6th Sep, 2024

Hillman - Grassl Bijection

Om Joglekar

Definition : A reverse plane partition of shape λ is a tableau of shape λ such that rows & colums are both weakly increasing

Notation : Given λ, by r_n we mean the number of reverse plane partitions of shape λ with sum of all entries equal to n

Proposition 1

Reverse plane partitions of λ are in 1:1 correspondence with SSYT

proof

Given an SSYT of shape λ, consider the sum of all entries & call this sum K

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$

Consider the rpp obtained from the SSYT replacing each x_{ij} by $x_{ij} - i$ (i is row number)

$(1 \le i \le m, \ 1 \le j \le \lambda_i)$

Rows still remain weakly decreasing.

Consider a column $y_1 < y_2 < \dots < y_p$

Then $y_1 - 1 \le y_2 - 2 \le y_3 - 3 \le \dots \le y_p - p$ & hence the resulting tableau is an rpp.

Conversely, add i to x_{ij} in rpp to get SSYT

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Corollary 2

Let q_n denote no. of SSYT of shape λ (λ fixed initially) such that sum of entries in n

Then, $\sum_{n=0}^{\infty} q_n x^n = x^K \sum_{n=0}^{\infty} r_n x^n$

where $K = \lambda_1 + 2\lambda_2 + \dots + n\lambda_n$

Definition : Consider the ferrer shape of $\lambda = (\lambda_1, \cdots, \lambda_m) \vdash n$

The positions are given by $x_{i, j(i)}$ where $1 \le i \le m$ and $1 \le j(i) \le \lambda_i$. Let us, for convenience, say j instead of $j(i)$

By the hook length of the cell $(i, j(i))$, we mean

$h_{ij} = (\lambda_i - j) + (m - i) + 1$

Example $\lambda = (5, 4, 4, 3, 1, 1, 1)$. Find h_{23}, h_{41}, h_{42}

$$\begin{array}{ccccc} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \\ \times & \times & \times & \times & \\ \times & \times & \times & & \\ \times & & & & \\ \times & & & & \\ \times & & & & \end{array}$$

$h_{23} = 4$

$h_{41} = 6$

$h_{42} = 2$

Theorem 3 (Hillmann Grasl Algorithm)

$$\sum_{n=0}^{\infty} r_n x^n = \prod_{(i,j) \in \lambda} \frac{1}{1 - x^{h_{ij}}}$$

where h_{ij} is the hook length associated with x_{ij}

Proof

We give a bijection between plane partitions whose sum of entries is n and sequence $(h_{i_1 j_1}, h_{i_2 j_2}, \cdots)$ of hook lengths whose sum is n

Given an rpp T, we give an algorithm to get a sequence of hook lengths

Start at the top rightmost non zero entry $\underset{\wedge}{}$ $(1, \lambda_1)$ initially. Move left if the entry is equal; else move down. We get a path P

Subtract 1 from each entry of the path P.

Claim : We get another rpp after subtraction

proof : Well, rows remain weakly increasing clearly. Suppose in some column j, $a_{ij} > a_{i+1,j}$ then the path would have moved as $\cdots \to a_{i+1, j+1} \longrightarrow a_{i+1, j} \to \cdots$

Now $a_{i+1,j} < a_{ij} \le a_{i,j+1} \le a_{i+1,j+1} \Rightarrow a_{i+1,j} < a_{i+1,j+1}$

and the path couldnt have moved left. Contradiction!

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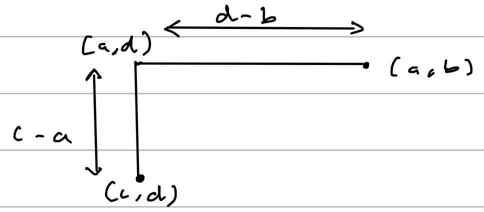
Claim : If P starts at (a,b) and ends at (c,d) then the no. of 1's subtracted is h_{ad}

proof : we just have a path using only S & W steps and no. of 1's subtracted = no. of points on path = length of path + 1 = $d-b + c-a + 1$

Since there is no block below the (c,d) block (else path would continue), $c-a$ is the leg length of the block (a,d).

Since (a,b) is the last block in the row, $c-a$ is the arm length & hence $d-b+c-a+1 = h_{a,d}$

Pictorially,



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Now it is clear that we can keep repeating the process and whittle down all entries to 0's.

Clearly, sum of the extracted hook lengths is n

(Note: If T is entirely zero to begin with, it corresponds to the constant term 1 in the RHS product)

Conversely, we build T from a sequence of hook lengths

lemma : In the decomposition of T into hook lengths, h_{ij} is removed before $h_{i'j'}$ iff $i < i'$ or $i = i', j \geq j'$

proof :

($\Leftarrow$) $i < i'$ directly implies h_{ij} is removed before $h_{i'j'}$ since h_{ij} has constant i until all entries of row i are 0

Suppose $i = i'$ & $j < j'$, we show that the path starting at (i,a) lies always to the left (weakly) of any path starting at (i,b) $\forall\, b > a$ (and hence, if the paths end at $(x_1, y_1), (x_2, y_2)$ we must have $y_1 < y_2$)

Suppose not, then we can find an intersection point (x,y) on both paths such that the first path moves S & the second path moves W

Since second path moved W, $(x,y) = (x, y-1)$

Equality holds even after subtracting the 1's

But then, the first path shouldnt have moved S.

($\Longrightarrow$) Suppose h_{ij} is removed before $h_{i'j'}$

Case 1 : $i < i'$ we are done

Case 2.1 : $i = i'$ $j \leq j'$ we are done

Case 2.2 : $i = i'$ $j > j'$ this leads to a contradiction because of the reasoning of paths lying weakly to the left

Case 3 ; $i > i'$ straight contradiction since as mentioned before i index remains constant until the entire row reduces to 0's

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We now describe how to construct the rpp.

Let $(h_{i_1 j_1}, h_{i_2 j_2}, \ldots)$ be a seq of hook lengths adding up to n

Start with the 0-tableau of shape λ.

Suppose we want to 'add' $h_{a,b}$ to our existing Tableau, we start at the last cell of column b & move E, N as follows : Move E if entry is equal else move north (even if there is no entry). Stop the path in the maximal possible way such that the path does NOT enter row $a+1$

Add 1's all along this path

It is clearly weakly increasing along columns by construction (since our path only moves N, E & inequality is preserved while moving N)

To claim that it is weakly increasing along rows, we must show that every path corresponding to h_{ab} ends at row a in the last box ie. (a, λ_a)

Claim : The path corresponding to $h_{a,b}$ ends at (a, λ_a)

proof : The proof goes by induction. Let π_k be the path for h_{i_k, j_k}. We wish to show that $(i_k, \lambda_{i_k}) \in \pi_k$. Suppose the seq of hook lengths is $(h_{i_1, j_1}, \ldots, h_{i_p, j_p})$, then by construction of π_p, since row i_p is all zeroes, we MUST reach its end.

Now for $k < p$, get π_k, π_{k+1}.

Induction hypothesis : π_{k+1} reaches end of Row i_{k+1}

By ordering of hook lengths, $i_k < i_{k+1}$ or $i_k = i_{k+1}$ & $j_k \geq j_{k+1}$

Suppose $i_k < i_{k+1}$, then row i_k has not been tampered with yet & is all 0's. Thus, r_k reaches the end.
Suppose $i_k = i_{k+1}$ & $j_k \geq j_{k+1}$, then by the argument we made earlier, r_k lies weakly to the right of r_{k+1} & since r_{k+1} reaches the end of row $i_k = i_{k+1}$, so does r_{k+1} ///

example

$$\begin{array}{ccccccc}
0 & 0 & 1 & 1 & 2 & 4 & 4 \\
0 & 1 & 2 & 2 & 3 & & \\
0 & 4 & 4 & 4 & 5 & & \\
3 & 5 & 6 & & & & \\
5 & 5 & & & & & \\
5 & & & & & & \\
6 & & & & & &
\end{array} \qquad h_{16}$$

$$\begin{array}{ccccccc}
0 & 0 & 1 & 1 & 2 & 3 & 3 \\
0 & 1 & 2 & 2 & 3 & & \\
0 & 4 & 4 & 4 & 5 & & \\
3 & 5 & 6 & & & & \\
5 & 5 & & & & & \\
5 & & & & & & \\
6 & & & & & &
\end{array} \qquad h_{16}$$

$$\begin{array}{ccccccc}
0 & 0 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 2 & 3 & & \\
0 & 4 & 4 & 4 & 5 & & \\
3 & 5 & 6 & & & & \\
5 & 5 & & & & & \\
5 & & & & & & \\
6 & & & & & &
\end{array} \qquad h_{15}$$

0	0	1	1	1	1	1
0	1	2	2	2		
0	4	4	4	4		
3	5	6				
5	5					
5						
6						

h_{11}

0	0	0	0	0	0	0
0	1	1	2	2		
0	3	3	4	4		
3	4	6				
4	4					
4						
5						

h_{24}

0	0	0	0	0	0	0
0	1	1	1	1		
0	3	3	3	4		
3	4	6				
4	4					
4						
5						

h_{21}

0	0	0	0	0	0	0
0	0	0	0	0		
0	2	3	3	4		
3	3	6				
3	3					
3						
4						

h_{35}

0 0 0 0 0 0 0

0 0 0 0 0

0 2 3 3 3

3 3 6

3 3

3

4

h_{33}

0 0 0 0 0 0 0

0 0 0 0 0

0 2 2 2 2

3 3 5

3 3

3

4

h_{31}

0 0 0 0 0 0 0

0 0 0 0 0

0 1 1 1 1

2 2 5

2 3

2

3

h_{31}

0 0 0 0 0 0 0

0 0 0 0 0

0 0 0 0 0

1 1 5 x4 times

1 3

1

2

h_{43}, h_{43}, h_{43}, h_{43}

o o o o o o o
o o o o o
o o o o o
1 1 1
1 3
1
2

h_{41}

o o o o o o o
o o o o o
o o o o o
o o o
o 3 ×3 times
o
1 ×1 time

h_{52}, h_{52}, h_{52}, h_{71}

o o o o o o o
o o o o o
o o o o o
o o o
o o
o
o

stop !

We have the sequence

h_{16} h_{16} h_{15} h_{11} h_{24} h_{21} h_{35} h_{33} h_{31} h_{31}
h_{43} h_{43} h_{43} h_{43} h_{41} h_{52} h_{52} h_{52} h_{71}

x x x x x x x
x x x x x
x x x x x
x x x
x x
x
x

Thus, the values are

2, 2, 5, 13, 3, 10, 1, 4, 9, 9, 1, 1, 1, 1, 6, 1, 1, 1, 1

The sum is : 72

The sum of all entries in the tableau is 72