INTRODUCTION

Q) Let $T = \{ z \in \mathbb{C} \mid |z| = 1 \}$, $\phi : T \to \mathbb{C}^\times$ be a continuous group homomorphism. Show that $\phi(T) \subseteq T$ and also that set of all group homorphisms from $T \to \mathbb{C}^\times$ is given by $\{ \varphi_n \mid n \in \mathbb{Z} \}$ where $\varphi_n : T \to T$ $\varphi_n(z) = z^n$, $z \in T$

Ans Consider an n^{th} root of unity $e^{\frac{i 2\pi k}{n}}$ $(0 < k < n)$

Then, say, $\phi(e^{\frac{2\pi i k}{n}}) = r e^{i\alpha}$

$\therefore$ $\phi(e^{2\pi i k}) = r^n e^{in\alpha} = \phi(1) = 1$

$\therefore$ $r = 1$, $e^{in\alpha} = 1 \Rightarrow n\alpha = 2m\pi$

Overall, every n^{th} root of unity maps to some point in T

Since the roots of unity are dense in T and ϕ is continuous, $\phi(T) \subseteq T$

Further, $\phi(e^{\frac{2\pi i k}{n}}) = e^{i\alpha} = e^{\frac{i 2\pi m}{n}}$

We must now show $k \mid m$

$$\phi(e^{\frac{2\pi i}{n}}) = e^{\frac{2\pi i m}{nk}}$$

$$\phi(1) = 1 \Rightarrow e^{2\pi i \frac{m}{k}} = 1 \Rightarrow k \mid m$$

Again, invoking density of the roots and continuity of ϕ, $\phi(x) = x^k$ for some k and all $x \mapsto x^k$ are continuous homomorphisms

Q) Classify all continuous group homomorphisms from $(\mathbb{R}, +)$ to $(\mathbb{T}, \times)$ ($\mathbb{T}$ as in above question)

Ans Firstly, $f(x) = e^{ikx}$ $\forall\, x \in \mathbb{R}$ is a continuous grp hom

we claim, if g is any cont. grp hom, $\exists$ k st. for all $x \in \mathbb{R}$, $g(x) = e^{ikx}$

Let $p : \mathbb{R} \to \mathbb{T}$ be $p(x) = e^{ix}$

Claim 1: $\forall$ $f : \mathbb{R} \to \mathbb{T}$ cont-grp hom, $\exists$ cont grp hom $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f = p \circ \tilde{f}$

claim 2: Every cont grp hom from $\mathbb{R}$ to $\mathbb{R}$ is of the form $f(x) = kx$ for any $k \in \mathbb{R}$

proof of claim 2:

Let $f(1) = k \Rightarrow f(n) = kn$ $\forall\, n \in \mathbb{N} \Rightarrow f(q) = kq$ $\forall$ $q \in \mathbb{Q}$

By density of rationals in reals & continuity of f, $f(x) = kx$ $\forall\, x \in \mathbb{R}$.

proof of claim 1:

Skipped (uses covering spaces and homotopy theory)

Using claim 1, 2 we get our result. ∎

Q) Let G be a finite cyclic group of order n. Let $\phi : G \to \mathbb{C}^{\times}$ be a group homomorphism. Show that if $G = \langle x \rangle$, then,

(i) $\phi(x) \in \{ e^{\frac{2\pi i k}{n}} : 0 \le k \le n-1 \}$

(ii) set of all group homomorphisms $G \to \mathbb{C}^{\times}$ is a group under point wise multiplication

(iii) The above group is isomorphic to G

Ans (i) Let $\phi(x) = z$

Then $\phi(x^n) = \phi(1) = z^n = 1$

$\therefore$ z is a root of unity

In particular, we have n group homomorphisms (assign x to each of the n roots of unity)

(ii) closure : ϕ_1, ϕ_2 are grp hom $G \to \mathbb{C}^{\times}$

Then $\phi_1 \phi_2 : G \to \mathbb{C}^{\times}$ is also a grp hom

since $\phi_1 \phi_2 (xy) = \phi_1(xy)\, \phi_2(xy) = \phi_1\phi_2(x)\, \phi_1\phi_2(y)$

associativity : $(\phi_1 \phi_2)\phi_3 = \phi_1(\phi_2\phi_3)$ is true for any functions $(\because (fg)h = f(gh))$

identity : $\phi_0 : G \to \mathbb{C}^{\times}$ as $\phi_0(x) = 1$ is the identity

since $\phi_1\phi_0(x^r) = \phi_1(x^r)\, \phi_0(x^r) = \phi_1(x^r)$

and $\phi_0\phi_1(x^r) = \phi_0(x^r)\, \phi_1(x^r) = \phi_1(x^r)$

$(\because \phi_0(x^r) = (\phi_0(x))^r = 1^r = 1)$

inverse : Let $\phi : G \to \mathbb{C}^{\times}$ as $\phi(x) = e^{\frac{2\pi i k}{n}}$ for some $0 \le k \le n-1$

Then, $f : G \to \mathbb{C}^{\times}$ as $f(x) = e^{\frac{2\pi i(n-k)}{n}}$ is the required inverse

$f\phi(x) = f(x)\,\phi(x) = e^{2\pi i} = 1 \Rightarrow f\phi = \phi_0$

(iii) We set up the isomorphism $\tau : G \to \hat{G}$ as follows

$\tau(x^r) = \phi_{x_r}$ where $\phi_x : G \to \mathbb{C}^\times$ as $\phi_{x_r}(y) = e^{\frac{2\pi i r}{n}}$

This is a homomorphism because,

$\tau(x^a)\,\tau(x^b) = \phi_{x_a}\,\phi_{x_b}$ $\qquad y \xrightarrow{\phi_{x_a}\phi_{x_b}} e^{\frac{2\pi i a}{n}}\, e^{\frac{2\pi i b}{n}}$

$\tau(x^a x^b) = \phi_{x_{a+b}}$ $\qquad y \xrightarrow{\phi_{x_{a+b}}} e^{\frac{2\pi i (a+b)}{n}}$

This is clearly a bijection since $|G| = |\hat{G}| = n$ and it is injective.

Follow up : what about infinite cyclic groups like $(\mathbb{Z}, +)$? What are the group homomorphisms from $\mathbb{Z} \to \mathbb{T}$

Ans Let $f(1) = z$ (s.t. $|z| = 1$). Then $f(n) = z^n$ and hence $f(1)$ determines the entire homomorphism

$\therefore\ \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \simeq \mathbb{T}$

Note that $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$ is a group under p.w. multiplication

($\because$ Let $f, g \in \mathrm{Hom}(\mathbb{Z}, \mathbb{T})$. Then $fg(1) = f(1)g(1)$ which is again some z s.t. $|z| = 1$)

Q) Classify group homomorphism from G to $\mathbb{C}^\times$ for a simple non-abelian group G which is finite

<u>Ans</u> Let $\phi : G \to \mathbb{C}^\times$ be a group homomorphism

$\ker \phi \lhd G$. But G is simple

$\therefore \ker \phi = \{1\}$ or G

If $\ker \phi = 1$, then $\phi(G) \cong G$ is a subgroup of $\mathbb{C}^\times$

But G is non abelian & every subgroup of $\mathbb{C}^\times$ is abelian

Thus, this cannot happen

Thus $\ker \phi = G$ and $\phi \equiv 1$

Q) Show the existence of two grp hom $S_3 \to \mathbb{C}^\times$. Also show the existence of an injective grp hom $S_3 \to GL_2(\mathbb{C})$

<u>Ans</u> $S_3 = \langle a, b, c \mid a^2 = b^2 = c^3 = abc = 1 \rangle$

Let $f : S_3 \to \mathbb{C}^\times$ $\quad f(a) = z_1$, $f(b) = z_2$, $f(c) = z_3$

$z_1^2 = z_2^2 = z_3^3 = 1$, $\quad z_1 z_2 z_3 = 1$

$\therefore z_3 = 1$, $\quad z_1 = \pm 1$, $\quad z_2 = \pm 1$

$\therefore$ 2 homomorphisms

$S_3 = \langle (12), (123) \rangle$. Let $\tau : S_3 \to \mathbb{C}^\times$ be inj hom

Let $\tau((12)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = A$

$\tau((123)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = B$ $\qquad \{1, a, b, b^2, ab, bab\}$

$A^2 = I$, $\quad B^3 = I$, $\quad (AB)^2 = I$

Since $S_3 = \langle a, b \mid a^2 = b^3 = (ab)^2 = 1 \rangle$, this is indeed a homomorphism and it is injective since $\operatorname{Im}(\tau) = \{I, A, B, B^2, AB, AB^2\}$ (all distinct)

Definition: A representation of a group G is a hom $\rho : G \to GL_n(\mathbb{C})$. n is called the degree of the rep

Definition: A class function on a group G is a function $f: G \to \mathbb{C}$ which is constant on each conjugacy class of G

Q) Check that $A \mapsto tr(A)$, $A \mapsto det(A)$ are class functions on $GL_n(\mathbb{C})$

Are there any others ?

Ans Two matrices are in the same conjugacy class iff they are similar and trace & det is same for similar matrices

We can go one step ahead and claim that $A \mapsto e_k$ is a class function where e_k = sum of all $k \times k$ principal minors (This is so, because e_k are coefficients (with sign) of the characteristic poly which is const on each conj class)

■

Definition: For a representation $\rho: G \to GL_n(\mathbb{C})$, the character of ρ is defined to be $\chi : G \to \mathbb{C}$ as $\chi(g) = trace(\rho(g))$

Proposition 0.1

The character of a representation is a class function.

proof

Let $b = gag^{-1}$ for some $g \in G$ $(a, b \in G)$

Then, $\chi(b) = tr(\rho(b)) = tr(\rho(g)\rho(a)\rho(g^{-1})) = tr(\rho(a)) = \chi(a)$

■

Definition: Let $G = \{g_1, \cdots, g_n\}$ be a finite group. Assign to each g_i some indeterminate x_i. Construct $A_{n \times n}$ with i,j^{th} entry being the indeterminate corresponding to $g_i g_j^{-1}$. $det(A)$ is the grp determinant

Proposition 0.2

For any finite group G of order n, $x_1 + x_2 + \cdots + x_n$ is always a factor of the group determinant

Proof

Consider the first row of the matrix A and hence, look at the set $\{g_1 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_1 g_n^{-1}\}$. This set must have cardinality n and hence must be G itself.

Thus every x_i appears in row 1

Similarly every x_i appears in every row and hence by adding all columns, $(x_1 + \cdots + x_n)$ is a common factor of the group determinant

■

Q) Find the factorisation of the group det for C_3 and C_4

Ans For $C_3 = \{1, x, x^2\}$, assign $\{x_1, x_2, x_3\}$ resp.

$$\therefore A_{C_3} = \begin{bmatrix} x_1 & x_3 & x_2 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \end{bmatrix}$$

$$\therefore \det(A_{C_3}) = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)$$

For $C_4 = \{1, x, x^2, x^3\}$, assign $\{a, b, c, d\}$ resp.

$$\therefore A_{C_4} = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}$$

$$\therefore \det(A_{C_4}) = (a+b+c+d)(a-b+c-d)((a-c)^2 + (b-d)^2)$$

■

Proposition 0.3

Let a group G act on a set X. Let S_X denote group of bijections of X under composition operation. There is a 1:1 correspondence between homomorphisms $\varphi : G \to S_X$ and set of actions of G on X

proof

Let $\varphi : G \to S_X$ be a homomorphism

Define a group action of G on X as $g * x := (\varphi(g))(x)$

This is indeed an action because $1 * x = (\varphi(1))(x) = (id)(x) = x$

and $g*(h*x) = g*((\varphi(h))(x)) = (\varphi(g))(\varphi(h)(x)) = (\varphi(g) \circ \varphi(h))(x)$

$= (\varphi(gh))(x) = (gh) * x$

Conversely, let there be an action of G on X. Define $\varphi : G \to S_X$ as $\varphi(g) : X \to X$ as $(\varphi(g))(x) := g * x$. This is indeed a homomorphism since $\varphi(g) \circ \varphi(h)(x) = \varphi(g)(h*x) = g*(h*x)$

$= (gh)*x = (\varphi(gh))(x) \Rightarrow \varphi(gh) = \varphi(g) \circ \varphi(h)$

∎

CHAPTER 1 : Basics, Maschke's theorem

Definition : A representation of a finite group G is a group homomorphism $\rho : G \to GL(V)$ where V is any vector space and $GL(V)$ is the set of all invertible linear maps from V to V.

We may denote $\rho(g) : V \to V$ by $\rho_g : V \to V$ and also even write $\rho_g(v)$ as $g \cdot v$ or simply gv.

If $\rho : G \to GL(V)$ is a representation, we say V is a G-module

Definition : The degree of a representation $\rho : G \to GL(V)$ is the dimension of V

Definition : By trivial representation of G, we mean $\rho_{triv} : G \to GL(V)$ which maps $\rho_{triv}(g) = id \quad \forall g \in G$ $\quad (id = I_{\dim(V)})$

Notice that in particular a one dimensional rep is the same as a group homomorphism $\rho : G \to \mathbb{C}^\times$ (assuming vector spaces over field $\mathbb{C}$)

Q) Classify degree one representations of $\mathbb{Z}/n\mathbb{Z}$

Ans Let $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a homomorphism

Then, all we need to do is define $f(1 \pmod{n})$ subject to $f(n \cdot 1 \pmod{n}) = f(1 \pmod{n})^n = 1$

Thus, $f(1 \pmod{n})$ must be an n^{th} root of unity and the homomorphism will be determined

Thus $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \simeq \{x \in \mathbb{C} \mid x^n = 1\}$

Q) Show that the Dihedral group D_{2n} viewed as $D_{2n} \subseteq \mathbb{C}$ (as a group of symmetries of a regular n-gon) itself, is its own degree 2 representation

Ans Let the vertices of the regular n-gon be at $1, \omega, \omega^2, \ldots, \omega^{n-1}$ for $\omega = \exp(2\pi i/n)$

Rotation through $\exp(2\pi i/n)$ & reflection about x-axis are both symmetries and they generate D_{2n}

Rotation through $\exp(2\pi i/n)$ is $\begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} = A$

Reflection through x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = B$

It can be checked easily that $B^2 = I$, $A^n = I$,

$A^j \neq I \;\; \forall\, j = 1, 2, \ldots, n-1$ and $BAB = A^{n-1} \Rightarrow AB = BA^{n-1}$

Thus, $\langle A, B \rangle = \{1, A, A^2, \ldots, A^{n-1}, B, BA, BA^2, \ldots, BA^{n-1}\}$

Q) Prove that there always exist two one-degree representations of S_n (upto commuting isomorphisms — this will be defined just after)

Ans $\rho_1 : S_n \to \mathbb{C}^*$ as $\rho_1(x) = 1 \;\; \forall\, x \in S_n$

is the trivial representation

$\rho_2 : S_n \to \mathbb{C}^*$ as $\rho_2(x) = \operatorname{sgn}(x) \;\; \forall\, x \in S_n$ is

another 1-degree rep, clearly "different" from ρ_1

Definition : Two reps $\phi : G \to GL(V)$, $\varphi : G \to GL(W)$ are said to be <u>equivalent</u> if $\exists$ an isomorphism $T : V \to W$ s.t. $\varphi_g \circ T = T \circ \phi_g \quad \forall g$

(Note : One such T should work for every $g \in G$)

Q) For $G = \mathbb{Z}/n\mathbb{Z}$, consider the two 2-degree representations

$$\phi([m]) = \begin{bmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{bmatrix}$$

$$\varphi([m]) = \begin{bmatrix} \exp(2\pi i m/n) & 0 \\ 0 & \exp(-2\pi i m/n) \end{bmatrix}$$

Show that these two are equivalent

<u>Ans</u>

$$\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -i\, e^{ix} & e^{ix} \\ i\, e^{-ix} & e^{-ix} \end{bmatrix}$$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} -i\, e^{ix} & e^{ix} \\ i\, e^{-ix} & e^{-ix} \end{bmatrix}$$

∎

Definition : The <u>standard rep of S_n</u> is a degree n rep obtained by permuting rows of I_n according to $\sigma \in S_n$

ex : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix} \xmapsto{\rho} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

<u>Proposition 1.1</u>

The above map is indeed a representation of S_n

proof

All we need to verify that the map is a group homomorphism.

i.e. $\rho(\sigma\tau) = \rho(\sigma)\cdot\rho(\tau)$

Firstly, $\rho(\sigma)\rho(\tau)$ is itself a permutation matrix being the product of two permutation matrices.

Consider the ij^{th} entry on the LHS $= \rho(\sigma\tau)_{ij}$

Suppose it is non zero, then $j \xmapsto{\sigma\tau} i$. Let $j \xmapsto{\tau} k \xrightarrow{\sigma} i$

On the RHS, $\rho(\sigma)\rho(\tau)_{ij} = \sum_{t=1}^{n} \rho(\sigma)_{it}\,\rho(\tau)_{tj}$

Since $j \xmapsto{\tau} k \xmapsto{\sigma} i$, $\rho(\tau)$, $\rho(\sigma)$ are both non-zero only when $t = k$ and hence $\rho(\sigma)\rho(\tau)_{ij} = 1$

Similarly, if $\rho(\sigma\tau)_{ij} = 0$ then $j \cancel{\xmapsto{\sigma\tau}} i$.

Suppose $j \xmapsto{\tau} k_1 \xmapsto{\sigma} k_2 \neq i$

$j \neq k_3 \xmapsto{\tau} k_4 \xmapsto{\sigma} i$

Then, $\sum_{t=1}^{n} \rho(\sigma)_{it}\,\rho(\tau)_{tj} = 0$

Indeed, if $\rho(\tau) \neq 0$, then $t = k_1$ but $k_1 \neq k_4$ else $j \xmapsto{\sigma\tau} i$

∎

Q.) Show that all permutation matrices have a common eigen vector, namely, the all 1's vector

Ans Let A_σ be the permutation matrix corresponding to $\sigma \in S_n$

Then $A_\sigma \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{i_1} \\ \vdots \\ x_{i_n} \end{bmatrix}$ where $\sigma(i_1, i_2, \dots, i_n) = (1, 2, \dots, n)$

Thus, $A_\sigma \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and hence we have a common eigen vector for all σ

∎

Definition : Let $\rho : G \to GL(V)$ be a representation. A vector subspace W of V is said to be G-invariant if $\forall\, g \in G$, $\rho_g(\omega) \in W$ $\forall\, \omega \in W$ i.e. $\rho_g(W) \subseteq W$ $\forall\, g \in G$

($\{0\}$, V themselves are G-inv subspaces of V)

Definition : $\rho : G \to GL(V)$ is said to be irreducible if the only G-invariant subspaces are $\{0\}$, V (trivial ones)

Q) Show that the standard representation of S_n is not irred.

Ans We saw in the previous question that $[1\ 1 \cdots 1]^T$ is a common eigen vector. Hence $W = \text{span}\{[1\ 1 \cdots 1]^T\}$ is a subspace of $\mathbb{C}^n$ such that $\rho_g(W) \subseteq W$

Q) Show that one-degree reps are irreducible

Ans Let $\rho : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ be a one degree rep.
$\nexists\, W$ st. $W \neq \{0\}, \mathbb{C}^\times$ and $W \underset{\text{Subspace}}{\subseteq} \mathbb{C}^\times$ since $\mathbb{C}^\times$ itself has dimension 1

Theorem 1.2

A two degree representation is reducible iff all ρ_g have a common eigen vector

proof

Suppose $\rho : G \to GL(V)$ $(\dim V = 2)$ is reducible, then there exists a G-inv subspace $W \neq V, \{0\}$ $\Rightarrow \dim W = 1$ $\Rightarrow W = \text{span}\{v\}$. Since $\rho_g(W) \subseteq W$ $\forall\, g \in G$, $\rho_g(v) = \lambda_g v$ $\forall\, g \in G$ (λ_g changes with g)

Conversely, if $\rho_g(v) = \lambda_g v$ then $\text{span}\{v\}$ is the G-invariant non trivial subspace of V

Proposition 1.3

All irreducible representations of a finite abelian group are one-dimensional

proof

Let $\rho : G \to GL(V)$ be a rep of G.

Then $\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g) \quad \forall\ g,h \in G$

Thus $\{\rho_g \mid g \in G\}$ is a commuting family of matrices. By a result from linear algebra, there is a common eigen vector

Thus, if ρ_g is irred, it must be of dim 1

■

Q) Let $X = \{1,2,\dots,n\}$. Consider the permutation module $\mathbb{C}[X]$

i.e. the vector space $\{c_1\cdot 1 + c_2\cdot 2 + \dots + c_n\cdot n \mid c_i \in \mathbb{C}\}$

List its subspaces closed under action of $G = S_n$ on $\mathbb{C}[X]$.

Also decompose into irreducibles (we $G = S_n$, $V = \mathbb{C}[X]$)

<u>Ans</u> There are two non-trivial subspaces closed under the action of S_n:

$W_1 = \{c_1\cdot 1 + \dots + c_n\cdot n \mid c_1 = c_2 = \dots = c_n\}$

$W_2 = \{c_1\cdot 1 + \dots + c_n\cdot n \mid c_1 + c_2 + \dots + c_n = 0\}$

W_1 has dimension 1 and is hence irreducible

Suppose W_2 has a G-invariant non-trivial subspace W.

Let $v = c_1\cdot 1 + \dots + c_n\cdot n \in W$ st. $(c_1, c_2, \dots, c_n) \neq (0,0,\dots,0)$

Assume WLOG that $c_1 \neq 0$

All c_i cannot be the same (else $c_1 + \dots + c_1 \neq 0$)

Hence assume WLOG $c_1 \neq c_2$

$(1,2)\cdot v = c_1\cdot 2 + c_2\cdot 1 + c_3\cdot 3 + \dots + c_n\cdot n$

$\therefore v - (1,2)\cdot v = (c_1 - c_2)\cdot 1 + (c_2 - c_1)\cdot 2 \in W$

$\therefore 1 - 2 \in W \implies (2,3)\cdot(1-2) = 1-3 \in W$

Similarly, $1-4, 1-5, \cdots, 1-n \in W$

$\therefore \dim(W) = n-1 \Rightarrow W = W_2$

$\therefore W_2$ is also irreducible

Q) Check that the 'natural' 2-deg representation of D_{2n} is irreducible

Ans The group of ρ_g's is generated by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

for $\theta = \frac{2\pi}{n}$

eigen vectors are $\begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}$; $\begin{matrix} -i \\ 1 \end{matrix}, \begin{matrix} i \\ 1 \end{matrix}$ resp

and hence there are no common e.vecs

$\therefore$ The rep is irreducible

Definition : $\rho : G \to GL(V)$ is said to be decomposable if there are G-invariant subspaces V_1, V_2 of V st $V_1 \neq 0$, $V_2 \neq 0$, $V_1 \oplus V_2 = V$

Proposition 1.4

decomposability implies reducibility but converse is false

proof

decomposable $\Rightarrow$ there is a non-trivial G-inv subspace (in fact, there are two) $\Rightarrow$ reducible

For the converse, let $G = \mathbb{Z}$, $\rho : G \to GL_2(\mathbb{C})$ be given by

$\rho(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. This is indeed a representation and

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a common eigenvector for all $\rho_g \Rightarrow$ reducible.

But this is not decomposable. Suppose it was, then

$V = V_1 \oplus V_2$ with $\dim V_1 = \dim V_2 = 1$ (since $\dim V = 2$)

Let $V_1 = \operatorname{span}\{v_1\}$, $V_2 = \operatorname{span}\{v_2\}$

Then v_1, v_2 are LI common e.vec of all ρ_g. But this is untrue since $\forall\, n \neq 0$, ρ_n are not diagonalisable and hence two such LI vectors cannot exist ■

Theorem 1.5

Let $\phi : G \to GL(V)$, $\varphi : G \to GL(W)$ be equivalent.

(i) ϕ decomp $\Rightarrow$ φ decomp

(ii) ϕ irred $\Rightarrow$ φ irred

proof

(i) Let $T : V \to W$ be the isomorphism st. $\varphi_g T = T \phi_g$.

ϕ is decomposable $\Rightarrow$ $V = V_1 \oplus V_2$ for G-invariant V_1, V_2

Let $W_1 = T(V_1)$, $W_2 = T(V_2)$

Firstly, we claim that $W = W_1 \oplus W_2$

To this extent, let $w \in W_1 \cap W_2 = T(V_1) \cap T(V_2) = T(V_1 \cap V_2)$

$= T(\{0\}) = \{0\}$ ($\because T$ is an isomorphism $\Rightarrow$ T is injective)

Let $w \in W = T(V_1 \oplus V_2)$

$\therefore w = Tv$ (some $v \in V$)

$\therefore w = Tv_1 + Tv_2$ (some $v_1 \in V_1$, $v_2 \in V_2$)

$Tv_1 \in W_1$, $Tv_2 \in W_2$

$\therefore w \in W_1 + W_2$

$\therefore W = W_1 \oplus W_2$

(we used $V = X \oplus Y$ iff $X \cap Y = \{0\}$ for subspaces X, Y of V)

Now we show W_1 & W_2 are G-invariant

Let $w \in W_1$. $\varphi_g w = T \phi_g T^{-1}(w)$. $T^{-1}(w) \in V_1$ & V_1 is G-inv.

$\therefore \varphi_g w \in T(V_1) = W_1$. Similarly W_2

(ii) Again, $\psi_g = T \phi_g T^{-1}$

Assume ϕ_g is irreducible

Suppose $\exists$ G-inv non trivial subspace W_0 of W, then $T^{-1}(W_0)$ is a non trivial subspace of V

It is G-invariant, too. Indeed if $v \in T^{-1}(W_0)$, then

$\phi_g w = T^{-1} \psi_g T(v) = T^{-1} \psi_g(w)$ for some $w \in W_0$

Since W_0 is g-inv., $T^{-1}(\psi_g(w)) = T^{-1}(w_1)$ for some $w_1 \in W_0$

$\therefore \phi_g w \in T^{-1}(w_1) \subseteq T^{-1}(W_0)$

This is not possible since V was irreducible

■

Definition: Let $\phi : G \to GL(V)$ be a rep with V being an inner product $\mathbb{C}$-vec space. ϕ is said to be unitary if each ϕ_g is unitary wrt $\langle , \rangle$ on V (i.e. $\langle \phi_g v, \phi_g w \rangle = \langle v, w \rangle \ \forall g$)

Q) Classify one dim unitary reps

Ans A 1×1 unitary matrix in $\mathbb{C}$ is just a unit complex number and hence one dim unitary reps are just any grp homomorphisms

$\phi : G \to \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$

■

Proposition 1.6

Let ϕ be unitary rep. Then ϕ is either irred or decomposable. Thus, reducibility implies decomposability for unitary reps

proof

Let ϕ be reducible. $\exists$ V_0 (non-trivial) subspace of V s.t. V_0 is G-inv.

We show that $V_0^{\perp}$ is also G-inv so that $V = V_0 \oplus V_0^{\perp}$

Let $y \in V_0^{\perp}$. $\phi_g^* = \phi_g^{-1} = \phi_{g^{-1}}$. $\phi_{g^{-1}}(x) \in V_0 \ \forall x \in V_0$

$\therefore \langle y, \phi_{g^{-1}}(x) \rangle = 0 \quad (\forall x \ \forall g)$

$\therefore \langle y, \phi_g^* x \rangle = 0$

$\therefore \langle \phi_g y, x \rangle = 0$

$\therefore \phi_g y \in V_0^\perp$

$\therefore V_0^\perp$ is G-invariant

■

Proposition 1.7

Let $\rho : G \to GL(V)$ be a rep of finite G with V being a finite dim $\mathbb{C}$-vector space. Then $\exists \langle , \rangle$ on V st all ρ_g are unitary wrt $\langle , \rangle$.

That is, there is a G-inv inner product

In particular, every rep is equivalent to a unitary rep

proof

Let $[,]$ be any product on V

Define $\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} [\rho_g v, \rho_g w]$

This is indeed an inner product

(i) $\langle 0, 0 \rangle = 0$

$\langle v, v \rangle = 0 \Rightarrow [\rho_g v, \rho_g v] = 0 \ \forall g \Rightarrow \rho_g v = 0 \ \forall g \Rightarrow v = 0$

(ii) $\langle v, aw_1 + bw_2 \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho_g v, \rho_g (aw_1 + bw_2)]$

linearity of ρ_g & linearity of $[,]$ gives $a\langle v, w_1 \rangle + b\langle v, w_2 \rangle$

(iii) $\langle w, v \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{[\rho_g v, \rho_g w]} = \overline{\langle v, w \rangle}$

(iv) $\langle v, v \rangle \geq 0$ is obvious since $[\rho_g v, \rho_g v] \geq 0 \ \forall g$

To verify G-invariance (i.e. to verify that ρ_g is unitary),

$\langle \rho_g v, \rho_g w \rangle = \frac{1}{|G|} \sum_{h \in G} [\rho_h \rho_g v, \rho_h \rho_g w] = \frac{1}{|G|} \sum_{t \in G} [\rho_t v, \rho_t w]$

$= \langle v, w \rangle$

■

Theorem 1.8

A three degree representation is reducible iff all ρ_g have a common eigen vector

proof

Suppose all ρ_g have a common eigen vector v, then $\text{span}\{v\}$ is then non-trivial G-invariant subspace.

Conversely, suppose there is a non trivial G-invariant subspace W, then either W is one-dimensional, in which case we are done (since $W = \text{span}\{v\}$ & v is the req. e.vec) or W is 2 dimensional in which case $W^\perp$ is 1 dimensional and we are done again.

(Note : $W^\perp$ is taken wrt the inner prod which makes ρ unitary)

∎

Definition : $\rho : G \to GL(V)$ is said to be completely reducible if V admits a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$ where each V_i is G-invariant and $\rho_i : G \to GL(V_i)$, the restriction of ρ_g to V_i is irred

Proposition 1.9

Let ϕ, φ be equivalent. If ϕ is completly red, so is φ

($\phi : G \to GL(V)$, $\varphi : G \to GL(W)$)

proof

Using the exact procedure as in proof of theorem 1.5 (need to use both parts here), we can get this result as a corollary

∎

Theorem 1.10 (Maschke's theorem)

Every rep of a finite group is completely reducible

proof

Let $\phi : G \to GL(V)$ be a rep of a finite group G

For $\dim V = 1$, ϕ is already irred and there is nothing to prove.

Assume that the theorem holds for all V such that $\dim(V) \leq n-1$

Let $\dim(V) = n$. ϕ is either irreducible in which case we are done

or it is decomposable as $V = V_1 \oplus V_2$. $\dim(V_1) < n$, $\dim(V_2) < n$

and by induction hypothesis, $V_1 = W_1 \oplus \cdots \oplus W_r$, $V_2 = U_1 \oplus \cdots \oplus U_{r'}$

and we are done

Note: All of $W_1, \ldots, W_r, U_1, \ldots, U_{r'}$ are irred and G-inv.

Q) Given $\rho : G \to GL(V)$, using an isomorphism $T : V \to \mathbb{C}^n$ ($\dim V = n$), we get an equivalent representation $\tilde{\rho} : G \to GL_n(\mathbb{C})$. Show that we can choose the basis of V in such a way so that

$$\tilde{\rho}(g) = \mathrm{Diag}(\rho_1(g), \ldots, \rho_k(g)) \quad \text{(block diagonal)}$$

where each $\rho_i(g)$ is an $n_i \times n_i$ block and $V = V_1 \oplus \cdots \oplus V_k$ and $\dim(V_i) = n_i$

Ans We assume $V = W \oplus U$ and show that $\tilde{\rho}(g) = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$

W is an r dimensional submodule of V ($\dim V = n$)

Let $w_1, \ldots, w_r$ form a basis of W. Extend this to a basis of V as $w_1, \ldots, w_r, v_1, \ldots, v_{n-r}$.

Let
$$\rho_g w_i = \sum_{j=1}^{r} a_{ji}(g)\, w_j$$

$$\rho_g v_i = \sum_{j=1}^{r} b_{ji}(g)\, w_j + \sum_{j=1}^{n-r} d_{ji}(g)\, v_j$$

$$\therefore \rho_g = \left[\begin{array}{cc|cc} a_{11}(g) & \cdots & b_{11}(g) & \cdots \\ \vdots & & \vdots & \\ a_{1r}(g) & \cdots & b_{1r}(g) & \cdots \\ \hline 0 & & d_{11}(g) & \cdots \\ \vdots & & \vdots & \\ 0 & & d_{1\,n-r}(g) & \cdots \end{array}\right] = \begin{bmatrix} A(g) & B(g) \\ 0 & D(g) \end{bmatrix}$$

$\rho_g \rho_h = \rho_{gh} \Rightarrow$ $A(g)A(h) = A(gh)$ — ①

$D(g)\,D(h) = D(gh)$ — ②

$A(g)\,B(h) + B(g)\,D(h) = B(gh)$ — ③

sizes: A is $r\times r$, D is $s\times s$, B is $r\times s$, O is $s\times r$

① is not new because A_g is just $\rho_g|_W$ and W is also a G-submodule (i.e. $\rho' : G \to GL(W)$ is a rep)

Also observe that ρ_g invertible $\Rightarrow$ A_g, D_g invertible

We want to show the existence of $T : V \to \mathbb{C}^n$ s.t.

$$T\rho_g T^{-1} = \begin{bmatrix} A_g & 0 \\ 0 & D_g \end{bmatrix}$$

Let $$T = \begin{bmatrix} I_r & Q \\ 0 & I_s \end{bmatrix}$$

Then $T\rho_g T^{-1} = \begin{bmatrix} A_g & 0 \\ 0 & D_g \end{bmatrix}$ tells us,

$$-A_g Q + B_g + Q D_g = 0$$

Now from 3, $A_g B_h D_h^{-1} + B_g = B_{gh} D_h^{-1}$

Fix g and sum over varying h:

$$A_g \sum_{h\in G} B_h D_h^{-1} + B_g |G| = \sum_{h\in G} B_{gh} D_h^{-1}$$

$$= \sum_{f\in G} B_f D_{f^{-1}g}$$

$$\therefore A_g \sum_{h\in G} B_h D_h^{-1} + B_g |G| = \Big(\sum_{f\in G} B_f D_f^{-1}\Big) D_g$$

$\therefore$ $Q = -\dfrac{\sum_{h\in G} B_f D_f^{-1}}{|G|}$ is the choice

Thus we have our required linear isomorphism T

Definition: Let $\phi : G \to GL_n(\mathbb{C})$ be a rep. The map $\bar{\phi} : G \to GL_n(\mathbb{C})$ given by $\bar{\phi}(g) = \overline{\phi(g)}$ is called the conjugate representation. The map $\phi^* : G \to GL_n(\mathbb{C})$ given by $\phi^*(g) = (\phi(g^{-1}))^T$ is called the contragredient representation

Proposition 1.11

The above defined reps are indeed reps

proof

$$\bar{\phi}(gh) = \overline{\phi(gh)} = \overline{\phi(g)\phi(h)} = \overline{\phi(g)}\;\overline{\phi(h)} \qquad (\because \overline{AB} = \bar{A}\bar{B})$$

$$\phi^*(gh) = \phi(h^{-1}g^{-1})^T = (\phi(h^{-1})\phi(g^{-1}))^T = \phi^*(g)\phi^*(h)$$

$\blacksquare$

Definition: For a group G, we define commutator $[g,h] = g^{-1}h^{-1}gh \;\forall\, g,h \in G$.

We further define the commutator subgroup as the group generated by all commutators and denote it by G'

Proposition 1.12

(i) $G' \trianglelefteq G$

(ii) G' is the smallest subgroup of G st G/G' is abelian

proof

(i) $$g x^{-1}y^{-1}xy g^{-1} = g x^{-1}g^{-1} g y^{-1} g^{-1} g x g^{-1} g y g^{-1} = [gxg^{-1}, gyg^{-1}] \in G'$$

$\therefore G' \trianglelefteq G$

In particular, G/G' is a group

(ii) Let $N \trianglelefteq G$ st. G/N is abelian

$xNyN = yNxN$

$\therefore xyN = yxN$

$\therefore x^{-1}y^{-1}xy \in N \Rightarrow [x,y] \in N \Rightarrow G' \subseteq N$

We also show that every $H \leq G$ st. $G' \subseteq H$ is normal

For $g \in G$, $h \in H$, $ghg^{-1}h^{-1} = h' \in G' \Rightarrow h' \in H$

$\therefore ghg^{-1} = h'h \in H$

$\blacksquare$

Theorem 1.13 (Lifting)

Let $N \triangleleft G$ and $\rho : G/N \to GL(V)$ be a rep. ρ can be lifted to $\tilde{\rho}$, a rep of G by defining $\tilde{\rho} : G \to GL(V)$ as $\tilde{\rho}(g) = \rho(gN)$

This lifting process preserves dimension and irreducibility

Proof

$\tilde{\rho}(gh) = \rho(ghN) = \rho(gN)\rho(hN) = \tilde{\rho}(g)\tilde{\rho}(h)$

$\therefore \tilde{\rho}$ is indeed a rep

Clearly, lifting preserves dimension since $\dim(V)$ hasn't changed

Consider the matrices of ρ and $\tilde{\rho}$ for each $gN \in G/N$, $g \in G$ respectively.

In general as we saw earlier, the matrix is of the form

$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. If the 0 does not exist, then the rep is irreducible. Since $\rho, \tilde{\rho}$ have the same matrices (ρ has some matrices, $\tilde{\rho}$ uses these with multiplicity), clearly, ρ is reducible iff $\tilde{\rho}$ is reducible.

■

Proposition 1.14

The number of one dim reps of G are in 1:1 correspondence with elements of G/G'

Proof

We prove the following 2 facts

(i) $\mathrm{Hom}(G, \mathbb{C}^\times) \simeq \mathrm{Hom}(G/G', \mathbb{C}^\times)$ for any group G

(ii) $\mathrm{Hom}(G, \mathbb{C}^\times) \simeq G$ for any abelian group G

This proves our proposition

proof of (i) :

Let $\rho : G/G' \longrightarrow \mathbb{C}^\times \in \text{Hom}(G/G', \mathbb{C}^\times)$

Construct $\varphi(\rho) : G \to \mathbb{C}^\times \in \text{Hom}(G, \mathbb{C}^\times)$ naturally as

$\varphi(\rho)(x) = \rho(xG') \quad \forall x \in G$

We claim φ is the required isomorphism

$\varphi(\rho\tau)(x) = \rho\tau(xG') = \rho(xG')\tau(xG')$

$\varphi(\rho)\cdot\varphi(\tau)(x) = \varphi(\rho)(x)\,\rho(\tau)x = \rho(xG')\,\tau(xG')$

$\therefore \varphi$ is a group homomorphism

$\ker(\varphi) = \{\rho : G/G' \to \mathbb{C}^\times \mid \rho(xG') = 1 \quad \forall x \in G\}$

$\rho(xG') = 1 \quad \forall x \in G \Rightarrow \rho(y) = 1 \quad \forall y \in G/G'$

$\therefore \ker(\varphi) = \{id\} \Rightarrow \varphi$ is one-one

To show φ is onto, let $\phi : G \to \mathbb{C}^\times$ be a grp hom

Clearly $G' \subseteq \ker(\phi)$. Define $\rho : G/G' \to \mathbb{C}^\times$ as

$\rho(xG') = \phi(x)$. If such a ρ is well defined, we

are done.

Indeed if $xG' = yG'$ then $xy^{-1} \in G' \Rightarrow xy^{-1} \in \ker\phi$

$\Rightarrow \phi(x)\phi(y^{-1}) = 1 \Rightarrow \phi(x) = \phi(y)$

proof of (ii) :

We have already seen at the very beginning that if G is finite cyclic,

$G \simeq \text{Hom}(G, \mathbb{C}^\times)$

By the structure theorem for Abelian groups, let G be isomorphic

to $C_{p_1^{a_1}} \times \dots \times C_{p_n^{a_n}}$

We already saw that $C_{p_i^{a_i}} \simeq \text{Hom}(C_{p_i^{a_i}}, \mathbb{C}^\times) := \widehat{C_{p_i^{a_i}}}$

Then, $G \simeq C_{p_1^{a_1}} \times \dots \times C_{p_n^{a_n}} \simeq \widehat{C_{p_1^{a_1}}} \times \dots \times \widehat{C_{p_n^{a_n}}}$

So we can hope and expect $\widehat{G} \simeq \widehat{C_{p_1^{a_1}}} \times \dots \times \widehat{C_{p_n^{a_n}}}$

This is indeed true

We prove this for $G \simeq C_1 \times C_2$ and rest follows by induction.

Let K, H be groups

Consider
$$j_K : K \to K \times H$$
$$g \mapsto (g, 1)$$

$$g_H : H \to K \times H$$
$$h \mapsto (1, h)$$

Construct $\phi : \mathrm{Hom}(K \times H, \mathbb{C}) \quad \mathrm{Hom}(K, \mathbb{C}) \times \mathrm{Hom}(H, \mathbb{C})$

as $\phi(\lambda) := (\lambda \circ j_K, \lambda \circ j_H)$

This is the required isomorphism $(\because \hat{G} \simeq \hat{C}_1 \times \hat{C}_2)$

Indeed
$$\phi(\sigma \cdot \tau) = ((\sigma \cdot \tau) \circ j, (\sigma \cdot \tau) \circ j_H)$$
$$= ((\sigma \circ j_K) \cdot (\tau \circ j_K), (\sigma \circ j_H) \cdot (\tau \circ j_H))$$
$$= (\sigma \circ j_K, \sigma \circ j_H)(\tau \circ j_K, \tau \circ j_H)$$
$$= \phi(\sigma) \cdot \phi(\tau)$$

Construct $\varphi : \mathrm{Hom}(K, \mathbb{C}) \times \mathrm{Hom}(H, \mathbb{C}) \longrightarrow \mathrm{Hom}(K \times H, \mathbb{C})$

as $\varphi(\sigma, \tau) : K \times H \to \mathbb{C}$ as

$$(k, h) \xmapsto{\varphi(\sigma, \tau)} \sigma(k) \cdot \tau(h)$$

we claim φ and ϕ are inverse

Indeed $\phi \circ \varphi(\sigma, \tau) = (\varphi(\sigma, \tau) \circ j_K, \varphi(\sigma, \tau) \circ j_H)$

$$(\varphi(\sigma, \tau) \circ j_K)(k) = \varphi(\sigma, \tau)(k, 1) = \sigma(k)$$
$$(\varphi(\sigma, \tau) \circ j_H)(h) = \varphi(\sigma, \tau)(1, h) = \tau(h)$$

$$\therefore \phi(\varphi(\sigma, \tau)) = (\sigma, \tau) \Rightarrow \phi \circ \varphi = id$$

Conversely, $(\varphi \circ \phi)(\lambda) = \varphi(\phi(\lambda))$

$$= \varphi((\lambda \circ j_K, \lambda \circ j_H))$$

$$\varphi((\lambda \circ j_K, \lambda \circ j_H))(k,h) = \lambda \circ j_K(k) \cdot \lambda \circ j_H(h)$$

$$= \lambda((k,1)) \cdot \lambda((1,h))$$

$$= \lambda((k,1) \cdot (1,h))$$

$$= \lambda(k,h)$$

$\therefore \varphi((\lambda \circ j_K, \lambda \circ j_H)) = \lambda$

$\therefore \varphi \circ \phi = id$

$\blacksquare$

CHAPTER 2 : Character Theory Basics

Definition : Let $\phi : G \to GL(V)$, $\psi : G \to GL(W)$ be reps. A morphism in the category of representations of G (a G-morphism) is a linear map $T : V \to W$ such that $T\phi_g = \psi_g T \quad \forall g \in G$

Note that in particular, if T is an isomorphism, then ϕ and ψ are equivalent representations

Definition : We denote the set of all G-morphisms between ϕ and ψ by $Hom_G(\phi, \psi)$

Q) Show that $Hom_G(\phi, \psi)$ is a $\mathbb{C}$-linear vector subspace of $Hom(V, W)$

Ans $Hom(V, W)$ is a $\mathbb{C}$ linear space and $Hom_G(\phi, \psi)$ also written as $Hom_G(V, W)$ is clearly a subspace.

It is also a vector space since if $\sigma, \tau \in Hom_G(\phi, \psi)$

Then $(\sigma + \tau)\phi_g = \sigma\phi_g + \tau\phi_g = \psi_g\sigma + \psi_g\tau = \psi_g(\sigma + \tau)$

$\therefore \sigma + \tau \in Hom_G(\phi, \psi)$

Similarly $k\sigma \in Hom_G(\phi, \psi)$ for any scalar $k \in \mathbb{C}$

$\blacksquare$

Proposition 2.1

Let $T : V \to W$ be a G-hom between $\phi : G \to GL(V)$, $\psi : G \to GL(W)$

Then $\ker T$ is a G-submodule of V and $Im(T)$ is a G-submodule of W

proof

Let $v \in \ker T \Rightarrow T(v) = 0 \Rightarrow \psi_g T(v) = 0 \Rightarrow T(\phi_g(v)) = 0$

$\therefore \phi_g(v) \in \ker T \Rightarrow \ker T$ is G-invariant. $\ker T$ is clearly also a subspace of V

Let $w \in \operatorname{Im} T \Rightarrow T(u) = w \Rightarrow \psi_g(w) = \psi_g(T(u)) = T(\phi_g(u))$

$\therefore \psi_g(w) \in \operatorname{Im} T \Rightarrow \operatorname{Im} T$ is G-invariant. $\operatorname{Im} T$ is clearly also a subspace of W

Theorem 2.2 (Schur's lemma)

Let V, W be irreducible G-modules and $T \in \operatorname{Hom}_G(V, W)$. Then either $T = 0$ or T is invertible (and hence an isomorphism)

proof

$\ker T$ is a G-submodule of V and hence must be $\{0\}$ or V

Similarly $\operatorname{Im} T$ must be $\{0\}$ or W

$\ker T = V \Leftrightarrow \operatorname{Im} T = \{0\}$ in which case $T = 0$

$\therefore \ker T = \{0\} \Leftrightarrow \operatorname{Im} T = W$ in which case T is invertible and hence an isomorphism (making the two steps equivalent)

Corollary 2.3

Let V be an irreducible G-module. Then $\operatorname{Hom}_G(V, V)$ can be characterized as $\{ kI \mid k \in \mathbb{C} \}$

proof

Let $\rho : G \to GL(V)$. Clearly $\{kI \mid k \in \mathbb{C}\} \subseteq \operatorname{Hom}_G(V, V)$

Let $T \in \operatorname{Hom}_G(V, V) \Rightarrow \rho_g T = T \rho_g \Rightarrow (T - kI)\rho_g = \rho_g (T - kI)$

By Schur's lemma, $T - kI = 0$ or is an isomorphism.

But $\det(T - kI) = 0$ (choose k to be an eigen value). Thus,

$T - kI = 0 \Rightarrow T = kI$

Q) Find the dimension of $\mathrm{Hom}_G(V, W)$ (V, W irred G-modules)

Ans If $V \simeq W$, $\mathrm{Hom}_G(V, W) \longleftrightarrow \{ kI \mid k \in \mathbb{C} \}$ where '$\longleftrightarrow$' indicates equal sets through change of basis (since $V \neq W$ but $V \simeq W$)

$\therefore \dim \mathrm{Hom}_G(V, W) = 1$

If $V \not\simeq W$, $\mathrm{Hom}_G(V, W) = \{0\} \Rightarrow \dim = 0$

Q) Show that if two reps ρ, φ are equivalent, they must be unitarily equivalent (i.e. if $T\rho_g = \varphi_g T$ then $\exists$ unitary U such that $U\rho_g = \varphi_g U$)

Ans $T \rho_g = \varphi_g T$

$\therefore \rho_g^* T^* = T^* \varphi_g^*$

Since every rep is equivalent to a unitary rep, ρ, φ may be taken unitary. Thus $\rho_g^* = (\rho_g)^{-1} = \rho_{g^{-1}}$

$\therefore \rho_{g^{-1}} T^* = T^* \varphi_{g^{-1}}$

$\therefore T^*$ also witnesses the equivalence of ρ, φ

$T \rho_g \rho_{g^{-1}} T^* = \varphi_g T T^* \varphi_{g^{-1}}$

$\therefore T T^* \varphi_g = \varphi_g T T^*$

By corollary of Schur's lemma, $T T^* = \lambda I$

Choose $U^* = \frac{1}{\sqrt{\lambda}} T^*$ ie. $U = \frac{1}{\sqrt{\lambda}} T$ $(\lambda \neq 0)$

Clearly U also witnesses equivalence of ρ, φ and

$UU^* = \frac{1}{\lambda} T T^* = I$

Q) Let V be a G-module with W, a G-submodule. Choose a basis $w_1, \cdots, w_r$ of W and extend it to a basis $w_1, \cdots, w_r, v_1, \cdots, v_s$ of V

Then we have seen that $\rho_g = \begin{bmatrix} A(g) & B(g) \\ 0 & D(g) \end{bmatrix}$ where $A(g)$ is

$r \times r$ and $D(g)$ is $s \times s$.

Find a basis of the s-dim subspace $W^\perp$ corresponding to which we get the matrix $D(g)$ (currently with $v_1, \cdots, v_s$ we get $B(g)$ also. We want a basis so that $B(g) = 0$)

Ans $\rho_g v_i = \sum_{j=1}^{r} b_{ji} w_j + \sum_{k=1}^{s} d_{ki} v_k$

Consider $V/W := \{ v + W \mid v \in V \}$

Equip it with $(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W$ and

$\alpha \cdot (v_1 + W) := \alpha v_1 + W$

Then V/W is the result of the equivalence $v \sim w$ iff $v - w \in W$ on V

Consider $\pi : V \to V/W$ sending $v \longmapsto v + W$

$\ker(\pi) = W$

$\therefore$ If $v = \sum_{i=1}^{r} \alpha_i w_i + \sum_{j=1}^{s} \beta_j v_j$, $\pi(v) = \sum_{j=1}^{s} \beta_j \pi(v_j)$

$\therefore \{\pi(v_j)\}_{j=1}^{s}$ is a basis for V/W

Clearly $\text{span}\{\pi(v_i)\} = V/W$

If $\sum_{i=1}^{s} b_i \pi(v_i) = 0$ then $\sum_{i=1}^{s} b_i v_i \in \ker(\pi) = W$

but $v_i \notin W \Rightarrow$ all b_i must be 0

Now we turn V/W into a G-module as $\rho_g(v + W) := \rho_g v + W$

This is well defined, because if $v_1 - v_2 \in W$ then since W is a G-submodule, $\rho_g v_1 - \rho_g v_2 \in W$

Thus $\rho_g(\pi(v_i)) = \rho_g(v_i + W) = \rho_g v_i + W = \pi(\rho_g v_i)$

$$= \pi\left(\sum_{j=1}^{r} b_{ji} w_j + \sum_{k=1}^{s} d_{ki} v_k\right) = \sum_{k=1}^{s} d_{ki}\, \pi(v_k)$$

Thus with respect to the basis $\{\pi(v_1), \dots, \pi(v_s)\}$ we get the matrix $D(g)$ corresponding to the s-dimensional G-submodule V/W

Definition: A class function on a group G is a function f, $f : G \to \mathbb{C}^\times$ such that f is constant on each conjugacy class of G

Definition: Given a rep $\rho : G \to GL(V)$, a character of the rep ρ is a function $\chi : G \to \mathbb{C}^\times$ defined as $g \longmapsto \text{trace}(\rho(g))$

Proposition 2.4

Equivalent reps have the same character. Further, for a given rep, the character is a class function

proof

Let ρ, φ be equivalent reps for G-modules V, W (isomorphic)

Then $\exists\ T : V \to W$ (invertible) such that $T\rho_g = \varphi_g T$. Let χ be character of ρ and χ' be character of φ

Then $\chi_g = \text{trace}(\rho_g) = \text{trace}(T^{-1}\varphi_g T) = \text{trace}(\varphi_g) = \chi'_g$

For the second part, let $x = gyg^{-1}$ for $x, y, g \in G$.

$\chi_x = \text{trace}(\rho(x)) = \text{trace}(\rho(gyg^{-1})) = \text{trace}(y) = \chi_y$

Q) Find the character of the natural representation of S_n

Ans Let $\rho : S_n \to GL_n(\mathbb{C})$ be the natural representation of S_n

Thus for $g = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$, $[\rho_g]_{ij} = \delta_{a_i j}$

$$\therefore \chi_g = \sum_{t=1}^{n} \delta_{a_t t}$$

$\delta_{a_t t} = 1$ iff g fixes t

$\therefore \chi_g =$ no. of elements fixed by g

In particular, $\chi_{id} = n$, $\chi_{\bar{id}} = \begin{cases} 1 & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$

where $id = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$, $\bar{id} = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$

■

Definition: Given a representation $\rho : G \to GL(V)$, we define $\mathcal{C}(\rho)$, the commutant algebra to be the set of all matrices over $\mathbb{C}$ such that $T\rho_g = \rho_g T$ $\forall g \in G$. (Basically $Hom_G(V, V)$)

- This is indeed an algebra since it is closed under addition, scalar multiplication, matrix multiplication (and also conjugate transpose)
- Also notice that if A is irreducible, by corollary to Schur's lemma, T must be a scalar multiple of the identity

Proposition 2.5

Let ρ, φ be equivalent. Then $\mathcal{C}(\rho) \cong \mathcal{C}(\varphi)$

Proof

Let $f : \mathcal{C}(\rho) \to \mathcal{C}(\varphi)$ as $f(A) = P^{-1}AP$ (where $\varphi_g = P^{-1}\rho_g P$)

Then $A \in \mathcal{C}(\rho) \iff A\rho_g = \rho_g A \iff AP\varphi_g P^{-1} = P\varphi_g P^{-1} A$

$\iff (P^{-1}AP)\varphi_g = \varphi_g(P^{-1}AP) \iff P^{-1}AP \in \mathcal{C}(\varphi)$

■

Q) Try to find $\mathcal{C}(\rho)$ in a general sense

<u>Ans</u> By virtue of the above prop, we may assume ρ is of the form $\rho_g = \begin{bmatrix} A_1(g) & & \\ & \ddots & \\ & & A_k(g) \end{bmatrix}$ (block diag) where the restricted representations $A_1 \cdots A_k$ are irreducible

Let $T\rho_g = \rho_g T$. We wish to characterize T

Let $T = \begin{bmatrix} T_{11} & T_{12} & \cdots \\ \cdots & & T_{kk} \end{bmatrix}$ where T_{ij} has size $m_i \times m_j$

(A_i has size $m_i \times m_i$)

By direct multiplication $T_{ij} A_j(g) = A_i(g) T_{ij} \quad \forall g, i, j$

(There are s^2 linear systems)

By corollary of Schur's lemma, $T_{ij} = \begin{cases} \lambda_{ij} I_{m_i} & A_i \overset{\text{equiv}}{\sim} A_j \\ 0 & A_i \not\sim A_j \end{cases}$

Conversely any such T also witnesses $T\rho_g = \rho_g T$

Thus if $\rho_g = \text{diag}\,(\underbrace{A_1(g)\, A_1(g) \cdots A_1(g)}_{e_1 \text{ times}} \cdots \underbrace{A_k(g) \cdots A_k(g)}_{e_k \text{ times}})$

where $A_i \not\sim A_j$ (for $i \neq j$) and A_i has size $m_i \times m_i$

then $T = \text{diag}\,(C_1, C_2, \cdots, C_k)$

where $C_i = \begin{bmatrix} \lambda_{11}^i I_{m_i} & \lambda_{12}^i I_{m_i} & \cdots & \lambda_{1e_i}^i I_{m_i} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{e_i 1}^i I_{m_i} & \lambda_{e_i 2}^i I_{m_i} & \cdots & \lambda_{e_i e_i}^i I_{m_i} \end{bmatrix}$

■

Q) find $\dim(\mathcal{C}(\rho))$ where $\rho(g) = \text{Diag}(\underbrace{A_1 \cdots A_1}_{e_1} \cdots \underbrace{A_k \cdots A_k}_{e_k})$

Ans we obtained $T = \text{diag}(C_1 \cdots C_k)$ where

$$C_i = \begin{bmatrix} \lambda^i_{11} I_{m_i} & \lambda^i_{12} I_{m_i} & \cdots & \lambda^i_{1e_i} I_{m_i} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^i_{e_i 1} I_{m_i} & \lambda^i_{e_i 2} I_{m_i} & \cdots & \lambda^i_{e_i e_i} I_{m_i} \end{bmatrix}$$

$\dim(C_i) = e_i^2$ and constants of a given C_i are independent of other C_j

$\therefore \dim(\mathcal{C}(\rho)) = e_1^2 + \cdots + e_k^2$

Q) Let $\rho : G \to GL_n(\mathbb{C})$ be a rep such that $\mathcal{C}(\rho) = \{\lambda I_n \mid \lambda \in \mathbb{C}\}$. Prove that ρ is irreducible

Ans $\dim \mathcal{C}(\rho) = 1 \Rightarrow e_1^2 + \cdots + e_k^2 = 1 \Rightarrow k = 1, e_1 = e = 1$

$\therefore \rho$ is irreducible

This is the converse of Schur's lemma

Definition : For a finite group G, define the function algebra on G to be the set containing all functions $f : G \to \mathbb{C}$. We denote it by $L(G)$

It is clear that this is a vector space and if we define fg to be pointwise product, we get an algebra.

The inner product on $L(G)$ is given by $\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) g(x^{-1})$

Definition : given a group G, a family of one irred rep out of each equivalence class of irred reps is called a parliament of irreducible representations

[we shall see later that the parliament of irreps is finite]

Proposition 2.6

Let A, B be ineq irred reps of G. Treating entries of A_g, B_g as elements of $L(G)$, every entry of A_g is orthogonal to every entry of B_g

proof

Let $[A_g] = a_{ij}$ $\quad 1 \le i, j \le m$

$[B_g] = b_{pq}$ $\quad 1 \le p, q \le n$

Let D be an arbitrary $m \times n$ matrix in $\mathbb{C}$

Define $C = \sum_{y \in G} A(y) D B(y^{-1})$

Then $C = \sum_{y \in G} A(xy) D B(y^{-1}x^{-1}) = A_x C B_{x^{-1}}$

$\therefore A_x C = C B_x$ holds $\forall x \in G$

By corollary of Schur's lemma, $C \equiv 0$ matrix

$\therefore \sum_{y \in G} A(y) D B(y^{-1}) \equiv 0$

$\therefore \sum_{y \in G} \sum_{t_1=1}^{m} \sum_{t_2=1}^{n} a^{y}_{it_1} d_{t_1 t_2} b^{y^{-1}}_{t_2 j} = 0 \quad \forall \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$

where $[A_y]_{ij} = a^y_{ij}$ $\quad 1 \le i, j \le m$

$[D]_{ij} = d_{ij}$ $\quad 1 \le i \le m, \quad 1 \le j \le n$

$[B_y]_{ij} = b^y_{ij}$ $\quad 1 \le i, j \le n$

Since this holds for every D, all coeffs of all d_{ij} must be zero

Thus $\sum_{y \in G} a^y_{ij} b^{y^{-1}}_{pq} = 0 \quad \forall \begin{array}{l} 1 \le i, j \le m \\ 1 \le p, q \le n \end{array}$

$\therefore \langle a^y_{ij}, b^y_{pq} \rangle = 0$

■

Q) Use above proof idea to show that if ρ_g is an irrep, then coeffs a_{ij} of $\rho_g = [A]$ satisfy, $\langle a_{ij}, a_{pq} \rangle = \frac{1}{m} \delta_{iq} \delta_{jp}$

Ans using Schur's lemma in previous proof we get that the matrix C (now of size $m \times m$) $= \lambda I_m$

$tr(C) = m\lambda$

Now $tr(C) = tr\left(\sum_y A_y D A_{y^{-1}}\right) = \sum_y tr(D) = |G| \, tr(D)$

$$\therefore \lambda = \frac{|G| \, tr(D)}{m}$$

$$\therefore \sum_y A_y D A_{y^{-1}} = \frac{|G| (d_{11} + \cdots + d_{mm})}{m} I_m$$

comparing coefficients of d_{ij}, we find

$$\sum_y a_{ij}^{y} a_{pq}^{y^{-1}} = 0 \quad \text{if } i \neq q \text{ or } j \neq p$$

If $i = q$ and $j = p$, $\sum_y a_{ij}^{y} a_{pq}^{y^{-1}} = \frac{|G|}{m}$

$$\therefore \langle a_{ij}, a_{pq} \rangle = \frac{1}{m} \delta_{iq} \delta_{jp}$$

■

Proposition 2.7

If χ, χ' are characters of irreps ρ, φ, then $\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho \sim \varphi \\ 0 & \rho \nsim \varphi \end{cases}$

Proof

If $\rho \nsim \varphi$, $\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi'(g^{-1})$

$$= \frac{1}{|G|} \sum_{g \in G} (a_{11}^{g} + a_{22}^{g} + \cdots + a_{mm}^{g})(b_{11}^{g^{-1}} + b_{22}^{g^{-1}} + \cdots + b_{nn}^{g^{-1}})$$

$$= \sum_{i=1}^{m} \sum_{p=1}^{n} \langle a_{ii}, b_{pp} \rangle = \sum_i \sum_p 0 = 0$$

If $\rho \sim \varphi$, $\langle \chi, \chi' \rangle = \sum_{i=1}^{m} \sum_{p=1}^{m} \langle a_{ii}, a_{pp} \rangle = m \cdot \frac{1}{m} = 1$

■

Proposition 2-8

$\langle \chi, \chi \rangle = 1$ iff ρ is irreducible

proof

(We use notion of tensor products. Reading about 'Kronecker product' will be sufficient)

For completeness,

$$\underset{(m \times n)}{A} \otimes \underset{(p \times q)}{B} := \begin{bmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \vdots & & & \\ a_{m1} B & a_{m2} B & \cdots & a_{mn} B \end{bmatrix}$$

Let $\rho_g = \bigoplus_{i=1}^{k} I_{r_i} \otimes \rho_i(g)_{d_i \times d_i}$

(ρ_g has r_1 copies of ρ_1 (size d_1), r_2 copies of ρ_2 (size d_2)...)

Then $\chi_g = \sum_{i=1}^{k} r_i \chi_{i,g}$

$\therefore \langle \chi, \chi_j \rangle = \sum_{i=1}^{k} r_i \langle \chi_i, \chi_j \rangle = r_j$

($\rho_1, \rho_2, \ldots$ are mutually inequivalent)

$\therefore \langle \chi, \chi \rangle = \langle \chi, \sum_{i=1}^{k} r_i \chi_i \rangle = \sum_{i=1}^{k} r_i \langle \chi, \chi_i \rangle$

$\therefore \langle \chi, \chi \rangle = \sum_{i=1}^{k} r_i^2$

(note: I have dropped suffix/prefix of g. Everything is happening for a fixed element g)

If ρ is irred, $\langle \chi, \chi \rangle = 1^2 = 1$ ($\because k=1, r_1 = 1$)

If $\sum_{i=1}^{k} r_i^2 = 1$ then $k=1, r_1 = r = 1 \Rightarrow \rho$ is irred

$\blacksquare$

Proposition 2.9

Any collection of distinct characters of irred reps is linearly indep

proof

Let $\chi_1, \chi_2, \dots \chi_n$ be such characters

Let $\sum_{i=1}^{n} c_i \chi_i = 0$

$\langle \chi_j, \chi_i \rangle = \delta_{ij}$ (the irreps have to be inequivalent else same χ)

$\therefore \sum_{i=1}^{n} c_i \delta_{ij} = 0 \Rightarrow c_j = 0$

Q) Show that if the degrees in the parliament of irreps are $d_1, \dots, d_k$, then $k \leq \sum d_i^2 \leq |G|$

Ans Let M_j be the matrix corresponding to the j^{th} irrep.

Let S_j be its entries $(S_j \in L(G))$

Then, $S_i \perp S_j$ $(i \neq j)$ $\Rightarrow$ $\{S_1, \dots, S_k\}$ form an orthogonal and hence LI set $\Rightarrow$ $|S_1| + |S_2| + \dots + |S_k| \leq \dim(L(G)) = |G|$

But $|S_i| = d_i^2$ and we are done

$d_i^2 \geq 1 \Rightarrow \sum d_i^2 \geq k$

■

Definition: Let $CA_G = \{ \sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \}$ (Σ is formal sum)

This is known as the group algebra

Proposition 2.10

CA_G is a $\mathbb{C}$-vector space with obvious addition and scalar multiplication definitions. Further, with $\alpha_1 g_1 \cdot \alpha_2 g_2 := \alpha_1 \alpha_2 (g_1 g_2)$ and extending linearly to get a product on CA_G, we get an algebra.

proof

showing that it is a $\mathbb{C}$-vector space is straightforward.

$$\sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g := \sum_{g\in G} (\alpha_g + \beta_g) g \in \mathbb{C}A_G$$

$$\lambda \sum_{g\in G} \alpha_g g := \sum_{g\in G} (\lambda \alpha_g) g \in \mathbb{C}A_G$$

Associativity, commutativity, identity & inverse wrt + are direct.

$(\lambda_1 \lambda_2) \sum \alpha_g g = \lambda_1 (\lambda_2 \sum \alpha_g g)$ also follows easily

along with $1 \cdot v = v$, $a(u+v) = au + av$, $(a+b)v = av + bv$

To check that this is an algebra:

$$\left(\sum \alpha_g g + \sum \beta_g g\right) \cdot \sum \gamma_g g = \sum (\alpha_g + \beta_g) g \cdot \sum \gamma_g g$$

$$= \sum_g \sum_{\substack{a,b \text{ s.t.} \\ ab = g}} (\alpha_a + \beta_a)\gamma_b \, g$$

$$= \sum_g \sum_{\substack{a,b \text{ st.} \\ ab = g}} \alpha_a \gamma_b \, g + \sum_g \sum_{\substack{a,b \text{ st.} \\ ab = g}} \beta_a \gamma_b \, g$$

$$= \left(\sum \alpha_g g \cdot \sum \gamma_g g\right) + \left(\sum \beta_g g \cdot \sum \gamma_g g\right)$$

left distributivity follows from $\sum \alpha_g g \cdot \sum \beta_g g = \sum \beta_g g \sum \alpha_g g$

$\lambda_1 x \cdot \lambda_2 y = \lambda_1 \lambda_2 \, x \cdot y$ is also straightforward

■

Definition: The group algebra $\mathbb{C}A_G$ can be turned into a G-module by defining $\rho : G \to \mathbb{C}A_G$ as $\rho_g : \mathbb{C}A_G \to \mathbb{C}A_G$ is given by $\rho_g\left(\sum_{h\in G} \alpha_h h\right) := \sum_{h\in G} \alpha_h (gh)$. This is denoted $\rho_{L\,reg}$ and called as the left regular representation. This clearly has degree $|G|$

Q) Find the matrix form of ρ_{Lreg} (or attempt to describe it)

Ans Let $G = \{g_1, \dots, g_n\}$ in that specific order so that it serves as an ordered basis of $\mathbb{C}A_G$

Then $$\rho_{Lreg}(g) = \begin{array}{cc} & \begin{array}{ccccc} g_1 & g_2 & g_3 & \cdots & g_n \end{array} \\ \begin{array}{c} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{array} & \begin{bmatrix} & & & & \\ & 1 & & & \\ & & & & 1 \\ 1 & & & & \\ & & 1 & & \end{bmatrix} \end{array}$$

The j^{th} column will have a 1 in the row that has the label gg_j

Alternatively, the i^{th} row will have a 1 in the column that has the label $g^{-1}g_i$

To verify, let $h = g_j \in \mathbb{C}A_G$

Then $\rho_{Lreg}(g)(h)$ by definition is gh (which is g_k, say)

Through our matrix $\rho_{Lreg}(g) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$ position $= j^{th}$ column of $\rho_{Lreg}(g)$

and this will have a 1 in the gg_j position ie. $g_k{}^{th}$ position

and hence this verifies the matrix form ∎

Proposition 2.11

$\rho_{Lreg}(g)$ is a permutation matrix

proof

Each column has a single 1 (else $gg_i = gg_j \Rightarrow g_i = g_j$ $\rightarrow\leftarrow$)

Each row has a single 1 (else $g^{-1}g_i = g^{-1}g_j \Rightarrow g_i = g_j$ $\rightarrow\leftarrow$)

(Thus ρ_{Lreg} is unitary) ∎

Q) Find the character of ρ_{Lreg}

Ans $tr(\rho_{Lreg}(id)) = tr(I_{|G|}) = |G|$

Claim: $tr(\rho_{Lreg}(g)) = 0 \quad \forall \; g \neq id$

If i^{th} column has a 1 in the i^{th} place, it will mean that $gg_i = g_i \Rightarrow g = id \rightarrow\leftarrow$

$$\therefore \chi_{Lreg}(g) = \begin{cases} |G| & g = id \\ 0 & \text{otherwise} \end{cases}$$

■

Theorem 2.12

Let $\{\phi_1, \dots, \phi_k\}$ be a parliament of irreps with degrees $d_1, \dots, d_k$.

Then $\rho_{Lreg} \sim d_1\phi_1 \oplus d_2\phi_2 \oplus \cdots \oplus d_k\phi_k$ ($\oplus$ is block diagonalizer)

proof

Since $\{\phi_1, \dots, \phi_k\}$ is a parliament of irreps, ρ_{Lreg} can be expressed as $m_1\phi_1 \oplus \cdots \oplus m_k\phi_k$

$$\rho_{Lreg} \sim m_1\phi_1 \oplus \cdots \oplus m_k\phi_k$$

$$\therefore \chi_{Lreg} = m_1\chi_1 + \cdots + m_k\chi_k$$

$\therefore \langle \chi_{Lreg}, \chi_i \rangle = m_i$ (since $\{\chi_1, \dots, \chi_k\}$ are orthogonal)

$$\therefore m_i = \frac{1}{|G|} \sum_{g \in G} \chi_{Lreg}(g)\,\overline{\chi_i(g)}$$

$$= \frac{1}{|G|} \cdot |G| \, \overline{\chi_i(id)} = d_i$$

■

Corollary 2.13

In accordance with the above theorem, $\sum_{i=1}^{k} d_i^2 = |G|$

proof

$$\langle \chi_{Lreg}, \chi_{Lreg} \rangle = \frac{1}{|G|} |G| \cdot |G| = |G| = \langle \chi_{Lreg}, \sum d_i\chi_i \rangle = \sum d_i^2$$

■

Proposition 2.14

Two reps of a finite group are equivalent iff they have same χ

proof

We have already seen the forward direction.

For the backward direction, let $\chi_1, \dots, \chi_r$ be complete set of distinct irreducible characters corresponding to the irred representations $\phi_1, \dots, \phi_r$ with degrees $d_1, \dots, d_r$

By Maschke's theorem, $\rho \sim m_1 \phi_1 \oplus \dots \oplus m_r \phi_r$

$\therefore \chi_\rho = m_1 \chi_1 + \dots + m_r \chi_r$ where $m_i = \langle \chi_\rho, \chi_i \rangle$

Let φ be another rep with $\chi_\varphi = n_1 \chi_1 + \dots + n_r \chi_r$

$\chi_\varphi = \chi_\rho \Rightarrow \sum m_i \chi_i = \sum n_i \chi_i$

Taking inner product with χ_j $\forall j$, $m_j = n_j$

thus completing the proof

■

Theorem 2.15 (Group Algebra lemma)

Let $\mathbb{C}A_G = V_1 \oplus \dots \oplus V_r$ be the complete decomposition into irred G-modules V_i (wrt ρ_{Lreg}, of course). If W is any irreducible G-module, W must be isomorphic to V_i for some i

(W can literally be any G-module i.e. any $\rho : G \to GL(W)$)

proof

lemma 1: If U, V are G-modules and $T : U \to V$ is a G-Hom i.e. $T \in \mathrm{Hom}(U, V)$, then $\exists$ G-submodule $W \leq U$ such that $U = \ker(T) \oplus W$ and W is G-isomorphic to $\mathrm{Im}(T)$

proof: $\ker(T)$ is a G-inv subspace of U as we have seen $\Rightarrow U = \ker T \oplus W$ $(W = (\ker T)^\perp)$

We now have to show $W \underset{G\text{-iso}}{\cong} \mathrm{Im}(T)$ (both are G-submodules of V)

Claim: Restriction of T works (the T which witnessed the equivalence of ρ_g, φ_g)

$\tilde{\rho}_g : G \to GL(W)$, $\tilde{\varphi}_g : G \to GL(\mathrm{Im}(T))$

Let $T\rho_g = \varphi_g T$

$$\therefore \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{\rho}}_g & 0 \\ 0 & \tilde{\rho}_g \end{bmatrix} = \begin{bmatrix} \tilde{\tilde{\varphi}}_g & 0 \\ 0 & \tilde{\varphi}_g \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

$$\therefore \quad T_4 \tilde{\rho}_g = \tilde{\varphi}_g T_4$$

$$\therefore \quad T_4 \in \mathrm{Hom}_G(\tilde{\rho}_g, \tilde{\varphi}_g)$$

By Schur's lemma corollary, T_4 is a scalar multiple of I and hence we have a G-isomorphism between W, $\mathrm{Im}(T)$

Lemma 2: Let V be a G-module. If $V = V_1 \oplus \cdots \oplus V_r$ where each V_i is irreducible, the this decomposition is unique upto isomorphism

proof: Let $W \leq V$ be any G-submodule.

Let $w \in W \setminus \{0\}$ be fixed.

Since $V = \bigoplus V_i$, $w = v_1 + \cdots + v_r$ (uniquely)

Let v_{k_0} be the first non zero v_i (exists since $w \neq 0$)

Construct $\varphi_w : W \to V_{k_0}$ as

$$\varphi_w(u_1 + u_2 + \cdots + u_r) := u_{k_0}$$

This is clearly linear and not the zero map $(\because \varphi_w(w) = v_{k_0} \neq 0)$

Let $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r \in W$

Then $\rho_g(\alpha) = \rho_g(\alpha_1) + \cdots + \rho_g(\alpha_r) \quad \forall g$

$$\therefore \varphi_w \rho_g(\alpha) = \rho_g(\alpha_{k_0})$$

Now $\varphi_w(\alpha) = \alpha_{k_0}$

and $\rho_g(\varphi_w(\alpha)) = \rho_g(\alpha_{k_0})$

$\therefore$ φ_w is indeed a G- hom

[Note: Strictly speaking we must use $\tilde{\rho}_g$ instead of ρ_g where $\tilde{\rho}_g$ denotes restriction to V_{k_0}. Similarly $\tilde{\tilde{\rho}}_g$ (restriction to W)]

Since it is not the zero map and is a map b/w irreducibles, by Schur's lemma, $W \underset{G\text{-iso}}{\simeq} V_i$

proof of theorem

Let W be any irred G-module $(\phi : G \to GL(W))$ (non zero. For zero, it is trivial)

Let $w \in W$. Note that $\phi_g w \in W \quad \forall g \in G$

For $r = \sum_{g \in G} \alpha_g g \in \mathbb{C}A_G$, $\left(\sum_{g \in G} \alpha_g \phi_g\right) w \in W$

Construct $\varphi_w : \mathbb{C}A_G \longrightarrow W$ as

$$\varphi_w(r) = \phi_r(w) \quad \left(\text{where } \phi_r = \sum \alpha_g \phi_g\right)$$

φ_w is clearly a linear map (check!)

φ_w is also a G- homomorphism. Let $r = \sum \alpha_g g \in \mathbb{C}A_G$

$$\begin{array}{ccc}
\sum \alpha_g g & \xmapsto{\ \varphi_w\ } & \sum \alpha_g \phi_g(w) \\
\Big\downarrow {\scriptstyle \rho_{Lreg}(h)} & & \Big\downarrow {\scriptstyle \phi_h} \\
\sum \alpha_g (hg) & \xmapsto{\ \varphi_w\ } & \sum \alpha_g \phi_h \phi_g(w)
\end{array}$$

By lemma 1, $\exists$ G-submodule U of $\mathbb{C}A_G$ s.t. $\mathbb{C}A_G = U \oplus \ker(\varphi_w)$

and $U \underset{G\text{-iso}}{\simeq} \operatorname{Im}(\varphi_w) = W$ (check $r = id$ to see that φ_w is onto)

But W is irreducible $\Rightarrow$ U is irreducible. By lemma 2 $U \simeq V_i$ for some i $\Rightarrow$ $W \simeq V_i$ for some i

$\blacksquare$

Definition: Two groups A, B are said to be anti-isomorphic if there exists an anti-isomorphism $\varphi : A \to B$ i.e. φ is a bijection such that $\varphi(gh) = \varphi(h)\varphi(g)$

Proposition 2.16

Let $A \underset{iso}{\overset{anti}{\simeq}} B$. Then $Z(A) \underset{iso}{\overset{anti}{\simeq}} Z(B)$ (Z denotes center)

proof

$Z(A) = \{ a \in A \mid g^{-1} a g = a \quad \forall g \in A \}$

$\varphi(Z(A)) \subseteq Z(B)$ (where $\varphi : A \to B$ is the anti iso)

Indeed if $a \in Z(A)$, $a = g^{-1} a g \quad \forall g \in A$

$\therefore \varphi(a) = \varphi(g^{-1} a g) = \varphi(g)\varphi(a)(\varphi(g))^{-1} \quad \forall g \in A$

Since φ is onto, $\{ \varphi(g) \mid g \in A \} = B$

$\therefore \varphi(a) \in Z(B)$

$(\varphi(id) = id \Rightarrow \varphi(g)^{-1} = \varphi(g^{-1}))$

using same stuff with φ^{-1}, $Z(B) \subseteq \varphi(Z(A))$
(since φ is a bijection)

$\therefore \varphi(Z(A)) = Z(B)$

Let $\tilde{\varphi} : Z(A) \to Z(B)$ be the restriction

φ is one-one $\Rightarrow \tilde{\varphi}$ is one-one

$\varphi(Z(A)) = Z(B) \Rightarrow \varphi$ is onto

$\tilde{\varphi}(gh) = \varphi(gh) = \varphi(h)\varphi(g) = \tilde{\varphi}(h)\tilde{\varphi}(g)$

$\forall g, h \in Z(A)$

$\therefore \tilde{\varphi}$ is the required anti-isomorphism

Theorem 2.17

Let G be a finite group. Upto equivalence, the total number of irreps of G is equal to the number of conjugacy classes of G

proof

Let $v \in \mathbb{C}A_G$. Define $\varphi_v : \mathbb{C}A_G \to \mathbb{C}A_G$ as $\varphi_v(w) = w \cdot v$

Let $\phi : \mathbb{C}A_G \to \mathrm{Hom}_G(\mathbb{C}A_G, \mathbb{C}A_G)$ as $\phi(v) = \varphi_v$

φ_v is clearly linear. It is indeed a G-hom since, if $w \in \mathbb{C}A_G$, then $\rho_{Lreg}(g)\varphi_v(w) = \rho_{Lreg}(wv)$ and

$\varphi_v(\rho_{Lreg}(g)(w)) = (\rho_{Lreg}(g)(w)) \cdot v$

Let $v = \sum \alpha_h h$

$w = \sum \beta_h h$

$$wv = \sum_h \Big(\sum_{\substack{a,b \in G \\ ab = h}} \beta_a \alpha_b \Big) h$$

$$\rho_{Lreg}(g)(wv) = \sum_h \Big(\sum_{\substack{a,b \in G \\ ab = h}} \beta_a \alpha_b \Big)(gh)$$

$$\rho_{Lreg}(g)(w) = \sum_h \beta_h (gh) = \sum_z \beta_{g^{-1}z} z$$

$$(\rho_{Lreg}(g)(w)) \cdot (v) = \sum_d \Big(\sum_{\substack{a,b \in G \\ ab = d}} \beta_{g^{-1}a} \alpha_b \Big) d$$

Let $d = gh$ (d varies over $G \Rightarrow h$ varies over G)

$\therefore g^{-1}ab = g^{-1}d = h$

$\therefore$ we get $\displaystyle\sum_h \Big(\sum_{\substack{a,b \in G \\ ab = gh}} \beta_{g^{-1}a} \alpha_b \Big) gh$

Now let $g^{-1}a = c \Rightarrow a = gc$

$\therefore ab = gh \equiv cb = h$

$\therefore$ Inner sum now reads $\displaystyle\sum_{\substack{c,b \in G \\ cb = h}} \beta_c \alpha_b$

Thus $\rho_{Lreg}(g)(w) \cdot v = \rho_{Lreg}(g)(w \cdot v)$

Now we claim that ϕ is an anti-isomorphism

$\phi(ab) = \varphi_{ab}$, $\phi(b)\phi(a) = \varphi_b \circ \varphi_a$

$\varphi_{ab}(w) = w(ab) = (wa)b = \varphi_b(wa) = (\varphi_b \circ \varphi_a)(w)$

(The multiplication in $\mathbb{C}A_G$ is commutative)

$\ker \phi = \{ v \in \mathbb{C}A_G \mid \varphi_v = id \}$

$= \{ v \in \mathbb{C}A_G \mid wv = w \quad \forall w \in \mathbb{C}A_G \}$

$wv = w \Rightarrow \sum_{\substack{a,b \in G \\ ab = h}} \beta_a \alpha_b = \beta_h$

For any given h, $\sum_{a \in G} \beta_a \alpha_{a^{-1}h} = \beta_h$

$\therefore \beta_{g_1} = \beta_{g_1}\alpha_{id} + \beta_{g_2}\alpha_{g_2^{-1}g_1} + \cdots + \beta_{g_n}\alpha_{g_n^{-1}g_1}$

$\beta_{g_2} = \beta_{g_1}\alpha_{g_1^{-1}g_2} + \beta_{g_2}\alpha_{id} + \cdots + \beta_{g_n}\alpha_{g_n^{-1}g_2}$

$\vdots$

Since this holds for every $\{\beta_a\}_{a \in G}$,

$$\begin{bmatrix} \alpha_{id} - 1 & \alpha_{g_2^{-1}g_1} & \cdots & \alpha_{g_n^{-1}g_1} \\ \alpha_{g_1^{-1}g_2} & \alpha_{id} - 1 & \cdots & \alpha_{g_n^{-1}g_2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_1^{-1}g_n} & \alpha_{g_2^{-1}g_n} & \cdots & \alpha_{id} - 1 \end{bmatrix} = 0$$

$\therefore \alpha_g = \begin{cases} 1 & g = id \\ 0 & \text{otherwise} \end{cases}$

$\therefore v = id$

$\therefore \ker \phi = \{id\}$

Now let $\theta \in \mathrm{Hom}_G(\mathbb{C}A_G, \mathbb{C}A_G)$

Let $\theta(id) = w = \sum \alpha_g g$

We claim $\theta = \varphi_w$

Indeed, $\varphi_w(v) = vw$

and $\theta(g) = \theta(\rho_{reg}(g)(id)) = \rho_{reg}(g)(\theta(id))$

$= \sum \alpha_h (gh)$

$$\theta(v) = \theta\left(\sum \beta_h h\right) = \sum \beta_h \, \theta(h)$$

$$= \sum_{h \in G} \beta_h \left(\sum_{t \in G} \alpha_t (ht) \right)$$

$$= \sum_{h \in G} \sum_{t \in G} \beta_h \alpha_t \; ht$$

$$= \left(\sum_{h \in G} \beta_h h \right) \left(\sum_{t \in G} \alpha_t t \right)$$

$$= v w$$

By previous proposition,

$$\text{Center}\left(\text{Hom}_G(\mathbb{C}A_G, \mathbb{C}A_G) \right) \underset{\text{iso}}{\overset{\text{anti}}{\simeq}} \text{Center}(\mathbb{C}A_G)$$

$\therefore$ Both have same dimensions

Let $w = \sum_{g \in G} \beta_g \, g \in \text{center}(\mathbb{C}A_G)$

Then $wv = vw \quad \forall \; v \in \mathbb{C}A_G$

In particular, $wh = hw \quad \forall \; h \in G$

$$\therefore \quad h^{-1} w h = w \quad \forall \; h \in G$$

$$\therefore \quad \sum_{g \in G} \beta_g (h^{-1} g h) = w \quad \forall \; h \in G$$

$$\therefore \quad \sum_{t \in G} \beta_{hth^{-1}} \; t = \sum_{g \in G} \beta_g \; g \quad \forall \; h \in G$$

$$\therefore \quad \beta_{hth^{-1}} = \beta_g \quad \forall \; h \in G$$

$\therefore$ coefficients of w are are class function

Let $C_1, \ldots, C_m$ be the conjugacy classes of G

Define $L_i = \sum_{g \in C_i} g$. Then $\{L_i\}_1^m$ forms a basis for Center$(\mathbb{C}A_G)$ [Clearly span $\{L_i\}$ = Center$(\mathbb{C}A_G)$

and $d_1 L_1 + \cdots + d_m L_m = 0 \Rightarrow \sum_{g \in C_1} d_1 g + \sum_{g \in C_2} d_2 g + \cdots = 0$

$\therefore$ Every $d_i = 0$ since all summations are 'disjoint']

$\therefore$ dim (center ($\mathbb{C}A_G$)) = m = no. of conjugacy classes

$\therefore$ dim (center ($\mathrm{Hom}_G(\mathbb{C}A_G, \mathbb{C}A_G)$)) = no. of conjugacy classes

Now we recall the commutant algebra $\mathcal{C}(\rho_g)$ which contains matrices T st. $T\rho_g = \rho_g T$

we had already seen that if ρ_g is of the form

$\mathrm{Diag}\,(\underbrace{A_1(g), \dots, A_1(g)}_{e_1 \text{ times}}, \dots, \underbrace{A_k(g), \dots, A_k(g)}_{e_k \text{ times}})$, then, (size $A_i = m_i \times m_i$)

T is of the form $\sum_{i=1}^{k}{}^{\oplus} X_i \otimes I_{m_i \times m_i}$ for any arbitrary $e_i \times e_i$ matrix X_i

If $Z = \sum_{i=1}^{k}{}^{\oplus} D_i \otimes I_{m_i \times m_i} \in \mathrm{Center}(\mathcal{C}(\rho))$, then

$$TZ = ZT \quad \forall T \Rightarrow X_i D_i = D_i X_i \quad \forall X_i$$

($\because (P \otimes Q)(R \otimes S) = PR \otimes QS$)

X_i can be any arbitrary matrix $\Rightarrow D_i = d_i I_{e_i \times e_i}$

$$\therefore Z = \sum_{i=1}^{k}{}^{\oplus} d_i I_{e_i m_i \times e_i m_i}$$

Conversely, any such $Z \in \mathrm{Center}(\mathcal{C}(\rho))$

$\therefore$ dim (center ($\mathcal{C}(\rho)$)) = k = no. of irreps afforded

Basically, $\mathcal{C}(\rho) = \mathrm{Hom}_G(V, V)$ (For $\rho : G \to GL(V)$)

In our case,

dim (center ($\mathrm{Hom}_G(\mathbb{C}A_G, \mathbb{C}A_G)$)) = k = no of irreps

$\|$

dim (center ($\mathbb{C}A_G$)) = m = no of conjugacy classes

$\blacksquare$

Q) Find possible degrees of the representation of S_3, S_4, D_{2n} $(n=4)$

Ans Characters are class functions and hence no. of elements in the parliament of irreps = no of conjugacy classes of G

(i) S_3 : class eqn is $1+3+2 \Rightarrow 3$ conjugacy classes

$\therefore d_1^2 + d_2^2 + d_3^2 = |S_3| = 6$

But $d_1 = 1$ (trivial rep)

$\therefore d_2^2 + d_3^2 = 5 \Rightarrow d_2 = 1, \; d_3 = 2$

($d_2 = 1$ corresponds to the sgn rep
$d_3 = 2$ is the natural rep)

(ii) S_4 : class eqn is $1+3+6+8+6$

$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 24$

$d_1 = 1, \; d_2 = 1$ (trivial, sgn)

$d_3^2 + d_4^2 + d_5^2 = 22 \Rightarrow d_3 = 2, \; d_4 = 3, \; d_5 = 3$

($d_3 = 2$ corresponds to the natural rep)

(iii) D_{2n}, $n=4$ class eqn is $1+1+2+2+2$

$\therefore d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$

$d_1 = 1$ (trivial)

$\therefore d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8 \Rightarrow d_2 = d_3 = d_4 = d_5 = 2$

Definition : For a representation ρ with k irreps, construct the $k \times k$ matrix C_G whose i,j^{th} entry is $\chi_i(C_j)$ (where $C_1 \cdots C_k$ are the conjugacy classes of G and $\chi_i(C_j) = \chi_i(g)$ for some $g \in C_j$)

Note 1 : no. of conjugacy classes will be equal to k by prev. theorem

Note 2 : $\chi_i(C_j)$ is well defined by prop 2.4

Proposition 2.18

The matrix N_G, an analog of C_G, defined by the i,j^{th} entry $\sqrt{\frac{|C_j|}{|G|}}\, \chi_i(C_j)$ is unitary

proof

$$\langle \chi_i , \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\, \overline{\chi_j(g)}$$

$$= \frac{1}{|G|} \sum_{t=1}^{k} |C_t|\, \chi_i(C_t)\, \overline{\chi_j(C_t)}$$

$$= \sum_{t=1}^{k} \left(\sqrt{\frac{|C_t|}{|G|}}\, \chi_i(C_t)\right)\left(\sqrt{\frac{|C_t|}{|G|}}\, \overline{\chi_j(C_t)}\right)$$

= dot product of row i, row j in N_G

But $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ (prop 2.7)

$\therefore$ N_G is indeed unitary

Q) Now that N_G is unitary, what result do we get using dot product of two columns?

Ans $$\delta_{ij} = \sum_{t=1}^{k} \frac{\sqrt{|C_i||C_j|}}{|G|}\, \chi_t(C_i)\, \overline{\chi_t(C_j)}$$

In summary,

$$\sum_{t=1}^{k} \frac{|C_t|}{|G|}\, \chi_i(C_t)\, \overline{\chi_j(C_t)} = \delta_{ij} \quad \text{(sum over conj classes)}$$

$$\sum_{t=1}^{k} \frac{\sqrt{|C_i||C_j|}}{|G|}\, \chi_t(C_i)\, \overline{\chi_t(C_j)} = \delta_{ij} \quad \text{(sum over irred } \chi\text{)}$$

Q) Show that C_G is invertible

Ans N_G is unitary $\Rightarrow$ $|\det(N_G)| = 1$ ($|\cdot|$ is complex mod)

But $\det(N_G) = \sqrt{\frac{|C_1|}{|G|}}\sqrt{\frac{|C_2|}{|G|}} \cdots \sqrt{\frac{|C_k|}{|G|}} \det(C_G)$

(each column of C_G was scaled to get N_G)

Q) Find $\det(N_G)$ for $G = S_3, S_4, D_{2n}$ $(n=4)$

Ans (i) S_3 :

k = no of conjugacy classes = 3

$d_1^2 + d_2^2 + d_3^2 = |S_3| = 6$

$d_1 = 1, \; d_2 = 1$ (trivial, sgn)

$\therefore d_3 = 2 \longrightarrow S_3 \simeq \underset{(n=3)}{D_{2n}}$ so this is the natural D_6 rep

$D_6 = \{1, r, r^2, s, sr, sr^2\}$ (with $rsr = s$)

$\{1\}, \; \{r, r^2\}, \; \{s, sr, sr^2\}$ all the conj classes

$$\therefore C_G = \begin{array}{c c} & \begin{array}{ccc} \{1\} & \{r, r^2\} & \{s, sr, sr^2\} \end{array} \\ \begin{array}{c} \chi_{triv} \\ \chi_{sgn} \\ \chi_2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \end{array}$$

(since $\rho_{triv}(g) = [1] \; \forall g$, $\rho_{sgn}(g) = [sgn(g)] \; \forall g$,

$\rho_2(r) = \begin{bmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{bmatrix}$, $\rho_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$)

$$\therefore N_G = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \end{bmatrix}$$

And $\det(N_4) = -1$

(ii) S_4 :

k = no of conjugacy classes = 5

$\{1\}$, $|12|$, $|123|$, $|1234|$, $|(12)(34)|$

$1 + 6 + 8 + 6 + 3 = 24$

$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 24 \quad (d_1, d_2 = 1)$

$\therefore$ sizes are $1, 1, 2, 3, 3$

$$C_G = \begin{array}{c} \\ \chi_{triv} \\ \chi_{sgn} \\ \chi_{stan} \\ \chi_4 \\ \chi_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \{1\} & |12| & |123| & |1234| & |(12)(34)| \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 3 & ? & ? & ? & ? \\ 2 & ? & 2 & 2 & 2 \\ 3 & ? & 2 & ? & ? \end{bmatrix} \end{array}$$

χ_{stan} is determined as follows

We have the natural perm matrix rep in $\mathbb{C}^4$ which need since span $\{e_1 + e_2 + e_3 + e_4\}$ is an S_4 - submodule. Take the orthogonal component : span $\{e_2 - e_1, e_3 - e_1, e_4 - e_1\}$

(Btw e_1, e_2, e_3, e_4 denote 4 standard basis vectors of $\mathbb{C}^4$)

This is clearly a S_4 - submodule (as it should be by Maschke's theorem)

Thus, if we use the basis $\{e_1 + \dots + e_4, e_2 - e_1, e_3 - e_1, e_4 - e_1\}$

for C_4, we get the matrices

$$\rho(id) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & I_3 \end{array}\right] \qquad \rho((12)) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rho((123)) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \rho((1234)) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rho((12)(34)) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ The third row of C_G reads $3, 1, 0, -1, -1$

Since $\langle \chi_{stan}, \chi_{stan} \rangle = 1$, χ_{stan} was indeed irred

There are 6 unknowns

Just use orthogonality relations to solve ~20 eqns in 6 variables

The trick is that usually tensor products seem to work.

$\chi_{sgn} \otimes \chi_{stan}$ is indeed irred (as can be checked)

$\chi_{stan} \otimes \chi_{stan}$ isnt but gives $\chi_{stan} \otimes \chi_{stan} = \chi_{triv} + \chi_{stan} + \chi_{sgn} \otimes \chi_{stan} + \chi_5$. χ_5 is the last one we need

(iii) D_{2n} with $n = 4$ $= \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ $(srs = r^3)$

Conjugacy classes : $\{1\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}$

$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$

$\therefore \{d_i\} = \{1, 1, 1, 1, 2\}$

$\rho_1(g) = [1]$ (trivial)

$$\rho_5(r) = \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix}, \quad \rho_5(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(standard rep of } D_{2n})$$

ρ_2, ρ_3, ρ_4 satisfy: $(\rho(r))^4 = 1$, $(\rho(s))^2 = 1$, $\rho(r) = \rho(r)^3$

($\because$ Technically $\rho(s)\rho(r)\rho(s) = \rho(r)^3$ but $\rho(s) \in \mathbb{C}$ & $\rho(s)^2 = 1$)

$\rho(s) = 1$, $\rho(r) = 1$ is trivial (ρ_1)

$\rho(s) = 1$, $\rho(r) = -1$ is ρ_2

$\rho(s) = -1$, $\rho(r) = 1$ is ρ_3

$\rho(s) = -1$, $\rho(r) = -1$ is ρ_4

$$\therefore \quad C_G = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & r & r^2 & s & sr \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & 0 & 0 \end{bmatrix} \end{array}$$

$$\therefore \quad N_G = \begin{bmatrix} 1/\sqrt{8} & \sqrt{2}/\sqrt{8} & 1/\sqrt{8} & \sqrt{2}/\sqrt{8} & \sqrt{2}/\sqrt{8} \\ 1/\sqrt{8} & -\sqrt{2}/\sqrt{8} & 1/\sqrt{8} & \sqrt{2}/\sqrt{8} & -\sqrt{2}/\sqrt{8} \\ 1/\sqrt{8} & \sqrt{2}/\sqrt{8} & 1/\sqrt{8} & -\sqrt{2}/\sqrt{8} & -\sqrt{2}/\sqrt{8} \\ 1/\sqrt{8} & -\sqrt{2}/\sqrt{8} & 1/\sqrt{8} & -\sqrt{2}/\sqrt{8} & \sqrt{2}/\sqrt{8} \\ 2/\sqrt{8} & 0 & -2/\sqrt{8} & 0 & 0 \end{bmatrix}$$

$\det N_G = -1$

Proposition 2.19

If $g \in G_n$ (finite) is such that $|\chi(g)| = \chi(e) = d$ (complex mod), then $\rho(g) \sim \lambda I_{d\times d}$

proof

Let $o(g) = m$

$$\rho(g)^m = \rho(e) = I_{d\times d} \Rightarrow \rho(g) \text{ is diagonalisable}$$

$(\because A^m = id \Rightarrow A$ is diagonalisable$)$

Let $\lambda_1, \dots, \lambda_d$ be e.vals

$$\therefore \chi_g = \sum_{i=1}^{d} \lambda_i \Rightarrow \left|\sum_{i=1}^{d} \lambda_i\right| = d \Rightarrow \sum_{i=1}^{d} |\lambda_i| \geq d$$

But $\lambda_1^m, \lambda_2^m, \dots, \lambda_d^m$ are e.vals of $\rho(g)^m = I$

$\therefore |\lambda_1|, |\lambda_2| \dots, |\lambda_d|$ are all 1

$\therefore d \leq \sum_{i=1}^{d} 1 = d \Rightarrow$ equality must hold in

$$\left|\sum_{i=1}^{d} \lambda_i\right| \leq \sum_{i=1}^{d} |\lambda_i| \quad (\text{all } \lambda_i \in \mathbb{C})$$

Equality in $\triangle$ inequality $\Rightarrow$ all λ_i are in the same direction. But $|\lambda_i| = 1 \ \forall i \Rightarrow$ all λ_i lie on the unit circle $\Rightarrow$ all λ_i are equal

$$\therefore \rho(g) = \lambda I_{d\times d} \qquad (\because PAP^{-1} = I \Rightarrow A = I)$$

■

Corollary 2.20

Let χ be given. Then $\ker(\rho) = \{g \in G \mid \chi(g) = \chi(e) = d\}$

i.e. $\ker(\rho) = \ker(\chi)$

proof

Let $K = \{g \in G \mid \chi(g) = d\}$

$\ker(\rho) = \{g \in G \mid \rho(g) = I_{d\times d}\}$

Clearly, $\ker(\rho) \subseteq K \quad (\rho(g) = I_{d\times d} \Rightarrow \chi_g = d)$

Now let $\chi_g = d$

$\therefore \ |\chi_g| = d \Rightarrow \rho_g = \lambda I_{d\times d} \Rightarrow \chi_g = \lambda d$

$\therefore \ \lambda = 1 \Rightarrow \rho_g = I_{d\times d}$

$\therefore \ g \in \ker(\rho) \Rightarrow K \subseteq \ker(\rho)$

■

Q) Using character table (or otherwise) show that if $g, h \in G$ are such that $\chi_i(g) = \chi_i(h)$ $\forall$ irred characters χ_i of G, then g, h are in the same conjugacy class

Ans Let $g \notin [h]$ = conjugacy class of h

$\therefore g, h$ correspond to different columns in C_G but $\chi_i(g) = \chi_i(h) \Rightarrow$ these two columns must be identical $\Rightarrow \det(C_G) = 0$. contradiction

■

We now look at the lifting process (1.12, 1.13) again, in the upcoming section below.

Proposition 2.21

Let $\rho : G \to GL(V)$ be a rep. Then $\overset{M}{\overset{\shortparallel}{\{g \in G \mid \chi_g = \chi_{id}\}}} \triangleleft G$

(And hence by 1.13, $\tilde{\rho} : G/M \to GL(V)$ $\quad xM \xrightarrow{\tilde{\rho}} \rho(x)$ is a well defined rep of G/M)

proof

Let $g_1 \in M$, $a \in G$

We must show that $\chi_{a g_1 a^{-1}} = \chi_{id} = \chi_{g_1}$

$a g_1 a^{-1}$ is in same conjugacy class as g_1 and hence we are done

■

Note that by virtue of the above proposition, we can just go along each row of C_G and take those elements for which $\chi_g = \chi_{id}$ to get normal subgrps of G

Proposition 2.22

Given $N \trianglelefteq G$, $\exists\ S \subseteq \{1, 2, \dots, k\}$ (k = no. of conj classes of G = no of inequivalent irreps) such that $N = \bigcap_{i \in S} \ker(\chi_i)$

(Before, given $\ker\{\chi_i\}$'s we got a normal subgrp. Now we are reversing the process and asking if given any $N \trianglelefteq G$, can it be written in terms of ker of characters)

proof

Let $\theta_1, \theta_2, \dots, \theta_r$ be the irred characters of G/N

We use the lifting process to lift them to characters of G (If ρ is rep of G/N, define $\varphi(g) = \rho(gN)$)

$\therefore\ \hat{\chi}_g = \chi_{gN}$ ($\hat{\chi}$ being the character of φ)

Clearly $N \subseteq \ker(\varphi) = \{g \in G \mid \varphi(g) = \rho(gN) = \rho(N)\}$

$\therefore\ N \subseteq \bigcap_{i=1}^{r} \ker(\hat{\theta}_i)$ ($\hat{\theta}_i$ are lifted from θ_i)

$\left(\because\ \ker(\varphi) = \bigcap_{i=1}^{r} \ker(\hat{\theta}_i)\right.$ can be shown with ease$\left.\right)$

Suppose $g \in \ker(\hat{\theta}_i)\ \forall i$

Then $\hat{\theta}_i(g) = \theta_i(gN) = \theta_i(N)\ \forall i$

$\therefore\ gN, N$ are in the same conjugacy class (have seen a question on this just after 2.20)

$\therefore\ g \in N$

Q) For any group G, what can you conclude about no. of normal subgroups

Ans By the above proposition, there are at most 2^k where k is no. of conjugacy classes

■

CHAPTER 3 : Induced representations

During the lifting process (1.13), we saw how we can construct the representation of a group from its normal subgroup.

Frobenius found a way to lift representations from arbitrary subgroups

Q) Illustrate Frobenius' idea of lifting the reps from any arbitrary subgroup

Ans Let $H < G$ be a subgroup of index $[G:H] = r$

Let ρ be a representation of H, $\rho : H \to GL(V)$, of degree d

Decompose G into left cosets : $Ht_1 \cup \cdots \cup Ht_r$

Extend ρ to G as $\rho(g) = 0_{d \times d}$ $\forall g \in G \setminus H$

Create a new representation φ as follows :

$\varphi(g)$ is an $rd \times rd$ matrix whose $d \times d$ blocks are indexed by $\{1, 2, \ldots, r\} \times \{1, 2, \ldots r\}$ such that the $(i, j)^{th}$ $d \times d$ block is given by $\rho(t_i g t_j^{-1})$ i.e.

$$\varphi(g) = \begin{array}{cc} & j \\ i & \begin{bmatrix} & \\ & \rho(t_i g t_j^{-1}) \\ & \end{bmatrix}_{r \times r} \end{array}$$

(blocks of $d \times d$)

We have to show that $\mathcal{U}$ is indeed a representation of G

The $(i,j)^{th}$ block of $\mathcal{U}(gh)$ is $\rho(t_i gh t_j^{-1})$

The $(i,j)^{th}$ block of $\mathcal{U}(g)\mathcal{U}(h)$ is $\sum_{y=1}^{r} \rho(t_i g t_y^{-1})\rho(t_y h t_j^{-1})$

Case 1 : $t_i g h t_j^{-1} \in H$

Now $t_i g \in G \Rightarrow t_i g$ is present in a unique right coset, say, $H t_s$

$\therefore t_i g t_s^{-1} \in H$

$\therefore t_i g t_m^{-1} \notin H \quad \forall \quad m \neq s$

Thus $\sum_{y=1}^{r} \rho(t_i g t_y^{-1})\rho(t_y h t_j^{-1})$

$= \rho(t_i g t_s^{-1})\rho(t_s h t_j^{-1})$

$= \rho(t_i g h t_j^{-1})$

Case 2 : $t_i g h t_j^{-1} \notin H$

Suppose for some y,

$t_i g t_y^{-1} \in H$ and $t_y h t_j^{-1} \in H$

Then $t_i g h t_j^{-1} \in H$ which is a contradiction

Hence, every term in the summation is zero

(since $\forall y$, either $t_i g t_y^{-1} \notin H$ or $t_y h t_j^{-1} \notin H$)

This finishes the proof

$\blacksquare$

Remarks

(i) For a fixed i, only one of $\rho(t_i g t_j^{-1})$ is non zero block (similarly j) (happens at i s.t. $t_i g t_j^{-1} \in H$)

(ii) Think about what happens when a different coset space is taken i.e. $\{t_1, \dots, t_r\}$ replaced by $\{s_1, \dots, s_r\}$

Definition : In accordance with the above, if $H \leq G$ and ρ is a rep of H which extends to $\mathcal{U}$, a rep of G, the character of $\mathcal{U}$ is called an induced character (character of G induced by H)

Proposition 3.1

Induced character is unique

proof

The non-uniqueness could only possibly arise by choosing a different coset space $\{s_1, \dots, s_r\}$

Suppose $G = Ht_1 \cup \dots \cup Ht_r = Hs_1 \cup \dots \cup Hs_r$ and

WLOG $Ht_i = Hs_i \quad \forall \; i = 1, 2, \dots, r$

Suppose $s_i = h_i t_i \quad$ (some $h_i \in H$)

Then $s_i g s_i^{-1} = h_i t_i g t_i^{-1} h_i^{-1}$

$s_i g s_i^{-1} \in H$ iff $t_i g t_i^{-1} \in H$ and since these are conjugates, $\rho(s_i g s_i^{-1}) = \rho(t_i g t_i^{-1})$

Thus $\chi(\mathcal{U}) = \chi(\mathcal{U}') = \sum_{i=1}^{r} \chi(s_i g s_i^{-1}) = \sum_{i=1}^{r} \chi(t_i g t_i^{-1})$

■

Proposition 3.2

If $H \leq G$ & $\tilde{\chi}$ is an induced character arising from a rep ρ of H,

$$\tilde{\chi}(g) = \frac{1}{|H|} \sum_{y \in G} \chi(y g y^{-1})$$

proof

Suppose $G = Ht_1 \cup \cdots \cup Ht_r$. Fix some $u \in H$ and choose the coset space $\{s_1, \ldots, s_r\}$ where $s_i = ut_i$

Then by the proof of the previous proposition,

$$\tilde{\chi}(g) = \sum_{i=1}^{n} \chi(u t_i g t_i^{-1} u^{-1})$$

$$\therefore \sum_{u \in H} \tilde{\chi}(g) = \sum_{i=1}^{n} \sum_{u \in H} \chi(u t_i g t_i^{-1} u^{-1})$$

$$\therefore |H| \tilde{\chi}(g) = \sum_{\text{vary } ut_i} \chi(ut_i g (ut_i)^{-1})$$

$$= \sum_{y \in G} \chi(y g y^{-1})$$

(varying u, ut_i varies over Ht_i)

■

Proposition 3.3

Using notation as above, if C is any particular conjugacy class of G,

$$\tilde{\chi}(g) = \frac{[G:H]}{[G:Z_g]} \sum_{y \in C} \chi(y) \quad (\text{for } g \in C)$$

(where $Z_g = \{a \in G \mid ag = ga\} = \text{Stab}(g)$)

Proof

$$\tilde{\chi}(g) = \frac{1}{|H|} \sum_{y \in G} \chi(ygy^{-1}) \qquad (g \in C)$$

Now $ygy^{-1} \in C$ as y varies over G. We get $|G|$ elements out of which $|C|$ of them are distinct

In fact $y_1 g y_1^{-1} = y_2 g y_2^{-1}$ iff $y_2^{-1} y_1 g = g y_2^{-1} y_1$

ie. $y_2^{-1} y_1 \in Z_g$

Also, by the orbit stabilizer theorem, $|C| = [G : Z_g]$

Thus $\frac{|G|}{|Z_g|} = |C|$ and as y varies in G,

ygy^{-1} runs through each element of C, $|Z_g|$ times

$$\therefore \tilde{\chi}(g) = \frac{1}{|H|} \left(|Z_g| \cdot \sum_{a \in C} \chi(a) \right)$$

∎

Q) (Recall the reverse procedure - taking a rep ρ of a group G and restricting it to a subgroup H). Show that restriction does not preserve irreducibility but does preserve reducibility

Ans $$\frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)} = 1 \quad (\text{since } \rho \text{ is irred})$$

But $$\frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\chi(h)} \neq 1 \quad \text{necessarily}$$

(Find an example!)

But if we have a ρ invariant subspace P of G, P is still $\rho|_H$ invariant. ■

Theorem 3.4 (Frobenius reciprocity theorem)

Let $H \leq G$ and φ, ϕ be characters of H, G respectively. Then,

$$\langle \tilde{\varphi}, \phi \rangle_G = \langle \varphi, \phi|_H \rangle_H$$

[where $\tilde{\varphi}$ is lifting of φ to G & $\phi|_H$ is restriction of ϕ to H. $\langle x_1, x_2 \rangle_P = \frac{1}{|P|} \sum_{g \in P} x_1(g) \overline{x_2(g)}$]

proof

$$\tilde{\varphi}(x) = \frac{1}{|H|} \sum_{y \in G} \varphi(y x y^{-1})$$

$$\therefore \langle \tilde{\varphi}, \phi \rangle_G = \frac{1}{|G|} \sum_{x \in G} \tilde{\varphi}(x) \overline{\phi(x)}$$

$$= \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{y \in G} \varphi(y x y^{-1}) \overline{\phi(x)}$$

$$= \frac{1}{|G|\,|H|} \sum_{z \in G} \sum_{y \in G} \varphi(z) \overline{\phi(y^{-1} z y)}$$

$$= \frac{1}{|G|\,|H|} \sum_{z \in G} \sum_{y \in G} \varphi(z) \overline{\phi(z)}$$

$$= \frac{1}{|H|} \sum_{z \in G} \varphi(z) \bar{\phi}(z)$$

$$= \frac{1}{|H|} \sum_{z \in H} \varphi(z) \bar{\phi}(z) \quad (\because \varphi(z) = 0 \text{ if } z \notin H)$$

$$= \langle \varphi, \phi|_H \rangle_H$$

■

Proposition 3.5

Given $H \triangleleft G$ and $\rho : H \to GL(V)$ a representation of H with character χ, $\rho_t(h) := \rho(tht^{-1})$ is also a rep of H $\forall\, t \in G$. Further, irreducibility is preserved.

proof

$$\rho_t(h_1)\,\rho_t(h_2) = \rho(th_1t^{-1})\,\rho(th_2t^{-1})$$

$$H \triangleleft G \Rightarrow th_1t^{-1},\ th_2t^{-1} \in H$$

$$\therefore\ \rho_t(h_1)\,\rho_t(h_2) = \rho(th_1h_2t^{-1}) = \rho_t(h_1h_2)$$

Now $\chi_t(h) = \chi(tht^{-1})$

Thus,
$$\begin{aligned}
\langle \chi_t, \chi_t \rangle_H &= \frac{1}{|H|}\sum_{h\in H} \chi_t(h)\,\overline{\chi_t(h)} \\
&= \frac{1}{|H|}\sum_{h\in H} \chi(tht^{-1})\,\overline{\chi}(tht^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in H} \chi(y)\,\overline{\chi}(y) \\
&= \langle \chi, \chi \rangle_H
\end{aligned}$$

Thus, in particular, $\langle \chi, \chi \rangle_H = 1 = \langle \chi_t, \chi_t \rangle_H$

and hence irreducibility is preserved

$\blacksquare$

Definition: Let $H \triangleleft G$, χ be an irred character of H. Let χ_t be as above. $W = \{ t \in G \mid \chi_t(h) = \chi(h)\ \forall\, h \in H \}$ is called the inertia group of χ

It is quite easy to observe that W is a subgroup of G containing H and hence $H \leq W \leq G$

Definition : In accordance with the above theory, we say χ, χ_t are conjugate characters in G

Proposition 3.6

The number of distinct conjugates χ (charcter of H ($\trianglelefteq G$)) has in G is given by $[G:W]$ where W is the inertia group

Proof

Suppose $\chi_{t_1} = \chi_{t_2}$, $\chi(t_1 h t_1^{-1}) = \chi(t_2 h t_2^{-1})$ $\forall\ h \in H$

$\Leftrightarrow\ \chi(t_1 t_2^{-1} h t_2 t_1^{-1}) = \chi(h) \quad \forall\ h \in H$

(As h varies in H, $t_2^{-1} h t_2$ varies in H)

$\Leftrightarrow\ \chi = \chi_{t_1 t_2^{-1}} \Leftrightarrow t_1 t_2^{-1} \in W \Leftrightarrow t_1 W = t_2 W$

$\therefore$ No. of distinct conjugates = no. of left cosets of W in G = $[G:W]$

■

Precisely, if $G = Wt_1 \cup Wt_2 \cup \cdots \cup Wt_m$, then the complete set of distinct conjugates of χ is $\chi_{t_1}, \ldots, \chi_{t_m}$

Q) Lift χ from H to G (using process for arbitrary subgroups) to $\tilde{\chi}$ and check that : ($G = Wt_1 \cup \cdots \cup Wt_m$)

$$\tilde{\chi}(h) = [W:H] \sum_{j=1}^{m} \chi_{t_j}(h) \quad \forall\ h \in H$$

Ans $\tilde{\chi}(g) = \frac{1}{|H|} \sum_{y \in G} \chi(y g y^{-1})$

$$= \frac{1}{|H|} \sum_{w \in W} \sum_{j=1}^{m} \chi(w t_j g t_j^{-1} w^{-1})$$

$$\therefore \tilde{\chi}|_H (h) = \frac{1}{|H|} \sum_{w \in W} \sum_{j=1}^{m} \chi(t_j h t_j^{-1}) \quad (\because \text{defn. of } W)$$

$$= [W:H] \sum_{j=1}^{m} \chi(t_j h t_j^{-1})$$

Q) Under what circumstances is $\tilde{\chi}$ an irred character of G ? (Assume χ irred)

Ans $\langle \tilde{\chi}, \tilde{\chi} \rangle_G = \langle \tilde{\chi}|_H , \chi \rangle_H$

(reciprocity theorem with $\psi = \chi$, $\phi = \tilde{\chi}$)

$$= \frac{1}{|H|} \sum_{y \in H} \tilde{\chi}|_H (y) \, \bar{\chi}(y)$$

$$= \frac{1}{|H|} [W:H] \sum_{y \in H} \sum_{j=1}^{m} \chi(t_j y t_j^{-1}) \, \bar{\chi}(y)$$

Assume WLOG that $t_1 = id$

Then, only $j=1$ term survives (all other χ orthogonal)

$\therefore$ we get $\frac{1}{|H|} [W:H] \sum_{y \in H} \chi(y) \bar{\chi}(y) = [W:H]$

$(\because \langle \chi, \chi \rangle = 1)$

$\therefore$ If χ is an irred character of $H \trianglelefteq G$, then $\tilde{\chi}$ is irred character of G iff $W = H$ ie. $\chi_t \neq \chi$ $\forall\, t \notin H$

Theorem 3.7 (Clifford's theorem)

Let $H \triangleleft G$ and ε be an irred character of G. Then there exists an irred character χ of H such that

$\varepsilon|_H = \lambda \sum_{j=1}^{m} \chi_{t_j}$ (χ_{t_j} is a complete set of conjugates of χ) for some positive integer λ

proof

Let $\varepsilon|_H$ be expressed in terms of irred characters of H. At least one of them is non-zero & call it χ and its coefficient λ

ie. $\langle \varepsilon|_H , \chi \rangle_H = \lambda$

$\therefore \langle \varepsilon , \tilde{\chi} \rangle_G = \lambda$

$\therefore \tilde{\chi} = \lambda \varepsilon + \lambda' \varepsilon' + \cdots$

where $\varepsilon, \varepsilon', \cdots$ are all irred characters of G

$\therefore \tilde{\chi}|_H = \lambda \varepsilon|_H + \lambda' \varepsilon'|_H + \cdots$

But $\tilde{\chi}|_H = [N : H] \sum_{j=1}^{m} \chi_{t_j}$

Suppose η is an irred char of H that is not of the form of χ_t, then $\langle \tilde{\chi}|_H , \eta \rangle_H = 0$

$\therefore \lambda \langle \varepsilon|_H , \eta \rangle + \lambda' \langle \varepsilon'|_H , \eta \rangle + \cdots = 0$

Since $\langle \chi_1 , \chi_2 \rangle$ always non negative,

in particular, $\langle \varepsilon|_H , \eta \rangle = 0$ $(\because \lambda \neq 0)$

$\therefore$ Expansion of $\varepsilon|_H$ doesn't involve any irred char of H which cannot be written as χ_t

$$\therefore \quad \varepsilon|_H = \sum_{j=1}^{m} \alpha_j \chi_{t_j}$$

Now, $\langle \chi_{t_j}, \varepsilon|_H \rangle = \frac{1}{|H|} \sum_{y \in H} \chi(t_j y t_j^{-1}) \overline{\varepsilon(y)}$

$$= \frac{1}{|H|} \sum_{v \in H} \chi(v) \overline{\varepsilon(t_j^{-1} v t_j)}$$

$$= \frac{1}{|H|} \sum_{v \in H} \chi(v) \overline{\varepsilon(v)}$$

$$= \langle \chi, \varepsilon|_H \rangle_H = \lambda$$

Tensor Products

We revisit tensor products

Recall the important property $(P \otimes Q)(R \otimes S) = PR \otimes QS$

Proposition 3.8

Tensor product of two reps is a rep (same group G)

Proof

Let ρ, φ be reps of G

Then $(\rho \otimes \varphi)(gh) := \rho(gh) \otimes \varphi(gh) = \rho(g)\rho(h) \otimes \varphi(g)\varphi(h)$

$$= (\rho(g) \otimes \varphi(g))(\rho(h) \otimes \varphi(h))$$

$$= (\rho \otimes \varphi)(g) \cdot (\rho \otimes \varphi)(h)$$

Q) Find the character of the tensor product rep

Ans $tr(\rho(g) \otimes \varphi(g)) = tr(\rho(g))\, tr(\varphi(g)) = \chi_1(g)\chi_2(g)$

[Thus if $\chi_1, \chi_2, \dots, \chi_k$ is a complete list of characters of G, then $\chi_i \chi_j = \sum_{t=1}^{k} \alpha_{ijt}\, \chi_t$]

Fact 3.9

Let V, W have bases $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$. Then the vector space $V \otimes W$ has basis $\{v_i \otimes w_j\}$ $1 \le i \le m$, $1 \le j \le n$

This fact is easy to prove (use bilinearity)

Q) Suppose $\alpha = \sum_{r,s} t_{rs}(v_r \otimes w_s) \in V \otimes W$. Consider the $m \times n$ matrix $T = [t_{rs}]$. (There is a direct correspondence between such matrices & elements of $V \otimes W$)

Suppose we now change bases of V, W, what happens to the matrix T?

Ans
$$\alpha = \sum_{r,s} t_{rs}(v_r \otimes w_s) = \sum_{r,s} t'_{rs}(v'_r \otimes w'_s)$$

Suppose
$$v'_r = \sum_{\alpha=1}^{m} a_{r\alpha} v_\alpha \quad (A = [a_{ij}]_{m\times m})$$
$$w'_s = \sum_{\beta=1}^{n} b_{s\beta} w_\beta \quad (B = [b_{ij}]_{n\times n})$$

Then,
$$v'_r \otimes w'_s = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{r\alpha} b_{s\beta}(v_\alpha \otimes w_\beta)$$

$$\therefore\ t_{\alpha\beta} = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} t'_{rs}\, a_{r\alpha}\, b_{s\beta}$$

$\therefore \quad T = A^t \, T' \, B$

Q) Turn $V \otimes W$ into a G-module

Ans Let V afford rep ρ, W afford φ

Define $\rho \otimes \varphi : G \longrightarrow GL(V \otimes W)$ as

$$(\rho \otimes \varphi)(g) := \rho(g) \otimes \varphi(g)$$

(easy to check this is a rep. We already checked this some while ago)

Definition : Consider $V \otimes V$. Fix a basis $\{v_i\}$ of V. If $\omega = \sum_{i,j} a_{ij} (v_i \otimes v_j) \in V \otimes V$, ω is said to be a symmetric tensor if $a_{ij} = a_{ji}$

Q) Show that this is well-defined

Ans Suppose $\omega = \sum_{i,j} a_{ij} (v_i \otimes v_j) = \sum_{i,j} b_{ij} (w_i \otimes w_j)$

(for bases $\{v_i\}_1^n$, $\{w_i\}_1^n$ of V), consider the matrices A, B

Suppose $A = A^T$, we need to show $B = B^T$

If the base change matrix is D (ie. $v_i = \sum d_{ij} w_j$)

then $B = D^T A D$

$$B^T = D^T A^T D = D^T A D = B$$

Proposition 3.10

The set of all symmetric tensors of rank 2 (i.e. in $V \otimes V$), forms a subspace of $V \otimes V$

proof

Let $\alpha = \sum_{i,j} a_{ij} (v_i \otimes v_j)$, $\beta = \sum_{i,j} b_{ij} (v_i \otimes v_j)$ be symm tensors of rank 2 & $\lambda \in \mathbb{R}$ be any scalar

$$\lambda \alpha = \sum_{ij} (\lambda a_{ij}) (v_i \otimes v_j)$$ is clearly symmetric. And so is,

$$\sum_{ij} a_{ij} (v_i \otimes v_j) + \sum_{ij} b_{ij} (v_i \otimes v_j) = \sum_{ij} (a_{ij} + b_{ij}) (v_i \otimes v_j)$$

Notation: We denote this vector subspace of symmetric tensors of rank 2 by $V^{(2)}$

Q) Show that $V^{(2)}$ is a G-module, too

Ans Well, if $\rho : G \to GL(V^{(2)})$

Proposition 3.11

If $\dim(V) = m$, $\dim(V^{(2)}) = \frac{m(m+1)}{2}$

proof

We explicitly construct a basis

Let E_{ij} be the $m \times m$ 0-1 matrix with 1 only in (i, j)

Then E_{ii}, $E_{ij} + E_{ji}$ are symmetric & linearly independent (this is easy to check)

$$\sum_{i=1}^{n} t_{ii} E_{ii} + \sum_{i<j} t_{ij}(E_{ij} + E_{ji}) = [t_{ij}] \Rightarrow t_{ij} = t_{ji}$$

Thus, they span the entire $V^{(2)}$

(If you want the tensor description of the basis, just use the correspondence $E_{ii} \leftrightarrow v_i \otimes v_i$, $E_{ij} \leftrightarrow v_i \otimes v_j$

Q) Given $\rho: G \to GL(V)$ $(\dim V = n)$, construct a rep $\varphi: G \to GL(V^{(2)})$ from this ρ

Ans $\varphi(g)$ is a $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix. Suppose we fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V. Let us index all our $\varphi(g)$ matrices in the lexicographic order:

$e_{11} \ldots e_{1n}, e_{22}, e_{23}, \ldots, e_{2n}, \ldots, e_{n-1,n-1}, e_{n-1,n}, e_{nn}$

$(e_{ij} = e_i \otimes e_j)$

Suppose $\rho(g) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$$\therefore \rho(g) e_i = a_{1i} e_i + a_{2i} e_i + \cdots + a_{ni} e_i = \sum_{t=1}^{n} a_{ti} e_i$$

As we noted before the 'natural' way to get φ is by just doing $\varphi(g) = \rho(g) \otimes \rho(g)$.

Now the matrix is determined by its action on the basis

$$(\rho(g) \otimes \rho(g))(e_{ii}) = \rho(g) e_i \otimes \rho(g) e_i$$

$$= \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha i} a_{\beta i} (e_\alpha \otimes e_\beta)$$

$$(\rho(g) \otimes \rho(g))(e_{ij} + e_{ji}) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} (a_{\alpha i} a_{\beta j} + a_{\alpha j} a_{\beta i})(e_\alpha \otimes e_\beta)$$

Thus the matrix $\mathcal{C}(g)$ is completely determined ∎

Definition: The above matrix $\mathcal{C}(g)$ (from $\rho(g)$) is called the second Schläfian matrix of $\rho(g)$

Q) Find the second Schläfian matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Ans $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A e_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = a e_1 + c e_2$$

$$A e_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = b e_1 + d e_2$$

Suppose B is the second Schläfian matrix of A

$$B e_{11} = A e_1 \otimes A e_1 = a^2 e_{11} + ac(e_{12} + e_{21}) + c^2 e_{22}$$

$$B(e_{12} + e_{21}) = (A e_1 \otimes A e_2) + (A e_2 \otimes A e_1)$$

$$= 2ab\, e_{11} + (ad + bc)(e_{12} + e_{21}) + 2cd\, e_{22}$$

$$B e_{22} = A e_2 \otimes A e_2 = b^2 e_{11} + bd(e_{12} + e_{21}) + d^2 e_{22}$$

$$\therefore B = \begin{array}{c} \\ E_{11} \\ E_{12} \\ E_{21} \end{array} \begin{array}{c} \begin{array}{ccc} E_{11} & E_{12} & E_{21} \end{array} \\ \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \end{array} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

∎

Proposition 3.12

The character of $\mathcal{C} : G \to GL(V^{(2)})$ is given by

$$\chi^{(2)}(g) = \frac{1}{2}\left[(\chi(g))^2 + \chi(g^2)\right]$$

(χ is character of $\rho : G \to GL(V)$)

Proof

We know character is basis independent. Choose basis of V appropriately so that $\rho(g) = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_m)$

i.e. $\rho(g)(e_i) = \alpha_i e_i$

$$\therefore \varphi(g)(e_{ii}) = \alpha_i^2 e_{ii}$$

$$\varphi(g)(e_{ij} + e_{ji}) = \alpha_i \alpha_j (e_i \otimes e_j + e_j \otimes e_i)$$

$$\therefore \varphi(g) = \text{diag}(\alpha_1^2, \alpha_1\alpha_2, \cdots, \alpha_1\alpha_m, \alpha_2^2, \alpha_2\alpha_3 \cdots, \alpha_2\alpha_m, \cdots \alpha_m^2)$$

$$\therefore \chi^{(2)}(g) = \sum_{i \le j} \alpha_i \alpha_j \qquad (i, j \text{ in } 1, 2, \cdots, m)$$

$$= \frac{1}{2}\left[(\alpha_1 + \alpha_2 + \cdots + \alpha_m)^2 + (\alpha_1^2 + \cdots + \alpha_m^2)\right]$$

$$= \frac{1}{2}\left[(\chi(g))^2 + \chi(g^2)\right]$$

$$\left(\because \rho(g)^2 = \rho(g^2) = \text{diag}(\alpha_1^2, \cdots, \alpha_m^2)\right)$$

∎

Note: Just like symmetric tensors, one can develop the theory of skew-symmetric tensors. We state the results below

→ The subspace of skew symmetric tensors is denoted $V^{(1^2)}$

→ A basis is $\{e_{ij} - e_{ji}\}_{i \neq j}$ & hence dim $= \frac{m(m-1)}{2}$

→ $\chi^{(1^2)}(g) = \frac{1}{2}[(\chi(g))^2 - \chi(g^2)]$

exercise Prove that $V \otimes V = V^{(2)} \oplus V^{(1^2)}$

(Hint: every matrix $A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$)

Q) Let $H, K < G$. For each $x \in G$, define the HK double coset as $HxK := \{ hxk \mid h \in H, k \in K \}$

Prove that :

(i) HxK is the union of $x_1 K, \dots, x_n K$ where $\{x_1, \dots, x_n\}$ is the orbit containing xK of H acting by left mult on the set of left cosets of K

(ii) HxK, HyK are the same set or are disjoint

Ans (i) Let $hxk \in HxK$. But $xhk \in hxK$

$\therefore \; HxK \subseteq \bigcup_{yK \in H \cdot xK} yK$

Let $g \in \bigcup_{yK \in H \cdot xK} yK$

Then $g = yK$ (some y st. $yK \in H \cdot xK$)

$yK = h_0 x K \Rightarrow g = h_0 x K \in HxK$

$\therefore \; \bigcup_{yK \in H \cdot xK} yK \subseteq HxK$

(ii) Suppose $HxK \cap HyK \neq \emptyset$

Let $\lambda \in HxK \cap HyK$

$\therefore \; \lambda = h_1 x k_1 = h_2 y k_2$

$\therefore \; x = h_1^{-1} h_2 y k_2 k_1^{-1}$, $y = h_2^{-1} h_1 x k_1 k_2^{-1}$

$\therefore \; HxK = \{ h_0 x k_0 \mid h_0 \in H, k_0 \in K \}$

$= \{ h_0 h_1^{-1} h_2 y k_2 k_1^{-1} k_0 \mid h_0 \in H, k_0 \in K \}$

$\subseteq HyK$

Similarly $HyK \subseteq HxK$

Lemma 3.13

$$|AtB| = \frac{|A|\,|B|}{|t^{-1}At \cap B|} \qquad (\text{for subgroups } A, B \text{ of finite grp } G)$$

Proof :

$$|AtB| = |t^{-1}AtB| = \frac{|A|\,|tBt^{-1}|}{|t^{-1}At \cap B|} \qquad \left(\because |HK| = \frac{|H|\,|K|}{|H\cap K|} \right)$$

$$= \frac{|A|\,|B|}{|t^{-1}At \cap B|}$$

■

Theorem II (Mackey's theorem)

Let $L, M \leq G$ $(|G| < \infty)$. Let $G = Lt_1M \sqcup \cdots \sqcup Lt_rM$.

Define $D_i = t_i^{-1} L t_i \cap M$ $(i = 1, 2, \ldots, r)$

Let χ_1, χ_2 be characters of L, M

Then ε_i defined as $\varepsilon_i(x) = \chi_1(t_i x t_i^{-1})\,\chi_2(x)$

is a character of D_i $(x \in D_i)$ and the lifted

characters satisfy :

$$\tilde{\chi}_1(g)\,\tilde{\chi}_2(g) = \sum_{i=1}^{r} \tilde{\varepsilon}_i(g)$$

Proof

The motivation is that we have $\tilde{\chi}_1$, $\tilde{\chi}_2$ and as

seen earlier in 3.2,

$$\tilde{\chi}_1(x)\,\tilde{\chi}_2(x) = \frac{1}{|L|\,|M|} \sum_{g,h \in G} \chi_1(gxg^{-1})\,\chi_2(hxh^{-1})$$

We wish to make the RHS nicer

(Btw, LHS is character of tensor product of ρ_L, ρ_M)

Coming onto the proof, if ρ_L, ρ_M are reps of L, M, then a rep of D_i is given by φ_i where

$$\varphi_i(x) = \rho_L(t_i x t_i^{-1}) \otimes \rho_M(x) \qquad (x \in D_i)$$

Thus $\varepsilon_i(x) = \chi_1(t_i x t_i^{-1})\, \chi_2(x)$ is a character of D_i

Then, $$\tilde{\varepsilon}_i(x) = \frac{1}{|D_i|} \sum_{g \in G} \varepsilon_i(g x g^{-1})$$

$$= \frac{1}{|D_i|} \sum_{g \in G} \chi_1(t_i g x g^{-1} t_i^{-1})\, \chi_2(g x g^{-1})$$

$$= \frac{1}{|D_i|} \sum_{g \in G} \chi_1(u t_i v g x g^{-1} v^{-1} t_i^{-1} u^{-1})\, \chi_2(g x g^{-1})$$

(replacing t_i by $u t_i$ $(u \in L)$, g by vg $(v \in M)$)

$$\therefore \sum_{u,v} \tilde{\varepsilon}_i(x) = |L|\,|M|\, \tilde{\varepsilon}_i(x)$$

$$= \frac{1}{|D_i|} \sum_{u \in L} \sum_{v \in M} \sum_{g \in G} \chi_1(u t_i v g x g^{-1} v^{-1} t_i^{-1} u^{-1})\, \chi_2(g x g^{-1})$$

By the lemma, $u t_i v$ cover D_i exactly $\frac{|L|\,|M|}{|t^{-1} L t \cap M|}$ times

$$\therefore \quad |L|\,|M|\, \tilde{\varepsilon}_i(x) = \sum_{g \in G} \sum_{w \in D_i} \chi_1(w g x g^{-1} w^{-1})\, \chi_2(g x g^{-1})$$

$$\therefore \sum_{i=1}^{r} \tilde{\varepsilon}_i(x) = \frac{1}{|L|\,|M|} \sum_{g \in G} \sum_{i=1}^{r} \sum_{w \in D_i} \chi_1(w g x g^{-1} w^{-1})\, \chi_2(g x g^{-1})$$

$$= \frac{1}{|L|\,|M|} \sum_{g \in G} \sum_{h \in G} \chi_1(h g x g^{-1} h^{-1})\, \chi_2(g x g^{-1})$$

$$= \frac{1}{|L|\,|M|} \sum_{g \in G} \sum_{h \in G} \chi_1(g x g^{-1})\, \chi_2($$

CHAPTER 4

(Permutation groups)

We recall some stuff from group theory (without proofs):

- A permutation group G is said to be transitive if $\forall$ pairs (α, β) of symbols, $\exists\ \sigma \in G$ s.t. $\sigma(\alpha) = \beta$
- G is transitive iff $\forall$ symbols α $\exists\ \sigma \in G$ st. $\sigma(1) = \alpha$
- The set of permutations which fix α is called the stabilizer of α (it is a subgroup of G)
- If G is transitive on $\{1, 2, \dots, n\}$ and $p_\alpha(\alpha) = 1$ $\forall \alpha$, then the stabilizer of α is $p_\alpha^{-1} H p_\alpha$ where H is the stabilizer of 1 (Thus $1 \xrightarrow{\sigma} \alpha$ iff $\sigma \in H p_\alpha$)
- $G = Hp_1 \sqcup \dots \sqcup Hp_n \Rightarrow [G : H] = n$
- The canonical /natural (dim n) rep of G is a perm matrix with trace = no of symbols fixed by σ

Proposition 4.1

Let G be a perm group with natural character χ. Then

$$\sum_{\sigma \in G} \chi(\sigma) = |G| \qquad (G \text{ transitive})$$

proof

Firstly, suppose ε is the 1 dimensional trivial character of H (mapping every σ to $I_{n \times n}$), then $\tilde{\varepsilon}(\sigma) = \sum_{i=1}^{n} \varepsilon(p_i \sigma p_i^{-1})$

where $G = Hp_1 \sqcup \cdots \sqcup Hp_n$ $\quad (p_i(i) = 1)$

But
$$\varepsilon(p_i \sigma p_i^{-1}) = \begin{cases} 1 & p_i \sigma p_i^{-1} \in H \text{ ie. } \sigma \text{ fixes } i \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \sum_{i=1}^{n} \varepsilon(p_i \sigma p_i^{-1}) = \text{no. of elements fixed by } \sigma$$
$$= \chi(\sigma)$$

Let ξ be the 1-dim trivial character of G.

Then $\xi|_H = \varepsilon$

$$\therefore \langle \xi|_H , \varepsilon \rangle_H = \frac{1}{|H|} \sum_{h \in H} \cancel{\varepsilon(h)}^{\,1} \; \cancel{\overline{\varepsilon(h)}}^{\,1}$$

$$= \frac{1}{|H|} \, |H| = 1$$

$$\therefore \langle \xi , \tilde{\varepsilon} \rangle_G = 1 \quad \text{(reciprocity theorem)}$$

$$\therefore \langle \xi , \chi \rangle = 1$$

$$\therefore |G| = \sum_{\sigma \in G} \cancel{\xi(\sigma)}^{\,1} \; \overline{\chi(\sigma)} = \sum_{\sigma \in G} \chi(\sigma) \quad \text{(everything is real)}$$

∎

Definition: A permutation group G acting on $\{1, 2, \ldots, n\}$ is said to be <u>doubly transitive</u> if corresponding to any two pairs (α, β), (γ, δ), $\exists\, \sigma$ st $\sigma(\alpha) = \beta$, $\sigma(\gamma) = \delta$ $(\alpha \neq \gamma,\ \beta \neq \delta)$

Proposition 4.2

If G is doubly transitive with natural character χ, then $\varphi(\sigma) := \chi(\sigma) - 1$ is an irred character of G

proof

Suppose H is the stabilizer of 1

For every $u = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \in H$, associate $v = \begin{pmatrix} 2 & \cdots & n \\ \alpha_2 & \cdots & \alpha_n \end{pmatrix}$

As u ranges over H, v ranges over H' (a perm group acting on the symbols $\{2, 3, \ldots, n\}$

Thus, we have $H \simeq H'$. Further, H' is transitive

(if $\beta, \gamma \neq 1$, then $\exists \ \sigma \in G$ st. $\sigma(1) = 1$, $\sigma(\beta) = \gamma$)

$\therefore \ \chi(u) = 1 + \chi'(v)$ (χ' is natural char of H')

$$\therefore \sum_{u \in H} \chi(u) = |H| + \sum_{v \in H'} \chi'(v)$$

$$= |H| + |H'| = 2|H|$$

$$\therefore \ \langle \varepsilon, \chi|_H \rangle_H = 2 = \langle \tilde{\varepsilon}, \chi \rangle_G$$

$$= \langle \chi, \chi \rangle_G$$

$\therefore$ If $\chi = m_1 \chi_1 + m_2 \chi_2 + \cdots$ is the fourier analysis of χ, we already know $m_1 = 1$ from proposition 4-1.

Now, $1 + m_2^2 + m_3^2 + \cdots = \langle \chi, \chi \rangle = 2$

$\therefore \ \chi - \chi_1$ is an irreducible character

∎

We shall now try to study representations of S_n

Recall that each conjugacy class in S_n has the same cycle type & hence no. of conjugacy classes of S_n = partitions of n

Further, suppose a conjugacy class is given by $\lambda \vdash n$, then its size is $\frac{n!}{z_\lambda}$

(If $\lambda = (1^{m_1}, 2^{m_2}, \dots)$, $z_\lambda = m_1!\, 1^{m_1}\, m_2!\, 2^{m_2} \dots$)

It will also be useful to note that for any character χ of S_n, $\overline{\chi(\sigma)} = \chi(\sigma^{-1}) = \chi(\sigma)$

Definition: Let $\chi_1, \dots, \chi_k$ be irred characters of G. The class function $\varepsilon = u_1 \chi_1 + \dots + u_k \chi_k$ $(u_i \in \mathbb{Z})$ is called a generalised character of G

Notice how $u_i \in \mathbb{Z}$ and not just $\{0, 1, \dots\}$

Proposition 4.3

Let ε be a generalised character s.t. $\langle \varepsilon, \varepsilon \rangle = 1$. Then $\varepsilon = \pm \chi$ where χ is an irred character. Further, $\varepsilon(1) > 0 \Rightarrow \varepsilon = \chi$

proof

Orthogonality relations continue to hold and if $\varepsilon = \sum_{i=1}^{k} c_i \chi_i$,

$1 = \sum_{i=1}^{k} c_i^2 \Rightarrow c_t = \pm 1$ for some t and all other c_j's are 0 $\Rightarrow \varepsilon = \pm \chi_t$

$\varepsilon(1) > 0 \Rightarrow \pm \chi_t(1) = \pm(\text{size of identity matrix}) > 0$

$\therefore \varepsilon = + \chi_t$

∎

Recall some theory of symmetric functions (see my symm functions notes (first few pages))

Let $P(x) = \sum_{t}^{\text{finite}} a_t \, x^t$

where $t = (t_1, t_2, \cdots, t_n) \in \{0, 1, 2, \cdots, n\}^n$

and $x = (x_1, x_2, \cdots, x_n)$ are indeterminates

and $a_t \in \mathbb{R} \quad \forall \, t$

P is symmetric if $P(\sigma(x)) = P(x) \quad \forall \, \sigma$

(in two variables, $P(x_1, x_2) = P(x_2, x_1)$. In three variables, $P(x_1, x_2, x_3) = P(x_1, x_3, x_2) = \cdots = P(x_3, x_2, x_1)$)

P is skew symmetric if $P(\sigma(x)) = \text{sgn}(\sigma) \, P(x)$

a) Show that in a skew symmetric polynomial $P(x)$ with sum over all possible t (setting all but finitely many $a_t = 0$), the summation may be restricted to n-tuples t containing distinct entries

Ans $\quad P(x) = \sum_{t} a_t \, x^t$

$$P(\sigma(x)) = \sum_{t} a_t \, (\sigma(x))^t = \text{sgn}(\sigma) \sum_{t} a_t \, x^t$$

Since t varies over all possible n-tuples, we may replace t by $\sigma(t)$

$$\therefore \sum_{t} a_{\sigma(t)} \, (\sigma(x))^{\sigma(t)} = \text{sgn}(\sigma) \sum_{t} a_t \, x^t$$

Since $(\sigma(x))^{\sigma(t)} = \sigma(x^t) = x^t$, we get $a_{\sigma(t)} = \text{sgn}(\sigma) \, a_t \quad \forall \, \sigma$

Suppose some $t_i = t_j$ in t, then, choosing $\sigma = (i\,j)$, we get $a_t = (-1)\, a_t \Rightarrow a_t = 0$

$\therefore$ t must have distinct entries in order for the a_t to be non-zero.

Definition: Given an n-tuple $t = (t_1, t_2, \dots, t_n)$, we define the alternant V_t to be the determinant of $[\, x_i^{t_j} \,]_{n \times n}$

Theorem 4.4

A basis of homogenous skew symm polynomials of degree m is given by $\{ V_\ell \mid \ell \vdash m,\ \ell \text{ is strictly decreasing } n \text{ tuple} \}$

$[\{ \ell : SDP_n - m \} \text{ in short}]$

proof

Consider $f(x_1, x_2, \dots, x_n) = \sum_{t:\, n\text{-tuple}} a_t x^t$

From the previous discussion, we have that

$$f(x_1, \dots, x_n) = \sum_{\substack{t\,:\, n \text{ tuple} \\ \text{with distinct parts}}} a_t\, x^t$$

$$= \sum_{\substack{\ell\,:\, \text{Strictly dec} \\ n\text{-tuple}}} \sum_{\sigma \in S_n} a_{\sigma(\ell)}\, x^{\sigma(\ell)}$$

$$= \sum_{\ell\,:\, SD\ n\text{-tuple}} a_\ell \sum_{\sigma \in S_n} \operatorname{Sgn}(\sigma)\, x^{\sigma(\ell)}$$

But since f is homogenous of degree m, we must restrict it to those ℓ with $\ell_1 + \ell_2 + \dots + \ell_n = m$

i.e. l is a strictly decreasing partition of m & an n-tuple. i.e. $l \in SDP_n - m$

Now $\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_i^{l_{\sigma(i)}} = V_l$ (by definition of determinant from first principles)

$\therefore f(x_1, \ldots, x_n) = \sum_{l : SDP_n - m} a_l V_l$

Thus our set clearly is a spanning set

Now suppose $\sum_{l : SDP_n - m} a_l V_l = 0$

We make use of the lexicographic order on partitions of m. Consider all those $a_l \neq 0$ & choose that l_0 such that $a_{l_0} \neq 0$ & l_0 is the biggest $SDP_n - m$ (lexico). Notice that $l_0 > \sigma(l_0)$ $\forall \sigma \in S_n$. Thus x^{l_0} is the biggest term in the expansion of V_{l_0} (order on x^l induced by order on the l's. Similarly we say $V^l > V^\mu$ if $l > \mu$). Since V_{l_0} is biggest, this x^{l_0} term never gets cancelled leading to a contradiction since RHS $= 0$ ■

Recall the power sum symmetric functions

$$P_r(x_1, x_2, \ldots, x_n) = x_1^r + x_2^r + \cdots + x_n^r$$

if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, we define : $\left(\lambda \vdash n \text{ i.e. } \sum_{i=1}^{n} i m_i = n\right)$

$$P_\lambda(x_1, \ldots, x_n) := P_1^{m_1} P_2^{m_2} \cdots P_n^{m_n}$$

Proposition 4.5

Coefficient of x^p in $p_\lambda(x_1, \ldots, x_n)$ $\quad (p = (p_1, \ldots, p_n))$ any n tuple $(p \vDash n)$

is given by $\displaystyle\sum_{A,B} \prod_{r=1}^{n} \binom{m_r}{\alpha_{1r} \cdots \alpha_{nr}}$ where, $(\lambda \vdash n)$

$$A : \alpha_{1r} + \alpha_{2r} + \cdots + \alpha_{nr} = m_r \qquad \forall r = 1, 2, \ldots, n$$

$$B : \alpha_{j1} + 2\alpha_{j2} + \cdots + n\alpha_{jn} = p_j \qquad \forall j = 1, 2, \ldots, n$$

Note : $p \vDash n$ else coeff of x^p is 0

proof

Firstly, $$p_r^{m_r} = (x_1^r + \cdots + x_n^r)^{m_r}$$

$$= \sum_{(A)} \binom{m_r}{\alpha_{1r}\, \alpha_{2r} \cdots \alpha_{nr}} \left(x_1^{\alpha_{1r}} \cdots x_n^{\alpha_{nr}}\right)^r$$

where the sum is over all possible $\alpha_{1r}, \ldots, \alpha_{nr}$ such that $\alpha_{1r} + \cdots + \alpha_{nr} = m_r \quad (A)$

$\therefore \; p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$

$$= \prod_{r=1}^{n} \sum_{(A)} \binom{m_r}{\alpha_{1r} \cdots \alpha_{nr}} x_1^{r\alpha_{1r}} \cdots x_n^{r\alpha_{nr}}$$

Then directly, the coefficient of $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ is clearly given by $\displaystyle\sum_{A,B} \prod_{r=1}^{n} \binom{m_r}{\alpha_{1r} \cdots \alpha_{nr}}$

where $B : \alpha_{j1} + 2\alpha_{j2} + \cdots + n\alpha_{jn} = p_j$

(Blindly collect all x_j terms from $\prod \sum$ above)

"Why is this sum of products so important?", you ask.

Definition : For some partition $p = (p_1, \cdots, p_n) \vdash n$ $(p_1 \geq \cdots \geq p_n \geq 0)$ consider the subgroup $H_p = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_n}$ of S_n (where S_{p_1} acts on $1, 2, \cdots, p_1$, S_{p_2} acts on $p_1+1, \cdots, p_1+p_2$, so on) (S_{p_i} is $\{id\}$ if $p_i = 0$). Consider the trivial character of H_p (1 dimensional). A compound character of S_n is the lifted simple character of H_p (denoted ϕ_p)

Proposition 4.6

$$\phi_p \big|_{C_\lambda} = \sum_{A, B} \prod_{j=1}^{n} \binom{m_j}{\alpha_{j1} \cdots \alpha_{jn}} \qquad (A, B \text{ as before})$$

where C_λ is a conjugacy class of S_n given by the cycle type $(1^{m_1}, 2^{m_2}, \cdots, n^{m_n}) = \lambda$ $(p \vdash n)$

proof

Let ε denote the trivial 1-dim character of H_p.

By prop 3.3, $\phi_p\big|_{C_\lambda}(\sigma) = \dfrac{[S_n : H_p]}{[G : Z_\sigma]} \sum_{z \in C_\lambda} \varepsilon(z)$

Now by the orbit stab theorem, $[G : Z_\sigma] = |C_\lambda| = \dfrac{n!}{z_\lambda}$

Also, $\sum_{z \in C_\lambda} \varepsilon(z) = \sum_{z \in C_\lambda \cap H_p} \varepsilon(z)$ (else $\varepsilon = 0$)

$$= |C_\lambda \cap H_p|$$

$$\therefore \phi_p \big|_{C_\lambda} = \frac{\cancel{n!}}{p_1! \cdots p_n!} \frac{z_\lambda}{\cancel{n!}} |C_\lambda \cap H_p|$$

We try to find an expression for $|C_\lambda \cap H_p|$

Now, suppose $u = u_1 u_2 \cdots u_n \in H_p \quad (u_i \in S_{p_i})$

and u_i is product of α_{i1} 1-cycles, α_{i2} 2-cycles, $\cdots$.

Firstly, $u_i \in S_{p_i} \Rightarrow p_i = \alpha_{i1} + 2\alpha_{i2} + \cdots + n\alpha_{in}$ (B)

Then, total no. of i-cycles $= m_i \quad (u \in C_\lambda)$

$\therefore \quad \alpha_{1i} + \alpha_{2i} + \cdots + \alpha_{ni} = m_i \qquad$ (A)

Now for a fixed i, B specifies a conjugacy class of S_{p_i}. The size of this class is $\dfrac{p_i!}{z_{\mu_i}}$

where $\mu_i = (1^{\alpha_{i1}}, 2^{\alpha_{i2}}, \cdots, n^{\alpha_{in}})$

Letting i vary, we have $\prod_{i=1}^{n} \dfrac{p_i!}{z_{\mu_i}}$ elements

in $C_\lambda \cap H_p$ all giving rise to the same matrix $[\alpha_{ij}]$

$$\therefore |C_\lambda \cap H_p| = \sum_{A,B} \prod_{i=1}^{n} \frac{p_i!}{z_{\mu_i}}$$

$$\therefore \phi_p\big|_{C_\lambda} = \frac{z_\lambda}{\cancel{\prod_{i=1}^{n} p_i!}} \sum_{A,B} \frac{\cancel{\prod_{i=1}^{n} p_i!}}{\prod_{i=1}^{n} z_{\mu_i}}$$

$$= \sum_{A,B} \prod_{i=1}^{n} \frac{\cancel{i^{m_i}}\, m_i!}{\cancel{1^{\alpha_{i1}} 2^{\alpha_{i2}} \cdots n^{\alpha_{in}}}\, \alpha_{i1}! \cdots \alpha_{in}!}$$

$$= \sum_{A,B} \prod_{i=1}^{n} \binom{m_i}{\alpha_{i1} \cdots \alpha_{in}}$$

Corollary

conjugacy class of S_n corresponding to cycle type of λ

$$P_\lambda(x_1, \dots, x_n) = \sum_{\rho \vdash n} \phi_\rho \Big|_{C_\lambda} x^\rho \qquad (\lambda \vdash n)$$

∎

Thus, we have found a generating function for the compound chars.

We want one for simple characters

$$P_\lambda(x_1, \dots, x_n)\, f(x_1, \dots, x_n) = \sum_{\rho \vdash n} \varepsilon_\rho^\lambda\, x^\rho$$

(for any poly f with integral coefficients)

The idea is to choose f in a way so that the generalized characters ε_ρ^λ satisfy the orthogonality relations

Frobenius found the required f and never revealed the motivation. Man just said "should be clear from the proof"

Theorem 4.7

$P_\lambda \cdot \Delta(x_1, \dots, x_n)$ is a generating function of the simple characters of S_n where Δ is the Vandermonde determinant

$$\Delta(x_1, \dots, x_n) = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}_{n \times n}$$

proof

[I personally find this theorem & proof very remarkable]

Since P_λ is symm of degree n & Δ is skew symm degree $\frac{n(n-1)}{2}$, the product $p_\lambda R$ is skew symm of degree $\frac{n(n+1)}{2} = N$

Thus, from the basis,

$$p_\lambda \Delta(x_1, \ldots, x_n) = \sum_{l : SDP_n - N} \varepsilon_\lambda^l V_l$$

We now introduce another set of indeterminates $y_1, \ldots, y_n$.

We also introduce the indeterminate t

$$\prod_{i,j} (1 - t x_i y_j)^{-1} = \exp\left(- \sum_{i,j} \log(1 - t x_i y_j) \right)$$

$$= \exp\left(t \sum_{ij} x_i y_j + \frac{1}{2} t^2 \sum_{ij} x_i^2 y_j^2 + \cdots \right)$$

$$= \exp\left(t\, p_1(x) p_1(y) + \frac{t^2}{2} p_2(x) p_2(y) + \cdots \right)$$

$$= \prod_{r=1}^{\infty} \exp\left(\frac{t^r}{r} p_r(x) p_r(y) \right)$$

$$= \prod_{r=1}^{\infty} \sum_{\alpha_r = 0}^{\infty} \left(\frac{(p_r(x))^{\alpha_r} (p_r(y))^{\alpha_r} \quad t^{r \cdot \alpha_r}}{r^{\alpha_r} \quad \alpha_r !} \right)$$

$$= \sum_{u=0}^{\infty} G_u(x, y) \; t^u$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be an infinite seq with non-neg integers

st. finitely many are non-zero

Then, define $p_\alpha(x) = \prod_{i=1}^{\infty} (p_i(x))^{\alpha_i}$

$$p_\alpha(y) = \prod_{i=1}^{\infty} (p_i(y))^{\alpha_i}$$

Let $g(\alpha) := \dfrac{1}{1^{\alpha_1} \alpha_1! \; 2^{\alpha_2} \alpha_2! \; \cdots}$

Let $\|\alpha\| := \alpha_1 + 2\alpha_2 + \cdots$

Now,

$G_u(x,y)$ = coeff of t^u in

$$\prod_{r=1}^{\infty} \sum_{\alpha_r=0}^{\infty} \left(\frac{(p_r(x))^{\alpha_r} (p_r(y))^{\alpha_r} \; t^{r\cdot\alpha_r}}{r^{\alpha_r} \; \alpha_r \, !} \right)$$

$$= \sum_{\|\alpha\|=u} p_\alpha(x) \, p_\alpha(y) \, g(\alpha)$$

$$\therefore \quad \Delta(x)\Delta(y) \prod_{i,j} (1 - t x_i y_j)^{-1} = \Delta(x)\Delta(y) \sum_{u=0}^{\infty} G_u(x,y) \, t^u \qquad \text{— (1)}$$

Now by a result on Cauchy determinants,

$$\det\left(\left[\frac{1}{1 - x_i y_j} \right]_{n\times n} \right) = \Delta(x)\Delta(y) \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1}$$

$$\therefore \quad \det\left(\left[\frac{1}{1 - t x_i y_j} \right] \right) = (\sqrt{t})^{\frac{n(n-1)}{2}} (\sqrt{t})^{\frac{n(n-1)}{2}} \prod_{i,j=1}^{n} (1 - t x_i y_j)^{-1} \, \Delta(x)\Delta(y) \qquad \text{— (2)}$$

$\quad \| $

$\quad K$ (say)

$$K = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (1 - t x_i y_{\sigma(i)})^{-1} \qquad \left(\text{definition of determinant}\right)$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(1 + t x_i y_{\sigma(i)} + t^2 x_i^2 y_{\sigma(i)}^2 + \cdots \right)$$

$$= \sum_{v=0}^{\infty} H_v(x,y) \, t^v \qquad \text{— (3)}$$

A typical term of $H_\nu(x,y)$ is of the form $\pm(x\,\sigma(y))^{\rho}$

where $\rho \models \nu$

Since K is skew-symm in x & also in y, only those ρ occur which have distinct parts

$$\therefore \quad H_\nu(x,y) = \sum_{\sigma\in S_n}\sum_{\rho\in S_n}\sum_{\substack{l:\text{ strictly}\\ \text{decreasing partition}\\ \text{of } \nu\\ (\text{ofc, } l \text{ is n tuple})}} \operatorname{sgn}(\sigma)\, x^{\rho(l)}\, \sigma(y)^{\rho(l)}$$

(convince yourself that ρ indeed varies of S_n & all such l occur, too)

Now $\sigma(y)^{\rho(l)} = y^{\sigma^{-1}(\rho(l))}$. Put $\pi = \sigma^{-1}\circ\rho$

$$\therefore \quad H_\nu(x,y) = \sum_{l:\,SDP_n-\nu}\ \sum_{\pi,\rho\in S_n} \operatorname{sgn}(\pi)\operatorname{sgn}(\rho)\, x^{\rho(l)}\, y^{\pi(l)}$$

$$= \sum_{l:\,SDP_n-\nu} V_l(x)\, V_l(y)$$

Using ①, ②, ③,

$$t^{\frac{n(n-1)}{2}}\,\Delta(x)\,\Delta(y)\sum_{u=0}^{\infty} G_u(x,y)\,t^u = \sum_{\nu=0}^{\infty} H_\nu(x,y)\,t^\nu$$

Comparing coefficients of t^N, $\left(N = \frac{n(n+1)}{2}\right)$

$$\sum_{\|\alpha\|=N} g(\alpha)\,p_\alpha(x)\,\Delta(x)\,p_\alpha(y)\,\Delta(y) = \sum_{l:\,SDP_n-N} V_l(x)\,V_l(y)$$

Now $$P_\alpha(x)\,\Delta(x) = \sum_{l : SDP_n - N} \varepsilon_\alpha^{l}\, V_l(x)$$

$$P_\alpha(y)\,\Delta(y) = \sum_{m : SDP_n - N} \varepsilon_\alpha^{m}\, V_m(y)$$

Thus, substituting it back we get,

$$\sum_{\|\alpha\|=N} g(\alpha) \sum_{l : SDP_n - N} \sum_{m : SDP_n - N} \varepsilon_\alpha^{l}\, \varepsilon_\alpha^{m}\, V_l(x)\, V_m(y)$$

$$= \sum_{l : SDP_n - N} V_l(x)\, V_l(y)$$

$$\therefore \sum_{\|\alpha\|=N} g(\alpha)\, \varepsilon_\alpha^{l}\, \varepsilon_\alpha^{m} = \delta_{lm}$$

$$\therefore \langle \varepsilon^{l}, \varepsilon^{m} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon^{l}(\sigma)\, \overline{\varepsilon^{m}(\sigma)}$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon^{l}(\sigma)\, \varepsilon^{m}(\sigma)$$

$$= \frac{1}{n!} \sum_{\|\alpha\|=N} |C_\alpha|\, \varepsilon^{l}\big|_{C_\alpha}\, \varepsilon^{m}\big|_{C_\alpha}$$

$$= \sum_{\|\alpha\|=N} \frac{|C_\alpha|}{n!}\, \varepsilon_\alpha^{l}\, \varepsilon_\alpha^{m}$$

$$= \sum_{\|\alpha\|=N} g(\alpha)\, \varepsilon_\alpha^{l}\, \varepsilon_\alpha^{m}$$

$$= \delta_{lm}$$

Now we invoke proposition 4.3 to conclude that our ε^l's are irred characters (upto sign)

We now show $\varepsilon^l(id) > 0$ completing the proof.

Let f^l denote $\varepsilon^l(id)$

The cycle pattern of conjugacy class of $id \in S_n$ is given by $\alpha_1 = n$, $\alpha_2 = \cdots = \alpha_n = 0$ (That is $1+1+1+\cdots+1$)

$$P_\alpha \cdot \Delta(x_1, \ldots, x_n) = \sum_{l : SDP_n - N} \varepsilon_\alpha^l V_l$$

$$\Rightarrow \quad P_{id}\, \Delta(x_1 \cdots x_n) = \sum_{l : SDP_n - N} f^l V_l$$

But $P_{id}(x_1 \cdots x_n) = P_{(n,0,0,\ldots,0)}(x_1, \ldots, x_n)$

$$= (x_1 + \cdots + x_n)^n$$

$$= \sum_{r_1 + \cdots + r_n = n} \binom{n}{r_1 \cdots r_n} x^r$$

Now, $\Delta = \underset{\substack{\| \\ \eta}}{V_{(n-1, n-2, \ldots, 1, 0)}} = \sum_{\sigma \in S_n} sgn(\sigma)\, x^{\sigma(\eta)}$

$$\therefore \quad P_{id} \cdot \Delta(x_1 \cdots x_n) = \sum_{r_1 + \cdots + r_n = n} \sum_{\sigma \in S_n} \binom{n}{r_1 \cdots r_n} sgn(\sigma)\, x^{r + \sigma(\eta)} \quad \text{--- (1)}$$

$$= \sum_{l : SDP_n - N} f^l V_l$$

Let l_0 be a $SDP_n - N$

The terms occurring in V_{l_0} are of the form $x^{\tau(l_0)}$ for arbitrary $\tau \in S_n$

We want to show that the coeff of V_l in (i) is positive

Firstly, $\sigma(\eta) + r = \tau(l) \Rightarrow r = \tau(l) - \sigma(\eta)$

To handle the negatives, set $\frac{1}{(-m)!} := 0 \quad \forall m > 0$

$$r_i = \tau(l)_i - \sigma(\eta)_i$$
$$= l_{\tau(i)} - \eta_{\sigma(i)}$$

$$\therefore P_{id} \cdot \Delta = \sum_{l : SDP_n - N} \sum_{\sigma, \tau \in S_n} \frac{n! \ \text{sgn}(\sigma)}{\prod_{i=1}^{n} (l_{\tau(i)} - \eta_{\sigma(i)})!} x^{\tau(l)}$$

$$= \sum_{l : SDP_n - N} \sum_{\pi \in S_n} \frac{n! \ \text{sgn}(\pi)}{\prod_{i=1}^{n} (l_i - \eta_{\pi(i)})!} V_l$$

(Putting $\pi = \tau^{-1} \circ \sigma$ & observing that sum with respect to σ yields V_l)

$$\therefore f^l = \sum_{\pi \in S_n} \frac{n! \ \text{sgn}(\pi)}{\prod_{i=1}^{n} (l_i - \eta_{\pi(i)})!} = n! \det(M)$$

where M is matrix whose i, j^{th} entry is $\dfrac{1}{(l_i - \eta_j)!}$

$$\therefore\ f^{l} = \frac{n!}{l_1! \cdots l_n!} \det\left(\left[\frac{l_i!}{(l_i - \eta_j)!} \right]_{n \times n} \right)$$

(just trust me bro)

$$= \frac{n!}{l_1! \cdots l_n!} \prod_{i<j} (l_i - l_j) > 0$$

(In case you are wondering what happened to η, recall that $\eta = (n-1, n-2, \dots, 3, 2, 1, 0)$ is known)

Finally,

$$p_\lambda(x_1, \dots, x_n) \cdot \Delta(x_1, \dots, x_n) = \sum_{l : SDP_n - N} \varepsilon_\lambda^{l} V_l$$

and all the ε^{l} are in correspondence with λ irreducible characters of the rep of S_n.

Now, we show that there is a bijection between $\{l : SDP\text{-}N\}$ and $\{\mu : \mu \vdash n\}$. given $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ construct $l_i = \mu_i + n - i$ and given $l = (l_1, l_2, \dots, l_n)$ construct $\mu_i = l_i - n + i$

Definition : With motivation from the above discussion (just above), we define W_p to be the shifted alternant as follows. If p is a partition $(p_1, \dots, p_n)$ of n, convert it to a strictly decreasing partition $(l_1, \dots, l_n)$ of $\frac{n(n+1)}{2}$ & define $W_p = V_l$

Definition: Given a partition $p \vdash n$ we define the p^{th} **Schur function** to be $F_p = \frac{W_p}{\Delta}$ (Δ is Vandermonde)

Proposition 4.8

Schur functions are symmetric polynomials

proof

Being the ratio of two skew symmetric polynomials, F_p is symmetric.

Now, W_p is zero whenever $x_i = x_j$ $(i \neq j)$. Thus W_p is divisible by $(x_i - x_j)$ $\forall$ $i \neq j$. Since $\Delta = \prod_{i<j}(x_i - x_j)$, $\Delta \mid W_p$ and F_p is a polynomial.

■

Q) Express $p_\lambda(x_1, \dots, x_n)$ $(\lambda \vdash n)$ in terms of Schur functions and simple characters

Ans

$$p_\lambda \cdot \Delta \; (x_1, \dots, x_n) = \sum_{l : SDP_n - N} \varepsilon^l_\lambda \; V_l$$

$$= \sum_{p \vdash n} \chi^p_\lambda \; W_p$$

$$\therefore \; p_\lambda (x_1, \dots, x_n) = \sum_{p \vdash n} \chi^p_\lambda \; F_p$$

■

Proposition 4.9 (Schur's formula)

$$F_p = \frac{1}{n!} \sum_{\alpha \vdash n} h_\alpha \chi_\alpha^p \, p_\alpha$$

proof

$$\langle \chi^p , \chi^q \rangle = \delta_{pq}$$

$$\parallel$$

$$\frac{1}{n!} \sum_{\sigma \in S_n} \chi^p(\sigma) \overline{\chi^q(\sigma)} = \frac{1}{n!} \sum_{\alpha \vdash n} \chi_\alpha^p \overline{\chi_\alpha^q}$$

$$= \frac{1}{n!} \sum_{\alpha \vdash n} h_\alpha \chi_\alpha^p \chi_\alpha^q$$

where h_α is the size of conjugacy class of α type $= \frac{n!}{z_\alpha}$

$$\frac{1}{n!} \sum_{\alpha \vdash n} h_\alpha \chi_\alpha^p s_\alpha$$

$$= \frac{1}{n!} \sum_{\alpha \vdash n} h_\alpha \chi_\alpha^p \left(\sum_{q \vdash n} \chi_\alpha^q F_q \right)$$

$$= \frac{1}{n!} \sum_{\alpha \vdash n} \sum_{q \vdash n} h_\alpha \chi_\alpha^p \chi_\alpha^q F_q$$

$$= \sum_{q \vdash n} \left(\frac{1}{n!} \sum_{\alpha \vdash n} \chi_\alpha^p \chi_\alpha^q \right) F_q$$

$$= \sum_{q \vdash n} \delta_{pq} F_q$$

$$= F_p$$

$\blacksquare$

We go over some basics of symmetric polynomial theory without giving proofs for those results not related to rep theory

Definition: We define the monomial symmetric functions as $m_t(x_1, \dots, x_n) = \sum_{\sigma} x^{\sigma(t)}$ where σ ranges over all distinct permutations of the n-tuple t $(m_0 = 1)$

Definition: We define the elementary symmetric functions as $e_r(x_1, \dots, x_n) = m_{(1^r, 0^{n-r})}(x_1, \dots, x_n)$ $(r \leq n)$

and extend it as $e_\lambda(x_1, \dots, x_n) = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$ $(\lambda \vdash n)$

Observe that $e_r(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$ (r at a time)

Definition: We define the complete homogenous symm functions as $h_r(x_1, \dots, x_n) = \sum_{\lambda \vdash r} m_\lambda(x_1, \dots, x_n)$ $(h_0 = 1)$ and extend to $h_\lambda(x_1, \dots, x_n) = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$ $(\lambda \vdash n)$

Observe that $h_r(x_1, \dots, x_n) = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r} = \sum_{\alpha_1 + \dots + \alpha_n = r} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

Definition: We define the power sum symmetric functions as $p_r(x_1, \dots, x_n) = x_1^r + \dots + x_n^r$, $p_0(x_1, \dots, x_n) = 1$

and extend as $p_\lambda(x_1, \dots, x_n) = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$ $(\lambda \vdash n)$

Fact 4.10

$\{m_\lambda \mid \lambda \vdash m\}$, $\{e_\lambda \mid \lambda \vdash m\}$, $\{h_\lambda \mid \lambda \vdash m\}$, $\{p_\lambda \mid \lambda \vdash m\}$ are all bases for homogenous symm poly in n variables of degree m

Fact 4.11

$\{ F_\lambda \mid \lambda \vdash m \}$ is also a basis for homogenous symmetric polynomials in n variables of degree m

Fact 4.12

$$\prod_{i=1}^{n} (1 + x_i t) = 1 + e_1 t + e_2 t^2 + e_3 t^3 + \cdots = f(t)$$

$$\prod_{i=1}^{n} \frac{1}{(1 - x_i t)} = 1 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots = \frac{1}{f(-t)}$$

$$\therefore \quad \sum_{k=0}^{n} (-1)^k e_k h_{r-k} = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases}$$

(we have extended definitions to $-$ve as $h_m = 0$ for $m < 0$ and $e_m = 0$ for $m > n$ or $m < 0$)

$$\log (f(-t)) = p_1 t - \frac{p_2}{2} t^2 + \frac{p_3}{3} t^3 - \cdots$$

$$\therefore \quad e_m = \sum_{\|\alpha\| = m} g(\alpha) (-1)^{\alpha_2 + \alpha_4 + \cdots} p_\alpha$$

$$\log \left(\frac{1}{f(t)} \right) = p_1 t + \frac{p_2}{2} t^2 + \cdots$$

$$\therefore \quad h_m = \sum_{\|\alpha\| = m} g(\alpha) p_\alpha$$

Fact 4.13 [Jacobi, 1841]

$$F_p = \det \left([h_{p_i - i + j}]_{n \times n} \right)$$

There is an easy trick to remember the RHS in the above fact

Suppose $p = (2, 2, 1, 0)$, the matrix is formed as

$$\begin{bmatrix} h_2 & h_3 & h_4 & h_5 \\ h_0 & h_2 & h_2 & h_3 \\ 0 & h_0 & h_1 & h_2 \\ 0 & 0 & 0 & h_0 \end{bmatrix}$$

Proposition 4.14

$$\chi_\alpha^p = 1 \quad \text{if} \quad p = (n, 0, 0, 0, \ldots)$$

proof

$$F_{(n,0,0,\ldots 0)} = \det \begin{pmatrix} h_n & & & & & \\ 0 & h_0 & & & & \\ 0 & 0 & h_0 & & & \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & h_0 \end{pmatrix} = h_n$$

$$\therefore \quad F_p = h_n = \sum_{\alpha \vdash n} \frac{1}{Z_\alpha} p_\alpha \qquad \left(\because g_\alpha = \frac{1}{Z_\alpha} \right)$$

From proposition 4.9 (Schur's formula), we have our required result that $\chi_\alpha^p = 1$ ie. χ^p is the trivial character (1 dim)

∎

Q) Now analyse $F_{(1,1,1,\dots,1)}$ and draw conclusions about $\chi_\alpha^{(1,1,\dots,1)}$

<u>Ans</u>

$$F_{(1,1,\dots,1)} = \begin{pmatrix} h_1 & h_2 & h_3 & \dots & h_{n-1} & h_n \\ 1 & h_1 & h_2 & \dots & h_{n-2} & h_{n-1} \\ 0 & 1 & h_1 & \dots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_1 & h_2 \\ 0 & 0 & 0 & \dots & 1 & h_1 \end{pmatrix} = ???$$

It is convenient to use the Vandermonde definition

$$F_{(1,1,\dots,1)} = \frac{\det\left(x_j^{\,n+1-i}\right)}{\det\left(x_j^{\,n-i}\right)} = e_n \qquad (\text{check!})$$

$$\therefore \quad \chi_\alpha^{(1,1,\dots,1)} = (-1)^{\alpha_2+\alpha_4+\dots}$$

$\blacksquare$

In the upcoming discussion we assume familiarity with Ferrer shapes and conjugate partitions

<u>Fact 4.15</u>

$p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_m)$ $\quad (p \vdash n,\ q \vdash n)$

are conjugate iff

(i) $p_1 = m$, $\quad q_1 = n$

(ii) $p_i + q_j \neq i + j - 1 \quad \forall\ 1 \leq i \leq n,\ 1 \leq j \leq m$

Further, if $m = n$, p, q are conjugates iff :

$$\{\, p_i + n - i \mid 1 \leq i \leq n \,\} \sqcup \{\, n - 1 + j - q_j \mid 1 \leq j \leq n \,\} = \underset{\substack{\| \\ \{0,1,\dots,2n-1\}}}{[2n]_0}$$

Fact 4.16

$\det(h_{p_i - i + j}) = \det(e_{q_i - i + j})$ where p, q are conjugate partitions of n

(This fact is deep and uses Jacobi's theorem on subdeterminants extracted using partitions)

Further the LHS matrix is significant because of the fact that every Schur function of S_n can be expressed as a subdeterminant of this matrix

Corollary (to facts 4.15, 4.16)

For $p, q \vdash n$ conjugate partitions,

$$\chi^{p} = \chi^{(1,1,1,\dots 1)} \chi^{q}$$