Real Analysis MA 403

Definition: Let S be a set. An order on S is a relation denoted by $<$, satisfying :-

(i) $x \in S$, $y \in S$ $\Rightarrow$ only one of the following is true : $x < y$, $x = y$, $y < x$

(ii) $x < y$, $y < z$ $\Rightarrow$ $x < z$

note : $x \leq y \Rightarrow x < y$ or $x = y$

Definition: Let S be an ordered set, $E \subset S$. If there exists a $\beta \in S$ such that $\forall x \in E$, $x \leq \beta$, then E is said to be bounded above by β. β is the upper bound of E

Definition: For an ordered set S, $E \subset S$ and E being bounded above, if there exists an $\alpha \in S$ satisfying

(i) α is an upper bound of E

(ii) $\gamma < \alpha \Rightarrow \gamma$ is not an upper bound of E

equiv (ii') $\alpha \leq \beta$ for every upper bound β of E

then α is said to be the least upper bound of E or the supremum of E (similarly we can define infimum using lower bound / bounded below)

Definition: An ordered set S is said to have the lub property if for every non empty subset E of S, which is bounded above, we have a supremum of E in S

THEOREM 1

Let S satisfy LUB property. Let $B \subset S$ be non empty and bounded below. If L is the set of all lower bounds of B, $\sup L = \inf B$ and both exist

Definition : A field F is a set F with two operations called addition and multiplication which satisfy :

(A1) $x \in F,\ y \in F \Rightarrow x+y \in F$

(A2) $x+y = y+x \quad \forall\ x, y \in F$

(A3) $(x+y)+z = x+(y+z) \quad \forall\ x, y, z \in F$

(A4) $\exists\ 0 \in F$ s.t. $x+0 = 0+x = x \quad \forall\ x \in F$

(A5) $\forall\ x \in F\ \exists\ -x \in F$ s.t. $x+(-x) = (-x)+x = 0$

(M1) $x \in F,\ y \in F \Rightarrow xy \in F$

(M2) $xy = yx \quad \forall\ x, y \in F$

(M3) $(xy)z = x(yz) \quad \forall\ x, y, z \in F$

(M4) $\exists\ 1 \in F$ s.t. $x \cdot 1 = 1 \cdot x = x \quad \forall\ x \in F$

(M5) $\forall\ x \in F \setminus \{0\},\ \exists\ x^{-1} \in F$ s.t. $x \cdot x^{-1} = x^{-1} \cdot x = 1$

(D) $x \cdot (y+z) = (xy) + (xz) \quad \forall\ x, y, z \in F$

THEOREM 2

(i) $x+y = x+z \Rightarrow y = z$

(ii) $x+y = x \Rightarrow y = 0$

(iii) $x+y = 0 \Rightarrow y = -x$

(iv) $-(-x) = x$

(v) $xy = xz,\ x \neq 0 \Rightarrow y = z$

(vi) $xy = x,\ x \neq 0 \Rightarrow y = 1$

(vii) $xy = 1,\ x \neq 0 \Rightarrow y = x^{-1}$

(viii) $x \neq 0 \Rightarrow (x^{-1})^{-1} = x$

(ix) $0 \cdot x = 0$

(x) $x \neq 0,\ y \neq 0 \Rightarrow xy \neq 0$

(xi) $(-x)y = x(-y) = -(xy)$

(xii) $(-x)(-y) = xy$

all of these hold for any field F and $x, y, z \in F$

Definition: An *ordered field* is a field F which is also an ordered set such that

(i) $x+y < x+z$ if $x,y,z \in F$ and $y<z$

(ii) $xy > 0$ if $x,y \in F$, $x>0$, $y>0$

THEOREM 3

(i) $x>0 \Rightarrow -x<0$

(ii) $x>0,\ y<z \Rightarrow xy < xz$

(iii) $x<0,\ y<z \Rightarrow xy > xz$

(iv) $x \neq 0 \Rightarrow x^2 > 0$

(v) $0<x<y \Rightarrow 0 < 1/y < 1/x$

hold for any ordered field F, $x,y,z \in F$

THEOREM 4 ($\mathbb{R}$ from $\mathbb{Q}$)

There exists an ordered field $\mathbb{R}$ which has the lub property and contains $\mathbb{Q}$ as a subfield

THEOREM 5 (Archimedean property)

(i) $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x>0 \Rightarrow \exists\ n \in \mathbb{N} = \{1,2,\dots\}$ s.t.

$nx > y$

(ii) $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x<y \Rightarrow \exists\ p \in \mathbb{Q}$ s.t. $x<p<y$

THEOREM 6

For every $x>0$ ($x \in \mathbb{R}$) and every integer $n>0$, there is a unique positive real y such that

$y^n = x$

THEOREM 7

$a, b \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $n \in \mathbb{N}$

Then $(ab)^{1/n} = a^{1/n} \cdot b^{1/n}$

THEOREM 8 (Greatest integer function)

$\forall x \in \mathbb{R}, \quad \exists!\, m \in \mathbb{Z}$ s.t. $m \leq x < m+1$

and we write $m = \lfloor x \rfloor$ and $m+1 = \lceil x \rceil$

Definition : The extended real number system consists of the real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$ such that $-\infty < x < +\infty$ hold $\forall x \in \mathbb{R}$

Conventions : The extended reals do not form a field but we use the conventions : $x + \infty = +\infty$, $x - \infty = -\infty$,

$\frac{x}{+\infty} = \frac{x}{-\infty} = 0$, $x > 0 \Rightarrow x \cdot (\pm\infty) = \pm\infty$, ~~[illegible]~~

$x < 0 \Rightarrow x \cdot (\pm\infty) = \mp\infty$

Definition : A complex number is an ordered pair (a, b) of reals a, b . We define $x + y = (a+c, b+d)$ and $x \cdot y = (ac - bd, ad + bc)$ for $x = (a, b)$ and $y = (c, d)$

THEOREM 9

The set of complex numbers forms a field $\mathbb{C}$ with the addition and multiplication operations as defined above

Definition: We define iota as $i = (0, 1) \in \mathbb{C}$

THEOREM 10

$i^2 = -1$ and for $a, b \in \mathbb{R}$, $(a, b) = a + bi$

Definition: if $z = a + bi \in \mathbb{C}$, we refer to a as the real part, b as the imaginary part and $\bar{z}$ as the conjugate of z where $\bar{z} := a - bi$

THEOREM 11

$z, w \in \mathbb{C}$

(i) $\overline{z+w} = \bar{z} + \bar{w}$

(ii) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(iii) $z + \bar{z} = 2\,\mathrm{Re}(z)$

$z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iv) $z\bar{z} \geq 0$ with equality iff $z = 0$

Definition: For $z \in \mathbb{C}$, we define $|z| = (z\bar{z})^{1/2}$ to be the absolute value of z

THEOREM 12

$z, w \in \mathbb{C}$

(i) $|z| \geq 0$ with equality iff $z = 0$

(ii) $|\bar{z}| = |z|$

(iii) $|zw| = |z||w|$

(iv) $|\mathrm{Re}\, z| \leq |z|$

(v) $|z + w| \leq |z| + |w|$

THEOREM 13 (Schwarz inequality)

$a_1, \ldots, a_n, b_1, \ldots b_n \in \mathbb{C}$

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right)$$

with equality iff a_j, b_j are proportional

Definition: For $k > 0$, $k \in \mathbb{N}$, we define $\mathbb{R}^k$ to be the set of all ordered k-tuples $\vec{x} = (x_1, x_2, \ldots, x_k)$ where $x_i \in \mathbb{R}$ $\forall$ $i = 1, 2, \ldots, k$ are known as the coordinates of $\vec{x}$. $\vec{x}$ is called a vector in $\mathbb{R}^k$

We define addition & multiplication as

$$(x_1, x_2, \ldots, x_k) + (y_1, \ldots, y_k) = (x_1 + y_1, \ldots, x_k + y_k)$$

$$\alpha \cdot (x_1, \ldots, x_k) = (\alpha x_1, \ldots, \alpha x_k)$$

$\mathbb{R}^k$ with these operations is turned into a vector space over the real field

We also define an inner product on $\mathbb{R}^k$ as follows:

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{k} x_i y_i$$

We define the norm of a vector as $\|\vec{x}\| = (\langle \vec{x}, \vec{x} \rangle)^{1/2}$

$\mathbb{R}^k$ equipped with a norm (and hence an inner product) is known as a Euclidean k-space

THEOREM 14

$x, y, z \in \mathbb{R}^k$, $\alpha \in \mathbb{R}$

(i) $\|x\| \geq 0$ with equality iff $x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$

(iii) $\|x \cdot y\| \leq \|x\| \|y\|$

(iv) $\|x + y\| \leq \|x\| + \|y\|$

(v) $\|x - y\| \leq \|x - y\| + \|y - z\|$

Definition: For some $n \in \mathbb{N}$, J_n denotes the set $\{1, 2, \ldots, n\}$.

we say, for any set A,

(i) A is finite if $\exists f : A \to J_n$ which is a bijection

(ii) A is infinite if A is not finite

(iii) A is countable if $\exists f : A \to \mathbb{N}$ which is a bijection

(iv) A is uncountable if A is neither finite nor countable

(v) A is at most countable if A is finite or countable

Definition: A sequence is a function defined on $\mathbb{N}$. If $f : \mathbb{N} \to B$, we get a sequence $f(1), f(2), \ldots$ in B

THEOREM 15

(i) Every infinite subset of a countable set A is countable

(ii) Let $\{E_n\}$, $n = 1, 2, \ldots$ be a sequence of countable sets. Then the countable union $\bigcup_{n=1}^{\infty} E_n$ is also countable

(iii) Let A be countable and B_n be the set of all n-tuples $(a_1, \ldots, a_n)$ where $a_k \in A$ and the elements a_i need not be distinct. Then B_n is countable $\forall n \in \mathbb{N}$

Definition: A set X, whose elements are referred to as points, is said to be a metric space if with any two points $p, q \in X$, there is a real number $d(p,q)$, called distance associated s.t. $d(p,q) \geq 0$ with equality iff $p = q$, $d(p,q) = d(q,p)$, $d(p,q) \leq d(p,r) + d(r,q)$.

Definition: For $\vec{x} \in \mathbb{R}^k$, $r > 0$, we define the open ball B with center at $\vec{x}$, radius r to be the set

$$S = \{ \vec{y} \in \mathbb{R}^k : \|\vec{y} - \vec{x}\| < r \}$$

(closed ball $\Leftrightarrow$ $\|\vec{y} - \vec{x}\| \leq r$)

Definition: A set $E \subset \mathbb{R}^k$ is said to be convex if

$$\lambda \vec{x} + (1-\lambda) \vec{y} \in E \quad \forall \ \vec{x}, \vec{y} \in E, \ \lambda \in (0,1)$$

Definition: Let X be a metric space

(i) A neighbourhood of $p \in X$ is a set $N_r(p)$ consisting of all $q \in X$ s.t. $d(p,q) < r$ for the given $r > 0$

(ii) A point $p \in X$ is a limit point of $E \subset X$ if every neighbourhood of p contains a point $q \neq p$ such that $q \in E$ ie. $(N_r(p) \setminus \{p\}) \cap E \neq \phi \quad \forall \ r > 0$

(iii) $p \in X$ is said to be an isolated point of $E \subset X$ if $p \in E$ but is not a limit point of E

(iv) E is said to be closed if it contains all its limit points

(v) $p \in X$ is said to be an interior point of E if there is a neighbourhood N of p such that $N \subsetneq E$

(vi) E is said to be open if every point of E is an interior point

(vii) E is perfect if E is closed and every point of E is a limit point of E

(viii) E is bounded if $\exists \ M \in \mathbb{R}$, $q \in X$ s.t. $d(p,q) < M \ \forall \ p \in E$

(ix) E is dense in X if every point of X is a limit point of E or is a point of E (or both)

(x) The complement of E (E^c) is the set of all $p \in X$ s.t. $p \notin E$

THEOREM 16

Every neighbourhood of any point in a metric space is an open set

THEOREM 17

If p is a limit point of E, then every neighbourhood of p contains infinitely many points of E

corollary: A finite set has no limit points

THEOREM 18

E is open $\Longleftrightarrow$ E^c is closed

THEOREM 19

(i) at most countable union of open sets is open

(ii) at most countable intersection of closed sets is closed

(iii) finite intersection of open sets is open

(iv) finite union of closed sets is closed

Definition: For $E \subset X$ (metric space), if E' denotes the set of all limit points of E, then the closure of E is defined to be $\overline{E} = E \cup E'$

THEOREM 20

(i) $\overline{E}$ is closed

(ii) $E = \overline{E}$ iff E is closed

(iii) $\overline{E} \subset F$ for every closed F s.t. $E \subset F$

THEOREM 21

Let $E \subset \mathbb{R}$ be bounded above. $\sup E \in E$ iff E is closed

THEOREM 22

Suppose $Y \subset X$.. $E \subset Y \subset X$. E is open relative to Y iff $E = Y \cap G$ for some open G in X

Definition: An open cover of a set $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets in X such that $E \subset \bigcup_\alpha G_\alpha$. A finite subcover of an open cover is a finite subcollection of $\{G_\alpha\}$ s.t $E \subset \bigcup_\beta G_\beta$ where β runs over the finite set.

Definition: A set K in X is said to be compact if every open cover of K admits a finite subcover

THEOREM 23

Let $K \subset Y \subset X$. K is compact relative to X iff K is compact relative to Y

THEOREM 24

(i) All compact sets are closed

(ii) All closed subsets of compact sets are compact

THEOREM 25 (Generalised nested interval theorem)

If $\{K_\alpha\}$ is a compact sets collection such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non empty, then $\bigcap_\alpha K_\alpha$ is also non empty

Corollary: If $\{K_n\}_{n \in \mathbb{N}}$ is compact $\forall n$, $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

THEOREM 26

If E is an infinite subset of a compact set K, then E has a limit point in K

THEOREM 27

We strengthen the corollary of theorem 25. (Do we? ☹)

If $\{I_n\}$ is any [redacted] seq of intervals in $\mathbb{R}$ s.t. $I_n \supset I_{n+1}$,

then $\bigcap\limits_{i=1}^{\infty} I_n \neq \emptyset$ (note: interval in $\mathbb{R}$ is $[a,b]$)

THEOREM 28 (Heine Borel)

For $E \subset \mathbb{R}^k$, TFAE

(i) E is closed and bounded

(ii) E is compact

(iii) Every infinite subset of E has a limit point in E

THEOREM 29 (Weierstrass)

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$

THEOREM 30

non empty perfect sets in $\mathbb{R}^k$ are uncountable

THEOREM 31

The cantor set is perfect (and is in fact an uncountable set of measure zero)

<u>Definition</u>: A & B are said to be <u>separated</u> if $A \cap \bar{B}$ and $\bar{A} \cap B$ are both [redacted] empty. $E \subset X$ is said to be <u>connected</u> if E cannot be written as a union of non empty separated sets

THEOREM 32

$E \subset \mathbb{R}$ is connected iff it has the property that if $x \in E$, $y \in E$, then $z \in E$ $\forall z$ s.t. $x < z < y$

i.e. only intervals are connected in $\mathbb{R}$

Definition: A sequence $\{p_n\} \subset X$ is said to converge if there is a point $p \in X$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ so that $n \geq N$ implies $d(p_n, p) < \varepsilon$. we say $\{p_n\} \to p$ or $p_n \to p$

If p_n does not converge, it is said to diverge

$\{p_n\}$ is said to be bounded if the set $\{p_1, p_2, \dots\}$ is bounded

THEOREM 33

(i) $\{p_n\} \to p \in X$ iff every neighbourhood of p contains p_n for almost all n

(ii) $p \in X$, $p' \in X$ and $\{p_n\} \to p$ and $\{p_n\} \to p'$ $\Rightarrow$ $p = p'$

(iii) all convergent sequences are bounded

(iv) $E \subset X$, p is a limit point of E $\Rightarrow$ $\exists \{p_n\} \subset E$ such that $\{p_n\} \to p$

(v) $s_n \to s$, $t_n \to t$ $\Rightarrow$
- $s_n \pm t_n \to s \pm t$
- $\alpha s_n \to \alpha s$ $\forall \alpha \in X$
- $s_n t_n \to st$
- $\frac{1}{s_n} \to \frac{1}{s}$ provided $s_n \neq 0$ $\forall n$, $s \neq 0$

(vi) Let $x_n = (\alpha_{1_n}, \alpha_{2_n}, \dots, \alpha_{k_n}) \in \mathbb{R}^k$

$x_n \to x = (\alpha_1, \dots, \alpha_k)$ iff $\lim\limits_{n\to\infty} \alpha_{j_n} = \alpha_j$ $\forall 1 \leq j \leq k$

(vii) $\{x_n\}, \{y_n\} \subset \mathbb{R}^k$, $\{\beta_n\} \subset \mathbb{R}$, $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then,

$x_n + y_n \to x + y$, $x_n \cdot y_n \to x y$, $\beta_n x_n \to \beta x$

Definition: Given a sequence $\{p_n\}$, a sequence $\{p_{n_i}\}$ is a <u>subsequence</u> of $\{p_n\}$ where $\{n_i\}$ is a sequence of positive integers s.t. $n_1 < n_2 < n_3 < \dots$

THEOREM 34

(i) $\{p_n\}$ converges to L iff all of its subsequence converge to L

(ii) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence

(iii) If $\{p_n\}$ is a sequence in a compact metric space X, there exists a subsequence of $\{p_n\}$ that converges in X

(iv) The subsequential limits of a given sequence form a closed space in X

Definition: A sequence $\{p_n\} \subset X$ is said to be <u>cauchy</u> if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(p_n, p_m) < \varepsilon \quad \forall n \geq N, m \geq N$

Definition: for $\phi \neq E \subset X$, let $S = \{x \in \mathbb{R} \mid x = d(p,q), p,q \in E\}$

We define the <u>diameter</u> of E to be $\sup_{x \in X} S$

THEOREM 35

(i) If $\{p_n\} \subset X$ and $E_N = \{p_N, p_{N+1}, \dots\}$, then $\{p_n\}$ is a cauchy sequence iff $\lim_{N \to \infty} \operatorname{diam} E_N = 0$

(ii) $\operatorname{diam} \bar{E} = \operatorname{diam} E$

(iii) K_n is a seq of compact sets in X s.t. $K_n \supset K_{n+1} \; \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} \operatorname{diam} K_n = 0$. Then $\bigcap_{k=1}^{\infty} K_k$ is singleton

THEOREM 36

Every convergent sequence is Cauchy (any metric space)

THEOREM 37

(i) Cauchy sequences in compact metric spaces converge in the same space

(ii) In R^k, every cauchy seq converges

Definition: Metric spaces where all cauchy sequences converge are known as complete metric spaces

Definition: A sequence $\{s_n\}$ of reals is monotonically increasing if $s_n \leq s_{n+1}$ $\forall$ $n \in \mathbb{N}$ (similarly monotonically decreasing)

THEOREM 38

Monotonic sequences converge iff they are bounded

Definition: Suppose $\{s_n\} \subset \mathbb{R}$ s.t. $\forall$ $M \in \mathbb{R}$, $\exists$ $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n \geq M$. we then write $s_n \to +\infty$. similarly with $\geq$ replaced by $\leq$ in $s_n \geq M$, we say $s_n \to -\infty$

Definition: Let E be the set of all subsequential limits including $+\infty$, $-\infty$ for a sequence $\{s_n\}$. The upper and lower limits of $\{s_n\}$ are denoted s^* and s_* and defined as

$$s^* := \sup E =: \limsup_{n\to\infty} s_n \quad , \quad s_* := \inf E =: \liminf_{n\to\infty} s_n$$

(s^* or $\limsup_{n\to\infty} s_n$ are both notations, $\sup E$ is definition)

THEOREM 39

(i) $s^* \in E$

(ii) $x > s^* \Rightarrow \exists$ $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n < x$

Further, s^* is unique with the above properties

THEOREM 40

$s_n \leq t_n$ for all $n \geq N$ (some fixed $N \in \mathbb{N}$)

Then $\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n$

$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n$

THEOREM 41

(i) $p > 0 \Rightarrow \lim_{n \to \infty} \frac{1}{n^p} = 0$

(ii) $p > 0 \Rightarrow \lim_{n \to \infty} p^{1/n} = 1$

(iii) $\lim_{n \to \infty} n^{1/n} = 1$

(iv) $p > 0,\ \alpha \in \mathbb{R} \Rightarrow \lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$

(v) $|x| < 1 \Rightarrow \lim_{n \to \infty} x^n = 0$

Definition: for a given sequence $\{a_n\}$, we associate it to another sequence $\{b_n\}$ called the series of $\{a_n\}$ defined as $b_n = \sum_{k=1}^{n} a_k$. To denote $\{b_n\}$, we write $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

THEOREM 42

$\sum a_n$ converges iff $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$

In particular, at $m=n$, $\sum a_n$ converges $\Rightarrow \lim_{n \to \infty} a_n = 0$

THEOREM 43

A non-negative series converges iff its partial sums form a bounded seq

THEOREM 44 (Comparison test)

(i) $|a_n| \leq c_n \ \forall n \geq N_0$. If Σc_n converges, then Σa_n converges

(ii) $a_n \geq d_n \geq 0 \ \forall n \geq N_0$, If Σd_n diverges, then Σa_n diverges

THEOREM 45

For $0 \leq x < 1$, $\Sigma x^n \to \frac{1}{1-x}$ i.e. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Further, for $x \geq 1$, the series diverges

THEOREM 46

(i) $\Sigma \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

(ii) If $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges and if $p \leq 1$ then diverges

(iii) Σa_n converges iff $\Sigma 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \dots$ converges

Definition: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

THEOREM 47

(i) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

(ii) e is irrational

(iii) e is not algebraic

THEOREM 48 (Root test)

Let $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$

$\alpha < 1 \Rightarrow \Sigma a_n$ converges

$\alpha > 1 \Rightarrow \Sigma a_n$ diverges

THEOREM 49 (Ratio test)

$\sum a_n$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$

for every $n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$

THEOREM 50 (Root test is stronger)

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Definition: Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series

THEOREM 51

Given $\{c_n\}$ a complex sequence, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$ ($\alpha = 0 \Rightarrow R = +\infty$, $\alpha = +\infty \Rightarrow R = 0$). Then the power series $\sum c_n z^n$ converges if $|z| < R$, diverges if $|z| > R$

THEOREM 52

(i) Let $A_n = \sum_{k=0}^{n} a_k$, $A_{-1} := 0$. Then for $0 \leq p \leq q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

(ii) Suppose $A_n = \sum a_n$ form a bounded sequence, $b_0 \geq b_1 \geq \dots$ is a monotonically decreasing sequence, $\lim_{n\to\infty} b_n = 0$. Then $\sum a_n b_n$ converges

THEOREM 53

Suppose $|c_1| \geq |c_2| \geq |c_3| \geq \dots$, $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ ($m \in \mathbb{N}$), $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n$ converges

THEOREM 54

Suppose $R = 1$ for $\sum c_n z^n$ and $c_0 \geq c_1 \geq \dots$ with $c_n \to 0$.

Then $\sum c_n z^n$ converges on $|z| = 1$ except possibly at $z = 1$

Definition: $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

THEOREM 55

absolute convergence $\Rightarrow$ convergence

THEOREM 56

$\sum a_n \to A$, $\sum b_n \to B$ $\Rightarrow$ $\sum a_n + b_n \to A + B$ and

$\sum c a_n \to cA$ for $c \in X$

Definition: given $\sum a_n$, $\sum b_n$, the cauchy product is a sequence $\{c_n\}$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

THEOREM 57 (Merten's theorem)

Suppose $\sum a_n$ converges absolutely, $\sum a_n = A$, $\sum b_n = B$,

c_n is the cauchy product of a_n, b_n, then,

$\sum c_n$ converges to AB.

Further, if $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C

and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$

Definition: $\{k_n\}$ is a sequence in which every positive integer appears exactly once (a bijection from $\mathbb{N}$ to $\mathbb{N}$). For a sequence $\{a_n\}$, the sequence $\{a_{k_n}\}$ is a 'rearrangement'

THEOREM 58 (Riemann)

For Σa_n which converges but not absolutely, there exists a rearrangement a_n' such that if s_n' is the sequence of its partial sums,

$$\liminf_{n \to \infty} s_n' = \alpha, \quad \limsup_{n \to \infty} s_n' = \beta \quad \text{for}$$

any $\alpha, \beta \in \mathbb{R} \cup \{+\infty, -\infty\}$

THEOREM 59

If Σa_n converges absolutely, any rearrangement converges to the same sum

Definition: Let X, Y be metric spaces and $E \subset X$. Let $f: E \to Y$ and p be a limit point of E. We say $\lim_{x \to p} f(x) = q$ if $\exists\, q \in Y$ such that $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $d_Y(f(x), q) < \varepsilon$ whenever $x \in E$ and $0 < d_X(x, p) < \delta$

THEOREM 60

In accordance with the above definition,

$\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E s.t. $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

THEOREM 61

$\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$

$\Rightarrow$ (i) $\lim_{x \to p} (f+g)(x) = A + B$

(ii) $\lim_{x \to p} (fg)(x) = AB$

(iii) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ provided $B \neq 0$

Definition: $f: E \to Y$ is said to be **continuous** at p if for every $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $d_Y(f(x), f(p)) < \varepsilon$ $\forall\ x \in E$ such that $d_X(x, p) < \delta$. If f is continuous at every point of E, then f is said to be continuous on E

THEOREM 62

Composition of continuous functions is continuous. i.e. If f is continuous at p, g is continuous at $f(p)$, then $g \circ f$ is continuous at p.

THEOREM 63

$f: X \to Y$ is continuous iff $f^{-1}(V)$ is open in X for every open V in Y

THEOREM 64

Let f, g be continuous. Then $f+g$, $f \cdot g$, $\frac{f}{g}$ all are continuous (in the last case, provided $g(x) \neq 0$ $\forall\ x$)

Definition: A mapping f of a set E into $\mathbb{R}^k$ is said to be **bounded** if there is a real number M s.t. $|f(x)| \leq M$ $\forall\ x \in E$

THEOREM 65

Let $f: X \to Y$ be continuous. $Im(f) = f(X)$ is compact. If X is a compact metric space

THEOREM 66

Let f be a continuous real valued function on a compact metric space X and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exist $p, q \in X$ s.t. $f(p) = M$, $f(q) = M$

THEOREM 67

inverse of a continuous 1-1 map $f : X \to Y$, where X is compact, is continuous.

Definition: Let $f : X \to Y$. We say f is <u>uniformly continuous</u> on X if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $d_Y(f(p), f(q)) < \varepsilon$ $\forall$ $p, q \in X$ s.t. $d_X(p, q) < \delta$

THEOREM 68

All uniformly continuous functions are continuous and the converse is true only on compact spaces.

THEOREM 69

Let E be a non compact set in $\mathbb{R}$

(i) there exists a continuous function on E which is not bounded

(ii) there exists a continuous & bounded function which has no maximum

(iii) If E is also bounded, $\exists$ a continuous function which is not uniformly continuous

THEOREM 70

continuous maps send connected sets to connected sets (as image)

THEOREM 71 (Intermediate value theorem)

Let f be continuous on $[a,b]$. If $f(a) < f(b)$ and c is such that $f(a) < c < f(b)$, then $\exists$ $x \in [a,b]$ such that $f(x) = c$

Definition: Let f be defined on (a,b). We write $f(x+) = q$ for $x \in [a,b)$ if $f(t_n) \to q$ as $n \to \infty$ $\forall$ $\{t_n\}$ in (x,b) s.t. $t_n \to x$

Definition: Let f be defined on (a,b). If f is discontinuous at some $x \in (a,b)$ and $f(x+)$ and $f(x-)$ exist, then f is said to have a simple discontinuity (or discontinuity of the first kind). The other types of discontinuities are discontinuities of second kind

THEOREM 72 (Structure of simple discontinuities)

$\lim_{t \to x} f(t)$ exists iff $f(x+) = f(x-) = \lim_{t \to x} f(t)$.

Further, discontinuities of the first kind can only arise if $f(x+) \neq f(x-)$ or $f(x+) = f(x-) \neq f(x)$

Definition: Let f be real on (a,b). f is said to be monotonically increasing on (a,b) if $a < x < y < b \Rightarrow f(x) \leq f(y)$ (similarly decreasing by reversing the last inequality)

THEOREM 73

For a monotonically increasing (decreasing) function on (a,b), $f(x+)$, $f(x-)$ exist $\forall x \in (a,b)$. Further,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

(Reverse inequalities & interchange 'sup', 'inf' for monotonically decreasing)

Further, for $a < x < y < b$,

$$f(x+) \leq f(y-) \qquad (\text{resp. } f(x+) \geq f(y-) \text{ for decr func})$$

THEOREM 74 (Discontinuities of monotonic functions)

Monotonic functions do not have discontinuities of the second kind and further if f is monotonic on (a,b), the set of points of (a,b) at which f is discontinuous is at most countable

Definition: We define a neighbourhood of $+\infty$ as $(c,+\infty)$ for any $c \in \mathbb{R}$ and similarly for any $a \in \mathbb{R}$, $(-\infty, a)$ is a neighbourhood of $-\infty$

Definition: Let f be a real function defined on $E \subset \mathbb{R}$. We say that $f(t) \to A$ as $t \to x$ where A, x are in $\mathbb{R} \cup \{+\infty, -\infty\}$, if for every neighbourhood U of A, $\exists$ neighbourhood V of x such that $V \cap E \neq \phi$ and $f(t) \in U$ $\forall$ $t \in (V \cap E) \setminus \{x\}$

Definition: Let f be defined on $[a,b]$. For any $x \in [a,b]$, we define $\phi(t) = \dfrac{f(t) - f(x)}{t - x}$ $\forall$ $t \in (a,b) \setminus \{x\}$.

We define $f'(x)$, the derivative of f at x as

$\lim_{t \to x} \phi(t)$ (if it exists)

THEOREM 75

If f is differentiable at x, it is continuous at x

THEOREM 75

f, g and differentiable at $x \in [a,b] \Rightarrow f+g$, fg, f/g are, too; with $(f+g)' = f' + g'$, $(fg)' = f'g + g'f$ and

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

(provided $g(x) \neq 0$ for the $1/g$ case)

THEOREM 76 (Chain Rule)

Let f be cont on $[a,b]$ and $f'(x)$ exists at some $x \in [a,b]$. g is defined on $I \supset \text{Range}(f)$ and g is differentiable at $f(x)$. Then, $h(t) = g(f(t)) \ \forall\ t \in [a,b]$ is diff at x with $h'(x) = g'(f(x)) \cdot f'(x)$

Definition: We say a function has a local maximum at a point p if $\exists\ \delta > 0$ such that $f(q) \leq f(p)$ for all q with $d(p,q) < \delta$ (f is defined on X, $p,q \in X$)

(Similarly local minima)

THEOREM 77 (Fermat's Extremum theorem)

If f (defined on $[a,b]$) has a local extrema (minima or maxima) at $x \in (a,b)$ and if $f'(x)$ exists, then $f'(x) = 0$

THEOREM 78 (Mean Value theorem - generalised)

If $f, g \in C[a,b]$ (continuous on $[a,b]$) and are diff in (a,b), then $\exists\ x \in (a,b)$ s.t.

$$(f(b) - f(a))\, g'(x) = f'(x)(g(b) - g(a))$$

Note: Using $g(x) = x$ gives the common version of the MVT (Lagranges MVT)

THEOREM 79 (Rolle's theorem)

If f is cont in $[a,b]$ and diff on (a,b), ~~then $\exists x \in (a,b)$ such that $f(b) - f$~~ and $f(a) = f(b)$, then,

$\exists\ x \in (a,b)$ s.t $f'(x) = 0$

THEOREM 80

Let f be differentiable in (a,b)

(i) $f'(x) \geq 0 \ \forall\ x \in (a,b) \Rightarrow f$ is monotonically increasing

(ii) $f'(x) \leq 0 \ \forall\ x \in (a,b) \Rightarrow f$ is monotonically decreasing

(iii) $f'(x) = 0 \ \forall\ x \in (a,b) \Rightarrow f$ is constant on (a,b)

THEOREM 81 (Darboux's theorem)

Suppose f is a real diff function on $[a,b]$ and $f'(a) < \lambda < f'(b)$. Then $\exists\ x \in (a,b)$ s.t. $f'(x) = \lambda$

(Note: It is NOT the IVT (theorem 71) since we never required f' to be continuous)

THEOREM 82

If f is diff on $[a,b]$, f' cannot have any simple discontinuities on $[a,b]$

THEOREM 83 (L'hospital's rule)

Let f, g be diff in (a,b) and $g'(x) \neq 0 \ \forall\ x \in (a,b)$

(Note: $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and assume WLOG $a < b$)

Suppose $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$

If $f(x) \to 0$, $g(x) \to 0$ as $x \to 0$ or also if $g(x) \to +\infty$ as $x \to a$,

then $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$

Definition: $f^{(n)}(x)$ is defined to be the n^{th} derivative of f at x where the process is defined inductively, that is, the n^{th} derivative of x is the derivative of $f^{(n-1)}(x)$ and $f^{(1)}(x) = f'(x)$

THEOREM 84 (Taylor's theorem)

Let f be real on $[a,b]$ and $f^{(n-1)}$ exist and be cont on $[a,b]$. Let $f^{(n)}$ exist on (a,b). For α, β, two distinct points in $[a,b]$, if

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k,$$

$\exists$ $c \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$$

Definition: Let $[a,b]$ be an interval. A partition P of $[a,b]$ is a finite set of points $x_0, x_1, \dots, x_n$ where

$$a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$$

We also further denote $\Delta x_i := x_i - x_{i-1}$ $\forall$ $i = 1, 2, \dots, n$

We denote
$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$$
$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \quad \forall\ i = 1, 2, \dots, n$$
(for f bounded on $[a,b]$)

Definition: We define the upper sum of f corresponding to a partition P of $[a,b]$ as $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$ and correspondingly lower sum $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$

Definition: We define the upper and lower Riemann integrals of a bounded function f on $[a,b]$ as

$$U(f) = \inf_P U(f,P)$$
$$L(f) = \sup_P L(f,P)$$

Definition: We say a bounded function f defined on $[a,b]$ is Riemann integrable if $L(f) = U(f)$ and we define

$$\int_{[a,b]} f = \int_a^b f = L(f) = U(f)$$

Definition: A partition P' is a refinement of P if every point in P is contained in P'

THEOREM 85

Given any partitions P, Q we can find a partition R which is a refinement of both P, Q

THEOREM 86

For any two partitions P, Q of $[a,b]$,

$$L(f,P) \le U(f,Q)$$

Further, for a refinement P' of P,

$$m(b-a) \le L(f,P) \le L(f,P') \le U(f,P') \le U(f,P) \le M(b-a)$$

where $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$

Remark: inf, sup may be replaced by min, max since $[a,b]$ is compact & inf, sup are achieved

Definition: Let α be a monotonic function on $[a,b]$ (Take increasing for convenience). Let $P = \{x_0, \dots, x_n\}$ be a partition

Let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$

We define
$$U(f, P, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$L(f, P, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

We define
$$U(f, \alpha) = \inf_P U(f, P, \alpha)$$
$$L(f, \alpha) = \sup_P L(f, P, \alpha)$$

If $U(f,\alpha) = L(f,\alpha) = \lambda$, we say that f is Riemann Stieltjes integrable and write
$$\int_a^b f \, d\alpha = \int_a^b f(x) \, d\alpha(x) = \lambda$$

Note: f is bounded on $[a,b]$

THEOREM 87

In accordance with the above,
$$L(f, P, \alpha) \leq L(f, P', \alpha)$$
$$U(f, P', \alpha) \leq U(f, P, \alpha)$$
$$L(f, P_1, \alpha) \leq U(f, P_2, \alpha)$$

where P, P_1, P_2 are any ~~of~~ partitions of $[a,b]$ and P' is a refinement of P

THEOREM 88

$f \in \mathcal{R}(\alpha)$ (Riemann-Stieltjes wrt α) iff $\forall \varepsilon > 0$,

$\exists P$ s.t. $U(f, P, \alpha) - L(f, P, \alpha) < \varepsilon$

THEOREM 89

f is continuous on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$

THEOREM 90

f is monotonic $\Rightarrow$ f is integrable

f is monotonic, α is continuous $\Rightarrow f \in \mathcal{R}(\alpha)$
(and α monotonic by defn)

THEOREM 91

Suppose f is bounded on $[a,b]$, has finitely many points of discontinuity & α is continuous at every point of discontinuity of f, then $f \in \mathcal{R}(\alpha)$

THEOREM 92

Suppose $f \in \mathcal{R}(\alpha)$, $m \le f \le M$, ϕ is cont on $[m,M]$ and $h(t) = \phi(f(t))$ on $[a,b]$, then $h \in \mathcal{R}(\alpha)$

THEOREM 93 (Properties of integral)

(i) $f_1, f_2 \in \mathcal{R}(\alpha) \Rightarrow f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf_1 \in \mathcal{R}(\alpha)$ for every constant c. In fact $\int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha = \int_a^b (f_1+f_2)\, d\alpha$ and $\int_a^b (cf)\, d\alpha = c\int_a^b f\, d\alpha$

(ii) $f_1 \le f_2$ on $[a,b] \Rightarrow \int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$

(iii) $f \in \mathcal{R}(\alpha)$ on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,c]$, $[c,b]$

and $\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$

(iv) $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f| \le M$ on $[a,b]$,

$\left|\int_a^b f\, d\alpha\right| \le M[\alpha(b) - \alpha(a)]$

(v) $\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$, $\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$

THEOREM 94

$f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$ on $[a,b]$ $\Rightarrow$

(i) $fg \in \mathscr{R}(\alpha)$

(ii) $|f| \in \mathscr{R}(\alpha)$ with $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$

Definition: The unit step function I is defined as

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

(a.k.a. Heaviside function)

THEOREM 95

If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$, then,

$$\int_a^b f \, d\alpha = f(s)$$

THEOREM 96

Let $c_n \geq 0 \ \forall n \in \mathbb{N}$, $\sum c_n$ converges. Let $\{s_n\}$ be a sequence in (a,b) (of distinct points).

Let $\alpha(x) := \sum_{n=1}^{\infty} c_n I(x - s_n)$

Let f be continuous on (a,b)

Then $\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

THEOREM 97

just riemann integrable (ie. $\alpha(x) = x$)

Let α increase monotonically and let $\alpha' \in \mathscr{R}$. Let f be a bounded real function on $[a,b]$. Then $f \in \mathscr{R}$ iff $f\alpha' \in \mathscr{R}$ and in that case, $\int_a^b f \, d\alpha = \int_a^b f(x) \alpha'(x) \, dx$

THEOREM 98 (Change of variable / substitution)

Let φ be a strictly increasing continuous function that maps $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define

$$\beta(y) = \alpha(\varphi(y)), \quad g(y) = f(\varphi(y)) \quad \forall\, y \in [A, B]$$

Then, $g \in \mathcal{R}(\beta)$ on $[A, B]$ with

$$\int_A^B g \, d\beta = \int_a^b f \, d\alpha$$

THEOREM 99 (FTC 1)

Let $f \in \mathcal{R}$ on $[a, b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t) \, dt.$$

Then F is unif cont on $[a, b]$ and further, if f is cont at $x_0 \in [a, b]$, F is diff at x_0 with

$$F'(x_0) = f(x_0)$$

THEOREM 100 (FTC 2)

Let $f \in \mathcal{R}$ on $[a, b]$, let F be a differentiable function on $[a, b]$ such that $F' = f$

Then $\displaystyle\int_a^b f(x) \, dx = F(b) - F(a)$

THEOREM 101 (Integration by parts)

Let F, G be differentiable on $[a, b]$ with $f = F' \in \mathcal{R}$ and $g = G' \in \mathcal{R}$. Then

$$\int_a^b F(x) G(x) \, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x) \, dx$$

Definition: A continuous mapping $\gamma : [a,b] \to \mathbb{R}^k$ is called as a curve in $\mathbb{R}^k$. If γ is one-one, we call it an arc and if $\gamma(a) = \gamma(b)$, we say closed curve

Definition: Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$ and γ be a curve in $\mathbb{R}^k$ over $[a,b]$. We associate a number to γ, P denoted $\Lambda(\gamma, P)$ which is defined as

$$\Lambda(\gamma, P) = \sum_{i=1}^{n} \| \gamma(x_i) - \gamma(x_{i-1}) \|_k$$

We define the length of γ as $\sup_P \Lambda(P, \gamma) = \Lambda(\gamma)$

Definition: If $\Lambda(\gamma) < +\infty$, we say γ is rectifiable

THEOREM 102

If γ' exists and is continuous on $[a,b]$, then γ is rectifiable with $\Lambda(\gamma) = \int_a^b \| \gamma'(t) \|_k \, dt$

Definition: Let $\{f_n\}$ be a sequence of functions defined on a set E. Suppose $\forall x \in E$, $\{f_n(x)\}$ converges, we define

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad \forall x \in E$$

We say $\{f_n\}$ converges pointwise to f on E

Suppose $\sum f_n(x)$ converges $\forall x \in E$, we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \forall x \in E$$

We say f is the sum function of the series $\sum f_n$

Definition: A sequence of functions $\{f_n\}$ converges uniformly on E to a function f if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in E$

We say $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $\{s_n\}$ given by $s_n(x) = \sum_{i=1}^{n} f_n(x)$ converges uniformly on E

THEOREM 103 (Cauchy criterion for uniform convergence)

$\{f_n\} \subset E$ converges uniformly on E iff $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $m, n \geq N$ and $x \in E \Rightarrow |f_n(x) - f_m(x)| \leq \varepsilon$

THEOREM 104

Suppose $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in E$. ~~Put M_n~~

Set $M_n = \sup_{x \in E} |f_n(x) - f(x)|$

Then $f_n \to f$ uniformly iff $M_n \to 0$. as $n \to \infty$

THEOREM 105 (Weirstrass test)

Let $\{f_n\} \subset E$. Suppose $|f_n(x)| \leq M_n \quad \forall x \in E, \quad n = 1, 2, \cdots$.

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges

THEOREM 106 (swap lim, lim)

Let $f_n \xrightarrow{\text{unif}} f$ on E (in a complete metric space). Let x be a limit point of E. Let $\lim_{t \to x} f_n(t) = A_n$. Then $\{A_n\}$ converges with $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$

i.e. $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t)$

THEOREM 107

If $\{f_n\}$ is a seq of continuous functions which converges uniformly to f, then f is continuous (everything in some set E)

THEOREM 108

Suppose K is compact and the following hold,

(i) $\{f_n\}$ is a seq of cont. functions on K

(ii) $\{f_n\}$ converges pointwise to a cont function f on K

(iii) $f_n(x) \geq f_{n+1}(x) \quad \forall x \in K, \; n = 1, 2, \ldots$

Then $f_n \to f$ uniformly on K (in some sense, converse of 107)

Definition: For a metric space X, let $\mathscr{C}(X)$ denote the set of all complex valued continuous bounded functions having domain X

Further, associate a norm to each $f \in \mathscr{C}(X)$ as follows

$\|f\| = \sup_{x \in X} |f(x)|$. This turns $\mathscr{C}(X)$ into a metric space

THEOREM 109

$(\mathscr{C}(X), \|\cdot\|)$ is a complete metric space

THEOREM 110 (swap lim, integral)

Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ $\forall n$ and $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ with $\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$

i.e. $$\int_a^b \lim_{n \to \infty} f_n(x) \, d\alpha = \lim_{n \to \infty} \int_a^b f_n(x) \, d\alpha$$

THEOREM 111 (swap sum, integral)

If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$

with series conv. uniformly on $[a,b]$, then,

$$\int_a^b f(x)\, d\alpha = \int_a^b \sum_{n=1}^{\infty} f_n(x)\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, d\alpha$$

THEOREM 112 (swap derivative, lim)

Let $\{f_n\}$ be a seq of functions that are diff on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

THEOREM 113

An example from the set of Weirstrass functions,

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) \quad \text{is continuous } \forall x \in \mathbb{R}$$

but not differentiable for any $x \in \mathbb{R}$

where $\varphi(x) = |x| \quad \forall x \in [-1,1]$

and $\varphi(x+2) = \varphi(x) \quad \forall x \in \mathbb{R}$

Definition: For a sequence $\{f_n\}$ on E, we say

(i) $\{f_n\}$ is pointwise bounded if $\{f_n(x)\}$ is bounded $\forall x \in E$

(ii) $\{f_n\}$ is uniformly bounded if $\exists M$ s.t. $|f_n(x)| < M \;\forall x \in E, \forall n \in \mathbb{N}$

THEOREM 114

If $\{f_n\}$ is pointwise bounded on E, then we can find a pointwise convergent subsequence $\{f_{n_k}\}$ on a countable subset $E_1 \subset E$ (directly from Bolzano-Weirstrass)

THEOREM* 115

(i) Even if $\{f_n\}$ is uniformly bounded on a compact set E and all f_n are continuous, there still need not exist a subsequence which converges pointwise on E

(ii) Even if we have a pointwise convergent seq $\{f_n\}$ which is uniformly bounded on a compact set E, we may not be able to find a uniformly convergent subsequence

<u>Definition</u>: A family $\mathcal{F}$ of complex functions f defined on E in a metric space X is said to be <u>equicontinuous</u> on E if for every $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ $\forall\ x \in E, y \in E, f \in \mathcal{F}$ (every member of $\mathcal{F}$ is uniformly continuous)

THEOREM 115

If $\{f_n\}$ is a pointwise bounded seq on E, we can find a pointwise convergent subsequence $\{f_{n_k}\}$ on E (extension of theorem 114 from E_1 to E)

* usually, negative results are not theorems (they are remarks)

THEOREM 116

cont, bounded functions
of

If K is compact metric space and $f_n \in \mathcal{C}(K)$ $\forall n \in \mathbb{N}$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous

THEOREM 117

If K is compact, $f_n \in \mathcal{C}(K)$ $\forall n \in \mathbb{N}$ and if $\{f_n\}$ is pointwise bounded & equicontinuous then

(i) $\{f_n\}$ is uniformly bounded on K

(ii) $\{f_n\}$ admits a uniformly convergent subsequence

THEOREM 118 (Stone - Weirstrass Theorem - OG form)

Let f be a continuous complex valued function defined on $[a,b]$. Then $\exists$ P_n (a seq. of polynomials) such that $\lim_{n\to\infty} P_n = f(x)$ uniformly on $[a,b]$

Note: Further, if f is real, P_n's may be chosen real

THEOREM 119

For every $[-a,a] \subsetneq \mathbb{R}$, $\exists$ seq of real polynomials P_n s.t. $P_n(0) = 0$ and $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$

Definition: A family $\mathcal{A}$ of complex functions on E is said to be an algebra if $\mathcal{A}$ is closed under addition, product and scalar multiplication (in $\mathbb{C}$)

Definition : If $\mathcal{A}$ is an algebra of polynomials such that if $f_n \in \mathcal{A}\ \forall n$ and $f_n \xrightarrow{unif} f$, then $f \in \mathcal{A}$, we say $\mathcal{A}$ is uniformly closed

THEOREM 120

Let $\mathcal{B}$ be the uniform closure of $\mathcal{A}$ (an algebra of bounded functions). Then $\mathcal{B}$ is uniformly closed

Definition : $\mathcal{A}$ is said to separate points on E if for every $(x_1, x_2) \in E \times E$ such that $x_1 \neq x_2$, there is a corresponding function $f \in \mathcal{A}$ s.t. $f(x_1) \neq f(x_2)$

Definition : If $\forall\, x \in E$, $\exists\, g \in \mathcal{A}$ s.t. $g(x) \neq 0$, we say $\mathcal{A}$ vanishes at no point of E

THEOREM 121

Suppose $\mathcal{A}$ is an algebra on E, $\mathcal{A}$ separates points of E and $\mathcal{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, c_1, c_2 are scalars in $\mathbb{C}$, then $\mathcal{A}$ contains a function f such that

$$f(x_1) = c_1 \quad , \quad f(x_2) = c_2$$

~~THEOREM 122 (Stone-Weierstrass - Stone's generalisation)~~

~~Let $\mathcal{A}$ be a real cont algebra of a compact set K. If $\mathcal{A}$ separates points of K & doesn't vanish on K, then $\bar{\mathcal{A}}$ consists of~~

THEOREM 122 (Generalised Stone-Weirstrass)

Let $\mathcal{A}$ be an algebra of real, continuous functions over a compact set K. If $\mathcal{A}$ separates points of K and vanishes at no point of K, then the uniform closure of $\mathcal{A}$, contains all real continuous functions on K

Highlights of proof :

Step 1 : $f \in \mathcal{B}$ (uniform closure of $\mathcal{A}$) $\Rightarrow |f| \in \mathcal{B}$

Step 2 : $f, g \in \mathcal{B} \Rightarrow \max(f,g), \min(f,g) \in \mathcal{B}$

where $\max(f,g) = \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & f(x) < g(x) \end{cases}$

Step 3 : Given a real cont f (on K) and $x \in K$ and any $\varepsilon > 0$, $\exists\ g_x \in \mathcal{B}$ s.t. $g_x(x) = f(x)$ and $g_x(t) > f(t) - \varepsilon \quad (\forall t \in K)$

Step 4 : Given a real cont f (on K) and $\varepsilon > 0$, $\exists\ h \in \mathcal{B}$ such that $|h(x) - f(x)| < \varepsilon \quad \forall x \in K$

(note : Step 4 is equivalent to the theorem)

(note : This theorem doesn't hold for complex algebras)

THEOREM 123

Suppose $\mathcal{A}$ is a self adjoint algebra (i.e. $f \in \mathcal{A} \Leftrightarrow \bar{f} \in \mathcal{A}$) then the generalised stone weierstrass theorem (122) holds for $\mathcal{A}$

complex conjugate