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# Communication Over Graphs

We have a graph $G$ with local information at every vertex of the graph. By this, we mean that every vertex only knows its own neighbours.
Suppose you want to send a parcel to, say, Chris Evans. You do not know his adress. You know your friends adress though and you send it to him requesting him to send it for you since he may know where Chris lives. If he does, we have solved our problem. If he doesnt, he passes it on to one of his friends, hoping that they do it for him and the path continues & may possibly reach Chris (assuming connectedness).
Of course, BFS & DFS complete the task but the runtime is huge. Graham gave a very elegant way to complete the task by embedding the graph in a squashed hypercube.

**Definition:** Define the squashed hypercube graph $SQ(n)$ on $3^n$ vertices where $V(G) = \{(x_1, \dots, x_n) \mid x_i \in \{0, 1, *\}\}$. For any two $u, v \in V(G)$, we define $d(u,v) = \sum_{i=1}^{n} \langle u_i, v_i \rangle$ where $\langle 1, 0 \rangle = \langle 0, 1 \rangle = 1$ and all other combinations are $0$ (ie. $\langle *, \_ \rangle = \langle \_, * \rangle = \langle 0, 0 \rangle = \langle 1, 1 \rangle = 0$). $u$ and $v$ are connected by an edge iff $d(u,v) = 1$

Observe how the vertex $(*, *, \dots, *)$ is not connected to anything else since its distance to every vertex is $0$

**Definition:** Given graphs $G$, $H$ with distances $d_G$, $d_H$ (shortest dist.) we say $G$ is isometrically embeddable in $H$ if there is a homomorphism $\varphi : G \to H$ st. $d_G(v_i, v_j) = d_H(\varphi(v_i), \varphi(v_j))$ $\forall\, v_i, v_j \in V(G)$
$\varphi$ is called the isometric embedding of $G$ in $H$

Q) Show that $K_n$ is isometrically embeddable in $SQ(m)$ for some $m$

Ans Let $v_1, \dots, v_n$ be the vertices of $K_n$

Choose $\varphi : K_n \longrightarrow SQ(n-1)$ mapping

$\varphi(v_1) = 0\ 0\ 0 \cdots 0$

$\varphi(v_2) = 1\ 0\ 0 \cdots 0$

$\varphi(v_3) = *\ 1\ 0 \cdots 0$

$\varphi(v_4) = *\ *\ 1 \cdots 0$

$\vdots$

$\varphi(v_n) = *\ *\ * \cdots 1$

Proposition 1

Every graph can be isometrically embedded in some squashed hypercube

proof

We embed $G$ into $SQ(N)$ where $N = \sum_{i<j} d_G(v_i, v_j)$

where $V(G) = \{v_1, \dots, v_n\}$

Initially we deal with the vertex $v_1$

Define $\varphi : G \longrightarrow SQ(N)$, "partially" as follows :

Firstly, write $\varphi(v)$ in blocks as $\boxed{\phantom{xxx}}\,\boxed{\phantom{xxx}} \cdots \boxed{\phantom{xxx}}$

where first block has size $d_G(v_1, v_2)$, next one has $d_G(v_1, v_3)$

and so on upto $d_G(v_1, v_n)$

Now construct

$\varphi(v_1) = 0 \cdots 0 \,|\, 0 \cdots 0 \,|\, \cdots \,|\, 0 \cdots 0$

$\varphi(v_2) = 1 \cdots 1 \,|\, * \cdots * \,|\, \cdots \,|\, * \cdots *$

$\varphi(v_3) = * \cdots * \,|\, 1 \cdots 1 \,|\, \cdots \,|\, * \cdots *$

$\vdots$

$\varphi(v_n) = * \cdots * \,|\, * \cdots * \,|\, \cdots \,|\, 1 \cdots 1$

This gives distance between $\varphi(v_1)$, $\varphi(v_j)$ as $d_G(v_1, v_j)$.

But other distances aren't okay yet.

Append all $*$'s to $\varphi(v_1)$ since we are done with it.

Now append $0 \cdots 0 \,|\, 0 \cdots 0 \,|\, \cdots \,|\, 0 \cdots 0$ to $\varphi(v_2)$ with block sizes being $d_G(v_2, v_3), \dots, d_G(v_2, \dots v_n)$ & repeat construction.

Once $v_2$ is done, append all $*$'s to $\varphi(v_2)$.

Q) How does isometrically embedding $G$ into $SQ(n)$ help with our requirement ?

Ans Suppose we want to send information from vertex $s$ to vertex $t$. Using the isometric embedding, one knows the address of $s$ & $t$ both in the squashed hypercube.

There must exist some neighbour $\varphi(v)$ of $\varphi(s)$ so that $\varphi(v)$ is closer to $\varphi(t)$ than $\varphi(s)$ (assuming connectedness of our graph) because of the definition of distance being the shortest length.

Since $s$ knows its neighbours, it can just calculate $d(\varphi(v), \varphi(t)) = d(v, t)$ $\forall v$, minimize this and pass it on to $t$.

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## Proposition 2

Let $\varphi : G \to SQ(n)$ be an isom. embedding. Define the matrix $A$ whose $(i, j)^{th}$ entry is given by $< \varphi(v_i)_t , \varphi(v_j)_t >$ for some $t \in \{1, 2, \ldots, n\}$.

Then $A$ is an adjacency matrix of a complete bipartite graph, upto 0 columns

proof

Firstly, $A$ is clearly symmetric since $< , >$ is symmetric by definition.

Consider the values $\varphi(v_1)_t , \varphi(v_2)_t , \ldots, \varphi(v_n)_t$.

If we have a value $*$, its distance from any other vertex is 0 and we don't care about these as they are the 0 rows and columns of $A$.

Now each 0 will be connected to each 1 & vice versa giving us the complete bipartite graph $K_{r,s}$ where $r$ = no. of 1's and $s$ = no. of 0's

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## Proposition 3

The eigenvalues of $K_{r,s}$ are $\sqrt{rs}$, $-\sqrt{rs}$, $0, \ldots, 0$

proof

Let the adjacency matrix be $A$.

Write $A = \begin{bmatrix} 0 & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix}$

rank $(A) = 2$ , $\text{tr} A = 0$ $\Rightarrow$ evals are $\lambda, -\lambda, 0, 0, \ldots, 0$

Consider the vector $v = (\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ ($\alpha$ occurs $r$ times and $\beta$ occurs $s$ times)

Then $Av = (s\beta, s\beta, \ldots, s\beta, r\alpha, r\alpha, \ldots, r\alpha)$ ($s\beta$ occurs $r$ times & $r\alpha$ occurs $s$ times)

If we force $s\beta = \lambda\alpha$, $r\alpha = \lambda\beta$ we get that $v$ is an eigenvector with eigenvalue $\lambda = \sqrt{rs}$

The result now follows

## Proposition 4 (Rank - canonical factorisation)

If $M_{n\times m}$ has rank $r$, then $r$ is the smallest integer such that $M = XY^t$ for some $X_{n\times r}$, $Y_{m\times r}$

proof

Convert $M$ to RREF & then to CREF to get

$EMF = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $E, F$ are invertible (order $n, m$ resp.)

Then if we decompose $E^{-1}$ into $\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$ & similarly $F^{-1}$, where $E_1$, $F_1$ have size $r\times r$, we get

$M = \begin{bmatrix} E_1F_1 & E_1F_2 \\ E_3F_1 & E_3F_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_3 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = XY^T$

Now if $M = XY^T$ for some $X_{n\times p}$, $Y_{m\times p}$ then every row of $M$ can be written as a linear combination of the $p$ rows of $Y^T$ and hence rank $M = r \leq p$ (spanning set size is always larger than the basis)

The result follows

## Corollary 5

rank $(A+B) \leq$ rank $(A)$ + rank $(B)$ (i.e. rank is sub-additive)

proof

Let $A_1, A_2, \ldots, A_n$ denote the columns of an $m\times n$ matrix $A$.

Then $M = XY^T$ for $X_{n\times r}$, $Y_{m\times r}$ ($M$ is $n\times m$ of rank $r$)

$= \sum_{i=1}^{r} X_i Y_i^t$

Thus rank of a matrix is the minimum number such that $M$ may be expressed as a sum of $r$ $1\times 1$ matrices

Suppose rank$(A) = \alpha$, rank$(B) = \beta$, then $A+B$ can be written as the sum of $\alpha+\beta$ rank 1 matrices & hence rank$(A+B)$ is at most $\alpha+\beta$

(Note that the fact that every rank 1 matrix can be written as $vw^t$ has been used)

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Definition : Let $A_{n\times n}$ be hermitian. If we can express $A = XY^* + YX^*$ when $X, Y$ are $n \times r$ where $r$ is as small as possible, we call $r$ the hermitian rank of $A$

Hermitian rank of a hermitian matrix $A$ is the smallest $r$ st. $A$ can be expressed as a sum of $r$ hermitian matrices (to prove this, do the column trick as before)

Proposition 6

$h(A) \leq r(A) \leq 2h(A)$ for $A$ hermitian ($h$ = hermitian rank)

proof

Let $h(A) = h$ ie. $A = \sum_{i=1}^{h} x_i y_i^* + \sum_{i=1}^{h} y_i x_i^*$

Then clearly $r \leq 2h$ follows since $A$ has been expressed as a sum of $2h$ rank 1 matrices

Let $r(A) = r$. Then $A = XY^t$ for some $X, Y$ $n\times r$ matrices. Taking $\overline{Y}$, we may let $A = XY^*$

Since $A$ is hermitian, $XY^* = YX^*$.

Thus, write $A = \frac{1}{2} XY^* + \frac{1}{2} YX^*$

Let $\frac{1}{2} X = U \Rightarrow A = UY^* + YU^*$

By definition of $h(A)$, $h \leq r$

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Definition : Let $A$ be hermitian. We associate the triple $(n_0(A), n_+(A), n_-(A))$ (no. of evals of given sign) to it. This triple is called the inertia of the matrix

Definition : We say two matrices $A, B$ are congruent and write $A \sim B$ if $P^* A P = B$ where $P$ is invertible

## Proposition 7

For any hermitian $A$ with inertia $(a, b, c)$, $A \sim \text{diag}(1^b, -1^c, 0^a)$

**proof**

Using the spectral theorem, we write

$P^* A P = \text{diag}(\lambda_1, \cdots, \lambda_n)$

Assume $\lambda_1, \cdots, \lambda_b > 0$, $\lambda_{b+1}, \cdots, \lambda_{b+c} < 0$, $\lambda_{c+1}, \cdots, \lambda_n = 0$

Then $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = Q \begin{bmatrix} I_b & & \\ & -I_c & \\ & & 0 \end{bmatrix} Q$

where $Q = \text{diag}(\sqrt{|\lambda_1|}, \sqrt{|\lambda_2|}, \cdots, \sqrt{|\lambda|_{b+c}}, 1, 1, \cdots, 1)$

Noticing that $Q^* = Q$, we may write

$\text{diag}(1^b, -1^c, 0^a) = S^* A S$ where $S = PQ^{-1}$

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## Theorem 8 (Sylvester's law of inertia)

Congruent hermitian matrices have the same inertia

**proof**

Let $A = P^* B P$ have inertia $(a, b, c)$

Then $S^* A S = (PS)^* B (PS) = \text{diag}(1^b, -1^c, 0^a) = D$ (say)

$\therefore B = Q^* D Q$ where $Q = PS$

$n_0(B) = a$ (since $Q$ is invertible, it doesn't change nullity)

Hence $n_+(B) + n_-(B) = \text{rank}(B) = b + c$

**lemma**: For a hermitian matrix $A$, we have $n_+(A)$ is equal to $\max\{\dim W \mid W \text{ is a subspace st- } x^* A x > 0 \ \forall\, x \in W, x \neq 0\}$

**proof of lemma**:

Let $W$ be a subspace st. $x^* A x > 0$ on $W \setminus \{0\}$

Write $A = Q D Q^*$ using the spectral theorem and change variables as $x = Qy$ so that $y^* D y > 0$ on $W \setminus \{0\}$ $(\because Q^{-1} W = W)$

ie. $\sum_{i=1}^n \lambda_i |y_i|^2 > 0$ holds $\forall\, (y_1, \cdots, y_n)$ in $W \setminus \{0\}$

We claim that all such $\lambda_i$ must be positive. Otherwise one may simply choose very small $y_j$ corresponding to positive $\lambda_j$ and get a contradiction for the positivity of the sum.

Thus all $y_i$ are $0$ corresponding to the $n-p$ zero & negative eigenvalues showing that $\dim W \leq p$ = no. of +ve evals

Conversely, we show $p \leq W$

Consider $V_+ = \text{span}\{v \mid v \text{ is an evec with +ve eval}\}$

Let the +ve evals be $\lambda_1, \cdots, \lambda_p$ with evecs $v_1, \cdots, v_p$

For any $x \in V^+ \setminus \{0\}$, write $x = \sum_{i=1}^{p} c_i v_i$

$x^* A x = (\sum_{i=1}^{p} \bar{c_i} v_i^*)(\sum_{i=1}^{p} c_i \lambda_i v_i) = \sum_{i=1}^{p} \lambda_i |c_i|^2 > 0$

This shows existence of dimension $p$ subspace

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Since multiplication by invertible matrices will not change the dimension, $n_+(B) = b$ and the proof is complete

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Proposition 9

$h(A) = \max(n_+(A), n_-(A))$ for any hermitian matrix $A$

proof

Let $B = PAP^*$ & $A = XY^* + YX^*$ be the decomp corresponding to $h(A)$. Then $B = PXY^*P^* + PYX^*P^* = PX(PY)^* + (PY)(PX)^*$ implying that $h(B) \le h(A)$. By invertibility of $P$, $h(A) \le h(B)$ can be shown.

Thus hermitian rank is constant for conjugate matrices & so is the RHS. Thus it suffices to prove the theorem for $\text{diag}(0^a, 1^b, -1^c)$ since $A$ is conjugate to this.

WLOG let $b > c$. Pair the $c$ number of $1$'s and $-1$'s together and write each of the $c$ blocks $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as some $xy^* + yx^*$ (very easy to come up with $x, y$)

The remaining $b-c$ $1$'s may be written as $\frac{1}{2} e_i e_i^* + e_i^* (\frac{1}{2} e_i)$

Thus, $A$ is written as a sum of $b-c+c$ hermitian matrices which gives $h(A) \le b-c+c = b$

We now show the other inequality

Write $A = XY^* + YX^*$ where $X, Y$ are $n \times h$

If $u \in \text{Null}(X^*)$, then $u^* A u = 0$

Let $A$ have the nullspace $E_0(A)$

Let $E_+(A)$ be the span of evecs corresponding to +ve e.vals

Claim: $\text{Null}(X^*) \cap (E_0(A) + E_+(A)) \subseteq E_0(A)$

Consider orthonormal bases $\{u_1, \dots, u_p\}$ & $\{w_1, \dots, w_q\}$ of $E_0(A)$, $E_+(A)$.

Write $u = \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j w_j$

$Au = \sum_{j=1}^{q} \beta_j w_j$

$u \in \text{Null}(X^*) \Rightarrow u^* A u = 0 \Rightarrow \sum_{i=1}^{q} \sum_{j=1}^{q} \beta_i \bar{\beta_j} w_j^* A w_i = 0$

(Note that $u_i^* A^* = u_i^* A = (Au_i)^* = 0$ since $u_i \in E_0(A)$)

Using orthonormality we get

$\sum_{i,j=1}^{q} \beta_i \bar{\beta}_j \lambda_i w_j^* w_i = \sum_{i=1}^{q} |\beta_i|^2 \lambda_i = 0$

By definition of $E_+(A)$, all $\lambda_i$ are positive $\Rightarrow \beta_j = 0$ and our claim is proved

Taking $\perp$ and using $A \subseteq B \Rightarrow A^\perp \supseteq B^\perp$ & $(A \cap B)^\perp = A^\perp + B^\perp$ (for a finite dim vector space), we get

$Null(X^*)^\perp + (E_0(A) + E_+(A))^\perp \supseteq E_0(A)^\perp$

$\therefore Col(X) + E_-(A) \supseteq Col(A^*) = col(A)$

Using dimensions, $n_+(A) + n_-(A) \leq rank(X) + n_-(A)$

$\therefore n_+(A) \leq rank(X) \leq h$ ($X$ is $n \times h$ matrix)

Similarly $n_-(A) \leq h$

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**Corollary 10**

$max(n_+(A), n_-(A))$ is subadditive

**Theorem 11** (Witsen - Hansen theorem)

If $G$ is isometrically embeddable in $SQ(d)$, then we have, $d \geq max\{n_+(D), n_-(D)\}$ where $D$ is the distance matrix of $G$ i.e. $[D_{ij}] = d(v_i, v_j)$ and $n_+$, $n_-$ denote the number of positive & negative eigenvalues of the matrix

**proof**

Let $\varphi : G \to SQ(d)$ be an isometric embedding of $G$

Define matrices $A_1, A_2, \dots, A_d$ as follows.

$A_t$ has $(i,j)^{th}$ entry as the dist. between the $t^{th}$ coordinates of $\varphi(v_i), \varphi(v_j)$

Observe that $A_1 + A_2 + \dots + A_d$ has its $(i,j)^{th}$ entry as the distance between $\varphi(v_i)$, $\varphi(v_j)$ & hence is just $D$

$A_i$ is the adjacency matrix of some complete bipartite graph by proposition 2. By proposition 3, $max\{n_+(A_i), n_-(A_i)\} = 1$.

$$\begin{aligned} \text{Thus, } d &= \sum_{i=1} 1 \\ &= \sum_{i=1}^{d} max(n_+(A_i), n_-(A_i)) \\ &\geq \sum_{i=}^{d} max(n_+(D), n_-(D)) \quad \text{(by corollary 10)} \end{aligned}$$

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