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Frobenius' Theorem on Simple Characters of S_n

The objective is to study the irreducible characters of S_n. Why do we care about irreducible characters of S_n? Primarily, it is because we want to study about arbitrary characters of S_n. These characters also play a crucial role in the understanding of representations of GL_n.

Let us recall some general facts about representations

Facts

1) Number of conjugacy classes of a group is equal to the number of irreducible representations of that group
2) If $\rho_1, \rho_2, \dots, \rho_k$ are the complete set of irreps of a group with degrees $d_1, \dots, d_k$, then $d_1^2 + \dots + d_k^2 = |G|$
3) In accordance with the point above, each $d_i \mid |G|$

But how does studying these irreducible representations give us information about any arbitrary representation of G?
We have a few nice theorems in this regard

Facts

1) [Maschke's theorem] Every representation of a finite group is completely reducible (i.e. $\rho : G \to GL(V)$ admits $V = \bigoplus V_i$ such that each V_i is G-invariant & $\rho_i : G \to GL(V_i)$ (restriction) is irred)
2) [Schur's lemma] If V, W are irred G-modules, then $\mathrm{Hom}_G(V, W)$ consists of the 0 map and isomorphisms
3) For any irreducible characters χ, χ', $\langle \chi, \chi' \rangle = 1$ if the representations are equivalent and 0 otherwise
4) $\langle \chi, \chi \rangle = 1$ iff χ is irreducible
5) If $\{\rho_1, \dots, \rho_k\}$ is the full set of irred characters with degrees d_i, then $\rho_{Lreg} \sim d_1 \phi_1 \oplus \dots \oplus d_k \phi_k$ ($\oplus$ meaning block diagonalisation)
6) Two reps are equivalent iff they have the same character
7) [Group Algebra Lemma] Let $\mathbb{C}A_G = V_1 \oplus \dots \oplus V_k$ be a complete decomp

into irreducible G-modules (wrt ρ_{Lreg}). If W is any irred G-module, then W must be isomorphic to V_i for some i

Induced characters

We discuss some theory & facts about lifting characters from subgroups to the group itself

Given $H \leq G$ and $\rho : H \longrightarrow GL(V)$ of dimension d, if $[G:H] = r$, we decompose $G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_r$ and define $\rho(g)$ to be the $d \times d$ zero matrix if $g \in G \setminus H$.

We create the rep $\mathcal{U} : G \longrightarrow GL(W)$ of dimension rd : $\mathcal{U}(g)$ is an $rd \times rd$ matrix. Group it into r^2 no of $d \times d$ blocks. The $(i,j)^{th}$ block is $\rho(t_i g t_j^{-1})$

Facts :

1) The induced character is unique
2) If $H \leq G$ & $\tilde{\chi}$ is the induced character arising from ρ (rep of H), then we have the following expressions for $\tilde{\chi}$

$$\tilde{\chi}(g) = \frac{1}{|H|} \sum_{y \in G} \chi(y g y^{-1}) \quad \forall \; g \in G$$

$$\tilde{\chi}(g) = \frac{[G:H]}{[G:Z_g]} \sum_{y \in C} \chi(y) \quad \forall \; g \in C$$ (C is a conjugacy class and Z_g is the stabilizer of G)

3) [Frobenius Reciprocity Theorem] Let $H \leq G$ and $\mathcal{U}, \phi$ be characters of H, G resp. Then $\langle \tilde{\mathcal{U}}, \phi \rangle_G = \langle \mathcal{U}, \phi|_H \rangle_H$

The permutation group S_n

We finally come to the actual content : irreps of S_n

We make a few quick remarks

1) If χ is any character of S_n, $\chi(g) = \chi(g^{-1}) = \bar{\chi}(g)$ and hence the values taken are real
2) If χ is the natural character of S_n, $\chi(g)$ is the no. of elements of $[n]$ fixed by g
3) For any permutation group G with natural character χ, we have $\sum_{g \in G} \chi(g) = |G|$, if G is transitive

We are now done with the recap and shall move on to the study of representations of S_n

Definition : Let $\chi_1, \dots, \chi_k$ be all the irred characters of G. The class function $\varepsilon = u_1 \chi_1 + \dots + u_k \chi_k$ $(u_i \in \mathbb{Z})$ is called a generalised character of G

Proposition 1

Let ε be a generalised character st. $\langle \varepsilon, \varepsilon \rangle = 1$. Then $\varepsilon = \pm \chi$ where χ is an irred character. Further, $\varepsilon(1) > 0$ $\Rightarrow \varepsilon = \chi$ ($\chi(1) < 0 \Rightarrow \varepsilon = -\chi$). More generally, if $\varepsilon_1, \dots, \varepsilon_s$ are generalised characters st $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$, $\varepsilon_i(1) > 0 \ \forall i, j$ then all of them are distinct irred characters

proof

Let $\varepsilon = u_1 \chi_1 + \dots + u_k \chi_k$

$\langle \varepsilon, \varepsilon \rangle = \sum u_i^2 = 1 \Rightarrow$ some $u_i^2 = 1$, others are 0

$\therefore \varepsilon = \pm \chi_i$ for some i

$\varepsilon(1) > 0 \Rightarrow \pm \chi_i(1) = \pm$ (size of identity matrix) > 0

$\therefore \varepsilon = + \chi_i$

$\langle \varepsilon_i, \varepsilon_i \rangle = 1$, $\varepsilon_i(1) > 1 \Rightarrow \varepsilon_i = \chi_{a_i}$ for some $a_i \in [k]$

$\langle \varepsilon_i, \varepsilon_j \rangle = 0 \ \forall j \neq i \Rightarrow a_i \neq a_j \ \forall i \neq j$

Thus $\{\varepsilon_i\}_{i=1}^{s} = \{\chi_{a_1}, \chi_{a_2}, \dots, \chi_{a_s}\} \subseteq \{\chi_1, \dots, \chi_k\}$ is indeed a collection of distinct irred characters

■

Definition : Given any n-tuple $(t_1, t_2, \dots, t_n)$, we define the alternant V_t to be the determinant of $[x_i^{t_j}]_{1 \le i, j \le n}$

Definition : For any $n \ge 1$, we define $p_n(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i^n$ and for $\lambda \vdash n$, we define $p_\lambda(x_1, \dots, x_k) = \prod_t p_{\lambda_t}(x_1, \dots, x_k)$

Fact

Given $\lambda \vdash n$, the coefficient of $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ in $p_\lambda(x_1, \dots, x_n)$ is $\sum_{A, B} \prod_{r=1}^{n} \binom{m_r}{\alpha_{1r}, \dots, \alpha_{nr}}$ (where $p_1 + \dots + p_n = |\lambda|$)

where

$A : \alpha_{1r} + \dots + \alpha_{nr} = m_r \quad \forall r = 1, \dots, n$

$B : \alpha_{j1} + 2\alpha_{j2} + \dots + n\alpha_{jn} = p_j \quad \forall j = 1, \dots, n$

$(\alpha_{ij} \ge 0 \ \forall i, j)$

■

Definition : For some $p = (p_1, \ldots, p_n) \models n$, we consider the group $S_{p_1} \times \cdots \times S_{p_n}$ and view it as a subgroup of S_n (S_{p_i} acts on the batch of p_i symbols ; $S_0 = \{id\}$). We consider H_p, the trivial (1D) character of this subgroup and lift this to, what is called, a compound character ϕ_p of S_n

Proposition 2

$\phi_p |_{C_\lambda}$ = coefficient of x^p in $p_\lambda(x_1, \ldots, x_n)$ $(p \models n)$

proof

Let ε denote the trivial character of H_p

$$\phi_p |_{C_\lambda}(\sigma) = \frac{[S_n : H_p]}{[S_n : Z_\sigma]} \sum_{z \in C_\lambda} \varepsilon(z) \qquad (\sigma \in C_\lambda)$$

$[S_n : Z_\sigma] = |C_\lambda|$ (orbit-stab theorem)

$\sum_{z \in C_\lambda} \varepsilon(z) = |C_\lambda \cap H_p|$ (if $z \notin H_p$, $\varepsilon(z) = 0$)

We try to find an expression for $|C_\lambda \cap H_p|$

Let $u = u_1 \cdots u_n \in H_p$ ($u_i \in S_{p_i}$) with u_i being a product of α_{i1} 1-cycles, α_{i2} 2-cycles, etc.

$u_i \in S_{p_i} \Rightarrow \alpha_{i1} + 2\alpha_{i2} + \cdots + n\alpha_{in} = p_i$ (B)

Total no. of i-cycles is m_i (say)

Then $\alpha_{1i} + \alpha_{2i} + \cdots + \alpha_{ni} = m_i$ (A)

For a fixed i, (B) specifies a conjugacy class of S_{p_i} whose underlying partition is $(1^{\alpha_{i1}}, 2^{\alpha_{i2}}, \ldots, n^{\alpha_{in}}) = \mu_i$ (say)

The size of this conjugacy class is $p_i! / z_{\mu_i}$ (where for $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$, $z_\lambda := m_1!\, 1^{m_1}\, m_2!\, 2^{m_2} \cdots$)

Letting i vary, we have $\prod_{i=1}^{n} p_i! / z_{\mu_i}$ elements of $C_\lambda \cap H_p$ giving the same matrix $[\alpha_{ij}]$

$$\therefore |C_\lambda \cap H_p| = \sum_{A, B} \prod_{i=1}^{n} \frac{p_i!}{z_{\mu_i}}$$

The result now follows using fact proved earlier

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Corollary 3

$$p_\lambda(x_1, \ldots, x_n) = \sum_{p \models n} \phi_p |_{C_\lambda} x^p \quad \text{for } \lambda \vdash n$$

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We have found a generating function for compound characters and we would like one for simple characters.

For any homogenous $f(x_1, \ldots, x_n)$ with integral coefficients, we have

$p_\lambda(x_1, \ldots, x_n) f(x_1, \ldots, x_n) = \sum_{p \vDash N} \varepsilon_\lambda^p x^p \qquad (N = n + \deg(f))$

If we choose f appropriately so that ε_λ^p satisfy the orthog relations from prop 1, we will get a generating function for simple characters (since no. of ε_λ^p's is equal to no. of possible λ's which is equal to no. of irreps of S_n)

Frobenius came up with $f(x_1, \ldots, x_n)$ to be the alternant V_δ where $\delta = (0, 1, \ldots, n-1)$. This is commonly also called the Vandermonde determinant

Theorem 4

$$p_\lambda(x_1, \ldots, x_n) V_{(0,1,\ldots,n-1)} = \sum_{p \vDash N} \chi_p \big|_{C_\lambda} x^p \qquad \left(N = \frac{n(n+1)}{2}\right)$$

where $\{\chi_p \mid p \vdash n\}$ form the irreducible characters of S_n

Before going to the proof directly, we need to lay some groundwork

Definition: By $SDP_n - m$, we mean strictly decreasing n-tuples whose entries add up to m

Proposition 5

$\{ V_\ell \mid \ell \in SDP_n - m \}$ is a basis for skew-symmetric polynomials which are homogeneous of degree m

proof

Let A be the set of all n tuples with distinct parts which add up to m

Let $f(x_1, \ldots, x_n) = \sum_{t \in A} a_t x^t$

$= \sum_{\ell \in SDP_n - m} \sum_{\sigma \in S_n} a_{\sigma(\ell)} x^{\sigma(\ell)}$

$= \sum_{\ell \in SDP_n - m} a_\ell \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x^{\sigma(\ell)}$

$= \sum_{\ell \in SDP_n - m} a_\ell V_\ell$

(Just expand the determinant : $V_\ell = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_i^{\ell_{\sigma(i)}}$)

Thus, our set is a spanning set

Now consider $\sum_{\ell \in SDP_n - m} a_\ell V_\ell = 0$

Consider all non-zero a_l and fix l_0 among the subscripts such that l_0 is the biggest among all SOP$_n$-m (wrt the lexicographic ordering)
Thus x^{l_0} is the biggest term in the expansion of V_{l_0} since $l_0 > \sigma(l_0)$ $\forall$ $\sigma \in S_n$
Since further, by choice, V_{l_0} is biggest, x^{l_0} term never gets cancelled giving a contradiction ■

Notation : We shall write Δ in place of $V_{(0,1,\dots,n-1)}$, the Vandermonde determinant. $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$

Symmetric Functions

We take a quick detour into the rich combinatorial theory of symmetric functions

Definition : We define some symmetric functions below. Here, λ is some partition given by $\lambda = (\lambda_1, \lambda_2, \dots)$ and $x = (x_1, x_2, \dots)$.
If $\alpha = (\alpha_1, \alpha_2, \dots)$ is some seq of integers, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots$

(i) Monomial symmetric functions :
$m_\lambda(x) := \sum x^{\sigma(\lambda)}$ where sum is over all possible distinct permutations σ of λ

(ii) elementary symmetric functions :
$e_n(x) := m_{1^n}(x) = \sum x_{i_1} \cdots x_{i_n}$ where sum is over $i_1, \dots, i_n$ such that $1 \le i_1 < \dots < i_n$
$e_\lambda(x) := e_{\lambda_1}(x) \cdot e_{\lambda_2}(x) \dots$

(iii) complete symmetric functions :
$h_n(x) := \sum_{\lambda \vdash n} m_\lambda(x) = \sum x_{i_1} \cdots x_{i_n}$ where sum is over $i_1, \dots, i_n$ such that $1 \le i_1 \le i_2 \dots \le i_n$
Incidentally, $h_n(x_1, \dots, x_m) = \sum_{\alpha_1 + \dots + \alpha_m = n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$
$h_\lambda(x) := h_{\lambda_1}(x) \cdot h_{\lambda_2}(x) \dots$

(iv) power sum symmetric functions :
$p_n(x) := m_{(n)}(x) = x_1^n + x_2^n + \dots$
$p_\lambda(x) := p_{\lambda_1}(x) \cdot p_{\lambda_2}(x) \dots$

Fact sheet

1) $\{m_\lambda \mid \lambda \vdash n\}$, $\{e_\lambda \mid \lambda \vdash n\}$, $\{h_\lambda \mid \lambda \vdash n\}$, $\{p_\lambda \mid \lambda \vdash n\}$ form basis for the symmetric functions of n variables

2) $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$ where $M_{\lambda\alpha}$ is the number of 0-1 matrices having Row sums λ, column sums α

3) $h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu$ where $N_{\lambda\alpha}$ is the number of $\mathbb{N}$ matrices having Row sums λ, column sums α

4) $p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu$ where $R_{\lambda\alpha}$ is the number of ordered partitions $(B_1, \dots, B_{\ell(\mu)})$ of $[\ell(\lambda)]$ s.t. $\mu_j = \sum_{i \in B_j} \lambda_i$ $(1 \le j \le k)$

5)
$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda \sum_\mu M_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y)$$
$$= \sum_\lambda e_\lambda(x)\, m_\lambda(y)$$
$$= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, p_n(y)\right)$$
$$= \sum_\lambda \frac{1}{n} p_\lambda(x)\, p_\lambda(y)$$

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda \sum_\mu N_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y)$$
$$= \sum_\lambda h_\lambda(x)\, m_\lambda(y)$$
$$= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x)\, p_n(y)\right)$$
$$= \sum_\lambda \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda(x)\, p_\lambda(y)$$

where if $\lambda = (1^{m_1}, 2^{m_2}, \dots)$, then $z_\lambda := \prod_i i^{m_i}(i!)$
and $\varepsilon_\lambda = (-1)^{m_2 + m_4 + \cdots}$

6)
$$e_n(x) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda(x)$$
$$h_n(x) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \cdot \varepsilon_\lambda \cdot p_\lambda(x)$$

Enough with the detours ... Back to the problem at hand ...

Proposition 6

Let C_0 be the det of the $n \times n$ matrix $[a_{ij}] = \frac{1}{1-x_i y_j}$. Then,
$C_0 = \Delta(x)\Delta(y) \prod_{i,j} (1-x_i y_j)^{-1}$

proof

Put $f(t) = \prod_{k=1}^{n} (1-t x_k) = 1 - c_1 t + c_2 t - \dots + (-1)^n c_n t^n$

From fact 5, $c_i = e_i(x_1, \dots, x_n)$

Let $f_i(t) = \dfrac{f(t)}{1-t x_i}$ $\forall\ i = 1, 2, \dots, n$

Let $f_i(t) = \sum_{r=1}^{n} b_{ir}(x)\, t^{n-r}$

Taking LCM along each column, we get,

$C_0 = \dfrac{1}{\prod_j f(y_j)} \det(f_i(y_j))$

<u>lemma</u> : Let $x_1, \dots, x_n, t$ be indeterminates. If we have
$g_i(t) = \sum_{k=1}^{n} a_{ik} t^{n-k}$ $(i = 1, \dots, n)$ polynomials of degree $< n$,
then $\det(g_i(x_j)) = \Delta(x) \det(a_{ij})$
(proof is straightforward : $[g_i(x_j)] = [a_{ij}][x_k^{n-i}]$) ///

Here, if we let $f_i(t) = \sum_{r=1}^{n} b_{ir}(x)\, t^{n-r}$, we get,
$\det(f_i(y_j)) = \det(b_{ir}(x))\, \Delta(y)$

Now, just expanding $(1-t x_i)^{-1}$ as a GP, we have,
$f_i(t) = \left(\sum_{p=0}^{n} (-1)^p c_p t^p\right)\left(\sum_{q=0}^{\infty} x_i^q t^q\right)$.

Thus, $b_{ir}(x) = \sum_{p=0}^{n-r} (-1)^p c_p\, x_i^{n-r-p} = \mathscr{C}_r(x_i)$ (say)

$\mathscr{C}_r$ is a monic polynomial of degree $n-r$

The coefficient matrix a_{ij} from the lemma corresponding to $\mathscr{C}$
is upper triangular & hence has $\det = 1$ ($\because$ monic)

$\therefore \det(b_{ir}(x)) = \det(A)\, \Delta(x) = \Delta(x)$

Thus $C_0 = \prod_j \dfrac{1}{f(y_j)} \Delta(x)\Delta(y)$

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Corollary 7

$$\det\left[\frac{1}{1-t x_i y_j}\right]_{n\times n} = t^{n(n-1)/2}\, \Delta(x)\Delta(y) \prod_{i,j} (1-t x_i y_j)^{-1}$$

■

Proposition 8

If $\det\left[\frac{1}{1-tx_iy_j}\right]_{n\times n} = \sum_{m=0}^{\infty} H_m(x,y)\, t^m$, then we have $H_m(x,y) = \sum_{\ell \in SDP_n - m} V_\ell(x)\, V_\ell(y)$

proof

Let us call the det on LHS K for brevity

$$K = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (1-t x_i y_{\sigma(i)})^{-1}$$

$$= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (1 + t x_i y_{\sigma(i)} + t^2 x_i^2 y_{\sigma(i)}^2 + \cdots)$$

A typical term of $H_m(x,y)$ is of the form $\pm (x\,\sigma(y))^p$ where $p \vDash v$

Skew symmetricity of K forces only those p to occur which have distinct parts

$$\therefore H_m(x,y) = \sum_{\sigma\in S_n} \sum_{\rho\in S_n} \sum_{\ell\in SDP_n - m} \mathrm{sgn}(\sigma)\, x^{\rho(\ell)}\, \sigma(y)^{\rho(\ell)}$$

Put $\pi = \sigma^{-1}\circ\rho$.

$$\therefore H_m(x,y) = \sum_{\ell\in SDP_n - m} \sum_{\pi,\sigma\in S_n} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)\, x^{\rho(\ell)}\, y^{\pi(\ell)}$$

Using definition of V_ℓ, we are done

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Proposition 9

If $\prod_{i,j} (1-t x_i y_j)^{-1} = \sum_{u=0}^{\infty} G_u(x,y)\, t^u$, then $G_u(x,y)$ is given by $\sum_{\|\alpha\|=u} p_{\tilde\alpha}(x)\, p_\alpha(y)\, g(\alpha)$ where :

$\|\alpha\| = \alpha_1 + 2\alpha_2 + \cdots$, $g(\alpha) = \frac{1}{1^{\alpha_1}\alpha_1!\; 2^{\alpha_2}\alpha_2!}$, $p_{\tilde\alpha} = p_{(1^{\alpha_1}, 2^{\alpha_2}, \ldots)}$

proof

$$\prod_{i,j} (1-t x_i y_j)^{-1} = \exp\left(-\sum \log(1-t x_i y_j)\right)$$

$$= \exp\left(t \sum x_i y_j + \tfrac{t^2}{2} \sum x_i^2 y_j^2 + \cdots\right)$$

$$= \exp\left(t\, p_1(x)\, p_1(y) + \tfrac{t^2}{2}\, p_2(x)\, p_2(y) + \cdots\right)$$

$$= \prod_{r=1}^{\infty} \exp\left(\frac{t^r}{r}\, p_r(x)\, p_r(y)\right)$$

$$= \prod_{r=1}^{\infty} \sum_{\alpha_r=0}^{\infty} \left(\frac{(p_r(x))^{\alpha_r}\,(p_r(y))^{\alpha_r}\, t^{r\alpha_r}}{r^{\alpha_r}\,\alpha_r!}\right)$$

The result follows from the definitions of $\|\alpha\|$, $g(\alpha)$ and $p_{\tilde\alpha}$

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Corollary 10

$$\sum_{\|\alpha\|=n} g(\alpha)\, p_{\tilde{\alpha}}(x)\, p_{\tilde{\alpha}}(y)\, \Delta(x)\Delta(y) = \sum_{\ell \in SDP_n - N} V_\ell(x)\, V_\ell(y) \qquad \left(N = \frac{n(n+1)}{2}\right)$$

proof

The proof follows by combining corollary 7 and propositions 8 and 9 & comparing coefficients of $t^{n(n+1)/2} = t^N$ on both sides

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proof of theorem 4

$$p_\lambda(x)\,\Delta(x) = \sum_{p \vDash N} \varepsilon_\lambda^p\, x^p$$
$$= \sum_{\ell \in SDP_n - N} \varepsilon_\lambda^p\, V_\ell$$

Applying in corollary 10, we get

$$\sum_{\|\alpha\|=n} g(\alpha) \sum_{l,m \in SDP_n - N} \varepsilon_{\tilde{\alpha}}^{\ell}\, \varepsilon_{\tilde{\alpha}}^{m}\, V_\ell(x)\, V_m(y) = \sum_{\ell \in SDP_n - N} V_\ell(x)\, V_\ell(y)$$

Thus, $\sum_{\|\alpha\|=n} g(\alpha)\, \varepsilon_{\tilde{\alpha}}^{\ell}\, \varepsilon_{\tilde{\alpha}}^{m} = \delta_{\ell m}$ (comparing coefficients)

Finally we compute the inner product

$$\langle \varepsilon^\ell, \varepsilon^m \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon^\ell(\sigma)\, \overline{\varepsilon^m(\sigma)}$$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon^\ell(\sigma)\, \varepsilon^m(\sigma)$$
$$= \frac{1}{n!} \sum_{\|\alpha\|=n} |C_{\tilde{\alpha}}|\, \varepsilon_{\tilde{\alpha}}^{\ell}\, \varepsilon_{\tilde{\alpha}}^{m}$$
$$= \sum_{\|\alpha\|=n} g(\alpha)\, \varepsilon_{\tilde{\alpha}}^{\ell}\, \varepsilon_{\tilde{\alpha}}^{m} = \delta_{\ell m}$$

All we need to show now is that $f^\ell := \varepsilon^\ell(id) > 0$

The cycle pattern of the conjugacy class of id is given by 1^n i.e. $\alpha = (n, 0, 0, \dots)$

$$p_\alpha \cdot \Delta = \sum_{\ell \in SDP_n - N} \varepsilon_\alpha^\ell\, V_\ell$$

$$\therefore\ p_{1^n} \cdot \Delta = \sum_{\ell \in SDP_n - N} f^\ell\, V_\ell$$

But $p_{1^n}(x_1, \dots, x_n) = (x_1 + \dots + x_n)^n$

$$= \sum_{r_1 + \dots + r_n = n} \binom{n}{r_1 \cdots r_n} x^r \qquad (r_i \geq 0)$$

$$\therefore \sum_{r_1+\dots+r_n = n} \sum_{\sigma \in S_n} \binom{n}{r_1 \cdots r_n} \operatorname{sgn}(\sigma)\, x^{r + \sigma(\delta)} = \sum_{\ell \in SDP_n - N} f^\ell\, V_\ell$$

where $\Delta = V_\delta$ ie. $\delta = (n-1, n-2, \ldots, 1, 0)$

Let $l_0 \in SDP_n - N$

The terms in V_{l_0} are of the form $x^{\tau(l_0)}$ $(\tau \in S_n)$.

We try to f^l explicitly by finding coefficient of V_l in LHS

$r + \sigma(\delta) = \tau(l) \Rightarrow r = \tau(l) - \sigma(\delta)$

ie. $r_i = \tau(l)_i - \sigma(\delta)_i = l_{\tau(i)} - \delta_{\sigma(i)}$

If $r_i < 0$ for some i, just ignore its contribution since the multinomial coefficient doesnt make sense

Thus, $$P_{1,n} \cdot \Delta = \sum_{l \in SDP_n - N} \sum_{\sigma, \tau \in S_n} \frac{n!}{\prod_i (l_{\tau(i)} - \delta_{\sigma(i)})!} \operatorname{sgn}(\sigma)\, x^{\tau(l)}$$

Putting $\pi = \tau^{-1} \cdot \sigma$, we pull out a V_l from thin air to get $$P_{1,n} \Delta = \sum_{l \in SDP_n - N} \sum_{\pi \in S_n} \frac{n! \operatorname{sgn}(\pi)}{\prod_i (l_i - \delta_{\pi(i)})!} V_l$$

Thus $$f_l = \sum_{\pi \in S_n} \frac{n! \operatorname{sgn}(\pi)}{\prod_i (l_i - \delta_{\pi(i)})!}$$

$$= n! \det(M_l)$$

where M_l is the matrix $\left[\dfrac{1}{(l_i - \delta_j)!}\right]_{n \times n}$

$$\det(M_l) = \frac{1}{l_1! \cdots l_n!} \det\left[\frac{l_i!}{(l_i - \delta_j)!}\right]_{n \times n}$$

If we show that the det on the RHS is positive, we will be done completely.

To this extent consider the n polynomials (of deg $< n$)

$\phi_1(t) = t(t-1)\cdots(t-n+2)$

$\phi_2(t) = t(t-1)\cdots(t-n+3)$

$\vdots$

$\phi_n(t) = t$

ie. $\phi_i(t) = t(t-1)\cdots(t-n+i+1)$ $\forall\ 1 \le i \le n-1$, $\phi_n(t) = t$

Writing $\phi_i(t) = \sum_{k=1}^n a_{ik} t^{n-k}$ $(i = 1, 2, \ldots, n)$, we get that $A = [a_{ij}] = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ and $\det A = 1$

Using the lemma in proof of prop 6, $\det[\phi_i(x_j)] = \Delta(x)$

But $\phi_i(l_j) = \dfrac{l_j!}{(l_j - \delta_i)!}$ where $\delta = (n-1, n-2, \ldots, 1, 0)$

Thus the det on RHS is just $\Delta(l) = \prod_{i<j}(l_i - l_j) > 0$ since l is strictly decreasing