**Definition:** A partition $P$ of a rectangle $R = [a,b] \times [c,d]$ is given by $P_1 \times P_2$ where $P_1$ is a partition of $[a,b]$ and $P_2$ is of $[c,d]$. Subsequently, if $R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, then $\bigcup_{j=0}^{m-1} \bigcup_{i=0}^{n-1} R_{ij} = R$

**Definition:** The norm of a partition $P$ is $\|P\| := \max \{ (x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \dots, (n-1), j = 0, \dots, (m-1) \}$

**Definition:** Let $f : R \to \mathbb{R}$ be bounded over $R$. We define upper and lower Darboux sums as

$$L(f, P) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} m_{ij}(f) \cdot \Delta_{ij}$$

$$U(f, P) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} M_{ij}(f) \cdot \Delta_{ij}$$

where $m_{ij}$, $M_{ij}$ are inf and sup of $f$ over $R_{ij}$ and $\Delta_{ij}$ is the area of $R_{ij}$

**Definition:** The upper and lower Darboux integrals are defined as $L(f) = \sup \{ L(f,P) \mid \forall P, \text{ partions of } R \}$ and $U(f) = \inf \{ U(f,P) \mid \forall P, \text{ partions of } R \}$

**THEOREM 1**

For any two partitions $P_1$, $P_2$ of $R$, we have $L(f, P_1) \leq U(f, P_2)$ and for any refinement $P'$ of a partition $P$ of $R$, $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$. In particular $L(f) \leq U(f)$

**Definition:** A bounded function is said to be Darboux integrable if $L(f) = U(f)$.

## THEOREM 2

Let $f: R \to \mathbb{R}$ be bounded. $f$ is integrable iff $\forall \varepsilon > 0$, $\exists$ a partion $P_\varepsilon$ of $R$ so that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

**Definition:** A tagged partition of a rectangle $R$ is given by $(P, t)$ where $P$ is a partition of $R$ and $t = \{ t_{ij} \mid t_{ij} \in R_{ij},\ i = 0, \dots, n-1,\ j = 0, \dots, m-1 \}$

**Definition:** The Riemann sum of a bounded function $f$ associated to the tagged partition $(P, t)$ is

$$S(f, P, t) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(t_{ij}) \cdot \Delta_{ij}$$

**Definition:** A bounded function $f: R \to \mathbb{R}$ is said to be Riemann integrable if $\exists\, S \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ so that $|S(f, P, t) - S| < \varepsilon$ for every $(P, t)$ satisfying $\|P\| < \delta$

## THEOREM 3

Darboux and Riemann integrability are equivalent

**Definition:** we define the regular partition $P_{reg}$ of any interval $[a, b]$ to be $\{x_0, x_1, \dots, x_n\}$ where $x_0 = a$, $x_n = b$, $x_t = a + \frac{(b-a) \cdot t}{n}$ $\forall\, t \in \{1, \dots, n-1\}$.

Consequently, a regular partition of a rectangle is the cartesian product of regular partitions of intervals making up the rectangles such that the number of parts in each of the regular partitions is the same, say $n$. We denote this regular partition of $n^2$ 'cells' by $P_n$

## THEOREM 4

A bounded function is Riemann integrable iff the Riemann sums $s(f, P_n, t) \to S \in \mathbb{R}$ as $n \to \infty$ for any $t$

Convention: if $a=b$ or if $c=d$ or if both happen, $I = \iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = 0$. If $a<b$ and $c<d$,

$$\iint_{[b,a]\times[c,d]} f(x,y)\,dx\,dy = -I$$

$$\iint_{[a,b]\times[d,c]} f(x,y)\,dx\,dy = -I$$

$$\iint_{[b,a]\times[d,c]} f(x,y)\,dx\,dy = I$$

## THEOREM 5 (Domain additivity)

Let $R$ be a rectangle consisting of sub rectangles. $f: R \to \mathbb{R}$, is a bounded function. $f$ is integrable over $R$ iff $f$ is integrable over each of the sub rectangles. Further, integral of $f$ over $R$ equals the ~~integral~~ sum of integrals of $f$ over the sub rectangles.

## THEOREM 6

(i) if $f(x,y) = c$, $\iint_R f(x,y)\,dx\,dy = c \cdot Ar(R)$

(ii) if $f, g$ are integrable over $R$, $\iint_R (f+g) = \iint_R f + \iint_R g$

(iii) if $f$ is integrable over $R$, $\iint_R c\cdot f(x,y)\,dx\,dy = c\iint_R f$

(iv) if $f(x,y) \leq g(x,y) \ \forall x, y \in R$, $\iint_R f \leq \iint_R g$

(v) if $f$ is integrable over $R$, so is $|f|$ over $R$ with

$$\left| \iint_R f \right| \leq \iint_R |f|$$

(vi) if $f$ & $g$ are integrable over $R$, so is $f \cdot g$ over $R$

## THEOREM 7 (Fubini's theorem)

Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be integrable. Let $I = \iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$.
If for each $x \in [a,b]$, the Riemann integral $\int_c^d f(x,y)\,dy$ exists, then $I = \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx$. Similar theorem with $x, y$ interchanged also exists

## THEOREM 8

If $f$ is bounded, monotonic in $x$, monotonic in $y$, then $f$ is integrable

## THEOREM 9

If $f : R \to \mathbb{R}$ is bounded and continuous except at countably many points, then $f$ is integrable.

## THEOREM 10 (Cavalieri's principle)

Suppose two solids are included between two parallel planes and if for every plane parallel to these planes, the plane cuts out equal areas in both solids, then the solids have equal volumes

Definition : Let $D$ be any bounded domain in $\mathbb{R}^2$. We define $\int_D f(x,y)\,dx\,dy = \int_R f^*(x,y)\,dx\,dy$ where $R$ is a rectangle containing $D$ and $f^*(x,y)$ is defined as
$$f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

THEOREM 11

If $D$ is a bounded domain in $\mathbb{R}^2$ so that the boundary of $D$ is a cont. closed curve, then any bounded and continuous function over $D$ is integrable.

Definition : Let $h_1, h_2 : [a,b] \to \mathbb{R}$ be cont. such that $h_1(x) \le h_2(x)$ $\forall x \in [a,b]$. Consider a region $D_1 = \{(x,y) \mid a \le x \le b,\ h_1(x) \le y \le h_2(x)\}$. Such a region is called a type 1 region, Let $k_1, k_2 : [c,d] \to \mathbb{R}$ be cont. such that $k_1(y) \le k_2(y)$ $\forall y \in [c,d]$. Then $D_2 = \{(x,y) \mid c \le y \le d,\ k_1(y) \le x \le k_2(y)\}$ is called a type 2 region. Together, we have elementary regions

THEOREM 12

Let $D_1$ be a type 1 region & $D_2$ be of type 2.

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_a^b \left[ \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy \right] dx$$

$$\iint_{D_2} f(x,y)\,dx\,dy = \int_c^d \left[ \int_{k_1(y)}^{k_2(y)} f(x,y)\,dx \right] dy$$

## THEOREM 13

Let $f$ be integrable over some region $D$ in $\mathbb{R}^2$ and $g(r,\theta) = f(r\cos\theta, r\sin\theta)$ be the polar representation of $f$. Then,

$$\iint_D f(x,y)\, dx\, dy = \iint_{D^*} g(r,\theta) \cdot r \cdot dr\, d\theta$$

where $D^*$ is the region $D$ in polar coordinates

## THEOREM 14

If $D$ is an elementary region in $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ is cont., $\exists\ (x_0, y_0) \in D$ so that

$$f(x_0, y_0) = \frac{1}{\text{Area}(D)} \cdot \iint_D f(x,y)\, dx\, dy$$

**Definition:** Let $\Omega$ be an open subset of $\mathbb{R}^2$ and $h: \Omega \to \mathbb{R}^2$ be one-one differentiable. We define the Jacobian matrix, $J(h)$, as $\begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix}$ where

$h := (h_1, h_2)$ and $h_1, h_2 : \Omega \to \mathbb{R}$ are constituents of $h$

## THEOREM 15

Let $D$ be closed and bounded in $\mathbb{R}^2$ so that the boundary of $D$ has content $0$. Let $f: D \to \mathbb{R}$ be continuous. Let $\Omega$ be an open subset of $\mathbb{R}^2$, $h: \Omega \to \mathbb{R}^2$ be one-one and differentiable. Let $D^*$ be so that $h(D^*) = D$

If $\det(J) \neq 0$, we have,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} (f \circ h)(u,v)\,|\det J(h)|\,du\,dv$$

THEOREM 16

Let $f$ be integrable over $D$ in $\mathbb{R}^3$ and $g(r,\theta,\phi) = f(r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi)$ be the new function obtained by changing coordinates to spherical. Then,

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} g(r,\theta,\phi)\cdot r^2\sin\phi\ dr\,d\theta\,d\phi$$

where $D^*$ is $D$ in polar coordinates

Definition: A scalar field on $D$ is a map $f: D \to \mathbb{R}$ and a vector field on $D$ is a map $f: D \to \mathbb{R}^n$ for some $n \geq 2$

Definition: The gradient operator acts on scalar fields and the result is a vector field. Given $f: \mathbb{R}^3 \to \mathbb{R}$, we define $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

This is also called as gradient vector field

Definition: A vector field is said to be conservative if it is the gradient of some scalar function

Definition: If $F$ is a vec field from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, then a flow line is a map $c: [a,b] \to D$ s.t. $c'(t) = F(c(t))$

**Definition:** A path is a cont. map $c:[a,b] \to \mathbb{R}^n$ while a curve is the image of a path

**Definition:** $c:[a,b] \to \mathbb{R}^n$ is a closed path if $c(a)=c(b)$ and is a simple path if $c$ is one-one with the exception of end-points. $c$ is a regular path if $c$ is $C^1$ with $c'(t) \neq 0 \ \forall t$

**Definition:** Let $f: D \subset \mathbb{R}^n \to \mathbb{R}^n$ be cont and let $c$ be regular. We define line integral as $\int_c F \cdot d\vec{s} = \int_a^b \overrightarrow{F(c(t))} \cdot \vec{c}'(t)\,dt$

THEOREM 17

If $c_1$ is a path joining $P_1$ & $P_2$ and $c_2$ joins the points $P_2$ and $P_3$ and $c_3$ is the path joining $P_1$ and $P_3$ through $P_2$, $\int_{c_3} F \cdot d\vec{s} = \int_{c_1} F \cdot d\vec{s} + \int_{c_2} F \cdot d\vec{s}$

THEOREM 18

Let $c$ be regular. Suppose we make a change of variables $t = h(u)$, where $h$ is a $C^1$ diffeomorphism from $[\alpha,\beta]$ to $[a,b]$ with $h(\alpha)=a$, $h(\beta)=b$. Let $\gamma(u) = c(h(u))$. Then, $\gamma$ is a reparameterisation of $c$ and $\int_c F \cdot d\vec{s} = \int_\gamma F \cdot d\vec{s}$

**Definition:** Let $c:[a,b] \to \mathbb{R}^3$. Let $s(t) = \int_a^t \| c'(u) \| \, du$.

Let $h:[0, l(c)] \to [a,b]$ be inverse of $s$. $\tilde{c}(u) = c(h(u))$ is the arc length parameterisation of $c$

## THEOREM 19

If $\tilde{c}$ is the arc length parameterisation, $\int_c F \cdot d\vec{s} = \int_{\tilde{c}} F \cdot d\vec{s}$ by theorem 18 and further,

$$\int_{\tilde{c}} F \cdot d\vec{s} = \int_0^{l(c)} F(c(h(u))) \cdot T(h(u)) \, du$$

where $T(z) = \dfrac{c'(z)}{\|c'(z)\|}$ is the unit tangent vector

**Definition:** For a scalar field $f : D \to \mathbb{R}$, we define the integral over a path as

$$\int_c f \, ds = \int_a^b f(c(t)) \, \|c'(t)\| \, dt$$

where $c : [a, b] \to D$ is a non singular path

## THEOREM 20 (FTC)

Let $n = 2, 3$, $D \subset \mathbb{R}^n$, $c : [a, b] \to D$ be a smooth path. Let $f : D \to \mathbb{R}$ be differentiable with $\nabla f$ being cont. on $c$

Then $\int_c \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$

**Definition:** A vector field $F$ is path independent if for any two different paths $c_1$ and $c_2$ with same initial and terminal points, $\int_{c_1} F \cdot d\vec{s} = \int_{c_2} F \cdot d\vec{s}$

## THEOREM 21

All conservative vector fields are path independent

Definition: A subset $D$ of $\mathbb{R}^n$ is connected if it cannot be written as the disjoint union of two non-empty sets $D_1 \cup D_2$ with $D_1 = D \cap U_1$, $D_2 = D \cap U_2$ for $U_1$ and $U_2$ being open sets ??

Definition: A domain $D \subset \mathbb{R}^n$ is path connected if for any two points in the domain, $\exists$ continuous path joining them

THEOREM 22

Every path connected domain is connected

THEOREM 23

If a domain is open and connected, it is path connected

THEOREM 24

Path independent fields defined on an open connected domain are conservative

Definition: $D \subset \mathbb{R}^n$ is called simply connected if for any closed curve lying in the domain, the interior of the curve also lies inside the domain

THEOREM 25

If $f = F_1 \hat{i} + F_2 \hat{j}$ is conservative, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. Further, converse holds if $F_1$, $F_2$ are defined on open simply connected domain

**Convention:** We take a curves orientation to be positive if it is anticlockwise. Physically, if one walks along the curve in the positive orientation, the region enclosed will lie to the left hand side of them.

**THEOREM 26** (Green's theorem)

Let $D$ be a region in $\mathbb{R}^2$ with a positively oriented boundary $\partial D$ consisting of a finite no. of non-intersecting simple $C^1$ curves. Let $\Omega$ be another region so that $D \cup \partial D \subset \Omega$. Let $f_1, f_2 : \Omega \to \mathbb{R}$ be $C^1$ functions.

$$\int_{\partial D} (f_1 \hat{i} + f_2 \hat{j}) \cdot d\vec{s} = \iint_D \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy$$

**THEOREM 27**

If $C$ is positively oriented and bounds a region $D$,

area $(D) = \frac{1}{2} \int_C F \cdot d\vec{s}$ where $F = -y\hat{i} + x\hat{j}$

**Defininition:** We define the curl of a vector field in $\mathbb{R}^3$ as follows. Let $F = (f_1, f_2, f_3)$ be a vec field in $\mathbb{R}^3$. $\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$ and

If $F = (f_1, f_2)$ is a vector field in $\mathbb{R}^2$, $\nabla \times F$ is defined as $\left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{k}$

THEOREM 28

The curl of a conservative field is zero. The converse is true if all components of F ~~$F_1$ and $F_2$~~ have continuous first order partial derivative and the domain is open and simply connected

Definition: we define the divergence of a $C^1$ vector field $F = (F_1, F_2, F_3)$ as $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

THEOREM 29

Divergence of curl is always zero

THEOREM 30 (Div form of Green)

$$\int_{\partial D} F \cdot \vec{n}\, ds = \iint_D \nabla \cdot F \, dx\, dy$$

Definition: Let D be path connected subset in $\mathbb{R}^2$. A parameterised surface is a cont. function $\phi : D \to \mathbb{R}^3$

Definition: We define the partial derivative of a vector valued function $\phi$ as $\phi_u(v_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\hat{i} + \frac{\partial y}{\partial u}(u_0, v_0)\hat{j} + \frac{\partial z}{\partial u}(v_0, v_0)\hat{k}$ where $\phi \equiv (x(u,v), y(u,v), z(u,v))$. One can define $\phi_v$ similarly.

## THEOREM 31

The normal to the surface at $\phi(u_0, v_0)$ has DR's given by $\phi_u(u_0, v_0) \times \phi_v(u_0, v_0)$

## THEOREM 32

The surface area of $\phi$ is given by $\iint_E \|\phi_u \times \phi_v\| \, du\,dv$ where $\phi : E \to \mathbb{R}^3$

**Definition:** We define the surface integral of a bounded vector field $F$ defined on the surface $\phi : E \to \mathbb{R}^3$ as $\iint_S F \cdot d\vec{S} = \iint_E F(\phi(u,v)) \cdot (\phi_u \times \phi_v) \, du\,dv$

where $d\vec{S} = \hat{n} \, dS$ and $dS = \|\phi_u \times \phi_v\| \, du\,dv$

**Definition:** A surface $S$ is orientable if $\exists$ a cont. vector field $F : S \to \mathbb{R}^3$ s.t. $\forall P \in S$, $F(P)$ is a unit normal vector

**Definition:** If the unit normal vector $\hat{n} = \dfrac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$ agrees with the given orientation of the specified surface, we say that the parameterisation $\phi$ is orientation preserving else it is orientation reversing

**Definition:** $U_1 \subset \mathbb{R}^n$, $U_2 \subset \mathbb{R}^m$. $\psi : U_1 \to U_2$ is a homeomorphism if it is cont, bijective, inverse is cont

**Definition:** $f : M \to N$ is a diffeomorphism if it is a bijection and $f$ and $f^{-1}$ both are differentiable

Definition: we say that a surface S has a boundary if for every point P in S, there exists an open set U in $\mathbb{R}^3$ such that $P \in U \cap S$ is homeomorphic to either $\mathbb{R}^2$ or the closed upper half plane. We say S does not have boundary if for every point P in S there exists an open set U in $\mathbb{R}^3$ such

THEOREM 33 (Stokes theorem)

Let S be a bounded oriented surface with a non empty boundary $\partial S$. Let $F = (F_1, F_2, F_3)$ be a $C^1$ vector field defined on an open set containing S:

$$\int_{\partial S} F \cdot d\vec{s} = \iint_S \nabla \times F \cdot d\vec{S}$$

THEOREM 34 (Gauss Div theorem)

Let $\omega$ be a simple solid region in $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface. Suppose S is positively oriented and F is a smooth vec field defined on some open set containing W;

$$\iint_S F \cdot d\vec{S} = \iiint_W (\nabla \cdot F)\, dx\, dy\, dz$$

# TUTORIAL 1

1) (a) $R = [0,1]^2$, $f(x,y) = \lfloor x \rfloor + \lfloor y \rfloor + 1 \ \forall (x,y) \in R$.
Show that $f$ is integrable over $R$ and find the value

(b) $R = [0,1]^2$, $f(x,y) = (x+y)^2 \ \forall (x,y) \in R$. Show that $f$ is integrable over $R$ & find its value

(c) $R = [a,b] \times [c,d]$, $f: R \to \mathbb{R}$ is integrable. Show that $|f|$ is also integrable

(d) Check integrability of $f$ over $[0,1]^2$

$$f(x,y) = \begin{cases} 1 & (x,y) \in \mathbb{Q}^2 \\ -1 & \text{otherwise} \end{cases}$$

Ans (a) Let $P_n$ be the regular partition of the unit square. Let $P_1 = \left[\frac{n-1}{n}, 1\right] \times \left[\frac{n-1}{n}, 1\right]$, $P_2 = \bigcup_{i=0}^{n-2} \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{n-1}{n}, 1\right]$, $P_3 = \bigcup_{i=0}^{n-2} \left[\frac{n-1}{n}, 1\right] \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$

$P_4$ is the remaining partition.

$$S(f, P_n, t) = \sum_{i=1}^{4} S(f, P_i, t)$$

But $S(f, P_4, t) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} 1 \cdot \Delta_{ij} = \left(\frac{n-1}{n}\right)^2$

$S(f, P_1, t) = f(t_1) \cdot \frac{1}{n^2}$, $t_1 \in \left[\frac{n-1}{n}, 1\right] \times \left[\frac{n-1}{n}, 1\right]$

$$S(f, P_2, t) = \sum_{i=0}^{n-2} \sum_{j=n-1}^{n-1} f(t_{ij}) \cdot \frac{1}{n^2}$$

$$S(f, P_3, t) = \sum_{i=n-1}^{n-1} \sum_{j=0}^{n-2} f(t_{ij}) \cdot \frac{1}{n^2}$$

Since $f(t_{ij}) = 1$ or $2$ or $3$,

by the sandwich theorem,

$$\lim_{n\to\infty} s(f, P_i, t) = 0 \quad \forall \; i = 1, 2, 3$$

and $\lim_{n\to\infty} s(f, P_4, t) = 1$

$\therefore \lim_{n\to\infty} s(f, P_n, t) = 1$

$\therefore$ integral exists and evaluates to 1

(b) we go with $P_n$ again.

$$s(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \cdot \frac{1}{n^2}$$

where $t_{ij} \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$

$$\therefore \frac{(i+j)^2}{n^2} \leq f(t_{ij}) \leq \left(\frac{i+j+2}{n}\right)^2$$

$$\therefore \frac{1}{6}\frac{n^2}{n^4}(7n^2 - 12n + 5) \leq s(f, P_n, t) \leq \frac{1}{6}\frac{n^2}{n^4}(7n^2 + 12n + 5)$$

By sandwich theorem, $\lim_{n\to\infty} s(f, P_n, t) = \underline{\underline{\frac{7}{6}}}$

(c) Let $P$ be any arbitary partition

$$U(f, P) - L(f, P) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (M_{ij} - m_{ij}) \cdot \Delta_{ij}$$

$$M_{ij} - m_{ij} = \sup\{ |f(x) - f(y)| : x, y \in R_{ij} \}$$

By triangle inequality,

$$\big|\, |f(x)| - |f(y)| \,\big| \leq |f(x) - f(y)|$$

$\therefore$ $M'_{ij} - m'_{ij} \leq M_{ij} - m_{ij}$ where the $'$ indicates sup and inf for $|f|$

$\therefore$ $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$

$\therefore$ $\forall\ \varepsilon > 0$, since we can find $P$ s.t.

$U(f, P) - L(f, P) < \varepsilon$, we can use the same $P$ to conclude that

$U(|f|, P) - L(|f|, P) < \varepsilon$

(d) Not integrable !

since $M_{ij}(f) - m_{ij}(f) = 1 - (-1) = 2$

$\therefore$ $U(f, P) - L(f, P) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2 \cdot \frac{1}{n^2} = 2$

(This holds for any partition !)

$\therefore$ Choosing $\varepsilon = 1$, we get a contradiction

2) (i) Sketch the solid bounded by $z = \sin y$, $x = 0, -1$, $y = 0, \frac{\pi}{2}$, $xy$ plane. Also find volume.

(ii) $\iint_R \sqrt{9 - y^2}\, dx\, dy$ where $R = [0, 3]^2$ represents the volume of a solid. Sketch the solid and find its volume

Ans (i) $\int_{-1}^{0} \int_{0}^{\pi/2} \sin y\, dy\, dx = \int_{-1}^{0} 1 \cdot dx = 1$

(ii) $\int_{0}^{3} \int_{0}^{3} \sqrt{9 - y^2}\, dy\, dx = \frac{27\pi}{4}$

3) $f : [0,1]^2 \to \mathbb{R}$ is defined as

$$f(x,y) = \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \text{ in lowest form }, y \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

Show that $f$ is integrable but iterated integrals do not always exist

Ans we first show iterated integrals do not exist

Fix some $y_0$. If $y_0 \notin \mathbb{Q}$, $f(x, y_0) = 1$ and this is integrable on $[0,1]$.

If $y_0 \in \mathbb{Q}$, $f(x, y_0) = g(x) = \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \\ 1 & \text{otherwise} \end{cases}$

This is the Thomae function and is integrable as seen in MA109

Thus in both cases, $\int_0^1 f(x, y_0)\, dx = 1$ and

hence $\int_0^1 \left( \int_0^1 f(x,y)\, dx \right) dy = 1$

Now we try to find $\int_0^1 \left( \int_0^1 f(x,y)\, dy \right) dx$.

Fix some $x_0$.

If $x_0 \notin \mathbb{Q}$, $f(x_0, y) = 1$ is integrable

If $x_0 \in \mathbb{Q}$, $g(y) = f(x_0, y) = \begin{cases} 1 - \frac{1}{q} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$

This is clearly not integrable on $[0,1]$

$\therefore \int_0^1 f(x,y)\, dy$ itself does not exist for $x \in \mathbb{Q}$

For the function as a whole, fix $\varepsilon > 0$

Define $S_y = \{ x \mid 1 - f(x,y) \geq \varepsilon , \ y \in \mathbb{Q}_{[0,1]} \}$

( o/c, $S_y$ is treated as $S_y \cap [0,1]$ )

Notice that $S_y$ only contains rationals and in particular only those $x = \frac{p}{q}$ where $\frac{1}{q} \geq \varepsilon$

i.e. $q \leq \frac{1}{\varepsilon}$. But the no. of such $q$ are finite. Thus, $S_y$ is finite. Let $|S_y| = L$

Take $\delta = \frac{\varepsilon}{L}$ i.e. consider a partition $P_\varepsilon$ of $[0,1]^2$ such that $\| P_\varepsilon \| < \frac{\varepsilon}{L}$

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (1 - m_{ij}) \cdot \Delta_{ij}$$

( since $M_{ij} = 1$ )

We split the sum as

$$\sum_{\substack{i=0 \\ S \cap [x_i, x_{i+1}] = \emptyset}}^{n-1} \sum_{j=0}^{m-1} (1 - m_{ij}) \Delta_{ij} + \sum_{\substack{i=0 \\ S \cap [x_i, x_{i+1}] \neq \emptyset}}^{n-1} \sum_{j=0}^{m-1} (1 - m_{ij}) \Delta_{ij}$$

For the second sum $S_2$,

$$S_2 \leq \sum_{\substack{i=0 \\ S \cap [x_i, x_{i+1}] \neq \emptyset}}^{n-1} \sum_{j=0}^{m-1} \Delta_{ij} \leq 2L \left( \frac{\varepsilon}{L} \cdot 1 \right) = 2\varepsilon$$

(at most $2L$ no. of rectangles with dimensions $\frac{\varepsilon}{L} \times 1$ containing a point from the set $S$ )

for the first sum,

$$S_1 \le \sum_{\substack{i=0 \\ S \cap [x_i, x_{i+1}] = \emptyset}}^{n-1} \sum_{j=0}^{m-1} \varepsilon \, \Delta ij \le \varepsilon$$

$\left( \because 1 - m_{ij} < \frac{1}{q} \le \varepsilon \right.$ since intervals

which do not contain $x$ in $S_y$ necessarily

have $\left. m_{ij} > 1 - \frac{1}{q} \right)$

$\therefore U(f,P) - L(f,P) < 3\varepsilon$ . Choosing $\varepsilon = \frac{\varepsilon}{3}$ does the job

4) $f : [0,1]^2 \to \mathbb{R}$ as

$$f(x,y) = \begin{cases} \frac{1}{x^2} & 0 < y < x < 1 \\ -\frac{1}{y^2} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Is $f$ integrable over $[0,1]^2$ ? Do both iterated integrals exist ? If they exist, do they agree ?

Ans The function is unbounded near the origin and hence not integrable over $R$

Let us evaluate the iterated integrals

Fix $y$.

If $y = 0$, $f(x,y) = 0$ & $\int_{x=0}^{1} f(x,y)\,dx = 0$

If $y \in (0,1]$, $\int_{x=0}^{1} f(x,y)\,dx$

$$= \int_{x=0}^{y} f(x,y)\,dx + \int_{x=y}^{1} f(x,y)\,dx$$

$$= \int_{x=0}^{y} \frac{-1}{y^2}\, dx + \int_{x=y}^{1} \frac{1}{x^2}\, dx$$

$$= \frac{-1}{y} + \frac{1}{y} - 1 = -1$$

$$\therefore A(y) = \begin{cases} -1 & y \in (0,1] \\ 0 & y = 0 \end{cases}$$

This is Riemann integrable with

$$\int_0^1 A(y)\, dy = -1$$

similarly $\int_0^1 A(x)\, dx = 1$

Both iterated integrals exist and are not equal (Another reason to say $f$ is not integrable)

15) Reverse the order & integrate & check if values are same

(i) $\int_0^1 \int_0^1 \log((x+1)(y+1))\, dx\, dy$

(ii) $\int_0^1 \int_0^1 (xy)^2 \cos(x^3)\, dx\, dy$

Ans (i) $\int_0^1 \int_0^1 \log((x+1)(y+1))\, dx\, dy$

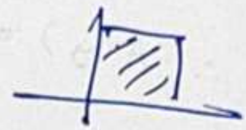
$$= \int_0^1 \int_0^1 \log((x+1)(y+1))\, dy\, dx$$

$$= 4 \ln 2 - 2$$

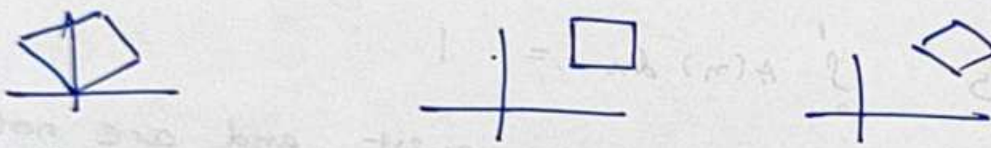
(ii) $\int_0^1 \int_0^1 x^2 y^2 \cos(x^3)\, dy\, dx$

$= \int_0^1 \int_0^1 x^2 y^2 \cos(x^3)\, dx\, dy$

$= \dfrac{\sin 1}{9}$



Note: when $\iint$ is on an ideal square [redacted], then order of integration does not matter



not ideal squares

b) $R = [a,b] \times [c,d]$, $f(x,y) = \phi(x)\psi(y)$ on $R$ with $\phi, \psi$ both continuous on $[a,b]$ and $[c,d]$.

Prove that $\iint_R f(x,y)\, dx\, dy = \left(\int_a^b \phi(x)\, dx\right)\left(\int_c^d \psi(y)\, dy\right)$

Find explicitly

(i) $\iint_{[1,2]\times[1,2]} x^9 y^5\, dx\, dy$

(ii) $\iint_{[0,1]^2} x y e^{x+y}\, dx\, dy$

Ans $\int_a^b \int_c^d \phi(x)\,\varphi(y)\,dy\,dx$

$= \int_a^b \left(\phi(x) \int_c^d \varphi(y)\,dy\right) dx$

$= \left(\int_a^b \phi(x)\,dx\right)\left(\int_c^d \varphi(y)\,dy\right)$

$$\iint_{[1,2]^2} x^r y^s \,dx\,dy = \frac{(2^{r+1}-1)(2^{s+1}-1)}{(r+1)(s+1)}$$

$$\iint_{[0,1]^2} xye^{x+y}\,dx\,dy = 1 \times 1 = 1$$

If you arent convinced with the Fubini proof,

$$S(f,P,t) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \overset{t_{ij}}{\overset{\parallel}{f(\alpha_i,\beta_j)}}\ \cancel{f(t_{ij})}\ \Delta_{ij}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \phi(\alpha_i)\,\varphi(\beta_j)\,(x_{i+1}-x_i)\cdot(y_{j+1}-y_j)$$

$$= \left(\sum_{i=0}^{n-1} \phi(\alpha_i)(x_{i+1}-x_i)\right)\left(\sum_{j=0}^{n-1} \varphi(\beta_j)(y_{j+1}-y_j)\right)$$

Since $\phi, \varphi$ are continuous, they are integrable and $f$ is given to be integrable.

$\therefore$ As $\|P\| \to 0$, $S(f,P,t) \to \left(\int_a^b \phi(x)\,dx\right)\left(\int_c^d \varphi(y)\,dy\right)$

$\therefore \iint_R f\,dx\,dy = \left(\int_a^b \phi(x)\,dx\right)\left(\int_c^d \varphi(y)\,dy\right)$

7) Find

(i) $\iint_{[-1,2]\times[0,2]} (x+2y)^2 \, dx \, dy$

(ii) $\iint_R \left( xy + \frac{x}{y+1} \right) dx \, dy$, $R = [1,4] \times [1,2]$

Ans (i) 50

(ii) Use theorem from Q6 to get 14.29

8) $f$ is defined over $[-1,1]^2$ as

$$f(x,y) = \begin{cases} x+y & x^2+y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find set of points at which $f$ is discont.
Is $f$ integrable over $[-1,1]^2$ ?

Ans Let $S_1$ be the given circular region, $S_2$ be the remaining region. Clearly $f$ is continuous inside $S_1$ and $S_2$

On the boundary, approaching from $S_2$ gives value $a+b$ and approaching from $S_1$ gives 0

$\therefore$ it is only continuous if $a+b = 0$

$\therefore f$ is discontinuous at all points $(\alpha, \beta)$

s.t. $\alpha^2 + \beta^2 = 1$ except at

$\left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}\right)$ where it is continuous

$$\iint_R f(x,y)\,dx\,dy = \iint_{S_1} x\,dx\,dy + \iint_{S_1} y\,dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$+ \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dx\,dy$$

$$= 0$$

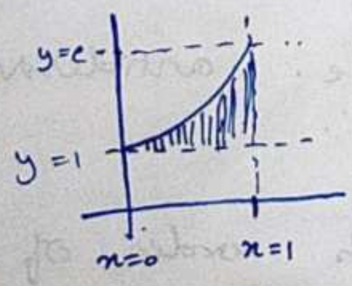
## TUTORIAL 2

1) switch order of integrals

(i) $\int_0^1 \int_1^{e^x} dy\,dx$

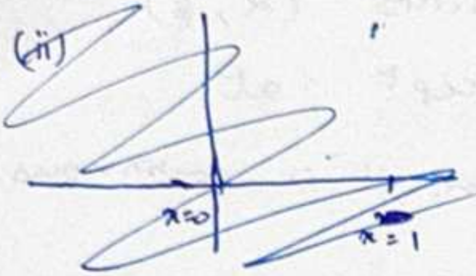
(ii) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy$

Ans (i)

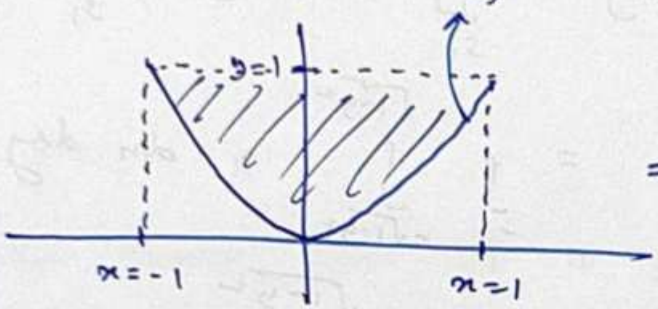



$$\Rightarrow \int_1^e \int_{\ln y}^{1} dx\,dy$$

(iii)




(ii)




$$\Rightarrow \int_{-1}^{1}\int_{x^2}^{1} dy\, dx$$

2) Evaluate the following

(i) $$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

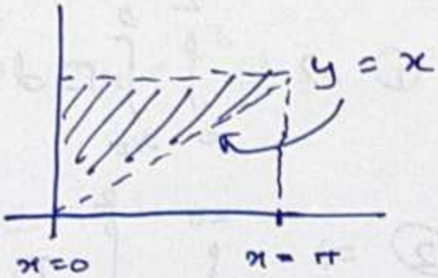
(ii) $$\int_0^{1}\int_y^{1} x^2 e^{xy}\, dx\, dy$$

(iii) $$\int_0^{2} (\tan^{-1}\pi x - \tan^{-1} x)\, dx$$

Ans (i) $\frac{\sin y}{y}$ is not expressible as $F'$ for any given $F$ i.e. antiderivative does not exist.

$\therefore$ So we switch order of $\iint$

we get

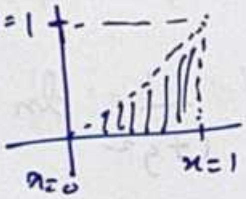



$\Rightarrow \int_0^{\pi}\int_0^{y}$

$\therefore \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = 2$

(ii) Same reason ... so switch

$\int_0^1 \int_y^1 dx\,dy \Rightarrow$

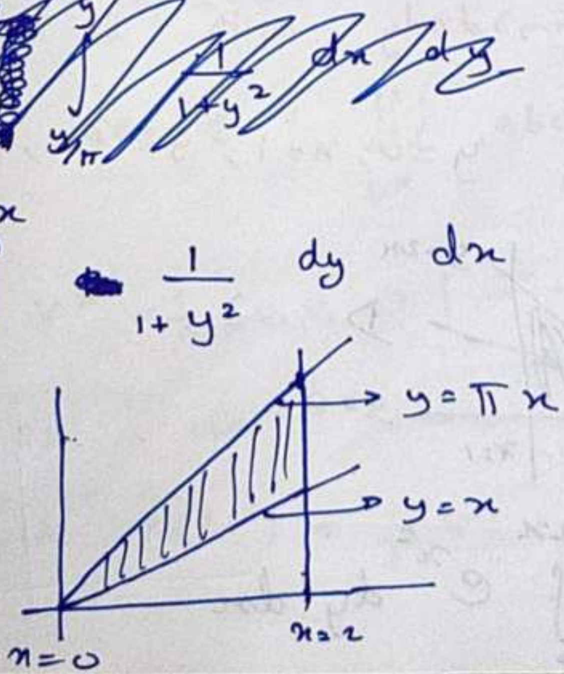


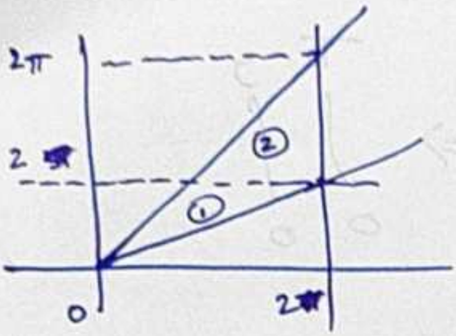

$\Rightarrow \int_0^1 \int_0^x dy\, dx$

$\therefore \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx = \frac{e-2}{2}$

(iii) $\int_0^2 \tan^{-1}\pi x - \tan^{-1} x \; dx$

$= \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2}\, dy\, dx$






① $\Rightarrow \int_0^2 \int_{y/\pi}^{y} \square \, dx\, dy$

② $\Rightarrow \int_2^{2\pi} \int_{y/\pi}^{2} \square \, dx\, dy$

$\therefore$ we get $\int_0^2 \int_{y/\pi}^{y} \frac{1}{1+y^2} \, dx\, dy$

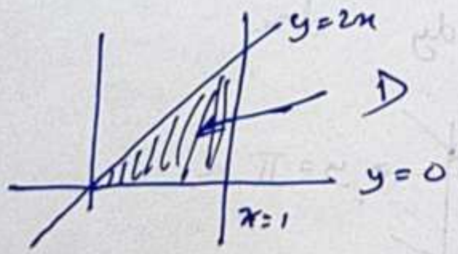
$$+ \int_0^2 \int_{y/\pi}^{2} \frac{1}{1+y^2} \, dx\, dy$$

$$= \frac{\ln 5}{2}\left(1 - \frac{1}{\pi}\right) + 2\left(\tan^{-1} 2\pi - \tan^{-1} 2\right)$$

$$- \frac{1}{2\pi} \ln\left(\frac{1+4\pi^2}{5}\right)$$

3) Find $\iint_D f(x,y)\, dx\, dy$ with $D$ being the region bounded by $y=0,\ x=1,\ y=2x,\ f(x,y) = e^{x^2}$

Ans




$$\int_0^1 \int_0^{2x} e^{x^2} \, dy\, dx$$

$$= \int_0^1 2x e^{x^2} dx$$

$$= e - 1$$

4) (i) Compute the volume of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a, b, c \in \mathbb{R}$$

(ii) Volume of $f(x, y) = e^{x+y}$ over $D$ where

$$D = \{ (x, y) \mid |x| + |y| \leq 1 \}$$

<u>Ans</u> (i) Convert to elliptical coordinates

$x = au \quad y = bv \quad z = cw.$

$$J = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

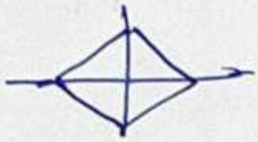
$\det J = abc$

$$\therefore \iiint_V dx\,dy\,dz = \iiint_{V^*} abc \; du\,dv\,dw$$

where $V^* = \{ (u, v, w) \mid u^2 + v^2 + w^2 \leq 1 \}$

This is a unit sphere and hence we get $(abc) * \frac{4}{3}\pi = \frac{4\pi abc}{3}$

(ii)



$\rightarrow$ $-1 \leq x+y \leq 1$

$-1 \leq x-y \leq 1$

use $u = x+y$, that is, $x = \frac{u+v}{2}$

$v = x-y$

$y = \frac{u-v}{2}$

$\therefore J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$\therefore |\det J| = \frac{1}{2}$

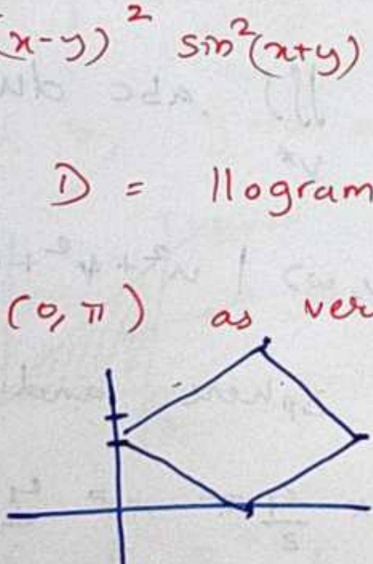
$\therefore \iint_D e^{x+y}\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2} e^{u}\, du\, dv$

$= e - \frac{1}{e}$

5) $\iint_D (x-y)^2 \sin^2(x+y)\, dx\, dy = ?$

when D = llogram with $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$ as vertices

Ans



$\pi \leq x+y \leq 3\pi$

$-\pi \leq x-y \leq \pi$

$\therefore$ we get $\displaystyle\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} \frac{1}{2} v^2 \sin^2 u \, du \, dv$

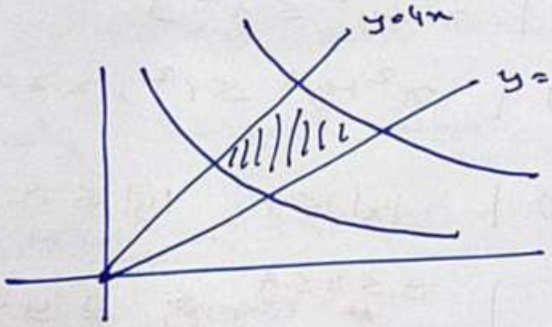
$$= \frac{\pi^4}{3}$$

( same sub as prev question )

6) Find $\iint_D dx\,dy$ for D being region bounded b/w $xy = 1$, $xy = 9$, $y = x$, $y = 4x$

(Do it by using $x = \frac{u}{v}$, $y = uv$ )

Ans




$x = \frac{u}{v}$ , $y = uv$

$$J = \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix}$$

$\Rightarrow |\det J| = \left|\frac{2u}{v}\right| = \frac{u}{v}$ (all +ve)

$xy = u^2$ , $\frac{y}{x} = v^2$

$1 \le \frac{x}{y} \le 9 \Rightarrow \quad 1 \le u \le 3$ (or $-3 \le u \le -1$)

$1 \le \frac{y}{x} \le 4 \Rightarrow \quad 1 \le v \le 2$ (or $-2 \le v \le -1$)

$\therefore$ we get $\left| \int_{\pm 1}^{\pm 3} \int_{\pm 1}^{\pm 2} \frac{2u}{v} \, dv \, du \right|$ area is positive

$= 8 \log 2$

7) Find $\lim_{r \to \infty} \iint_{D(r)} e^{-(x^2+y^2)} \, dx \, dy$

where $D(r)$ is

(i) $\{(x,y) \mid x^2+y^2 \le r^2\}$

(ii) $\{(x,y) \mid x^2+y^2 \le r^2, x \ge 0, y \ge 0\}$

(iii) $\{(x,y) \mid |x| \le r, |y| \le r\}$

(iv) $\{(x,y) \mid 0 \le x \le r, 0 \le y \le r\}$

Ans (i) $\int_0^{2\pi} \int_0^{r} e^{-R^2} R \, dR \, d\theta$

$= \pi(1 - e^{-r^2})$

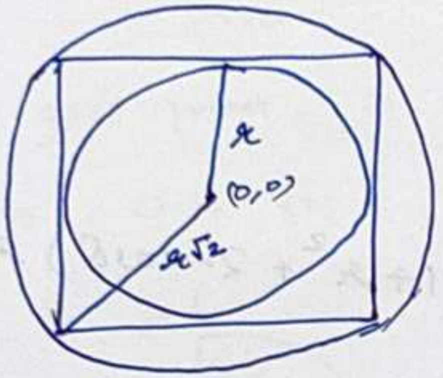
As $r \to \infty$, limit is $\pi$

(ii) $1/4^{th}$ of above $= \pi/4$

(iii) We make use of sandwich theorem

$$\int_0^{2\pi}\int_0^{r} e^{-R^2} R\,dR\,d\theta \le I(r) \le \int_0^{2\pi}\int_0^{r\sqrt{2}} e^{-R^2} R\,dR\,d\theta$$

By sandwich theorem we get limit $\pi$




(iv) $1/4^{th}$ of above $= \pi/4$

8) Find the volume between cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ using double integral

Ans simply integrate over $D_1$ → first octant of the cylinder $x^2+y^2=a^2$ & multiply volume by 8 times.

we have $$\int_0^{1}\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx = \frac{2a^3}{3}$$

$\therefore$ net volume $= \dfrac{16a^3}{3}$

9) Find vol under $z=x^2+y^2$ & over $x^2+y^2=2x$ (in XY plane)

Ans $\iint\limits_{V} (x^2+y^2)\, dx\, dy$

$x-1 = r\cos\theta$

$y = r\sin\theta$ (polar sub)

$|\det J| = |r|$

$\therefore V = \int_0^{2\pi} \int_0^{1} (1+r^2+2r\cos\theta)\, r\, dr\, d\theta$

$\therefore V = \dfrac{3\pi}{2}$

10) Express the solid $\{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le 1\}$

as $\{xyz \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x),\ \varphi_1(x,y) \le z \le \varphi_2(x,y)\}$

Ans $\{(x,y,z) \mid -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ \sqrt{x^2+y^2} \le z \le 1\}$

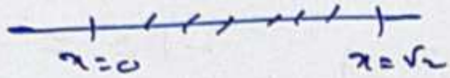
11) Find $\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x\, dz\, dy\, dx$ and sketch the solid of integration.

(Hint: Change to $dx\, dy\, dz$.)

Ans Notice that

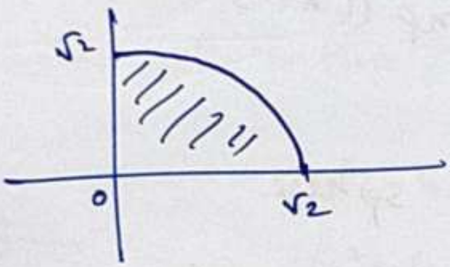
"$x$ goes from 0 to $\sqrt{2}$"

$\Rightarrow$



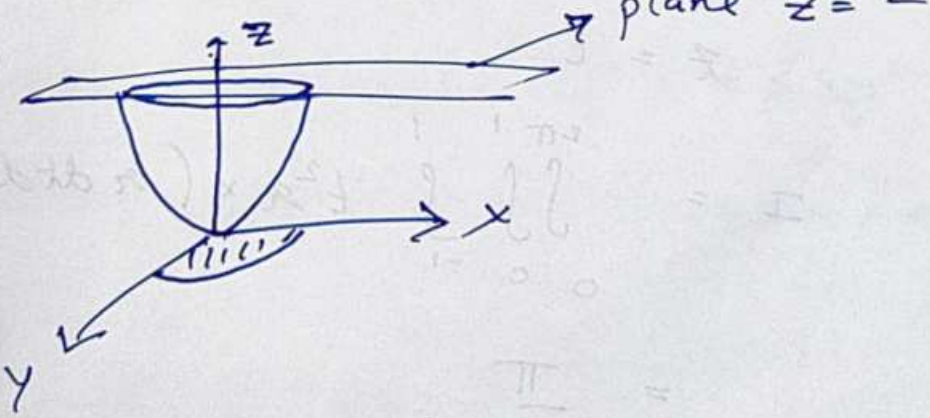

"$y$ goes from 0 to $\sqrt{2-x^2}$ for each $x$ in 0 to $\sqrt{2}$"

$\Rightarrow$




"$z$ goes from $x^2+y^2$ to 2 over this above region"

$\Rightarrow$




$$\Rightarrow \int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} x \, dx \, dy \, dz$$

$$= \frac{8\sqrt{2}}{15}$$

12) (i) $\iiint_D x^2z^2+y^2z^2 \, dx\,dy\,dz$

$D = x^2+y^2 \le 1, \; -1 \le z \le 1$

(capped cylinder)

(ii) $\iiint_D \exp((x^2+y^2+z^2)^{3/2}) \, dx\,dy\,dz$

$D$ = unit sphere

Ans (i)

$x = r\cos\theta$
$y = r\sin\theta$
$z = t$

$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 t^2 r \times (r \, dt\,dr\,d\theta)$$

$$= \frac{\pi}{3}$$

(ii) $x = r\cos u \sin v$
$y = r\sin u \sin v$
$z = r\cos v$

$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{r^3} \times r^2 \sin v \, dr\,du\,dv = \frac{4\pi}{3}(e-1)$$

# TUTORIAL 3

1) Find volume of solid under $z = x^2 + y^2$, above $x^2+y^2 = 2x$

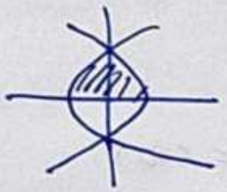
Ans Q9 Tut 2 repeated

2) using suitable change of variables, evaluate

$\iint_D y \, dx\,dy$ with $D$ being region bounded by $x$-axis

and parabolas $y^2 = 4 - 4x$, $y^2 = 4 + 4x$, $y \geq 0$

Ans



$y^2 = 4 - 4x$

$y^2 = 4 + 4x$

Obvious change : $y = \sqrt{v}$, $x = 1 + u$

Initially, $D = \left\{ (x,y) \mid \frac{y^2-4}{4} \leq x \leq \frac{4-y^2}{4},\ y \geq 0 \right\}$

$D^* = \left\{ (u,v) \mid 0 \leq v \leq 4,\ -\frac{v}{4} \leq u \leq \frac{v}{4} - 2 \right\}$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix} \Rightarrow |\det J| = \left| \frac{1}{2\sqrt{v}} \right|$$

$$= \frac{1}{2\sqrt{v}}$$

(on $D^*$)

$$\therefore I = \int_0^4 \int_{-\frac{v}{4}}^{\frac{v}{4}-2} \sqrt{v} \, \frac{1}{2\sqrt{v}} \, du\,dv = 2$$

3) use spherical coordinates to find vol above cone $z = \sqrt{x^2+y^2}$ and inside $x^2+y^2+z^2 = z$

Ans

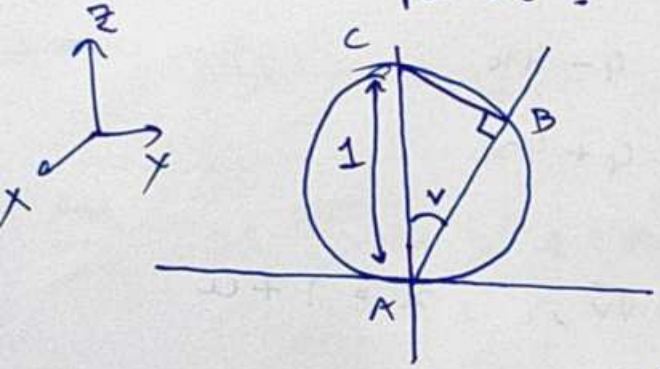
$$x = r \cos u \sin v$$

$$y = r \sin u \sin v$$

$$z = r \cos v$$

$$0 \le v \le \frac{\pi}{4}, \quad 0 \le u \le 2\pi, \quad 0 \le r \le \cos v$$

limits for $r$ ?




$\Rightarrow \quad AB = \cos v$

$$\Rightarrow I = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos v} r^2 \sin v \, dr\, du\, dv$$

$$= \pi/8$$

$\cancel{4) \iiint_W x(x^2+y^2)}$

4) $\iiint_W (x^2+y^2)\, dV$ with

$$W = \{ xyz \mid -2 \le x \le 2, \ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \ \sqrt{x^2+y^2} \le z \le 2 \}$$

Ans use cylindrical coordinates

$z = t \qquad x = r\cos\theta \qquad y = r\sin\theta$

$\theta \in [0, 2\pi], \quad r \in [0, 2], \quad t \in [r, 2]$

$$\therefore I = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2 \, r\,dr\,dt\,d\theta = \frac{16\pi}{5}$$

5) Describe solid whose volume is given by

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$ and

evaluate it

Ans evaluation is pretty straightforward and

answer is $\frac{7\pi}{6}$

The figure is the shell between spheres of radii 1, 2 in first octant

6) $\iiint_F \frac{1}{(x^2+y^2+z^2)^{n/2}} dV$ where F is shell

between spheres of radii $r$, $R$ $(R > r)$

Ans Use spherical coordinates. we get

$$\int_0^{\pi}\int_0^{2\pi}\int_r^{R} \frac{1}{r^n} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \begin{cases} 4\pi \ln\left(\frac{R}{r}\right) & \text{if } n = 3 \\ \frac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{otherwise} \end{cases}$$

7) Let $f, g$ be differentiable on $\mathbb{R}^2$. Show that

(i) $\nabla(fg) = f\nabla g + g\nabla f$

(ii) $\nabla f^n = n f^{n-1} \nabla f$

(iii) $\nabla(f/g) = \dfrac{(g\nabla f - f\nabla g)}{g^2}$ $(g \neq 0)$

Ans (i) $\nabla fg = \dfrac{\partial(fg)}{\partial x}\hat{i} + \dfrac{\partial(fg)}{\partial y}\hat{j}$

$= (\cdots\cdots) \longrightarrow$ product rule

$= f\nabla g + g\nabla f$

(ii) use part (i) & induction on $n$ as follows:

Base case: $n = 1$ $\therefore$

$\nabla f^1 = \nabla f = 1 f^0 \nabla f = \nabla f$

verified ✓

Assume $\nabla f^{k-1} = (k-1) f^{k-2} \nabla f$

Then $\nabla f^k = \nabla(f^{k-1} f)$

$= f^{k-1}\nabla f + f\nabla f^{k-1}$

$= f^{k-1}\nabla f + f((k-1)f^{k-2}\nabla f)$

$= k f^{k-1}\nabla f$

Thus we are done by induction

(iii) $\nabla\left(\frac{f}{g}\right) = \nabla\left(f \times \frac{1}{g}\right)$

$= f\left(\nabla \frac{1}{g}\right) + \frac{1}{g}\nabla f$

$= f\left(\frac{\partial(\frac{1}{g})}{\partial x}\hat{i} + \frac{\partial(\frac{1}{g})}{\partial y}\hat{j}\right) + \frac{1}{g}\nabla f$

$= \ldots$ (some differentiation JEE stuff)

$= \dfrac{g\nabla f - f\nabla g}{g^2}$

8) Let $\vec{a}, \vec{b}$ be two vectors. $\vec{r} = (x, y, z)$.

$|r|^2 = x^2+y^2+z^2$. Prove that

(i) $\nabla(|r|^n) = n|r|^{n-2}\vec{r}$

(ii) $\vec{a}\cdot\nabla\left(\frac{1}{|r|}\right) = -\left(\frac{\vec{a}\cdot\vec{r}}{|r|^3}\right)$

(iii) $\vec{b}\cdot\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{|r|}\right)\right) = \dfrac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{|r|^5} - \dfrac{\vec{a}\cdot\vec{b}}{|r|^3}$

Ans (i) $|r|^n = (x^2+y^2+z^2)^{n/2}$

$\therefore \dfrac{\partial |r|^n}{\partial x} = nx(x^2+y^2+z^2)^{\frac{n-2}{2}}$

$\therefore \nabla(|r|^n) = n|r|^{n-2}\vec{r}$

(ii) $\vec{a}\cdot\nabla\left(\frac{1}{|r|}\right)$

$= \vec{a}\cdot\left(-\dfrac{\nabla(|r|)}{|r|^2}\right)$

$$= -\frac{\vec{a} \cdot \nabla(|r|)}{|r|^2}$$

$$= -\frac{\vec{a} \cdot \left(\frac{1}{|r|}\vec{r}\right)}{|r|^2}$$

$$= -\frac{\vec{a} \cdot \vec{r}}{|r|^3}$$

(iii) $$\vec{b} \cdot \nabla\left(-\left(\frac{\vec{a} \cdot \vec{r}}{|r|^3}\right)\right)$$

$$= -\left(\vec{b} \cdot \nabla\left(\frac{\vec{a} \cdot \vec{r}}{|r|^3}\right)\right)$$

$$= -\left(\vec{b} \cdot \nabla\left(\frac{a_1 x + a_2 y + a_3 z}{(x^2+y^2+z^2)^{3/2}}\right)\right)$$

$= -$

some brutal calculations

$$= \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{|r|^5} - \frac{(\vec{a} \cdot \vec{b})}{|r|^3}$$

9) Find line integral of $F(x,y) = (x^2 - 2xy, y^2 - 2xy)$ from $(-1, 1)$ to $(1, 1)$ along $y = x^2$

**Ans** Parameterise as $c(t) = (t, t^2)$

$$\int_C F \cdot d\vec{s} = \int_{t_1}^{t_2} F(c(t)) \cdot c'(t)\, dt$$

$c(t) = (t, t^2) \qquad t \in [-1, 1]$

$c'(t) = (1, 2t)$

$$\therefore \int_C F \cdot d\vec{s} = \int_{-1}^{1} (t^2 - 2t^3, t^4 - 2t^3) \cdot (1, 2t)\, dt$$

$$= -\frac{14}{15}$$

10) Line integrate $(x^2+y^2, x-y)$ around $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

in counter clockwise direction (once)

Ans $\quad c(t) = (a\cos t, b\sin t) \qquad t \in [0, 2\pi]$

Final answer $\rightarrow \pi ab$

11) Line integrate $\left( \frac{x+y}{x^2+y^2}, \frac{-(x-y)}{x^2+y^2} \right)$ around $x^2+y^2 = a^2$

in counter clockwise direction (once)

Ans $\quad c(t) = (a\cos t, a\sin t) \qquad t \in [0, 2\pi]$

Final answer $\rightarrow -2\pi$

~~12) Line integrate $(x^2+y^2, (x-y))$ around same stuff as in Q10~~

12) Line integral $(y, z, x)$ around the curve of intersection of $z = xy$, $x^2+y^2=1$ travelled once in direction which is counter clockwise from viewing from $z = +\infty$

<u>Ans</u> $c(t) = (\cos t, \sin t, \cos t \cdot \sin t)$

Final answer $\longrightarrow$ $-\pi$

13) $C$ is the curve $x^2+y^2=1$, $z=0$. $c_1$ is the parameterisation $(\cos t, \sin t)$ $t \in [0, 2\pi]$. $c_2$ is $(\cos t, -\sin t)$ $t \in [0, \pi]$. Find $\int_{c_1} F$, $\int_{c_2} F$ with $F = (-y, x)$

<u>Ans</u> $c_1 \longrightarrow 2\pi$

$c_2 \longrightarrow -\pi$

14) Show that a constant force field does 0 work on a particle that moves once uniformly around $x^2+y^2=1$. Is it also true for $F = (\alpha x, \alpha y, \alpha z)$ for some constant $\alpha$

<u>Ans</u> Let $F = (a, b)$

$c(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

we get line integral 0

For $F = (\alpha x, \alpha y, \tau z)$ and same ~~parameter~~ parametrisation. (note : $c(t) = (\cos t, \sin t, 0)$ )

we still get $0$

15) For $C := x^2+y^2 = 1$, find

$$\oint_c \text{grad}\,(x^2-y^2) \cdot d\vec{s}$$

Ans By FTC, answer is $0$

16) Find $\int_c \text{grad}(x^2-y^2) \cdot d\vec{s}$ with $C$ being

$y = x^3$ from $(0,0)$ to $(2,8)$

Ans By FTC, we get $(x^2-y^2)\Big|_{(2,8)} - (x^2-y^2)\Big|_{(0,0)}$

$$= (2^2-8^2) - 0$$

$$= -60$$

17) Find $\oint_c \frac{dx+dy}{|x|+|y|}$ with ~~c being square~~ joining

$(0,\pm 1)$, $(\pm 1, 0)$

Ans Shortcut instead of computing 4 different line integrals :

$|x| + |y| = 1$ on the curve

$\therefore$ We have $\oint_c dx+dy = \oint_c \nabla(x+y) \cdot d\vec{s} = 0$

(FTC)

18) $\vec{F} = xy\,\hat{i} + x^6 y^2\,\hat{j}$ is a force which moves a particle from $(0,0)$ onto the line $x=1$ along $y = ax^b$ $(a, b > 0)$. If work is independent of $b$, find value of $a$

Ans $\quad c(t) = (t, at^b) \quad t \in [0,1]$

$$\therefore I = \frac{3a + ba^3}{3b+6}$$

This is independent of $b$ if $\dfrac{\partial I}{\partial b} = 0$

$$\therefore a = \sqrt{\frac{3}{2}}$$

($a$ is not usually a constant; coincidentally, $a$ is not dependent on $b$)

# TUTORIAL 4

1) Determine and classify into open/closed, path connected/not, simply connected/not

(a) $\{(x,y) \mid 0 < y < 3\}$

(b) $\{(x,y) \mid 1 < |x| < 2\}$

(c) $\{(x,y) \mid 1 \le x^2+y^2 \le 4,\ y \geq 0\}$

(d) $\{(x,y) \mid (x,y) \neq (1,4)\}$

Ans (a) open, path conn, simply conn

(b) open, not path conn, not simply conn

(c) Not open, path conn, simply conn

(d) open, path conn, not simply conn

2) Check if $\vec{F} = (3xy, -x^3y)$ is a gradient on some open subset of $\mathbb{R}^2$

Ans $\nabla \times \vec{F} = \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}$

$= 3x^2y - 3x \neq 0$

$\therefore$ Not a gradient

3) Show that $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$ is path independent & evaluate it for any $C$ joining $(1,0)$ to $(2,1)$

Ans $F = (2xe^{-y},\ 2y - x^2e^{-y})$

$\nabla \times F = -2xe^{-y} + 2xe^{-y} = 0$

$\therefore F$ is a gradient

Observe that $\nabla(x^2e^{-y} + y^2 + c) = \vec{F}$

$\therefore \int_C F \cdot \vec{ds} = x^2e^{-y} + y^2 + c \Big]_{1,0}^{2,1} = \dfrac{4}{e}$

4) Is $\int_C y\,dx + x\,dy + xyz\,dz$ path independent in $\mathbb{R}^3$;

Ans $\quad \text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & xyz \end{vmatrix}$

$$= (xz, -yz, 0) \neq 0$$

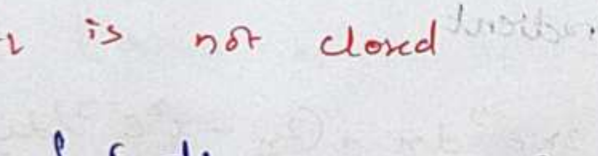
$\therefore$ No !

5) Let $F = \nabla f$, $f = \sin(x-2y)$. Find $C_1$, $C_2$ so that $\int_{C_1} F\cdot ds = 0$, $\int_{C_2} F\cdot ds = 1$ and $C_2$ is not closed

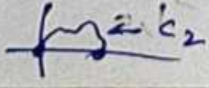
Ans $\quad \int_{C_1} F\cdot ds = \sin(x-2y)\Big|_{t_1}^{t_2} = 0$

$$\int_{C_2} F\cdot ds = \sin(x-2y)\Big|_{t_3}^{t_4} = 1$$

Choose $t_1 = (0,0)$ & $t_2 = (2\pi, 0)$




choose $t_3 = (0,0)$, $t_4 = (\frac{\pi}{2}, 0)$



You can literally choose any path since integral will only depend on end points

6) Check if F is conservative or not. If it is, find f so that $F = \nabla f$

(i) $(y^2 e^{xy}, (1+xy)e^{xy})$.

(ii) $(ye^x + \sin y, e^x + x\cos y)$

(iii) $\left(2xy + \frac{1}{y^2}, x^2 - \frac{2x}{y^3}\right)$

Ans (i) curl = 0 $\Rightarrow$ conservative

$$\frac{\partial f}{\partial x} = y^2 e^{xy} \Rightarrow f(x,y) = ye^{xy} + g(y)$$

$$\therefore \frac{\partial f}{\partial y} = (1+xy)e^{xy} + g'(y) = (1+xy)e^{xy}$$

$$\therefore g(y) = K$$

$$\therefore f(x,y) = ye^{xy} + K$$

(ii) curl = 0 $\Rightarrow$ conservative

$$f = ye^x + x\sin y + c$$

(iii) curl = 0 $\Rightarrow$ conservative

$$f = x^2 y + \frac{x}{y^2} + C$$

7) Find $f$ so that $F = \nabla f$ & evaluate $\int_C F \cdot ds$

(i) $(2xyz + \sin x, x^2 z, x^2 y)$ , $c(t) = (\cos^5 t, \sin^3 t, t^4)$
$t \in [0, \pi]$

(ii) $((1 + xy)e^{xy}, x^2 e^{xy})$ , $c(t) = (\cos t, 2\sin t)$
$t \in [0, \pi/2]$

(iii) $(yz, xz, xy + 2z)$ , $c(t) =$ line joining $(1, 0, -2)$ to $(4, 6, 3)$

Ans Same as prev. but one additional step of FTC

Final answers:

$(i)$ $0$

$(ii)$ $-1$ (Check !)

$(iii)$ $77$

8) For $v = (2xy + z^3, x^2, 3xz^2)$, show that $\nabla \phi = v$ for some $\phi$ & hence find $\oint_C v \cdot ds$ where $c$ is any arbitrary smooth closed curve

Ans Same as above question...

$\phi = x^2 y + xz^3 + K$

$\oint_C v \cdot ds = 0$ by FTC

9) $S = \mathbb{R}^2 \setminus \{0,0\}$. Let

$$F(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

(i) Prove $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} (=) 0$

(ii) Find $\int_C F \cdot d\vec{s}$ for the unit circle $C$

(pick any orientation of your choice)

(iii) Is $F$ conservative on $S$ ?

Ans (i) just do it ... ? ✓

(ii) $c(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

(anti clock wise)

$c'(t) = (-\sin t, \cos t)$

$$\int_C F \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = \underline{\underline{2\pi}}$$

(iii) No! Curl is zero but $\oint_C F \cdot d\vec{s}$ is not zero for some closed loop $C$

10) A radial force field is one which can be expressed as $\vec{F}(x,y,z) = f(r)\vec{r}$ where $\vec{r} = (x,y,z)$ and $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$. Prove that $f$ cont $\Rightarrow F$ conservative

Ans we want to prove that $F$ is conservative in $\mathbb{R}^3$

$\therefore$ we need $\phi$ so that $\nabla \phi = F$

$$\nabla \phi(r) = f(r)\,\vec{r}$$

But

$$\nabla \phi(r) = \phi'(r)\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \frac{\phi'(r)}{r}\,\vec{r}$$

$$\frac{\phi'(r)}{r}\,\vec{r} = f(r)\,\vec{r}$$

$$\therefore \quad \phi'(r) = r\,f(r)$$

$$\therefore \quad \phi(r) = \phi(a) + \int_a^{\vec{r}} s\,f(s)\,ds$$

for some $a \in \mathbb{R}^3$

$\therefore$ $\phi$ exists & hence $F = \nabla\phi$ and hence $F$ is conservative.

(we used that $f$ is continuous & hence $s f(s)$ is continuous & hence integrable)

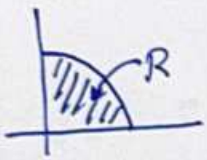
# TUTORIAL 5

1) Verify Greens theorem :

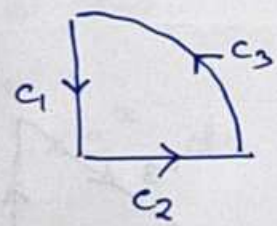
(i) $F = (-xy, xy)$ $\quad R: x \geq 0, 0 \leq y \leq 1-x^2$

(ii) $F = (2xy, e^x + x^2)$ $\quad R: \Delta$ joining $(0,0)$ $(1,0)$ $(1,1)$

Ans (i) line integral :




$$L = \int_{\partial R} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} + \int_{C_3} \vec{F} \cdot d\vec{s}$$




$C_1 : (0, t) \quad t \in [1, 0]$

$C_2 : (t, 0) \quad t \in [0, 1]$

$C_3 : (t, 1-t^2) \quad t \in [1, 0]$

$$\therefore L = \int_1^0 (0+0) \cdot c'(t)\, dt$$

$$+ \int_0^1 (0+0) \cdot c'(t)\, dt$$

$$+ \int_1^0 ((t^2-1)t,\ t(1-t^2)) \cdot (1, -2t)\, dt$$

$$= \int_1^0 t^3 - t - 2t^2 + 2t^4 \, dt = \frac{31}{60}$$

- Double integral :

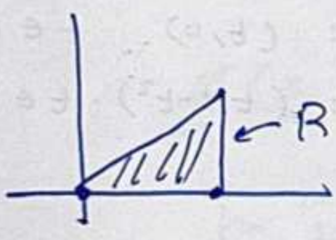
$$D = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx\, dy$$

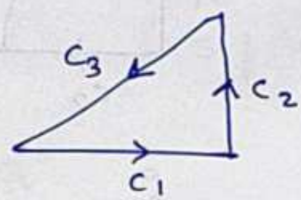
$$= \iint_R (y + x)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x^2} (x+y)\, dy\, dx$$

$$= \int_0^1 \frac{x^4 - 2x^3 - 2x^2 + 2x + 1}{2} = \frac{31}{60}$$ ✓ verified

(ii)




$$L = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} + \int_{C_3} \vec{F} \cdot d\vec{s}$$

$$= \int_0^1 (0, e^t + t^2) \cdot (1, 0)\, dt$$

$$+ \int_0^1 (2t, e+1) \cdot (0, 1)\, dt$$

$$+ \int_1^0 (2t^2, e^t + t^2) \cdot (1, 1)\, dt$$

$$= (0) + (e+1) + \int_1^0 3t^2 + e^t \, dt$$

$$= e + 1 - e$$

$$= 1$$

$$D = \iint_R e^x + 2x - 2x \, dx\, dy$$

$$= \boxed{\iint_R e^x \, dA}$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} e^x \, dy\, dx$$

$$= \int_0^1 x e^x \, dx$$

$$= 1$$ verified ✓

2) Use greens theorem to find $\oint_{\partial R} y^2 dx + x\, dy$

where R is

(i) square : 00 20 22 02

(ii) square : $\pm 1, \pm 1$

(iii) disc of radius 2 around (0,0) (clockwise)

Ans First of all, the line integral becomes

$$\iint_R (1-2y)\, dx\, dy$$

integration on squares is pretty direct

(i) $\pm \int_0^2 \int_0^2 (1-2y)\, dx\, dy = \pm 4$

(ii) $\pm \int_{-1}^{1} \int_{-1}^{1} (1-2y)\, dx\, dy = \pm 4$

$\boxed{\text{orientation of green in question}}$

(iii) $x = r\cos\theta$

$y = r\sin\theta$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} (1 - 2r\sin\theta)\, r\, dr\, d\theta$$

$$= 4\pi$$

3) Show using greens theorem that area enclosed by a curve in polar coordinates is given by $A = \frac{1}{2} \oint_C r^2 d\theta$. Find area of

(i) cardiod $r = a(1-\cos\theta) \quad \theta \in [0, 2\pi]$

(ii) lemniscate $r^2 = a^2 \cos 2\theta \quad \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

Ans (i) $\text{Area} = \iint_R 1 \, dA$

$$= \frac{1}{2} \int_{\partial R} x\,dy - y\,dx$$

$x = r\cos\theta$

$y = r\sin\theta$

Note: $x(t) = r(t)\cos\theta(t)$ $\quad t \in [a, b]$

$y(t) = r(t)\sin\theta(t)$

$$A = \frac{1}{2} \oint_{\partial R} x\,dy - y\,dx$$

$$= \frac{1}{2} \oint_{\partial R} \overset{(-y,\, x)}{} \cdot d\vec{s}$$

$$= \frac{1}{2} \int_a^b \left(-r(t)\sin\theta(t),\ r(t)\cos\theta(t)\right) \cdot c'(t)\, dt$$

Note: all wrt $t$ 2

$$= \frac{1}{2} \int_a^b (-r\sin\theta, r\cos\theta) \cdot (-r\sin\theta\,\theta' + r'\cos\theta,\ r'\sin\theta + r\cos\theta\,\theta')\, dt$$

$$= \frac{1}{2} \int_a^b (-r\sin\theta, r\cos\theta) \cdot (-r\sin\theta + r'\cos\theta,\ r'\sin\theta + r\cos\theta) \frac{d\theta}{dt}\, dt$$

$$= \frac{1}{2} \int_a^b r^2\cos^2\theta + r^2\sin^2\theta \frac{d\theta}{dt} dt = \frac{1}{2} \int_a^b r^2(t) \frac{d\theta}{dt}\, dt$$

$$= \frac{1}{2} \oint_{\partial R} r^2 \, d\theta$$

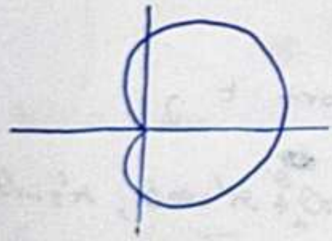
(i) $$\frac{1}{2} \oint_C r^2 \, d\theta = \frac{a^2}{2} \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \frac{3\pi a^2}{2}$$

(ii) $$\frac{1}{2} \oint_C r^2 \, d\theta = \frac{a^2}{2} \int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta = \frac{a^2}{2}$$

<u>Some figures for reference</u>

1) <u>Cardiod</u> :



" rolling circle on circle "

parametric → $x(t) = 2a(1-\cos t)\cos t$

$y(t) = 2a(1-\cos t)\sin t$

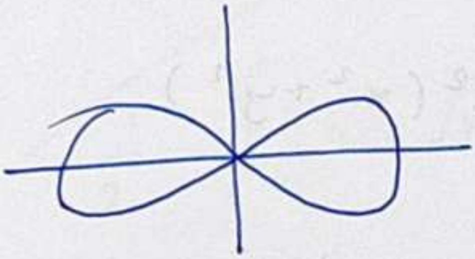
polar → $r(t) = 2a(1-\cos t)$

Can be found in tea cups, radio waves, etc.

2) lemniscate

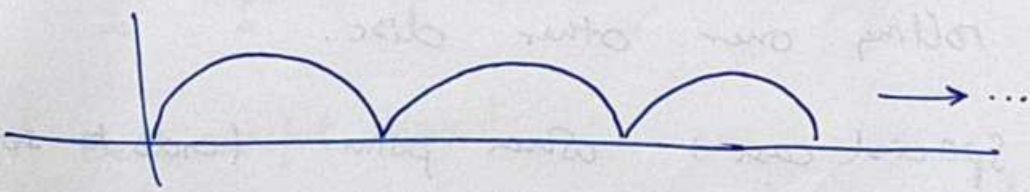
Zero set of $y^2 - x^2(a^2 - x^2)$

$r^2 = a^2 \sin 2\theta$



3) Cycloid

trajectory of point on a car wheel



$x(t) = \alpha(t - \sin t)$

$y(t) = \alpha(1 - \cos t)$

($\alpha$ represents circles radius)

cartesian:

$$\alpha \cos\left(\frac{x + \sqrt{y(2\alpha - y)}}{\alpha}\right) + y = \alpha$$

## 4) Limacon

(1) $r(t) = b + a \cos t \rightarrow$ (parametric)

(2) $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$

(3) $x(t) = (b + a \cos t) \cos t$

$y(t) = (b + a \cos t) \sin t$

Trajectory of random point inside disc rolling over other disc.

Special case: when point tends to some point on circumference, we get cardiod.

No diagram since so many thing possible!

4) Find area of

(i). $r = a(1 - \cos \theta)$ in $Q_I$

(ii) $\vec{r}(t) = a(t - \sin t)\,\hat{i} + a(1 - \cos t)\,\hat{j}$ $t \in [0, 2\pi]$

(iii) $r(t) = 1 - 2\cos \theta$ $0 \le \theta \le \frac{\pi}{2}$

~~Ans (i) $\frac{a^2}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = a^2\left(\frac{3\pi}{8}-1\right)$~~

~~(ii) $|r| = r(t) = \sqrt{a^2(t-\sin t)^2 + a^2(1-\cos t)^2}$~~

~~$= a\sqrt{t^2 - 2t\sin t + 1 + 1 - 2\cos t}$~~

~~$\frac{1}{2}\int_0^{2\pi}\left(a\sqrt{t^2-2t\sin t+2-2\cos t}\right)^2 dt$~~

~~$= \frac{1}{2}\int_0^{2\pi} a^2\left(t^2-2t\sin t+2-2\cos t\right)dt$~~

---

Ans (i) $\frac{a^2}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = a^2\left(\frac{3\pi}{8}-1\right)$

(ii) Be careful here!

we are given $(x(t), y(t))$ and

not $r(t)$ directly.

Do NOT get confused between

polar & parametric

$\downarrow$ has to be $x(t), y(t) = r\cos t,\ r\sin t$

(when $r$ also depends on $t$)

$\longrightarrow$ can be anything

So $|r(t)|$ is not necessarily

$$(x(t))^2 + (y(t))^2 .$$

This is only for polar. We are given parametric.

Well, it is true but you needn't bother with that.

To clarify, let us go back to the cardiod

parametric : $x(t) = 2a(1-\cos t)\cos t$

$y(t) = 2a(1-\cos t)\sin t$

polar : $r(t) = 2a(1-\cos t)$

Note: just a coincidence that $r(t) = x^2(t) + y^2(t)$.

~~What if $x(t) = \cos^2 t$~~

~~$y(t) = \cos t \sin t$ ?~~

~~Then $r(t)$ is .~~

if $x(t) = 2a(1-\cos t)\cos 4t$

$y(t) = 2a(1-\cos t)\sin 4t$

Then $r(t) = 2a(1-\cos(t))$ as per $(x(t))^2 + (y(t))^2$

but this $r(t)$ is not the polar $r(t)$

So here, we use that

$$A = \oint_c -y\,dx$$

(you could also use $\frac{1}{2}\oint x\,dy - y\,dx$)

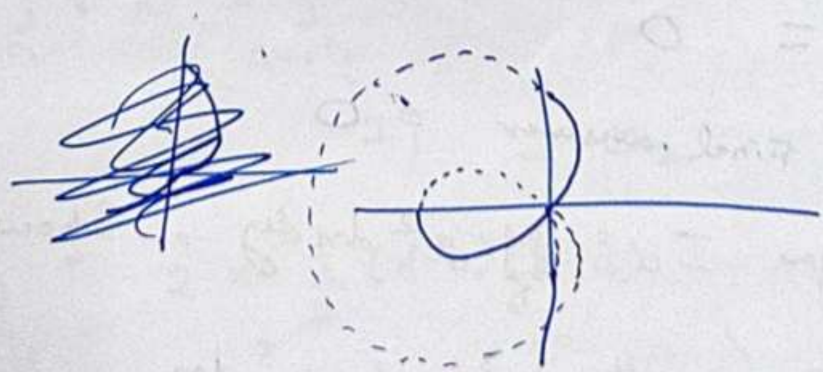
(but it is longer :( )

$$\oint_c -y\,dx = \int_0^{2\pi} a(1-\cos t)\frac{dx}{dt}\,dt$$

$$= a^2\int_0^{2\pi}(1-\cos t)^2\,dt$$

$$= 3\pi a^2$$

(iii) $\frac{1}{2}\int_0^{\pi/2}(1-2\cos t)^2\,dt = \frac{3\pi}{4} - 2$



5) $D = \{(x,y) \mid a^2 \le x^2+y^2 \le b^2\}$

$$\int_{\partial D} xe^{-y} dx + \left(-x^2 y e^{-y^2} + \frac{1}{x^2+y^2}\right) dy = ?$$

Ans split as

$$\int_{\partial D} xe^{-y} dx - x^2 y e^{-y^2} dy$$

$$+ \int_{\partial D} \frac{1}{x^2+y^2} dy$$

First part is 0 by FTC!

Second part becomes $\iint_D \frac{-2x}{(x^2+y^2)^2} dx\,dy$

$$= \int_0^{2\pi} \int_a^b - \frac{2r\cos\theta}{r^4(\cos^2\theta + \sin^2\theta)^2} \times r\,dr\,d\theta$$

$$= 0$$

$\therefore$ Final answer $= 0$

6) Define $I_0 = \iint_D r^2 \,dx\,dy$. Show that

$$3 I_0 = \int_{\partial D} x^3 dy - y^3 dx$$

Ans greens: $I_0 = \iint_D x^2 \, dx \, dy$

$$= \frac{1}{3} \oint_{\partial D} x^3 dy - y^3 dx$$

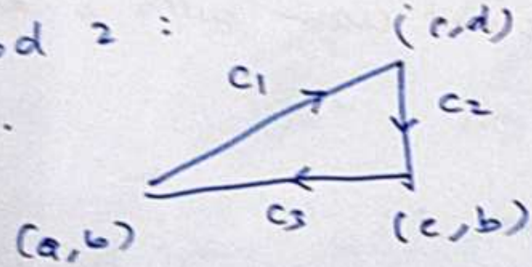
Done! LOL

7) C joins (a,b) to (c,d) (st. line). Show that

$$\int_C x\,dy - y\,dx = ad - bc$$

Ans Method 1 : directly do by parametrising the line (bit messy)

Method 2 :




Complete it to a triangle

$$\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{C_1 C_2 C_3} \vec{F}\cdot d\vec{s} - \int_{C_2} \vec{F}\cdot d\vec{s} - \int_{C_3} \vec{F}\cdot d\vec{s}$$

$$= I_1 - I_2 - I_3$$

$$I_1 = 2 \times \frac{1}{2}(d-b)(c-a)$$

$$\left(\because \text{Area} = \frac{1}{2} \oint_{C} \vec{F}\cdot d\vec{s}\right)$$

$I$ [illegible]

$I_2 = c(d-b)$ (parameterize & do)

$I_3 = b(c-a)$ ( " )

$\therefore I_1 - I_2 - I_3$

$= \underline{\underline{ad - bc}}$ (check!)

8) Find $\oint_C \nabla(x^2-y^2) \cdot \hat{n}\, ds$ for any anticlockwise closed curve with $\hat{n}$ as outward unit normal.

<u>Ans</u> Divergence form:

$$\iint_D \nabla \cdot F \, dx\, dy$$

$$= \iint_D \operatorname{div}(\operatorname{grad}(x^2-y^2))\, dx\, dy = 0$$

9) D satisfies greens hypothesis. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function.

(i) Show that $\nabla^2 \phi = \operatorname{div}(\operatorname{grad} \phi)$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(ii) Show $\iint_D \nabla^2 \phi \, dx\,dy = \int_{\partial D} \frac{\partial \phi}{\partial \hat{n}}$. where $\frac{\partial \phi}{\partial \hat{n}}$ is

the directional derivative

(iii) Find $\int_C \frac{\partial \phi}{\partial \hat{n}}$ for $\phi = e^x \sin y$, D is

the triangle : (0,0) (4,2) (0,2)

Ans (i) $\text{grad } \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j}$

$$\text{div}(\text{grad } \phi) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\phi)$$

(ii) Recall from MA109 ... (traumatic flashbacks)

$$\frac{\partial \phi}{\partial \hat{n}} = \nabla \phi \cdot \hat{n}$$

$\therefore$ By div form of greens

$$\int_{\partial D} \frac{\partial \phi}{\partial \hat{n}} = \int_{\partial D} \nabla \phi \cdot \hat{n} \, ds$$

$$= \iint_D \text{div}(\nabla \phi) \, dA = \iint_D \nabla^2 \phi \, dx\,dy$$

(iii) $\frac{\partial^2 \phi}{\partial x^2} = e^x \sin y$

$$\frac{\partial^2 \phi}{\partial y^2} = -e^x \sin y$$

$$\therefore \oint_c \frac{\partial \phi}{\partial \hat{n}} \, ds = \iint 0 = 0$$

10) $\Omega = \{ (x,y) \mid x^2+y^2 > 1 \}$. consider all curves anticlockwise and find:

(i) $\int_c \frac{y\,dx - x\,dy}{x^2+y^2}$ where $C$ is any simple closed curve enclosing the origin

(ii) same as above but not enclosing origin

(iii) Find $\int_c \frac{\partial \ln r}{\partial y} dx - \frac{\partial \ln r}{\partial x} dy$ for any simple closed curve $c$ in $\Omega$

Ans (i) see tut 4 Q9 but sign has changed so answer is $-2\pi$

(ii) $0$ (domain is open & simply connected) ($\text{curl} = 0$) (green can be applied)

(iii) $\frac{\partial \ln r}{\partial x} = \frac{\partial}{\partial x} \ln\left(\sqrt{x^2+y^2}\right) = \frac{x}{x^2+y^2}$

$\therefore$ If $0 \notin C$, answer $= 0$

If $0 \in C$, answer $= -2\pi$

11) Is there a $\vec{G}$ so that

(i) $\text{curl } G = (x \sin y, \cos y, z - xy)$

(ii) $\text{curl } G = (x, y, z)$

Ans (i) div curl $\neq 0$

(ii) div curl $\neq 0$

$\therefore$ No such $G$ :(

12) Show that any vector field of the form

$\overrightarrow{F(x,y,z)} = (f(x), g(y), h(z))$ has curl $= 0$

Show that any vector field of the form

$\overrightarrow{F(x,y,z)} = (f(y,z), g(x,z), h(x,y))$ has div $= 0$

Ans Just do it

# TUTORIAL 6

1) Parameterise

(i) plane $x - y + 2z + 4 = 0$

(ii) right circular cylinder $y^2 + z^2 = a^2$

Ans (i) $z = \frac{y - x - 4}{2}$

$\therefore \left( u, v, \frac{v - u - 4}{2} \right)$

(ii) $x$ can be anything. Say $u$

Then $(u, a\cos v, a \sin v$

2) Find tangent plane to surface with parameterisation

$x = u^2$, $y = v^2$, $z = u + 2v$ at $1, 1, 3$

Ans surface $\rightarrow (u^2, v^2, u + 2v)$

$$\phi_u = (2u, 0, 1)$$

$$\phi_v = (0, 2v, 2)$$

$$\phi_u \times \phi_v = (-2v, -4u, 4uv)$$

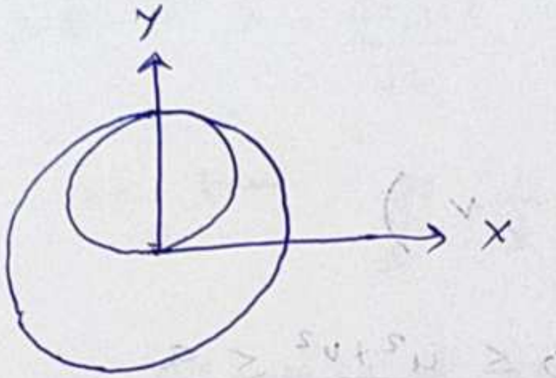
$\therefore$ DR's of normal are $(-2, -4, 4)$

point is $1, 1, 3$. DR's are known.

eqns is $(x - 1) + 2(y - 1) - 2(z - 3) = 0$

5) surface area of portion of $x^2+y^2+z^2=a^2$ which lies within $x^2+y^2=ay$ (for some $a>0$)

Ans




we find SA lying above XY plane and later double it.

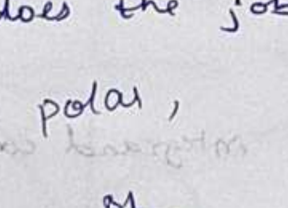
big sphere : $(u, v, \sqrt{a^2-u^2-v^2})$

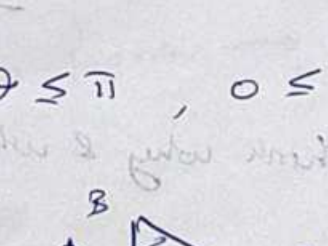
But now we want to restrict $u, v$ to only a certain area.

$0 \le v \le a$ , $-\sqrt{av-v^2} \le u \le \sqrt{av-v^2}$

does the job

In polar, $0 \le \theta \le \pi$ , $0 \le r \le a\sin\theta$



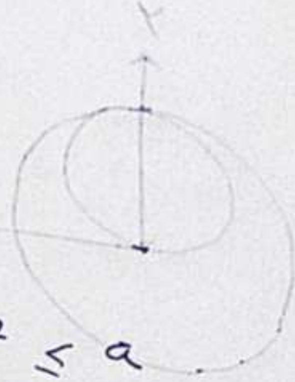

$\therefore$ we get $\int_0^{\pi}\int_0^{a\sin\theta} \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta = (\pi-2)a^2$

$\therefore$ net area $= 2a^2(\pi-2)$

4) Area of paraboloid $x^2 + z^2 = 2ay$ which lies within $y = 0$, $y = a$ planes is ___

Ans Parameterise :

$$\left( u, \frac{u^2+v^2}{2a}, v \right).$$



Now restrict $0 \le \frac{u^2+v^2}{2a} \le a$

In polar, $0 \le \theta \le 2\pi$,

$0 \le r \le \sqrt{2}\, a$

$$\therefore \text{Area} = \int_0^{2\pi} \int_0^{\sqrt{2}a} \frac{1}{a} \sqrt{1 + \frac{r^2}{a^2}}\; r \, dr \, d\theta$$

$$= \frac{3\sqrt{3} - 1}{3} \cdot 2\pi$$

(Think why & where the integrand comes from)

5) S is plane ~~-~~ containing 100 010 001

Let $\vec{F} = (x, y, z)$, $\hat{n}$ denote unit normal to S ~~-~~ having non negative z component. Evaluate the surface integral $\iint_{S^*} \vec{F} \cdot \hat{n} \, ds$

where $s^*$ is just the triangular part of S

Ans $\hat{n} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

$\therefore \vec{F} \cdot \hat{n} = \frac{x+y+z}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ (on the plane)

$\therefore$ we have $\iint_S \frac{1}{\sqrt{3}} \, dx\, dy = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}$

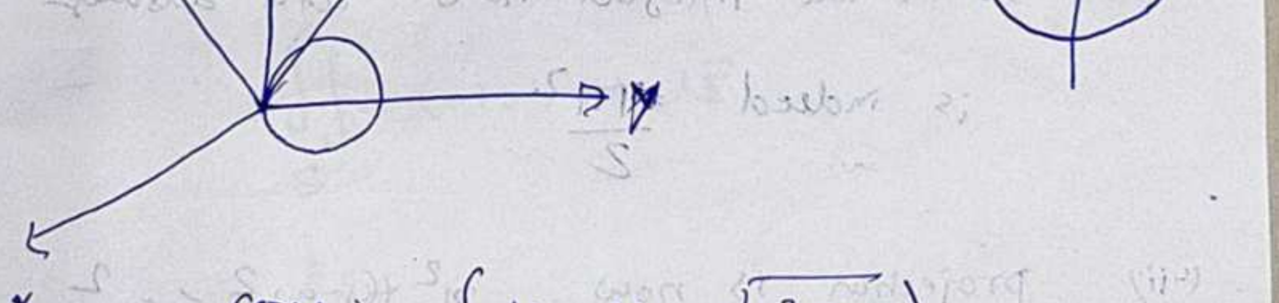
Boring method → see the later solutions
(use parametrisation $(u+v, u-v, 1-2u)$ )

6) Verify Stokes theorem for : $F = (x-y, x+z, y+z)$ over surface of cone $z^2 = x^2+y^2$ intercepted by

(i) $x^2+(y-a)^2+z^2 = a^2$, $z \geq 0$

(ii) $x^2+(y-a)^2 = a^2$

Ans (i)




Cone : $\left(u, v, \sqrt{u^2+v^2}\right)$

$\therefore \phi_u \times \phi_v = \left( \frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \right)$

Further note : curl F = (0, 0, 2)

Now $\iint_R \text{curl}F \cdot \hat{n}\,ds$

$= \iint_R 2\, ds$

$= 2\iint_R ds = 2 \times \frac{\pi a^2}{4} = \frac{\pi a^2}{2}$

( ∵ projection of surface on xy plane is :

$x^2 + (y - \frac{a}{2})^2 \leq \frac{a^2}{4}$

projection of C onto xy plane is

$(\frac{a}{2}\cos\theta, \frac{a}{2} + \frac{a}{2}\sin\theta)$

$\theta \in [0, 2\pi]$

Do the line integral now and answer is indeed $\frac{\pi a^2}{2}$

(ii) projection is now $x^2 + (y-a)^2 \leq a^2$

∴ Surface integral is $2\iint_R ds = 2 \times \pi a^2$

The $c(t)$ for line integral is

$(a\cos t,\ a + a\sin t) \qquad t \in [0, 2\pi]$

and line integral also comes out to be $2\pi a^2$

7) Using stokes, evaluate $\int_c yz\,dx + xz\,dy + xy\,dz$ where $c$ is curve of intersection of $x^2 + 9y^2 = 9,\ z = y^2 + 1$ with clockwise orientation (viewed from origin)

Ans curl of given field is zero! LOL!

$\therefore$ answer is zero!

($c$ is simple closed curve).

8) integrate $(z, -x, -y)$ around $\Delta$ with vertices

000 020 002

Ans curl is $-1, 1, -1$

$$\therefore \iint_S (-1, 1, -1) \cdot d\vec{S}$$

$$= \iint_S (-1, 1, -1) \cdot (1, 0, 0)\, dS$$

$$= \int_0^2 \int_0^{2-y} -dz\, dy = \underline{\underline{-2}}$$

9) C is curve of intersection of unit cylinder $x^2+y^2=1$, plane $x+y+z=1$.
[ project C on XY plane & use find ACW direction.

Find $\int_C -y^3 dx + x^3 dy - z^3 dz$

Ans $\text{curl } F = (0, 0, 3x^2+3y^2)$

Surface : $(u, v, 1-u-v)$

$\therefore$ normal is $(1, 1, 1)$

$$\therefore \iint_S \nabla\times F \cdot d\vec{S} = \iint_{S^*} 3x^2+3y^2\, dx\, dy$$

where $S^*$ is projection of S onto XY plane which in our case is the unit circle

$$\therefore \iint_{\substack{\text{unit}\\\text{circle}}} 3x^2+3y^2\, dx\, dy = 3\int_0^{2\pi}\int_0^1 r^2 \cdot r\, dr\, d\theta$$

$$= 2\pi \times \frac{1}{4} \times 3 = \frac{3\pi}{2}$$

10) $F = (y, -x, e^{xz})$, $S = \{(x,y,z) \mid x^2+y^2+(z-\sqrt{3})^2 = 4, z \geq 0\}$

Find $\iint_S \text{curl } F \cdot d\vec{S}$

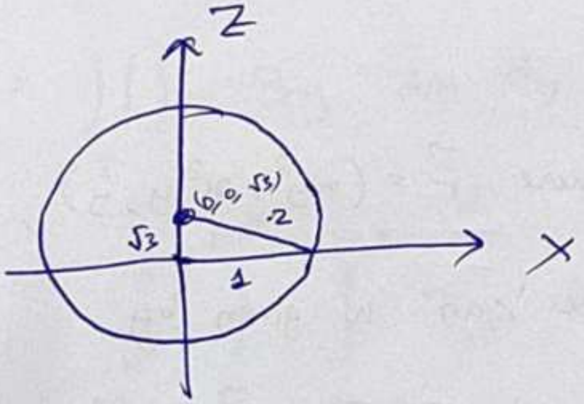
Ans: using stokes we convert to a line integral

The surface is a portion of sphere above XY plane & hence its boundary is circle in XY plane

Here, center of sphere = $0\ 0\ \sqrt{3}$

radius = 2

i.e. $(\cos t, \sin t, 0)$ is the boundary.




Do the line integral:

$$\int_0^{2\pi} (\sin t, -\cos t, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$e^{0 \times \cos t} = 1$

$= -2\pi$

11) Find flux of $(x^3, y^3, z^3)$ through unit sphere.

Ans $\nabla\cdot F = 3(x^2+y^2+z^2)$

we want
$$\iiint_{\text{unit sphere}} 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 3r^2\; r^2\sin\theta\; dr\,d\phi\,d\theta$$

$$= \frac{12}{5}\pi$$

12) Find $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = (xy^2, x^2y, y)$, and S is surface of the 'can' W given by $x^2+y^2 \le 1$, $-1 \le z \le 1$

Ans
$$\int_0^{2\pi}\int_{-1}^{1}\int_0^{1} r^2\cdot r\; dr\,dt\,d\theta$$

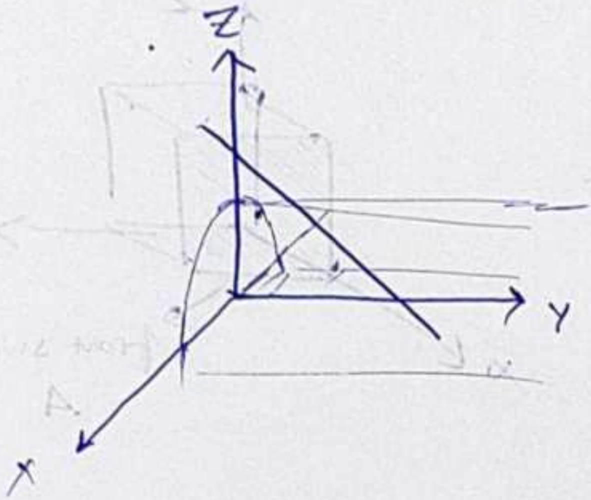
$= \pi$ (cylindrical)

$(\because \operatorname{div} F = x^2+y^2)$

13) Find $\iint_S \vec{F}\cdot d\vec{S}$ for $F = (xy,\, y^2+e^{xz^2},\, \sin xy)$

S being surface bounded by $z = 1-x^2$, $z=0$, $y=0$, $y+z=2$

Ans




$$\iiint_S \operatorname{div} F \, dx\,dy\,dz$$

$$= \iiint_S 3y \, dx\,dy\,dz$$

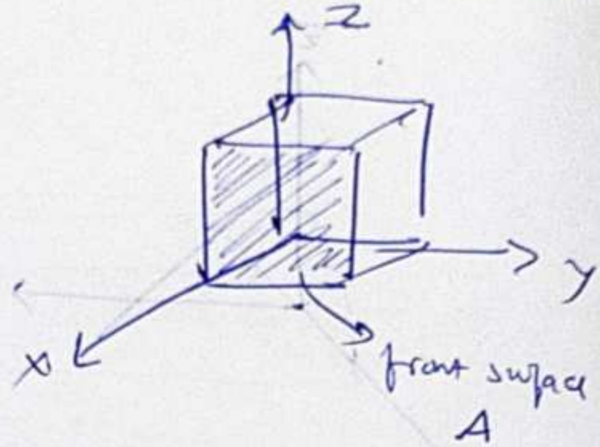
$$= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y \, dy\,dz\,dx$$

$$= \frac{98}{35}$$

14) flux of $F = (xy, yz, zx)$ through standard unit cube

Ans we only add flux from 3 surfaces of cube since on the other three, flux is zero since F itself is zero

By symmetry, we only find 1 and triple it




$$\iint_A (xy, yz, zx) \cdot (1,0,0)\, ds$$

$$= \int_0^1\int_0^1 y \, dy\, dz = \frac{1}{2} \quad (\because x = 1)$$

$\therefore$ Final result $= \dfrac{3}{2}$

15) is $(x, -2y, z)$ a curl? find the field if it is.

Ans $\operatorname{div}((x, -2y, z))$

$= 1 - 2 + 1 = 0$ !

It is a curl.

Big equations to find the field

Overall, $\nabla \times (-yz, 0, xy) = (x, -2y, z)$

→ hard to get :(