Matrix - Ball construction

This is in context of the RSK algorithm which gives a bijection between finitely supported matrices c with $R_c = \alpha$, $C_c = \beta$ and pairs of SSYT (P, Q) with type $(Q) = \alpha$, type $(P) = \beta$

We now give an algorithm to convert a matrix into a numbered matrix ball diagram

Step 1 : Given the matrix A, construct $\tilde{A}$ by replacing every entry a_{ij} with a_{ij} number of balls

Rule 1 : All balls in the same cell are numbered contiguously as $i, i+1, i+2, \cdots$

Step 2 : If $a_{11} \neq 0$, number the balls in the $(1,1)^{th}$ cell as $1, 2, 3, \cdots, a_{11}$

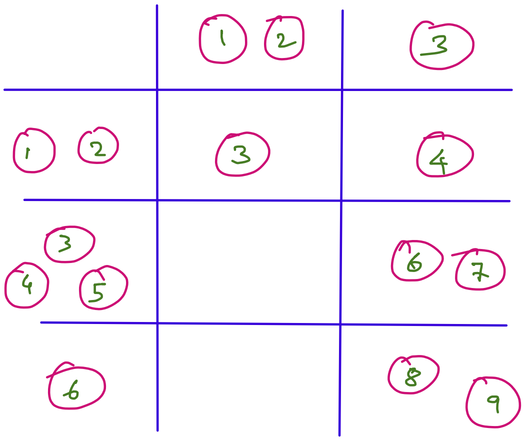
If $a_{11} = 0$, obtain a_{1p}, a_{q1} $(p, q \neq 1)$, the first non zero entries in the 1st row & 1st column resp. (at least one of them exists) and number the balls in these cells as $1, 2, \cdots a_{1p}$ and $1, 2, \cdots, a_{1q}$

Rule 2 : Corresponding to cell (i, j), the lowest numbered ball is numbered by the smallest number that appears after the biggest numbered ball when all balls are considered to the left & above the cell (i, j) (weakly)

Step 3 : Fill all cells from top left corner to bottom right corner following rule 1 & 2 example

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

The empty ball diagram is



This a well-defined algorithm which will allow us to convert A into (P, Q) without the need to keep writing multiple tableux & without bumping & insertion algorithm

Let c_i be the least column no. in which the no. i appears & similary r_i is for rows

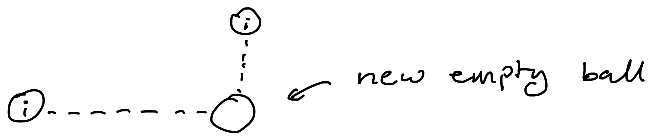
We construct a pair of Tableux (P, Q) using the algorithm described below

Step 1 : First row of P is $c_1 \; c_2 \; c_3 \cdots c_m$
First row of Q is $r_1, r_2 \; r_3 \cdots r_m$
(where m is the biggest numbered ball)

Step 2 : Consider balls with label i. They cannot be in the same row & column, starting from the leftmost ball i, consider pairs of such balls.

Suppose one such pair occurs in cells (α, β) and (γ, δ) (with $\alpha > \gamma$ & $\beta < \delta$), place a new empty ball in cell (α, δ)

Pictorially :



Do this for each such pair & hence if we have k balls labelled i ; if $k = 1$, dont create any new empty ball and if $k > 1$, considering adjacent pairs bottom left to top right, create $k-1$ new balls.

Thus if m is the largest numbered ball, we have m less balls in our new ball-matrix

Step 3 : Number the new ball matrix & obtain c_i, r_i & repeating step 1, we get the second rows of P & Q

exercise Give an algo for recovering the matrix uniquely thereby proving that it is a bijection

The claim is that it is in fact the exact same bijection as the RSK algorithm !

Theorem

The matrix-ball bijection is the same as the RSK algo. In other words the tableux P, Q created coincide

proof

The proof is by induction on no. of entries of matrix A. if A has a single non-zero entry, it is clear that both ways give the same P & Q

If $A = [\lambda]$, $w_A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ & hence $P = Q = 111\cdots1$

Through matrix ball we have ① ② --- ⓝ & $P = Q = 11 \cdots 1$

Let us introduce some notation

Let A be the matrix with generalised permutation w_A

Let $A^{(1)}$ be the numbered matrix – ball diagram of A

Let $A^{(2)}$ denote the matrix obtained from $A^{(1)}$ by deleting m number of balls (m is largest no. in $A^{(1)}$)

Let A^{α} be the matrix corresponding to $A^{(2)}$

By 'last entry' we mean the last entry of $\hat{R}_c$ (reading out the matrix row wise) (ofc non zero is meant)

Let A_β denote the matrix obtained by subtracting 1 from the last entry of the original matrix A

Let $P(w)$, $Q(w)$ denote the RSK P & Q while $P(A)$, $Q(A)$ denote the matrix ball P & Q

It is clear that if $w_A = \begin{pmatrix} u_1 & u_2 & \cdots & u_j \\ v_1 & v_2 & \cdots & v_j \end{pmatrix}$, then

$w_{A_\beta} = \begin{pmatrix} u_1 & \cdots & u_{j-1} \\ v_1 & \cdots & v_{j-1} \end{pmatrix}$

Assume the induction hypothesis that $P(A_\beta) = P(w_{A_\beta})$ and $Q(A_\beta) = Q(w_{A_\beta})$ ie., $P(A_\beta)$ can be obtained by inserting $v_1, v_2, \cdots v_{j-1}$ & $Q(A_\beta)$ is obtained by placing $u_1 \cdots u_{j-1}$ in the new boxes

Now we only need to show that $P(A)$ is obtained by inserting v_j to $P(A_\beta)$ & $Q(A)$ is obtained by placing u_j in the new box

For notational convenience, $\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$A^{(1)}$ has one extra ball in the $(x, y)^{th}$ box as compared to $A_\beta^{(1)}$. Let this ball be numbered 'p'.

<u>Case 1</u> : There is only one ball labelled p in $A^{(1)}$

Then when $A^{(2)}$ & hence A^{α} is obtained, it is same as $A_\beta^{(2)}$ & A_β^{α} ie. $A^{\alpha} = A_\beta^{\alpha}$

[$A^{(1)}$ & $A_\beta^{(1)}$ are same except for a single ball labelled

p. Hence, they get 'processed' to $A_\beta^{(2)}$ & $A^{(2)}$ in the same way as the 'p' is simply irrelevant]

Thus, $P(A)$ & $P(A_\beta)$ are same second row onwards & $Q(A)$ & $Q(A_\beta)$ are same second row onwards

The first row of $P(A)$ is $c_1, c_2, \ldots, c_p$ and $c_p = y$

The first row of $P(A_\beta)$ is $c_1, c_2, \ldots, c_{p-1}$ and $c_{p-1} \leq y$

Clearly $P(A_\beta) \leftarrow y$ gives $P(A)$ & in the new box at the end of row 1, x is inserted

Hence $Q(A) = Q(\omega_A)$

This concludes case 1

<u>Case 2</u>: There are ≥ 2 no. of p labelled balls in $A^{(1)}$

Now, there cant be any entries below (x, y) since it is the last & the other p's have to be strictly to the right (else this p would have been $\geq p+1$)

Let the next p be at x', y'

Claim: When y is inserted into $P(A_\beta)$, y' is bumped from the p^{th} box

The first $p-1$ entries for $P(A)$ & $P(A_\beta)$ are all the same & are just $c_1, c_2, \ldots, c_{p-1}$ (where c_i is calculated in any one of $A^{(1)}$ or $A_\beta^{(1)}$ (it is the same))

The p^{th} entry in $P(A_\beta)$ is y' & that in $P(A)$ is y

Thus our claim is proved

(Note that $c_1 \leq \cdots \leq c_{p-1} < y$)

Now we look at $P(A^\alpha)$ & $P(A_\beta{}^\alpha)$: the portions of P below the first row

We now need to show that $P((A_\beta)^\alpha) \leftarrow y' = P(A^\alpha)$

We also need to show that the new box created is the box in $Q(A^\alpha)$ that is not in $Q((A_\beta)^\alpha)$

We now claim that the last position of A^α is (x, y') &

$(A_\beta)^\alpha = (A^\alpha)_\beta$

Suppose the claim is proved then by induction hypothesis

$P((A^\alpha)_\beta) \leftarrow y' = P(A^\alpha)$ & $P((A^\alpha)_\beta) = P((A_\beta)^\alpha)$

so we are done with P

Also $x = u_{j-1}$ is inserted into the new box of $Q((A^\alpha)_\beta)$ which is not in $Q(A)^\alpha$ by induc hypo & if claim is proved, we get what we want.

We restate & prove the claim

Claim : last position of A^α is (x, y') & $(A_\beta)^\alpha = (A^\alpha)_\beta$

We have a p in (x, y) & (x', y') in $A^{(1)}$ & hence in $A^{(2)}$ there is a ball in (x, y') position. We now clearly have that no entry below (x, y') in A^α is non-zero since the (x, y) p of $A^{(1)}$ was last entry

Suppose there is a ball in the x^{th} row of $A^{(2)}$ in some column strictly to the right of y', say w

It must have occured because of some $l < p$ in the x^{th} row of $A^{(1)}$ weakly to the left of the y^{th} column & the same l in some position (γ, w) where $\gamma < x$

But this position of l contradicts the position of p in the (x', y') position since $y' < w$ But $p > l$

Hence the first part of our claim is proved

To see $(A^\alpha)_\beta = (A_\beta)^\alpha$, we obviously only need to see what happens to the p-labelled balls since the rest of the balls are same in A & A_β and are 'processed' the same way

To get $(A^\alpha)_\beta$, we remove the (x, y') position ball labelled p (since (x, y') is last position)

To get A_β, we delete the $(x, y)^{th}$ ball from A & to get $(A_\beta)^\alpha$ from it, we anyway delete the (x, y') position while processing A_β. Thus, the claim is proved

Theorem

$$A \xrightarrow{RSK} (P, Q) \implies A^T \xrightarrow{RSK} (Q, P)$$

Proof

The matrix ball diagram for A^T is just the transpose diagram of A, including labelling, since the construction is symmetric wrt rows & columns.

Hence, when we read off $r_1, r_2, \ldots$ & $c_1, c_2, \ldots$, they are $c_1, c_2, \ldots$ & $r_1, r_2, \ldots$ of A.

Suppose $A^T \to (P', Q')$

Then by our observation above, first row of P' = first row of Q & first row of Q' = first row of P

We now claim that $(A^T)^\flat = (A^\flat)^T$ completing the proof by induction

Suppose in A, there are two balls numbered l in positions (a,b) & (c,d) $(d > b,\ a > c)$. Then a new blank ball is created in position (a,d)

In A^T, the l-balls are present in (b,a) & (d,c) and hence the new ball is created at (d,a)

This proves our claim

[Note : $(\alpha,\beta) + (\gamma,\delta)$ gives new ball at $(\max(\alpha,\gamma), \max(\beta,\delta))$]

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