SYMMETRIC FUNCTIONS

(Theory of symmetric polynomials)

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Symmetric Functions

Notation: Let $x = (x_1, x_2, \dots)$ be a set of indeterminates.

If $\alpha = (\alpha_1, \alpha_2, \dots)$ is a weak composition of n (i.e. some of the α_i may be 0), by x^α we mean $\prod_{i=1}^{\infty} x_i^{\alpha_i}$

Definition: By a homogeneous symmetric function of degree n over R (comm ring), we mean a FPS $f(x) = \sum_{\alpha \models n} c_\alpha x^\alpha$ (where $c_\alpha \in R$) such that $f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x)$ for every possible permutation $\sigma : \{1, 2, \dots\} \to \{1, 2, \dots\}$

A few examples:

$f(x_1, x_2) = 2x_1x_2 - x_1^2x_2^2$ $\qquad R = \mathbb{Z}$, degree $= 2$

$f(x_1, x_2, \dots) = k$ (some $k \in \mathbb{R}$) $\qquad R = \mathbb{R}$, degree $= 0$

$f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i$ $\qquad R = \mathbb{F}_2$, degree $= 1$

Definition: The set of all homogeneous symmetric functions of degree n over R is denoted by Λ_R^n

Observe: Λ_R^n is an R-module (obvious way to sum & to scalar multiply FPS)

We shall mostly work with $R = \mathbb{Q}$ thus rendering Λ_R^n as a $\mathbb{Q}$-vector space

Also observe that with our usual FPS product, we can turn $\Lambda_{\mathbb{Q}} := \Lambda^1_{\mathbb{Q}} \oplus \Lambda^2_{\mathbb{Q}} \oplus \cdots$ into a $\mathbb{Q}$-algebra

Definition : $\Lambda_{\mathbb{Q}}$ is called as the algebra of symm functions

observe : $\Lambda_{\mathbb{Q}}$ is commutative with identity ($1 \in \Lambda^0_{\mathbb{Q}}$)

Further, $\Lambda_{\mathbb{Q}}$ is infact a graded algebra.

Notation : We shall write Λ^n to mean $\Lambda^n_{\mathbb{Q}}$ and similarly Λ to mean $\Lambda_{\mathbb{Q}}$

(Familiarity with partitions of n and ferrer's shapes is assumed)

Notation : By $\mathbb{P}_n$, we mean the collection of all partitions $T \vdash n$. By $\mathbb{P}$, we mean $\bigcup_{n \geq 0} \mathbb{P}_n$ ($\mathbb{P}_0 = \{\emptyset\}$, $\mathbb{P}_1 = \{1\}$)

Note that we shall treat the partition $4+3+3+1$ (of $n = 11$) as $(4, 3, 3, 1, 0, 0, 0, \ldots)$

This will be convenient for defining some orders on $\mathbb{P} / \mathbb{P}_n$

Definitions :

- For $\lambda, \mu \in \mathbb{P}$, $\lambda \leq_1 \mu$ if $\lambda_i \leq \mu_i \; \forall i$

  (Basically containment of Ferrer shapes → we get Young's lattice)

- For $\lambda, \mu \in \mathbb{P}_n$, $\lambda \leq_2 \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \; \forall i$

- For $\lambda, \mu \in \mathbb{P}_n$, $\lambda \leq_3 \mu$ if $\lambda = \mu$ OR $\exists i \geq 0$ st $\lambda_k = \mu_k$ $\forall k = 1, \ldots, i$ and $\lambda_{i+1} \leq \mu_{i+1}$ (dictionary)

Proposition 1

$a \leq_2 b \Rightarrow a \leq_3 b$ (i.e. $\leq_3$ extends $\leq_2$)

proof

If $a = b$ we are done. Suppose $a \neq b$.

$\forall i$, $a_1 + \cdots + a_i \leq b_1 + \cdots + b_i$

In particular, $a_1 \leq b_1$

Thus, in the dictionary order, $a \leq_3 b$

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Definition: For $\lambda \in P$, define $\text{rank}(\lambda)$ to be $\max_i \{ \lambda_i \geq i \}$

Observe: This is the length of the main diagonal ($\searrow$) of the Ferrer shape of λ

Definition: given $\lambda \vdash n$ define a symmetric function $m_\lambda(x) \in \Lambda^n$ as $m_\lambda(x) = \sum_\sigma x^{\sigma(\lambda)}$ where the sum is over all σ s.t. all of $\sigma(\lambda)$ are distinct (monomial symmetric functions)

ex Let $\lambda = (2, 1, 0, 0, \ldots) \vdash 3$

Then $m_\lambda(x_1, x_2, x_3, \ldots) = \sum_{i,j} x_i x_j^2$

Let $\lambda = \emptyset \vdash 0$

Then $m_\lambda(x_1, x_2, \ldots) = 1$

Let $\lambda = (1, 1, 0, \ldots) \vdash 2$

Then $m_\lambda(x_1, x_2, \ldots) = \sum_{i<j} x_i x_j$

(keep distinctness of σ in mind)

Proposition 2

The above defined $m_\lambda(x)$ is indeed a symmetric function which is homogeneous

proof

Homogenity is obvious since $|\sigma(\lambda)| = |\lambda| \ \forall \sigma$

To verify that it is a symmetric function, we need to verify the permutation condition.

Let τ be a permutation of x

$$m_\lambda(\tau(x)) = \sum_\sigma (\tau(x))^{\sigma(\lambda)} = \sum_\sigma (\tau(x))^{\tau \circ \sigma(\lambda)}$$

($\because$ For a fixed τ, if σ runs over all possible permutations of $\{1, 2, \cdots\}$, so does $\tau \circ \sigma$)

$$= \sum_\sigma x^{\sigma(\lambda)} = m_\lambda(x)$$

($\because$ After all, x^λ is a product of the x & hence $(\tau(x))^{\tau(\lambda)} = \tau(x^\alpha) = x^\lambda$)

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Proposition 3

$\{m_\lambda : \lambda \vdash n\}$ is a basis of Λ^n

proof

Let $f(x) = \sum_\alpha c_\alpha x^\alpha \in \Lambda^n$.

Consider $g(x) = \sum_{\lambda \vdash n} c_\lambda m_\lambda = \sum_{\lambda \vdash n} c_\lambda \sum_\sigma x^{\sigma(\lambda)}$

$$= \sum_{\lambda \vdash n} \sum_\sigma c_\lambda x^{\sigma(\lambda)}$$

$$f(\tau(x)) = \sum_{\alpha} c_\alpha (\tau(x))^{\alpha} = \sum_{\alpha} c_\alpha x^{\tau^{-1}(\alpha)} = \sum_{\alpha} c_\alpha x^{\alpha}$$

$\therefore$ In the FPS, λ and its permutations have the same coefficient c_α which we call c_λ now.

Thus, taking c_λ common & summing over all possible λ, we get $f(x) = g(x)$

$\therefore$ $\{ m_\lambda \mid \lambda \vdash n \}$ spans Λ^n

Suppose $\sum_{i=1}^{r} \beta_i m_{\lambda_i}(x) = 0$.

Then, $\sum_{i=1}^{r} \sum_{\sigma} \beta_i x^{\sigma(\lambda_i)} = 0$

Clearly each β_i must be zero (check that none of the β_i can be 'clubbed' together)

Corollary

$\dim(\Lambda^n)$ = no. of partitions of n

Definition: We define the elementary symmetric functions as

$$e_k(x) = m_{\underbrace{(1,1,\ldots,1}_{k \text{ times}},0,0,\ldots)}(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$ and

$$e_\lambda(x) := e_{\lambda_1}(x)\, e_{\lambda_2}(x) \cdots$$

Remarks:

- Product of symmetric functions is symmetric
- $\deg(e_{\lambda_i}(x)) = \lambda_i \Rightarrow \deg(e_\lambda(x)) = |\lambda|$

Proposition 4

Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \dots) \vDash n$. Then the coefficient of x^α in e_λ is equal to the number of 0-1 matrices having $R_e = \lambda$, $C_c = \alpha$

proof

Call this coefficient $M_{\lambda\alpha}$

Then $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} \, m_\mu$

($\because$ m_μ has many α's of type μ but each $\alpha \vDash n$ corresponds to a unique $\mu \vdash n$)

Consider $X = \begin{bmatrix} x_1 & x_2 & \cdots \\ \vdots & & \end{bmatrix}$ ($\infty \times \infty$ matrix)

To obtain a term of e_λ we choose λ_i entries from row i $\forall i$ and take their product. Suppose product is x^α

($\because$ $e_\lambda = (\Sigma \cdots)(\Sigma \cdots)(\Sigma \cdots) \cdots$)

Turning the chosen entries to 1's & non chosen to 0's, we get a 0-1 matrix with row sums $(\lambda_1, \lambda_2, \dots) = \lambda$ and column sums $(\alpha_1, \alpha_2, \dots) = \alpha$

Conversely, just reversing the process, we get x^α from such a 0-1 matrix

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Corollary

$M_{\lambda\mu} = M_{\mu\lambda}$. (We shall see later that $\{e_\lambda\}$ is also a basis & hence base change matrix $\{m_\lambda\} \leftrightarrow \{e_\lambda\}$ is symmetric)

Q) Suppose we have n balls — λ_1 of which are of colour 1, and so on.

We also have boxes labelled $1, 2, 3, \ldots$

We want to distribute the balls into the boxes so that no box has two balls of the same colour and box i has exactly μ_i balls.

[$(\lambda_1, \lambda_2, \ldots)$ & $(\mu_1, \mu_2, \ldots)$ are provided]

Find the number of ways of doing this

Ans WLOG, $\lambda_1 \geq \lambda_2 \geq \ldots$ ie. $\lambda \vdash n$. Thus $\mu \vDash n$

We claim that the answer is $M_{\lambda\mu}$.

Suppose a 0-1 matrix is provided ($R_c = \lambda$, $C_c = \mu$).

Consider the i^{th} column & if we have a 1 in the j^{th} place, we put a single ball of colour j into the box i (total number of balls in box $i = \mu_i$ and since it is a 0-1 matrix, at most 1 ball of each colour is put in any given box)

Conversely, it is not difficult to construct such a matrix from the distribution ∎

Definition: Let $\{u_\lambda\}$ be a basis for Λ and let $f = \sum_\lambda a_\lambda u_\lambda$ be some element in Λ. If $a_\lambda \geq 0 \ \forall \lambda$, we say f is <u>u-positive</u> and if $a_\lambda \in \mathbb{Z} \ \forall \lambda$, we say f is <u>u-integral</u>

Proposition 5

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda, \mu} M_{\lambda\mu} \, m_\lambda(x) \, m_\mu(y)$$

$$= \sum_{\lambda} m_\lambda(x) \, e_\lambda(y)$$

proof

The second equality follows directly since,

$$e_\lambda(y) = \sum_\mu M_{\lambda\mu} \, m_\mu(y)$$

Suppose we want the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots y_1^{\beta_1} y_2^{\beta_2} \cdots$ from the LHS ie. $\prod_{i,j} (1+x_i y_j)$. This can be done by choosing a 0-1 matrix A with finitely many 1's satisfying

$$\prod_{i,j} (x_i y_j)^{a_{ij}} = x^\alpha y^\beta = x_1^{\alpha_1} x_2^{\alpha_2} \cdots y_1^{\beta_1} y_2^{\beta_2} \cdots$$

(Note that choice of matrix is not unique since if we take $x_1 x_2 y_1 y_2$, we can associate it to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ both)

Now $\prod_{i,j} (x_i y_j)^{a_{ij}} = x^{R_c} y^{C_c}$

$\therefore$ coefficient of $x^\alpha y^\beta$ in $\prod_{i,j} (1 + x_i y_j)$ is no. of 1-0 matrices having $R_c = \alpha$, $C_c = \beta$ which is exactly $M_{\alpha\beta} = M_{\beta\alpha}$

$$\therefore \prod_{i,j} (1 + x_i y_j) = \sum_{\lambda, \mu} M_{\lambda\mu} \, m_\lambda(x) \, m_\mu(y)$$

(Note that $\alpha \vDash n$ so $M_{\alpha\beta}$ doesnt make sense. We need to 'correct' α to $\tilde{\alpha}$ first & then say $M_{\tilde{\alpha}\beta}$ but we abuse notation)

Theorem 6 (Fundamental theorem of symmetric functions)

Let $\lambda, \mu \vdash n$. Then $M_{\lambda\mu} = 0$ unless $\mu \leq \lambda'$ and $M_{\lambda\lambda'} = 1$

Hence, $\{e_\lambda : \lambda \vdash n\}$ is a basis of Λ^n ($\leq$ refers to $\leq_2$)

Proof

Suppose $M_{\lambda\mu} \neq 0$. Then $\exists$ 0-1 matrix A with $R_c = \lambda$, $C_c = \mu$. Construct A' from A by left justifying all 1's i.e. $R_c = 1$ & $A'_{ij} = 1$ for $1 \leq j \leq \lambda_i$.

For any i, no. of 1's in first i columns of $A' \geq$ no of 1's in first i columns of A

$\therefore C_c(A') \geq C_c(A) = \mu$ (dominance order)

$\therefore \lambda' \geq \mu$

($\because$ The 1's form a ferrer shape of λ & hence $C_c(A') = \lambda'$)

Now A' is the only matrix having $R_c = \lambda$, $C_c = \lambda'$ (by uniqueness of Ferrer's shape & its transpose)

$\therefore M_{\lambda\lambda'} = M_{\lambda'\lambda} = 1$

Now, from this seemingly unrelated first part of the theorem, we prove the main stuff: $\{e_\lambda\}$ is a basis!

Let $\lambda^1, \lambda^2, \ldots, \lambda^k$ be an ordering of P_n compatible with $\leq$ (where k = no. of partitions of n) (dominance order)

(Note that $(\lambda^k)', \ldots, (\lambda^1)'$ is also compatible with $\leq$)

Then construct a matrix $A_{k\times k}$ whose rows are indexed by $\lambda^1, \ldots, \lambda^k$ & columns indexed by $(\lambda^k)', \ldots (\lambda^1)'$

and the $(\lambda^i, (\lambda^j)')^{th}$ entry is $M_{\lambda^i, (\lambda^j)'}$

Firstly, the diagonal is all 1's.

Next, entries above it are all 0 since $M_{\lambda^i, (\lambda^j)'}$ is non zero only if $\lambda^i \leq ((\lambda^j)')' = \lambda^j$ which will happen iff $i \leq j$

Thus, we have an invertible lower triangular matrix (with determinant 1)

Since $e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu$ & the matrix of $M_{\lambda\mu}$ is invertible, m_μ can be expressed in terms of e_λ & since $\{m_\mu\}$ is a basis, it follows that $\{e_\lambda\}$ is a basis

Definition: We define complete symmetric functions h_λ $(\lambda \in P)$ as follows: $h_n = \sum_{\lambda \vdash n} m_\lambda$, $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$

Observe: $\sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$ (note that n also varies)

and h_n is hence the sum of monomials of degree n

Proposition 7

Let $h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu$ i.e. $N_{\lambda\alpha}$ be the coefficient of x^α in h_λ. Then $N_{\lambda\alpha}$ = no. of matrices over $\mathbb{N}$ with $R_c = \lambda$ and $C_c = \alpha$

Proof

Same as proposition 4.

Choosing x^α from $h_{\lambda_1} h_{\lambda_2} \cdots$ is same as choosing $x_1^{a_{i_1}} x_2^{a_{i_2}} \cdots$ from each h_{λ_i} such that $\prod_i (x_1^{a_{i_1}} \cdots) = x^\alpha$

Thus $C_c = ((a_{1_1} + a_{2_1} + \cdots), (a_{1_2} + a_{2_2} + \cdots), \cdots) = \alpha$

$R_c = ((a_{11} + a_{12} + \cdots), (a_{2_1} + a_{2_2} + \cdots), \cdots) = \lambda$

Corollary

$N_{\mu\lambda} = N_{\lambda\mu}$ (content as above)

Remark: A similar combinatorial interpretation as with $M_{\mu\lambda}$ exists. We just relax the condition of "at most one ball of a given colour in a given box"

Q) Justify the duality of $N_{\mu\lambda}$, $M_{\mu\lambda}$ ie. e_λ, h_λ

Ans Consider the partition $1^n \vdash n$ ie. $(\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, 0, \ldots)$

$$e_k(1^n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} 1$$

$$= \binom{n}{k}$$

$$h_k(1^n) = \sum_{1 \le i_1 \le i_2 < \cdots \le i_k \le n} 1$$

$$= \binom{n+k-1}{k}$$

Just as a remark,

$$\left(\!\!\binom{n}{k}\!\!\right) = \text{no. of } k \text{ multisets of } [n]$$

$$= \binom{n+k-1}{k} \qquad \left(\because\ a_1\ 1's,\ a_2\ 2's, \ldots, a_n\ n's \text{ and } a_1 + a_2 + \cdots + a_n = k \right)$$

$$= (-1)^k \cdot \binom{-n}{k} := (-1)^k \cdot \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$$

■

Proposition 8

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda, \mu} N_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y)$$

$$= \sum_{\lambda} m_\lambda(x)\, h_\lambda(y)$$

proof

The second equality follows from $\sum_{\mu} N_{\lambda\mu} m_\mu(y) = h_\lambda(y)$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \prod_{i,j} \left(1 + x_i y_j + (x_i y_j)^2 + \cdots\right)$$

Suppose we want the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots\ y_1^{\beta_1} y_2^{\beta_2} \cdots$,

this can be done by choosing a matrix A with entries from $\mathbb{N}$ such that

$$\prod_{(i,j)} (x_i y_j)^{a_{ij}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots\ y_1^{\beta_1} y_2^{\beta_2} \cdots$$

ie., associate to $x_1^{\alpha_1} \cdots y_1^{\beta_1} \cdots$, all such matrices A

But $\prod_{(i,j)} (x_i y_j)^{a_{ij}} = x^{R_c}\, y^{C_c}$

$\therefore$ We want all those matrices A s.t.

$$x^{\alpha} y^{\beta} = x^{R_c} y^{C_c} \quad \text{i.e.} \quad R_c = \alpha, \; C_c = \beta$$

Number of such matrices is just $N_{\alpha\beta}$

$\therefore$ The result follows

Theorem 9

$\{ h_\lambda \mid \lambda \vdash n \}$ is a basis for Λ^n

proof

We know that $\{ e_\lambda \mid \lambda \vdash n \}$ is a basis and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$. Thus, an algebra endomorphism $f : \Lambda \to \Lambda$ is completely determined by $f(e_1), f(e_2), \ldots$ and converse is also true.

Consider alg. endo $\omega : \Lambda \to \Lambda$ as $\omega(e_n) = h_n$. If we show ω is invertible we are done. (ω preserves mult)

We claim $\omega^{-1} = \omega$ i.e. $\omega^2 = 1$

Consider
$$f(t) = \sum_{n=0}^{\infty} h_n t^n$$
$$g(t) = \sum_{n=0}^{\infty} e_n t^n$$

Now we claim that $\prod_{n=1}^{\infty} (1 + x_n t) = g(t)$

Well, coefficient of t^k in LHS = sum over all x taken k at a time $= m_{1^k}(x) = e_k$

Similarly, one can check that $\prod_{n=1}^{\infty} (1 - x_n t)^{-1} = f(t)$

$\therefore \quad f(t)\, g(-t) = 1$

$$\therefore \quad \sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0 \quad \forall\, n = 1, 2, \cdots$$

Conversely, if $\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0 \quad \forall\, n = 1, 2, \dots$

with $u_0 = 1$, we have $u_n = e_n$ (prove this fact by induction on n)

Now $\omega\left(\sum_{i=0}^{n} (-1)^i e_i h_{n-i}\right) = \omega(0) = 0 \quad \forall\, n \geq 1$

$$\therefore \quad \sum_{i=0}^{n} (-1)^i h_i\, \omega(h_{n-i}) = 0 \qquad \forall\, n \geq 1$$

$$\therefore \quad \sum_{i=0}^{n} (-1)^{n-i} h_{n-i}\, \omega(h_i) = 0 \qquad \forall\, n \geq 1$$

$$\therefore \quad \sum_{i=0}^{n} (-1)^i \omega(h_i)\, h_{n-i} = 0 \qquad \forall\, n \geq 1$$

$$\therefore \quad \omega(h_i) = e_i$$

$\blacksquare$

Definition: We define power sum symmetric functions as follows

$$p_n = m_n = \sum_{i=1}^{\infty} x_i^n \quad \text{and} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

Proposition 10

Let $\lambda = (\lambda_1, \dots) \vdash n$ with $l = \ell(\lambda)$ (i.e. $\lambda = (\lambda_1, \dots, \lambda_l)$)

Suppose $p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu$. Let $k = \ell(\mu)$. Then,

$R_{\lambda\mu}$ = no. of ordered partitions $(B_1, \dots, B_k)$ of $[l]$ such

that $\mu_j = \sum_{i \in B_j} \lambda_i \quad \forall\ 1 \leq j \leq k$

proof

$R_{\lambda\mu}$ is the coefficient of x^μ in $(\sum x_i^{\lambda_1})(\sum x_i^{\lambda_2})\cdots$

Consider $(x_1^{\lambda_1} + x_2^{\lambda_1} + \cdots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \cdots)\cdots$. We want the coefficient of $x_1^{\mu_1} x_2^{\mu_2}\cdots$. Choose $x_{i_j}^{\lambda_j}$ from each $\sum x_i^{\lambda_j}$. $\prod_{j=1}^{\infty} x_{i_j}^{\lambda_j}$ should be $x_1^{\mu_1} x_2^{\mu_2}\cdots$

ie. $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} x_{i_3}^{\lambda_3} \cdots x_{i_\ell}^{\lambda_\ell} = x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$

Let $B_r = \{ j \mid i_j = r \}$ $\quad (r = 1, 2, \ldots, k)$

Clearly $(B_1, \ldots B_k)$ is an ordered partition of $[\ell]$.

Further $\forall\, j \in B_r$, $\sum_{j \in B_r} \lambda_j$ must be μ_r

Conversely, every such partition will give x^μ

Theorem 11

$R_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ and $R_{\lambda\lambda} = \prod_i (m_i(\lambda)!)$ where

$$\lambda = (\underbrace{1, 1, \ldots, 1}_{m_1 \text{ times}}, \underbrace{2, 2, \ldots, 2}_{m_2 \text{ times}}, \ldots)$$

Thus, $\{ p_\lambda \mid \lambda \vdash n \}$ is a basis for Λ^n

proof

Suppose $R_{\lambda\mu} \neq 0$. Obtain some block partition $(B_1, \ldots, B_k)$ as in previous proposition

Given some $r \in \{1, 2, \ldots, \ell\}$, let $B_{i_1}, B_{i_2}, \ldots, B_{i_s}$ be the blocks from $\{B_1, \ldots, B_k\}$ such that they contain at least one of $1, 2, \ldots, r$

(Pick 1. It is in B_{i_1}. Pick 2. Maybe in B_{i_1} or B_{i_2}

and so on)

$$\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_s} \geq \lambda_1 + \cdots + \lambda_r$$

($\because$ condition of $(B_1, \ldots, B_k)$)

$\therefore \quad \mu_1 + \cdots + \mu_r \geq \mu_{i_1} + \cdots + \mu_{i_s} \geq \lambda_1 + \cdots + \lambda_r$

$\therefore \quad \mu \geq \lambda$ ($\geq$ in sense of $\geq_2$)

Now suppose $\mu = \lambda$

Then we want $(B_1, \ldots, B_\ell)$ to be a partition of $[\ell]$.

Thus each block is singleton.

Reverse the order of the partition and write $\lambda = (1, 1, \ldots, 1, 2, 2, \ldots)$

Then, $B_1, \ldots, B_{m_1}$ can be interchanged without any problem since $\lambda_t = \sum_{\alpha \in B_t} \lambda_\alpha = \lambda_1 \quad \forall\ t = 1, 2, \ldots, m_1$

(assumed that $\lambda_1, \ldots, \lambda_{m_1} = 1$)

$\therefore$ we have $m_1!$ ways

Similarly others. Thus total number of ways to write the singletons is $m_1!\, m_2! \cdots$

Choose to index the matrix A by $\lambda_1, \ldots, \lambda^k$ (both the rows & the columns) (k = no. of partitions of n)

We have again a lower triangular matrix with none of the diagonal entries zero. Thus it is invertible.

Hence if $p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu$, m_μ can also be expressed in terms of p_λ via A^{-1}. Thus $\{p_\lambda \mid \lambda \vdash n\}$ is a basis

Definition: Define for $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$, $z_\lambda = \prod_i i^{m_i} m_i!$

To simplify this, we introduce two new notations:

$\|\lambda\| = m_1!\, m_2! \cdots$ ($R_{\lambda\lambda} = \|\lambda\|$) and $\lambda^* = \prod \lambda_i$

examples: Let $\lambda = (5,5,4,3,3,2,2,2,2,2,2,2,2,1,0,0,\dots)$

Then $\|\lambda\| = 2!\; 1!\; 2!\; 8!\; 1!$

$\lambda^* = 5 \times 5 \times 4 \times 3 \times 3 \times 2^8 \times 1$

$z_\lambda = \lambda^* \|\lambda\|$

Notation: If $\sigma \in S_n$, by type(σ), we mean the partition $(\lambda_1, \dots, \lambda_k) \vdash n$ where k = no. of cycles (writing σ in the canonical cycle form involving all of $1, 2, \dots, n$ exactly once) where λ_i = length of the i^{th} cycle (writing largest cycle first)

Q) Find number of permutations $\sigma \in S_n$ of type(σ) $= (1^{m_1}, 2^{m_2}, \dots n^{m_n}) = \lambda \vdash n$

Ans: Consider any permutation $t_1 t_2 \cdots t_n$ of $1, 2, \dots n$

Paranthesize the first m_1 elements as 1-cycles (i.e. $(t_1)(t_2)\cdots(t_{m_1})$). Paranthesize the next $2m_2$ elements as 2-cycles (i.e. $(t_{m_1+1}\, t_{m_1+2})\cdots$) we get a permutation σ of the desired cycle type.

We claim that there are multiple such $t_1 \cdots t_n$ giving rise to the same σ (in particular z_λ of them)

Well, we can interchange all m_i i-length cycles among themselves & nothing will change.

Thus we divide by $\prod_i (m_i!) = \|\lambda\|$.

Further, we can choose the first element of each i-cycle in i ways. Thus, we divide by $\prod_i i^{m_i} = \lambda^x$

$\therefore$ Required answer is $\dfrac{n!}{\|\lambda\| \lambda^x} = \dfrac{n!}{z_\lambda}$

Q) Verify that for $\sigma \in S_n$, $\operatorname{sgn}(\sigma) = (-1)^{m_2 + m_4 + m_6 + \cdots}$

where $\operatorname{type}(\sigma) = (1^{m_1}, 2^{m_2}, \ldots)$

Ans σ has m_i number of i-cycles

Each i cycle can be written as a product of $i-1$ transpositions

$\therefore$ we have in total, $\sum_{i=1}^{n} m_i (i-1)$ number of transpositions

i.e. $m_2 + 3m_4 + \cdots + m_n (n-1)$ which has the same parity as $m_2 + m_4 + \cdots$

Definition : For $\lambda = (1^{m_1}, \ldots)$, define $\varepsilon_\lambda = (-1)^{m_2 + m_4 + \cdots}$

Proposition 12

(i) $$\prod_{i,j} (1 - x_i y_j)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x) p_n(y)\right)$$

(ii) $$\prod_{i,j} (1 + x_i y_j) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n-1} p_n(x) p_n(y)\right)$$

proof

(i) $\log(\text{LHS}) = \sum_{i,j} \log\left[(1 - x_i y_j)^{-1}\right]$

$$= \sum_{i,j} \sum_{n=1}^{\infty} \frac{1}{n} x_i^n y_j^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} p_n(x) p_n(y)$$

(ii) $\log(\text{LHS}) = \sum_{i,j} \log\left[(1 + x_i y_j)\right]$

$$= \sum_{i,j} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x_i^n y_j^n$$

$$= \sum_{n=1}^{\infty} \sum_{i,j} \frac{(-1)^{n-1}}{n} x_i^n y_j^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y)$$

$\blacksquare$

We make use some results of exponential generating functions.

Result 1 : Let $f, g : \mathbb{N} \to \mathbb{C}$. Define $h : \mathbb{N} \to \mathbb{C}$ as

$$h(\# X) = \sum_{(S,T)} f(\# S)\, g(\# T) \quad \text{where} \quad S \sqcup T = X$$

and X is some finite set. Then $E_h = E_f \cdot E_g$

Result 2 : Let $f : \mathbb{N} \setminus \{0\} \to \mathbb{C}$, $g : \mathbb{N} \to \mathbb{C}$ with $g(0) = 1$. Define $h : \mathbb{N} \to \mathbb{C}$

as $h(\# S) = \sum_{(B_1, \dots, B_k) \vDash S} f(\# B_1) \cdots f(\# B_k)\, g(k)$, $h(0) = 1$.

Then $E_h = E_g \circ E_f$

Result 3 : given $f : \mathbb{P} \to \mathbb{C}$ define $h : \mathbb{N} \to \mathbb{C}$ as follows:

$$h(\# S) = \sum_{(B_1, \dots, B_k) \vDash S} f(\# B_1) \cdots f(\# B_k), \quad h(0) = 1.$$

Then $E_h = \exp(E_f)$

Result 4 : Let $f : \mathbb{P} \to \mathbb{C}$, $h : \mathbb{N} \to \mathbb{C}$ st. $E_h = \exp(E_f)$

Then, $h(n+1) = \sum_{k=0}^{n} \binom{n}{k} h(k) f(n+1-k)$

Result 5 : Let $f : \mathbb{P} \to \mathbb{C}$, $g : \mathbb{N} \to \mathbb{C}$ with $g(0) = 1$. Define h as

$$h(\# X) = \sum_{\sigma \in S_X} f(\# c_1) f(\# c_2) \cdots f(\# c_k)\, g(k), \quad h(0) = 1$$

where σ has cycle decomposition $c_1 c_2 \cdots c_k$. Then,

$$E_h = E_g\left(\sum_{n=1}^{\infty} f_n \frac{x^n}{n} \right)$$

Result 6 : Let $f : \mathbb{P} \to K$. Define $h(\# X) = \sum_{\sigma \in S_X} f(\# c_1) \cdots f(\# c_k)$

and $h(0) = 1$. Then $E_h = \exp\left(\sum_{n=1}^{\infty} f_n \frac{x^n}{n} \right)$

Q) Can you "extend" proposition 12 now ?

Ans (i) $\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, p_n(y) \right) = ?$

well, in result 6, choose $f_n = p_n(x)\, p_n(y)$, $x = 1$

(there are two different x, mind you !)

$$\therefore \quad h(n) = \sum_{\sigma \in S_n} p_{\text{type}(\sigma)}(x)\, p_{(\text{type}\,\sigma)}(y)$$

$$= \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda(x)\, p_\lambda(y)$$

$$\therefore \quad \frac{h(n)}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, p_n(y) \right) =$$

$$= \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda(x)\, p_\lambda(y)$$

(ii) $\exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x)\, p_n(y) \right) = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda(x)\, p_\lambda(y)$

is similar

Proposition 13

With ω as in theorem 9, ε_λ is an e.val of ω with the corresponding evec p_λ

proof

Well, $\omega\left(\sum_{\lambda \vdash n} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}\right) = \omega\left(\prod_{i,j} (1 - x_i y_j)^{-1}\right)$

$$= \omega\left(\sum_{\lambda \vdash n} m_\lambda(x)\, h_\lambda(y)\right)$$

$$= \sum_{\lambda \vdash n} m_\lambda(x) \quad \omega(h_\lambda(y))$$

($\because$ ω is treated as operator on functions in y)

$$= \sum_{\lambda \vdash n} m_\lambda(x)\, e_\lambda(y)$$

$$= \prod_{ij} (1 + x_i y_j)$$

$$= \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda(x)\, p_\lambda(y)$$

$$\therefore \omega(p_\lambda(y)) = \varepsilon_\lambda \, p_\lambda(y)$$

■

Corollary:

In particular $\omega(p_n(x)) = (-1)^{n-1} p_n(x) = -p_n(-x)$

Q) Restrict ω to Λ^n. Find the characteristic polynomial of $\omega : \Lambda^n \to \Lambda^n$

(No need to justify that $\omega(\Lambda^n) = \Lambda^n$)

Ans $\quad \omega p_\lambda = \varepsilon_\lambda p_\lambda$

$\therefore \{p_\lambda \mid \lambda \vdash n\}$ is a set of eigen vectors of ω.

Since it is LI and has size $k = \dim \Lambda^n$,

we have an eigenbasis of Λ^n.

Thus the exhaustive list of evals is $\{\varepsilon_\lambda \mid \lambda \vdash n\}$

Thus, the characteristic polynomial is $(x-1)^{e(n)}(x+1)^{o(n)}$

where $e(n)$ = no of partitions $\lambda \vdash n$ with an even number of even parts

$o(n)$ = no of partitions $\lambda \vdash n$ with an odd number of even parts

■

Q) Summarise m, e, h, p. Write h, e in terms of p

Ans We summarise :

(i) $m_\lambda(x) = \sum_\sigma x^\sigma$ (σ permutation of λ)

(ii) $e_n(x) = m_{1^n}(x) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$

(summation over x_i n at a time)

$e_\lambda(x) = e_{\lambda_1}(x)\, e_{\lambda_2}(x) \cdots$

(iii) $h_n(x) = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$

$h_\lambda(x) = h_{\lambda_1}(x)\, h_{\lambda_2}(x) \cdots$

(iv) $p_n(x) = \sum_{i=1}^{\infty} x_i^n$, $p_\lambda(x) = p_{\lambda_1}(x)\, p_{\lambda_2}(x) \cdots$

Then,

$$e_\lambda(x) = \sum_{\mu \vdash n} M_{\lambda\mu}\, m_\mu(x) \qquad \text{— } ①$$

$$h_\lambda(x) = \sum_{\mu \vdash n} N_{\lambda\mu}\, m_\mu(x) \qquad \text{— } ②$$

$$p_\lambda(x) = \sum_{\mu \vdash n} R_{\lambda\mu}\, m_\mu(x) \qquad \text{— } ③$$

$$\sum_{\lambda \vdash n} m_\lambda(x)\, e_\lambda(y) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda}\, p_\lambda(x)\, p_\lambda(y) \qquad \text{— } ④$$

$$\sum_{\lambda \vdash n} m_\lambda(x)\, h_\lambda(y) = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda}\, p_\lambda(x)\, p_\lambda(y) \qquad \text{— } ⑤$$

Substituting $y = (t, 0, 0, \dots)$ in ④,

$$e_n(y) = \sum ((y_i))_n = \begin{cases} 0 & n \geq 2 \\ t & n = 1 \\ 1 & n = 0 \end{cases}$$

$$\therefore e_\lambda(y) = \begin{cases} t^n & \lambda = 1^n \\ 0 & \text{otherwise} \end{cases}$$

$$p_n(y) = t^n \Rightarrow p_\lambda(y) = t^{\lambda_1} t^{\lambda_2} \cdots = t^n$$

$$\therefore m_{1^n}(x)\, \cancel{t^n} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda}\, p_\lambda(x)\, \cancel{t^n}$$

$$\therefore e_n(x) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda}\, p_\lambda(x)$$

Similarly, $h_n(x) = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda}\, p_\lambda(x)$

Definition : Let R be a comm $\mathbb{Q}$-algebra with identity. A specialisation of Λ is a hom $\varphi : \Lambda \to R$ with $\varphi(\mathrm{id}) = \mathrm{id}$

A quick example is that of the substitution specialization ie. assign some values to $x_1, x_2, \cdots$ from R

Reducing the number of variables

$r_n : \Lambda \to \Lambda_n$ as $r_n(f) = f(x_1, x_2, \cdots, x_n, 0, 0, 0, \cdots)$

[Here Λ_n is set of all poly $f \in \mathbb{Q}[x_1, \cdots, x_n]$ in variables $x_1, \cdots, x_n$, with rational coeffs s.t. $f(x) = f(\sigma(x))$]

Proposition 14

Let $\mathrm{Par}_n = \{ \lambda \in \mathbb{P} \mid \ell(\lambda) \leq n \}$

(i) $\{ r_n(m_\lambda) \mid \lambda \in \mathrm{Par}_n \}$, $\{ r_n(e_\lambda) \mid \lambda' \in \mathrm{Par}_n \}$, $\{ r_n(h_\lambda) \mid \lambda' \in \mathrm{Par}_n \}$, $\{ r_n(p_\lambda) \mid \lambda' \in \mathrm{Par}_n \}$ are all $\mathbb{Q}$-bases of Λ_n

(ii) Define $\omega_n : \Lambda_n \to \Lambda_n$ as $\omega_n(e_\lambda) = h_\lambda$ (identify e_λ, h_λ in Λ_n as $r_n(e_\lambda), r_n(h_\lambda)$). $(\lambda' \in \mathrm{Par}_n)$

Then ω_n is an algebra automorphism, an involution and $\omega_n(p_\lambda) = \varepsilon_\lambda p_\lambda$ for $\lambda' \in \mathrm{Par}_n$

proof

Skipped! It is essentially the same stuff as before (Just that there is finiteness in most places (for ex: matrices which are in correspondence with $M_{\lambda\mu}$, $N_{\lambda\mu}$) $\blacksquare$

Proposition 15

If $\lambda \notin \mathrm{Par}_n$, $\quad \pi_n(m_\lambda) = \pi_n(e_{\lambda'}) = 0$

proof

$\lambda \notin \mathrm{Par}_n \Rightarrow \ell(\lambda) > n$

$\therefore$ Every monomial of $m_\lambda(x)$ involves at least $n+1$ terms

$x_{i_1} x_{i_2} \cdots x_{i_{n+1}} \cdots$

On substituting $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$, everything vanishes

Coming to $e_{\lambda'}$, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ $\quad (k > n)$

Then $\lambda' = (k, u_2, u_3, \ldots, u_{\lambda_1})$

Then $\pi_n(e_{\lambda'}(x)) = \pi_n(e_k(x)) \prod_{j=2}^{\lambda_1} \pi_n(e_j(x))$

But $\pi_n(e_k(x)) = \pi_n(m_{1^k}(x)) = 0$ since $\ell(1^k) > n$

$\blacksquare$

Principal specializations

$ps_n : \Lambda \to \mathbb{Q}[q]$

$f(x_1, \ldots) \longmapsto f(1, q, q^2, \ldots, q^{n-1}, 0, 0, 0, \ldots)$

$ps : \Lambda \to \mathbb{Q}[q]$ (called stable principal stabilization)

$f \longmapsto \lim_{n \to \infty} ps_n(f)$

$ps_n^1 : \Lambda \to \mathbb{Q}$

$f \longmapsto f(\underbrace{1, 1, 1, \ldots, 1}_{n \text{ 1's}}, 0, 0, 0, \ldots)$

Q) Describe the action of these 3 specializations on the basis vectors m_λ, e_λ, h_λ, p_λ

(Provide only direct results)

Ans

	ps_n	ps	ps^1_n
m_λ	I know the answer but this box is too small to contain it ;)	I know the answer but this box is too small to contain it ;)	$\binom{n}{\ell(\lambda)}\binom{\ell(\lambda)}{m_1, m_2, \dots}$
e_λ	$\prod_{i=1}^{\infty} q^{\binom{\lambda_i}{2}}\binom{n}{\lambda_i}$	$\prod_{i=1}^{\infty} \frac{q^{\binom{\lambda_i}{2}}}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_{i=1}^{\infty}\binom{n}{\lambda_i}$
h_λ	$\prod_{i=1}^{\infty}\binom{n+\lambda_i-1}{\lambda_i}$	$\prod_{i=1}^{\infty}\frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_{i=1}^{\infty}\left(\!\binom{n}{\lambda_i}\!\right)$
p_λ	$\prod_{i=1}^{\ell}\frac{1-q^{n\lambda_i}}{1-q^{\lambda_i}}$	$\prod_{i=1}^{\ell}\frac{1}{1-q^{\lambda_i}}$	$n^{\ell(\lambda)}$

Exponential Specialization

$$ex : \Lambda \longrightarrow \mathbb{Q}[t]$$

$$ex(p_n) = t\,\delta_{1n} = \begin{cases} t & n = 1 \\ 0 & n \neq 1 \end{cases}$$

It can be checked easily that this is a $\mathbb{Q}$-algebra morphism

Proposition 16

$$ex(f) = \sum_{n \geq 0} [x_1 \cdots x_n]_f \frac{t^n}{n!} \qquad \left([x^\alpha]_f = \text{coeff of } x^\alpha \text{ in } f \right)$$

Proof

Well, $ex(p_\lambda) = ex(p_{\lambda_1})\, ex(p_{\lambda_2}) \cdots$

$$= \begin{cases} t^n & \lambda = 1^n \\ 0 & \text{otherwise} \end{cases} = \sum_{n \geq 0} [x_1 \cdots x_n]_{p_\lambda} \frac{t^n}{n!}$$

Since $\sum_{n \geq 0} [x_1 \cdots x_n]_f \frac{t^n}{n!}$ is linear, the result follows

Q) Find $ex(m_\lambda)$, $ex(h_\lambda)$, $ex(e_\lambda)$

Ans using the above proposition,

$$ex(m_\lambda) = \begin{cases} \frac{t^n}{n!} & \lambda = 1^n \\ 0 \end{cases}$$

$$ex(h_\lambda) = ex(e_\lambda) = \frac{t^{|\lambda|}}{\lambda_1!\,\lambda_2!\cdots}$$

Definition : Define a bilinear map $\langle\, , \,\rangle$ on Λ as follows.

$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ (clearly it is bilinear)

Proposition 17

$\langle f, g \rangle = \langle g, f \rangle$

proof

$$\langle h_\lambda, h_\mu \rangle = \left\langle \sum_\alpha N_{\lambda\alpha} m_\alpha , h_\mu \right\rangle$$

$$= N_{\lambda\mu}$$

$$= N_{\mu\lambda}$$

$$= \langle h_\mu, h_\lambda \rangle$$

Works for bases $\{h_\lambda\}$, $\{h_\mu\}$ (the same bases), and hence by linearity $\langle f, g \rangle = \langle g, f \rangle$

Proposition 18

Let $\{u_\lambda\}$, $\{v_\lambda\}$ be basis of Λ st. $\forall\, \lambda \vdash n$, $u_\lambda, v_\lambda \in \Lambda^n$.

The bases are dual to each other iff

$$\sum_\lambda u_\lambda(x)\, v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

Proof

Let $m_\lambda = \sum_\rho a_{\lambda\rho} u_\rho$, $h_\mu = \sum_\nu b_{\mu\nu} v_\nu$

$$\delta_{\lambda\mu} = \sum_\rho \sum_\nu a_{\lambda\rho} \cdot b_{\mu\nu} \langle u_\rho , v_\nu \rangle$$

Construct a matrix indexed by $\mathbb{P}$ (for each n)

(ordered in an obvious manner ($\mathbb{P}_1$ then $\mathbb{P}_2$ ...))

Then $I = A M B^T$

where A has λ, ρ th entry $a_{\lambda\rho}$

B has μ, ν th entry $b_{\mu\nu}$

M has ρ, ν th entry $\langle u_\rho, v_\nu \rangle$

$\therefore$ The bases are dual iff $M = I$

iff $I = AB^T$ (iff $I = A^T B$)

$I = A^T B$ iff $\delta_{\rho\nu} = \sum_\lambda a_{\lambda\rho} b_{\lambda\nu}$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(x) h_\lambda(y)$$

$$= \sum_\lambda \left(\sum_\rho a_{\lambda\rho} u_\rho(x) \right) \left(\sum_\nu b_{\lambda\nu} v_\nu(y) \right)$$

$$= \sum_{\rho,\nu} \left(\sum_\lambda a_{\lambda\rho} b_{\lambda\nu} \right) u_\rho(x) v_\nu(y)$$

$\therefore$ They are a dual bases iff $\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\rho,\nu} \delta_{\rho\nu} u_\rho(x) v_\nu(y)$

$$= \sum_\rho u_\rho(x) v_\rho(y)$$

$\blacksquare$

Proposition 19

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$$

proof

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

$\therefore \{p_\lambda\}, \left\{\frac{p_\mu}{z_\mu}\right\}$ are dual

$\therefore \left\langle p_\lambda, \frac{p_\mu}{z_\mu} \right\rangle = \delta_{\lambda\mu}$ and the result follows

$\blacksquare$

Q) Show that $\langle f, f \rangle > 0 \quad \forall f \in \Lambda \setminus \{0\}$ (which will show that $\langle\, ,\, \rangle$ is an inner product)

Ans Let $f = \sum_\lambda c_\lambda p_\lambda$

Then $\langle f, f \rangle = \sum_\lambda c_\lambda^2 z_\lambda \geqslant 0$

$(\because z_\lambda \geqslant 0)$

$\langle f, f \rangle = 0 \iff$ either all c_λ's zero (thus $f = 0$)

or $z_\lambda = 0$ (for those λ for which $c_\lambda \neq 0$)

Now if $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$, $z_\lambda = 0 \Rightarrow (1^{m_1} 2^{m_2} \ldots)(m_1! \, m_2! \ldots) = 0$

$\therefore m_1 = m_2 = \cdots = 0 \Rightarrow \lambda = (0, 0, 0, \cdots)$ ie. λ is empty.

This is not possible

$\blacksquare$

Some Questions from Stanley

7.2) Consider $\mathbb{P}_n$ with the dominance order $\le_2$

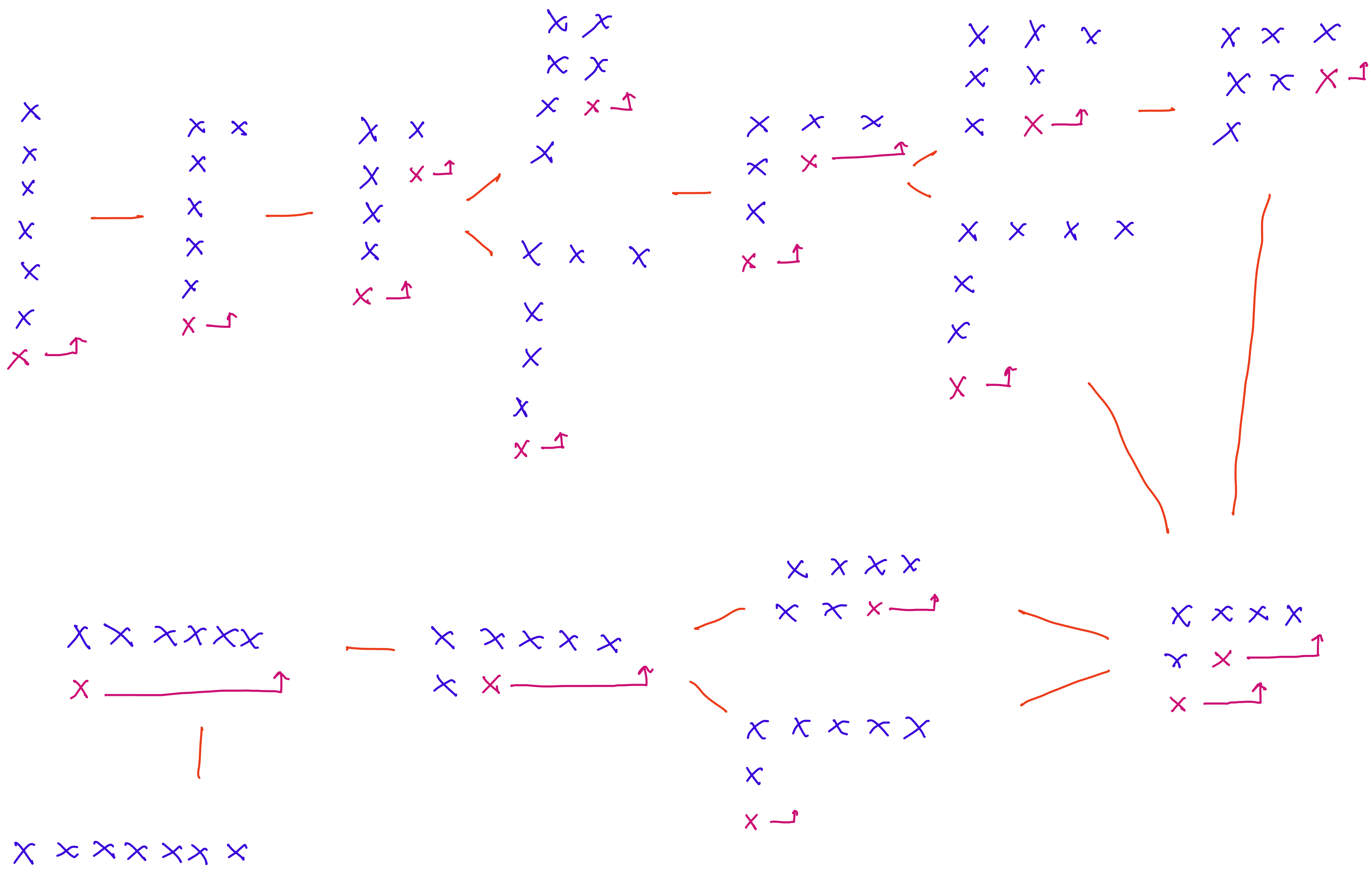
(i) Draw $\mathbb{P}_7$

(ii) Show that $\mathbb{P}_n$ is a lattice (existence of lub & glb for any given pair (λ, μ))

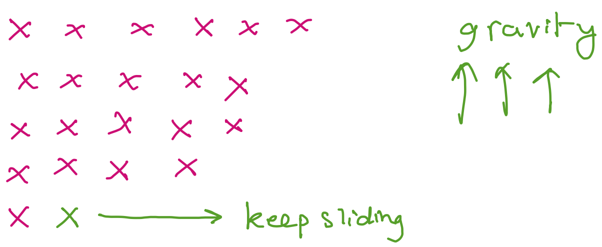
Ans (i) There is this really nice idea of stacking up to reach a bigger partition.

Of course, the smallest one is $(1, 1, 1, \ldots 1)$ (it has the lowest possible n^{th} partial sums)

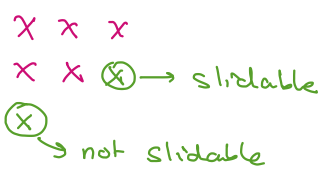
We "build up" from here



Rule:



exception:



(ii) The fact that it is a lattice readily follows now because $\times\times\times\times\times \cdots \times$ (n times) is an upper bound for both λ, μ & considering all diagrams between $\lambda, (n)$ & $\mu, (n)$; these are finitely many & hence we must have the lub

Similarly argue with $(1,1,\ldots,1)$ being the lower bound for any λ, μ pair

(easy to show $\lambda \leq (n)$ $\forall \lambda \in \mathbb{P}_n$ (follows by sliding construction))

7.3) Expand $\prod_{i \geq 1} (1 + x_i + x_i^2)$ in terms of e_λ

Ans $\prod_{i \geq 1} (1 + x_i + x_i^2)$ (reminds us of $1 + \omega + \omega^2$)

$$= \prod_{i \geq 1} (1 - \omega x_i)(1 - \omega^2 x_i)$$

Now in $\prod_{i \geq 1} (1 - \omega x_i)$ there are no higher order terms like $x_1^2, x_2^2, \ldots$ so on.

Coefficient of $x_{i_1} x_{i_2} \cdots x_{i_m}$ in $\prod_{i \geq 1} (1 - \omega x_i) = (-\omega)^m$

$$\therefore \prod_{i \geq 1} (1 - \omega x_i) = \sum_{m=0}^{\infty} (-\omega)^m e_m$$

IIly $$\prod_{i \geq 1} (1 - \omega^2 x_i) = \sum_{m=0}^{\infty} (-\omega^2)^m e_m$$

$$\therefore \prod_{i\geq 1}(1-x_i-x_i^2) = \left(\sum_{m=0}^{\infty}(-\omega)^m e_m\right)\left(\sum_{m=0}^{\infty}(-\omega^2)^m e_m\right)$$

$$= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}(-\omega)^a(-\omega^2)^b e_a e_b$$

$$= \sum_{a=b=0}^{\infty} e_a e_a + \sum_{a=0}^{\infty}\sum_{b=a+1}^{\infty}(-\omega^2)^a(-\omega)^b e_a e_b$$

$$+ \sum_{a=0}^{\infty}\sum_{b=0}^{a-1}(-\omega^2)^a(-\omega^2)^b e_a e_b$$

$$= \sum_{a=0}^{\infty} e_{(a,a)} + \sum_{\substack{a,b\\ a<b}}(-1)^{a+b}\left[\omega^{2a+b}+\omega^{a+2b}\right]e_{(b,a)}$$

$\blacksquare$

7.4) Prove that $h_r(x_1,\dots,x_n) = \sum_{k=1}^{n} x_k^{\,n-1+r}\prod_{i\neq k}\frac{1}{(x_k-x_i)}$

Ans we induct on n

$$h_r(x_1) = \sum_{1\leq i_1\leq\cdots\leq i_r\leq 1} x_{i_1}x_{i_2}\cdots x_{i_r} = x_1^r$$

$$\sum_{k=1}^{1} x_1^r\prod_{i\neq k}\frac{1}{(x_k-x_i)} = x_1^r$$

$$h_r(x_1,x_2) = \sum_{1\leq i_1\leq\cdots\leq i_r\leq 2} x_{i_1}x_{i_2}\cdots x_{i_r}$$

$$= \sum_{a_1+a_2=r} x_1^{a_1}x_2^{a_2}$$

$$= x_1^r + x_1^{r-1}x_2 + \cdots + x_1x_2^{r-1} + x_2^r$$

$$\sum_{k=1}^{2} x_k^{r+1} \prod_{i \neq k} \frac{1}{(x_k - x_i)} = \frac{x_1^{r+1}}{(x_1 - x_2)} + \frac{x_2^{r+1}}{x_2 - x_1}$$

$$= \frac{x_1^{r+1} - x_2^{r+1}}{x_1 - x_2}$$

$\therefore$ Base case $(n = 1, 2)$ are verified

Assume that the result holds for $n = 2, 3, \ldots, m-1$

$$h_r(x_1, \ldots, x_m) = \sum_{a_1 + \cdots + a_m = r} x_1^{a_1} \cdots x_m^{a_m}$$

$$= \sum_{t=0}^{r} \sum_{a_{m-1} + a_m = t} \sum_{a_1 + \cdots + a_{m-2} = r-t} x_1^{a_1} \cdots x_m^{a_m}$$

$$= \sum_{t=0}^{r} \sum_{a+b=t} h_{r-t}(x_1, \ldots, x_{m-2})\, x_{m-1}^{a}\, x_m^{b}$$

$$= \sum_{t=0}^{r} h_{r-t}(x_1, \ldots, x_{m-2}) \sum_{a+b=t} x_{m-1}^{a}\, x_m^{b}$$

$$= \sum_{t=0}^{r} h_{r-t}(x_1, \ldots, x_{m-2}) \frac{x_m^{t+1} - x_{m-1}^{t+1}}{x_m - x_{m-1}}$$

Now $\displaystyle\sum_{t=0}^{r} h_{r-t}(x_1, \ldots, x_{m-2}) \frac{x_m^{t+1}}{x_m - x_{m-1}} = \frac{1}{x_m - x_{m-1}} h_{r+1}(x_1, \ldots, x_{m-2}, x_m)$

$\therefore$ we have,

$$h_r(x_1, \ldots, x_m) = \frac{1}{(x_m - x_{m-1})} \left[h_{r+1}(x_1, \ldots, x_{m-2}, x_m) - h_{r+1}(x_1, \ldots, x_{m-1}) \right]$$

Invoking our induction hypothesis, the $[\cdots]$ becomes

$$\sum_{\substack{k=1\\ k\neq m-1}}^{m} x_k^{m-1+r} \prod_{\substack{i=1\\ i\neq m-1\\ i\neq k}}^{m} \frac{1}{(x_k - x_i)} - \sum_{k=1}^{m-1} x_k^{m-1+r} \prod_{\substack{i=1\\ i\neq k}}^{m-1} \frac{1}{(x_k - x_i)}$$

$$= \sum_{k=1}^{m-2} \left(\left(x_k^{m+r-1} \prod_{\substack{i=1\\ i\neq k}}^{m-2} \frac{1}{x_k - x_i} \right) \left(\frac{1}{x_k - x_m} - \frac{1}{x_k - x_{m-1}} \right) \right)$$

$$+ \frac{x_m^{m+r-1}}{(x_m - x_1)(x_m - x_2)\cdots(x_m - x_{m-2})} - \frac{x_{m-1}^{m+r-1}}{(x_{m-1} - 1)\cdots(x_{m-1} - x_{m-2})}$$

$$= \sum_{k=1}^{m-2} x_k^{m+r-1} \prod_{\substack{i=1\\ i\neq k}}^{m-2} \frac{1}{(x_k - x_i)} \times \frac{(x_m - x_{m-1})}{(x_k - x_m)(x_k - x_{m-1})}$$

$$+ \frac{x_m^{m+r-1}}{(x_m - x_1)(x_m - x_2)\cdots(x_m - x_{m-2})} - \frac{x_{m-1}^{m+r-1}}{(x_{m-1} - 1)\cdots(x_{m-1} - x_{m-2})}$$

Finally, with the $\dfrac{1}{x_m - x_{m-1}}$, we get

$$\sum_{k=1}^{m-2} x_k^{m+r-1} \prod_{\substack{i=1\\ i\neq k}}^{m} \frac{1}{(x_k - x_i)}$$

$$+ \frac{x_m^{m+r-1}}{(x_m - x_1)(x_m - x_2)\cdots(x_m - x_{m-2})(x_m - x_{m-1})} + \frac{x_{m-1}^{m+r-1}}{(x_{m-1} - 1)\cdots(x_{m-1} - x_{m-2})(x_{m-1} - x_m)}$$

Thus our claim is proved via induction !

7.5) Prove :

$$\left(1-\sum_{n=1}^{\infty} p_n t^n\right)^{-1} = \frac{\sum_{n=0}^{\infty} h_n t^n}{1-\sum_{n=1}^{\infty} (n-1) h_n t^n}$$

Ans After some algebraic simplifications, we conclude that we need to prove :

$$\sum_{n=1}^{\infty} p_n t^n = \frac{\sum_{n=0}^{\infty} n h_n t^n}{\sum_{n=0}^{\infty} h_n t^n}$$

What we know :

$$\begin{aligned}\prod_{i,j} (1-x_i y_j)^{-1} &= \sum_{\lambda \vdash n} m_\lambda(x)\, e_\lambda(y) \\ &= \sum_{\lambda \vdash n} \sum_{\mu \vdash n} M_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y) \quad (1) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, p_n(y)\right) \\ &= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda(x)\, p_\lambda(y)\end{aligned}$$

$$\begin{aligned}\prod_{i,j} (1+x_i y_j) &= \sum_{\lambda \vdash n} m_\lambda(x)\, h_\lambda(y) \\ &= \sum_{\lambda \vdash n} \sum_{\mu \vdash n} N_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y) \quad (2) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x)\, p_n(y)\right) \\ &= \sum_{\lambda} \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda(x)\, p_\lambda(y)\end{aligned}$$

In $\prod_{i,j} (1 - x_i y_j)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x) \, p_n(y) \right)$, substitute

$y = (t, 0, 0, \dots)$ to get

$$\prod_{i=1}^{\infty} (1 - x_i t)^{-1} = \exp\left(\sum_{n=1}^{\infty} p_n(x) \frac{t^n}{n} \right)$$

We have seen earlier that $\sum_{n=0}^{\infty} h_n t^n = \prod_{i=1}^{\infty} (1 - t x_i)^{-1}$

(Just compare coefficients of t^k on both sides)

$$\therefore \log\left(\sum_{n=0}^{\infty} h_n t^n \right) = \sum_{n=1}^{\infty} p_n(x) \frac{t^n}{n}$$

Differentiating wrt t :

$$\sum_{n=1}^{\infty} p_n(x) \, t^{n-1} = \frac{\sum_{n=0}^{\infty} n h_n t^n}{\sum_{n=0}^{\infty} h_n t^n}$$

Schur Polynomials

Definition : A semi standard young tableaux (SSYT) of shape λ is an array, having the same shape as the Ferrer's shape of λ, containing positive integers that weakly increase along columns & strictly increase along rows

Definition : We say an SSYT T has type $\text{type}(T) = \alpha$ if α_i parts of T are equal to i

Notation : For some SSYT T, $x^T := x^{\text{type}(T)} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$

example

$$T = \begin{array}{cccccc} 1 & 1 & 1 & 3 & 4 & 4 \\ 2 & 4 & 4 & 5 & 5 & \\ 5 & 5 & 7 & & & \\ 6 & 9 & 9 & & & \end{array}$$

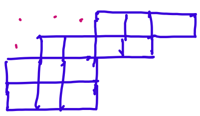
$$sh(T) = (6, 5, 3, 3)$$

$$\text{type}(T) = (3, 1, 1, 4, 4, 1, 1, 0, 2)$$

$$x^T = x_1^3 x_2 x_3 x_4^4 x_5^4 x_6 x_7 x_9^2$$

Definition : Given λ, μ st. $\mu_i \leq \lambda_i \ \forall i$, we define the SSYT of shape λ/μ to be the array indexed by $1 \leq i \leq \ell(\lambda)$, $\mu_i < j \leq \lambda_i$ that is weakly increasing in rows & strictly increasing in columns.

ex $(6, 5, 3, 3)/(3, 1)$ describes the shape



type (T), x^T are extended naturally from before

Definition : Given a skew shape λ/μ, the skew schur function $s_{\lambda/\mu}$ is defined as $s_{\lambda/\mu}(x) = \sum_{T:\, sh(T) = \lambda/\mu} x^T$

If $\mu = \phi = (0, 0, \cdots, 0)$, we call s_λ as the schur function indexed by λ

Theorem 20

$s_{\lambda/\mu}$ is a symmetric polynomial

proof

We show that swapping x_i, x_{i+1} doesnt do anything.

Suppose there are n boxes to be filled in λ/μ and $\alpha = (\alpha_1, \cdots)$ is a weak composition of n

Let $\tilde{\alpha} = \alpha$ with α_i, α_{i+1} swapped

Let $T_{\lambda/\mu, \alpha}$ be those SSTY's with shape λ/μ, type α.

If we establish a bijection between $T_{\lambda/\mu, \alpha}$ & $T_{\lambda/\mu, \tilde{\alpha}}$ we are done

Let $T \in T_{\lambda/\mu, \alpha}$. Consider those parts in T which are i or $i+1$ (they will be $\alpha_i + \alpha_{i+1}$ in number)

Three types of columns : (i) one copy of i & one copy of $i+1$
(ii) either only 1 copy of i OR 1 copy of $i+1$
(iii) neither i nor $i+1$

We ignore types i, iii and operate on ii

Now in each row, there will a no. of i's, b no of $i+1$'s change it to a no. of $i+1$'s & b no. of i's (remember to only operate on columns of type ii) and arrange in appropriate order in the $a+b$ spaces

The resulting array will have α_i no of $i+1$'s & α_{i+1} no. of i's and will hence belong to $T_{\lambda/\mu, \tilde{\alpha}}$

Clearly, our operation is a bijection (existence of an inverse)

∎

Definition : Given $\lambda \vdash n$, α weak composition of n, we define the Kostka number $K_{\lambda\alpha}$ to be the number of SSTY of type α, shape λ

Q) Find an expression for s_λ in terms of m_λ

Ans
$$s_\lambda = \sum_{T:\text{shape } \lambda} x^T$$

Corresponding to every given SSTY of type α, we have an x^T term

$$\therefore s_\lambda = \sum_{\alpha \vDash n} (\text{no of SSTY of type } \alpha, \text{shape } \lambda)\ x^\alpha$$

$$= \sum_{\alpha \vDash n} K_{\lambda\alpha}\, x^\alpha$$

$$= \sum_{\alpha \vdash n} \sum_{\sigma} K_{\lambda\,\sigma(\alpha)}\, x^{\sigma(\alpha)}$$

where σ runs over all distinct permutations of α

$$= \sum_{\alpha \vdash n} K_{\lambda\alpha}\, m_\alpha \qquad (\because K_{\lambda\,\sigma(\alpha)} = K_{\lambda\alpha})$$

∎

Exercise Define Skew Kostka numbers $K_{\lambda/\mu,\alpha}$ and prove that

$$s_{\lambda/\nu} = \sum_{\mu \vdash n} K_{\lambda/\nu,\mu} \; m_\mu$$

Definition : A standard Young Tableau (SYT) of shape λ is a λ shaped array ($\lambda \vdash n$) containing entries from the set $\{1,2,\ldots,n\}$ exactly once.

The careful reader must note that no. of SYT of shape λ is $K_{\lambda,1^n}$ ($\lambda \vdash n$)

This number is also denoted f^λ at times

Proposition 21

(i) Number of paths that can be taken to reach the shape of λ from $\emptyset$ in the Young's lattice is f^λ

(ii) Number of ways in which n voters can vote for the candidates $A_1, A_2, \ldots$ s.t. A_i receives λ_i votes & A_i is always ahead of A_{i+1} in number of votes is f^λ (partial-sequences of votes)

(iii) Number of lattice paths in $\mathbb{R}^{\ell(\lambda)}$ from $(0,\ldots,0)$ to $(\lambda_1,\ldots,\lambda_{\ell(\lambda)})$ which stay in the region $x_1 \geq x_2 \geq \cdots \geq x_\ell \geq 0$ is f^λ

proof

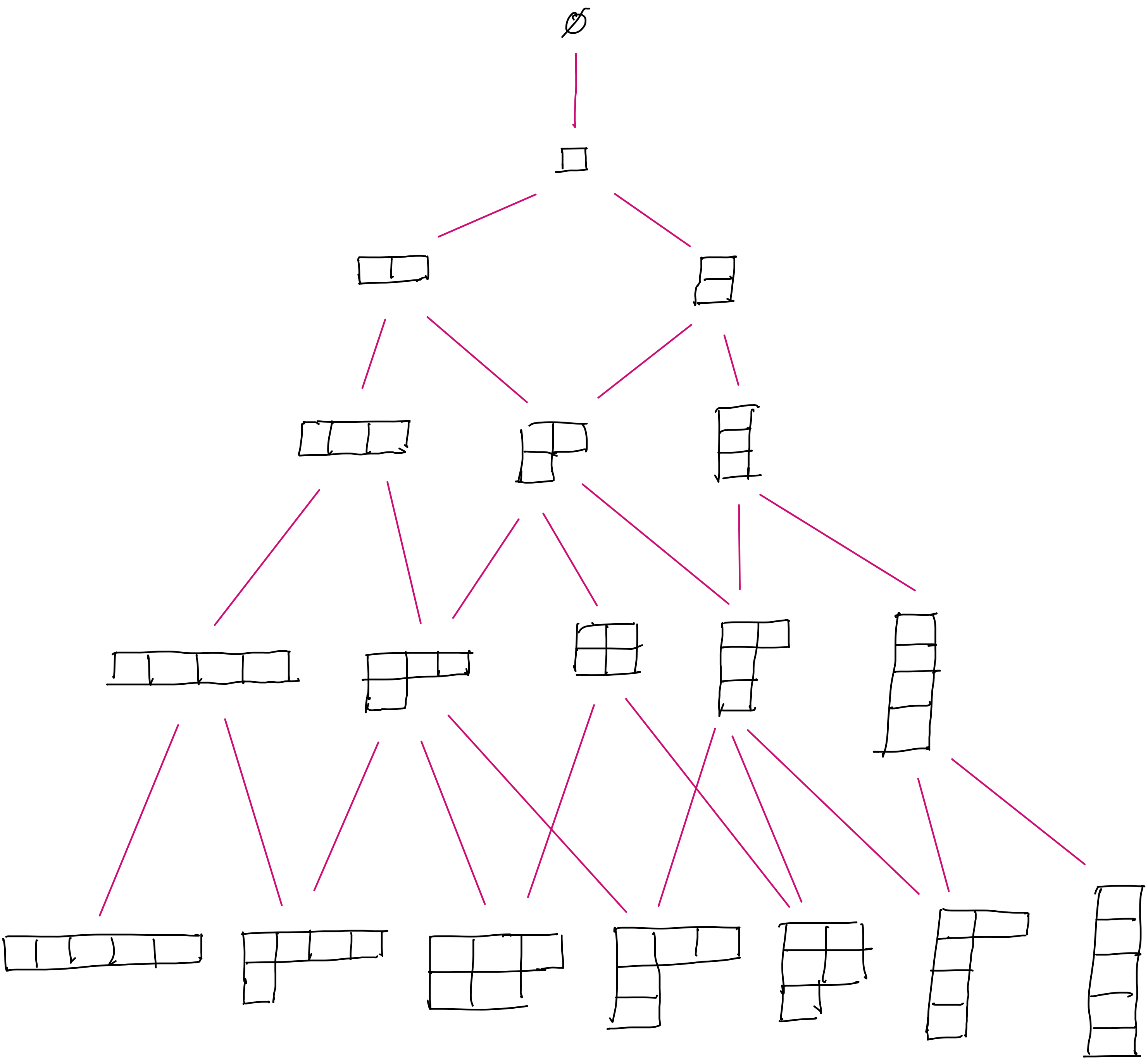
(i) Start with $\emptyset$. We can add a square to the Ferrer shape to move up. Assign the number i to the square added to λ^{i-1} in order to obtain λ^i (starting with $\boxed{1}$ corresponding to adding a box to $\emptyset$ to get (1))

Clearly, every row & column will be strictly increasing and we have used all numbers 1 through n to get to n boxes.

(convince yourself that adding box with label n to an SYT of size $n-1$ will still keep it as an SYT)

Conversely, given a SYT on $[n]$, delete boxes backwards to get a path

For convenience we draw the Young's lattice upto rank 5



(ii) If the k^{th} voter votes for A_i, put k in the i^{th} row of the shape λ

example Suppose $\lambda = (3,2)$ then 5 voters are present and a voting sequence could be 1 2 1 1 2 but not 2 2 1 1 1 (since after 1st vote is revealed, 2nd guy is leading albeit temporarily)

For 1 2 1 1 2 we construct

1	3	4
2	5	

Trivially, each row must be increasing by order of voters (1st voter, then 2nd, and so on)

Suppose in a column we have a discrepancy i.e. α is above β but $\alpha > \beta$. Let α be in row R_α, β in row R_β (note that R_α is above R_β).

When α votes are counted, the rows R_α, R_β will have the same no. of filled boxes which leads to a contradiction (since after $\alpha - 1$ votes, $\# R_\alpha < \# R_\beta$)

(convince yourself that R_β cannot have more boxes)

Conversely, reversing the process, it is clear that we can get a voting seq back

(iii) Suppose $a_1 a_2 \cdots a_n$ is a voting seq for $\lambda \vdash n$. Let $v_i - v_{i-1}$ be the a_i^{th} coordinate vector $= (0,0,\ldots,\underset{\uparrow}{1},0,0,\ldots 0)$

a_i^{th} position

Then we have a lattice path that stays within the region ($x_1 \geq \cdots \geq x_\ell \geq 0$ describes the condition that A_i th candidate never lags behind A_{i+1})

Conversely, using the reverse process, we discover a voting sequence.

Definition : A <u>Gelfand - Tsetlin pyramid</u> is a pyramid array

$$\begin{array}{ccccccc}
a_{11} & & a_{12} & & \cdots & & a_{1n} \\
& a_{22} & & a_{23} & \cdots & a_{2n} & \\
& & \ddots & & \ddots & & \\
& & & a_{nn} & & &
\end{array}$$

such that every row is weakly decreasing and every triangle triple satisfies : $a_{ij} \geq a_{i+1,j+1} \geq a_{i,j+1}$ ($\rightarrow$ decrease, $\searrow\nearrow$ decrease)

example

$$\begin{array}{ccccccccccc}
7 & & 5 & & 3 & & 3 & & 2 & & 1 \\
& 6 & & 5 & & 3 & & 2 & & 2 & \\
& & 5 & & 4 & & 3 & & 2 & & \\
& & & 4 & & 4 & & 2 & & & \\
& & & & 4 & & 3 & & & & \\
& & & & & 3 & & & & &
\end{array}$$

These structures were introduced to enable the study of representations of GL_n

Proposition 22

GT pyramids are in bijection with SSTY's

proof

We describe the forward construction. The reverse process will use a 'wild' example to illustrate, so that the reader may convince themselves or better still, prove it.

The first row of the GTP describes the shape of our SSTY. Consider the skew shape described by row i, $i+1$ of the GTP. Fill all boxes of this shape with the number $n-i+1$. (Fill n in R_1/R_2, $n-1$ in R_2/R_3, ..., 1 in R_{n-1}/R_n)

This results in an SSYT of shape R_1. The reasoning is provided below:

We know that the shapes of the rows satisfy the containment $R_n \subseteq R_{n-1} \subseteq \cdots \subseteq R_2 \subseteq R_1$ by definition of GTP.

Now the number i is to be filled in the skew shape given by R_{n-i+1} / R_{n-i+2} $(i \geq 2)$ (R_n for $i=1$)

and hence the number i cannot possibly be in the shape of R_{n-i+2} (has to be outside this shape)

(and hence in any other 'lower' R_j)

Suppose $j < i$. Then. $R_{n-j+1} \subseteq R_{n-i+1}$. Thus, j, which is being filled in R_{n-j+1} / R_{n-j+2} $(j \neq 1)$ or R_n $(j=1)$, cannot "leak" outside R_{n-j+1} $(\forall j)$

and hence the rows must be weakly decreasing

The fact that the columns are strictly increasing is subtle.

It is clear from the containment $R_n \subseteq \cdots \subseteq R_1$ that the columns must b eakly increasing.

Suppose we have the number α in position (m, p) and $(m+1, p)$ of the Tableau.

Clearly $\alpha \neq 1$ since $\underline{1}$ is filled in a horizontal partition with a single row since R_n is just a single number.

Now α is filled into $R_{n-\alpha+1} / R_{n-\alpha+2}$ which is just

$$\begin{matrix} a_{n-\alpha+1, n-\alpha+1} & a_{n-\alpha+1, n-\alpha+2} & \cdots & a_{n-\alpha+1, n} \\ & a_{n-\alpha+2, n-\alpha+2} & \cdots & a_{n-\alpha+2, n} \end{matrix}$$

in the GTP

$\therefore$ α is filled in all boxes indexed as

$$1 \leq i \leq \alpha$$

$$a_{n-\alpha+2, n-\alpha+i+1} < j \leq a_{n-\alpha+1, n-\alpha+i} \qquad (\forall \ i = 1, 2, \ldots, \alpha-1)$$

$$1 \leq j \leq a_{n-\alpha+1, n} \qquad (i = \alpha)$$

As per assumption α is filled into boxes labelled (m, p) & $(m+1, p)$

By restriction on i, $1 \leq m \leq \alpha$, $1 \leq m+1 \leq \alpha$

and hence $m \neq \alpha$

From (m, p) being filled by α,

$$a_{n-\alpha+2,\, n-\alpha+m+1} < p \leq a_{n-\alpha+1,\, n-\alpha+m}$$

From $(m+1, p)$ being filled by α,

$$a_{n-\alpha+2,\, n-\alpha+m+2} < p \leq a_{n-\alpha+1,\, n-\alpha+m+1}$$

But $a_{n-\alpha+1,\, n-\alpha+m+1} \leq a_{n-\alpha+2,\, n-\alpha+m+1}$ leading to a contradiction since no such p can exist

Example for the converse :

Suppose we have the lattice

$$\begin{array}{lllllll} 1 & 1 & 1 & 3 & 5 & 5 & 69 \\ 2 & 3 & 5 \\ 3 & 4 \\ 5 & 7 \end{array}$$

Consider the highest number : 69

Hence our pyramid has height 69

The first row has 69 entries & describes the shape of the SSYT & is hence $7, 3, 2, 2, \overbrace{0, 0, \ldots, 0}^{65 \text{ times}}$

The second row describes positions of 69 & is hence

$6, 3, 2, 2, \overbrace{0, 0, \dots, 0}^{64 \text{ times}}$ (Notice how the 0's are forced since both parents are 0's $\begin{smallmatrix} 0 & & 0 \\ & 0 & \end{smallmatrix}$)

This continues till the 63rd row which is $6, 3, 2, 2, 0, 0, 0$

The i^{th} row (along with $i+1^{th}$) determines filling of $n-i+1$ & hence 63rd (along with 64^{th}) determines filling of $69 - 63 + 1 = 7$. Thus 64^{th} row is $6, 3, 2, 1, 0, 0$

Thus we have the pyramid

$$
\begin{array}{c}
7 \quad 3 \quad 2 \quad 2 \quad 0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\
6 \quad 3 \quad 2 \quad 2 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\
6 \quad 3 \quad 2 \quad 2 \quad 0 \quad 0 \quad \cdots \quad 0 \\
\ddots \qquad \qquad \qquad \qquad \qquad \qquad \iddots \\
6 \quad 3 \quad 2 \quad 2 \quad 0 \quad 0 \quad 0 \\
6 \quad 3 \quad 2 \quad 1 \quad 0 \quad 0 \\
6 \quad 3 \quad 2 \quad 1 \quad 0 \\
4 \quad 2 \quad 2 \quad 0 \\
4 \quad 2 \quad 1 \\
3 \quad 1 \\
3
\end{array}
$$

* The reader in their proof must give an explicit description of the i^{th} row (starting from 1st row)

Proposition 23

Suppose λ, μ are partitions of n and $K_{\lambda\mu} \neq 0$. Then $\mu \leq_2 \lambda$

further, $\mu = \lambda \Rightarrow K_{\lambda\lambda} = 1$

This proves that $\{s_\lambda \mid \lambda \vdash n\}$ is a basis of Λ^n

proof

$K_{\lambda\mu} \neq 0 \Rightarrow \exists$ SSYT of shape λ, type μ. Suppose a part k appears in below the k^{th} row, then we have a contradiction in that particular strictly increasing column

Hence $1, 2, \dots, k$ all appear in first k rows

$\therefore \mu_1 + \mu_2 + \dots + \mu_k \leq$ no. of boxes in first k rows $= \lambda_1 + \dots + \lambda_k$

$\therefore \mu \leq_2 \lambda$

$\mu = \lambda \Rightarrow$ entire row i filled with i's $\Rightarrow K_{\lambda\lambda} = 1$

(no other choice)

■

RSK Algorithm

Operation "$P \leftarrow k$" : P is a non-skew SSYT (P_{ij}) and k is a non-negative integer. Let r be the smallest +ve integer st. $P_{1,r} > k$. If r doesn't exist, place k at the end of the first row. If r exists, place k in $P_{1,r}$ and bump this element $P_{1r} = k'$ into second row.

We are now dealing with $P \leftarrow k'$ (excluding row 1)

The resulting array is $P \leftarrow k$ (will have an extra box)

example Insert '4' into the SSYT :

$$P = \begin{array}{ccccccc} 1 & 1 & 2 & 4 & 5 & 5 & 6 \\ 2 & 3 & 3 & 6 & 6 & 8 & \\ 4 & 4 & 6 & 8 & & & \\ 6 & 7 & & & & & \\ 8 & 9 & & & & & \end{array}$$

$x = 5$. So the first 5 is bumped to the second row which bumps the 6 which bumps the 8

$\therefore$ we have

$$\begin{array}{ccccccc} & & & & 4 & & \\ 1 & 1 & 2 & 4 & 5 & 5 & 6 \\ 2 & 3 & 3 & 6 & 6 & 8 & \\ 4 & 4 & 6 & 8 & & & \\ 6 & 7 & \square & & & & \\ 8 & 9 & & & & & \end{array} = \begin{array}{ccccccc} 1 & 1 & 2 & 4 & 4 & 5 & 6 \\ 2 & 3 & 3 & 5 & 6 & 8 & \\ 4 & 4 & 6 & 6 & & & \\ 6 & 7 & 8 & & & & \\ 8 & 9 & & & & & \end{array}$$

Definition : By an insertion path we mean the indices of the bumped elements including the new box positions. It is usually denoted by $I(P \leftarrow k)$

ex In our case, $I(P \leftarrow 4) = \{(1\,5), (2\,4), (3\,4), (4\,3)\}$

Proposition 24

If $I(P \leftarrow k) = \{(1\,x_1), (2\,x_2), \ldots, (m\,x_m)\}$, then

$x_1 \geq x_2 \geq \cdots \geq x_m$

proof

Let $(r\ x_r) \in I(P \leftarrow k)$. Then either $P_{r+1, x_r} > P_{r, x_r}$ or P_{r+1, x_r} is not a possible box in P

In the first case, the bumped P_{r,x_r} cannot go to the right of column x_r since $P_{r+1, x_r} > P_{r, x_r}$

In the second case, $x_{r+1} < x_r$ since we can't have gaps

■

Proposition 25

Let $j \le k$

If $(r, x_r) \in I(P \leftarrow j)$, $(r, y_r) \in I((P \leftarrow j) \leftarrow k)$ then $x_r < y_r$.

Moreover, $|I((P \leftarrow j) \leftarrow k)| \le |I(P \leftarrow j)|$

proof

In row 1, k is inserted strictly to the right j (in $P \leftarrow j$) since $k \ge j$

$\therefore$ j bumped an element from Row 1 of P which is smaller than the element k bumped from Row 1 of $P \leftarrow j$

Thus applying induction to the bumped elements, we have our result.

Now, the last pair in $I(P \leftarrow j)$ describes a box at the end of a row. Let it be $(m\ x_m)$

Hence if there is a pair $(m\ y_m)$ in $I((P \leftarrow j) \leftarrow k)$, $x_m < y_m$ & hence process of y_m stops making $m\ y_m$

also the last element of the m^{th} row.

Thus size of $I((P \leftarrow j) \leftarrow k)$ is at most $|I(P \leftarrow j)|$ ∎

exercise Show that $P \leftarrow k$ is an SSYT

Definition: A generalised permutation is a two line array

$$w = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix} \quad (\text{some } m \geq 0) \text{ such that}$$

(i) $i_1 \leq i_2 \leq \cdots \leq i_m$

(ii) $i_r = i_s$, $r \leq s \Rightarrow j_r \leq j_s$

Here $i_t, j_t \in \{1, 2, \cdots\} \ \forall t$

Proposition 26

Matrices with entries in $\mathbb{N}$ having finitely many non-zero entries are in bijection with generalised permutations

Proof

Given a matrix with finitely many non-zero entries, we read it row wise and for the (i,j)-th entry a_{ij} read, we append $\begin{bmatrix} i \\ j \end{bmatrix}$ to our generalised permutation a_{ij} times

Since it is read row-wise, in the top row of our perm, we have weakly increasing i values & for the same i value (which means we are in the same row), the j entries weakly increase since we read the row left to right

Conversely, given a generalised permutation, we create the matrix whose (i,j) th entry is the no. of times $\binom{i}{j}$ shows up in the generalised permutation ∎

The Algorithm

The algorithm described below converts a matrix with finite support (ie. finitely many entries non-zero) to a pair of Young Tableaux.

We shall that this is a bijection later on.

Step 1 : Start with a finitely supported $\mathbb{N}$-matrix A

Step 2 : Convert A into a generalised permutation w_A

Step 3 : Start with empty Young Tableaux P_1, Q_1

Step 4 : Given P_t, Q_t, if $w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix}$, do the operation $P_t \leftarrow j_t$ to obtain P_{t+1}

Step 5 : Consider the position of the new cell added to P_t to obtain P_{t+1} & create a cell in the same position with entry i_t & append it to Q_t to obtain Q_{t+1}

Remarks

Firstly note that we begin with P_1, Q_1 empty & hence of the same shape. Assuming P_t, Q_t are of the same shape, since the new cell of Q_{t+1} (wrt Q_t) is at the same position as that of P_{t+1} (wrt P_t), the pair P, Q has the same shape at every stage making step 5 well defined

Secondly, by construction, P_t is always an SSYT but the same conclusion cannot be drawn for Q_t so easily. This needs an elaborate argument

Proposition 27

The Q_t's constructed are always SSYT's

Proof

Base case is checked trivially. Now suppose Q_t is an SSYT, then since the new box is inserted only at the end of rows & the top row in w_A is weakly increasing, we can ensure that rows & columns of Q_{t+1} are weakly increasing.

Now suppose $i_{t-1} = i_t$. We claim that when i_t is added to Q_t to get Q_{t+1}, it doesnt end up in the same column as the i_{t-1}

Now $j_{t-1} \leq j_t$ & hence the insertion path of j_t (in P_t to obtain P_{t+1}) lies strictly to the right of that of j_{t-1} & hence the new box created is in a column strictly to the right ensuring that no two entries of a column in Q_{t+1} are equal

■

Theorem 28 (RSK bijection)

The RSK algo described above is a bijection

Proof

We need to reverse the algo & construct A or w_A from (P, Q), a pair of SSYT of the same shape

[Recall that equal entries of Q are inserted strictly left to right]

Let $Q_{r,s}$ be the rightmost largest entry of Q (This is the freshly inserted entry). This is i.

Consider the corresponding cell of P (i.e. $P_{r,s}$) & the entry in it. This is j.

Simply delete i & its box from Q to get Q'. To get P' from P, we reverse the insertion algo $(P' \leftarrow j) = P$

The algorithm for reversing the insertion process has been left as an easy exercise to the reader.

Thus, given (P, Q), we obtained (P', Q') & $\binom{i}{j}$.

Iteratively, we obtain w_A (in reverse order)

To show that w_A is indeed a gen perm, we argue as follows.

Firstly, the top row of w_A is clearly weakly increasing.

Next, suppose $i_t = i_{t+1}$

Let i_t be $Q_{r,s}$, i_{t+1} be Q_{uv}

Then $s < v$ & $u \leq r$ (insertion path moves rightward)

Thus inverse insertion path from $P_{r,s}$ lies strictly to the left of the inv. ins. path from $P_{u,v}$ [argue same as in prop 25)

Hence, $j_t \leq j_{t+1}$

Corollary

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$$

(Generalised permutations when restricted to permutations will simply produce SYT instead of SSYT. That is, we consider only permutation matrices)

We now study some consequences of the RSK algorithm, the first being the Cauchy identity

Proposition 29 (Cauchy Identity)

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) \, s_\lambda(y)$$

proof

As seen, countlessly many times previously, the coefficient of $x^\alpha y^\beta$ on LHS is the no. of $\mathbb{N}$-matrices with row sum α & column sum β

The coefficient of $x^\alpha y^\beta$ on RHS is no. of pairs (P, Q) of SSYT of same shape with type $(P) = \alpha$, type $(Q) = \beta$

It is quite easy to restrict the RSK algo to this & see the bijection unfold.

■

Theorem 30

The Schur functions form an orthonormal basis of Λ

proof

From prop 23, it follows that the matrix relating m_λ, s_λ is invertible (with det 1, in fact) & hence s_λ is a basis.

Further, from prop 18, $\{s_\lambda\}$ is its own dual basis & hence

$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

■

Proposition 31

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda \qquad \left(\text{note: } s_\mu = \sum_\lambda K_{\mu\lambda} m_\lambda \right)$$

proof

Let $h_\mu = \sum_\lambda a_{\lambda\mu} s_\lambda$

Then, $a_{\lambda\mu} = \langle h_\mu, s_\lambda \rangle = \sum_\beta K_{\lambda\beta} \langle h_\mu, m_\beta \rangle$

$= K_{\lambda\beta} \qquad (\because \langle h_\mu, m_\beta \rangle = \delta_{\mu\beta})$

■

Proposition 32

$$(h_1)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$$

proof

LHS is $(x_1 + x_2 + \cdots + x_n)^n$

Coefficient of $x_1 x_2 x_3 \cdots x_n$ on LHS is $n!$

RHS is $\sum_{\lambda \vdash n} f_\lambda \; s_\lambda(x_1, \ldots, x_n)$

Coefficient of $x_1 \cdots x_n$ on RHS is $\sum_{\lambda \vdash n} f_\lambda \, \gamma_\lambda$

where γ_λ = coeff of $x_1 \cdots x_n$ in $s_\lambda(x_1, \ldots, x_n)$

But $\gamma_\lambda = f_\lambda$

$\therefore$ The coeff is $\sum_{\lambda \vdash n} f_\lambda^2 = n!$ & we are done

■

Theorem 33

If $A \xrightarrow{RSK} (P, Q)$, then $A^T \xrightarrow{RSK} (Q, P)$

proof

Refer 'Matrix – Ball construction' file

■

Q) Give a formula for $\sum_{\lambda \vdash n} f^\lambda$ in 'terms' of S_n

Ans $(P, Q) \xleftrightarrow{RSK} \omega_A$

To get $\sum_{\lambda \vdash n} f^\lambda$, we restrict LHS to $P = Q$ &

$P = Q$ being SYT

The latter condition forces $\omega \in S_n$ and $P = Q$

implies that $A^T = A$ and a perm matrix A_σ corresponding

to $\sigma \in S_n$ is symmetric iff $\sigma^2 = id$

Thus $\sum_{\lambda \vdash n} f^\lambda$ = no of $\sigma \in S_n$ st. $\sigma^2 = id$

■

We give a quick intro to the dual RSK algorithm

We proceed very similar to the RSK algo. Obtain ω_A from a 0-1 matrix A of finite support. But now, an element i is inserted into P differently. Instead of the usual bumping, we bump the leftmost element $\geq i$ (instead of the usual $> i$)

One can go through the proof of RSK algo & give a proof of the following RSK^* bijection

Theorem 29 (Dual RSK or RSK^* bijection)

The RSK^* algorithm gives a bijection between 0-1 matrices of finite support and pairs (P, Q) with P^T & Q being SSYT of same shape. Further, type $(P) = C_c$ and type $(Q) = R_c$

Proposition 30 (Dual Cauchy identity)

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y)$$

There are quite a few consequences as we shall see

Proposition 31

$$\omega(s_\lambda) = s_{\lambda'}$$

proof

Let ω_y act on y variables

Then
$$\omega_y \left(\prod (1 - x_i y_j)^{-1} \right) = \omega_y \left(\sum_\lambda m_\lambda(x) h_\lambda(y) \right)$$
$$= \sum_\lambda m_\lambda(x) e_\lambda(y)$$
$$= \prod (1 + x_i y_j)$$

Thus,

$$\sum_\lambda s_\lambda(x)\, s_{\lambda'}(y) = \prod (1 + x_i y_j)$$
$$= \omega_y \left(\sum_\lambda s_\lambda(x)\, s_\lambda(y) \right)$$
$$= \sum_\lambda s_\lambda(x)\, \omega_y(s_\lambda(y))$$

The result follows

■

We now state one of the most important theorems for Schur functions. The proof can be found in the file 'Schur poly defn'

Theorem 32 (Definitions of s_λ)

The following are equivalent definitions of the Schur polynomial in n variables $x = (x_1, \dots, x_n)$

1) [Littlewood] $s_\lambda(x) = \sum_T x^{\text{type}(T)}$ where T is any SSYT of shape λ with entries from the range $\{1, 2, \dots, n\}$
2) [Cauchy] $s_\lambda(x) = V_{\lambda+\delta} / V_\delta$ where V_μ is the det of the alternant matrix (defined after this theorem)
3) [Jacobi-Trudi] $s_\lambda(x) = \det(H)$ where H is an $n \times n$ matrix with $(i,j)^{th}$ entry $h_{\lambda_i + j - i}$
4) [Dual Jacobi-Trudi] $s_\lambda(x) = \det(E)$ where E is an $n \times n$ matrix with $(i,j)^{th}$ entry $e_{\lambda'_i + j - i}$

■

Definition: Given $\lambda = (\lambda_1, \dots, \lambda_n)$, we define V_λ to be the alternant of λ defined as the determinant of an $n \times n$ matrix whose $(i,j)^{th}$ entry is $x_i^{\lambda_j}$. We define a generic $\delta = (n-1, n-2, \dots, 2, 1, 0)$

Proposition 33

$$s_\nu e_\mu = \sum_\lambda K_{\lambda'/\nu',\mu} s_\lambda$$

proof

Recall that $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$

$$\therefore \quad e_\mu = \sum_\lambda K_{\lambda,\mu} s_{\lambda'} = \sum_\lambda K_{\lambda',\mu} s_\lambda$$

Since $s_\lambda = \dfrac{V_{\lambda+\delta}}{V_\delta}$, we have $V_\delta e_\mu = \sum_\lambda K_{\lambda',\mu} V_{\lambda+\delta}$

This proposition claims $V_{\nu+\delta} e_\mu = \sum_\lambda K_{\lambda'/\nu',\mu} V_{\lambda+\delta}$

which is indeed a generalisation

The proof is similar to the $\nu = 0$ case proved in the supplement

$\blacksquare$

We show an example of how to construct our skew tableau

Let $\lambda = (5,4,4,4,3,1)$, $\mu = (3,2,2,2,2,1,1)$, $\nu = (3,2,2,1)$

We want the coefficient of $x^{\lambda+\delta} = x_1^{10} x_2^8 x_3^7 x_4^6 x_5^4 x_6^1$ in

$V_{\nu+\delta} e_\mu = V_{(8,6,5,3,1,0)} e_3 e_2^4 e_1^2$ (By skew suffices to only consider this term. The other 23 terms are just permutations)

$V_{\nu+\delta}$ supplies us with the initial $x_1^8 x_2^6 x_3^5 x_4^3 x_5$ (any other term is bound to get cancelled)

One possible route is

$$x_1^8 x_2^6 x_3^5 x_4^3 x_5 \xrightarrow{x_1x_2x_3,\ x_1x_2,\ x_3x_5,\ x_4x_5,\ x_4,x_6,\ x_4,\ x_7} x_1^{10} x_2^8 x_3^7 x_4^6 x_5^4 x_6^1$$

The corresponding tableau is

$$\begin{array}{cccccc} \times & \times & \times & \times & 3 & 5 \\ \times & \times & \times & 4 & 4 & \\ \times & 1 & 1 & 5 & 7 & \\ 1 & 2 & 3 & 6 & & \\ 2 & & & & & \end{array}$$

Proposition 34

For any $f \in \Lambda$, $\langle f s_\mu , s_\lambda \rangle = \langle f , s_{\lambda/\mu} \rangle$

proof

$$s_\nu e_\mu = \sum_\lambda K_{\lambda'/\nu', \mu} \, s_\lambda$$

$$\therefore \; s_{\nu'} h_\mu = \sum_\lambda K_{\lambda'/\nu', \mu} \, s_{\lambda'}$$

$$\therefore \; s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu, \mu} \, s_\lambda$$

$$\therefore \; \langle s_\nu h_\mu , s_\lambda \rangle = K_{\lambda/\nu, \mu} = \langle h_\mu , s_{\lambda/\nu} \rangle$$

Since $\{h_\mu\}$ is a basis, the result follows by linearity. ∎

Proposition 35

$$\omega (s_{\lambda/\nu}) = s_{\lambda'/\nu'}$$

proof

$$\langle s_\mu s_\nu , s_\lambda \rangle = \langle s_\mu , s_{\lambda/\nu} \rangle$$

claim: $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$

By bilinearity, it suffices to choose $f = p_\lambda$, $g = p_\mu$

$\langle p_\lambda , p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ and $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$

$$\therefore \; \langle \omega(p_\lambda), \omega(p_\mu) \rangle = \varepsilon_\lambda \varepsilon_\mu z_\lambda \delta_{\lambda\mu} = (\varepsilon_\lambda)^2 z_\lambda \delta_{\lambda\mu} = z_\lambda \delta_{\lambda\mu}$$

This proves the claim

Thus applying ω, we get $\langle s_{\mu'} s_{\nu'} , s_{\lambda'} \rangle = \langle s_{\mu'} , \omega(s_{\lambda/\nu}) \rangle$

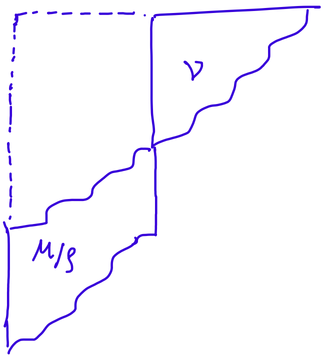
But $\langle s_{\mu'} s_{\nu'} , s_{\lambda'} \rangle = \langle s_{\mu'} , s_{\lambda'/\nu'} \rangle$ ∎

Definition: $\langle s_\lambda , s_\mu s_\nu \rangle = \langle s_{\lambda/\nu} , s_\mu \rangle = \langle s_{\lambda/\mu} , s_\nu \rangle$ is an integer called the Littlewood - Richardson coefficient which is denoted by $c^\lambda_{\mu\nu}$

Q) Show that $\langle s_{\lambda/\nu}, s_{\mu/\rho} \rangle$ is also an LR-coeff

Ans $\langle s_{\lambda/\nu}, s_{\mu/\rho} \rangle = \langle s_\lambda, s_\nu s_{\mu/\rho} \rangle$

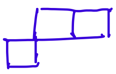
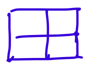
Now we claim $s_\nu s_{\mu/\rho}$ is itself some $s_{\alpha/\beta}$ as is clear by the following diagram



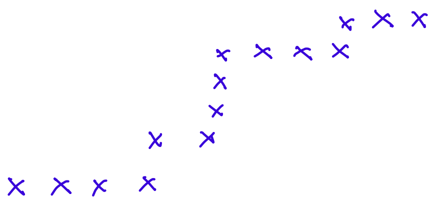
We now state the relevant facts about Murnaghan-Nakayama rule & proceed along.

Details can be found in 'Mur-Nak & S_n'

Definition : A skew shape λ/μ is said to be connected if it is connected as a region in $\mathbb{R}^2$

example  is not connected while  is

Definition : A border strip / rim hook / ribbon is a connected skew shape with no 2×2 square in it

example  is a border strip.

The idea is just to move along the "border" in a single "path"

Note that if we start from bottom left corner, we are only allowed to move N or E (Why not S? Why not W?)

Fact 36

$$S_\mu \, p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} s_\lambda$$

The sum is over all $\lambda \supseteq \mu$ s.t. λ/μ is a border strip of r boxes

Here ht is height of border strip which is total no. of rows minus one

Definition : A border-strip tableau of shape λ/μ and type α (where $\alpha \vDash n = |\lambda/\mu|$) is an assignment of positive integers to the squares of λ/μ st.

(i) Rows & columns are weakly increasing

(ii) i appears α_i times

(iii) squares occupied by i form a border strip

example

×	×	×	×	1	3	3	5	5	5
×	×	1	1	1	3	4	5	10	
×	×	1	2	2	3	4	8		
×	×	1	2	3	3	4			
×	2	2	2	4	4	4			

This is a border strip tableau of shape $(10, 9, 8, 7, 7)/(4, 2, 2, 2, 1)$ and type $(5, 6, 6, 6, 4, 0, 0, 1, 0, 1)$

Definition : Just as for a border strip λ/μ, $ht(\lambda/\mu)$ is defined to be the no. of rows minus one, we define the height of a border-strip tableau to be the sum of heights of the border-strips formed in it

example In our example, height $= 3 + 2 + 3 + 3 + 1 + 0 + 0 = 12$

Proposition 37

$$s_\mu \, p_\alpha = \sum_\lambda \chi^{\lambda/\mu}(\alpha) \, s_\lambda \quad , \quad \text{where} \, ,$$

$\chi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{ht(T)}$ summed over border-strip tableau T of shape λ/μ & type α

proof

Directly follows from fact 36 & definition of height

$\blacksquare$

Proposition 38 (Murnaghan-Nakayama lemma)

$$s_{\lambda/\mu} = \sum_\nu \frac{1}{z_\nu} \chi^{\lambda/\mu}(\nu) \, p_\nu$$

proof

From prop 37, $\chi^{\lambda/\mu}(\nu) = \langle s_\mu p_\nu , s_\lambda \rangle$

$$= \langle p_\nu , s_{\lambda/\mu} \rangle$$

Let $s^{\lambda/\mu} = \sum_\nu c_\nu \, p_\nu$

$$\therefore \quad \chi^{\lambda/\mu}(\alpha) = \sum_\nu c_\nu \langle p_\alpha , p_\nu \rangle$$

$$= \sum_\nu c_\nu \, z_\alpha \, \delta_{\alpha\nu}$$

$$= c_\alpha z_\alpha$$

$$\therefore \quad c_\alpha = \frac{1}{z_\alpha} \chi^{\lambda/\mu}(\alpha)$$

Thus $s_{\lambda/\mu} = \sum_\nu \frac{1}{z_\nu} \chi^{\lambda/\mu}(\nu) \, p_\nu$

$\blacksquare$

Proposition 39 (orthogonality properties)

$$\sum_\lambda \chi^\lambda(\alpha) \, \chi^\lambda(\beta) = z_\alpha \, \delta_{\alpha\beta}$$

$$\sum_\lambda \frac{1}{z_\lambda} \chi^\alpha(\lambda) \, \chi^\beta(\lambda) = \delta_{\alpha\beta}$$

Proof

We know that $s_\lambda = \sum_\nu \frac{1}{z_\nu} \chi^\lambda(\nu) \, p_\nu$

Since the base change matrix with $(\lambda, \nu)^{th}$ entry $\frac{1}{z_\nu} \chi^\lambda(\nu)$ is orthogonal, we get our result ■

We shall now just state some interesting results without proof.
Relevant references shall be cited

Fact (column insertion)

Analogous to row insertion $P \leftarrow i$, one may want to define a column insertion $i \rightarrow P$

We have the following 2 results

(i) $j \rightarrow (P \leftarrow i) = (j \rightarrow P) \leftarrow i$

(ii) $w \xrightarrow{RSK} (P, Q) \Rightarrow w^{\text{reversed}} \xrightarrow{RSK} (P^t, Q^*)$

(what happens to Q is a little weird)

[Prop 3.3.2 & 3.2.3 Sagan - Symmetric group 2nd edition]

Fact (Greene's theorem (not Green!))

For $\sigma \in S_n$, let $I_k(\sigma)$ denote length of largest k-subseq of σ
A k-subsequence of σ is a subsequence of σ which can be cut into k increasing parts

Analogously, one can define $D_k(w)$ (decreasing k-subseq)

Let $w \xrightarrow{RSK} (P, Q)$. Then $sh(w) := $ shape of $P = \lambda$ (say)

Then $\lambda_1 + \dots + \lambda_k = I_k(w)$, $\lambda_1' + \dots + \lambda_k' = D_k(w)$

example $I_3(7,1,4,6,3,2,5) = 6$ [take $7,1,4,6,3,5 \rightarrow$ 7, 135, 46 split into 3 inc seq]

[Section 3.5 Sagan - Symmetric group 2nd edition]

Fact (Knuth Transformations)

Any permutation of the following 4 forms

$\cdots a\ c\ b \cdots$ $\quad$ $\cdots c\ a\ b \cdots$ $\quad$ $\cdots b\ a\ c \cdots$ $\quad$ $\cdots b\ c\ a \cdots$

$\cdots c\ a\ b \cdots$ $\quad$ $\cdots a\ c\ b \cdots$ $\quad$ $\cdots b\ c\ a \cdots$ $\quad$ $\cdots b\ a\ c \cdots$

is called a Knuth transformation if $a < b < c$

(other entries same, only a c swap with some $b \in (a, c)$ next to them)

σ is said to be Knuth equivalent to τ ($\sigma \overset{K}{\sim} \tau$) if τ can be obtained from σ using Knuth transformations

We have the following fact [Enum Combi 2 – Stanley – A1.1.4]

$\sigma \overset{K}{\sim} \tau$ iff $\sigma \xrightarrow{RSK} (P, Q_1)$ & $\tau \xrightarrow{RSK} (P, Q_2)$

Fact (Jeu De Taquin, Schützenburger)

Just google "15 sliding puzzle" and get the gist of how to play it. It is quite a neat puzzle that I enjoyed solving as a kid.

We now illustrate how to shift a skew tableux into a desired box naturally

Suppose T =

x	1	3	7	10
2	5	6	9	
4	8	11		

and we want to shift T into the x to make it an SYT, we do the following. Between 1 & 2, we obviously slide 1 into the x mark. Between 3 4 5, we slide 3. Between 6 & 7 we slide 6 up. Between 9 & 11, 9 slides.

Thus we get

1	3	6	7	10
2	5	9		
4	8	11		

We say two tableux T, T' are equivalent if one can be obtained from the other using jeu de taquin shifts

We have the following fact

(i) Each Jeu de taquin class has exactly one tableu which isnt skew

For Q, an SYT of shape λ, define $\Delta(Q) = jdt_b(\tilde{Q})$ where b is the box with entry 1 & $\tilde{Q}$ is tableu obtained from Q by deleting the box bearing 1 & reducing all entries by 1

Let us now pass from Q to $\emptyset$ as

$$Q \longrightarrow \Delta(Q) \longrightarrow \Delta(\Delta(Q)) \longrightarrow \cdots \longrightarrow \emptyset$$

Let Q have n boxes (so it is filled by $1, 2, \ldots, n$)

Then $\Delta(Q)$ has 1 box less than Q.

Put n in this box

Put $n-1$ in the box $\Delta^2(Q) \setminus \Delta(Q)$

and so on.

We get a new tableu $\text{evac}(Q)$ called evacuation tableu of Q

Here are the facts

(ii) $\text{evac}(Q)$ is an SYT (obviously of same shape as Q)

(iii) $\text{evac}(\text{evac}(Q)) = Q$

This map $Q \longmapsto \text{evac}(Q)$ is called the Schützenberger involution

Now let $w \in S_n$. Define w^c to be the 'bit-complement' of w. That is, if $w = (w_1, \ldots, w_n)$ then w^c has i^{th} bit as $n+1-w_i$

Let $w^{\#} = (w^c)^{\text{reversed}} = (n+1-w_n, \ldots, n+1-w_1)$

Here is the final result [Enum Combi 2 - Stanley - A1.2]

(iv) $w \xrightarrow{RSK} (P, Q) \Rightarrow w^{\#} \xrightarrow{RSK} (\text{evac}(P), \text{evac}(Q))$

(v) $w \xrightarrow{RSK} (P, Q) \Rightarrow w^{\text{reversed}} \xrightarrow{RSK} (P^T, (\text{evac}(Q))^T)$

Fact (Littlewood – Richardson)

Recall, $c^{\lambda}_{\mu\nu} = \langle s_\mu s_\nu , s_\lambda \rangle = \langle s_\nu , s_{\lambda/\mu} \rangle = \langle s_\mu , s_{\lambda/\nu} \rangle$

is the Littlewood - Richardson coefficient

(i) Fix SYT P of shape ν. $c^{\lambda}_{\mu\nu}$ is the no. of SYT of shape λ/μ which are jeu de taquin equivalent to P

(ii) $c^{\lambda}_{\mu\nu}$ is equal to the no. of SSYT of shape λ/μ and type ν whose $\hat{\mathcal{R}}_c$ is a lattice permutation

Now $\hat{R}_c$ is reading tableux entrywise along rows but $\hat{\mathcal{R}}_c$ reads it entry wise left to right starting from bottom row first

Also recall that a lattice permutation / ballot word is a seq $a_1 \cdots a_n$ s.t. in any initial chunk $a_1 \cdots a_j$ $(j \leq n)$, the no. of i's is $\geq$ no. of $i+1$'s