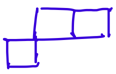
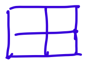
# The Murnaghan - Nakayama rule

This theorem was developed individually by both around the year 1937

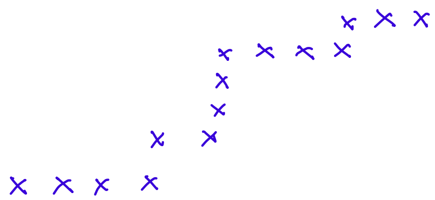
It basically gives the base change matrix between the bases $\{s_\lambda\}_{\lambda \vdash n}$ and $\{p_\lambda\}_{\lambda \vdash n}$

**Definition** : A skew shape $\lambda/\mu$ is said to be connected if it is connected as a region in $\mathbb{R}^2$

example  is not connected while  is

**Definition** : A border strip / rim hook / ribbon is a connected skew shape with no $2 \times 2$ square in it

example



is a border strip.

The idea is just to move along the "border" in a single "path"

Note that if we start from bottom left corner, we are only allowed to move N or E (why not S? why not W?)

**Definition** : We define $ht(\lambda/\mu)$, the height of a border strip to be the no. of rows minus 1

In the above example, the height is $6 - 1 = 5$

## Proposition 1

$$S_\mu P_r = \sum_\lambda (-1)^{ht(\lambda/\mu)} s_\lambda$$

The sum is over all $\lambda \supseteq \mu$ st. $\lambda/\mu$ is a border strip of $r$ boxes

Proof

Using definition of s, we want to show

$$V_{\mu+\delta}\, p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} V_{\lambda+\delta}$$

By definition, $V_\alpha = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\alpha)}$

$$\therefore \quad V_{\mu+\delta}\, p_r = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\mu+\delta)} (x_1^r + x_2^r + \dots + x_n^r)$$

$$= \sum_{j=1}^{n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\mu+\delta)} x_j^r$$

$$= \sum_{j=1}^{n} V_{\mu+\delta+r e_j} \qquad (r \xrightarrow{\sigma} r' \text{ (unique)})$$

where $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0, 0)$ (1 in $j^{th}$ place)

Now pick a $V_{\mu+\delta+re_j}$

Arrange $\mu+\delta+re_j$ in decreasing order

If two terms are equal, the $V$ evaluates to $0$

Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$

Then $\mu+\delta = \lambda = (\mu_1 + n-1, \mu_2 + n-2, \dots, \mu_n + 0)$

$\therefore \lambda_i = \mu_i + n - i$

We are adding $r$ to some term $\lambda_i$

If no two terms are equal, we obtain $p$ & $q$ such that

$\lambda_{p-1} > \lambda_q + r > \lambda_p \quad (p \le q)$

Now the order of our partition is disturbed but $V_\alpha$ is defined for compositions

Thus we correct it with a sign & write

$V_{\mu+\delta+re_j} = (-1)^{q-p} V_{\beta+\delta}$ where

$\beta+\delta = (\lambda_1, \lambda_2, \dots, \lambda_{p-1}, \lambda_q + r, \lambda_p, \dots, \lambda_{q-1}, \lambda_{q+1}, \dots, \lambda_n)$

(The $q^{th}$ one was added by $r$ & it moved forward) such $\beta$ are precisely those for which $\beta \setminus \mu$ is a border strip of size $r$

Let us first see why size is $r$

$$\beta + \delta = \lambda_1 \; \cdots \; \lambda_{p-1} \quad \lambda_q + r \quad \lambda_p \; \cdots \; \lambda_{q-1} \quad \lambda_{q+1} \; \cdots \; \lambda_n$$

$$\mu + \delta = \lambda_1 \; \cdots \; \lambda_{p-1} \quad \lambda_p \quad \lambda_{p+1} \; \cdots \; \lambda_q \quad \lambda_{q+1} \; \cdots \; \lambda_n$$

Clearly $\beta + \delta$ has $\lambda_q + r - \lambda_q = r$ boxes more than $\mu + \delta$ & hence $\beta$ has $r$ more boxes than $\mu$

Checking that $\beta \setminus \mu$ is a border strip has been left as an (notation bash) exercise

$$\therefore \; V_{\mu+\delta} \, P_r = \sum_{\beta} (-1)^{ht(\beta/\mu)} \, V_{\beta+\delta}$$

We have used the fact that $q - p = ht(\beta/\mu)$ ∎

<u>Definition</u>: A <u>border-strip tableau</u> of shape $\lambda/\mu$ and type $\alpha$ (where $\alpha \vDash n = |\lambda/\mu|$) is an assignment of positive integers to the squares of $\lambda/\mu$ s.t.

(i) Rows & columns are weakly increasing

(ii) $i$ appears $\alpha_i$ times

(iii) squares occupied by $i$ form a border strip

<u>example</u>

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| × | × | × | × | 1 | 3 | 3 | 5 | 5 | 5 |
| × | × | 1 | 1 | 1 | 3 | 4 | 5 | 10 | |
| × | × | 1 | 2 | 2 | 3 | 4 | 8 | | |
| × | × | 1 | 2 | 3 | 3 | 4 | | | |
| × | 2 | 2 | 2 | 4 | 4 | 4 | | | |

This is a BST of shape $(10, 9, 8, 7, 7) / (4, 2, 2, 2, 1)$ and type $(5, 6, 6, 6, 4, 0, 0, 1, 0, 1)$

<u>Definition</u>: We define the <u>height of a BST</u> to be the sum of heights of the individual border strips contained in it

## Proposition 2

$$s_\mu p_\alpha = \sum_\lambda \chi^{\lambda/\mu}(\alpha)\, s_\lambda \quad , \quad \text{where} ,$$

$$\chi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{ht(T)}$$ summed over border-strip tableau $T$ of shape $\lambda/\mu$ & type $\alpha$

**proof**

Directly follows from prop 1 & definition of $p_\alpha$ ■

## Theorem 3 (Murnaghan-Nakayama lemma)

$$s_{\lambda/\mu} = \sum_\nu \frac{1}{z_\nu} \chi^{\lambda/\mu}(\nu)\, p_\nu$$

**proof**

From prop 37, $\chi^{\lambda/\mu}(\nu) = \langle s_\mu p_\nu , s_\lambda \rangle$

$$= \langle p_\nu , s_{\lambda/\mu} \rangle$$

Let $s^{\lambda/\mu} = \sum_\nu c_\nu p_\nu$

$$\therefore \quad \chi^{\lambda/\mu}(\alpha) = \sum_\nu c_\nu \langle p_\alpha , p_\nu \rangle$$

$$= \sum_\nu c_\nu z_\alpha \delta_{\alpha\nu}$$

$$= c_\alpha z_\alpha$$

$$\therefore \quad c_\alpha = \frac{1}{z_\alpha} \chi^{\lambda/\mu}(\alpha)$$

Thus $$s_{\lambda/\mu} = \sum_\nu \frac{1}{z_\nu} \chi^{\lambda/\mu}(\nu)\, p_\nu$$ ■

Now we move on to discuss some rep theoretic aspects of the mur-nak rule.

We first see some identities similar to the orthogonality properties of simple characters

## Proposition 4 (orthogonality properties)

(i) $\sum_{\lambda} \chi^{\lambda}(\alpha)\, \chi^{\lambda}(\beta) = z_{\alpha}\, \delta_{\alpha\beta}$

(ii) $\sum_{\lambda} \frac{1}{z_{\lambda}} \chi^{\alpha}(\lambda)\, \chi^{\beta}(\lambda) = \delta_{\alpha\beta}$

**Proof**

We know that $s_{\lambda} = \sum_{\nu} \frac{1}{z_{\nu}} \chi^{\lambda}(\nu)\, p_{\nu}$

Since the base change matrix with $(\lambda, \nu)^{th}$ entry $\frac{1}{z_{\nu}} \chi^{\lambda}(\nu)$ is orthogonal, we get our result ∎

We recall a theorem of Frobenius, the proof of which can be found in 'rep theory' notes

**Fact**

$$p_{\lambda}(x_1, \dots, x_n) = \sum_{\mu \vdash n} \phi_{\mu}\Big|_{C_{\lambda}} x^{\mu} \quad \text{for } \lambda \vdash n$$

where $\phi_{\mu}$ is a lifted simple character of $H_{\mu} = S_{\mu_1} \times \dots \times S_{\mu_n}$ ($S_{\mu_1}$ acts on $1, \dots, \mu_1$, $S_{\mu_2}$ acts on next $\mu_2$ symbols and so on with $S_0 = \{id\}$)

$\phi_{\mu}$ is called a compound character of $S_n$ & is given by $\phi_{\mu}\Big|_{C_{\lambda}}$ = value of $\phi_{\mu}$ on conjugacy class of $S_n$ given by cycle type $\lambda = \sum_{A, B} \prod_{j=1}^{n} \binom{m_j}{\alpha_{j1} \cdots \alpha_{jn}}$ with

Btw this is just the coeff of $x^{\mu}$ in $p_{\lambda}$

$\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ & $A$, $B$ as

$A$ : $\alpha_{1r} + \dots + \alpha_{nr} = m_r \quad \forall\, r = 1, 2, \dots, n$

$B$ : $\alpha_{j1} + 2\alpha_{j2} + \dots + n\,\alpha_{jn} = \mu_j \quad \forall\, j = 1, 2, \dots, n$

(Find this around prop 4.5 (pg 87) of rep thry notes) ∎

Thus, we have been able to find the "generating function" for the compound characters $\phi_\mu$

Let $p_\lambda \cdot f = \sum_{\mu \vdash n} \varepsilon_\mu^\lambda x^\mu$ (for any $f \in \mathbb{Z}[x_1, \dots, x_n]$)

The idea is to choose $f$ so that the $\varepsilon_\mu^\lambda$ satisfy the orthogonality conditions

**Fact**

The required $f$ is the Vandermonde $V_\delta = \prod_{i<j} (x_i - x_j)$

ie. $p_\lambda V_\delta = \sum_{\mu \vdash n} \chi^\mu\big|_{C_\lambda} V_{\mu+\delta}$ where $\chi^\mu$ is the irred character of $S_n$ indexed by $\mu \vdash n$

From proposition 2, it is clear that $\chi^{\lambda\mu}(\alpha) = \chi^{\lambda\mu}(\sigma(\alpha))$ for any $\sigma \in S_n$ since $p_\alpha = p_{\sigma(\alpha)}$ so LHS stays the same

We want to show that $\chi^\lambda(\alpha) = \chi^\lambda\big|_{C_\alpha}$ where the RHS is the simple character of $S_n$ indexed by $\lambda$ and evaluated on the conjugacy class given by cycle type $\alpha$

Rewriting the above fact, we have,

$$p_\lambda = \sum_{\mu \vdash n} \chi^\mu\big|_{C_\lambda} s_\mu$$

But by proposition 2, we get $\chi^\mu\big|_{C_\lambda} = \chi^\mu(\lambda)$ & we are done!