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Moore Graphs

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In this presentation I discuss Moore graphs & their classification. Undergraduate knowledge of graph theory will suffice.
We assume all graphs are simple and undirected.

Definition: A k-regular graph with diameter d is called distance regular if for any two vertices x, y with $d(x,y) = i$, the following cardinalities are constant:

(i) $\{ z \in V \mid d(x,z) = i,\ d(y,z) = 1 \}$ $(= a_i)$

(ii) $\{ z \in V \mid d(x,z) = i+1,\ d(y,z) = 1 \}$ $(= b_i)$

(iii) $\{ z \in V \mid d(x,z) = i-1,\ d(y,z) = 1 \}$ $(= c_i)$

(b_d and c_0 do not exist)

One can then associate the matrix

$$\begin{bmatrix} * & c_1 & c_2 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & * \end{bmatrix}$$

There are many different ways to define Moore graphs. The two most common ways are through girth and through number of vertices.
We will go through vertices & prove the statement about the girth.

Proposition 1
The maximum number of vertices in a k-regular of diameter d is $1 + k(k-1) + \cdots + k(k-1)^{d-1}$

Proof
Start with a vertex v_0. It has at most k neighbours. Each of these have at most $k-1$ neighbours (excluding v_0). Each of those have at most $k-1$ neighbours and so on.

The no. of vertices at distance i from v_0 is at most $k(k-1)^{i-1}$ and since diameter is d, the bound is thus:

$1 + \sum_{i=1}^{d} k(k-1)^{i-1}$

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<u>Definition</u>: A k-regular graph on $1+\sum_{i=1}^{d} k(k-1)^{i-1}$ number of vertices having diameter d is called a <u>Moore graph</u>

We have the following proposition about tightness of k and d which simplifies the above definition

<u>Proposition 2</u>

If G is a k-regular graph on $1+\sum_{i=0}^{d-1} k(k-1)^{i}$ vertices then $\operatorname{diam}(G) \geq d$

<u>Proof</u>

Let $\operatorname{diam}(G) = \delta$

If $\delta < d$ then by proposition 2 we can have at most $1 + k + k(k-1) + \cdots + k(k-1)^{\delta-1}$ vertices which is fewer than the number of vertices given. Thus we have a contradiction

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Indeed it is possible to have $\operatorname{diam}(G) > d$. Let us take a 3-regular graph $(k=3)$ on $1+3+3(3-1) = 10$ $(d=2)$ vertices given by a 10 cycle & connecting diametrically opposite points. One can see directly that diameter is $3 > d = 2$.

Also, if G is a graph on $1+\sum_{i=1}^{d} k(k-1)^{i-1}$ vertices and has diameter d, then it might not even be regular as seen by taking the star $K_{1,4}$ which corresponds to the $k=2$, $d=2$ case.

However, if we add the condition that the graph is r-regular then similar to proposition 2, we have $r \geq k$ and indeed we can produce a graph with $r > k$:

Take $k=3, d=2$ case $\Rightarrow$ 10 vertices. Label them with $\mathbb{Z}_{10}$.

Connect i to $i+1$, $i-1$, $i+4$, $i-4$. This is indeed 4-regular with diameter 2 and $k = 3 < 4$

We now have the girth characterisation of Moore graphs. Recall that girth (G) is the size of the smallest cycle in G as a subgraph (it can only be $0, 3, 4, \dots$)

Proposition 3

For a graph which is not a tree, girth $(G) \leq 2\,\mathrm{diam}(G) + 1$

Proof

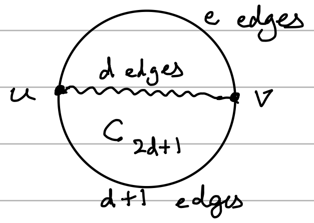
Let girth & diameter be denoted g and d.

Suppose $g > 2d+1$ & let C be a cycle of length g.

Pick u and v on the cycle such that $d_C(u,v) = d+1$ (possible in a cycle). But $\mathrm{diam}(G) = d \Rightarrow \exists$ path in G from u to v of length at most d.

This path along with one arc of the cycle gives a smaller cycle of length $2d+1$ which is a contradiction.

The figure below depicts the situation



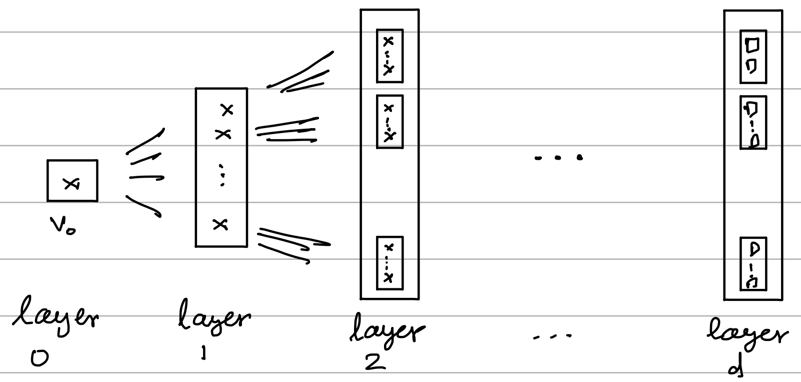
$(d+1+e = g > 2d+1 \Rightarrow e > d$ i.e. $e \geq d+1)$

Proposition 4

Equality is achieved in prop. 3 iff G is a Moore graph

Proof

Suppose G is a Moore graph. The graph looks like:



Layer 1 has k vertices

Layer 2 has k boxes each having $k-1$ vertices

$\vdots$

Layer d has k boxes each having $(k-1)^{d-1}$ vertices

Degrees of every vertex are satisfied except those in layer k.
In the figure above, vertices of layer d have degree 1 & hence they must be connected amongst each other.
Clearly we can demonstrate a $2d+1$ cycle by starting with v_0 and going to u, w in layer d st. u, v are connected.
Now suppose a cycle of length $l < 2d+1$ exists
Form the Moore graph starting with vertex w_0 of the cycle
Since no two vertices of the same layer are connected (except last layer) and no two vertices of a layer have edges to a single vertex of the next layer, it is clear that such a cycle cant exist.
Thus $g = 2d+1$
Conversely, let $g = 2d+1$. First we show that G is regular
Given any two vertices u, v in the graph, $d(u,v) \leq d$ and since $g = 2d+1$, there can exist only one path from u to v else we get a cycle of length $\leq 2d$
Let $x \in N(u)$. Obtain the shortest path P_{xv}. Map x to the neighbour of v lying along this path.
It is clear that two distinct neighbours of x cannot map to the same neighbour of v. Given a neighbour of v we can reverse the process and obtain a neighbour of u.
This sets up a bijection $N(u) \longleftrightarrow N(v)$.
Thus $d(u) = \text{const} \quad \forall u \in V$
Now we have a k-regular graph with diameter d such that $g = 2d+1$
By proposition 1, the number of vertices is at most $1 + \sum_{i=1}^{d} k(k-1)^{i-1}$
Now start with any vertex u.
There exist unique paths from u to every vertex x
If $d(u,x) = 1$, no. of possible $x = k$
Assume no of vertices at distance $i (\geq 1)$ from u is $k(k-1)^{i-1}$
Let us find no. of vertices at distance $i+1$. For this, we need to go through all the $k(k-1)^{i-1}$ vertices & for each of these, there are $k-1$ new neighbours (we cant form cycles because $g = 2d+1$) Thus we get $k(k-1)^{i}$ such x proving that no. of x at distance i from u is given by

$$\begin{cases} 1 & i=0 \\ k(k-1)^{i-1} & 1 \le i \le d \end{cases}$$

Thus total no. of vertices $= 1 + \sum_{i=1}^{d} k(k-1)^{i-1}$

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We now come back to the topic of distance regularity.
Observe that if G is distance regular with parameters given by $\{c_1, c_2, \dots, c_d, a_0, \dots, a_d, b_0, \dots, b_{d-1}\}$ ($diam(G) = d$) then $d(x,y) = i$, $z \in N(y) \Rightarrow i-1 \le d(z,x) \le i+1$ by the triangle inequality $\Rightarrow a_i + b_i + c_i = k \ \forall i$ (set $c_0 = b_d = 0$)

Proposition 5

Moore graphs are distance regular with parameters

$$\begin{bmatrix} * & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & k-1 \\ k & k-1 & k-1 & \dots & k-1 & * \end{bmatrix}$$

proof

Let us compute the c_i's first
Suppose x and y are arbitrary at distance i ($i > 0$)
We want to find no. of neighbours of y which are at a distance of $i-1$ from x.
Certainly, along the unique path of length i from x to y we can get hold of a neighbour
Suppose there are two such neighbours. Then we have two different paths from x to y contradicting the uniqueness of paths (since girth $= 2d+1$). Thus no. of such neighbours of y is exactly $1 \Rightarrow c_i = 1$ (irrespective of x, y)
Now let $0 \le j < d$, we try to compute a_j
Let $d(x,y) = j$. We want neighbours of y at a distance of j from x. Suppose such a neighbour exists, then we get a $2j+1$ cycle but $j < d \Rightarrow 2j+1 < 2d+1 = $ girth
This is not possible $\Rightarrow a_j = 0 \ \forall \ 0 \le j < d$
If $d(x,y) = d$ & we want neighbours of y at distance d from these, these will be $k-1$ in number because all neighbours of y lie in the d^{th} layer, except one of them in the previous layer. Thus $a_d = k-1$

Since $a_i + b_i + c_i = k$ (all neighbours of y must be at distance $i-1$ or i or $i+1$ from x, necessarily), we are done (set $c_0 = b_d = 0$)

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Definition : A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that
if $\{x, y\}$ is any edge, $|N(x) \cap N(y)| = \lambda$ = const.
if $\{x, y\}$ is not an edge then $|N(x) \cap N(y)| = \mu$ = const.

Before seeing that they are special cases of distance regular graphs, we give an important rule for the parameters. These are not free to be chosen.

Proposition 6

If G is strongly regular with parameters (n, k, λ, μ) then $k(k-\lambda-1) = \mu(n-k-1)$

proof

Fix $x \in V$. Consider $S_x = \{ \{y, z\} \in E \mid \{x, y\} \in E, \{x, z\} \notin E, z \neq x \}$

We count $|S_x|$ in two ways

For a fixed y st. $\{x, y\} \in E$, the number of neighbours z of y st. $z \neq x$ & $\{x, z\} \notin E$ is $k-1-\lambda$ (all $k-1$ neighbours of y except those that are also neighbours of x). The number of such y is k.

For a fixed z st. $z \neq x$ & $\{x, z\} \notin E$, the no. of neighbours y of z st. $\{x, y\} \in E$ is precisely μ and then no. of $z \neq x$ st. $z \notin N(x)$ is clearly $n-(k+1)$ (we exclude x & all its neighbours)

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Proposition 7

$$A^2 = kI + \lambda A + \mu(J - A - I)$$

where A is the adjacency matrix of an srg graph with parameters (n, k, λ, μ) and J is the all 1's matrix (all matrix sizes are $n \times n$)

proof

The $(x,y)^{th}$ entry of A^2 is

$$A^2(x,y) = \begin{cases} k & x = y \\ \lambda & \{x,y\} \in E \\ \mu & \{x,y\} \notin E, \; x \neq y \end{cases}$$

This is also the $(x,y)^{th}$ entry on the RHS, clearly ■

Since $\mathbb{1} = (1,1,\dots,1) \in \mathbb{R}^n$ is an Eigen vector for A with eigenvalue k & also an Eigen vector for J with eigenvalue n, applying to this equation gives an alternative algebraic proof of proposition 6.

Proposition 8

Let G be srg with parameters (n,k,λ,μ). If $\mu = 0$ then G is either disconnected or the complete graph. If $\mu \neq 0$ then G is a connected srg with $d = 2$ & parameters

$$\begin{bmatrix} * & 1 & \mu \\ 0 & \lambda & k-\mu \\ k & k-\lambda-1 & * \end{bmatrix}$$

proof

If $\mu = 0$, from prop 6, $k(k-\lambda-1) = 0 \Rightarrow k = \lambda + 1$

Now take an edge $\{x,y\}$. $|N(x) \cap N(y)| = \lambda$.

This means that there are no other vertices (if G is connected) and x is connected to all vertices & so is y. Thus we get the complete graph $K_{\lambda+2}$

If $\mu \neq 0$, any two x,y which are not connected have at least one common neighbour $\Rightarrow d(x,y) = 2$. If they are connected, $d(x,y) = 1$. Thus $d = diam(G) = 2$.

Now we figure out the parameters

Filling in the parameter matrix is a simple task. For instance, lets say we want to find b_1. Let $d(x,y) = 1$. We want no. of neighbours of y at dist 2 from x. Out of the k neighbours of y, one of them is x, λ of them are also neighbours of x & the rest are not connected to x. There are $k-\lambda-1$ in no. ■

We have one more result on eigenvalues of srg graphs after which we can characterise all Moore graphs (almost)

Theorem 9

If $G \neq K_n$ is a connected srg with parameters (n, k, λ, μ), then G has 3 distinct eigenvalues : k, r_1, r_2 with multiplicities $1, m_1, m_2$ where (use same parity for (r_1, f) & (r_2, g))

$$r_1, r_2 = \frac{\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2}$$

$$f, g = \frac{1}{2}\left(n - 1 \pm \frac{(\mu-\lambda)(v-1) - 2k}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}} \right)$$

Proof

It is a well known result that multiplicity of k as an e.val of a k-reg graph is 1 iff the graph is connected.

Now if the eigenvalues are $k, \theta, \theta, \ldots, \theta$ then from trace of the matrix being 0, $\theta = -k/n-1$ but the characteristic polynomial is monic & hence all rational roots are necessarily integers.

Thus $n-1 \mid k$ but $1 \leq k \leq n-1 \Rightarrow k = n-1$ & hence we have an $n-1$ regular graph on n vertices which can be uniquely identified as K_n

Let $\{ \mathbb{1}, v_2, \ldots, v_n \}$ be an eigenbasis of A with eigenvalues $\{k, \lambda_2, \ldots, \lambda_n\}$ resp.

We know $A^2 = kI + \lambda A + \mu(J - A - I)$

$\therefore A^2 v_i = k v_i + \lambda A v_i + \mu (J - A - I)(v_i)$

$\therefore \lambda_i^2 v_i = k v_i + \lambda \lambda_i v_i + \mu (0 v_i - \lambda_i v_i - v_i)$ $\quad (\because v_i \perp \mathbb{1})$

$\therefore \lambda_i^2 = k - \mu + \lambda_i(\lambda - \mu)$

$\therefore \lambda_i^2 - \lambda_i(\lambda-\mu) - (k-\mu) = 0$

The roots of the quadratic are :

$$r_1, r_2 = \frac{\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2}$$

We have only claimed that if $\lambda \neq k$ is an eigenvalue, then it must be a root of this quadratic equation. The converse is not true.

Thus, there can be at most 3 eigenvalues (distinct)

Of course we can't have only 1 eval $k, k, \ldots, k$

We already showed that if we have exactly 2, then it must be the complete graph.
Thus if $G \neq K_n$ we have 3 evals k, r_1, r_2 with multiplicities $1, m_1, m_2$
Now $1 + m_1 + m_2 = n$ & $\text{trace}(A) = 0 \Rightarrow k + m_1 r_1 + m_2 r_2 = 0$
Solving for m_1, m_2 we get what we want

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Corollary 10

For a connected srg of type (n, k, λ, μ) one of the following must necessarily hold (if it is not K_n):

(i) $n = 2k+1$, $\mu = \lambda + 1 = k/2$, $m_1 = m_2 = k$

(ii) $D = (\lambda-\mu)^2 + 4(k-\mu)$ is a perfect square. Further

* $\sqrt{D}$ divides $2k + (\lambda-\mu)(v-1)$ for odd v
* $2\sqrt{D}$ divides $2k + (\lambda-\mu)(v-1)$ for even v

proof

Follows by imposing that m_1, m_2 are in $\{1, 2, 3, \dots\}$
We need two cases: $(\mu-\lambda)(n-1) - 2k = 0$ & $\neq 0$.

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So far, Moore graphs are k-regular, have diameter d, have girth $2d+1$ & no. of vertices $= 1 + \sum_{i=0}^{d-1} k(k-1)^i$
Below is a summary of the classification
$k = 2 \Rightarrow G = C_{2d+1}$ & $(2, d)$ Moore graphs exist $\forall d \geq 1$
$d = 3 \Rightarrow k = 2, 3, 7, 57^*$.
$d \geq 3$ & $k \geq 3$ Moore graphs do not exist
$(57, 3)$ Moore graph is an open problem. Not known if it exists or not. The others exist for $d = 3$.

Theorem 11 (a)

If $k = 2$, then $\forall d \geq 1$, Moore graphs exist and they are simply C_{2d+1} i.e. all odd cycles

Proof

2-regular connected graphs (Moore graphs are connected, obviously, otherwise $d = \infty$) are just cycles and since girth $= 2d+1$, Moore graphs of degree 2 are necessarily odd cycles and conversely all $2d+1$ cycles are (k,d)-Moore graphs.

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Theorem 11 (b1)

If G is a $(k,2)$-Moore graph, then k must be $2, 3, 7$ or 57. Conversely, $(2,2)$, $(3,2)$ & $(7,2)$ Moore graphs exist

Proof

Moore graphs are srg of diameter 2 with paramets $(n, k, 0, 1)$ and from the constraint of vertices, $n = 1+k+k(k-1) = 1+k^2$

Now by theorem 9, if case 1 occurs, then

$k = n-k-1 \Rightarrow 2k+1 = 1+k^2 \Rightarrow k = 2$

If case 2 holds, then $D = (\lambda-\mu)^2 + 4(k-\mu) = 4k-3$ is a perfect square and $\sqrt{4k-3}$ divides $2k+(\lambda-\mu)(n-1) = 2k-k^2$

$$\therefore \quad 256 \times \frac{(2k-k^2)^2}{4k-3} = 64k^3 - 208k^2 + 100k + 75 + \frac{225}{4k-3}$$

is an integer $\Rightarrow 4k-3 \mid 225 \Rightarrow k = 2, 3, 7$ or 57

but for $k=2$, $4k-3 = 5$ is not a perfect square.

$\therefore k = 3, 5, 57$

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Theorem 11 (b2)

$(2,2)$, $(3,2)$ & $(7,2)$ Moore graphs exist & are unique

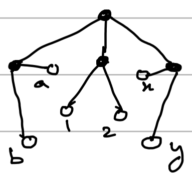
Proof

The pentagon is the unique $(2,2)$ Moore graph (uniqueness follows from theorem 11 (a))

The petersen graph is 3-regular, has diameter 2 and is on 10 vertices $\Rightarrow$ it is a $(3,2)$ Moore graph.

Conversely, let G be a $(3,2)$ Moore graph

The layer wise structure of G is

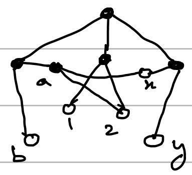


solid vertices have satisfied the degree requirements.

WLOG connect a to x. $a \not\sim y$ else we get a 4 cycle but the girth is $2d+1 = 5$. Similarly $a \not\sim b$ else 3 cycle

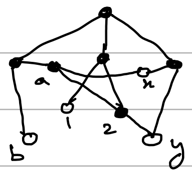
WLOG connect a to 2.

We now have the picture :



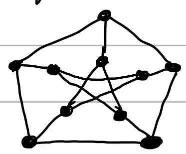
2 needs only one more edge. $2 \nsim 1$ & $2 \nsim x$ else we get a 3-cycle. $2 \nsim b$ else we get a 4-cycle. $\therefore 2 \sim y$

Now we have



$y \nsim x$ (3-cycle) & $y \nsim 1$ (4-cycle) $\Rightarrow y \sim b$

Going this way we indeed get the petersen graph



Now we come to (3, 7) Moore graphs. The proof is very detailed & requires an in-depth analysis. We need another separate presentation for this.

One may refer : "On Moore graphs with diameters 2 and 3" by Hoffman & Singleton (1960)

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Theorem 11(c)

Moore graphs with $k \geq 3$, $d \geq 3$ do not exist

proof

This is again a major breakthrough theorem and one may refer "On finite Moore graphs" by Bannai and Ito or "On Moore graphs" by Damerell (both are independent works)

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