Definition : A sequence in a set X, is a function $a : \mathbb{N} \to X$. We usually represent a seq with its range and hence denoted as $\{a_n\}_{n=1}^{\infty}$ or simply a_n

Definition : A series is a sequence generated by taking the partial sums of the first k terms of a given sequence. For a sequence a_n, its series is b_n where $b_k = \sum_{i=1}^{k} a_i$

Definition : A sequence is said to be strictly increasing if $\forall n \in \mathbb{N}$, $a_n < a_{n+1}$ and is said to be non-decreasing if $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$. Similarly we define strictly decreasing and non-increasing. Together, these four are called monotonic type of sequences

Definition : A sequence a_n is said to converge to a limit l if $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $|a_n - l| < \varepsilon$ holds $\forall n > N_0$. We then write $\lim_{n\to\infty} a_n = l$

A sequence which does not converge is said to diverge

THEOREM 1

If a_n and b_n are sequences converging to l_1 and l_2, then, $\lim_{n\to\infty} (a_n \pm b_n) = l_1 + l_2$, $\lim_{n\to\infty} (a_n \cdot b_n) = l_1 \cdot l_2$ and $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{l_1}{l_2}$ (provided $l_2 \neq 0$)

THEOREM 2 (Sandwich Theorem)

(ver. 1) If a_n, b_n, c_n are convergent sequences so that $a_n \le b_n \le c_n$ holds for all n, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \le \lim_{n\to\infty} c_n$

(ver. 2) If a_n and c_n are two sequences which converge to the same limit and b_n is a sequence such that $a_n \le b_n \le c_n$ $\forall n$, then b_n converges and further, all three sequences have the same limit

Definition : A sequence a_n is bounded if $\exists M$ in X so that $|a_n| \le M$ $\forall n \in \mathbb{N}$. Any sequence which is not bounded is unbounded

THEOREM 3

Every convergent sequence is bounded

THEOREM 4

A non-decreasing sequence which is bounded above converges and a non-increasing sequence which is bounded below converges

Definition: The supremum of a sequence a_n is a number M with the properties :(i) $a_n \le M$ $\forall n$ and (ii) if M_1 is such that $a_n \le M_1$ $\forall n$, then $M \le M_1$.

Similarly, the infimum is a ~~sequ~~ number N so that $a_n \ge N$ $\forall n$ and if N_1 is such that $a_n \ge N_1$ $\forall n$, then $N \ge N_1$

Definition: A sequence is said to be **Cauchy** if $\forall\ \varepsilon > 0\ \exists\ N_0 \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon\ \forall\ m, n > N_0$

THEOREM 5

Every convergent sequence is Cauchy

Definition: A space X is **complete** if every Cauchy sequence in X, converges in X

THEOREM 6

The reals are complete

Definition: Let a_n, a sequence in $\mathbb{R}^k$, be represented as $a_n = (a_{1_n}, a_{2_n}, a_{3_n}, \dots, a_{k_n})$. This sequence is said to **converge** to $(l_1, l_2, \dots, l_k)$ if $\forall\ \varepsilon > 0$ $\exists\ N_0 \in \mathbb{N}$ s.t. $\sqrt{(a_{1_n} - l_1)^2 + \dots + (a_{k_n} - l_k)^2} < \varepsilon$

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to a **limit** l at point $x_0 \in [a, b]$ if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - x_0| < \delta$.

We then write $\lim_{x \to x_0} f(x) = l$

THEOREM 7

If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, then,

$\lim_{x \to x_0} (f+g)(x) = l_1 + l_2$, $\lim_{x \to x_0} (f \cdot g)(x) = l_1 \cdot l_2$

and $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{l_1}{l_2}$ (g is non zero is some neighbourhood of x_0)

THEOREM 8 (Sandwich Theorem)

(ver 1) As $x \to x_0$, if $f(x) \to l_1$; $g(x) \to l_2$, $h(x) \to l_3$ and $f(x) \leq g(x) \leq h(x)$ $\forall x \in (a,b)$, then $l_1 \leq l_2 \leq l_3$

(ver 2) As $x \to x_0$, if $f(x)$ and $h(x)$ tend to the same value l and g is such that $f(x) \leq g(x) \leq h(x)$ $\forall$ $x \in (a,b)$, then g converges and further to the same l

THEOREM 9

Let $f:(a,b) \to \mathbb{R}$ be such that $\lim_{x \to c} f(x)$ exists for some $c \in (a,b)$. Then $\exists\ \eta$ such that f is bounded on the interval $(c-\eta, c+\eta) \subset (a,b)$. If $c = a$, $\exists\ \eta$ such that f is bounded on $(a, a+\eta) \subset (a,b)$ and if $c = b$, $\exists\ \eta$ such that f is bounded on $(b-\eta, b) \subset (a,b)$

Definition: we say $f: \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ if $\forall\ \varepsilon > 0$, $\exists\ X \in \mathbb{R}$ s.t. $|f(x) - l| < \varepsilon$ whenever $x > X$

Definition: The left-hand limit of a function $f:(a,b) \to \mathbb{R}$ at $x = c$ is a number l^- such that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - l^-| < \varepsilon$ whenever $c - x < \delta$ and $x \in (a,c)$ and the right hand limit of a function $f:(a,b) \to \mathbb{R}$ at $x = c$ is a number l^+ such that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - l^+| < \varepsilon$ whenever $x - c < \delta$ and $x \in (c,b)$

Definition: If $f:[a,b] \to \mathbb{R}$ is a function and $c \in [a,b]$ then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

THEOREM 10:

Let $f:(a,b)\to(c,d)$ and $g:(c,d)\to(e,f)$ be functions such that f is continuous at x_0 in (a,b) and g is continuous at $y_0=f(x_0)\in(c,d)$. Then, $g\circ f$ is continuous at x_0

THEOREM 11: (Intermediate Value Theorem)

Suppose $f:[a,b]\to\mathbb{R}$ is continuous. Then for every u between $f(a)$ and $f(b)$, $\exists\ c\in[a,b]$, s.t., $f(c)=u$

THEOREM 12

Every polynomial of odd degree has at least one root ~~and further~~; ~~every n degree polynomial with all real roots~~

THEOREM 13 (Extreme value theorem)

A continuous function on a closed interval is bounded and attains its infimum and supremum on that interval.

THEOREM 14

A function f is continuous at a point a in its domain iff for every a_n s.t. $a_n\to a$ as $n\to\infty$, we have $\{f(a_n)\}$ converging to $f(a)$

Definition: A function $f:(a,b)\to\mathbb{R}$ is <u>differentiable</u> at a point $c\in(a,b)$ if $\lim_{h\to0}\frac{f(c+h)-f(c)}{h}=f'(c)$ exists

Definition: $f:(a,b)\to\mathbb{R}$ is said to Lipschitz continuous with exponent α (usually $\alpha=1$) if $|f(x+h)-f(x)|\leq C|h|^{\alpha}$ $\forall\, x, x+h \in (a,b)$ and some constant C

THEOREM 15

If f is Lipschitz continuous with $\alpha>1$, the f is differentiable with zero derivative

THEOREM 16 (Caratheodory lemma)

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. f is differentiable at c iff $\exists\, f_1:(a,b)\to\mathbb{R}$ s.t. f_1 is continuous and satisfies $f(x)-f(c)=(x-c)f_1(x)$, $\forall\, x\in(a,b)$. Further, if such an f_1 exists, it is unique and $f'(c)=f_1(c)$

THEOREM 17

If f is differentiable at c, it is continuous at c

THEOREM 18

If f and g are differentiable at c, so are $f\pm g$; $f\cdot g$ at $x=c$. Further, $(f\pm g)'(c)=f'(c)\pm g'(c)$ and $(f\cdot g)'(c)=f'(c)g(c)+g'(c)f(c)$

THEOREM 19 (Chain Rule)

Let $f:(a_1,b_1)\to\mathbb{R}$ and $g:(a_2,b_2)\to\mathbb{R}$ such that $f((a_1,b_1))\subset(a_2,b_2)$. If $c\in(a_1,b_1)$ and $f(c)\in(a_2,b_2)$, and f is differentiable at c and g is differentiable at $f(c)$,

then $g \circ f$ is differentiable at c. Further, $(g \circ f)'(c)$
$= g'(f(c)) \cdot f'(c)$

Definition : $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. f is said to achieve a maximum at $x_0 \in X$ if $f(x_0) \geq f(x)$ $\forall x \in X$

Similarly we define minimum

Definition : Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. Suppose there is a sub interval s.t. $x_0 \in (c,d) \subset X$ s.t. $f(x_0) \geq f(x)$ $\forall$ $x \in (c,d)$, then f is said to have local maxima at $x = x_0$

THEOREM 20 (Fermat's theorem)

Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be differentiable, having a local maxima or minima at x_0. Then $f'(x_0) = 0$

THEOREM 21 (Rolle's Theorem)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous, differentiable and satisfies $f(a) = f(b)$. Then $\exists\, c \in (a,b)$ such that $f'(c) = 0$

THEOREM 22 (Mean Value Theorem)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous and f is diff in (a,b). Then $\exists\, c \in (a,b)$ s.t. $\frac{f(b) - f(a)}{b-a} = f'(c)$

THEOREM 23

If f satisfies the hypothesis of MVT and $f'(x) = 0$ $\forall x \in (a,b)$, then f is the constant function.

THEOREM 24 (Darboux's Theorem)

Let $f:(a,b)\to\mathbb{R}$ be differentiable. If $c,d\in(a,b)$ s.t. $c<d$, then for every u between $f'(c)$ and $f'(d)$, $\exists\, x\in[c,d]$ s.t. $f'(x)=u$

Definition: Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). We call $x_0\in(a,b)$ a stationary point of f if $f'(x_0)=0$

THEOREM 25

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and diff on (a,b). Let x_0 be a stationary point of f. Let f' be differentiable at x_0. If $f''(x_0)>0$, f has local minima at x_0. If $f''(x_0)<0$, f has local maxima at x_0 and if $f''(x_0)=0$, no conclusion can be drawn

Definition: Let I be an interval. Let $f:I\to\mathbb{R}$. f is said to be concave down (= convex up) if $f(tx_1+(1-t)x_2)\geq tf(x_1)+(1-t)f(x_2)$ $\forall\, x_1,x_2\in I$, $t\in[0,1]$. Replacing '$\geq$' with '$\leq$', '$<$', '$>$' we get concave up (= convex down), strictly concave up and strictly concave down

In general, convex = convex down and concave = concave down.

THEOREM 26

Every convex function is Lipschitz cont with exponent 1

THEOREM 27

A conven function is differentiable at all but at most countably many points

THEOREM 28

A differentiable function is a conven function iff the first derivative is increasing. Further, if the function is diff and conven, then it is continuously differentiable (ie. derivative is continuous)

Definition: An inflection point of a function is a point where the curvature changes its sign. That is, the function changes from being concave to conven

THEOREM 29

If x_0 is a point of inflection, $f''(x_0) = 0$ and if $f^{(n)}(x_0) = 0 \ \forall \ n = 2, 3, \ldots, (k-1)$ and $f^{(k)}(x_0) \neq 0$ with k being odd, $(k \geq 3)$ then x_0 is a point of inflection

Definition: $C^k(I)$ is defined as the set of all functions which are real valued, whose domain is I and are k-times continuously differentiable

Definition: For a function which is n times differentiable at a point $x_0 \in I$, we associate it to a sequence of polynomials called the Taylor polynomials, $P_0(x)$, $P_1(x), \ldots, P_n(x)$ of degrees $0, 1, \ldots, n$ defined as

$$P_i(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + f^{(2)}\frac{(x_0)(x-x_0)^2}{2!} + \cdots + f^{(i)}\frac{(x_0)\ (x_1-x_0)^i}{i!}$$

THEOREM 30

$P_i(x)$ has the same first i derivatives as the function f at point x_0.

That is, $P_i^{(k)}(x_0) = f^{(k)}(x_0) \quad \forall\ 0 \le k \le i$

THEOREM 31 (Taylor's Theorem)

Let I be an open interval so that $[a,b] \subset I$. Let $f \in C^n(I)$ and suppose $f^{(n)}$ is diff on I. Then $\exists\ c \in (a,b)$. st.

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Definition: A partition P of an interval $[a,b]$ is defined as a collection of points $\{ a = x_0 < x_1 < \cdots < x_n = b \}$. Naturally $[a,b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}]$

Definition: A partion P' is the refinement of a partion P if every point in P is contained in P'

THEOREM 32

Given any two partitions P and Q we can find R such that R is a refinement of both P and Q

Definition: given a partion $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a,b]$

and a function $f:[a,b]\to\mathbb{R}$, we define

$M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ $\forall i\in[n]$

Further, we define lower and upper sums of f over partition P as:

$$L(f,P) = \sum_{j=1}^{n} m_j\cdot(x_j - x_{j-1})$$

$$U(f,P) = \sum_{j=1}^{n} M_j\cdot(x_j - x_{j-1})$$

THEOREM 33

For any two partitions P_1 and P_2 of the same interval, we have, $L(f,P_1)\le U(f,P_2)$ and for any refinement P' of a partition P, we have $L(f,P)\le L(f,P')$ and $U(f,P')\le U(f,P)$

Definition: A tagged partition is a pair (P,t) where P is the partition of an interval $[a,b]$ and t is a collection of points $\{t_j\}_{j=1}^{n}$ where each t_j is a point in the interval $[x_{j-1},x_j]$ $\forall j=1,2,\dots,n$. Further, we define the Riemann sum as $R(f,P,t)=\sum_{j=1}^{n} f(t_j)\cdot(x_j - x_{j-1})$

THEOREM 34

Given any partition P and its tagged partition (P,t), $L(f,P)\le R(f,P,t)\le U(f,P)$

Definition: The <u>norm</u> of a partition P is defined as $\|P\| := \max_j \{ |x_j - x_{j-1}| \} \; \forall \; 1 \leq j \leq n$

Definition: $f : [a,b] \to \mathbb{R}$ is <u>Riemann integrable</u> if $\exists \; R \in \mathbb{R}$ s.t. $\forall \; \varepsilon > 0, \; \exists \; \delta > 0$ s.t. $|R(f, P, t) - R| < \varepsilon$ whenever $\|P\| < \delta$. We denote $R = \int_a^b f(x)\,dx$

THEOREM 35

$f : [a,b] \to \mathbb{R}$ is Riemann integrable if $\exists \; R \in \mathbb{R}$ s.t. $\forall \; \varepsilon > 0$ $\exists \; \delta > 0$ and a partition P such that for every tagged refinement (P', t') of P with $\|P'\| < \delta$, we have $|R(f, P, t) - R| < \varepsilon$

THEOREM 36

The Riemann integral exists if and only if the Darboux integral exists and in this case both values are equal

THEOREM 37

Every function bounded on a closed interval which is discontinuous at countably many points is integrable over that interval

THEOREM 38

Suppose f is RI on $[a,b]$ and $c \in [a,b]$, then,

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

THEOREM 39 (FTC)

Let $f:[a,b]\to\mathbb{R}$ be continuous and $F(x)=\int_a^x f(t)\,dt$ be defined $\forall\, x\in[a,b]$. Then F is continuous on $[a,b]$, diff on (a,b) and $F'(x)=f(x)\ \forall\, x\in(a,b)$

THEOREM 40 (FTC part 2)

Let $f:[a,b]\to\mathbb{R}$ be given. There exists $g:[a,b]\to\mathbb{R}$ which is differentiable on (a,b) and satisfies $g'(x)=f(x)\ \forall\, x\in(a,b)$. Then f is integrable on $[a,b]$ with $\int_a^b f(t)\,dt = g(b)-g(a)$

THEOREM 41 (Int. by parts)

Let $f:[a,b]\to\mathbb{R}$ be differentiable such that f' is integrable. suppose $g:[a,b]\to\mathbb{R}$ is integrable and has an anti-derivative G on $[a,b]$, then,

$$\int_a^b f(x)g(x)\,dx = f(b)G(b)-f(a)G(a)-\int_a^b f'(x)G(x)\,dx$$

THEOREM 42

Let $f:[a,b]\to\mathbb{R}$ be continuous and let $\phi:[\alpha,\beta]\to\mathbb{R}$ be continuously differentiable such that $\phi([\alpha,\beta])=[a,b]$. Then $(f\circ\phi)\circ\phi'$ is integrable and if $\phi'(t)\neq 0\ \forall\, t\in(\alpha,\beta)$, then,

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\phi(t))\cdot|\phi'(t)|\,dt$$

Definition: we define $\underline{\ln(x)} := \int_1^x \frac{1}{t}\,dt \quad \forall\, x\in(0,\infty)$ and we define $\underline{\exp(x)}$ to be its inverse

Definition: we define $\arctan(x) := \int_0^x \frac{1}{1+t^2}\,dt \ \forall x \in \mathbb{R}$ and $\tan x$ to be its inverse

Definition: Let f_1, f_2 be integrable on $[a,b]$ with $f_1(x) \le f_2(x) \ \forall x \in [a,b]$. For $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ f_1(x) \le y \le f_2(x)\}$, we define the area between the curves as $\text{area}(R) = \int_a^b (f_2(x) - f_1(x))\,dx$

THEOREM 43

Let R denote the region bounded by $r = f(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$ where $-\pi \le \alpha < \beta \le \pi$. Let $R = \{(r\cos\theta, r\sin\theta) \mid \alpha \le \theta \le \beta,\ 0 \le r \le f(\theta)\}$. The area of R is $\frac{1}{2}\int_\alpha^\beta (f(\theta))^2\,d\theta$

THEOREM 44 (Slice method for volume)

Let D be a bounded subset of $\mathbb{R}^3$. Let $a<b$ and suppose D lies between $x=a$ and $x=b$. For any $s \in [a,b]$, we denote the area of the slice by $A(s) = \text{ar}\{(x,y,z) \mid x=s\}$. The volume of the solid is then given by $\text{vol}(D) = \int_a^b A(s)\,ds$

THEOREM 45 (Washer method)

Let D be the solid obtained by revolving the region bounded by $y = f_1(x)$, $y = f_2(x)$, $x=a$, $x=b$ about the x-axis. The volume of D is given by $\left|\pi \int_a^b (f_2^2(x) - f_1^2(x))\,dx\right|$

THEOREM 46 (Shell method)

Let D be a bounded subset of $\mathbb{R}^3$. A cross section of D obtained by piercing a cylinder through D is called a <u>sliver</u> of D. If D is obtained by rotating ($y = f_1(x)$, $y = f_2(x)$, $x = a$, $x = b$) about Y-axis, the volume of D is $\left| 2\pi \int_a^b x \cdot (f_2(x) - f_1(x))\, dx \right|$

Definition: A <u>parameterised curve</u> or path C in $\mathbb{R}^2$ is given by $(x(t), y(t))$ where $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuous.

Definition: In accordance with the above, suppose x, y are continuously differentiable, we define arc length of C as $l(C) = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. Suppose C is given in polar coordinates as $(x(\theta), y(\theta))$ where $x(\theta) = \rho(\theta) \cdot \cos\theta$ and $y(\theta) = \rho(\theta) \cdot \sin\theta$ then we define, $l(C) = \int_\alpha^\beta \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}\, d\theta$

THEOREM 47

Let C be a smooth curve parameterised by $(x(t), y(t))$ $\forall\, t \in [\alpha, \beta]$. Suppose C doesnt cross the line L given by $ax + by + c = 0$, then area of surface generated by revolving C about L is $2\pi \int_\alpha^\beta \gamma(t) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

where $\gamma(t) = \left| \dfrac{a \cdot x(t) + b \cdot y(t) + c}{\sqrt{a^2 + b^2}} \right|$ $\forall\, t \in [\alpha, \beta]$

Definition: The <u>natural domain</u> of a function is defined as the subset of the specified domain over which the function is well defined

Definition: The level set of $f: \mathbb{R}^2 \to \mathbb{R}$ is given by $\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$. we can define contour lines of the same function as $\{(x,y,z) \mid f(x,y) = c,\ z = c\}$

Definition: Let $U \subseteq \mathbb{R}^2$. A function $f: U \to \mathbb{R}$ is said to be continuous at c if $\lim_{x \to c} f(x) = f(c)$

Definition: Let $U \subseteq \mathbb{R}^2$. The partial derivative of $f: U \to \mathbb{R}$ wrt the first coordinate variable x_1 is given by

$$\frac{\partial f}{\partial x_1}(a,b) := \lim_{h \to a} \frac{f(a+h,b) - f(a,b)}{h}$$

Definition: Let $U \subset \mathbb{R}^2$. $f: U \to \mathbb{R}$ is said to be differentiable at $(a,b) \in U$ if

$$\lim_{(h,k) \to (0,0)} \frac{\left| f(a+h,b+k) - f(a,b) - h \cdot \frac{\partial f}{\partial x}(a,b) - k \frac{\partial f}{\partial y}(a,b) \right|}{\| h,k \|}$$

exists and is equal to zero

Definition: Let $v = (v_1, v_2)$ be a unit vector in $\mathbb{R}^2$. The directional derivative of f in the direction specified by v at a point (a,b) is

$$\nabla_v = \lim_{t \to 0} \frac{f(a+tv_1, b+tv_2) - f(a,b)}{t}$$

THEOREM 48

If $f: U (\subset \mathbb{R}^2) \to \mathbb{R}$ is such that (WLOG) $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$ exists at all points in U and is continuous, then if $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ exists, it is continuous and further, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ holds at all points in the domain U

THEOREM 49

If for a function all partial derivatives exist at some point and are continuous in a neighbourhood of that point, then the function is diff at that point

THEOREM 50

Let $z = f(x,y)$ where $x = x(t)$ and $y = y(t)$. If z is differentiable at $(x(t), y(t))$, then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

THEOREM 51

we define the gradient $\nabla f := \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$. The normal vector to an implicit surface S in $\mathbb{R}^3$ is given by ∇g where S is given by $g(x,y,z) = 0$

Definition: we say that $f(x,y)$ attains its local minimum at (x_0, y_0) if there is a disk $D_r(x_0, y_0) = \{(x,y) \mid \|(x,y) - (x_0,y_0)\| < r\}$ such that $f(x,y) \geq f(x_0, y_0)$ $\forall (x,y) \in D_r$

Definition: (x_0, y_0) is called a critical point of f if both the partial derivatives of f are zero at (x_0, y_0)

THEOREM 52

If (x_0, y_0) is a local extremum and $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then (x_0, y_0) is a critical point where $f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$ and $f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$

Definition: we define the <u>Hessian matrix</u> of $f: \mathbb{R}^2 \to \mathbb{R}$ as

$$\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$ at the point (a,b)

where $f_{x_1 x_2}(a,b)$ means $\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} f(a,b)\right)$

we denote its determinant by D

THEOREM 53

Let (a,b) be a critical point of f.

If $D > 0$, $f_{xx}(a,b) > 0$, then (a,b) is local minima

If $D > 0$, $f_{xx}(a,b) < 0$, then (a,b) is local maxima

If $D < 0$, (a,b) is a saddle point

If $D = 0$, no conclusions can be drawn

Definition: A <u>saddle point</u> is a point on the curved surface where the curvatures in two mutually perpendicular planes are of opposite signs.

THEOREM 55 (Taylor's in 2D)

If $f \in C^2(D_r(x_0,y_0))$ for some $r > 0$, we have

$$f(x_0+h, y_0+k) = f(x_0,y_0) + h \cdot f_x + k \cdot f_y + \frac{1}{2!}\left(h^2 \cdot f_{xx} + k^2 f_{yy} + 2hk\, f_{xy}\right) + \tilde{R}_2(x_0,y_0)$$

where $\frac{\tilde{R}_2(x_0,y_0)}{\|(h,k)\|} \to 0$ as $\|h,k\| \to 0$

THEOREM 56

A continuous function on a compact set in $\mathbb{R}^2$ will attain its extreme values ??

TUTORIAL 1

1) Prove the following

(i) $\lim_{n\to\infty} \frac{10}{n} = 0$

(ii) $\lim_{n\to\infty} \frac{5}{3n+1} = 0$

(iii) $\lim_{n\to\infty} \frac{n^{2/3} \sin(n!)}{n+1} = 0$

(iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

Ans (i) Let $N_0 = \left\lfloor \frac{10}{\varepsilon} \right\rfloor + 1 \in \mathbb{N}$

Then $\forall\, n > N_0$, $n \in \mathbb{N}$,

$$n > \left\lfloor \frac{10}{\varepsilon} \right\rfloor + 1 \Rightarrow n > \frac{10}{\varepsilon}$$

$$\therefore \frac{10}{n} < \varepsilon \Rightarrow \left| \frac{10}{n} - 0 \right| < \varepsilon$$

$\therefore \forall\, \varepsilon > 0$, $\exists\, N_0 = \left\lceil \frac{10}{\varepsilon} \right\rceil \in \mathbb{N}$ s.t. $\left| \frac{10}{n} - 0 \right| < \varepsilon$ whenever $n > N_0$

(ii) Let $N_0 = \left\lceil \frac{1}{3}\left(\frac{5}{\varepsilon} - 1\right) \right\rceil \in \mathbb{N}$

$\forall\, n > N_0$, $n > \frac{1}{3}\left(\frac{5}{\varepsilon} - 1\right) \Rightarrow \left| \frac{5}{3n+1} - 0 \right| < \varepsilon$

(iii) Pick $N_0 = \lceil \frac{1}{\varepsilon^3} \rceil$

$\forall\, n > N_0,\quad n > \frac{1}{\varepsilon^3} \Rightarrow \frac{1}{n^{1/3}} < \varepsilon \quad (\because \varepsilon, n > 0)$

$\therefore \left| \frac{n^{2/3} \sin(n!)}{n+1} - 0 \right| \leq \left| \frac{n^{2/3}}{n+1} \right| < \left| \frac{1}{n^{1/3}} \right| < \varepsilon$

(iv) we claim $\lim_{n\to\infty} \frac{n}{n+1} = 1$

Choosing $N_0 = \lceil \frac{1}{\varepsilon} - 1 \rceil$, we have,

$n > N_0 \Rightarrow n > \frac{1}{\varepsilon} - 1 \Rightarrow \frac{1}{n+1} < \varepsilon$

$\Rightarrow \left| \frac{n}{n+1} - 1 \right| < \varepsilon$

2) Show that limit exists and find it.

(i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \cdots + \frac{n}{n^2+n} \right)$

(ii) $\lim_{n\to\infty} \left(\frac{n!}{n^n} \right)$

(iii) $\lim_{n\to\infty} \frac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$

(iv) $\lim_{n\to\infty} n^{1/n}$

(v) $\lim_{n\to\infty} \frac{\cos(\pi\sqrt{n})}{n^2}$

(vi) $\lim_{n\to\infty} \left(\sqrt{n} \left(\sqrt{n+1} - \sqrt{n} \right) \right)$

Ans (i) $\sum_{i=1}^{n} \frac{n}{n^2+n} \leq \sum_{i=1}^{n} \frac{n}{n^2+i} \leq \sum_{i=1}^{n} \frac{n}{n^2}$

$\therefore \frac{n}{n+1} \leq \sum_{i=1}^{n} \frac{n}{n^2+i} \leq 1$

By sandwich theorem, limit exists and is 1

(ii) $0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = \frac{1}{n}$

By sandwich theorem, limit exists and is 0

(iii) $0 \leq \frac{n^3+3n^2+1}{n^4+8n^2+2} \leq \frac{5n^3}{n^4} = \frac{5}{n}$

By sandwich theorem, limit exists and is 0

(iv) $1 \leq n^{1/n} \leq ?$

We know, $\left(1+\sqrt{\frac{2}{n}}\right)^n \geq n$ (why?)

Thus $n^{1/n} \leq 1+\sqrt{\frac{2}{n}}$

By sandwich theorem, limit exists and is 1

(v) $-\frac{1}{n^2} \leq \frac{\cos(\pi\sqrt{n})}{n^2} \leq \frac{1}{n^2}$

By sandwich theorem, limit exists and is 0

(vi) $\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+1}} \leq \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}}$

By sandwich theorem, limit exists and is $1/2$

(We need to prove here extra that $\lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n+1}} = \frac{1}{2}$)

3) Show that the following are divergent

(i) $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$ (ii) $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

Ans (i) All convergent sequences are bounded. FTSOC

let $\left\{\frac{n^2}{n+1}\right\}_{n\geq 1}$ be convergent.

$\therefore \exists M, \forall n \in \mathbb{N} \quad \frac{n^2}{n+1} \leq M$.

Hence, $\frac{n^2}{n+n} \leq \frac{n^2}{n+1} \leq M$

$\therefore$ Choosing $n = 4M$,

$\frac{16M^2}{8M} \leq M \Rightarrow 2M \leq M$ (contradiction)

(ii) Suppose it is convergent.

$\left\{\frac{(-1)^n}{n}\right\}_{n\geq 1}$ is convergent

$\therefore \left\{\frac{(-1)^n}{2}\right\}_{n\geq 1}$ is convergent.

$\therefore \{(-1)^n\}$ is convergent. Let it converge to L

Choose $\varepsilon = 0.5$

Then $\exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $|(-1)^n - L| < 0.5$

Choose $n = 2N_0$

$\therefore ||1-L| < 0.5$

choose $n = 2N_0 + 1$

$\therefore |1+L| < 0.5$

Contradiction.

4) Check increasing or decreasing

(i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$

(ii) $\left\{ \frac{2^n 3^n}{5^{n+1}} \right\}_{n \geq 1}$

(iii) $\left\{ \frac{1-n}{n^2} \right\}_{n \geq 2}$

Ans (i) $m^2 + m > 1 \quad \forall m \in \mathbb{N}$

$\therefore m^3 + 2m^2 + 2m > m^3 + m^2 + m + 1 \quad \forall m \in \mathbb{N}$

$\therefore m((m+1)^2 + 1) > (m^2+1)(m+1) \quad \forall m$

$\therefore \frac{m}{m^2+1} > \frac{m+1}{(m+1)^2+1} \quad \forall m \in \mathbb{N}$

$\therefore a_m > a_{m+1} \quad \forall m \in \mathbb{N}$

$\therefore$ Decreasing

(ii) $\frac{6}{5} > 1$

$\therefore \frac{2^{n+1} 3^{n+1}}{5^{n+2}} > \frac{2^n 3^n}{5^{n+1}} \quad \forall n \in \mathbb{N}$

$\therefore a_{n+1} > a_n \quad \forall n \in \mathbb{N}$

$\therefore$ increasing

(iii) $a_{n+1} - a_n = \frac{n(n-1) - 1}{n^2 (n+1)^2} > 0 \quad \forall n \geq 2$

$\therefore$ increasing

5) Prove convergence using monotonicity & boundedness. Also find limits

(i) $a_1 = \frac{3}{2}, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall n \geq 1$

(ii) $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n} \quad \forall n \geq 1$

(iii) $a_1 = 2, \quad a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1$

Ans (i) $a_1 = \frac{3}{2}$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2} \quad \forall n \geq 1$

$a_{n+1} - a_n = \frac{1}{a_n} - \frac{a_n}{2} = \frac{2 - (a_n)^2}{2a_n} \leq 0 \ \forall n \geq 2$

Thus, the sequence is monotonically decreasing except for a_1.

Clearly, it is bounded below by $\sqrt{2}$

Thus, it converges. (say, to L)

$L = \frac{1}{2}\left(L + \frac{1}{L}\right) \Rightarrow L = \sqrt{2}$

(ii) $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}} > a_1$

So we hypothesize & claim that a_n is increasing

Assume true for some k ie: $a_{k+1} > a_k$

$a_{k+2} = \sqrt{2 + a_{k+1}} > \sqrt{2 + a_k} = a_{k+1}$

thus we are done by induction

we claim $a_n < 2 \ \forall n \in \mathbb{N}$.

Base case: $a_1 = \sqrt{2} < 2$

Let it be true for a_k

$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+2} = 2$

Thus we are done by induction

Hence limit exists (say, L)

$L = \sqrt{2 + L} \Rightarrow L = 2$

(iii) claim it is increasing

Base case: $a_2 = 3 + \frac{2}{2} = 4 > 2 = a_1$

assume true for k, ie., $a_{k+1} > a_k$

Now, $a_{k+2} = 3 + \frac{a_{k+1}}{2} > 3 + \frac{a_k}{2} = a_{k+1}$.

Claim: bounded above by 6

Base case: $a_1 = 2 < 6$

Assume true for a_k i.e. $a_k < 6$

$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$

Thus, we get convergence.

say limit is L

$L = 3 + \frac{L}{2} \Rightarrow L = 6$

6) If $\lim_{n\to\infty} a_n = L$, find $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$

Ans first one will follow from question 10* (I modified this one : Prove that a seq. converges to a limit L iff all its subseq converge to L)

For the second one, we claim limit is $|L|$

Choose N_0 as the N_0 used in ~~while~~ ~~proof~~ $\lim_{n\to\infty} a_n = L$.

we know, $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$, s.t. whenever $n > N_0$, $|a_n - L| < \varepsilon \Rightarrow ||a_n| - |L|| < \varepsilon$

7) If $\lim_{n\to\infty} a_n = L \neq 0$, show that $\exists n_0 \in \mathbb{N}$ s.t.

$|a_n| \geq \frac{|L|}{2} \quad \forall n \geq n_0$

Ans choose $\varepsilon = \frac{|L|}{2}$. Thus $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$,

$|a_n - L| < \frac{|L|}{2} \Rightarrow ||a_n| - |L|| < \frac{|L|}{2} \Rightarrow |a_n| \geq \frac{|L|}{2}$

(open with -ve)

8) Prove that $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_n^{1/2} = \sqrt{L}$ (assuming $a_n \geq 0 \ \forall n \in \mathbb{N}$)

Ans $\forall \varepsilon' > 0$, $\exists N_0 \in \mathbb{N}$ s.t. whenever $n > N_0$, $|a_n - L| < \varepsilon'$

Choose $\varepsilon' = \varepsilon \cdot \sqrt{L}$

$\therefore \exists N_1 \in \mathbb{N}$, s.t. $|\sqrt{a_n} + \sqrt{L}|\,|\sqrt{a_n} - \sqrt{L}| < \varepsilon \cdot \sqrt{L}$

$\forall n > N_1$

$$\therefore |\sqrt{a_n} - \sqrt{L}| < \frac{\varepsilon \cdot \sqrt{L}}{|\sqrt{a_n} + \sqrt{L}|} < \frac{\varepsilon \cdot \sqrt{L}}{|\sqrt{L}|} = \varepsilon$$

ε' was arbitrary $\Rightarrow$ ε is arbitrary

$\Rightarrow \forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $|\sqrt{a_n} - \sqrt{L}| < \varepsilon \ \forall n > N_1$

$$\therefore \lim_{n\to\infty} a_n^{1/2} = \sqrt{L}$$

9) Prove or disprove for seq. $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$

(i) $\{a_n b_n\}$ is convergent if ~~$\{a_n\}$~~ $\{a_n\}$ is convergent

(ii) $\{a_n b_n\}$ is convergent if $\{a_n\}$ is convergent and $\{b_n\}$ is bounded

Ans (i) False: $a_n = 1/n$, $b_n = n^2$

(ii) False. $a_n = 1$, $b_n = (-1)^n$

10) Prove that a seq. converges to L iff all its subsequences converge to L

Ans ($\Rightarrow$) Let $a_n \to L$ as $n \to \infty$

Let $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of a_n

$\forall \varepsilon > 0$, $\exists N_0$ so that $\forall n > N_0$, $|a_n - L| < \varepsilon$.

Choose k so that $n_k > N_0$. Such a k exists, else the sub seq would be finite.

For this k, $|a_{n_k} - L| < \varepsilon$ and hence the same N_0 works.

(Note: $n_k \geq k > n_0$)

($\Leftarrow$) All subseq converges to L. A seq is a subseq of itself and we are done

TUTORIAL 2

1) Let $a, b, c \in \mathbb{R}$ with $a < c < b$. Let $f, g : (a,b) \to \mathbb{R}$ with $\lim_{x \to c} f(x) = 0$. Prove or disprove

(i) $\lim_{x \to c} f(x) g(x) = 0$

(ii) $\lim_{x \to c} f(x) g(x) = 0$ if g is bounded

(iii) $\lim_{x \to c} f(x) g(x) = 0$ if $\lim_{x \to c} g(x)$ exists

Ans (i) False. Choose $f(x) = x$, $g(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

(ii) True: $|g(x)| \leq M \Rightarrow 0 \leq |f(x) g(x)| \leq M |f(x)|$

By sandwich theorem $\lim_{x \to 0} f(x) g(x) = 0$

(iii) True. g is bounded near c and hence by above part it is true. (OR) use product law of limits

2) Let $\lim_{x \to \alpha} f(x)$ exist for some $\alpha \in \mathbb{R}$. Show that

$\lim_{h \to 0} [f(\alpha + h) - f(\alpha - h)] = 0$. Analyse the converse

Ans $\lim_{x\to\alpha} f(x) = L \Rightarrow \lim_{h\to 0^+} f(\alpha+h) = L$, $\lim_{h\to 0^+} f(\alpha-h) = L$

$\therefore \lim_{h\to 0^+} f(\alpha+h) - f(\alpha-h) = 0$

Also putting h as $-h$,

$\lim_{h\to 0^-} (\alpha-h) = L$, $\lim_{h\to 0^-} f(\alpha+h) = L$

$\therefore \lim_{h\to 0^-} f(\alpha+h) - f(\alpha-h) = 0$

$\therefore \lim_{h\to 0} f(\alpha+h) - f(\alpha-h) = 0$

The converse asks us if $\lim_{x\to\alpha} f(x)$ exists given that the RHL and LHL exist and are equal. which is not true. ~~[illegible]~~ Choose $f(x) = \begin{cases} x^2 & x \neq 4 \\ 15 & x = 4 \end{cases}$

3) Discuss continuity

(i) $f(x) = \sin\frac{1}{x}$ $x \neq 0$, $f(0) = 0$

(ii) $f(x) = x\sin\frac{1}{x}$ $x \neq 0$, $f(0) = 0$.

(iii) $f(x) = \begin{cases} \frac{x}{[x]} & 1 \le x < 2 \\ 1 & x = 2 \\ \sqrt{6-x} & 2 < x \le 3 \end{cases}$

Ans (i) For $x \neq 0$, $\frac{1}{x}$, $\sin x$ both are continuous and hence f is continuous

(ii) At $x = 0$, choose $x_n = \frac{2}{(4n+1)\pi}$ so that $f(x_n) = 1$ with $f(0) = 0$

(iii) For $x \in (2,3]$ it is cont' due to composition of $\sqrt{x}$ and $6-x$ which are both cont.

For $x \in [1,2)$ also it is cont since $\frac{x}{\lfloor x \rfloor} = x$

At 2, $\lim_{x\to 2^-} f(x) = \lim_{h\to 0^+} f(2-h) = \lim_{h\to 0^+} \frac{2-h}{1} = 2$ but

$f(2) \neq 2$

4) Let $f: \mathbb{R}\to\mathbb{R}$ s.t. $f(x+y) = f(x) + f(y)$, $\forall\ x, y \in \mathbb{R}$. If f is continuous at 0, show that f is cont at every $c \in \mathbb{R}$. Further, show $f(kx) = k f(x)$

Ans $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} f(h)$$

$$= f(c) + f(0) \quad (\because \text{cont at } 0)$$

$$= f(c)$$

Hence, it is continuous $\forall\ c \in \mathbb{R}$

For $m \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$f(mx) = f(x) + \cdots + f(x) = m f(x)$$

$$f(x) = f\left(\frac{x}{n}\right) + \cdots + f\left(\frac{x}{n}\right) = n f\left(\frac{x}{n}\right)$$

$$\therefore f(kx) = k f(x) \quad \forall\ k \in \mathbb{Q}$$

Since f is cont and $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$,

$$f(kx) = k f(x)\ \forall\ k \in \mathbb{R}$$

5) $f(x) = x^2 \sin\frac{1}{x}$, $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$ and check if f' is cont

Ans For $x \neq 0$, $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0 \Rightarrow f \text{ is diff everywhere}$$

But f' isnt cont at $x=0$ since $\lim_{x\to 0} \cos\frac{1}{x}$ DNE

~~6) $0 \le \left|\frac{f(x+h)-f(x)}{h}\right| \le C|h|^{\alpha-1}$ and by [illegible] Sandwich theorem,~~

6) $f:(a,b)\to\mathbb{R}$ s.t. $|f(x+h)-f(x)| \le C|h|^{\alpha}$ where $\alpha>1$.
Show that f is diff and find $f'(x)$ $\forall\, x\in(a,b)$

Ans $0 \le \left| \frac{f(x+h)-f(x)}{h} \right| \le C|h|^{\alpha-1}$ $\Rightarrow$ By sandwich theorem, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$

$\therefore f'(x) = 0 \quad \forall\, x\in(a,b)$

7) If f is diff at $c\in(a,b)$, p.t. $\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h}$ exists and equals $f'(c)$. Analyse the converse

Ans $\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h} = \frac{1}{2}\left[\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c)}{h} - \lim\limits_{h\to 0} \frac{f(c-h)-f(c)}{h}\right]$

$= \frac{1}{2}\left[f'(c) - (-f'(c))\right] = f'(c)$

Converse is clearly false for $|x|$ with $c=0$

8) $f:\mathbb{R}\to\mathbb{R}$ with $f(x+y)=f(x)f(y)$. If f is diff at 0, show that f is diff everywhere with $f'(c)=f'(0)f(c)$
Further show that $f\in C^{\infty}(\mathbb{R})$

Ans $f(0)=(f(0))^2$

If $f(0)=0$, $f(x+0)=f(x)f(0)=0$

$\Rightarrow f$ is the zero function which is trivially differentiable

If $f(0)=1$, $f'(c) = \lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h} = \left(\lim\limits_{h\to 0}\frac{f(h)-1}{h}\right) f(c)$

$= f'(0)\,f(c)$

9) Find derivative of

(i) $\cos^{-1}x$: $-1 < x \le 1$

(ii) $\operatorname{cosec}^{-1} x$ $|x| > 1$

Ans Derivative of f^{-1} at $x = c$ is $\dfrac{1}{f'(f^{-1}(c))}$

using this for $\cos^{-1}x$, derivative is $\dfrac{-1}{\sqrt{1-x^2}}$

using for $\operatorname{cosec}^{-1}x$, derivative is $-\dfrac{1}{x\sqrt{x^2-1}}$

10) Find $\dfrac{dy}{dx}$ if $y = f\left(\dfrac{2x-1}{x+1}\right)$ with $f'(x) = \sin(x^2)$

Ans JEE level

11) Find f which is cont everywhere & diff everywhere except at 2 points

Ans $f(x) = |x-1| + |x+1|$

12) Show that $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is discont everywhere

Ans For a given $c \in \mathbb{R}$, $\exists \{a_n\} \to c$ where a_n are all rational and $\{b_n\} \to c$ where b_n are all irrational. $\{f(a_n)\}$ & $\{f(b_n)\}$ converge to different values. Hence f is discontinuous at c

13) Let $g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 1-x & x \notin \mathbb{Q} \end{cases}$. Show that

g is cont only at $x = \frac{1}{2}$

Ans Similar to 12, we get $\lim_{x \to c} g(x)$ is both $1-c$ and c. For continuity, $c = 1-c \Rightarrow c = \frac{1}{2}$

14) $f:(a,b)\to\mathbb{R}$, $c\in(a,b)$, $\lim_{x\to c} f(x) > \alpha$. Prove that $\exists\, \delta>0$ s.t. $f(c+h)>\alpha$ $\forall$ $0<|h|<\delta$

Ans $\lim_{h\to 0} f(c+h) = L$

$\forall \varepsilon>0$, $\exists\, \delta>0$ such that $0<|h|<\delta$ implies $|f(c+h)-L|<\varepsilon$ (hence $f(c+h)>L-\varepsilon$)

Choose $\varepsilon = L-\alpha \Rightarrow f(c+h)>\alpha$ and we are done.

(Make sure to write $\varepsilon>0$ is valid since $L-\alpha>0$)

15) Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. Prove that TFAE

(i) f is diff at c

(ii) $\exists\, \delta>0$ and $\varepsilon_1:(-\delta,\delta)\to\mathbb{R}$ s.t. $\lim_{h\to 0}\varepsilon_1(h)=0$ and $f(c+h)=f(c)+\alpha h+h\,\varepsilon_1(h)$ $\forall$ $h\in(-\delta,\delta)$

(iii) $\exists\, \alpha\in\mathbb{R}$ s.t. $\lim_{h\to 0}\dfrac{|f(c+h)-f(c)-\alpha h|}{|h|}=0$

Ans (i) $\to$ (ii) Choose δ so that $(c-\delta, c+\delta)\subset(a,b)$ and with $\alpha=f'(c)$, define

$$\varepsilon_1(h) := \frac{f(c+h)-f(c)-\alpha h}{h} \text{ if } h\neq 0 \;\&\; \varepsilon_1(0)=0$$

(ii) $\to$ (iii) $\lim_{h\to 0}\varepsilon_1(h)=0$ will see this through

(iii) $\to$ (i) $\lim_{h\to 0}\left|\dfrac{f(c+h)-f(c)}{h}-\alpha\right|=0$ implies that f is differentiable at c with derivative α

TUTORIAL 3

1) Show that $x^3 - 6x + 3$ has all roots real

Ans $f(-4) = -37$

$f(0) = 3$

$f(2) = -1$, $f(4) = 43$

By IVT we have a real root between $(-4, 0)$ $(0, 2)$ and $(2, 4)$ $\Rightarrow$ 3 distinct real roots

2) Show that $x^3 + px + q$ has a single real root for $p, q \in \mathbb{R}$, $p > 0$

Ans $f'(x) = 3x^2 + p > 0$

Hence, f is a strictly increasing function and can have at most one real root by Rolle's theorem.

Since f is of odd degree, it has at least one root.

Thus f has exactly one real root.

3) Let f be cont on $[a, b]$, diff on (a, b). If $f(a) \cdot f(b) < 0$ and $f'(x) \neq 0 \; \forall \; x \in (a, b)$, prove that $\exists! \; x_0 \in (a, b)$ so that $f(x_0) = 0$

Ans $f(a) \cdot f(b) < 0 \Rightarrow$ By IVP we get at least one such x_0.

By Rolle's theorem there cannot be more than 1 such x_0 and we are done.

4) $f(x) = x^3 + px + q$

If f has 3 real roots (all distinct), show that $4p^3 + 27q^2 < 0$ by proving

(i) $p < 0$

(ii) f has maximum / minimum at $\pm\sqrt{-p/3}$

(iii) maxima and minima are of opposite signs

Ans (i) $p > 0$ is not possible because of question 2

$\therefore p \leq 0$

If $p = 0$, $f(x) = x^3 + q$ which has a single real root

Hence $p < 0$

(ii) $f'(x) = 0 \Rightarrow 3x^2 + p = 0 \Rightarrow x = \pm\sqrt{-p/3}$

(iii) $f''(x) = 6x$

$\therefore \sqrt{\frac{-p}{3}}$ is a minima, $-\sqrt{\frac{-p}{3}}$ is a maxima

$f'(x) = 3x^2 + p < 0 \quad \forall x \in \left(-\sqrt{\frac{-p}{3}}, \sqrt{\frac{-p}{3}}\right)$

Since f has 3 distinct real roots, f has one of the roots between $x = -\sqrt{\frac{-p}{3}}$ and $x = +\sqrt{\frac{-p}{3}}$

(think why?)

By converse of IVT we have extrema of opposite signs

Now use $f\left(-\sqrt{\frac{-p}{3}}\right) f\left(+\sqrt{\frac{-p}{3}}\right) < 0$ to prove $4p^3 + 27q^2 < 0$

5) use MVT to prove $|\sin a - \sin b| \leq |a-b|, \forall a, b \in \mathbb{R}$

Ans Let $f(x) = \sin x$

Applying MVT on $[a,b]$ since f satisfies the hypothesis:

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\sin b - \sin a}{b-a}$$

$$\therefore \cos c = \frac{\sin b - \sin a}{b-a}$$

$|\cos c| \leq 1$ gives the result.

6) Let f be cont on $[a,b]$, diff on (a,b) with $f(a)=a$, $f(b)=b$. Prove that $\exists\, c_1, c_2\ (c_1 \neq c_2) \in (a,b)$ such that $f'(c_1)+f'(c_2) = 2$

Ans MVT to f on $(a, \frac{a+b}{2})$ yields

$$f'(c_1) = \frac{f(\frac{a+b}{2}) - f(a)}{(\frac{b-a}{2})}$$

MVT to f on $(\frac{a+b}{2}, b)$ yields

$$f'(c_2) = \frac{f(b) - f(\frac{a+b}{2})}{(\frac{b-a}{2})}$$

$$\therefore f'(c_1)+f'(c_2) = \frac{f(b)-f(a)}{(\frac{b-a}{2})} = 2$$

7) Let $a>0$, f be cont on $[-a,a]$. Suppose $f'(x)$ exists and $f'(x) \le 1$ $\forall x \in (-a,a)$. If $f(a)=a$ and $f(-a)=-a$, prove that $f(0)=0$. Further prove $f(x)=x$ $\forall x$

Ans

$$f'(c_1) = \frac{f(0)-f(-a)}{0-(-a)} = \frac{f(0)+a}{a}$$

$$f'(c_2) = \frac{f(a)-f(0)}{a-0} = \frac{a-f(0)}{a}$$

using $f'(x) \le 1$,

$$f(0)+a \le a \Rightarrow f(0) \le 0$$

$$a-f(0) \le a \Rightarrow f(0) \ge 0$$

$$\therefore f(0)=0.$$

Let $g(x) = f(x) - x$

$g(a)=0, \quad g(-a)=0$

and $g'(x) = f'(x) - 1 \le 0 \Rightarrow g$ is non-increasing

but $g(a) = g(-a) \Rightarrow g$ is constant

$\therefore g(x)=0 \quad \forall x$

8) Find f explicitly or prove no such f exists

(i) $f''(x) > 0$ $\forall x \in \mathbb{R}$, $f'(0)=1$, $f'(1)=1$

(ii) $f''(x) > 0$ $\forall x \in \mathbb{R}$, $f'(0)=1$, $f'(1)=2$

(iii) $f''(x) \ge 0$ $\forall x \in \mathbb{R}$, $f'(0)=1$, $f(x) \le 100$ $\forall x>0$

(iv) $f''(x) > 0$ $\forall x \in \mathbb{R}$, $f'(0)=1$, $f(x) \le 1$ $\forall x<0$

Ans (i) not possible by Rolle

(why can we apply Rolle ?) (Think)

(ii) $\frac{x^2}{2} + x + K \quad \forall K \in \mathbb{R}$

(iii) MVT to f' on $(0, x)$ for some $x > 0$ gives

$f'(x) - 1 > 0$

$\therefore f(x) > x + c_0$ (still thinking how without $\int$)

If $c_0 \geq 100$, straightaway contradiction

since $f(x) > x + c_0 > c_0 \geq 100$

If $c_0 < 100$, for $x_0 = 100 - c_0 > 0$,

$f(x_0) > 100 \Rightarrow$ again contradiction

Better way!

$f''(x) \geq 0 \Rightarrow f'$ is increasing

$\frac{f(x) - f(0)}{x - 0} = f'(c_1) \geq f'(0)$ (increasing)

$\therefore f(x) - f(0) \geq x$

$\therefore f(x) \geq x + f(0)$ (work with $c_0 = f(0)$)

(iv) $f(x) = \begin{cases} \frac{1}{1-x} & x \leq 0 \\ 1 + x + x^2 & x > 0 \end{cases}$

just worry about $x \leq 0$ and patch a function on positive side keeping $f'(0) = 1$ in mind and we are done.

TUTORIAL 4

1) Sketch the following after analysing minima/maxima, concavity, points of inflection, asymptotes, etc.

(i) $y = 2x^3 + 2x^2 - 2x - 1$

(ii) $y = \dfrac{x^2}{x^2+1}$

(iii) $y = 1 + 12|x| - 3x^2 \qquad x \in [-2, 5]$

Ans (i) $y' = 6x^2 + 4x - 2$

$\therefore$ points of extrema satisfy $3x^2 + 2x - 1 = 0$

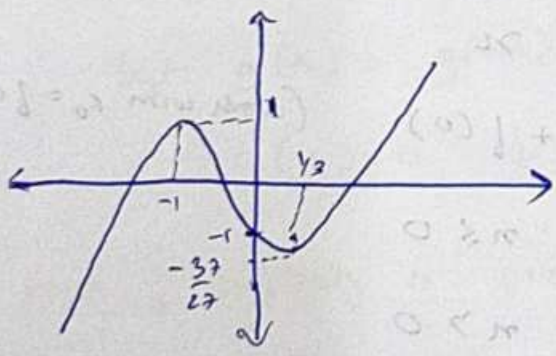
$\therefore \quad (x+1)(3x-1) = 0$

$\therefore \quad x = -1, \frac{1}{3}$

$y'' = 12x + 4 = 4(3x+1)$

$\therefore (-1, 1)$ is maxima

$\left(\frac{1}{3}, \frac{-37}{27}\right)$ is minima $\qquad f(0) = -1$



(ii) $y = \dfrac{x^2}{x^2+1} \geq 0 \quad \forall x \in \mathbb{R}$

$$\lim_{|x|\to\infty} \frac{x^2}{x^2+1} = 1$$

Also $\quad x^2 < x^2+1 \quad \forall \ x \in \mathbb{R}$

$\therefore \ y < 1 \quad \forall \ x \in \mathbb{R}$

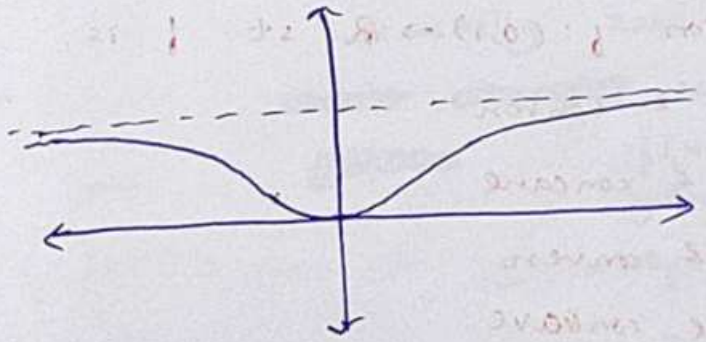
$\therefore$ 1 is the horizontal asymptote

$$y' = \frac{(x^2+1)(2x) - (x^2)(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$\therefore$ 0 is an extrema

$$y'' = \frac{2(1-3x^2)}{(1+x^2)^3}$$

$\therefore$ 0 is a minima



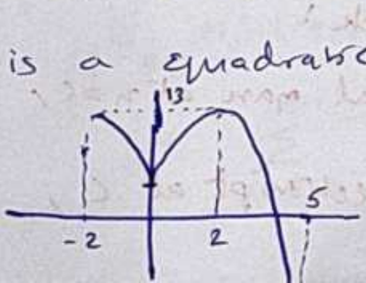
(iii) $\quad y = \begin{cases} 1+12x-3x^2 & x \in [0,5] \\ 1-12x-3x^2 & x \in [-2,0] \end{cases}$

$$y' = \begin{cases} 12-6x & x \in [0,5] \\ -12-6x & x \in [-2,0] \end{cases}$$

$\therefore \ x=2, \ x=-2$ are extremas

$y(2) = y(-2) = 13$

It is a quadratic



2) Sketch f satisfying:

$f(-2) = 8$ $\quad$ $f'(2) = f'(-2) = 0$

$f(0) = 4$ $\quad$ $f'(x) > 0$ for $|x| > 2$

$f(2) = 0$ $\quad$ $f'(x) < 0$ for $|x| < 2$

$f''(x) < 0$ for $x < 0$, $f''(x) > 0$ for $x > 0$

<u>Ans</u> f has local maxima at $x = -2$ and local minima at $x = 2$

f is concave in $(-\infty, 0)$ and convex in $(0, \infty)$

$x = 0$ is a point of inflection

Now draw it yourself

3) Give example of function $f : (0,1) \to \mathbb{R}$ s.t. f is

(i) strictly increasing & convex

(ii) " & concave

(iii) strictly decreasing & convex

(iv) " & concave

<u>Ans</u> (i) x^2

(ii) $\sqrt{x}$

(iii) $-\sqrt{x}$

(iv) $-x^2$

Remember

$f''(x) > 0 \Rightarrow$ convex

4) $f, g : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) \geq 0$, $g(x) \geq 0$ $\forall x \in \mathbb{R}$. Define $h(x) = f(x) g(x)$. Check:

(i) If f, g have local max at $x = c$, so does h at c

(ii) If f, g have inflection pt at c, so does h at c

Ans (i) True.

$f'(c) = 0$, $g'(c) = 0$, $f''(c) < 0$, $g''(c) < 0$

$h'(x) = f'(x)g(x) + g'(x)f(x)$

$h''(x) = f''(x)g(x) + g''(x)f(x) + 2f'(x)g'(x)$

$h'(c) = 0$

$h''(c) = f''(c)g(c) + g''(c)f(c) < 0$

$(\because g, f > 0)$

(ii) False.

$f''(c) = g''(c) = 0$

$\therefore h''(c) = 2f'(c)g'(c)$ which is not necessarily zero.

~~counter example~~ counter:

~~[illegible]~~ $f(x) = 1 + \sin x$

$g(x) = 1 + \sin x$

$c = 0$

5) Let $f(x) = 1$ if $x \in [0,1]$, $f(x) = 2$ if $x \in (1,2]$. Show that f is RI on $[0,2]$ and find the integral

Ans Let $P = \{0 = x_0 < x_1 < \dots < x_n = 2\}$ be a partition of $[0,2]$

1 lies in one of the partitions say $[x_j, x_{j+1}]$

Assume $1 \neq x_j$ (case 1)

$\therefore L(f,P) = \sum_{i=0}^{j-1} m_i (x_{i+1} - x_i) + m_j \cdot (x_{j+1} - x_j)$

$+ \sum_{i=j+1}^{n-1} m_i \cdot (x_{i+1} - x_i)$

$$\therefore L(f,P) = \sum_{i=0}^{j-1} (x_{i+1} - x_i) + \sum_{i=j+1}^{n-1} 2(x_{i+1} - x_i)$$

$$+ \quad (x_{j+1} - x_j)$$

$$= \cancel{x_j} - x_0 + x_{j+1} - \cancel{x_j} + 2\left(x_n - x_{j+1}\right)$$

$$= 2x_n - x_0 - x_{j+1}$$

$$= 4 - x_{j+1}$$

Similarly $U(f,P) = x_j - x_0 + 2\left(\cancel{x_{j+1}} - x_j + x_n - \cancel{x_{j+1}}\right)$

$$= 2x_n - x_0 - x_j$$

$$= 4 - x_j$$

Note : $x_j < 1 < x_{j+1}$

$\therefore \sup_P L(f,P) = L(f) = 3 \quad (\because x_{j+1} \in (1,2])$

and $\inf_P U(f,P) = U(f) = 3 \quad (\because x_j \in [0,1))$

$\therefore L(f) = U(f) = 3$

Case 2 : $x_j = 1$ for some j

$$L(f,P) = \sum_{i=0}^{j-1} m_i (x_{i+1} - x_i) + \sum_{i=j}^{n-1} m_i (x_{i+1} - x_i)$$

$$= \sum_{i=0}^{j-1} (x_{i+1} - x_i) + \sum_{i=j}^{n-1} 2(x_{i+1} - x_i)$$

$$= x_j - x_0 + 2(x_n - x_j)$$

$$= 4 - x_j = 3$$

$U(f,P) = 4 - x_j = 3$

Again, we reach $U(f) = L(f) = 3$

6) (i) Let $f : [a,b] \to \mathbb{R}$ be R I with $f(x) \geq 0 \; \forall x \in [a,b]$ Show that $\int_a^b f(x)\,dx \geq 0$. Further, if f is also continuous and $\int_a^b f(x)\,dx = 0$, show that $f(x) = 0$ $\forall x \in [a,b]$

(ii) Give function which ~~[illegible]~~ satisfies $f(x) \geq 0$ on $[a,b]$ but $\int_a^b f(x)\,dx = 0$ (also given that $f(x) \neq 0$ for some $x \in [a,b]$)

Ans (i) $f \geq 0 \Rightarrow m_j \geq 0 \Rightarrow L(f,P) \geq 0$

Similarly $U(f,P) \geq 0$

Also, $\int_a^b f(x)\,dx \geq L(f,P) \geq 0$

For the next part,

$f(x) \geq 0$ on $[a,b]$, f is cont, $\int_a^b f(x)\,dx = 0$

suppose $f(c) > 0$. Since f is continuous, $\exists\, \delta > 0$ s.t. $f(x) > q \;\; \forall x \in (c-\delta, c+\delta)$ where $q > 0$. Choose $q = \frac{f(c)}{2}$.

Thus $U(f,P) > \delta \cdot \frac{f(c)}{2} \Rightarrow \int_a^b f(x)\,dx \geq \frac{\delta f(c)}{2}$

Contradiction since $\int_a^b f(x)\,dx = 0$

(ii) $f(x) = \begin{cases} 1 & x = c \in (a,b) \\ 0 & \text{otherwise} \end{cases}$

7) Evaluate $\lim_{n\to\infty} S_n$ showing that S_n is an appropriate Riemann sum for a suitable function over a suitable interval

(i) $S_n = \frac{1}{n^{5/2}} \sum^{n} i^{3/2}$

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$

(iii) $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$

(iv) $S_n = \frac{1}{n}\left\{ \sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^2 \right\}$

(v) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$

Ans I will only show one of them in detail.

(i) $\int_0^1 x^{3/2}\, dx = \frac{2}{5}$

(ii) we claim that the given S_n represents the Riemann sum of the integral

$$\int_0^1 \frac{dx}{x^2+1} \quad \left(= \frac{\pi}{4} \text{ by the way} \right)$$

Partition $[0,1]$ as $\left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \right\}$

The Riemann sum over this partition is

$$R(f,P,t) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} \frac{1}{t_j^2+1} \times \frac{1}{n}$$

(where $t_j \in [x_{j-1}, x_j]$)

Choosing $t_j = x_j = \frac{j}{n}$,

$$R(f,P,t) = \sum_{j=1}^{n} \frac{1}{\frac{j^2}{n^2}+1} \times \frac{1}{n}$$

$$= \sum_{j=1}^{n} \frac{n}{j^2+n^2} = S_n$$

(iii) $\int_0^1 \cos(\pi x)\,dx = 0$

(iv) $\int_0^1 x\,dx + \int_1^2 x^{3/2}\,dx + \int_2^3 x^2\,dx = \frac{8\sqrt{2}}{5} + \frac{193}{30}$

(v) $\int_0^1 \frac{dx}{\sqrt{1+x}} = 2\sqrt{2} - 2$

8) compute :

(i) $\frac{d^2y}{dx^2}$ if $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

(ii) $\frac{dF}{dx}$ if $F(x) = \int^{2x} \cos t^2\,dt$, $F(x) = \int_0^{x^2} \cos t\,dt$

Ans JEE stuff (DIY)

9) Let $p \in \mathbb{R}$, f be continuous on $\mathbb{R}$ with period p.
Show that $\int_a^{a+p} f(t)\,dt$ has the same value for every real number a

Ans
$$F(a) = \int_a^{a+p} f(t)\,dt$$

$$F'(a) = f(a+p)\frac{d(a+p)}{da} - f(a)\frac{da}{da}$$

$$= f(a+p) - f(a)$$

$$= 0$$

$\therefore F(a)$ is the constant function

10) Let $f : \mathbb{R} \to \mathbb{R}$ be cont, $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, define $g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin(\lambda x - \lambda t)\,dt$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ $\forall x \in \mathbb{R}$, $g(0) = g'(0) = 0$

Ans JEE stuff.
Be careful with Newton-Leibniz.

TUTORIAL 5

1) Find the area of the region bounded by

(i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, $y = 0$

(ii) $y = x^4 - 2x^2$, $y = 2x^2$

(iii) $x = 3y - y^2$, $x + y = 3$

Ans (i) $\int_0^1 (1+x-2\sqrt{x})\,dx = \frac{1}{6}$

(ii) $\int_{-2}^{2} (2x^2-(x^4-2x^2))\,dx = \frac{128}{15}$

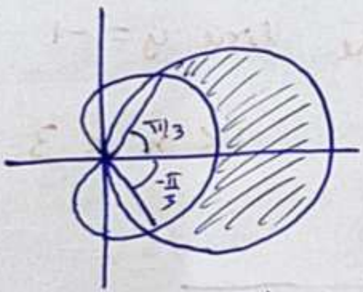
(iii) $\int_1^2 (3y-y^2-3+y)\,dy = \frac{4}{3}$

2) Let $f(x) = x-x^2$, $g(x) = ax$. Find a so that region above g and below f has area 4.5

Ans $\int_0^{1-a} (x-x^2-ax)\,dx = 4.5 \Rightarrow a = -2$

3) Find the area inside $r = 6a\cos\theta$ and outside the cardiod $r = 2a(1+\cos\theta)$

Ans



$$\text{Area} = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta$$

$$= 2a^2 \int_{-\pi/3}^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\,d\theta$$

$$= 4\pi a^2$$

4) Find the arc length of each of the curves given below :

(i) cycloid $x = t - \sin t$, $y = 1 - \cos t$ $\quad 0 \le t \le 2\pi$

(ii) $y = \int_0^x \sqrt{\cos 2t}\, dt$ $\quad 0 \le x \le \pi/4$

Ans (i) $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$= 8$$

(ii) $\int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx = \int_0^{\pi/4} \sqrt{1+\cos 2x}\, dx = 1$

5) Find arc length as well as area of surface generated by revolving about the line $y = -1$

$$y(x) = \frac{x^3}{3} + \frac{1}{4x} \qquad 1 \le x \le 3.$$

Ans $l(c) = \int_1^3 \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}}\, dx = \frac{53}{6}$

$$\text{area}(S) = 2\pi \int_\alpha^\beta \gamma(t) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

where $\gamma(t)$ is distance of $(x(t), y(t))$ from the line L and is hence $\dfrac{|a \cdot x(t) + b y(t) + c|}{\sqrt{a^2+b^2}}$

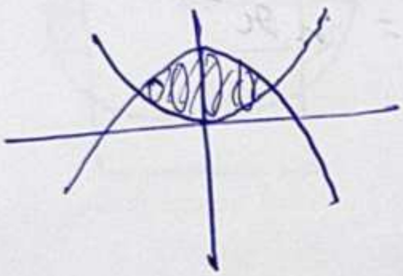
and $t \in [\alpha, \beta]$

Here, $ar(S) = 2\pi \int_1^3 (y-(-1)) \sqrt{1+(y')^2}\, dx$

$= \frac{1823\pi}{18}$

6) The cross sections of a solid by plane $\perp$ to x-axis are circles with diameters from $y = x^2$ to $y = 8 - x^2$. The solid lies b/w their points of intersection. Find the volume.

Ans



area of cross section $= 4 - x^2$

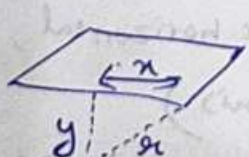
By Fubini's theorem,

$$Vol = \int_{-2}^{2} \pi (4-x^2)^2\, dx = \frac{512\pi}{15}$$

7) Find volume common to $x^2+y^2=a^2$, $y^2+z^2=a^2$

Ans see video on YT

the solid consists of planes that are squares!



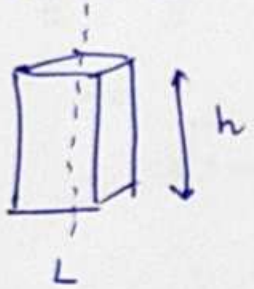
$\therefore x = \sqrt{r^2-y^2}$

$\therefore$ Area of a square $= 4(r^2-y^2)$

$\therefore$ volume $= \int_{-r}^{r} 4(r^2-y^2)\, dy = \frac{16}{3} r^3$

8) A fixed line L in 3-space and a square of side r in a plane $\perp$ to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution about L. Find the volume of the solid generated.

<u>Ans</u> By Fubini, the volume is same as the below solid



$$V = h r^2$$

9) Find vol of solid generated when region bounded by $y = 3 - x^2$, $y = -1$ is revolved about line $y = -1$ (by both washer & shell methods)

<u>Ans</u> Washer method (discs $\perp$ to x-axis):

Area of washer $= \pi (1+y)^2 = \pi (4 - x^2)^2$

$$\therefore V = \int_{-2}^{2} \pi (4 - x^2)^2 \, dx = \frac{512\pi}{15}$$

Shell method (cylinders are horizontal, i.e., common axis is parallel to x-axis)

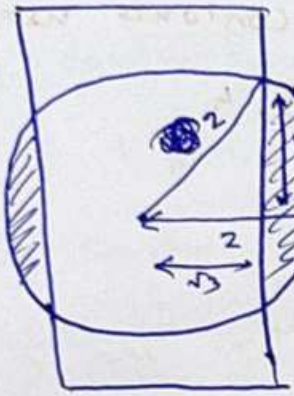
Area of shell $= 2\pi (y - (-1)) \cdot 2x$

$= 4\pi (1+y) \sqrt{3-y}$

$\therefore\ V = \int_{-1}^{3} 4\pi(1+y)\sqrt{3-y} = \frac{512\pi}{15}$

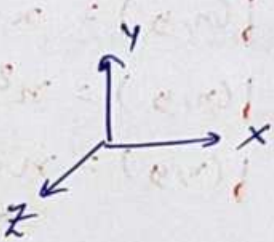
10) A round hole of radius $\sqrt{3}$ is bored through the centre of a solid ball of radius 2. Find vol. cut out

Ans



Sphere: $x^2+y^2+z^2=4$

hole: $x^2+z^2=3$



Vol. of shaded part by washer method

$$= \int_{-1}^{1} \pi\left(x^2-(\sqrt{3})^2\right) dy$$

$$= \int_{-1}^{1} \pi\left(4-y^2-3\right) dy = \frac{4\pi}{3}$$

$\therefore$ vol removed $= \frac{28\pi}{3}$ $\left(\because \text{total} = \frac{4}{3}\pi r^3\right)$ $(r=2)$

TUTORIAL 6

1) Find the natural domains of

(i) $\frac{xy}{x^2-y^2}$ (ii) $\ln(x^2+y^2)$

Ans (i) $\mathbb{R} \setminus \{(x,y) \mid x^2 = y^2\}$

$= \mathbb{R} \setminus \left(\{(x,x) \mid x \in \mathbb{R}\} \cup \{(x,-x) \mid x \in \mathbb{R}\}\right)$

(ii) $\mathbb{R} \setminus \{0\}$

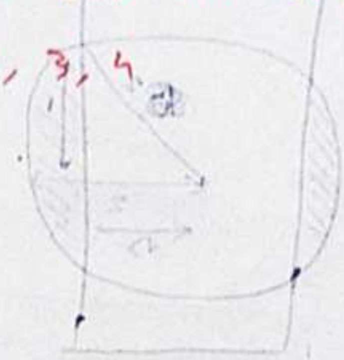
2) Describe level curves & contours using

$c = -3, -2, -1, 0, 1, 2, 3, 4.$

(i) $f(x,y) = x - y$

(ii) $f(x,y) = x^2 + y^2$

(iii) $f(x,y) = xy$



Ans For any c,

level curve is $\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

contour line is $\{(x,y,z) \in \mathbb{R}^3 \mid f(x,y) = c = z\}$

3) Examine continuity at $(0,0)$ taking $f(0,0) = 0$

(i) $\dfrac{x^3 y}{x^6 + y^2}$ (ii) $xy \, \dfrac{x^2 - y^2}{x^2 + y^2}$

(iii) $\big|\, |x| - |y| \,\big| - |x| - |y|$

Ans (i) Consider the sequences

$a_n = \left(\frac{1}{n}, \frac{1}{n^3}\right)$

$b_n = \left(\frac{1}{n}, \frac{2}{n^3}\right)$

$a_n, b_n \to 0$ as $n \to \infty$

but $f(a_n) \to \frac{1}{2}$, $f(b_n) \to \frac{1}{3}$ as $n \to \infty$

$\therefore$ discontinuous

(ii) $0 \le \left| xy \frac{x^2-y^2}{x^2+y^2} \right| \le |xy|$

By sandwich, it is continuous

(iii) $0 \le |f(x,y)| \le 2(|x|+|y|)$

By sandwich, it is continuous

4) If f, g are continuous, and $f, g : \mathbb{R} \to \mathbb{R}$, show that the following are cont

(i) $f(x) \pm g(y)$

(ii) $f(x)\, g(y)$

(iii) $\max\{f(x), g(y)\}$

(iv) $\min\{f(x), g(y)\}$

Ans (i) let $(x_n, y_n) \to (p, q)$; f, g are both continuous $\Rightarrow f(x_n) \to f(p)$, $g(y_n) \to g(q)$

$\therefore f(x_n) \pm g(y_n) \to f(p) \pm g(q)$

(ii) same as above

(iii) $\max\{a,b\} = \frac{a+b}{2} + \frac{|b-a|}{2}$

(iv) $\min\{a,b\} = \frac{a+b}{2} - \frac{|b-a|}{2}$

5) Let $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$.

Show that iterated limits exists and are both zero but $\lim_{x,y\to 0,0} f(x,y)$ doesnt exist

error?

Ans $\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} = 0$

$\therefore \lim_{x\to 0} \lim_{y\to 0} f(x,y) = \frac{1}{2}$

IIly $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \frac{1}{2}$

But taking $\left(\frac{1}{n}, \frac{1}{n}\right)$ as the seq., we get that it converges to 1 and taking $\left(\frac{1}{n}, 0\right)$ we get $1/2$

Hence $\lim_{xy\to 00}$ DNE

6) Examine the following for existence of partial derivatives at $(0,0)$, Take $f(0,0)=0$

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$

(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

Ans (i) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = 0$

IIly $f_y(0,0) = 0$

(ii) $f_x(0,0) = \lim_{h \to 0} \dfrac{\left(\frac{\sin^2 h}{|h|}\right)}{h}$ DNE

$|||^{ly}$ $f_y(0,0)$ DNE

7) $f(x,y) = (x^2+y^2) \sin \dfrac{1}{x^2+y^2}$, $f(0,0) = 0$

Show that f is cont at $(0,0)$ and p.d's exist but are not bounded in any disc around 0

Ans $0 \leq |f(x,y)| \leq |x^2+y^2|$

$\Rightarrow$ continuous at 0 by sandwich theorem

$f_x(a,b) = \lim_{h \to 0} \dfrac{f(a+h, b) - f(a,b)}{h}$

$= 2a\left(\sin\left(\dfrac{1}{a^2+b^2}\right) - \dfrac{1}{a^2+b^2}\cos\left(\dfrac{1}{a^2+b^2}\right)\right)$

$f_y(a,b) = 2b\left(\sin\left(\dfrac{1}{a^2+b^2}\right) - \dfrac{1}{a^2+b^2}\cos\left(\dfrac{1}{a^2+b^2}\right)\right)$

We show f_x isnt bounded around origin.

Approach 0 using $\left(\dfrac{1}{\sqrt{n\pi}}, 0\right)$

Then $\lim_{n \to \infty} \dfrac{2}{\sqrt{n\pi}}\left(\sin n\pi - n\pi \cos n\pi\right)$

$= \lim_{n \to \infty} \dfrac{2}{\sqrt{n\pi}}\left(-n\pi(\pm 1)\right)$ DNE

8) $f(0,0)=0$ and

$$f(x,y)=\begin{cases} x\sin\frac{1}{x}+y\sin\frac{1}{y} & x\neq 0,\ y\neq 0 \\ x\sin\frac{1}{x} & x\neq 0,\ y=0 \\ y\sin\frac{1}{y} & x=0,\ y\neq 0 \end{cases}$$

show that none of the p.d's exist at $(0,0)$ although f is cont at 0

Ans $0 \le |f(x,y)| \le |x|+|y|$

$\therefore$ continuous at $(0,0)$

$f_x(0,0)=\lim_{h\to 0}\sin\frac{1}{h}$ DNE

Illy $f_y(0,0)$ DNE

9) Examine existence of directional derivative and also differentiability at $(0,0)$. $f(0,0)=0$

(i) $xy\,\dfrac{x^2-y^2}{x^2+y^2}$

(ii) $\dfrac{x^3}{x^2+y^2}$

(iii) $(x^2+y^2)\sin\dfrac{1}{x^2+y^2}$

Ans (i) $D_{\vec{v}} f(0,0) = \lim_{h\to 0} \frac{f(ha, hb)}{h}$

$$= \lim_{h\to 0} \frac{h^2 ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0$$

$\therefore$ exists for every $\vec{v} = a\hat{i} + b\hat{j}$

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left|\frac{hk(h^2-k^2)}{(h^2+k^2)^{3/2}}\right|$$

Now $0 \leq \left|\frac{hk(h^2-k^2)}{(h^2+k^2)^{3/2}}\right| \leq \left|\frac{hk}{(h^2+k^2)^{3/2}}\right|$

$$\leq \sqrt{h^2+k^2}$$

$\therefore$ Total derivative at $(0,0) = 0$

(ii) $D_{\vec{v}} f(0,0) = \lim_{h\to 0} \frac{h^3 a^3}{h(h^2(a^2+b^2))} = a^3$

~~Tot~~ Total derivative limit is ,

$$\lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}} \quad \text{DNE}.$$

(Degree Nr = Degree Dr)

(iii) $$D_{\vec{v}}(a,b) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left(\frac{1}{h^2(a^2+b^2)}\right)}{h} = 0$$

Total derivative limit is

$$\lim_{h,k\to 0,0} \frac{(h^2+k^2)\sin\left(\frac{1}{h^2+k^2}\right)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^2+h^2}\right) = 0$$

(Sandwich)

10) Let $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$

Show that f is cont at $(0,0)$, $D_{\vec{v}} f(0,0)$ exists $\forall \vec{v}$, f is not diff at 0

Ans $0 \leq |f(x,y)| \leq \sqrt{x^2+y^2}$

$\therefore$ cont by sandwich

Let $\vec{v} = a\hat{i} + b\hat{j}$ with $b \neq 0$

$$D_{\vec{v}} f(0,0) = \lim_{h\to 0} \frac{1}{h} \frac{hb}{|hb|} \sqrt{h^2a^2+h^2b^2} = \frac{b\sqrt{a^2+b^2}}{|b|}$$

If $b=0$, $D_{\vec{v}} f(0,0) = 0$

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)|}{\sqrt{h^2+k^2}}$$

$$= \lim_{h,k\to 0,0} \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right|$$

DNE (deg Nr = deg Dr)

TUTORIAL 7

1) $F(x,y,z) = x^2+2xy+z^2-y^2$. Find ∇F at $(1,-1,3)$ and eqns of tangent plane and normal to surface $F(x,y,z)=7$ at the same point

Ans $\nabla F = (2x+2y,\ 2x-2y,\ 2z)$

$\nabla F \big|_{1,-1,3} = (0,4,6)$

DR's of normal to surface

Now use 3D geometry.

tangent plane : $2y+3z-7=0$

normal line : $\frac{x-1}{0} = \frac{y+1}{4} = \frac{z-3}{6}$

2) Find $D_{\vec{u}}\, F(2,2,1)$ with $F(x,y,z) = 3x - 5y + 2z$ and $\bar{u}$ is unit vector in direction of outward normal to $x^2+y^2+z^2 = 9$ at $(2,2,1)$

Ans

$$\bar{u} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\nabla F = (3, -5, 2)$$

$$D_{\bar{u}}\, F(2,2,1) = \nabla F\big|_{(2,2,1)} \cdot \vec{u}$$

$$= \frac{6}{3} - \frac{10}{3} + \frac{2}{3} = -\frac{2}{3}$$

3) $\sin(x+y) + \sin(y+z) = 1$, find $\frac{\partial^2 z}{\partial x \partial y}$ given $\cos(y+z) \neq 0$

Ans Partial wrt x :

$$\cos(x+y) + \cos(y+z)\frac{\partial z}{\partial x} = 0 \quad — (1)$$

Partial wrt y :

$$\cos(x+y) + \cos(y+z)\left(1 + \frac{\partial z}{\partial y}\right) = 0 \quad — (2)$$

Partial of (1) wrt y

$$-\sin(x+y) - \sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0$$

We have 3 eqns in 3 variables — (3)

so we solve for $\frac{\partial^2 z}{\partial x \partial y}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z) \frac{\cos^2(x+y)}{\cos^2(y+z)}$$

4) $f(0,0) = 0$, $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ for $(x,y) \neq (0,0)$,

Show that f_{xy}, f_{yx} exist at $(0,0)$ but they are not equal. Are they continuous?

Ans

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

Now $f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = -k$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1$$

Similarly $f_{yx}(0,0) = 1$

For $(x,y) \neq 0$,

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Illy $$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Both are not cont. at $(0,0)$ (since deg Nr = deg Dr, you can do the $\left(\frac{1}{n}, \frac{m}{n}\right)$ seq approach)

5) Show that there is local minima at the indicated point

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7$ @ $(-1,2)$

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$ @ $(0,0)$

Ans (i) $f_x(-1,2) = 0 = f_y(-1,2)$

$H(-1,2) = \begin{pmatrix} 12 & 0 \\ 0 & 48 \end{pmatrix}$

$\det(H) > 0$ with $12 > 0$

$\therefore$ minima

(ii) $f_x(0,0) = 0 = f_y(0,0)$

$H(0,0) = \begin{pmatrix} 6 & -2 \\ -2 & 10 \end{pmatrix}$

$\det(H) > 0$ with $6 > 0$

$\therefore$ minima

6) Analyse for minima/maxima/saddle

(i) $f(x,y) = (x^2-y^2)\cdot e^{-\left(\frac{x^2+y^2}{2}\right)}$

(ii) $f(x,y) = x^3 - 3xy^2$

Ans (i) $f_x = e^{-\frac{(x^2+y^2)}{2}}(2x - x^3 + xy^2)$

$f_y = e^{-\frac{(x^2+y^2)}{2}}(-2y + y^3 - x^2y)$

critical points : $(0,0), (\pm\sqrt{2}, 0), (0, \pm\sqrt{2})$

$H(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ saddle

$H(\pm\sqrt{2},0) = \begin{pmatrix} -\frac{4}{e} & 0 \\ 0 & -\frac{4}{e} \end{pmatrix} \Rightarrow$ Maxima

$H(0, \pm\sqrt{2}) = \begin{pmatrix} \frac{4}{e} & 0 \\ 0 & \frac{4}{e} \end{pmatrix} \Rightarrow$ minima

(ii) $f_x = 3x^2 - 3y^2$

$f_y = -6xy$

$\therefore (0,0)$ is only critical point

$H(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

no conclusions can be drawn.

However $f(\delta, 0) = \delta^3$

$\therefore$ rises on one side, falls on other

$\therefore$ saddle.

7) Find absolute minima and maxima of

$f(x,y) = (x^2 - 4x)\cos y$

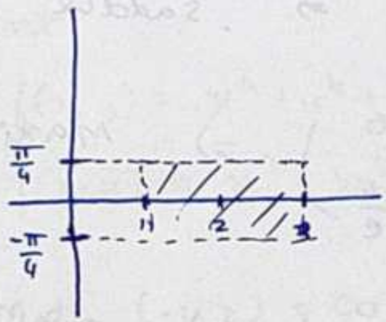
for $1 \le x \le 3$

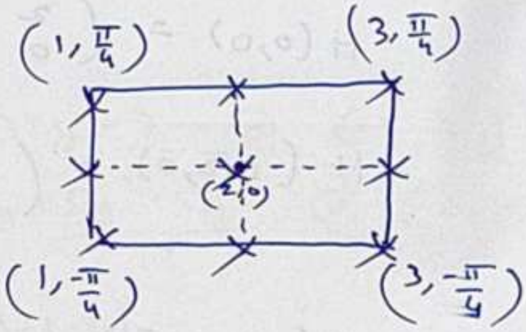
$-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

Ans $f_x = (2x-4)\cos y$

$f_y = -(x^2-4x)\sin y$

$\therefore (2,0)$ is the only critical point





Check at all 9 points and see max/min.

8) $T(x,y,z) = 400xyz$. Maximize subject to

$x^2+y^2+z^2 = 1$

Ans $\nabla f = \lambda \nabla g$

$(400yz, 400zx, 400xy) = \lambda \cdot (2x, 2y, 2z)$

$\therefore 400xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

If $\lambda \neq 0$, then $x = \pm y$, $y = \pm z$, $z = \pm x$

But $x^2+y^2+z^2=1$

$\therefore$ critical points are $\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$

$\uparrow$

8 points.

maxima is obviously $\dfrac{400}{(\sqrt{3})^3}$

if $\lambda=0$, $xy=yz=zx=0$

But $x^2+y^2+z^2=1$

$\therefore (\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$ are

critical points

$\therefore$ max $=400$

$\therefore$ overall max $=400$

9) Maximize xyz subject to $x+y+z=40$, $y+x=z$

Ans $x+y+z=40$

$y+x=z$

$\therefore z=20$

$\therefore$ we maximize $20xy$ subject to $x+y=20$

$x+y \geq 2\sqrt{xy}$

$\therefore (xy)_{max} = \left(\dfrac{20}{2}\right)^2 = 100$

(at $x=y=10$)

Using lagrange multipliers,

$\nabla(f + \lambda g_1 + \mu g_2) = 0$

$\therefore (yz + \lambda + \mu,\ zx + \lambda + \mu,\ xy + \lambda - \mu) = 0$

$\therefore\ yz = zx$ but $z \neq 0$ since $z = 20$

$\therefore x = y$

$g_1(x, y, z) = x + y + z - 40$

$g_2(x, y, z) = x + y - z$

$g_1 = g_2 = 0$ (constraint is given)

But $x = y$

$\therefore$ we get $x = y = 10$

(10) minimize $x^2 + y^2 + z^2$ subject to

$$\begin{cases} x + 2y + 3z = 6 \\ x + 3y + 4z = 9 \end{cases}$$

Ans $\nabla(f + \lambda g_1 + \mu g_2) = 0$ —— (1)

$$\left.\begin{aligned} \therefore\ 2x + \lambda + \mu &= 0 \\ 2y + 2\lambda + 3\mu &= 0 \\ 2z + 3\lambda + 4\mu &= 0 \end{aligned}\right\} \text{from (1)}$$

$$\left.\begin{aligned} x + 2y + 3z - 6 &= 0 \\ x + 3y + 4z - 9 &= 0 \end{aligned}\right\} \begin{matrix} g_1 \\ g_2 \end{matrix}$$

solving,

$\lambda = 10$, $\mu = -8$, $x = -1$, $y = 2$, $z = 1$

$\therefore f(-1,2,1) = 6$ is minimum

(why minimum & not maximum ? Find $f(0,0,0)$)