

derivative matrix

THEOREM 39

The eq. soln $x(t) = \varepsilon_0$ of $x' = F(x)$ is asymptotically stable if corresponding linearised system is also ~~asymp~~ asymptotically stable.

(asymptotically stable can be replaced by unstable also)

Definition: The stable solution $x(t) = \varepsilon_0$ is called hyperbolic if $\forall$ λ eigenvalues of $DF(\varepsilon_0)$, $Re(\lambda) = 0$

THEOREM 40 (Hartman - Grobman theorem)

Suppose $F : U \to \mathbb{R}^n$ is C^1 and $\varepsilon_0 \in U$ is a hyperbolic point of $x' = F(x)$

Then $\exists$ Ω_1 a neighbourhood of ε_0 and Ω_2 a neighbourhood of 0 and a homeomorphism $\varphi : \Omega_2 \to \Omega_1$ ~~s~~. that transforms the solutions of the linearised system to those of the original system.

Definition: Let $L : U \to \mathbb{R}$ $(U \subseteq \mathbb{R}^n)$ be a C^1 function.

L is called a Lyapunov function for $x' = F(x)$ if

(i) $L(\xi_0) = 0$ for some ξ_0 s.t $F(\xi_0) = 0$

(ii) $L(x) > 0 \quad \forall\ x \in U \setminus \{\xi_0\}$

(iii) $\frac{d}{dt}(L(\gamma(t))) \leq 0$ for $\gamma'(t) = F(x)$

equivalently, (iii) is

(iii)' $\langle \nabla L(x), F(x) \rangle \leq 0 \quad \forall\ x \in U$

THEOREM 41 (Lyapunov stability)

(i) If $\exists$ a lyapunov function for $x' = F(x)$, then the equilibrium solution $x(t) = \xi_0$ is stable

(ii) If $\langle \nabla L(x), F(x) \rangle < 0 \quad \forall\ x \in U \setminus \{\xi_0\}$, then the equilibrium solution is asymptotically stable

THEOREM 42 (Lyapunov instability)

Consider $x' = F(x)$ where F is lipschitz and $F(\xi_0) = 0$. Suppose $\exists\ L : U \subseteq \mathbb{R}^n \to \mathbb{R}$ which is C^1, $L(\xi_0) = 0$, $L(x) > 0 \quad \forall\ x \in U \setminus \{\xi_0\}$ and $\langle \nabla L(x), F(x) \rangle > 0 \quad \forall\ x \in U \setminus \{\xi_0\}$, then the stable point solution is unstable.

Definition: $u''(x) + f(x)u' + g(x)u = h(x)$, $x \in [a,b]$

given along with

$$\alpha_1 u(a) + \alpha_2 u'(a) = \eta_1$$

$$\beta_1 u(b) + \beta_2 u'(b) = \eta_2$$

is called a boundary value problem

if $u(a), u(b)$ are given, we call it a dirichlet boundary problem and if $u'(a), u'(b)$ are given we call it a Neumann boundary value problem

THEOREM 43

Suppose $\alpha_1^2 + \alpha_2^2 > 0$, $\beta_1^2 + \beta_2^2 > 0$ then either unique solution or

$$\begin{cases} u'' + fu' + gu = 0 \\ \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0 \end{cases}$$

has infinitely many solutions

Definition: $Lu := (p(x)u')' + q(x)u$

we wish to solve the Strum Liouville eigen value problem given by

$L_u = \lambda u$ (we solve for both λ, u)

and $\alpha_1 u(a) + \alpha_2 u'(a) = \eta_1$ $\alpha_1^2 + \alpha_2^2 > 0$

$\beta_1 u(b) + \beta_2 u'(b) = \eta_2$ $\beta_1^2 + \beta_2^2 > 0$

THEOREM 44

(i) $v L_u - u L_v = p(x) (u'v - v'u)'$

(ii) $\int_a^b v(x) L_u(x)\, dx = \int_a^b u(x) L_v(x)\, dx$

provided $\eta_1 = \eta_2 = 0$

(iii) We have countably infinitely many solutions to λ possible $\lambda_0 < \lambda_1 < \lambda_2 < \dots \longrightarrow +\infty$

(iv) The eigen functions corresponding to λ_k denoted by ω_k have exactly k zeroes in (a, b)

(v) Between two successive zeroes of ω_k, there is a zero of ω_{k+1}

(vi) Any continuous f satisfying the boundary conditions converges absolutely when expressed as $\sum_{k=0}^{\infty} c_k \omega_k$

and consequently $\int_a^b \omega_i \omega_j\, dx = \delta_{ij}$

Problem set 1

1) Find the general solution of the given ODE's

(i) $\frac{dx}{dt} = t^3 + \cos t$

<u>Ans</u> By theorem 1,

$$x(t) = \frac{t^4}{4} + \sin t + C \quad \text{for some } C \in \mathbb{R}$$

(ii) $\frac{dx}{dt} = -2x$

<u>Ans</u> By theorem 2,

$$x(t) = C e^{-2t} \quad \text{for some } C \in \mathbb{R}$$

(iii) $\frac{dx}{dt} = x^2$

<u>Ans</u> By theorem 3,

$$f(x) = x^2$$
$$g(t) = 1$$

$\therefore$ $x(t) = 0$ is always a solution and

suppose $0 \in I$ = interval / domain of solution,

then $|f(x)| > 0$

$$\therefore \varphi(\phi(t)) = \int_{t_0}^{t} 1 \, ds + C = t + K$$

$$\varphi(u) = \int_{u_0}^{u} \frac{1}{f(y)} dy = \int_{u_0}^{u} \frac{1}{y^2} dy = \frac{1}{u_0} - \frac{1}{u}$$

$$\therefore \frac{1}{u_0} - \frac{1}{\phi(t)} = t + K$$

$$\Rightarrow \phi(t) = \frac{1}{K' - t}$$

(iv) $\frac{dx}{dt} = x - x^2$

Ans using same theorem 3 as before,

$$\varphi(\phi(t)) = t + K$$

$$\varphi(u) = \int_{u_0}^{u} \frac{1}{y(1-y)} dy = \ln\left|\frac{y}{1-y}\right|\Bigg]_{u_0}^{u}$$

$$\therefore \ln\left(\left|\frac{\phi(t)}{1-\phi(t)}\right|\right) = t + K$$

$$\therefore \left|\frac{\phi(t)}{1-\phi(t)}\right| = C e^t$$

$$\therefore \frac{\phi(t)}{1-\phi(t)} = C' e^t \quad \text{(constant absorbs sign)}$$

$$\therefore \phi(t) = \frac{C' e^t}{1 + C' e^t} = \frac{e^t}{K + e^t}$$

(v) $\frac{dx}{dt} = x(1-x) - c$ for some constant c

Ans same as before,

$$\varphi(\phi(t)) = t + K$$

$$\varphi(u) = \int_{u_0}^{u} \frac{1}{y(1-y)+c}\, dy$$

$$= \int_{u_0}^{u} \frac{1}{c + y - y^2}\, dy$$

$$= \int_{u_0}^{u} \frac{dy}{\left(\frac{1}{4}+c\right) - \left(y-\frac{1}{2}\right)^2}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{\frac{1}{4}+c}} \ln\left|\dfrac{u-\frac{1}{2}-\sqrt{\frac{1}{4}+c}}{u-\frac{1}{2}+\sqrt{\frac{1}{4}+c}}\right| + K' \\ \quad \text{if } \frac{1}{4}+c > 0 \\ -\dfrac{1}{\sqrt{\left|\frac{1}{4}+c\right|}} \tan^{-1}\left(\dfrac{u-\frac{1}{2}}{\sqrt{\left|\frac{1}{4}+c\right|}}\right) + K' \\ \quad \text{if } \frac{1}{4}+c < 0 \end{cases}$$

∴ $\phi(t)$ can be determined by equating the above to $t + K$ (replacing u with $\phi(t)$)

Also include the solutions $x(t) = \alpha, \beta$ for roots of $x(1-x)+c = 0$

(vi) $\frac{dx}{dt} = e^x \sin t$

$f(x) = e^x$

$g(t) = \sin t$

$(f(x)) > 0 \quad \forall x \in \mathbb{R} \Rightarrow$ no static solution

$$\varphi(\phi(t)) = -\cos s \Big]_{s=t_0}^{s=t} + C$$

$$= -\cos t + K$$

$$\varphi(u) = \int_{u_0}^{u} \frac{1}{e^y}\, dy = e^{-u} - K'$$

$$\therefore e^{-\phi(t)} = -\cos t + C'$$

$c' \leq -1 \Rightarrow$ no solutions

$c' < 2 \Rightarrow -\phi(t) = \ln(c' - \cos t)$

whenever $c' - \cos t > 0$

$\Rightarrow \phi(t) = \ln\left(\frac{1}{c' - \cos t}\right)$

∴ the solution defined on an interval

where $\cos t < c'$

$-\cos t + c' > 0 \Rightarrow c' > \cos t$

for $c' > 1$ solution exists and is given

by $\phi(t) = \ln \dfrac{1}{c' - \cos t} \quad \forall\, t \in \mathbb{R}$

for $c' \leq -1$ no solution exists

for $c' = 1$,

$$\phi(t) = \ln \frac{1}{1 - \cos t} \quad \forall\, t \in \mathbb{R} \setminus \{t : \cos t = 1\}$$

for $|c'| < 1$

$\phi(t) = \ln \dfrac{1}{c' - \cos t}$ exists on some very 'bad' interval.

(vii) $\dfrac{dx}{dt} = (1+x^2)t$

__Ans__ $\quad \varphi(\phi(t)) = \dfrac{t^2}{2} + K$

$$\varphi(u) = \int_{u_0}^{u} \frac{dy}{1+y^2} = \tan^{-1} u + K'$$

ie $\tan^{-1}\left((\phi(t))\right) = \dfrac{t^2}{2} + c$

$\Rightarrow \phi(t) = \tan\left(\dfrac{t^2}{2} + c\right)$

2) solve $\begin{cases} \dfrac{dx}{dt} = \dfrac{-2tx}{1+t^2} + 1 \\ x(0) = 1 \end{cases}$

<u>Ans</u> using theorem 2,

$$a(t) = \int_{t_0}^{t} \alpha(s)\,ds = \int_{0}^{t} \frac{-2s}{1+s^2}\,ds$$

$$= -\ln|1+s^2| \Big]_{s=0}^{s=t}$$

$$= -\ln|1+t^2|$$

$$\therefore \phi(t) = e^{-\ln|1+t^2|}\left(\int_0^t e^{\ln|1+s^2|}\,1 \cdot ds + c\right)$$

$$\therefore \phi(t) = \frac{1}{|1+t^2|}\left(\int_0^t |1+s^2|\,ds + c\right)$$

At $t=0$, L.H.S $= 1$, R.H.S $= C \Rightarrow C = 1$

$$\therefore \phi(t) = \frac{1}{(1+t^2)}\left(t + \frac{t^3}{3}\right) + 1$$

$(\because 1+t^2 > 0 \ \forall t \in \mathbb{R})$

$$\therefore \phi(t) = \frac{3t + t^3 + 3}{3(1+t^2)}.$$

TUTORIAL 2

·) Find the general solution of

(i) $\frac{dx}{dt} = x \log\left(\frac{1}{x}\right)$

Ans $\varphi(\sigma(t)) = \int_{t_0}^{t} 1\, ds + C' = t + K$

$$\varphi(u) = \int_{u_0}^{u} \frac{1}{y \log\left(\frac{1}{y}\right)}\, dy$$

$$= -\log(|\log y|)\Big]_{u_0}^{u}$$

$$\therefore -\log(|\log \phi(t)|) = t + C$$

$$\therefore \quad |\log \phi(t)| = C'' e^{-t}$$

$$\therefore \quad \phi(t) = e^{C'' e^{-t}}$$

($\because C''$ absorbs sign)

(ii) $\frac{dx}{dt} = \left(\frac{t+x+1}{t+2}\right) - \exp\left(\frac{t+x+1}{t+2}\right)$

Ans Let $y = \frac{t+x+1}{t+2}$ &

$$\frac{dy}{dt} = \frac{(t+2)\left(1+\frac{dx}{dt}\right) - (t+x+1)}{(t+2)^2}$$

$$\therefore \frac{dy}{dt} = \frac{1+y-e^{-y}}{t+2} - \frac{y}{t+2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1-e^{-y}}{t+2}$$

$$\therefore \varphi(y(t)) = \int_{t_0}^{t} \frac{1}{s+2}\, ds + C''$$

$$= \log(|t+2|) + K$$

$$\varphi(u) = \int_{u_0}^{u} \frac{dy}{1-e^{-y}} = \ln|e^{u}-1| + K'$$

$$\therefore \ln|e^{y(t)}-1| = \ln|t+2| + C$$

$$\therefore |e^{y(t)}-1| = |C(t+2)|$$

$$\Rightarrow e^{y(t)} = C(t+2)+1$$

$$\Rightarrow y(t) = \ln(C(t+2)+1)$$

(c is chosen appropriately so that argument is positive. Hence choice of c affects the interval over which $y(t)$ exists)

$$\Rightarrow x(t) = (t+2)\ln(C(t+2)+1) - t - 1$$

(iii) $\frac{dx}{dt} = \frac{t+2x+1}{2t+x+2}$

Ans

$$\left.\begin{array}{l} 2t+4x+2=0 \\ 2t+x+2=0 \end{array}\right\} \Rightarrow \begin{array}{l} x=0 \\ t=-1 \end{array}$$

Substitute $X = x$

$T = t+1$

$\therefore \frac{dX}{dT} = \frac{dx}{dt} \times \frac{dt}{dT} = \frac{t+2x+1}{2t+x+2} = \frac{T+2X}{2T+X}$

Substitute $X = vT$

$\therefore \frac{dX}{dT} = v + T\frac{dv}{dT}$

$\therefore v + T\frac{dv}{dT} = \frac{1+2v}{2+v}$

$\therefore T\frac{dv}{dT} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v}$

$\therefore T\frac{dv}{dT} = \frac{1-v^2}{2+v}$

$\therefore \frac{dv}{dT} = \left(\frac{1-v^2}{2+v}\right)\frac{1}{T}$

$\therefore$ After applying separable and solving, we get an extremely huge solution.

So just leave it here.

(iv) $\frac{dx}{dt} = 3|x|^{2/3}$

Ans case 1 : $x(t) > 0$,

$$\frac{dx}{dt} = 3x^{2/3}$$

$$\therefore \varphi(\phi(t)) = t + K$$

and $\varphi(u) = \int_{u_0}^{u} \frac{1}{3y^{2/3}} \, dy$

$$= \sqrt[3]{u} + K'$$

$$\therefore \sqrt[3]{\phi(t)} = t + C$$

$$\therefore \phi(t) = (t + c)^3$$

but $\phi(t) > 0$

$$\Rightarrow t + c > 0$$

$$\Rightarrow t > -c$$

for $x(t) < 0$ also we get the same

$$\phi(t) = (t+c)^3$$

since $|x|^{2/3} = x^{2/3}$:)

$$\therefore \phi(t) = (t+c)^3 \quad \forall \ t \in R$$

~~(since at 0 it is not differentiable)~~

(v) $\frac{dx}{dt} = (t - x + 3)^2$

An $y = t - x + 3$

$$\frac{dy}{dt} = 1 - y^2$$

$$\therefore \; y(t) = \frac{e^{2t} - K}{K + e^{2t}}$$

$$\therefore \; x(t) = \frac{-(e^{2t} - K)}{e^{2t} + K} + t + 3$$

2) Solve

(i) $\begin{cases} \frac{dx}{dt} = \frac{tx}{1+t} + 1 \\ x(0) = 0 \end{cases}$

Ans $\alpha(t) = \int_0^t \frac{s}{1+s} \, ds$

$$= t - \log(|1+t|)$$

$$\therefore \phi(t) = e^{t - \log(|1+t|)} \left(\int_0^t e^{\log|1+s| - s} \, 1 \, ds + C \right)$$

$$\phi(t) = \frac{e^t}{|1+t|} \left(\int_0^t \frac{|1+s|}{e^s} \, ds + C \right)$$

$\therefore \phi(t) = \dfrac{e^t}{|1+t|}\left(2 - (t+2)e^{-t} + c\right)$

At $t = 0$,

$0 = 1\,(2 - 2 + c) \Rightarrow c = 0$

$\therefore \phi(t) = \dfrac{e^t}{|1+t|}\left(2 - (t+2)e^{-t}\right)$

(ii) $\begin{cases} \dfrac{dx}{dt} = \dfrac{e^{-x^2}}{x\,(2t+t^2)} \\ x(2) = 0 \end{cases}$

Ans $\varphi(\phi(t)) = \displaystyle\int_2^t \frac{1}{2s+s^2}\,ds + C$

$\therefore \varphi(\phi(t)) = \dfrac{\ln|t|}{2} - \dfrac{\ln|t+2|}{2} + K$

$\varphi(u) = \displaystyle\int_{u_0}^{u} y\,e^{y^2}\,dy$

$= \dfrac{e^{u^2}}{2} + K'$

$\therefore \dfrac{e^{(\phi(t))^2}}{2} = \dfrac{\ln|t|}{2} - \dfrac{\ln|t+2|}{2} + K''$

$\phi(2) = 0 \Rightarrow \dfrac{1}{2} = \dfrac{\ln 2}{2} - \dfrac{\ln 4}{2} + K''$

(iii) $\begin{cases} \dfrac{dx}{dt} = \dfrac{\cos t}{\cos^2 x} \\ x(\pi) = \dfrac{\pi}{4} \end{cases}$

Ans $\quad \mathcal{U}(\phi(t)) = \sin t + K'$

$$\mathcal{U}(u) = \int_{u_0}^{u} \cos^2 y \, dy$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + C$$

$$\therefore \quad \frac{x(t)}{2} + \frac{\sin(2x(t))}{4} = \sin t + K$$

$$x(\pi) = \pi/4$$

$$\Rightarrow \quad \frac{\pi}{8} + \frac{1}{4} = K$$

3) give a first order ODE for the family $y = cx^2$

$$\frac{dy}{dx} = 2cx = 2\frac{y}{x^2}x = \frac{2y}{x}$$

4) Given $t^2 \dfrac{d^2x}{dt^2} + 3t\dfrac{dx}{dt} + x = \dfrac{2}{t}$, change

coordinates as $y = x$, $s = \log t$

$$\frac{dy}{ds} = \frac{dx}{ds} = \frac{d(x(t))}{ds} = e^s \frac{dx}{dt} = t\frac{dx}{dt}$$

$$\frac{d}{ds}\left(\frac{dy}{ds}\right) = \frac{d}{ds}\left(e^s \frac{dx}{dt}\right) = e^s\left(\frac{d}{ds}\left(\frac{dx}{dt}\right) + \frac{dx}{dt}\right)$$

$$\frac{d}{ds}\left(\frac{dx}{dt}\right) = \frac{d^2}{dt^2}(x(t)) \cdot \frac{dt}{ds} = t\frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2y}{ds^2} = t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt}$$

$$\therefore \quad t^2\frac{d^2x}{dt^2} + 3t\frac{dx}{dt} + x = \frac{2}{t}$$

$$\Rightarrow \quad \frac{d^2y}{ds^2} + 2\frac{dy}{ds} + y = \frac{2}{e^s}$$

5) Show that $\frac{dx}{dt} = t^{n-1} f\left(\frac{x}{t^n}\right)$ can

be solved using $y = \frac{x}{t^n}$

Ans $\quad y = \frac{x}{t^n} \Rightarrow \frac{dy}{dt} = \frac{1}{t^n} t^{n-1} f(y)$

$$+ x(-n)t^{-n-1}$$

$$\therefore \frac{dy}{dt} = \frac{f(y)}{t} - \frac{ny}{t} = \frac{g(y)}{t}$$

$\therefore$ variable separable

6) Solve $x\frac{dy}{dx} + 3x = 2y$ using IF

Ans from theorem 7,

here $P(x,y) = x$

$Q(x,y) = 3x - 2y$

$$Q_y - P_x = -2 - 1 = -3$$

$$\frac{Q_y - P_x}{P} = -\frac{3}{x}$$ is only a function of x

$$\therefore \mu'(x) = \mu(x)\left(\frac{-3}{x}\right)$$

From variable separable, the solution is

$$\mu(x) = \frac{c}{x^3}$$

$$\therefore \frac{1}{x^2}\frac{dy}{dx} + \frac{3}{x^2} - \frac{2y}{x^3} = 0$$ is

an exact equation

Consider $F(x,y) = \frac{y}{x^2} - \frac{3}{x}$

Then $$\frac{dF(x,y)}{dy} = \frac{1}{x^2}\frac{dy}{dx} + \frac{3}{x^2} - \frac{2y}{x^3}$$

$$\therefore \frac{dF(x,y)}{dy} = 0$$

$\therefore f(x,y) = K$

$\therefore \dfrac{y}{x^2} - \dfrac{3}{x} = K$ is the solution

TUTORIAL 3

1) write out few terms of the Picard integration and if possible, find the solution

(i) $\dfrac{dx}{dt} = x + 2, \quad x(0) = 2$

Ans By Picard's integrates,

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} x_n(s)\, ds$$

Here $t_0 = 0, \quad x_0 = 2$

$$\therefore x_0(t) = 2$$

$$x_1(t) = 2 + \int_0^t (2+2)\, ds$$

$$= 2 + 4t$$

$$x_2(t) = 2 + \int_0^t (2 + 4s + 2)\, ds$$

$$= 2 + 4t + 2t^2$$

$$x_3(t) = 2 + 4t + 2t^2 + \frac{2t^3}{3}$$

$$x_4(t) = 2 + 4t + 2t^2 + \frac{2t^3}{3} + \frac{t^4}{6} + \cdots$$

$$\therefore\ x_n(t) = 2 + 4t + 2t^2 + \frac{2t^3}{3} + \frac{t^4}{6} + \cdots + \frac{4t^n}{n!}$$

$$\therefore\ x(t) = \lim_{n\to\infty} x_n(t) = 4e^t - 2$$

(ii) $\dfrac{dx}{dt} = \cos x \quad , \quad x(0) = 0$

<u>Ans</u> $x_0(t) = x_0 = 0$

$$x_n(t) = \int_0^t x_{n-1}(s)\, ds$$

$$\therefore\ x_1(t) = \int_0^t \cos(0)\, ds = t$$

$$x_2(t) = \int_0^t \cos(s)\, ds = \sin t$$

$$x_3(t) = \int_0^t \cos(\sin s)\, ds = ???$$

(iii) $\dfrac{dx}{dt} = x^{4/3} \quad , \quad x(0) = 0$

<u>Ans</u> $x_0 = 0$

$$x_1(t) = \int_0^t 0\, ds = 0$$

$$x_n(t) = \int_0^t x_{n-1}(s)\, ds$$

By induction all are 0

$\therefore\ x(t) = 0$

(iv) $\frac{dx}{dt} = x^{4/3}$, $x(0) = 1$

Ans

$x_0 = 1$

$x_1(t) = 1 + \int_0^t 1 \, ds = 1 + t$

$x_2(t) = 1 + \int_0^t (1+s)^{4/3} \, ds$

$= 1 + \frac{3(1+t)^{7/3}}{7} - \frac{3}{7}$

2) Are the following Lipschitz near $x = 0$

(i) $f(x) = \sin\left(\frac{2}{x}\right)$

Ans No!

$|f(x) - f(y)| = \left|\sin\left(\frac{2}{x}\right) - \sin\left(\frac{2}{y}\right)\right|$

$= \left|2\cos\left(\frac{2}{x} + \frac{2}{y}\right)\sin\left(\frac{2}{x} - \frac{2}{y}\right)\right|$

Suppose

$\left|2\cos\left(\frac{2}{x} + \frac{2}{y}\right)\sin\left(\frac{2}{x} - \frac{2}{y}\right)\right| \leq K|x-y|$

(for x, y near 0)

we get an imediate contradiction

since already $f'(x) = -\frac{2}{x^2}\cos\left(\frac{2}{x}\right)$ which is

not continuous at $x = 0$.

(OR you could get away saying f doesn't exist at 0)

(ii) $f(x) = \frac{1}{2-x^2}$

Ans f is C^1 at all points in some small neighbourhood of 0

and hence Lipschitz continuous

(iii) $f(x) = \sqrt{x} + 2x$

Ans $\exists$ no constant C s.t.

$$|\sqrt{x} - \sqrt{0}| \le C|x - 0|$$

Hence $\sqrt{x}$ is not lipschitz near 0 & we are done

(iv) $f(x) = x^2 \cos\frac{1}{x}$

A function with bounded derivative is lipschitz

Ans $f(x)$ is continuous everywhere

$$f'(x) = 2x\cos\frac{1}{x} - \sin\frac{1}{x}$$

which isn't continuous at 0

$$\lim_{x\to 0} f(x) = 0$$

$$\left|\frac{x^2\cos\frac{1}{x}}{x}\right| = \left|x\cos\frac{1}{x}\right| \le |x| \le \delta$$

for some $x \in (-\delta, \delta)$

Hence we have lipschitz continuity

TUTORIAL 4

1) (i) $\frac{dx}{dt} = x + \sin t$

Ans $\varphi(t) = \int_{t_0}^{t} 1 \, ds = t - t_0$

$$\phi(t) = e^{\varphi(t)} \left(\int_{t_0}^{t} e^{-\varphi(s)} \beta(s) \, ds + K \right)$$

$$= e^{t - t_0} \left(\int_{t_0}^{t} e^{-s + t_0} \sin s \, ds + K \right)$$

$$\therefore \phi(t) = e^{t} \int_{t_0}^{t} e^{-s} \sin s \, ds + K e^{t}$$

(after the DIY)

(ii) $\begin{cases} \frac{dx}{dt} = -y - t \\ \frac{dy}{dt} = x + t \end{cases}$, $x(0) = 1$, $y(0) = 0$

Ans $$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t \\ t \end{bmatrix}$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$= \sum_{k=0,4,8,\dots} \frac{t^k}{k!} I$$

$$+ \sum_{k=1,5,9,\dots} \frac{t^k}{k!} A$$

$$+ \sum_{k=2,6,10,\dots} \frac{t^k}{k!} A^2$$

$$+ \sum_{k=3,7,11,\dots} \frac{t^k}{k!} A^3$$

$$= \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{bmatrix} + \begin{bmatrix} -e^{4t+2} & 0 \\ 0 & -e^{4t+2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -e^{4t+1} \\ e^{4t+1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{4t+3} \\ -e^{4t+3} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t} - e^{4t+2} & -e^{4t+1} + e^{4t-1} \\ e^{4t+1} - e^{4t-1} & e^{4t} - e^{4t+2} \end{bmatrix}$$

By Duhamels formula,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{4t} - e^{4t+2} & -e^{4t+1} + e^{4t-1} \\ e^{4t+1} - e^{4t-1} & e^{4t} - e^{4t+2} \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_0}$$

$$+ \int_0^t \Big[\quad " \quad \Big] \begin{bmatrix} -(t-s) \\ t-s \end{bmatrix} ds$$

DIY (pretty long)

(iii) $x'(t) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} x(t)$

Ans $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow$ e vals are $4, 2$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$$

$\Rightarrow 3x + y = 4x$

$\Rightarrow x = y$ } evec $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$\therefore 3x + y = 2x$

$\therefore x = -y$ } evec $= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\therefore x(t) = \alpha e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(iv) $x'(t) = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 5t \end{pmatrix}$

Ans $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \to$ evals are $-2, 3$

and $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}^n = \begin{bmatrix} \frac{(-1)^n 2^n + 4\cdot 3^n}{5} & \frac{-(-1)^n \cdot 2^n + 3^n}{5} \\ \frac{-4(-1)^n \cdot 2^n + 4\cdot 3^n}{5} & \frac{4\cdot(-2)^n + 3^n}{5} \end{bmatrix}$$

$$\therefore \exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$= \begin{bmatrix} \frac{e^{-2t} + 4e^{3t}}{5} & \frac{-e^{2t} + e^{3t}}{5} \\ \frac{-4e^{-2t} + 4e^{3t}}{5} & \frac{4e^{-2t} + e^{3t}}{5} \end{bmatrix}$$

Apply Duhamel & we are done

(v) $x'(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t)$

$$A^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \forall \; n > 1$$

$$\therefore \exp(tA) = \left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\right] A^k$$

$$= I + tA = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$\therefore$ solution is $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$\therefore x(t) = \begin{bmatrix} \alpha + \beta t \\ \beta \end{bmatrix}$$

(vi) $x'' + x' + x = 0$

~~using the sum of $a_n(t)$~~ $\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$

e vals are ω, ω^2 (evecs are $\begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix}$)

$$\therefore \begin{bmatrix} x \\ x' \end{bmatrix} = \alpha e^{\omega t} \begin{bmatrix} 1 \\ \omega \end{bmatrix} + \beta e^{\omega^2 t} \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix}$$

$$\therefore x(t) = \alpha e^{\omega t} + \beta e^{\omega^2 t}$$

$$\therefore x(t) = \alpha e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + \beta e^{-\frac{1}{2}t} \sin \sqrt{\frac{3}{2}}\, t$$

(vii) $x'' + 2x' + 2x = e^t$

Ans consider $x'' + 2x' + 2 = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

evals are $-1-i$, $-1+i$

with evecs $\begin{bmatrix} -1 \\ i+1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1-i \end{bmatrix}$.

$\therefore x(t) = \alpha e^{(-1-i)t} + \beta e^{(-1+i)t}$

$\therefore x(t) = \alpha e^{-t} \cos t + \beta e^{-t} \sin t$

2) Determine $X'(t) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} X(t)$ has sink, source, saddle or center at origin. Also determine stability

Ans stability:

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 0 = b$

$\therefore (0,0)$ is a stable point

e vals are $-\frac{\sqrt{33}+5}{2}$, $\frac{\sqrt{33}+5}{2}$

$\Gamma_{max} = \max \{ Re(\lambda) \mid \lambda \text{ is an eval} \}$

$= \frac{\sqrt{33}+5}{2} > 0$

$\therefore (0,0)$ is unstable

Now $\lambda_1 < 0 < \lambda_2 \Rightarrow$ saddle

3) Consider $x'' + bx' + cx = 0$. Convert to first order autonomous and find evals of A.

Ans
$$\begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\det \begin{bmatrix} 0-x & 1 \\ -c & -b-x \end{bmatrix}$$

$$= x(b+x) + c$$

$\therefore$ evals satisfy

$$x^2 + bx + c = 0$$

$$\therefore x = \frac{-b \pm \sqrt{b^2-4c}}{2}$$

~~TUTORIAL 7~~

TUTORIAL 5

1) Show that the trivial solution of $x' = \beta x^3$ is

(i) assymp stable if $\beta < 0$

(ii) stable if $\beta \le 0$

(iii) unstable if $\beta > 0$

Ans Consider $L(x) = x^4$

Then $L(0) = 0$

$L(x) > 0 \quad \forall\ x \neq 0$

$L(x) = 0$ iff $x = 0$

$\langle \nabla L, f \rangle$

$= \langle 4x^3, \beta x^3 \rangle$

$= 4\beta x^6$

$\therefore$ If $\beta > 0$, $\langle \nabla L, f \rangle > 0 \Rightarrow$ unstable

If $\beta < 0$, $\langle \nabla L, f \rangle < 0 \Rightarrow$ assymp stable

If $\beta \le 0$, $\langle \nabla, f \rangle \le 0 \Rightarrow$ stable

2) Study stability of equilibrium solution $\binom{0}{0}$ of

(i) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y + x^3 \\ x \end{pmatrix}$

Ans we linearise it to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3x^2 & -1 \\ 1 & 0 \end{pmatrix}\Bigg|_{x,y=0,0} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\downarrow$

evals are $i, -i$

$\Gamma_{max} = 0 \Rightarrow$ linearised system is stable at (0,0)

$\therefore$ cant say anything

we need Lyapunov

Take $L = x^2 + y^2$

so that $L > 0 \quad \forall\ x,y \neq 0,0$

and $L = 0$ iff $x,y = 0,0$

$$\langle \nabla L, F \rangle = \langle (2x, 2y), (-y+x^3, x) \rangle$$

$$= 2x^4 \geq 0 \quad \forall\ x,y \neq 0,0$$

$\therefore$ unstable

(ii)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2\sin x - 4y \\ \sin x - 3y \end{pmatrix}$$

Ans Linearise to,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2\cos x & -4 \\ \cos x & -3 \end{pmatrix}\Bigg|_{x,y = 0,0} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\downarrow$

evals are $\dfrac{5 \pm i\sqrt{15}}{2}$

$\therefore \Gamma_{max} = 5/2 > 0$

$\therefore$ unstable

(iii) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sin(6x - 3y) \\ \sin(2x + y) \end{pmatrix}$

Ans $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6\cos(6x-3y) & -3\cos(6x-3y) \\ 2\cos(2x+y) & 1\cos(6x-3y) \end{pmatrix}\Bigg|_{x,y = 0,0} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\downarrow$

e vals are

3, 4

$\therefore \Gamma_{max} > 0$

$\therefore$ unstable

3) find all eq. solns & discuss stability

Ans $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8x - y^2 \\ -y + x^2 \end{bmatrix}$

Ans $\left. \begin{array}{l} 8x = y^2 \\ y = x^2 \end{array} \right\} \to 8x = x^4$

$\Rightarrow x = 0, 2$

$\Rightarrow y = 0, 4$

for $(0,0)$,

linearised matrix is $\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$

e. vals are $-1, 8$

$T_{max} > 0 \Rightarrow$ unstable

for $(2,4)$,

linearised matrix is $\begin{bmatrix} 8 & -8 \\ 4 & -1 \end{bmatrix}$

e vals are $\dfrac{7 \pm i\sqrt{47}}{2}$

$T_{max} > 0 \Rightarrow$ unstable

4) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y + x^3 \\ x + y^3 \end{pmatrix}$

(i) Show $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable but for linearised its stable

Ans linearised is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\downarrow$

evals are $i, -i$

$r_{max} = 0$

Also $AM = GM$ $\forall$ evals with $Re(\lambda) \leq 0$

$\therefore$ unstable

But

$$\left\langle \begin{pmatrix} -y + x^3 \\ x \end{pmatrix}, \begin{pmatrix} x + y^3 \\ y \end{pmatrix} \right\rangle$$

$$= -xy + x^4 + xy + y^4 = x^4 + y^4$$

$\therefore$ choosing $L = x^2 + y^2$ gives

unstable equilibrium

5) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} (y-1)(y^2 - 2y + 2) \\ (1-x)(x^2 - 2x + 2) \end{pmatrix}$

Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is stable

Ans choosing $\nabla L = -(y-1)(y^2 - 2y + 2), -(1-x)(x^2 - 2x + 2)$

we are done

i.e. choose $L = ??$

Let $Y = y - 1$

$X = x - 1$

$$\therefore \begin{bmatrix} Y' \\ X' \end{bmatrix} = \begin{bmatrix} Y(Y^2+1) \\ -X(X^2+1) \end{bmatrix}$$

~~Choose~~ we need to analyse $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for this system

Choose $L =$

6) Study eq. soln $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$

Ans choose $L = \frac{1}{2} y^2 + \int_0^x \sin s \, ds$

$= \frac{1}{2} y^2 - \cos x + 1$

so that $L > 0 \quad \forall \; x, y \neq (0,0)$

$L = 0$ at $x, y = 0, 0$

$\nabla L, F = (y, \sin x) \cdot (y, -\sin x) = 0$

$\therefore$ stable