Theory of Automata & Formal Languages

(Theory of Computation)

Compiled By

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Languages

Languages

- A language is a set of strings
- String: A sequence of letters
  - Examples: "cat", "dog", "house", …
  - Defined over an alphabet:

$$\Sigma = \{a, b, c, \ldots, z\}$$

Alphabets and Strings

- We will use small alphabets: $\Sigma = \{a, b\}$

- Strings

$$a$$
$$ab$$
$$abba$$
$$baba$$
$$aaabbbaabab$$

$$u = ab$$
$$v = bbbaaa$$
$$w = abba$$

String Operations (Concatenation)

$$w = a_1 a_2 \cdots a_n$$

$$v = b_1 b_2 \cdots b_m$$

$abba$

$bbbaaa$

$$wv = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$$

$abbabbbaaa$

String Operations (Reverse)

$$w = a_1 a_2 \cdots a_n$$

$$ababaaabbb$$

$$w^R = a_n \cdots a_2 a_1$$

$$bbbaaababa$$

String Length

$$w = a_1 a_2 \cdots a_n$$

- Length:

$$|w| = n$$

- Examples:

$$|abba| = 4$$

$$|aa| = 2$$

$$|a| = 1$$

Length of Concatenation

$$|uv| = |u| + |v|$$

- Example:

$$u = aab, \quad |u| = 3$$

$$v = abaab, \quad |v| = 5$$

$$|uv| = |aababaab| = 8$$

$$|uv| = |u| + |v| = 3 + 5 = 8$$

Empty String

- A string with no letters:

- Observations:

$$|\lambda| = 0$$

Null (λ)

$$\lambda w = w\lambda = w$$

$$\lambda abba = abba\lambda = abba$$

Substring

- Substring of string:
  - a subsequence of consecutive characters
-

String	Substring
$abbab$	ab
$abbab$	$abba$
$abbab$	b
$abbab$	$bbab$

Prefix and Suffix

$abbab$

- Prefixes

λ

a

ab

abb

$abba$

$abbab$

Suffixes

$abbab$

$bbab$

bab

ab

b

λ

$w = uv$

prefix

suffix

Another Operation

$$w^n = \underbrace{ww \cdots w}_{n}$$

$$(abba)^2 = abbaabba$$

- Example:

$$w^0 = \lambda$$

- Definition:

$$(abba)^0 = \lambda$$

The * (Closure)Operation

- Σ^*: the set of all possible strings from alphabet Σ

$$\Sigma^1 = \{a, b\} \qquad \Sigma^0 = \{\lambda\} \qquad \Sigma^2 = \{a,b\} \cdot \{a,b\}$$

$$= \{aa, ab, ba, bb\}$$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^\infty$$

The + (Positive Closure)Operation

Σ^+ : the set of all possible strings from alphabet Σ except λ

$$\Sigma = \{a, b\}$$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^\infty$$

$$\Sigma^+ = \Sigma^* - \lambda$$

$$\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Language

- A language is any subset of Σ^*

- Example: $\Sigma = \{a, b\}$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, \ldots\}$$

- Languages: $\{\lambda\}$ = L_1

$\{a, aa, aab\}$ – L_2

Finite Languages

$\{\lambda, abba, baba, aa, ab, aaaaaa\}$

Another Example

$$L = \{a^n b^n : n \geq 0\}$$

- An infinite language

$\Sigma = \{a, b\}$

$L \subseteq \Sigma^*$

$\left.\begin{array}{l} \lambda \\ ab \\ aabb \\ aaaaabbbbb \end{array}\right\} \in L$

$abb \notin L$

$n=0 \quad a^0 b^0 = \lambda$

$n=1 \quad a^1 b^1 = ab$

$n=2 \quad a^2 b^2 = aabb$

Operations on Languages

- The usual set operations

$$\{a, ab, aaaa\} \cup \{bb, ab\} = \{a, ab, bb, aaaa\}$$

$$\{a, ab, aaaa\} \cap \{bb, ab\} = \{ab\}$$

$$\{a, ab, aaaa\} - \{bb, ab\} = \{a, aaaa\}$$

- Complement: $\overline{L} = \Sigma^* - L$ $(L' = \overline{L})$

$$\overline{\{a, ba\}} = \{\lambda, b, aa, ab, bb, aaa, \ldots\}$$

Finite ↑ ↳ Infinite

Reverse

$$L^R = \{w^R : w \in L\}$$

- Definition:

$$\{ab, aab, baba\}^R = \{ba, baa, abab\}$$

- Examples:

$$L = \{a^n b^n : n \geq 0\}$$

$$L^R = \{b^n a^n : n \geq 0\}$$

Concatenation

$$L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$$

- Definition:

- Example:

$$\underbrace{\{a, ab, ba\}}_{L_1}\underbrace{\{b, aa\}}_{L_2} \qquad |L_1 \cdot L_2| = 6$$

$$= \{ab, aaa, abb, abaa, bab, baaa\}$$

Another Operation

- Definition: $$L^n = \underbrace{LL \cdots L}_{n}$$

$$\{a,b\}^3 = \{a,b\}\{a,b\}\{a,b\} =$$
$$\{aaa, aab, aba, abb, baa, bab, bba, bbb\}$$

- Special case: $$L^0 = \{\lambda\}$$

$$\{a, bba, aaa\}^0 = \{\lambda\}$$

More Examples

- $L = \{a^n b^n : n \geq 0\}$

$L^2 = \{\lambda, ab, aabb, aaabbb\}\{\lambda, ab, aabb, aaabbb\}$

$$L^2 = \{a^n b^n a^m b^m : n, m \geq 0\}$$

$L^2 = \{\lambda, ab, aabb, aaabbb, abab, abaabb, abaaabbb$

$$aabbaaabbb \in L^2$$

$L^3 = \{a^n b^n a^m b^m a^p b^p, m, n, p \geq 0\}$

Star-Closure (Kleene *)

$$L^* = L^0 \bigcup L^1 \bigcup L^2 \cdots$$

- Definition:
- Example:

$$\{a,bb\}^* = \left\{ \begin{array}{l} \lambda, \longrightarrow L^0 \\ a, bb, \longrightarrow L^1 \\ aa, abb, bba, bbbb, \longrightarrow L^2 \\ aaa, aabb, abba, abbbb, \ldots \end{array} \right\}$$

$L = L^1$

Positive Closure

- Definition:

$$L^{+} = L^{1} \cup L^{2} \cup \cdots$$
$$= L^{*} - \{\lambda\}$$

$$\{a,bb\}^{+} = \left\{\begin{array}{l} a,bb, \quad \to L_1 \\ aa,abb,bba,bbbb, \to L_2 \\ aaa,aabb,abba,abbbb,\ldots \end{array}\right\}$$

$L = L^2$

Examples

- Let L be a language. Under which circumstances L^+ is equal to L^* ?

(Assume L is not equal to $\{\Lambda\}$)

$\Sigma = \{a, b\}$ $\Sigma^* = \{\Lambda, a, b, aa, ab, ba, bb, \ldots\}$

$L = \{\Lambda, a\}$

$L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$

$= \{\Lambda, a, aa, \ldots\}$

$L^+ = L^1 \cup L^2 \cup L^3 \cup \ldots$

$= \{\Lambda, a, aa, aaa, \ldots\}$

$\boxed{L^+ = L^*}$

Examples

- Let L be a language. Under which circumstances L^+ is equal to L^* ? $(L = L^+ = L^*)$
- $L^+ = L^* - \{\lambda\}$
- If L contains 'λ' in it then it will be part of L^1 and so it will be part of L
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- L={x belongs to{a, b}*| length of x is even}

$L = \{\lambda, aa, ab, ba, bb, aaaa, aaab \ldots\}$