Schur Polynomials - 4 equivalent definitions

Definition : Given a ferrer shape of λ, we can fill its boxes with positive numbers such that each row is weakly increasing and each column is strictly increasing. Such an object is called a semi-standard Young's Tableaux of shape λ

example : An SSYT of shape $\lambda = (5,3,2,2,1) \vdash 13$ is

1 1 3 5 7
2 3 6
5 5
6 7
9

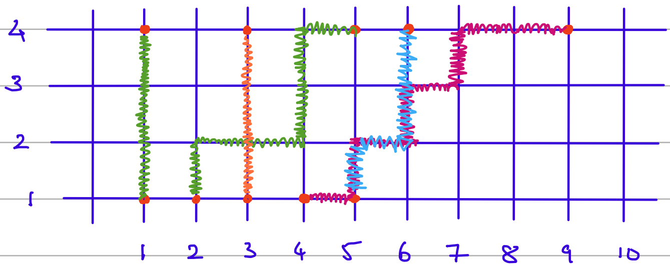
Definition : Let $P = \{(1,1), (2,1), \dots, (m,1)\}$ and Q be any collection of m points on the line $y = h$ in $\mathbb{Z}^2$. We define a lattice path from P to Q to be an m-tuple consisting of strings made up using the alphabet $\{U, R\}$ (up, right) which describe how each point of P connects to a point of Q (matching)

example : $P = \{(1,1), (2,1), \dots, (5,1)\}$

$Q = \{(1,4), (3,4), (5,4), (6,4), (9,4)\}$

A lattice path could be {UUU, URRUUR, UUU, RURURURR, URUU}

The diagram is as follows



Notice how first UUU is fixed and because p_2 matched with q_3, the $p_3 q_2$ match UUU was also fixed

Definition : We note the y coordinate height of each 'R' in each of the m strings of a lattice path from P to Q and associate the monomial $\prod x_i$ to it where i varies over heights corresponding to 'R'. Suppose P_i gets mapped to Q_j, gathering $\{j_1, j_2, \dots, j_m\}$ corresponding to the order $\{1, 2, \dots, m\}$ for i, we get a permutation σ of $[m]$ for Q. Impose a $\operatorname{sgn}(\sigma)$ to our monomial.

Definition : Summing over all possible monomials with signs we obtain a polynomial which we call as the PQ-lattice path polynomial

example : In our running example we have the monomial given by

$x_2 x_2 x_4 x_2 x_1 x_2 x_3 x_4 x_4 = x_1 x_2^4 x_3 x_4^3$

Proposition 1 :

Suppose the x coordinates in Q are $x_1, x_2, \dots, x_m$, the degree of the PQ polynomial is $(x_1 - 1) + (x_2 - 2) + \dots + (x_m - m)$

proof :

Suppose $p_1, p_2, \dots, p_m$ are connected to $q_{\sigma(1)}, \dots, q_{\sigma(m)}$, then total no. of rights from p_i to $q_{\sigma(i)}$ i.e. $(i, 1)$ to $(x_{\sigma(i)}, m)$ is clearly $x_{\sigma(i)} - i$. Summing over all i, we get our result ■

Definition : We define some symmetric functions below. Here, λ is some partition given by $\lambda = (\lambda_1, \lambda_2, \dots)$ and $x = (x_1, x_2, \dots)$.
If $\alpha = (\alpha_1, \alpha_2, \dots)$ is some seq of integers, $x^{\alpha} := x_1^{\alpha_1} x_2^{\alpha_2} \dots$

(i) Monomial symmetric functions :

$m_\lambda(x) := \sum x^{\sigma(\lambda)}$ where sum is over all possible distinct permutations σ of λ

(ii) elementary symmetric functions :

$e_n(x) := m_{1^n}(x) = \sum x_{i_1} \cdots x_{i_n}$ where sum is over $i_1, \dots, i_n$ such that $1 \le i_1 < \dots < i_n$

$e_\lambda(x) := e_{\lambda_1}(x) \cdot e_{\lambda_2}(x) \dots$

(iii) complete symmetric functions :

$h_n(x) := \sum_{\lambda \vdash n} m_\lambda(x) = \sum x_{i_1} \cdots x_{i_n}$ where sum is over $i_1, \dots, i_n$ such that $1 \le i_1 \le i_2 \dots \le i_n$

Incidentally, $h_n(x_1, \dots, x_m) = \sum_{\alpha_1 + \dots + \alpha_m = n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$

$h_\lambda(x) := h_{\lambda_1}(x) \cdot h_{\lambda_2}(x) \dots$

(iv) power sum symmetric functions :

$p_n(x) := m_{(n)}(x) = x_1^n + x_2^n + \dots$

$p_\lambda(x) := p_{\lambda_1}(x) \cdot p_{\lambda_2}(x) \dots$

There is another unconventional rep theory definition. Suppose $\lambda = (1^{m_1}, 2^{m_2}, \dots)$, then,

$p_\lambda(x) := (p_1(x))^{m_1} (p_2(x))^{m_2} \dots$

(Convince yourself that these are one and the same)

Instead of providing the 4 equivalent definitions, we give a single one and state the remaining as identities of Schur polynomials.

Definition: Let $\lambda \vdash n$. Consider an SSYT of shape λ filled with positive integers $\{1,2,\dots\}$. Count the frequency m_i of each number $i \in \{1,2,\dots\}$. Associate the polynomial $x_1^{m_1} x_2^{m_2} \cdots$ to this SSYT. When summed over all possible SSYT of shape λ, we get a polynomial known as the Schur polynomial. It is denoted commonly by $s_\lambda(x)$
(When restricted to n variables $s_\lambda(x_1,\dots,x_n)$ is sum over all SSYT polynomials where the tableau is filled by $\{1,2,\dots,n\}$. In this if $\ell(\lambda) =$ length of $\lambda > n$ we can't achieve strictness of columns and the polynomial is 0. Thus we assume $\lambda = (\lambda_1,\dots,\lambda_n)$)

Theorem 2 (Jacobi - Trudi identity)

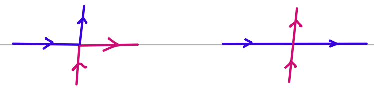
$$s_{\lambda+\delta}(x_1,\dots,x_n) = \det H_{n\times n}$$

where the $(i,j)^{th}$ entry of H is $h_{ij} = h_{\lambda_i - i + j}(x_1,\dots,x_n)$ and $\delta = (n-1, n-2, \dots, 2, 1, 0)$ and $\lambda = (\lambda_1,\dots,\lambda_n)$

proof

We try to find a formula for the PQ-lattice polynomial where,
$P = \{(1,1),\dots,(n,1)\}$, $Q = \{(1+\lambda_n, n), \dots, (n+\lambda_1, n)\}$.
Now as noted in proposition 1, the k^{th} string contributes towards a degree of $\sigma(k) + \lambda_{n-\sigma(k)+1} - k$.
Reading from n to 1, the k^{th} string now contributes a degree of $n - \sigma(k) + 1 + \lambda_{\sigma(k)} - (n-k+1) = \lambda_{\sigma(k)} + k - \sigma(k) = \beta(k)$
This monomial could be any possible monomial in $x_1,\dots,x_n$ since we can take a right at any possible height.
Thus, the net contribution is $\sum_{\alpha_1+\dots+\alpha_n = \beta(k)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = h_{\beta(k)}(x_1,\dots,x_n)$
Hence, finally,
PQ-polynomial is $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} h_{\beta(k)}(x_1,\dots,x_n)$ (if σ is not possible for the path, the polynomial is to be taken to be 0. You can indeed check that this is the case)
By definition of determinant this is just $\det(H^T) = \det(H)$
Now we interpret the PQ polynomial in a combinatorial way.
Consider the point (x_0, y_0) such that x_0, y_0 are chosen

as large as possible so that (x_0, y_0) is an intersection point of two paths in a given lattice path. By pigeonhole, exactly 2 paths intersect at (x_0, y_0) (else we get another maximal point) Let the paths be from p_{i_1} to q_{j_1}, p_{i_2} to q_{j_2}. We have another lattice path PQ which looks like this one (diagramatically) but these particular paths are interpreted in the other way ie. p_{i_1} to q_{j_2} & p_{i_2} to q_{j_1}. A coloured example is shown below



The parity differs by one since only a transposition is involved meanwhile the non-signed monomial part stays the same. Hence, these cancel off.

Thus, the PQ-polynomial is just considered to be summed over all possible non intersecting lattice path. If p_i is connected to a q_j for $j \neq i$, it is an easy excercise to conclude that there is an intersection. Thus, we want p_i to be connected to q_i ie. $\sigma = id$ & hence all signs involved are positive

Thus, the monomial weight for the k^{th} part is $\beta(k) = \lambda_k$

Construct an SSYT as follows to record the complete monomials. Corresponding to p_n, q_n, we have a monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ of degree λ_1 (ie. $\beta(1)$, n^{th} from beginning, 1st from end). Write this out with unit powers $x_1 x_1 \cdots x_n x_n$ (there will λ_1 symbols involved). Put the subscripts in the λ_1 boxes of the SSYT of shape λ

As an example, if we have $x_1^0 x_2^3 x_4^1 x_5^2$, Put 222455 in the row.

Continue till the last pair $p_1 q_1$.

We need to justify why this is an SSYT. The rows are clearly weakly increasing. The strictness of columns arises from the fact that the paths are non intersecting. We use induction for showing this.

Suppose row 1 is filled as $a_1 \, a_2 \cdots a_{\lambda_1}$. We first show that $b_1 > a_1$. This is clear because, by definition of a_1, the path from p_1 to q_1 takes its first right at height a_1 and hence if $b_1 < a_1$, the paths will crash ( ) and if $b_1 = a_1$ we have an intersection point ( ). Now subtract off a_1 from both sequences to get $0, a_2 - a_1, \cdots a_{\lambda_1} - a_1$ & $b_1 - a_1, b_2 - a_1, \ldots$. Forget about the first term. Then just thinking of the paths as starting from height $\max\{a_1, b_1\}$ and using the crashing logic already given we have by induction that $b_2 - a_1 > a_2 - a_1 \Rightarrow b_2 - a_2$. By total induction we have an SSYT

Conversely, given an SSYT, constructing the lattice diagram is direct and it is easily seen that the paths are non intersecting because columns are strictly increasing

Thus the PQ polynomial $\det(H)$ is just sum over polynomials associated with SSYT (as above).

Thus by definition, $s_\lambda(x) = \det(H)$ ∎

Theorem 3 (Dual Jacobi Trudi identity)

$$s_{\lambda+\delta}(x_1, \dots, x_n) = \det E_{n\times n}$$

where the $(i,j)^{th}$ entry of E is $e_{ij} = e_{\lambda'_i - i + j}(x_1, \dots, x_n)$

and $\delta = (n-1, n-2, \dots, 2, 1, 0)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$

Here $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ is the conjugate partition of λ (i.e., partition corresponding to the transpose of the Ferrer's shape of λ)

proof

We imitate the above proof and make minor modifications to it.

Now P is the same and $Q = \{(1+\lambda'_n, n), \dots, (n+\lambda'_1, n)\}$

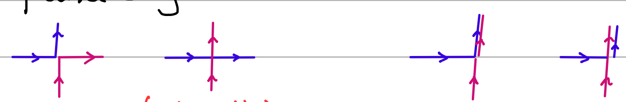
we now further restrict the paths such that we cannot have two consecutive R in any of the n strings

Now, naturally, the restricted lattice polynomial will have summation over monomials with degree $\beta(k)$ but each x_i will have degree at most 1. This is just $e_{\beta(k)}(x)$

Thus the poly is $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} e_{\lambda'_{\sigma(k)} - \sigma(k) + k}(x_1, \dots, x_n)$.

Again, this is just $\det(E^T) = \det(E)$

Combinatorially, observe that intersection points are now okay but intersection segments aren't. The diagrams below should be self explanatory



(not possible)

Also note that a horizontal overlap is always followed by a vertical one & one may thus choose the rightmost & topmost vertical overlap & cancel off pairs in the sum

Thus we want non overlapping lattice paths P to Q (restricted)

It is easy to see that this is just corresponding to the matching $p_1 q_1 \cdots p_n q_n$ (ie. $\sigma = id$).

Reading backwards, firstly from the $p_n q_n$ pair, we obtain monomial of degree $\beta(1) = \lambda_1'$ st. each x_i occurs at most once.

Constructing the "SSYT" as before, observe that rows will be strictly increasing now.

We now justify why columns must be weakly increasing.

Suppose the first row ($p_n q_n$) is $a_1 a_2 \cdots a_{\lambda_1'}$ & second row is $b_1 b_2 \cdots b_{\lambda_2'}$. Then $b_1 < a_1$ is not possible else the paths crash ( ) but $b_1 = a_1$ is okay because after turning right at b_1, by restriction it must go up & the $p_n q_n$ path has already gone to the right. Hence we have an intersection point but no overlap. ( )

Clearly, transposing this "SSYT" of shape λ' gives an actual SSYT of shape λ (" " because it isn't an SSYT by definition)

Conversely, every "SSYT" gives a non-overlapping path (just reverse the procedure. very easy)

Thus by definition $s_\lambda(x_1, \dots, x_n) = \det(E)$

∎

Definition: Given any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ we define the alternant V_λ to be the determinant of the $n \times n$ matrix whose $(i,j)^{th}$ entry is $x_i^{\lambda_j}$

(Note that all λ_i must be distinct for a non-zero alternant ie. λ must be a strictly decreasing partition)

Proposition 4:

Strictly decreasing partitions of $n(n+1)/2$ are in one-one correspondence with partitions of n

proof:

Let $(\mu_1, \dots, \mu_n)$ be a partition of n. Construct a partition $(\lambda_1, \dots, \lambda_n)$ where $\lambda_i = \mu_i + n - i$. Firstly, $\lambda_1 + \cdots + \lambda_n$ is just $n + \sum_{i=1}^{n} n - i = n + n(n-1)/2 = n(n+1)/2$ and clearly $\lambda_1 > \lambda_2 > \cdots > \lambda_n$

Conversely, given such a λ, construct $(\mu_1, \dots, \mu_n)$ as $\mu_i = l_i - n + i$

Clearly $\sum \mu_i = n$ and $l_i > l_{i+1} \Rightarrow l_i - n + i > l_{i+1} - n + i$

$\therefore \mu_i > \mu_{i+1} - 1 \Rightarrow \mu_i \geq \mu_{i+1}$

■

Definition: Given a partition $\lambda \vdash n$, convert it to a strictly decreasing partition of n ie. $\mu = \lambda + \delta$ $(\delta = (n-1, n-2, \dots, 1, 0))$

Define the shifted alternant W_λ to be $V_\mu = V_{\lambda+\delta}$

Definition: Define the Vandermonde determinant Δ_n to be the determinant of the $n \times n$ matrix whose $(i,j)^{th}$ entry is given by x_j^{i-1} ie. $\Delta_n = V_\delta$ $(\delta = (n-1, n-2, \dots, 0))$

Fact 5:

$$\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j) \qquad \text{(proved using induction on } n\text{)}$$

Proposition 6:

The ratio of W_λ to Δ_n is a symmetric polynomial

proof:

In the expression for $i \neq j$, if $x_i = x_j$, $W_\lambda = V_{\lambda+\delta} = 0$

$\therefore \Delta_n$ divides W_λ

Δ_n & W_λ are both skew symmetric & hence their ratio is symmetric (exchanging x_i with x_{i+1} changes two columns of the det in numerator & denominator & signs cancel)

■

Theorem 7:

The ratio described above is exactly $s_\lambda(x_1, \dots, x_n)$

Proving this will require developing some machinery

I shall only state the required results and proceed. Detailed proofs & analysis can be found in my symmetric functions full notes.

The theory I will be stating pertains to the 4 symmetric functions m, e, h, p. The RSK-algorithm, too, will be treated as a black box here

Fact sheet

1) $\{m_\lambda \mid \lambda \vdash n\}$, $\{e_\lambda \mid \lambda \vdash n\}$, $\{h_\lambda \mid \lambda \vdash n\}$, $\{p_\lambda \mid \lambda \vdash n\}$ form basis for the symmetric functions of n variables

2) $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$ where $M_{\lambda\alpha}$ is the number of 0-1 matrices having Row sums λ, column sums α

3) $h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu$ where $N_{\lambda\alpha}$ is the number of $\mathbb{N}$ matrices having Row sums λ, column sums α

4) $p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu$ where $R_{\lambda\alpha}$ is the number of ordered partitions $(B_1, \ldots, B_{\ell(\mu)})$ of $[\ell(\lambda)]$ st. $\mu_j = \sum_{i \in B_j} \lambda_i$ $(1 \le j \le k)$

5)
$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda \sum_\mu M_{\lambda\mu} m_\lambda(x) m_\mu(y)$$
$$= \sum_\lambda e_\lambda(x) m_\mu(y)$$
$$= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x) p_n(y)\right)$$
$$= \sum_\lambda \frac{1}{n} p_\lambda(x) p_\lambda(y)$$

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda \sum_\mu N_{\lambda\mu} m_\lambda(x) m_\mu(y)$$
$$= \sum_\lambda h_\lambda(x) m_\mu(y)$$
$$= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y)\right)$$
$$= \sum_\lambda \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda(x) p_\lambda(y)$$

Where if $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$, then $z_\lambda := \prod_i i^{m_i} (i!)$ and $\varepsilon_\lambda = (-1)^{m_2 + m_4 + \cdots}$

6) Define $\omega : \Lambda^n \longrightarrow \Lambda^n$ (Λ^n = symm functions in n-variables), a $\mathbb{Q}$-algebra morphism st $\omega(e_n) := h_n$. Then

(i) ω is invertible and $\omega(h_n) = e_n$

(ii) $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$

7) $e_n(x) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda(x)$

$h_n(x) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \cdot \varepsilon_\lambda \cdot p_\lambda(x)$

8) Define $\langle \, , \rangle$ on $\Lambda := \bigoplus_{n \geq 1} \Lambda^n$ as $\langle m_\lambda , h_\mu \rangle = \delta_{\lambda\mu}$.

We have

(i) $\langle \, , \rangle$ is an inner product

(ii) $\langle p_\lambda , p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$

9) Let $\{u_\lambda\}$, $\{v_\lambda\}$ be bases of Λ s.t. $\forall \lambda \vdash n$, $u_\lambda, v_\lambda \in \Lambda^n$.
Then, they are dual iff $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$

10) [RSK algorithm] Every infinite matrix A over $\mathbb{N}$ having finitely many non zero entries is in one-one correspondence with pairs (P, Q) of SSYT's of the same shape

11) $\{s_\lambda \mid \lambda \vdash n\}$ is a basis of Λ^n and if $s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$, $K_{\lambda\mu}$ is no. of SSTY of shape λ, type α i.e. α_i no. of i's in the tableu

12) $K_{\lambda, 1^n} = f^\lambda$ = no. of paths from ϕ to λ in Young's lattice

13) $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$

Proposition 8 [Cauchy's identity]

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

proof

LHS $= \prod_{i,j} \sum_{a_{ij}=0}^{\infty} (x_i y_j)^{a_{ij}}$

Thus coefficient of $(x_1^{\alpha_1} x_2^{\alpha_2} \cdots)(y_1^{\beta_1} y_2^{\beta_2} \cdots) = x^\alpha y^\beta$ is the number of $\mathbb{N}$-matrices with column sum α, row sum β.
Coefficient of $x^\alpha y^\beta$ in RHS is no. of pairs of SSYT

(P, Q) of shape λ each s.t. type $(P) = \alpha$, type $(Q) = \beta$

Via the RSK algo, these coefficients are the same and hence, we are done

Corollary 9

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

proof

From fact sheet (5) and proposition 8,

$$\sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

Coefficient of $m_\lambda(y)$ on LHS $= h_\lambda(x)$

$$\text{RHS} = \sum_\lambda s_\lambda(x) \left(\sum_\mu K_{\lambda\mu} m_\mu(y) \right)$$

coefficient of $m_\lambda(y)$ on RHS $= \sum_\mu K_{\lambda\mu} s_\mu(x)$

Fact 10

By virtue of the dual RSK algorithm,

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda'}(y)$$ where λ' denotes conjugate

Proposition 11

$$\omega_y \left(\prod_{i,j} (1 - x_i y_j)^{-1} \right) = \prod_{i,j} (1 + x_i y_j)$$

proof

$$\begin{aligned} \text{LHS} &= \omega_y \left(\sum_\lambda m_\lambda(x) h_\lambda(y) \right) \\ &= \sum_\lambda m_\lambda(x) e_\lambda(y) \\ &= \text{RHS} \end{aligned}$$

Corollary 12

$$\omega(s_\lambda) = s_{\lambda'}$$

proof

$$\begin{aligned} \sum_\lambda s_\lambda(x) s_{\lambda'}(y) &= \prod_{i,j} (1 + x_i y_j) \\ &= \omega_y \left(\prod_{i,j} (1 - x_i y_j)^{-1} \right) \\ &= \omega_y \left(\sum_\lambda s_\lambda(x) s_\lambda(y) \right) \end{aligned}$$

$$= \sum_{\lambda} s_\lambda(x)\, \omega_y(s_\lambda(y))$$

From linear indepence, $\omega(s_\lambda) = s_{\lambda'}$

proof of theorem 7

From corollary 9, $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$

Applying ω on both sides, we get,

$e_\mu = \sum_\lambda K_{\lambda\mu} s_{\lambda'} = \sum_\lambda K_{\lambda'\mu} s_\lambda$ (since we sum over all λ anyways, this is fine)

Since $\{s_\lambda\}$, $\{m_\lambda\}$ are both bases, $K_{\lambda\mu}$ is invertible and it suffices to show:

$$\Delta_n e_\mu = \sum_\lambda K_{\lambda'\mu} W_\lambda = \sum_\lambda K_{\lambda'\mu} V_{\lambda+\delta}$$

Both sides being skew symm, it suffices to show that the coefficient of $x^{\lambda+\delta}$ in LHS is $K_{\lambda'\mu}$

By definition, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$.

We multiply $\Delta_n e_\mu$ term by term as $\Delta_n e_{\mu_1}$, $\Delta_n e_{\mu_1} e_{\mu_2}, \ldots$

Each partial product is skew-symm & hence every x^i appearing must have all $i_1, \ldots, i_n$ distinct. When we multiply by $e_{\mu_{j+1}}$ to get the next partial product, we are essentially multiplying by sum of terms of the form $x_{m_1} x_{m_2} \cdots x_{m_p}$ $(p = \mu_{j+1})$. On multiplying 2 exponents become equal, in which case the term disappears or all remain distinct.

Thus, to get $x^{\lambda+\delta}$, we must start from x^δ and multiply it by x^{γ_1} from e_{μ_1}, x^{γ_2} from e_{μ_2} ... to finally end up with $x^{\lambda+\delta}$ in $\Delta_n e_\mu$

Given $x^{\gamma_i}, \ldots$ as above, create SSYT T as follows:

Put i in column j of T if x_j occurs in x^{γ_i}

Since $\mu_1 \geq \mu_2 \geq \cdots$, the rows of T will be weakly increasing. Given the info about columns of T, there will be a unique way to put the i's so that we have an SSYT

Note: we constructed an SSYT of shape λ' and type μ and there are $K_{\lambda'\mu}$ such. Thus, the proof is done. ∎

We give an example to clarify the last few statements made in the above proof

example : Find the coefficient of $x^{\lambda+\delta}$ in $\Delta_n e_\mu$ for $\lambda = (5,3,3,2)$, $\mu = (3,2,2,2,2,1,1)$, $n = 4$

Ans :

$$e_\mu = e_3 \, e_2^4 \, e_1^2 = (x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_4) \times (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)^4 \times (x_1 + x_2 + x_3 + x_4)^2$$

$$\Delta_4 = (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)$$

Δ_4 supplies us with our initial $x^\delta = x_1^3 x_2^2 x_3^1$ (if we chose something else like $x_2 x_3^2 x_4^3$, we will reach 0 eventually because of the symmetric nature of e)

Now we want to start with $x_1^3 x_2^2 x_3^1$ & multiply termwise from e_μ (1+4+2) partial products) so that we reach $x_1^8 x_2^5 x_3^4 x_4^2$ so that powers are always strictly decreasing.

Now, from $(x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_4)$ we may choose only $x_1x_2x_3$ (others dont have strictly decreasing coeffs)

Then $x_1^3 x_2^2 x_3^1 \longrightarrow x_1^4 x_2^3 x_3^2 x_4^0$

From $(x_1x_2 + \cdots + x_3x_4)$ suppose we choose x_1x_4 reach $x_1^5 x_2^3 x_3^2 x_4^1$

Then from $(x_1x_2 + \cdots + x_3x_4)$ we chose x_2x_3 to get $x_1^5 x_2^4 x_3^3 x_4^1$ and so on ...

The chain is shown below

$$x_1^3 x_2^2 x_3^1 \xrightarrow{x_1x_2x_3} x_1^4 x_2^3 x_3^2 \xrightarrow{x_1x_4} x_1^5 x_2^3 x_3^2 x_4^1 \xrightarrow{x_2x_3} x_1^5 x_2^4 x_3^3 x_4^1 \xrightarrow{x_1x_4}$$

$$x_1^8 x_2^5 x_3^4 x_4^2 \xleftarrow{x_3} x_1^8 x_2^5 x_3^3 x_4^2 \xleftarrow{x_1} x_1^7 x_2^5 x_3^3 x_4^2 \xleftarrow{x_1x_2} x_1^6 x_2^4 x_3^3 x_4^2$$

Based on the orange terms we create the SSYT :

For column 1, observe x_1 in x^{γ_1} , x_1 in x^{γ_2} ...

So x_1 is present in $x^{\gamma_1}, x^{\gamma_2}, x^{\gamma_4}, x^{\gamma_5}, x^{\gamma_6}$ so that column 1 will contain 1, 2, 4, 5, 6

The SSYT is

1	1	1	2
2	3	3	4
4	5	7	
5			
6			