Om Joglekar 3rd Feb, 2025

Connectness of $O(p,q)$

Recall that $O(p,q)$ is the set of $(p+q)\times(p+q)$ sized matrices A such that $A^T J A = J$ where $J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$

This group arises from the non-degenerate bilinear form $b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \longrightarrow \mathbb{R}$ given by

$b((x_1, \dots, x_{p+q}), (y_1, \dots, y_{p+q})) = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_{p+j} y_{p+j}$

We wish to analyse the connectedness of $O(p,q)$ (which gives us for free, the connectedness properties of $SO(p,q)$)

Notation :

For $g \in O(p,q)$, let us decompose $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A is $p\times p$ and D is $q \times q$. We shall refer to these blocks as A_g, B_g, C_g, D_g whenever necessary.

Proposition 1

For $g \in O(p,q)$, A, D are invertible

proof

From $g^T J g = J$, we get the following :

$A^TA - C^TC = I_p$, $D^TD - B^TB = I_q$, $A^TB = C^TD$

Suppose A isnt invertible, then $\exists\ x \neq 0$ s.t. $Ax = 0$ $(x \in \mathbb{R}^p)$

$\therefore\ C^TCx = -I_p x = -x$

C^TC is positive-semi definite & cannot have e-val -1. Contradiction.

Similarly D is also invertible

■

Proposition 2

If $g \in O(p,q)$, $\det(g) = \det(A) / \det(D)$

proof

$\det(g) = \det(A) \det(D - CA^{-1}B)$

But $D^TD - D^TCA^{-1}B = D^TD - (B^TA)(A^{-1}B) = D^TD - B^TB = I_q$

$\therefore\ \det g = \det(A) \det((D^T)^{-1}) = \det(A)/\det(D)$

■

Definition : We define the pseudo-euclidean space $\mathbb{R}(p,q)$ to be the space $\mathbb{R}^{p+q}$ equipped with the inner product given by
$\langle x, y \rangle = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_{p+j}\, y_{p+j}$
We define the pseudo unit sphere $S(p,q) = \{ x \in \mathbb{R}(p,q) \mid \langle x, x \rangle = 1 \}$

Note that for $q=0$, $\mathbb{R}(p,0)$ is just $\mathbb{R}^p$ with standard inner product and $S(p,0)$ is S^{p-1} (the p-dim sphere sitting in $\mathbb{R}^p$)

Proposition 3

$S(p,q) \cong \mathbb{R}^p \times S^{q-1}$ $\qquad\qquad \nu,\ n+1-\nu$

proof

$$
\begin{aligned}
S(p,q) &\cong \{ (x,y) \mid x \in \mathbb{R}^p,\ y \in \mathbb{R}^q,\ \|y\|^2 = 1 + \|x\|^2 \} \\
&= \{ (x, \sqrt{1+\|u\|^2}\, y) \mid u \in \mathbb{R}^p,\ \|y\| = 1 \} \\
&= \{ (x, \sqrt{1+\|u\|^2}\, y) \mid u \in \mathbb{R}^p,\ y \in S^{q-1} \} \\
&\cong \mathbb{R}^p \times S^{q-1}
\end{aligned}
$$

∎

Corollary 4

$S(p,q)$ is path connected provided $q \neq 1$

proof

$S(p,q)$ is path connected iff $\mathbb{R}^p$, S^{q-1} are so too.
$\mathbb{R}^p$ is path connected $\forall p$ but S^0 is not connected

∎

Definition : Let $S(p,q)$ be a pseudo sphere. At any $x \in S(p,q)$ we define the tangent space $T_x(S(p,q)) = \{ y \in \mathbb{R}^{p+q} \mid \langle x, y \rangle = 0 \}$

Definition : A frame at a point $x \in S(p,q)$ an ordered basis for the tangent space at x.

Definition : The frame bundle on $S(p,q)$ is just the 'collection' of all frames at all points defined as below
$F(S(p,q)) = \{ (x, e_1, \dots, e_{p+q-1}) \mid x \in S(p,q),\ e_i \in T_x(S(p,q)),\ \langle e_i, e_j \rangle = \varepsilon_i \delta_{ij} \}$
where $\varepsilon_i \in \{1, -1\}$ are chosen so that exactly $p-1$ of them are 1
One may define such concepts in general for manifolds

Proposition 5

$O(p,q) \cong F(S(p,q))$

proof

Let $g \in O(p,q)$

Consider the columns $g_1, g_2, \cdots, g_{p+q}$

We claim that $(g_1, (g_2, \cdots, g_{p+q})) \in F(S(p,q))$

Firstly, $g^T J g = J$. Comparing the $(1,1)^{th}$ entry on both sides gives $\langle g_1, g_1 \rangle = 1 \Rightarrow g_1 \in S(p,q)$

Now we want to show $\langle g_1, g_j \rangle = 0 \quad \forall j = 2, 3, \cdots$

This follows by comparing the $(1,j)^{th}$ entries on both sides

Further $\langle g_i, g_j \rangle = \varepsilon_i \delta_{ij} \quad \forall \; 2 \leq i, j \leq p+q$ follows by comparing $(i,j)^{th}$ entries on both sides

Further this is clearly continuous (we didnt even do any operation)

Next, if $(x, e_1, \cdots, e_{p+q-1}) \in F(S(p,q))$, construct the matrix g whose columns are $x, \cdots, e_{p+q-1}$ in that order

We claim $g \in O(p,q)$

Note that $h \in O(p,q)$ iff $\langle h_i, h_j \rangle = \varepsilon_i \delta_{ij}$ where exactly p of the ε_i are 1 and others are -1

$\langle x, x \rangle = 1 \Rightarrow$ first column is of unit length

$\langle g_i, g_j \rangle = \varepsilon_i \delta_{ij}$ where $p-1$ of the ε_i are 1 & rest are -1 (of course x isnt included here)

Further $\langle x, g_i \rangle = 0$

Thus $g \in O(p,q)$

This map is also continuous.

Thus we have a homeomorphism

$\blacksquare$

Definition: Let $g \in O(p,q)$. Based on the sign of $\det(A)$ and $\det D$, put g into one of the four subsets $O^{++}(p,q)$, $O^{+-}(p,q)$, $O^{-+}(p,q)$, $O^{--}(p,q)$

Proposition 6

All four of the above sets are clopen

proof

Since the functions $\det_1(g) = \det(A)$, $\det_2(g) = \det(D)$ are continuous, $O^{++}(p,q) = \det_1^{-1}((0,\infty)) \cap \det_2^{-1}((0,\infty))$ is open

but similarly, O^{--}, O^{+-}, O^{-+} are all open.
$(O^{++})^c = O^{--} \cup O^{+-} \cup O^{-+}$ is open
Thus O^{++} is closed $\Rightarrow$ O^{++} is clopen
Similarly all other subsets are clopen

■

Theorem 7

$O^{++}(p,q)$ is connected $\forall\ p,q \geq 1$

proof

The proof is by induction on $p+q$
For $p+q = 2$,

$$
\begin{aligned}
O^{++}(1,1) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,d > 0,\ ab = cd,\ a^2 - c^2 = d^2 - b^2 = 1 \right\} \\
&= \left\{ \begin{bmatrix} a & c \\ c & a \end{bmatrix} \,\middle|\, a > 0,\ a^2 - c^2 = 1 \right\} \\
&= \left\{ \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} \\
&\cong \mathbb{R}
\end{aligned}
$$

Thus $O^{++}(1,1)$ is connected
Assume the result holds for $p+q = 2, 3, \ldots, n$
Let $p+q = n+1$
It is easy to show $O(p,q) \cong O(q,p)$ by shuffling rows & their corresponding columns around.
Thus, WLOG we assume $p \leq {}^{n+1}/_{2}$
Recall the homeomorphism $O(p,q) \cong F(S(p,q))$ of prop 5.
Now $S(p,q)$ is connected by corollary 4.
We want a path from some fixed arbitrary $g \in O^{++}(p,q)$ to $I \in O^{++}(p,q)$
Consider the frame corresponding to g at g_1. Move this to a frame at e_1 by translating.
This is done by exploiting the path in $S(p,q)$ joining g_1 & e_1.
Correspondingly, back in $O(p,q)$, we get a path joining the matrix g to a matrix $\begin{bmatrix} 1 & * \\ 0 & B \end{bmatrix}$. Using that this is an element of $O(p,q)$, $*$ is immediately found to be 0
Note that the path remains in $O^{++}(p,q)$ since the maps $\det_i$ are continuous & as the entries change continuously to produce the end matrix, $\det_1$ & $\det_2$ start out positive and by continuity, cannot cross 0 to go over to negative.
$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \in O^{++}(p,q) \Rightarrow B \in O^{++}(p-1,q)$

But B can be joined to I in $O^{++}(p-1, q)$ by our induction hypothesis thus completing the proof

■

Corollary 8

(i) $SO(p,q) = O^{++}(p,q) \cup O^{--}(p,q)$

(ii) $O^{++}(p,q)$ is the connected component of I

(iii) $O^{++}(p,q), \dots, O^{--}(p,q)$ are the connected components of $O(p,q)$

Proof

(i) follows from proposition 2

(ii) O^{++} is clopen, connected & has I

(iii) Consider $d(\pm1, \pm1) = \operatorname{diag}(\pm1, 1, \dots, 1, \pm1)$ to be four different matrices of order $(p+q) \times (p+q)$

Note that each of them is in a different subset $O^{**}(p,q)$

Claim : $g \longmapsto d(\pm1, \pm1)g$ is continuous

This is clear since it just scales first & last rows by $\pm1, \pm1$

Thus O^{++} is diffeomorphic to O^{+-}, O^{-+}, O^{--} since these matrices are invertible.

■

Conclusion :

$O(p,q)$ has 4 connected components $O^{++}, O^{+-}, O^{-+}, O^{--}$

$SO(p,q) = O^{++} \cup O^{--}$ has 2 connected components

$I \in O^{++}(p,q)$

■

Recall that $U(p,q)$ is defined like $O(p,q)$ except with $U^* J U = J$

$SU(p,q) \subseteq U(p,q)$ is those matrices with determinant 1.

Now we would like to discuss the connectivity of $U(p,q)$ and hence $SU(p,q)$

To do this, we will first need to find the Lie Algebra because we have a powerful theorem which is essentially a strengthening of the polar decomposition in $GL_n(\mathbb{C})$

Definition: A subgroup $G \leq GL_n(\mathbb{C})$ is pseudo-algebraic if there is a finite set of polynomials in $2n^2$ variables with real coefficients $\{P_j(x_1, \dots, x_{n^2}, y_1, \dots, y_{n^2})\}_{j=1}^{t}$ so that $A = (x_{kl} + i y_{kl}) \in G$ iff $P_j(x_{11}, \dots, x_{nn}, y_{11}, \dots, y_{nn}) = 0$ $\forall j = 1, \dots, t$

Basically, the conditions on elements of A should be polynomial in nature

Since every pseudo-algebraic group is the zero set of a set of polynomials, it is closed and hence a lie group

Proposition 9

Let $P(x_1, \dots, x_n)$ be a polynomial with real coefficients. For any $(a_1, \dots, a_n) \in \mathbb{R}^n$, let $P(e^{ka_1}, \dots, e^{ka_n}) = 0$ $\forall k \in \mathbb{N}$. Then $P(e^{ta_1}, \dots, e^{ta_n}) = 0$ $\forall t \in \mathbb{R}$

proof

Suppose we have a monomial $\alpha x_1^{i_1} \cdots x_n^{i_n}$. When evaluated, it is $\alpha e^{t(\sum_j a_j i_j)}$

$\therefore P(e^{ta_1}, \dots, e^{ta_n}) = \sum_{k=1}^{N} \alpha_k e^{tb_k}$

$= \alpha_N e^{tb_N} + \sum_{k=1}^{N-1} \alpha_k e^{tb_k}$

This vanishes for all $t \in \mathbb{N}$ is known

We may assume all α_j are non zero else they drop out.

Also assume $b_1 < \dots < b_N$

Suppose $N > 0$ for contradiction

$\therefore e^{-tb_N} P(e^{ta_1}, \dots, e^{ta_n}) = \alpha_N + \sum_{k=1}^{N-1} \alpha_k e^{t(b_k - b_N)}$

Now all $b_k - b_N$ are negative

Letting $t \to \infty$ (in $\mathbb{N}$), $\sum_{k=1}^{N-1} \alpha_k e^{t(b_k - b_N)}$ goes to 0 while the LHS is a constant 0 by assumption

Thus $\alpha_N = 0$ gives a contradiction

$\blacksquare$

Theorem 10

Let G be pseudo-algebraic s.t. $A \in G \Rightarrow A^* \in G$. Then, $G \cong (U(n) \cap G) \times (H(n) \cap \mathrm{Lie}(G))$ where $U(n)$, $H(n)$ are the $n \times n$ unitary & hermitian matrices & $\mathrm{Lie}(G)$ is the Lie algebra proof

Let $A \in G \subseteq GL_n(\mathbb{C})$

Using the polar decomposition (which is a unique decomposition), $A = US$ where $U \in U(n)$, S is hermitian & pos definite.

Let $\lambda_1, \dots, \lambda_n > 0$ be its eigenvalues

Write $S = P^* \operatorname{diag}(\lambda_i) P$ using the spectral theorem

Then $S = e^H$ for some unique hermitian matrix H

(in fact $H = P^* \operatorname{diag}(\log \lambda_i) P$).

Thus $A = Ue^H$ for $U \in U(n)$, $H \in H(n)$.

If we show that $H \in \operatorname{Lie}(G)$, then $e^H \in G$ so that $U = A(e^H)^{-1} \in G$ thus proving the bijection

Now $A \in G \Rightarrow A^* \in G$ i.e. $(e^H)^* U^* \in G$ i.e. $e^H U^* \in G$

Thus $e^H U^* U e^H = e^{2H} \in G$

Since $2H$ is hermitian, diagonalise it as

$2H = V \operatorname{diag}(\lambda_1, \dots, \lambda_n) V^*$ and λ_i are all real

Note that G is pseudo-algebraic & hence so is VGV^{-1}.

Thus we may work with just $2H = \operatorname{diag}(\lambda_i)$

To say $e^{2H} \in G$ means that $e^{\lambda_1}, \dots, e^{\lambda_n}$ satisfy a set of algebraic equations.

Since G is a group, $e^{k(2H)} \in G \Rightarrow e^{k\lambda_1}, \dots, e^{k\lambda_n}$ satisfy the same set of equations. This is for $k \in \mathbb{N}$

By prop 9, it generalises to $k \in \mathbb{R}$

Thus $e^{2tH} \in G \ \forall \ t \in \mathbb{R} \Rightarrow e^{tH} \in G \ \forall \ t \in \mathbb{R}$

Hence $H \in \operatorname{Lie}(G)$

Now we have to show that the map $(U, H) \longmapsto Ue^H$ is a homeomorphism

The polar decomposition is a homeomorphism is a well known fact

We only need to show that $H(n) \cong HPD(n)$. This fact has been proved in Mneimné and Testard's "Introduction à la théorie des groupes de Lie classiques" as theorem 3.3.4

■

Now we compute the Lie Algebras of $O(p,q)$, $SO(p,q)$, $U(p,q)$, $SU(p,q)$

Proposition 11

The lie algebra of $O(p,q)$ is the set
$$\left\{ \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} : A_{p\times p},\ D_{q\times q} \text{ are skew-symm}, B_{p\times q} \text{ arbitrary} \right\}$$

proof

Suppose X is such that $A(t) = e^{tX} \in O(p,q)$ $\forall\ t \in \mathbb{R}$.
$A(t)^T J A(t) = J$

Now $e^{tX^t} J e^{tX}$ can be differentiated with respect to t to get $e^{tX^t}(X^t J + J X^t) e^{tX}$ ($\because$ product rule and $(e^{tX})' = X e^{tX}$)

Evaluating at $t=0$, we have $X^t J + J X^t = 0$

Conversely, let $X^t J + J X^t = 0$

Then if $f(t) := e^{tX^t} J e^{tX}$, we have $f'(t) = 0$

Thus f is constant. $f(0) = J$ & we have $f(t) = J$ $\forall\ t$

Thus, all we need to do is characterise X such that $X^T J + J X = 0$
$$\begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} + \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0$$

This gives $A^T + A = 0$, $D^T + D = 0$, $B^T - C = 0 = -C^T + B$

This establishes our claim

Proposition 12

$\mathrm{Lie}(SO(p,q)) = \mathrm{Lie}(O(p,q))$

proof

It is clear that LHS $\subseteq$ RHS

Conversely, let $X \in \mathrm{Lie}(O(p,q))$

ie. $X = \begin{bmatrix} A & B \\ B^t & D \end{bmatrix}$ with A_p, D_q skew symmetric, B arbitrary

Clearly $\mathrm{tr}(X) = 0$ since A, D are skew-symmetric

Thus $\det(e^{tX}) = e^{t\,\mathrm{tr}(X)} = e^0 = 1$

Since $e^{tX} \in O(p,q)$ & $\det(e^{tX}) = 1$, $X \in \mathrm{Lie}(SO(p,q))$

Proposition 13

$\mathrm{Lie}(U(p,q)) = \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mid A_{p\times p},\ D_{q\times q} \text{ skew hermitian}, B_{p\times q} \text{ is arbitrary} \right\}$

proof

The proof is exactly along the lines of the proof of prop 11. ■

Proposition 14

Lie $(SU(p,q))$ = Lie $(U(p,q))$

proof

This follows just like in the proof of prop 12. We clearly have that trace $\begin{pmatrix} A & B \\ B^* & D \end{pmatrix} = 0$ if A, D are skew hermitian & hence e^{tX} has determinant 1 for such an $X \in$ Lie $(U(p,q))$ ■

Proposition 15

$O(p,q) \cong O(p) \times O(q) \times \mathbb{R}^{pq}$

proof

We invoke theorem 10 with $G = O(p,q)$.

Firstly we find $O(p,q) \cap U(p+q)$

Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O(p,q)$

Then $A^TA - C^TC = I_p$, $D^TD - B^TB = I_q$, $A^TB = C^TD$

If $g \in U(p+q)$, then $g^Tg = I = gg^T$

This gives $AA^T + BB^T = I_p$, $CC^T + DD^T = I_q$, $AC^T + BD^T = 0$,

$A^TA + C^TC = I_p$, $B^TB + D^TD = I_q$, $A^TB + C^TD = 0$

Thus $C^TC = B^TB = 0 \Rightarrow C, B$ are 0 matrices

$\therefore O(p,q) \cap U(p+q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \mid A \in O(p), D \in O(q) \right\}$

$\cong O(p) \times O(q)$

Lie $(O(p,q)) = \mathfrak{g} = \left\{ \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} \mid A, D \text{ skew symm} \right\}$

Now if $X \in \mathfrak{g}$ is also hermitian (i.e. symmetric), then

$\begin{bmatrix} A & B \\ B^t & D \end{bmatrix} = \begin{bmatrix} A^t & B \\ B^t & D^t \end{bmatrix} \Rightarrow A, D$ symmetric

But A, D are skew symm $\Rightarrow A = D = 0$

$\therefore \mathfrak{g} \cap H(p+q) = \left\{ \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \mid B \in M_{p \times q}(\mathbb{R}) \right\}$

$\cong \mathbb{R}^{pq}$

Thus by theorem 10, $O(p,q) \cong O(p) \times O(q) \times \mathbb{R}^{pq}$ ■

This also neatly shows that $O(p,q)$ has 4 connected components

Corollary 16

$SO(p,q) \cong S(O(p) \times O(q)) \times \mathbb{R}^{pq}$ where

$S(O(p) \times O(q)) = \{ (A,B) \mid A \in O(p),\ B \in O(q),\ \det(A)\det(B) = 1 \}$

$\blacksquare$

Proposition 17

$U(p,q) \cong U(p) \times U(q) \times \mathbb{C}^{pq}$

proof

Let $\mathfrak{g} = \mathrm{Lie}(U(p,q)) = \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mid A, D \text{ skew-hermitian} \right\}$

Then $\mathfrak{g} \cap H(p+q) = \left\{ \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \right\}$

$\cong \mathbb{C}^{pq}$

Also, as in prop 15, one can easily see that

$U(p,q) \cap U(p+q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \mid A \in U(p),\ B \in U(q) \right\}$

$\cong U(p) \times U(q)$

Theorem 10 finishes the job

$\blacksquare$

This shows that $U(p,q)$ is connected!

Corollary 18

$SU(p,q) \cong S(U(p) \times U(q)) \times \mathbb{C}^{pq}$ where

$S(U(p) \times U(q)) = \{ (A,B) \mid A \in U(p),\ B \in U(q),\ \det(A)\det(B) = 1 \}$

Proposition 19

$SU(p,q)$ is connected

proof

By virtue of corollary 18, we only need to show that the set $S(U(p) \times U(q))$ is connected

Let $(A,B) \in S(U(p) \times U(q))$

Let $\det(A) = z$, $\det(B) = z^{-1}$

Since $A \in U(p)$ and $U(p)$ is path connected, obtain a path $\gamma : [0,1] \to U(p)$ joining A to I_p

Let $z(t) := \det(\gamma(t))$ $\quad (z(0) = z,\ z(1) = 1)$

Observe that $z(t)$ is a continuous path in $\mathbb{C}^{\times}$

We want a path $\delta : [0,1] \to U(q)$ s.t. $\delta(0) = B$, $\delta(1) = I_q$ and $\det(\delta(t)) = z(t)^{-1}$ $\forall t$

Write $B = \varsigma \tilde{B}$ where $\tilde{B} \in SU(q)$

Since $\det(B) = z^{-1}$, choose ς st. $\varsigma^q = z^{-1}$

Let $\omega : [0,1] \to \mathbb{C}^\times$ be st. $(\omega(t))^q = z(t)^{-1}$

(Since $[0,1]$ is contractible, one can pick a continuous branch of the q^{th} root of a cont function in $\mathbb{C}^\times$)

Now $B = \omega(0) \tilde{B}$ with $\tilde{B} \in SU(q)$

By path connectedness of $SU(q)$, obtain a path Ω joining $\tilde{B}$ to I_q and define

$\delta(t) = \omega(t)\Omega(t)$

Indeed $\delta(0) = \omega(0)\Omega(0) = \omega(0)\tilde{B} = B$

$\delta(1) = \omega(1)\Omega(1) = 1\, I_q = I_q$

$\det(\delta(t)) = \omega(t)^q \det(\Omega(t)) = z(t)^{-1} \cdot 1 = z(t)^{-1}$

Thus the path $(\gamma, \delta) : [0,1]^2 \to U(p) \times U(q)$ joins (A, B) to (I_p, I_q) where $(A,B) \in S(U(p) \times U(q))$

Thus $S(U(p) \times U(q))$ is connected. $\blacksquare$