Definition: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation

Definition: An ordinary differential equation is a differential equation containing one or more derivatives of an unknown function y

Definition: An equation involving partial derivatives of one or more dependent variables wrt a single independent variable is a partial diff eqn

Definition: The order of a differential eqn is the order of the highest derivative in the equation

Definition: The degree of a differential eqn is the power to which the highest order derivative is raised to

Definition: Let $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ be a differential equation of order n. It is said to be linear if F is linear in each $y, y', \ldots, y^{(n)}$ and has some function of x with it.

THEOREM 1

Every linear ODE is of the form $a_0(x)\, y^{(n)} + \cdots + a_n(x)\, y = b(x)$

Definition: A linear ODE is said to be homogenous if $b(x)$ (as in theorem 1) is zero

Definition: An explicit solution of $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$ on the interval (α, β) is a function $\phi(x)$ defined on (α, β) such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist on (α, β) and $\phi^{(n)}(x) = f(x, \phi, \phi', \ldots, \phi^{(n-1)})$

Definition: A relation $g(x, y) = 0$ is called implicit solution of $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$ if this relation defines at least one function ϕ on some (α, β) such that ϕ is an explicit solution

Definition: An initial value problem (IVP) is a differential equation along with an initial condition

Definition: An isocline is a series of lines plotted on the diff eqn vector graph having the same slope

Definition: A first order linear ODE is called separable if it has the form $M(x) + N(y) \cdot y' = 0$

Definition: A function $f(x_1, \ldots, x_n)$ is called homogenous if $f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n) \ \forall t \neq 0$
(d is called degree)

Definition: The first order ODE $M(x,y) + N(x,y)y' = 0$ is called homogenous if M & N are both homogeneous of the same degree

Clarification: We had seen another definition of homogenous ODE but we shall use this above one whenever we write homogenous ODE and the previous one when referring to a linear homogenous ODE

THEOREM 2 (homogeneous to separable)

Let $M(x,y) + N(x,y)y' = 0$ be a homogenous ODE of degree d. This can be reduced to a separable ODE by substituting $y = vx$, to yield

$$\frac{dx}{x} + \frac{N(1,v) \cdot dv}{M(1,v) + N(1,v) \cdot v} = 0$$

Definition: A first order ODE $M(x,y) + N(x,y)y' = 0$ is called exact if $\exists\, u(x,y)$ so that

$$\frac{\partial u}{\partial x} = M \quad , \quad \frac{\partial u}{\partial y} = N$$

THEOREM 3 (solving exact ODE)

Let $M(x,y) + N(x,y)y' = 0$ be exact with u as above. The solution is $u(x,y) = c$ for some constant c (which may be determined using initial conditions)

THEOREM 4 (testing exact)

Let M, N have first order partial derivatives and let them be continuous on $D \subset \mathbb{R}^2$.

$M(x,y) + N(x,y)\, y' = 0$ is exact iff $M_y = N_x$

THEOREM 5 (Integrating factor)

For $M(x,y)\,dx + N(x,y)\,dy = 0$ and assuming existence & continuity of partial derivatives on some $D \subseteq \mathbb{R}$, define $\alpha := M_y - N_x$

we have :

(i) $\alpha = 0 \Rightarrow$ ODE is exact.

(ii) $\alpha \neq 0$, $\frac{\alpha}{N}$ is a function of x alone $\Rightarrow$

multiply ODE with $e^{\int \frac{\alpha}{N} dx}$ to get an exact ODE

(iii) $\alpha \neq 0$, $\frac{\alpha}{M}$ is a function of y alone $\Rightarrow$

multiply ODE with $e^{\int -\frac{\alpha}{M} dy}$ to get an exact ODE

THEOREM 6 (Bernoulli reduction)

To solve $y' + P(x)\, y = Q(x)\, y^n$, we substitute $v = y^{1-n}$ to get a linear ODE

$$\frac{dv}{dx} + (1-n)\, P(x)\, v = Q(x)\,(1-n)$$

Definition: If two families of curves intersect at right angles always, they are called orthogonal trajectories of each other

THEOREM 7 (finding orthogonal trajectories)

Let $F(x, y, c) = 0$ be a family of curves

Let the diff eqn be $\frac{dy}{dx} = f(x, y)$

The orthogonal trajectory is obtained by solving

$$\frac{dy}{dx} = \frac{-1}{f(x,y)}$$

Definition: Let $D \subset \mathbb{R}^2$ and f be defined on D. f is said to satisfy Lipschitz condition wrt y if $\exists M \in \mathbb{R}$ s.t. $|f(x, y_1) - f(x, y_2)| \leq M \cdot |y_1 - y_2|$ $\forall (x, y_1), (x, y_2) \in D$

THEOREM 8

Let f be Lipschitz continuous wrt y. Then for every fixed x_0, the function $f(x_0, y)$ is continuous

(converse false)

THEOREM 9

If f is such that $\frac{\partial f}{\partial y}$ exists and is bounded, then f satisfies Lipschitz condition wrt y.

Further, $M = \inf_{(x,y) \in D} \left| \frac{\partial f}{\partial y}(x, y) \right|$

(converse false)

THEOREM 10 (existence-uniqueness)

Let $(x_0, y_0) \in \mathbb{R}^2$. Let R be a rectangle $|x - x_0| < a$ $|y - y_0| < b$. If f is continuous in R and bounded in R (with constant K), then the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has a solution $y(x)$ defined for all x s.t. $|x - x_0| < \alpha$ where $\alpha = \min\{a, \frac{b}{K}\}$. Further, the solution is unique if f is lipschitz in R

THEOREM 11 (Picard integration)

Let $y' = f(x, y)$, $y(x_0) = y_0$ be given generate $y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1})\, dt$ $\forall\, n \geq 1$. Then $y(x) = \lim_{n \to \infty} y_n(x)$ is the required solution to the IVP

Definition: The second order linear ODE (SOLODE)

$y'' + p(x) y' + q(x) y = r(x)$ is said to be homogeneous if $r(x) = 0$. (else non homogeneous)

Definition: The HSOLODE IVP is of the form

$$\begin{cases} y'' + p(x) y' + q(x) y = 0 \\ y(x_0) = a, \quad y'(x_0) = b \end{cases}$$

THEOREM 12

If p, q, r in the above definitions are continuous on some I, then the SOLODE has a unique solution on I

Definition: Two functions f, g are said to be linearly independent on I if $c_1 f(x) + c_2 g(x) = 0 \ \forall x \in I \Rightarrow c_1 = c_2 = 0$

Definition: The Wronskian det of two diff functions f, g is defined as

$$W(f,g)(x) := \det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}$$

THEOREM 13 (Wronskian properties)

(i) $W(f,g) = -W(g,f)$

(ii) $W(\alpha f, \beta g) = \alpha\beta\, W(f,g)$

(iii) If f_1, f_2 are LD over I, then $W(f_1, f_2) = 0$ everywhere on I

(iv) If $W(f_1; f_2)$ is non zero at some point of I then f_1, f_2 are LI on I

(v) If y_1, y_2 solve $y'' + p(x)y' + q(x)y = 0$ on I, then $W(y_1, y_2)$ is entirely zero on I or is entirely non zero on I (LD in former, LI in latter)

THEOREM 14

$y'' + p(x)y' + q(x)y = 0$, $y(x_0) = 0$, $y'(x_0) = 0$ has $y = 0$ (trivial solution) as the only solution

Definition: A basis of $y'' + p(x)y' + q(x)y = 0$ is a set of LI solutions y_1, y_2

THEOREM 15

If p, q are continuous (in above defn), then a basis of solutions exist

THEOREM 16

If y_1, y_2 form a basis of a HSOLODE, then every solution is of the form $c_1 y_1 + c_2 y_2$ for any c_1, c_2

THEOREM 17 (finding y_2 given y_1)

Let y_1 be a solution of $y'' + p(x)y' + q(x)y = 0$

Then $y_2 = v \cdot y_1$ completes the basis with y_1

where $v(x) = \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$

THEOREM 18

The wronskian of $y'' + p(x)y' + q(x)y = 0$ satisfies $W'(x) = -p(x)W(x)$ and hence, $W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(t)dt}$ (for $x_0 \in I$)

THEOREM 19 (Const. coefficients ODE)

The solutions to $y'' + py' + q = 0$ $(p, q \in \mathbb{R})$ are obtained by examining roots of $m^2 + pm + q = 0$.

Case 1: Real unequal m_1, m_2

$\Rightarrow$ Basis is $\{e^{m_1 x}, e^{m_2 x}\}$

Case 2: Equal $m_1 = m_2 = m = -\frac{p}{2}$

$\Rightarrow$ Basis is $\{e^{-\frac{px}{2}}, x e^{-\frac{px}{2}}\}$

Case 3: complex roots $m_1 = a + ib$, $m_2 = a - ib$

$\Rightarrow$ Basis is $\{e^{ax}\cos bx, e^{ax}\sin bx\}$

THEOREM 20 (Cauchy Euler ODE)

The solutions to $x^2 y'' + axy' + by = 0$ $(a, b \in \mathbb{R})$ are obtained by examining roots of $m(m-1) + am + b = 0$

Case 1: Real unequal m_1, m_2

$\Rightarrow$ Basis is $\{x^{m_1}, x^{m_2}\}$

Case 2: Equal $m_1 = m_2 \Rightarrow \frac{1-a}{2}$

$\Rightarrow$ Basis is $\{x^{\frac{1-a}{2}}, \ln x \; x^{\frac{1-a}{2}}\}$

Case 3: complex $m_1 = c + id$, $m_2 = c - id$

$\Rightarrow$ Basis is $\{x^c \cos(d \ln x), x^c \sin(d \ln x)\}$

THEOREM (21) (NHSOLODE)

Let y_p be a known soln to $y'' + p(x) y' + q(x) y = r(x)$ & y_1, y_2 are solns to homo part. $y = c_1 y_1 + c_2 y_2 + y_p$ is general soln.

THEOREM 22 (Variation of parameters)

Let $y'' + p(x)y' + q(x)y = r(x)$ be a NHSOLODE

If y_1, y_2 are known (solns to homo part i.e. $y'' + p(x)y' + q(x)y = 0$) then,

$$y_p(x) = y_2(x) \int \frac{y_1(x) \cdot r(x)}{W(y_1, y_2)(x)} dx - y_1(x) \int \frac{y_2(x) \cdot r(x)}{W(y_1, y_2)(x)} dx$$

THEOREM 23 (Method of undetermined coeffs)

The general form of y_p is guessed based on the RHS $r(x)$ in accordance with the table:

$r(x)$	y_p to be guessed
poly of degree n	general poly of degree n
$a\cos bx + c\sin bx$	$m\cos bx + n\sin bx$
$ae^{\beta x}$	$ke^{\beta x}$

where m, n, k are to be determined

Preference is given to the higher placed $r(x)$ type in the table

Definition: An n^{th} order linear ODE (nOLODE) is given by $a_0(x)y^{(n)} + \dots + a_n(x)y = g(x)$

In standard form,

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = r(x)$$

Definition: In accordance with the above definition, if $r(x) = 0$, we call it a HnOLODE (homogeneous)

Definition: A HnOLODE IVP takes the form

$$y^{(n)} + p_1(x)\, y^{(n-1)} + p_2(x)\, y^{(n-2)} + \cdots + p_n(x)\, y = 0$$

$$y^{(i)}(x_0) = k_i \qquad \forall\ i = 0, 1, 2, \ldots, n-1$$

THEOREM 24

On a given interval I, if all $p_i(x)$ are continuous, then the solution is unique on I

Definition: The wronskian for n functions is

$$W(y_1, \ldots, y_n)(x) := \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & & & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

THEOREM 25

(i) $W(y_1, \ldots, y_i, y_{i+1}, \ldots, y_n) = -W(y_1, \ldots, y_{i+1}, y_i, \ldots, y_n)$

(ii) $W(y_1, \ldots y_i, \ldots y_j \ldots y_n) = -W(y_1 \ldots y_j \ldots y_i \ldots y_n)$

(iii) $W(a_1 y_1 \ldots a_n y_n) = \left(\prod_{i=1}^{n} a_i\right) W(y_1 \ldots y_n)$

(iv) If on a given I, $W(y_1 \ldots y_n) \neq 0$ at some $x \in I$, then the functions are LI

(v) If $y_1 \ldots y_n$ are LD on some interval I, then $W(y_1 \ldots y_n) \equiv 0$ entirely on I

(vi) In accordande with theorem 13,

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0)\, e^{-\int_{x_0}^{x} p_1(t)\,dt}$$

for some $x_0 \in I$

THEOREM 26

If $p_i(x)$ is continuous $\forall i$ in $y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = 0$ then we can find a basis of n elements $\{y_1, \dots, y_n\}$. Further, the general solution has the form $\sum_i c_i y_i$

THEOREM 27

If y_p is a soln of a NHnOLODE, and $\{y_i\}_1^n$ is a basis for the corresponding HnOLODE, then a general soln for the former is given by

$$y_p + \sum_{i=1}^{n} c_i y_i$$

THEOREM 28 (const coeffs)

To solve $y^{(n)} + p_1 y^{(n-1)} + \dots + p_n y = 0$, we solve the eqn $m^n + p_1 m^{n-1} + \dots + p_n = 0$ and the basis is (in general) given by $\{e^{m_i x}\}_{i=1}^{n}$

In case of repeated roots attach factors $1, x, x^2, \dots$ to the $e^{m_i x}$ of the repeated m_i

In case of complex roots, take real & imaginary pair of e^{zx} (or $e^{\bar{z}x}$)

THEOREM 29 (Variation of parameters)

If $y_1, \dots, y_n$ is a basis for the corresp. HnOLODE of a given NHnOLODE $y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = r(x)$,

$$y_p = \sum_{i=1}^{n} v_i(x) y_i(x) \text{ where, } \begin{bmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(x) \end{bmatrix}$$

THEOREM 30 (Method of undetermined coeffs)

same as theorem 23 (exactly the same)

Definition: We say an operator Θ annihilates f

if $\Theta(f(x)) = 0$

THEOREM 31 (Annihilator method)

Applying an appropriate operator on LHS & RHS, an NHnOLODE can be made HnOLODE. We solve the new HnOLODE and drop solutions that match with basis solutions of corresponding HnOLODE of the NHnOLODE. A linear combination of the non-dropped factors gives us y_p

The choice of operator is made as follows

polynom of degree n $\longrightarrow$ D^{n+1}

$e^{\alpha x}$ $\longrightarrow$ $D-\alpha$

$x^n e^{kx}$ $\longrightarrow$ $(D-k)^{n+1}$

$\cos bx$ or $\sin bx$ $\longrightarrow$ D^2+b^2

$x^n \cos bx$ or $x^n \sin bx$ $\longrightarrow$ $(D^2+b^2)^{n+1}$

$Ae^{rt}+Be^{-rt}$ $\longrightarrow$ D^2-r^2

$e^{\lambda t}\cos\omega t + e^{\lambda t}\sin\omega t$ $\longrightarrow$ $(D-\lambda)^2+\omega^2$

(poly of degree k) $\cdot$ $(e^{\lambda t}\cos\omega t+e^{\lambda t}\sin\omega t)$ $\longrightarrow$ $((D-\lambda)^2+\omega^2)^{k+1}$

Definition: For a function $f(x)$, its Laplace transform is given by $L(f)(s) := \int_0^\infty e^{-st} f(t)\, dt$

$$\left(= \lim_{a \to \infty} \int_0^a e^{-st} f(t)\, dt \right)$$

Definition: f is said to be of exponential order if $\exists\, a \in \mathbb{R}$, positive constants t_0, K so that

$|f(t)| \leq K e^{at} \quad \forall\, t \geq t_0 > 0$

(ensure that $f(t) e^{-at}$ is bounded for some a)

THEOREM 32 (existence of Laplace)

If f is at least piecewise continuous and of exponential order, then Laplace transform of f exists

THEOREM 33 (Laplace chart)

$f(t)$	$F(s)$
c	c/s
e^{at}	$1/s-a$
$\sin(at)$	a/a^2+s^2
$\cos(at)$	s/a^2+s^2
t^n	$n!/s^{n+1}$
$\cosh(at)$	s/s^2-a^2
$\sinh(at)$	a/s^2-a^2
$t\cos(at)$	$s^2-a^2/(s^2+a^2)^2$
$t\sin(at)$	$2as/(s^2+a^2)^2$

THEOREM 34 (Laplace properties)

Let $F(s)$ denote the LT of $f(t)$. Then,

(i) $L(\alpha f(t) + \beta g(t)) = \alpha L(f(t)) + \beta L(g(t))$

(ii) $L(e^{at} f(t)) = F(s-a)$

(iii) $L(f(ct)) = \frac{1}{c} F\left(\frac{s}{c}\right)$

(iv) $L(f'(t)) = sF(s) - f(0)$

(v) $L(f''(t)) = s^2 F(s) - s f(0) - f'(0)$

(vi) $L(f^{(n)}(t)) = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$

(vii) $L\left(\int_0^t f(u)\, du\right) = \frac{F(s)}{s}$

(viii) $L(t^n f(t)) = (-1)^n F^{(n)}(s)$

(ix) $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(v)\, dv$

THEOREM 35 (Lerch's theorem)

Suppose $f, g : [0,\infty) \to \mathbb{R}$ are continuous so that $L(f(t)) = L(g(t))$, then $f \equiv g$

Definition: The Heaviside function $u_c : \mathbb{R} \to \{0,1\}$ is defined as $u_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$

THEOREM 36

Let $u_c(t)$ denote the Heaviside function. We have $L(u_c(t)) = \frac{e^{-cs}}{s}$, $L(u_c(t) f(t-c)) = e^{-cs} F(s)$

Definition: For functions f, g we define their convolution product as $(f * g)(t) := \int_0^t f(t-v) g(v) \, dv$

THEOREM 37

$$L(f * g) = L(f) \cdot L(g)$$

THEOREM 38

Let f have period p (assume $L(f)$ exists). Then

$$L(f(t)) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t) \, dt$$

Definition: The gamma function is defined as $\Gamma : (0, \infty) \to \mathbb{R}$ as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$

THEOREM 39

The above integral converges $\forall x \in (0, \infty)$

THEOREM 40

We have $\Gamma(x+1) = x\Gamma(x)$ and if $x \in \mathbb{Z}^+$, then $\Gamma(x+1) = x!$

THEOREM 41

If $L(s)$ is the laplace transform of $f(t)$, we have $\lim_{s \to \infty} L(s) = 0$

TUTORIAL 1

1) Classify the following (order, linear or not)

(i) $y''' + 4y'' = y$

(ii) $y' + 2y = \sin x$

(iii) $y\,y'' + 2x\,y' + y = 0$

(iv) $y'''' + \sin x \cdot y' + x^2 y = 0$

(v) $(1+y^2)\,y'' + t\,y^{(6)} + y = e^t$

Ans (i) order 3, linear

(ii) order 1, linear

(iii) order 2, non linear

(iv) order 4, linear

(v) order 6, non linear

Recall: An ODE is linear iff it has the form

$$a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \cdots + a_n(x)\,y = b(x)$$

for any functions $a_i(x)$, $b(x)$

2) Formulate the differential equations

(i) $y = ax^2$ (ii) $y - a^2 = a(x - b^2)$

(iii) $x^2 + y^2 = a^2$ (iv) $(x-a)^2 + (y-b)^2 = a^2$

(v) $y = a\sin x + b\cos x + a$

(vi) $y = a(1 - x^2) + bx + cx^3$

(vii) $y = cx + f(c)$

Ans (i) $y = ax^2$

$y' = 2ax$

$\therefore xy' = 2y$

(ii) $y - a^2 = a(x-b)^2$

$\therefore y' = 2a(x-b)$

$\therefore y'' = 2a$

$$y - a^2 = \cancel{a}\,\frac{(y')^2}{4a^{\cancel{2}}}$$

$$\therefore (y - a^2)\,a = \frac{(y')^2}{4}$$

$$\therefore \left(4y - (y'')^2\right)\frac{y''}{2} = (y')^2$$

(iii) $x + yy' = 0$

(iv) $(x-a)^2 + (y-b)^2 = a^2$

$(x-a) + (y-b)y' = 0$

$1 + (y-b)y'' + (y')^2 = 0$

Solving the three,

$$(y-b)^2(y')^2 + (y-b)^2 = a^2$$

Also $$y - b = -\frac{\left(1 + (y')^2\right)}{y''}$$

$$\therefore x - a = \left(\frac{1+(y')^2}{y''}\right) y'$$

$$[1+(y')^2]\frac{[1+(y')^2]^2}{(y'')^2} = a^2 = \left(x - \frac{(1+y')^2 y'}{y''}\right)^2$$

(v) $y = a\sin x + b\cos x + a$

$y' = a\cos x - b\sin x$

$y'' = -a\sin x - b\cos x$

$\therefore\ y + y'' = a$

$\therefore\ b = -\left(\frac{a\sin x + y''}{\cos x}\right)$

$\therefore\ y' = (y+y'')\cos x + (\sin x)\left(\frac{y'' + (y+y'')\sin x}{\cos x}\right)$

(vi) $y = a(1-x^2) + bx + cx^3$

$y' = -2ax + b + 3cx^2$

$y'' = -2a + 6cx$

$y''' = 6c$

Solve for a, b, c from first 3 and substitute in the last (or just work backwards)

$y'' = -2a + y'''x$

$b = y' + 2ax + \frac{y'''}{2}x^2$

$= y' + (y'''x - y'')x + \frac{y'''}{2}x^2$

$\therefore y = (1-x^2)\left(\frac{y'''x - y''}{2}\right) + \left(y' + (y'''x - y'')x + \frac{y'''x^2}{2}\right)x$

$+ \frac{y'''}{6}x^3$

(vii) $y = cx + f(c)$

$y' = c$

$\therefore y = cy' + f(y')$

3) Solve $x^3 \sin y \, y' = 2$ and find a particular solution so that $\lim_{x \to \infty} y(x) = \frac{\pi}{2}$

<u>Ans</u> $\sin y \, y' = \frac{2}{x^3}$

$\therefore -\cos y + c = -\frac{1}{x^2}$

$\Rightarrow \frac{1}{x^2} - \cos y = K$

As $x \to \infty$, $y \to \frac{\pi}{2}$

$\Rightarrow K = -\cos\frac{\pi}{2} = 0$

$\therefore \cos y = \frac{1}{x^2}$

(4) Prove that a curve with the property that all normals pass through a common point is a circle

<u>Ans</u> Let point be (a, b)

$-\frac{1}{y'} = \frac{x-a}{y-b}$ $\forall x, y$ on curve

$\Rightarrow (x-a)^2 + (y-b)^2 = c$

(on solving)

$\Rightarrow$ circle (for $c > 0$)

5) find m for which

(i) e^{mx} is a solution of

(a) $y'' + y' - 6y = 0$

(b) $y''' - 3y'' + 2y' = 0$

(ii) x^m is a solution of

(a) $x^2 y'' - 4xy' + 4y = 0$

(b) $x^2 y''' - xy'' + y' = 0$

Ans (i) (a) $m^2 + m - 6 = 0$

$\therefore m = -3, +2$

(b) $m^3 - 3m^2 + 2m = 0$

$\therefore m = 0, 1, 2$

(ii) (a) $m(m-1) - 4m + 4 = 0$

$\therefore m = 1, 4$

(b) $m(m-1)(m-2) - m(m-1) + m = 0$

$\therefore m = 0, 2, 2$

6) Verify that the given function is a solution

(i) $y'' + 4y = 5e^x + 3\sin x$; $y = a\sin 2x + b\cos 2x + e^x + \sin x$

(ii) $y'' - 5y' + 6y = 0$; $y = c_1 e^{3x} + c_2 e^{2x}$

(iii) $y''' + 6y'' + 11y' + 6y = e^{-2x}$; $y = ae^{-x} + be^{-2x} + ce^{-3x}$
$+ xe^{-2x}$

(iv) $y''' + 8y = 9e^{x} + 65\cos x$; $y = ae^{-2x} + e^{x}(b\cos\sqrt{3}x$
$+ c\sin\sqrt{3}x) + 8\cos x - \sin x + e^{x}$

<u>Ans</u> ? Just put it and check ... ?

7) Let φ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$. Show that $\varphi_1 + \varphi_2$ satisfies $y' + ay = (b_1 + b_2)(x)$. Use this to find solutions to $y' + y = \sin x + 3\cos 2x$, passing through origin

<u>Ans</u>
$$\varphi_1' + a\varphi_1 = b_1(x)$$
$$\varphi_2' + a\varphi_2 = b_2(x)$$
$$(\varphi_1 + \varphi_2)' + a(\varphi_1 + \varphi_2) = (\varphi_1' + a\varphi_1) + (\varphi_2' + a\varphi_2)$$
$$= b_1(x) + b_2(x)$$

Hence proved

$$y' + y = \sin x$$
$$\Rightarrow y = Ce^{-x} + \frac{1}{2}(\sin x - \cos x)$$

(use class 12 method)

$$y' + y = 3\cos 2x$$
$$\Rightarrow y = C'e^{-x} + \frac{3}{5}(2\sin 2x + \cos 2x)$$

Using the result, the soln to $y' + y$ [redacted] $=$
$\sin x + 3\cos 2x$ is

$$y = c'' e^{-x} + \frac{1}{2}(\sin x - \cos x) + \frac{3}{5}(2\sin 2x + \cos 2x)$$

passing through $(0,0)$

$$\Rightarrow 0 = c'' + \frac{1}{2}(-1) + \frac{3}{5}(1)$$

$$\Rightarrow c'' = -\frac{1}{10}$$

8) Solve the following

(i) $(x^2+1)\,dy + (y^2+4)\,dx = 0$, $y(1) = 0$

(ii) $y' = y\cot x$, $y(\pi/2) = 1$

(iii) $y' = y(y^2-1)$, $y(0) = 0$ or 1 or 2

(iv) $(x+2)y' - xy = 0$, $y(0) = 1$

(v) $y' = \dfrac{x-y}{x+y}$, $y(1) = 1$

(vi) $y' = (y-x)^2$, $y(0) = 2$

(vii) $2(y\sin 2x + \cos 2x)\,dx = \cos 2x\, dy$

Ans (i) separable

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = -\tan^{-1}(x) + C$$

$$C = \pi/4$$

(ii) separable

$$\ln|y| = \ln|\sin x| + C$$

$$C = 0$$

(iii) separable

$$x = -\ln|y| + \frac{1}{2}\ln|y^2-1| + C$$

In the case $y(0) = 2$, $C = \ln 2 - \frac{1}{2}\ln 3$

For $y(0) = 0$ or 1, $y' = 0 \Rightarrow y = 0$ or 1

is the constant function

(iv) separable

$$\ln|y| = x - 2\ln|x+2| + C$$

$$C = 2\ln 3 - 1$$

(v) use $y = vx$

$$xv' = \frac{1-2v-v^2}{1+v}$$

$$\therefore \ln|x| = -\frac{1}{2}\ln|v^2+2v-1| + C$$

$$C = \frac{1}{2}\ln 2$$

(vi) use $y - x = t$

$\Rightarrow y' - 1 = t'$

$y' = t^2$

$$\Rightarrow x = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$C = \frac{1}{2}\ln 3$$

(vii) separable ; $y\cos 2x = \sin 2x + C$

9) Find solutions of the homogeneous ODE's

(i) $y' = \dfrac{y^2 - xy}{x^2 + xy}$

(ii) $x^2 y' = y^2 + xy + x^2$

(iii) $x y' = y + x\cos^2\left(\frac{y}{x}\right)$

(iv) $x y' = y(\ln y - \ln x)$

Ans use $y = vx$ in all. I will only write final answers :

(i) $\ln|x| = -\dfrac{y}{2x} - \dfrac{1}{2}\ln\left|\dfrac{y}{x}\right|$

(ii) $\tan^{-1}\dfrac{y}{x} = c + \log|x|$

(iii) $\tan\left(\dfrac{y}{x}\right) = c + \log|x|$

(iv) $\log\left|\dfrac{y}{x}\right| = cx + 1$

10) Reduce to homogeneous

(i) $(17 + x - 2y) + (4x - 3y - 6)y' = 0$

(ii) $(x - y - 1) + (y - x + 5)y' = 0$

(iii) $x + 2y + 3 = (1 - 2x - 4y)y'$

Ans (i) use $X = x - 3$

$Y = y - 2$

$Y' = \dfrac{X - 2Y}{3Y - 4X}$

(ii) use $y - x = t$ to get $t' = -\dfrac{4}{t + 5}$

(iii) use $x+2y=t$ to get $\frac{t'-1}{2} = \frac{3+t}{1-2t}$

Note : (ii) & (iii) cannot be reduced to homogeneous but can be made separable

11) Solve $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$ with the conditions $y(0) = \pm\frac{\sqrt{3}}{2}$. Sketch & show the solns are part of the same ellipse. Also show that the part of the ellipse which is not a part of the ~~arcs~~ doesn't satisfy the diff eqn

Ans separable

$$\sin^{-1}x + \sin^{-1}y = c \quad — \quad (1)$$

$$y(0) = \frac{\sqrt{3}}{2} \Rightarrow c = \frac{\pi}{3}$$

$$y(0) = -\frac{\sqrt{3}}{2} \Rightarrow c = \frac{-\pi}{3}$$

in general $\sqrt{(1-x^2)(1-y^2)} = \cos C + xy$

(solving (1))

$$\therefore \quad x^2+y^2+2xy\cos C = \sin^2 C$$

For $c = \pm\frac{\pi}{3}$, we get $x^2+y^2+xy = \frac{3}{4}$

which is an ellipse.

Now, $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \geqslant 0$

The parts of the ellipse with $y' < 0$ clearly do not satisfy this

$x^2 + y^2 + xy = \frac{3}{4}$

12) $y = xy' + f(y')$ is called Clairaut's eqn. Show that the general soln is $y = cx + f(c)$.

Also show the existence of a special solution $f'(y') = -x$

Ans $y = xy' + f(y')$

$\Rightarrow y' = xy'' + y' + f'(y')\, y''$

$\Rightarrow (x + f'(y'))\, y'' = 0$

$\therefore f'(y') = -x$

OR

$y'' = 0$

$\Rightarrow y' = c$

$\Rightarrow y = cx + d$

But $y = xy' + f(y')$

$\Rightarrow cx + d = xc + f(c)$

$\Rightarrow d = f(c)$

$\Rightarrow y = cx + f(c)$

13) Find all solns

(i) $y = xy' + \frac{1}{y'}$

(ii) $y = xy' - \frac{y'}{\sqrt{1+(y')^2}}$

Ans (using 12)

(i) $y = cx + f(c) = cx + \frac{1}{c}$

OR

$x - \frac{1}{(y')^2} = 0 \Rightarrow y = \pm 2\sqrt{x} + c$

$y = xy' + \frac{1}{y'}$

$\Rightarrow \pm 2\sqrt{x} + c = x\left(\frac{\pm 1}{\sqrt{x}}\right) + (\mp\sqrt{x})$

$c = 0$ in both cases

$\therefore y = \pm 2\sqrt{x}$ $(\Rightarrow y^2 = 4x)$

(ii) $y = cx + f(c) \Rightarrow y = cx + \left(\frac{-c}{\sqrt{1+c^2}}\right)$

OR

$x - \frac{1}{(1+(y')^2)^{3/2}} = 0$

$\Rightarrow y' = \pm\sqrt{\frac{1}{x^{2/3}} - 1}$

$\Rightarrow y^2 = (1 - x^{2/3})^3$

(solve it)

14) For $y = x^2$ find eqn of tangent at (c, c^2) and find ODE for family of tangents

<u>Ans</u> $y - c^2 = 2c(x-c)$

$\Rightarrow y = 2xc - c^2$ → Clairaut !

The DE is $y = xy' - \frac{(y')^2}{4}$

Amazing part : This works for any tangent family to $y = f(x)$

(we always end up with Clairaut)

15) skipped (too annoying & easy)

16) Show that $y' - y^3 = 2x^{-3/2}$ has 3 solns of the form $\frac{A}{\sqrt{x}}$ but only one is real

<u>Ans</u> Putting $\frac{A}{\sqrt{x}}$, we get $A^3 + \frac{A}{2} + 2 = 0$

This cubic has only 1 real root

(Check using AOD from JEE !)

TUTORIAL 2

1) State for the following, the conditions under which we get an exact ODE

(i) $(f(x) + g(y))\,dx + (h(x) + k(y))\,dy = 0$

(ii) $(x^3 + xy^{\frac{1}{2}})\,dx + (ax^2y + bxy^2)\,dy = 0$

(iii) $(ax^2 + 2bxy + cy^2)\,dx + (bx^2 + 2cxy + gy^2)\,dy = 0$

Ans we only need to force $M_y = N_x$ where

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(i) $g'(y) = h'(x)$

Further, we can say

$g'(y) = h'(x) = \text{constant}$

(ii) $2xy = 2axy + 2by^2 \Rightarrow a = 1,\ b = 0$

(iii) $2bx + 2cy = 2bx + 2cy \Rightarrow 0 = 0$

$\therefore$ no conditions. It is already exact

2) Solve the exact equations

(i) $3x(xy-2)\,dx + (x^3 + 2y)\,dy = 0$

(ii) $(\cos x \cos y - \cot x)\,dx - \sin x \sin y\,dy = 0$

(iii) $e^x y(x+y)\,dx + e^x(x+2y-1)\,dy = 0$

Ans General process: M, N are provided.

Integrate $M(x,y)$ wrt x to get $G(x,y) + f(y)$

Now $\frac{\partial G}{\partial y} + f'(y) = \frac{\partial M}{\partial y} = N \therefore$ solve for f

The required solution is $G(x,y) + f(y)$

. I will write the required u

(i) $x^3 y - 3x^2 + y^2$

(ii) $\sin x \cos y - \ln|\sin x|$

(iii) $ye^x(x+y-1)$

$$\begin{tikzcd}
 & u(x,y) \arrow[dl, "\partial_x"'] \arrow[dr, "\partial_y"] & \\
M \arrow[dr, "\partial_y"'] & & N \arrow[dl, "\partial_x"] \\
 & M_y = N_x &
\end{tikzcd}$$

3) Determine suitable IF by inspection to make it exact

(i) $y\,dx + x\,dy = 0$

(ii) $d(e^x \sin y) = 0$

(iii) $dx + \left(\frac{y}{x}\right)^2 dy = 0$

(iv) $y e^{x/y} dx + (y - x e^{x/y})\, dy = 0$

(v) $(2x + e^y) + x e^y\, dy = 0$

(vi) $(x^2 + y^2)\, dx + xy\, dy = 0$

Ans use theorem 5 to solve

(i) $\alpha = 0$ (already exact i.e. $\mu = 1$)

(ii) $\alpha = 0$ (already exact)

(iii) $\alpha = \frac{2y^2}{x^3}$

$\frac{\alpha}{N} = \frac{2}{x}$ (function of x alone)

$\therefore \mu = e^{\int \frac{2}{x} dx} = x^2$

(iv) $\alpha = 2e^{x/y}$

$\frac{\alpha}{M} = \frac{2}{y}$ (function of y alone)

$\therefore \mu = e^{\int -\frac{2}{y} dy} = \frac{1}{y^2}$

(v) $\alpha = 0$ (already exact)

(vi) $\alpha = y$

$\frac{\alpha}{N} = \frac{1}{x}$ (function of x alone)

$\therefore \mu = e^{\int \frac{1}{x} dx} = x$

4) Verify that $Mdx + Ndy = 0$ can be expressed as

$$\frac{1}{2}(Mx+Ny)\,d(\ln xy) + \frac{1}{2}(Mx-Ny)\,d\left(\ln\left(\frac{x}{y}\right)\right) = 0$$

Hence show that $Mx \pm Ny = 0 \Rightarrow \frac{1}{Mx \mp Ny}$ is an IF

~~If $Mx - Ny = 0$ then $\frac{1}{Mx+Ny}$ is an IF~~

Ans $\frac{1}{2}(Mx+Ny)\,d(\ln xy)$

$$= \frac{1}{2}(Mx+Ny)\left(\frac{1}{xy}(y\,dx + x\,dy)\right)$$

$$= \frac{1}{2}(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

$$= \frac{1}{2}(M\,dx + N\,dy) + \frac{1}{2}\left(M\frac{x}{y}dy + N\frac{y}{x}dx\right)$$

$$\frac{1}{2}(Mx-Ny)\,d\left(\ln\left(\frac{x}{y}\right)\right)$$

$$= \frac{1}{2}(Mx-Ny)\left(\frac{y}{x}\left(\frac{1}{y}dx - \frac{x}{y^2}dy\right)\right)$$

$$= \frac{1}{2}(Mx-Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)$$

$$= \frac{1}{2}(M\,dx + N\,dy) - \frac{1}{2}\left(M\frac{x}{y}dy + N\frac{y}{x}dx\right)$$

$\therefore$ sum of the two is $M\,dx + N\,dy$

Now suppose $Mx + Ny = 0$

Then $\frac{1}{2}(Mx-Ny)\,d\left(\ln\left(\frac{x}{y}\right)\right) = 0$

Multiplying by $\frac{1}{Mx - Ny}$ gives

$d\left(\ln\left(\frac{x}{y}\right)\right) = 0$ which is exact

Similarly for $-$, $+$

4.1) $Mdx + Ndy = 0$. If M, N are homo of same degree, then $\frac{1}{Mx+Ny}$ is an I.F.

Prove this fact

Multiply by $\frac{1}{Mx+Ny}$ to get

$$d(\ln xy) + \left(\frac{Mx - Ny}{Mx + Ny}\right) d\left(\ln\left(\frac{x}{y}\right)\right) = 0$$

First part is exact. We want to check if second part is also exact

$$\frac{M(x,y)\cdot x - N(x,y)\cdot y}{M(x,y)\cdot x + N(x,y)\cdot y} d\left(\ln\left(\frac{x}{y}\right)\right) = 0$$

Put $y = vx$

$$\frac{M(1,v) - N(1,v)\cdot v}{M(1,v) + N(1,v)\cdot v} d\left(\ln\left(\frac{1}{v}\right)\right) = 0$$

Open everything up & check that $M'_v = N'_x$ in the new DE :

$$M'(x,v)\,dx + N'(x,v)\,dv = 0$$

5) If μ is an IF of $M\,dx + N\,dy = 0$, prove that

$$M_y - N_x = N\frac{\partial}{\partial x}\ln|\mu| - M\frac{\partial}{\partial y}\ln|\mu|.$$

use this to prove theorem 5.

Further, if $M_y - N_x = N\cdot f(x) - M\cdot g(y)$, then show that $\mu(x,y) = \exp\left(\int f(u)\,du + \int g(t)\,dt\right)$

Ans $\mu M\,dx + \mu N\,dy = 0$ is exact

$$\Rightarrow (\mu N)_x = (\mu M)_y$$

$$\Rightarrow \frac{\partial \mu}{\partial x}\cdot N + \mu N_x = \mu M_y + \frac{\partial \mu}{\partial y}\cdot M$$

$$\therefore M_y - N_x = \frac{\mu_x N - \mu_y M}{\mu}$$

$$= N\frac{\partial}{\partial x}\ln|\mu| - M\frac{\partial}{\partial y}\ln|\mu|$$

$\therefore$ If $\dfrac{M_y - N_x}{N} = f(x)$,

then $\dfrac{\partial}{\partial x}\ln|\mu| - \dfrac{M}{N}\dfrac{\partial}{\partial y}\ln|\mu| = f(x)$

But $\dfrac{M}{N}$ is a function of both x, y

$\therefore \mu$ must be a function of x only

and hence $\dfrac{\partial}{\partial x}\ln|\mu| = f(x)$

$$\Rightarrow \mu(x) = \exp\left(\int f(x)\,dx\right)$$

similarly $\dfrac{M_y - N_x}{M} = g(y)$

$$\Rightarrow \mu(y) = \exp\left(\int g(y)\,dy\right)$$

Now if $M_y - N_x = N f(x) - M g(y)$,

~~then $M\mu_y - N\mu_x = \mu(Mg(y))$.~~

then from $M\mu_y + \mu M_y = N\mu_x + \mu N_x$,

$$M\mu_y - N\mu_x = -\mu\left(N f(x) - M g(y)\right)$$

$$\therefore M\mu_y - N\mu_x = \mu\left(M g(y) - N f(x)\right)$$

This is indeed true for the given μ

6) Find general solutions

(i) $y - xy' = a(1 + x^2 y')$

(ii) $(y + x f(x^2+y^2))\,dx + (y f(x^2+y^2) - x)\,dy = 0$

(iii) $(x^2 + y^2 + \sqrt{x^2+y^2})\,dx + (xy\sqrt{x^2+y^2})\,dy = 0$

(iv) $(x+y)^2\, y' = 1$

(v) $y' - \frac{y}{x} = \frac{y^2}{x}$

(vi) $x^2 y' + 2xy = \sinh 3x$

(vii) $y' + y\tan x = \cos^2 x$

(viii) $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

Ans (i) separable $\quad y = \dfrac{a - x}{x - a + C}$

(ii) $\alpha = M_y - N_x = 2$

Trying $\alpha = 2 = f(x)\cdot N - g(y)\cdot M$

gets you nowhere

The integrating factor is $\dfrac{1}{x^2+y^2}$

(comes with practice & observation)

$$\frac{y + x f(x^2+y^2)}{x^2+y^2} = M, \quad \frac{-x + y f(x^2+y^2)}{x^2+y^2} = N$$

$$M_y = \frac{(x^2+y^2)(1 + 2xy f'(x^2+y^2)) - (y + x f(x^2+y^2))(2y)}{(x^2+y^2)^2}$$

$$N_x = \frac{(x^2+y^2)(-1 + 2xy f'(x^2+y^2)) - (-x + y f(x^2+y^2))(2x)}{(x^2+y^2)^2}$$

$$M_y - N_x = \frac{2(x^2+y^2) - (2y^2 + 2x^2)}{(x^2+y^2)^2} = 0$$

$$\int M \, \partial x = \tan^{-1}\left(\frac{x}{y}\right) + \int x \frac{f(x^2+y^2)}{x^2+y^2} \, \partial x$$

$$x^2 + y^2 = t$$

$$2x \, \partial x = dt$$

$$\int x \frac{f(x^2+y^2)}{x^2+y^2} \, \partial x = \frac{1}{2} \int \frac{f(t)}{t} \, dt$$

$$\frac{\partial}{\partial y}\left(\int M \partial x\right) = \frac{y^2}{x^2+y^2} \cdot x \cdot \left(-\frac{1}{y^2}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2} \int \frac{f(t)}{t}\right)$$

$$= N$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{1}{2} \int \frac{f(t)}{t} \, dt\right) = \frac{y f(x^2+y^2)}{x^2+y^2}$$

which is indeed true

$$\therefore u(x,y) = \int M \, \partial x = \tan^{-1}\frac{x}{y} + \int \frac{x f(x^2+y^2) \, \partial x}{x^2+y^2}$$

~~The solution is given by~~

$\cancel{\tan^{-1}\frac{x}{y} + \int \frac{x\, f(x^2+y^2)}{x^2+y^2}\, dx}$

$$\int N\,dy = \tan^{-1}\left(\frac{x}{y}\right) + \int y\, \frac{f(x^2+y^2)}{x^2+y^2}\, dy$$

Since the equation is exact, these must be the same.

Hence $\tan^{-1}\left(\frac{x}{y}\right) + \int y\, \frac{f(x^2+y^2)}{x^2+y^2}\, \partial y = C$

is a solution of the equation.

(iii) $\alpha = 4y\sqrt{x^2+y^2}$ $\quad \frac{\alpha}{N}$ is a function of x

(iv) use $t = x + y^2$

(v) separable $\quad y = \frac{e^{kx}}{1 - x e^{kx}}$

(vi) linear (solve using class 12)

note: $\sinh 3x = \frac{e^{3x} - e^{-3x}}{2}$

(vii) see above (vi)

(viii) Done in tut 1

7) Solve

(i) $(x^3+y^3)\,dx - 3xy^2\,dy = 0$

(ii) $(x^2+6y^2)\,dx + 4xy\,dy = 0$

(iii) $xy' = y\ln y - y\ln x$

(iv) $xy' = y + x\cos^2(y/x)$

Ans use $y = vx$ & solve

(already done similar stuff in tut 1)

8) Solve the first order linear ODE's

(i) $xy' - 2y = x^4$

(ii) $y' + 2y = e^{-2x}$

(iii) $y' = 1 + 3y \tan x$

(iv) $y' = \operatorname{cosec} x + y \cot x$

(v) ~~$y' = \operatorname{cosec} x + y \cot x$~~

(vi) $y' - my = c_1 e^{mx}$

Ans class 12 stuff!

I hope you understand now, why we used $IF = e^{\int p(x)\,dx}$ in class 12

and wrote solution as $y \cdot IF = \int IF \cdot q(x)\,dx$

9) Solve the Bernoulli eqns

(i) $e^y y' - e^y = 2x$

(ii) $2(y+1)y' - \frac{2}{x}(y+1)^2 = xy$

(iii) $xy' = 1 - y - xy$

(iv) $(xy + x^3y^3)\,y' = 1$

(v) $y' = xy + x^3y^3$

(vi) $xy' + y = 2x^6y^4$

(vii) $6y^2\,dx - x(2x^3 + y)\,dy = 0$

Ans I will only write what to substitute.

(i) $t = e^y$

(ii) $y+1 = t$ then $t^2 = u$

(iii) No sub. Divide ~~throught~~ throughout by x

(iv) ~~divide~~ write as $\frac{dx}{dy} = \dots$ & divide by $\frac{1}{x^3}$

Then make the sub $\frac{1}{x^2} = t$

(v) same as above

(vi) divide ~~throught~~ throughout by x & $t = \frac{1}{y^3}$

(vii) Bernoulli in x. $t = \frac{1}{x^3}$

10) (i) Solve $(x^2+6y^2)\,dx - 4xy\,dy = 0$ as Bernoulli

(ii) Take $y' = y(1-y)$; $y(0) = 0$. ICQ: it be solved using separation. What about solving as Bernoulli

(iii) Solve $2y\,dx + x(x^2 \ln y - 1)\,dy = 0$

(iv) Solve $\cos y \sin 2x\,dx = (\cos^2 x - \cos^2 y)\,dy$

Ans (i) Bernoulli in y, $t = x^2$

(ii) Bernoulli has problems since we can't divide by y ($\because y(0) = 0$)

similarly for separable, we can't divide by $y(1-y)$ directly

use $y = 1-u \Rightarrow -u' = u(1-u)$

$u' + u = u^2$. sub $\frac{1}{u} = t$

$(u(0) = 1)$ to get $t = 1 + Ke^x$

$\therefore u(x) = 1 \Rightarrow y(x) = 0 \;\forall x$

(iii) Bernoulli in y . use $t = \frac{1}{x^2}$

(iv) $\cos^2 x = t$ reduces it to class 12 problem

11) Find orthogonal trajectories

(i) $x^2 - y^2 = c$ (vi) $y^2 = 4x^2(1 - cx)$

(ii) $y = ce^{-x^2}$ (vii) $y^2 = x^3/(a-x)$

(iii) $e^x \cos y = c$ (viii) $y = c(\sec x + \tan x)$

(iv) $x^2 + y^2 = c$ (ix) $xy = c(x+y)$

(v) $y^2 = 4(x+h)$ (x) $x^2 + (y-c)^2 = 1 + c^2$

Ans Step 1 : find ODE

Step 2 : Substitute $-\frac{1}{y'}$ in place of y'

Step 3 : solve the new ODE

Final answers :

(i) $xy = c$ (vi) <complicated>

(ii) $y^2 = c + \log x$ (vii) <complicated>

(iii) $e^x \sin y = c$ (viii) $y^2 = c - 2\sin x$

(iv) $y = cx$ (ix) $x^3 - y^3 = c$

(v) $y = ce^{-x/2}$ (x) $y^2 = cx - x^2 - 1$

12) Find ODE for $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ (λ is parameter)

and find orthogonal trajectories also

Ans $\frac{(xy' - y)(x + yy')}{y'} = a^2 - b^2$

Replacing y' with $\frac{-1}{y'}$ gives us the same equation again!

these ellipse family is orthogonal to itself
(a.k.a. confocal conic sections)

13) $y' = P(x) + Q(x)y + R(x)y^2$ is Riccati eqn.
If a solution y_1 is known, then general solution is $y = y_1 + u$ where u satisfies
$\frac{du}{dx} = Ru^2 + Qu + 2Ry_1u$. Solve

(i) $y' + x^3y - x^2y^2 = 1$, $y_1 = x$
(ii) $y' = x^3(y-x)^2 + x^{-1}y$, $y_1 = x$

Ans Do as you're told and you'll end up with Bernoulli equations. (need to sub $v = \frac{1}{u}$)
(Note: for 1st part, don't bother solving the integral. It is non elementary)

14) Using Picard's integration, solve
(i) $y' = 2\sqrt{y}$, $y(1) = 0$
(ii) $y' = xy + 1$, $y(0) = 1$
(iii) $y' = x - y^2$, $y(0) = 1$

Ans $$y_{n+1} = y_0 + \int_{x_0}^{x} f(t, y_n)\, dt$$
Need to use recursion to get the sequences & take their limit

(i) keep getting 0 everytime

$\therefore y = 0$

Important note:

What happened to $y(x) = (x-1)^{2i}$?

well, $\frac{\partial f}{\partial y}$ is not continuous on any domain which has $(1,0)$ as an interior point and hence Picard's theorem cannot be applied

(the integral process is valid, though)

(ii) we get

$1, \quad 1+t+\frac{t^2}{2}, \quad 1+t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{8}, \ldots$

In general, $y_n = \sum_{k=0}^{2m} \frac{t^k}{k(k-2)(k-4)\ldots 1 \text{ or } 0}$

$\therefore$ Our $y = \sum_{k=0}^{\infty} \frac{t^k}{k(k-2)(k-4)\ldots}$

(iii) $y_0 = 1$

$y_1 = 1 - t + \frac{t^2}{2}$

$y_2 = 1 - t + \frac{3t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{4} - \frac{t^5}{20}$

$y_3 = 1 - t + \frac{3t^2}{2} - \frac{4t^3}{3} + \frac{13t^4}{12} - \frac{49t^5}{60} + \ldots$

(Complicated)

$y' = x - y^2$ isn't even "solvable" in some sense

Maybe typo in question

15) Show that $|\sin y| + x$ satisfies Lipschitz with $M=1$ but f_y DNE at $y=0$

Ans Clearly $\lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h}$

$$= \lim_{h\to 0} \frac{|\sin h| + x - x}{h}$$

$$= \lim_{h\to 0} \frac{|\sin h|}{|h|} \frac{|h|}{h} \quad \text{DNE}$$

$$|f(x,y_2) - f(x,y_1)| = \big|\, |\sin y_2| - |\sin y_1| \,\big|$$

$$\le |\sin y_2 - \sin y_1|$$

MVT

$$\le |y_2 - y_1|$$

$\therefore$ Lipschitz with $M = 1$

16) Check for Lipschitz (wrt y). Does f_y exist?

(i) $f = |x| + |y|$ over $\mathbb{R}^2$

(ii) $f = 2\sqrt{y}$ in $|x| \le 1$, $y \in [0,1]$

(iii) $f = x^2 |y|$ in $|x| \le 1$ $|y| \le 1$

(iv) $f = x^2 \cos^2 y + y^2 \sin^2 x$ in $|x| \le 1$, $y \in \mathbb{R}$

Ans

(i) $\big|\, |y_1| - |y_2| \,\big| \le |y_1 - y_2|$

(Lipschitz ✓)

$\lim_{h\to 0} \frac{|h|}{h}$ DNE (f_y ✗)

(ii) $|2\sqrt{y_1} - 2\sqrt{y_2}| = 2|\sqrt{y_1} - \sqrt{y_2}| \not\le M|y_1 - y_2|$

No!

Let $y_1 = 1/n$, $y_2 = 0$

Then $|\sqrt{\tfrac{1}{n}} - \sqrt{0}| \leq C \, |\tfrac{1}{n} - 0|$

$\Rightarrow \sqrt{\tfrac{1}{n}} \leq \tfrac{C}{n}$

$\Rightarrow \sqrt{n} \leq C \;\; \forall n$

This $\Rightarrow$ a ~~contr~~ contradiction

$\lim_{h \to 0} \dfrac{2\sqrt{h}}{h}$ clearly DNE

(iii) $\big| x^2 |y_1| - x^2 |y_2| \big|$ TVM

$= |x^2| \big(|y_1| - |y_2| \big)$

$\leq |x^2| \, |y_1 - y_2|$

$\leq |y_1 - y_2| \quad (\because |x| \leq 1)$

$\therefore$ lipschitz ✓

$\lim_{h \to 0} \dfrac{x^2 |h|}{h}$ DNE on $|y| \leq 1$, $|x| \leq 1$

(iv) $\lim_{h \to 0} \dfrac{x^2 \cos^2 h + h \sin^2 x - (x^2)}{h}$

$= \lim_{h \to 0} \dfrac{h \sin^2 x}{h} - \dfrac{x^2 \sin^2 h}{h}$

$= \sin^2 x - 0 = \sin^2 x$

$\Rightarrow$ l_y exists

$\Rightarrow$ lipschitz also.

TUTORIAL 3

1) Find curve through origin for which $y'' = y'$ & tangent at origin is $y = x$

Ans $y'' - y' = 0 \Rightarrow m^2 - m = 0 \Rightarrow 1, e^x$ are basis

solutions $\Rightarrow$ general is $a + be^x$

Tangent at origin is $y = bx \Rightarrow b = 1$

Passing throug origin $\Rightarrow a = -1$

$\therefore -1 + e^x$

2) Find gen sols of $y'' - y' - 2y = 0$, $y'' - 2y' + 5y = 0$

Ans (i) $m^2 - m - 2 = 0 \Rightarrow ae^{2x} + be^{-x}$

(ii) $m^2 - 2m + 5 = 0 \Rightarrow e^x(a\cos 2x + b\sin 2x)$

3) Find a, b in $y'' + ay' + by = 0$ if eqn has solns as $\{e^{-2x}, 1\}$ and (separately) $\{e^{zx}, e^{\bar{z}x}\}$

Ans (i) $m^2 + am + b$ has roots $-2, 0$

$\therefore a = 2, \quad b = 0$

(ii) $m^2 + am + b$ has roots $z, \bar{z}$

$\therefore a = -(z + \bar{z}), \quad b = |z|^2$

4) True or false (give proof or counter example). Let Ly denote $y'' + P(x)y' + Q(x)y$. & let y_1, y_2 be solns

(i) f, g LI on $I \Rightarrow$ LI on J containing I

(ii) f, g LD on $I \Rightarrow$ LD on J contained in I

(iii) y_1, y_2 LI on $I \Rightarrow$ LI on J contained in I

(iv) y_1, y_2 LD on $I \Rightarrow$ LD on J contained in I

Ans (i) True

f, g LI on I

$\Rightarrow f, g$ LI on J

suppose $c_1 f(x) + c_2 g(x) = 0 \ \forall x \in J$

Then $c_1 f(x) + c_2 g(x) = 0 \ \forall x \in I$ also

$\Rightarrow c_1 = c_2 = 0$

(ii) True

contrapositive of above

(iii) True

LI $\Leftrightarrow$ $W(y_1, y_2) \neq 0$ everywhere on I

$\Rightarrow$ LI since $W(y_1, y_2) \neq 0$ everywhere on J $(J \subset I)$

(iv) $W = 0$ everywhere on I

$\Rightarrow W = 0$ everywhere on $J \subset I$

$\Rightarrow y_1, y_2$ LD on J

5) check LI or not

(i) $\sin 2x, \cos(2x + \frac{\pi}{2})$ $(x > 0)$

(ii) $x^3, x^2|x|$ $(-1 < x < 1)$

(iii) $x|x|, x^2$ $(0 \leq x < 1)$

(iv) $\log x, \log x^2$ $(x > 0)$

(v) $x, x^2, \sin x$, $x \in \mathbb{R}$

Ans (i) No! Choose $a = b = 1$

(ii) $ax^2 + bx^2|x| = 0 \quad \forall x \in (-1, 1)$

True for $x = 0.1$, $x = -0.1$

$\Rightarrow a + b = 0, \quad a - b = 0$

$\Rightarrow a = b = 0$

$\therefore$ LI

(iii) No! Choose $a = 1, b = -1$

(iv) No! Choose $a = -2, b = 1$

(v) $W = \det \begin{bmatrix} x & x^2 & \sin x \\ 1 & 2x & \cos x \\ 0 & 2 & -\sin x \end{bmatrix}$

At $x = \pi/2$, $W = 2 - \left(\frac{\pi}{2}\right)^2 \neq 0$

$\therefore$ LI

6) Solve

(i) $y'' - 4y' + 3y = 0$

(ii) $y'' = 2y'$

Ans (i) $m^2 - 4m + 3 = 0 \Rightarrow ae^{3x} + be^{x}$

(ii) $m^2 = 2m \Rightarrow a + be^{2x}$

7) Solve

(i) $(D^2 + 5D + 6)y = 0$

(ii) $(D+1)^2 y = 0$

(iii) $(D^2 + 2D + 2)y = 0$

Ans (i) $m^2+5m+6=0 \Rightarrow ae^{-2x}+be^{-3x}$

(ii) $(m+1)^2=0 \Rightarrow ae^{-x}+bxe^{-x}$

(iii) $m^2+2m+2=0 \Rightarrow e^{-x}(a\cos x+b\sin x)$

8) Solve

(i) $(x^2D^2-4xD+4)y=0$

(ii) $(4x^2D^2+4xD-1)y=0$

(iii) $(x^2D^2-5xD+8)y=0$

Ans (i) $m(m-1)-4m+4=0 \Rightarrow ax+bx^4$

(ii) $4m(m-1)+4m-1=0 \Rightarrow ax^{1/2}+bx^{-1/2}$

(iii) $m(m-1)-5m+8=0 \Rightarrow ax^4+bx^2$

9) Solve (preferably using annihilators)

(i) $y''+2y'+3y=27x$

(ii) $y''+y'-2y=3e^x$

(iii) $y''+4y'+4y=18\cosh x$

(iv) $y^{(4)}+y=6\sin x$

(v) $y''+4y'+3y=\sin x+2\cos x$

(vi) $y''-2y'-2y=2e^x\cos x$

(vii) $y''+y=x\cos x+\sin x$

(viii) $2y^{(4)}+3y''+y=x^2+3\sin x$

(ix) $y^{(3)}-y'=2x^2e^x$

(x) $y'''-5y''+8y'-4y=2e^x\cos x$

(xi) $y''+y'-2y=14+2x-2x^2$

(xii) $y''+y'-2y=-6\sin 2x-18\cos 2x$

(xiii) $y''-4y'+3y=4e^{3x}$

Ans I will only do 2 of them (one using annihilator, other using undet coeff)

(x) $y''' - 5y'' + 8y' - 4y = 2e^{x}\cos x$

RHS form is $2e^{x}\cos x$

from theorem 31 we use

$$(D-1)^2 + 1$$

$$((D-1)^2+1)(e^{x}\cos x)$$

$$= (D-1)^2 e^{x}\cos x + e^{x}\cos x$$

$$= (D^2+2-2D)(e^{x}\cos x)$$

$$= -2e^{x}\sin x + 2e^{x}\cos x - 2(e^{x}(\cos x - \sin x))$$

$$= 0$$

$$(D^2+2-2D)(y''' - 5y'' + 8y' - 4y) = 0$$

$$\therefore [y^{(5)} - 5y^{(4)} + 8y^{(3)} - 4y''] + [2y''' - 10y'' + 16y' - 8y] + [-2y^{(4)} + 10y^{(3)} - 16y'' + 8y'] = 0$$

$$\Rightarrow y^{(5)} - 7y^{(4)} + 20y^{(3)} - 30y'' + 24y' - 8y = 0$$

$\therefore m = 1,\ 2,\ 2,\ 1-i,\ 1+i$

$1 \to e^{x}$, $2, 2 \to e^{2x}, xe^{2x}$, $1-i \to e^{x}\cos x$, $1+i \to e^{x}\sin x$

$\underbrace{e^{x} \quad e^{2x}, xe^{2x}}_{\text{also solutions of homo part}}$

$y_p = c_1 e^x \cos x + c_2 e^x \sin x$

Put it back in to see that $c_2 = 0$, $c_1 = 1$

$\therefore y_p = e^x \cos x$

$\therefore$ general soln is $e^x \cos x$ + (soln to homo)

$= e^x \cos x + c_0 e^x + c_1 e^{2x} + c_2 x e^{2x}$

(x) (same using method of UC)

RHS is $2e^x \cos x$

$\therefore$ we guess y_p as

$e^x (a \cos x + b \sin x)$

$(D^3 - 5D^2 + 8D - 4) y = 0$

Expand [and solve for a, b, c]

we get $a = 1$, $b = 0$

$\therefore y_p = e^x \cos x$

$\therefore$ gen soln is $e^x \cos x + c_1 e^x + c_2 e^{2x} + c_2 x e^{2x}$

10) Guess y_p using method of U.C but dont find coeffs

(i) $y'' + y = x^3 \sin x$

(ii) $y'' + 2y' + y = 2x^2 e^{-x} + x^3 e^{2x}$

(iii) $y' + 4y = x^3 e^{-4x}$

(iv) $y^{(4)} + y = x e^{\frac{1}{\sqrt{2}}x} \sin \frac{1}{\sqrt{2}} x$

Ans (i) $(ax^3 + bx^2 + cx + d)\sin x + (ex^3 + fx^2 + gx + h)\cos x$

(ii) $e^{-x}(ax^2 + bx + c) + e^{2x}(dx^3 + ex^2 + fx + g)$

(iii) $e^{-4x}(ax^3 + bx^2 + cx + d)$

(iv) $e^{\frac{1}{\sqrt{2}}x}\left((ax+b)\sin \frac{x}{\sqrt{2}} + (cx+d)\cos \frac{x}{\sqrt{2}} \right)$

11) Repeated type question

12) Solve $x^2y'' - xy' - 3y = 0$ s.t. $y(1) = 0$, $y(x) \to 0$ as $x \to \infty$

Ans $m(m-1) - m - 3 = 0$

$\Rightarrow m = -1, 3$

$\Rightarrow y = \frac{a}{x} + bx^3$

$y(1) = 0 \Rightarrow a + b = 0$

$y \to 0$ as $x \to \infty \Rightarrow b = 0 \Rightarrow a = 0$

[i.e. $y = 0$]

13) Show that every soln of const coeff equation $y'' + ay' + by = 0$ tends to zero as $x \to \infty$ iff real parts of roots of characteristic poly are -ve

Ans $y = a e^{m_1 x} + b e^{m_2 x}$ (case of distinct roots)

$m_1 \; m_2 < 0$

$\Longleftrightarrow y \to 0$ as $x \to \infty$

if roots are equal,

$y = e^{mx}(a + bx)$

Again $y \to 0$ as $x \to \infty$ iff $m < 0$

If roots are complex,

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$a < 0$ iff $y \to 0$ as $x \to \infty$

TUTORIAL 4

1) Use variation of parameters to find a solution

(i) $y'' - 5y' + 6y = 2e^x$

(ii) $y'' + y = \tan x \qquad x \in (0, \pi/2)$

(iii) $y'' + 4y' + 4y = x^{-2} e^{-2x}$ for $x > 0$

(iv) $y'' + 4y = 3 \operatorname{cosec} 2x$ for $x \in (0, \pi/2)$

(v) $x^2 y'' - 2xy' + 2y = 5x^3 \cos x$

(vi) $xy'' - y' = (3+x) x^3 e^x$

Ans

$$y_p = y_2 \int \frac{y_1 \cdot r(x)}{W(x)} dx - y_1 \int \frac{y_2 \cdot r(x)}{W(x)} dx$$

(i) $m^2 - 5m + 6 \Rightarrow y_1 = e^{2x}, \; y_2 = e^{3x}$

$$W = \left| \begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{bmatrix} \right| = e^{5x}$$

$$\therefore y_p = e^{3x} \int \frac{e^{2x} \cdot 2e^x}{e^{5x}} dx - e^{2x} \int \frac{e^{3x} \cdot 2e^x}{e^{5x}} dx$$

$$= e^x \Rightarrow y = e^x + ae^{2x} + be^{3x}$$

(ii) $y_p = \cos x \log\left(\left|\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right|\right)$

(iii) $y_p = -e^{-2x} \log x$

(iv) $-\frac{3}{2} x \cos 2x + \frac{3}{4} \sin 2x \log \sin 2x = y_p$

(v) $-5x \cos x = y_p$

(vi) $e^{x}(x^3 - 2x^2 + 3x - 3) = y_p$

2) Let y_1, y_2 be solns of $y'' + p(x) y' + q(x) y = 0$ on $x \in (a, b)$. Show that $W'(x) = -p(x) W(x)$. If $W(x_0) = 0$, show that $W(x) \equiv 0$

Ans $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

$W' = y_1 y_2'' - y_1'' y_2 + \cancel{y_1' y_2'} - \cancel{y_1' y_2'}$

$= -y_1 (p(x) y_2' + q(x) y_2) + y_2 (p(x) y_1'$
$+ q(x) y_1)$

$= -p(x)(y_1 y_2' - y_2 y_1')$

$= -p(x) W$

$\therefore W' = -p W$

$\Rightarrow W(x) = W(x_0) \exp\left(\int_{x_0}^{x} -p(t)\, dt\right)$

. for any $x_0 \in (a, b)$

$\therefore$ if $W(x_0) = 0$, $W(x) \equiv 0$

3) Let y_1 be a soln to $y'' + p(x) y' + q(x) y = 0$. Let I be an interval where y_1 does not vanish and let $a \in I$. Prove that the general soln is

given by $y = y_1(x)\,(c_2 + c_1\,\varphi(x))$

where $\varphi(x) = \int_a^x \frac{\exp\left(-\int_a^t p(u)\,du\right)}{y_1^2(t)}\,dt$

<u>Ans</u> y_1 is a known solution

$y_2 = \varphi(x)\,y_1$ is the other solution

$y_2'' + p(x)\,y_2' + q(x)\,y_2 = 0$

$y_2' = y_1\varphi' + \varphi y_1'$

$y_2'' = y_1\varphi'' + 2\varphi' y_1' + \varphi y_1''$

$\therefore\ y_1\varphi'' + 2\varphi' y_1' + \varphi y_1'' + p y_1\varphi' + p\varphi y_1' + q y_1\varphi = 0$

$\therefore\ y_1\varphi'' + 2\varphi' y_1' + p y_1\varphi' = 0$

$\therefore\ \frac{\varphi''}{\varphi'} = -\frac{2y_1'}{y_1} - p$

$\therefore\ \ln|\varphi'| = \ln\left|\frac{1}{y_1^2}\right| - \int p\,dx$

$\Rightarrow\ \varphi' = \frac{e^{-\int p(t)\,dt}}{y_1^2}$

$\Rightarrow\ \varphi = \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dx$

Putting limits makes no change.

Check the above fact by observing that the constant of integration doesn't affect anything

4) One soln is given. Find the other LI soln

(i) $4x^2y'' + 4xy' + (4x^2-1)y = 0$, $y_1 = \frac{\sin x}{\sqrt{x}}$

(ii) $y'' - 4xy' + 4(x^2-2)y = 0$, $y_1 = e^{x^2}$

(iii) $x(x-1)y'' + 3xy' + y = 0$, $y_1 = \frac{x}{(x-1)^2}$

(iv) $xy'' - y' + 4x^3y = 0$, $y_1 = \cos x^2$

(v) $x^2(1-x)^2y'' - x^3y' - \left(\frac{3-x^2}{4}\right)y = 0$, $y_1 = \sqrt{\frac{1-x^2}{x}}$

(vi) $x(1+3x^2)y'' + 2y' - 6xy = 0$, $y_1 = 1+x^2$

(vii) $(\sin x - x\cos x)y'' - x\sin x\, y' + \sin x\, y = 0$, $y_1 = x$

Ans use 3rd question to solve. Remember that to get $p(x)$, we need eqn in its standard form.

I will write down $Q(x)$ (NOT y_2)

(i) $-\cot x$

(ii) x

(iii) $\ln x + \frac{1}{x}$

(iv) $\frac{1}{2}\tan^2 x$

(v) $\frac{1}{\sqrt{1-x^2}}$

(vi) $-\frac{1}{x(1+x^2)}$

(vii) $\frac{1}{x\sin x}$

5) Prove that $\{e^{r_i x} \mid i = 1, 2, \dots, n\}$ is LI for distinct r_i

Ans Compute wronskian & check that it is non zero

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} \times e^{r_1+\cdots+r_n}$$

$$= \left[\prod_{i>j} (r_i - r_j)\right] e^{r_1+\cdots+r_n} \neq 0$$

since all r_i are distinct

6) generalised question 1

7) Three solns of a NHSOLODE are given by

$1+e^{x^2}$, $1+xe^{x^2}$, $(1+x)e^{x^2}-1$

Find the general soln.

Ans f & g are solns to NHSOLODE

$\Rightarrow f = y_p + z_1$

$g = y_p + z_2$

where z_1, z_2 solve HSOLODE

$\therefore f-g$ solves HSOLODE

$\therefore y_1 - y_2 = e^{x^2} - xe^{x^2}$ solves HSOLODE

$y_2 - y_3 = 2 - e^{x^2}$ solves HSOLODE

These two are LI & hence form the basis

$\therefore$ gen soln is $a(e^{x^2} - xe^{x^2}) + b(2-e^{x^2}) + 1 + e^{x^2}$

8) [Repeated type questions]

(i) $(1+x^2)y'' - 2xy' + 2y = x^3 + x$, $y_1 = x$

(ii) $xy'' - y' + (1-x)y = x^2$, $y_1 = e^x$

(iii) $(2x+1)y'' - 4(x+1)y' + 4y = e^{2x}$, $y_1 = e^{2x}$

(iv) $(x^3 - x^2)y'' - (x^3 + 2x^2 - 2x)y' + (2x^2 + 2x - 2)y$
$= (x^3 - 2x^2 + x)e^x$, $y_1 = x^2$

9) Reduce the order knowing that $y_1 = x$ is a soln

(i) $x^3 y''' - 3x^2 y'' + (6 - x^2)xy' - (6 - x^2)y = 0$

(ii) $y'''' + (x^2+1)y'' - 2x^2 y' + 2xy = 0$

Ans Put $y = ux$

Then put $u' = v$

Note: these questions are 'fabricated' and very special. Not very general. The way the question is made, the u term vanishes and only higher derivatives of u persist.

10) Not in syllabus

11) Solve

(i) $x^2 y'' + 2xy' + y = x^3$

(ii) $x^4 y^{(4)} + 8x^3 y^{(3)} + 16x^2 y'' + 8xy' + y = x^3$

(iii) $x^2 y'' + 2xy' + \frac{y}{4} = \frac{1}{\sqrt{x}}$

Ans Not gonna do same stuff again & again!

12) Find y_p

(i) $x^2 y'' - 6y = \ln x$

(ii) $x^2 y'' + 2xy' - 6y = 10x^2$

Ans ~~[illegible]~~ Easy

Note: For part (i) use variation of parameters since $\ln(x)$ doesn't fit into our template of annihilator / UC

13) [Same stuff again]

(i) $(x^2 - x)\, y'' + (x+1)\, y' - y = 0$

$y_1 = 1 + x$

(ii) $(2x+1)\, y'' - 4(x+1)\, y' + 4y = 0$

$y_1 = e^{2x}$

14) Find homo LODE on $(0,\infty)$ whose gen soln is $c_1 x^2 e^x + c_2 x^3 e^x$. Does there exist one with const coeffs?

Ans $W = x^4 e^x$

$W(x) = -p(x)\, W'(x) \Rightarrow p(x) = -2 - \frac{4}{x}$

$y'' + py' + qy = 0 \Rightarrow q = 1 + \frac{6}{x^2}$

$\therefore\ y'' - \left(2 + \frac{4}{x}\right) y' + \left(1 + \frac{6}{x^2}\right) y = 0$

Const coeff DE having $x^2 e^x$, $x^3 e^x$ must also have e^x, xe^x as solns.

(Think why)

TUTORIAL 5

1) Find the Laplace transforms

(i) $t\cos\omega t$ — $s^2-\omega^2/(s^2+\omega^2)^2$

(ii) $t\sin\omega t$ — $2s\omega/(s^2+\omega^2)^2$

(iii) $e^{-t}\sin^2 t$ — $2/(s^3+3s^2+7s+5)$

(iv) t^2e^{-at} — $2/(a+s)^3$

(v) $(1+te^{-t})^3$ — $3/(s+1)^2 + 6/(s+2)^3 + 6/(s+3)^4 + 1/s$

(vi) $(5e^{2t}-3)^2$ — $4(s^2+4s+18)/s(s^2-6s+8)$

(vii) $te^{-2t}\sin\omega t$ — $2(s+2)\omega/((s+2)^2+\omega^2)^2$

(viii) $t^n e^{\alpha t}$ — $n!\,(s-\alpha)^{-n-1}$

(ix) $t^2e^{-at}\sin bt$ — $2b(3(a+s)^2-b^2)/((a+s)^2+b^2)^3$

(x) $\cosh at\cos at$ — $\dfrac{s^3}{4a^4+s^4}$

Ans (i) $L(\cos\omega t) = \dfrac{s}{s^2+\omega^2}$

$$\Rightarrow L(-t\sin\omega t) = \frac{-2s\omega}{(s^2+\omega^2)^2}$$

(for uniformly convergent integral, by Dominant convergence theorem, $\int$ and $\frac{d}{d\omega}$ can be swapped)

(ii) similar to above

(iii) $L(\sin^2 t) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)$

(iv) $L(t^2) = \dfrac{2}{s^3}$

(v) to (viii) Similar to above after expanding

(ix) $L(t^2 \sin bt) = \frac{d}{db}\left(\frac{b^2-s^2}{(b^2+s^2)^2}\right)$

(x) $\cosh at = \frac{e^{at}+e^{-at}}{2}$

2) Find inverse laplace transforms

(i) $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

(vi) $\frac{1}{s^4-2s^3}$

(ii) $\frac{2sa}{(s^2-a^2)^2}$

(vii) $\frac{1}{s^4(s^2+\pi^2)}$

(iii) $\frac{1}{(s^2+a^2)^2}$

(viii) $\frac{s^2+a^2}{(s^2-a^2)^2}$

(iv) $\frac{s^3}{s^4+4a^4}$

(ix) $\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$

(v) $\frac{s-2}{s^2(s+4)^2}$

(x) $\frac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^2}$

Ans <u>All</u> of these can be done using partial fractions. Let me do one

(i) $\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{1}{2}\left(\frac{1}{(s-i\omega)^2}+\frac{1}{(s+i\omega)^2}\right)$

$L^{-1}\left(\frac{1}{(s-i\omega)^2}\right) = e^{i\omega x} L^{-1}\left(\frac{1}{s^2}\right) = x e^{i\omega x}$

$L^{-1}\left(\frac{1}{(s+i\omega)^2}\right) = x e^{-i\omega x}$

3) Solve using Laplace

(i) $y'' + y = 3\sin t$, $y(0) = y'(0) = 0$

(ii) $y'' + 3y' + 2y = e^{-t}$, $y(0) = y'(0) = 0$

(iii) $y'' + 2y' + y = 2\cos t$, $y(0) = 3$, $y'(0) = 0$

(iv) $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y'(0) = 8$

(v) $y'' - 2y' + 5y = 8\sin t - 4\cos t$, $y(0) = 1$, $y'(0) = 3$

(vi) $y'' - 2y' - 3y = 10\sinh(2t)$, $y(0) = 0$, $y'(0) = 4$

Ans (i) $(1+s^2)\,L(y) = \dfrac{3}{s^2+9}$

$\therefore y = \dfrac{3}{8}\sin x - \dfrac{1}{8}\sin 3x$

Do the rest similarly

(ii) $xe^{-x} - e^{-x} + e^{-2x}$

(iv) $2e^{2x} - e^{-4x}$

(iii) $3e^{-x} + 3xe^{-x} + \dfrac{\sin x}{2} - \dfrac{xe^{-x}}{2}$

(v) $e^{x}\cos 2x + 2\sin x$

(vi) $e^{-2x} - \dfrac{e^{3x}}{2} - \dfrac{e^{-x}}{6} - \dfrac{5}{3}e^{2x}$

4) Solve using Laplace

(i) $x' = x + y$, $y' = 4x + y$

(ii) $x' = 3x + 2y$, $y' = -5x + y$

(iii) $x'' - x + y' = y - 1$, $y'' + y + x' - x = 0$

(iv) $x' = 5x + 8y + 1$, $y' = -6x - 2y + t$, $x(0) = 4$, $y(0) = -3$

(v) $y_1' + y_2 = 2\cos t$, $y_1 + y_2' = 0$, $y_1(0) = 0$, $y_2(0) = 1$

(vi) $y_1'' + y_2 = -5\cos 2t$, $y_2'' + y_1 = 5\cos 2t$, $y_1(0) = 1$

$y_1'(0) = 1$, $y_2(0) = -1$, $y_2'(0) = 1$

(vii) $2y_1' - y_2' - y_3' = 0$, $y_1' + y_2' = 4t + 2$, $y_2' + y_3$

$= t^2 + 2$, $y_1(0) = y_2(0) = y_3(0) = 0$

(viii) $y_1'' = y_1 + 3y_2$, $y_2'' = 4y_1 - 4e^t$,

$y_1(0) = 2$, $y_1'(0) = 3$, $y_2(0) = 1$, $y_2'(0) = 2$

Ans Procedure

Take Laplace on all given eqns to get

$L(y)$ & $L(x)$, Solve for these from

the simultaneous system

Invert to get y, x

(v) Taking laplace, we get

$$sL(y_1) + L(y_2) - y_1(0) = \frac{2s}{s^2+1}$$

$$L(y_1) + sL(y_2) - y_2(0) = 0$$

Solving, $L(y_1) = \frac{1}{s^2+1}$

$L(y_2) = \frac{s}{s^2+1}$

$\Rightarrow$ $y_1 = \sin t$

$y_2 = \cos t$

5) Assuming that for a power series in $\frac{1}{s}$ with no const. term, the Laplace transform can be obtained term by term, prove that

(i) $L^{-1}\left(\frac{1}{s-1}\right) = e^{t}$

(ii) $L^{-1}\left(\frac{1}{1+s^2}\right) = \sin t$

(iii) $L^{-1}\left(\frac{1}{s}e^{-b/s}\right) = J_0(2\sqrt{bt})$

(iv) $L^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right) = J_0(at)$

(v) $L^{-1}\left(\frac{e^{-b/s}}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi t}}\cos(2\sqrt{bt})$

(vi) $L^{-1}\left(\tan^{-1}\frac{1}{s}\right) = \frac{\sin t}{t}$

Ans (i) $\frac{1}{s-1} = \frac{1}{s(1-\frac{1}{s})} = \sum_{j=0}^{\infty}\frac{1}{s^{j+1}}$

$$\therefore L^{-1}\left(\frac{1}{s-1}\right) = \sum_{j=0}^{\infty} L^{-1}\left(\frac{1}{s^{j+1}}\right)$$

$$= \sum_{j=0}^{\infty}\frac{t^j}{j!}$$

$$= e^{t}$$

(ii) $\frac{1}{s^2+1} = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right)$

$$= \frac{1}{2i}\left(\sum_{j=0}^{\infty}\frac{i^j}{s^{j+1}} - \sum_{j=0}^{\infty}\frac{(-i)^j}{s^{j+1}}\right)$$

$$= \frac{1}{i}\sum_{\text{odd } j}\frac{i^j}{s^{j+1}} = \sum_{t=0}^{\infty}\frac{(-1)^t}{s^{2t+2}}$$

$\therefore L^{-1}\left(\frac{1}{s^2+1}\right)$

$$= \sum_{t=0}^{\infty} L^{-1}\left(\frac{(-1)^t}{s^{2t+2}}\right)$$

$$= \sum_{t=0}^{\infty} \frac{(-1)^t\, t^{2t+1}}{(2t+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$$

$$= \sin t$$

(iii) $$\frac{1}{s} e^{-\frac{b}{s}} = \sum_{j=0}^{\infty} \frac{(-1)^j\, b^j}{j!\, s^{j+1}}$$

$$\therefore L^{-1}\left(\frac{1}{s} e^{-b/s}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j\, b^j\, t^j}{j! \times j!}$$

$J_0(x)$ = Bessel function of first kind (of order 0)

$$= \sum_{i=0}^{\infty} \frac{(-1)^i\, x^{2i}}{2^{2i}\, (i!)^2}$$

$$\therefore L^{-1}\left(\frac{1}{s} e^{-b/s}\right) = J_0(2\sqrt{bt})$$

(iv) $$\frac{1}{\sqrt{a^2+s^2}} = \sum_{i=0}^{\infty} \frac{(-1)^i\, (2i)!\, a^{2i}}{2^i\, 2^i\, (i!)(i!)\, s^{2i+1}}$$

$$\therefore L^{-1}\left(\frac{1}{\sqrt{a^2+s^2}}\right) = \sum_{i=0}^{\infty} \frac{(-1)^i\, \cancel{(2i)!}\, a^{2i}\, t^{2i}}{\cancel{(2i)!}\, (2^i)^2\, (i!)^2}$$

$$= J_0(at)$$

(v) Same as iii but use $J_m(x)$ for $m = \pm 1/2$ definition

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

(vi) Use $\tan^{-1}$ series

6) Find Laplace transform of a periodic function with period p

Ans
$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \sum_{k=0}^{\infty} \int_{kp}^{(k+1)p} e^{-st} f(t)\, dt$$

$$= \sum_{k=0}^{\infty} \int_0^p e^{-s(kp+u)} f(kp+u)\, du$$

(using $t = kp + u$)

$$= \sum_{k=0}^{\infty} e^{-kps} \int_0^p e^{-su} f(u)\, du$$

$$= \frac{1}{1-e^{-sp}} \int_0^p e^{-su} f(u)\, du$$

7) Find laplace transform of $\lfloor x \rfloor$

Ans $\lfloor x \rfloor = x - \{x\}$

where $\{\ \}$ is the fractional part function

$\{x\}$ is periodic with period 1

$$\therefore L(\{x\}) = \frac{1}{1-e^{-s}} \int_0^1 e^{-su} \{t\} \, du$$

$$= \frac{1}{1-e^{-s}} \int_0^1 u e^{-su} \, du$$

$$= \frac{(1 - se^{-s} - e^{-s})}{s^2 (1-e^{-s})}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$$

$$= \frac{1}{s^2} - \frac{1}{s(e^{s}-1)}$$

$$L(x) = \frac{1}{s^2}$$

$$\therefore L(\lfloor x \rfloor) = \frac{1}{s(e^{s}-1)}$$

8) Find $L^{-1}\left(\frac{e^{-s} - e^{-2s} - e^{-3s} + e^{-4s}}{s^2}\right)$

Ans use theorem 36.

$$L^{-1}(f \cdot e^{-as})(t) = u_a(t) \cdot L^{-1}(f)(t-a)$$

$$\therefore L^{-1}\left(\frac{e^{-as}}{s^2}\right) = u_a(t) \cdot (t-a)$$

1) Find laplace transforms of

(i) $f(t) = u_\pi(t) \sin t$

(ii) $f(t) = u_1(t)\, e^{-2t}$

where $u_k(x)$ is heaviside function

Ans $L(u_c(t) \cdot f(t-c)) = e^{-cs} F(s)$

(i) use $f(t) = \sin(t+c)$

so that $f(t-c) = \sin t$

$\therefore L(u_\pi(t) \sin t)$

$= L(u_\pi(t) \cdot f(t-\pi))$

$= e^{-\pi s} F(s)$

where $F(s) = L(\sin(t+\pi))$

$= L(-\sin t)$

$= \frac{-1}{1+s^2}$

$\therefore$ result is $\frac{-e^{-\pi s}}{1+s^2}$

(ii) DIY (answer. (ii) $\frac{e^{-s-2}}{s+2}$)

10) $L^{-1}\left(\ln\left(\frac{s^2+4s+5}{s^2+2s+5}\right)\right)$

Ans we want $\mathcal{L}^{-1}\left(\ln\left(\frac{s+a}{s+b}\right)\right)$

$$\ln\left(\frac{s+a}{s+b}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j (a-b)^{j+1}}{(j+1)(s+b)^{j+1}}$$

$\therefore L^{-1}$ is $$e^{-bx} \sum_{j=0}^{\infty} \frac{(b-a)^{j+1} x^j}{(j+1)!}$$

$$= \frac{e^{-bx} - e^{-ax}}{x}$$

Here, $\ln\left(\frac{s^2+4s+5}{s^2+2s+5}\right)$

$$= \ln\left(\frac{(s-\alpha)(s-\beta)}{(s-\gamma)(s-\delta)}\right)$$

where $\alpha, \beta, \gamma, \delta$ are roots.

Now split & use above formula

Answer is $$\frac{2\left(e^{-t}(\cos 2t) - e^{-2t}(\cos 2t)\right)}{t}$$

11) $L(f(t)) = F(s)$, $L(g(t)) = G(s)$. Prove that $L(f*g(t)) = F(s)G(s)$

Ans ~~Repeat~~ Refer to proof of convolution.

$$F(s)\,G(s) = \left(\int_0^\infty e^{-st} f(t)\,dt\right)\left(\int_0^\infty e^{-su} g(u)\,du\right)$$

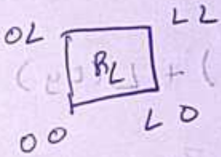
$$= \lim_{L\to\infty}\left(\int_0^L e^{-st} f(t)\,dt\right)\left(\int_0^L e^{-su} g(u)\,du\right)$$

$$= \lim_{L\to\infty}\int_0^L\int_0^L e^{-s(t+u)} f(t)\,g(u)\,du\,dt$$

(Fubini's theorem)

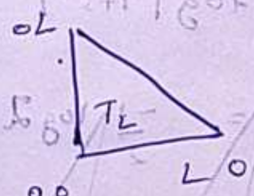
$$= \lim_{L\to\infty}\iint_{R_L} e^{-s(t+u)} f(t)\,g(u)\,dt\,du$$

where R_L is the square region

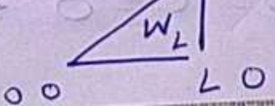


$$= \lim_{L\to\infty}\iint_{T_L} e^{-s(t+u)} f(t)\,g(u)\,du$$

when T_L is



$$= \lim_{L\to\infty}\iint_{A_L} e^{-sw} f(w-u)\,g(u)\,dw\,du$$



$$= \int_0^{L} \int_0^{\omega} e^{-s\omega} f(\omega - u)\, g(u)\, du\, d\omega$$

$$= \int_0^{L} e^{-s\omega} \left(\int_0^{\omega} f(\omega-u)\, g(u)\, du \right) d\omega$$

$$\underset{f * g(\omega)}{\shortparallel}$$

12) Find laplace transform of soln of $ty'' + y' + ty = 0$ satisfying $y(0) = k$, $\cancel{y}(1) = 1/\sqrt{2}$ where Y denotes laplace transform of y

Ans $\quad L(ty'') + L(y') + \boxed{L(ty)} = 0$

$$\therefore -2sY - s^2 Y' + \cancel{y(0)} + sY - y(0) - Y' = 0$$

$$\therefore -sY - (1+s^2)Y' = 0$$

$$\therefore \frac{Y'(s)}{Y(s)} = \frac{2s}{1+s^2} \times \frac{-1}{2}$$

$$\therefore \log Y(s) = \frac{-1}{2} \log|1+s^2| + C$$

$$\cancel{Y(1) = \frac{1}{\sqrt{2}} = \frac{-1}{2}\log 2 + C}$$

$$\cancel{\therefore Y(s) = \frac{-1}{2}\log|1+s^2| + \log\sqrt{2} + \frac{1}{\sqrt{2}}}$$

$$\therefore \log \frac{1}{\sqrt{2}} = -\frac{1}{2}\log 2 + C$$

$$\therefore C = 0 \Rightarrow Y(s) = \frac{1}{\sqrt{1+s^2}}$$

13) Find convolution of $t^{a-1}u(t)$, $t^{b-1}u(t)$

Ans Let h be their convolution

Then $h(x) = \int_0^x t^{a-1} u(t) \cdot (x-t)^{b-1} u(x-t)\, dt$

$h(x) = 0$ if $x < 0$

But for $x > 0$ we have

$$h(x) = \int_0^x t^{a-1} (x-t)^{b-1}\, dt$$

$$= x \int_0^1 (xu)^{a-1} (x-xu)^{b-1}\, du$$

$$(t = xu)$$

$$= x^{a+b-1} \int_0^1 t^{a-1} (1-t)^{b-1}\, dt$$

$$\therefore h(x) = u(x)\, x^{a+b-1} \int_0^1 t^{a-1} (1-t)^{b-1}\, dt$$

$$= u(x)\, x^{a+b-1} \underbrace{B(a,b)}_{\text{beta function}}$$

14) Find $f*f$ for $L(f(t)) = \frac{1}{\sqrt{s^2+1}}$

Ans $L(f*f) = \frac{1}{s^2+1} \Rightarrow f*f = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin x$

15) Evaluate using their laplace transform

(i) $\int_0^\infty \frac{\sin tx}{x} dx$

(ii) $\int_0^\infty \frac{\cos tx}{x^2+a^2} dx$

(iii) $\int_0^\infty \sin(tx^a) dx$

(iv) $\int_0^\infty \frac{1}{x^2}(1-\cos tx) dx$

(v) $\int_0^\infty \frac{\sin^4 tx}{x^3} dx$

(vi) $\int_0^\infty \left(\frac{x^2-b^2}{x^2+b^2}\right) \frac{\sin tx}{x} dx$

Ans (i) if $t > 0$,

$$L(I) = \int_0^\infty \frac{1}{x} L(\sin tx) dx$$

$$= \int_0^\infty \frac{dx}{x^2+s^2} = \frac{\pi}{2}$$

Similarly for $t < 0$ we get $\frac{-\pi}{2}$

(ii) → (vi) DIY

(ii) $\pi/2a \; e^{-at}$

(iii) $\frac{\Gamma(\frac{1}{a}) \sin(\frac{\pi}{2a})}{a \cdot t^{1/a}}$

(iv) $\frac{\pi t}{2}$

~~E~~

(vi) $\frac{\pi}{2}(2e^{-bt}-1)\ \frac{|t|}{t}$

(v) $t^2 \log 2$

16) Solve

(i) $y(t) = 1 - \sinh t + \int_0^t (1+x)\, y(t-x)\, dx$

(ii) $A = \int_0^t \frac{y(x)}{\sqrt{t-x}}\, dx$

(iii) $\frac{dy}{dt} = 1 - \int_0^t y(t-u)\, du \quad , \quad y(0) = 1$

Ans (i) $L(y) = \frac{1}{s} - \frac{1}{s^2-1} + \frac{(s+1)}{s^2} L(y)$

$\therefore L(y) = \frac{s^2(s^2-2)}{(s^2-1)(s^2-s-1)}$

$\Rightarrow y = \frac{e^t + e^{-t}}{2}$

(ii) $\frac{A}{s} = L(y) \sqrt{\frac{\pi}{s}} \Rightarrow y = \frac{A}{\pi \sqrt{t}}$

(iii) $sL(y) - y(0) = \frac{1}{s} - \frac{L(y)}{s}$

$\Rightarrow L(y) = \frac{s+1}{s^2+1}$

$\Rightarrow y = \sin t + \cos t$

17) No way ! Not again !

18) out of syllabus

19) $f(t) = \frac{1}{1+t^2} \Rightarrow L(f) = F$ satisfies $F'' + F = \frac{1}{s}$

Further show $F = \int_0^\infty \frac{\sin\lambda}{(\lambda+s)}\, d\lambda$

Ans $L(f) = F(s) = \int_0^\infty e^{-st} \frac{dt}{1+t^2}$

Differentiate by hand two times to get $F''(s) = \int_0^\infty t^2 \frac{e^{-st}\, dt}{1+t^2}$

$\therefore F + F'' = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$

$F'' + F = \frac{1}{t}$

$\cancel{y'' + y = \frac{1}{t}}$

$\cancel{\therefore s^2 L - s\, y(0) - y'(0)}$

We are solving now for y in

$y'' + y = \frac{1}{t}$

$\{\cos t, \sin t\}$ is basis for homo part.

By variation of parameters,

$y_p = \sin t \int \frac{\cos t}{t}\, dt - \cos t \int \frac{\sin t}{t}\, dt$

I have no idea how they got that form !