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# The Traveling Salesman Problem:

→ We are given a complete undirected graph $G=\langle V,E\rangle$ that has a non-negative integer cost $c(u,v)$ associated with each edge $(u,v)\in E$

→ We must find a Hamiltonian cycle of $G$ with minimum cost.

→ $c(A)$: Total cost of the edges in the Subset $A\subseteq E$.

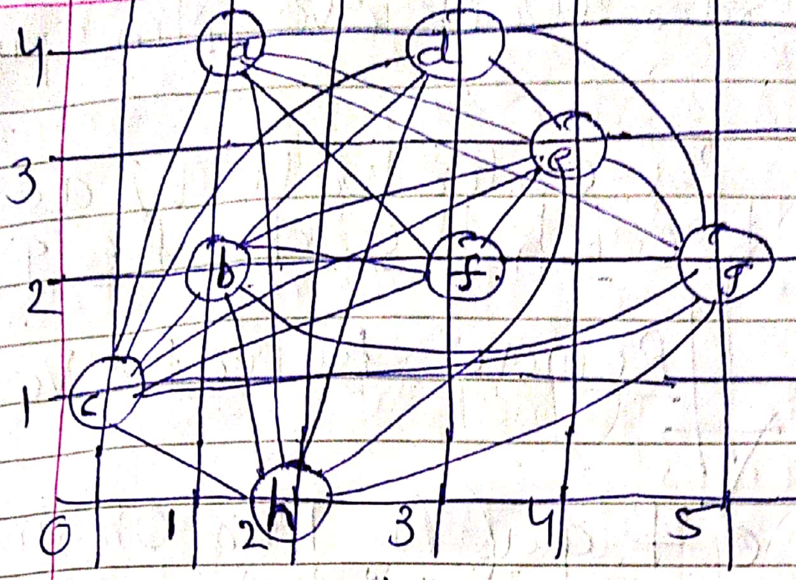
$$c(A)=\sum_{(u,v)\in A} c(u,v)$$

→ Least costly way to go from a place $u$ to a place $w$ is to go directly, with no intermediate steps.

→ We formalize this notion by saying that the cost function $c$ satisfies the Triangle Inequality if, for all vertices $u,v,w\in V$

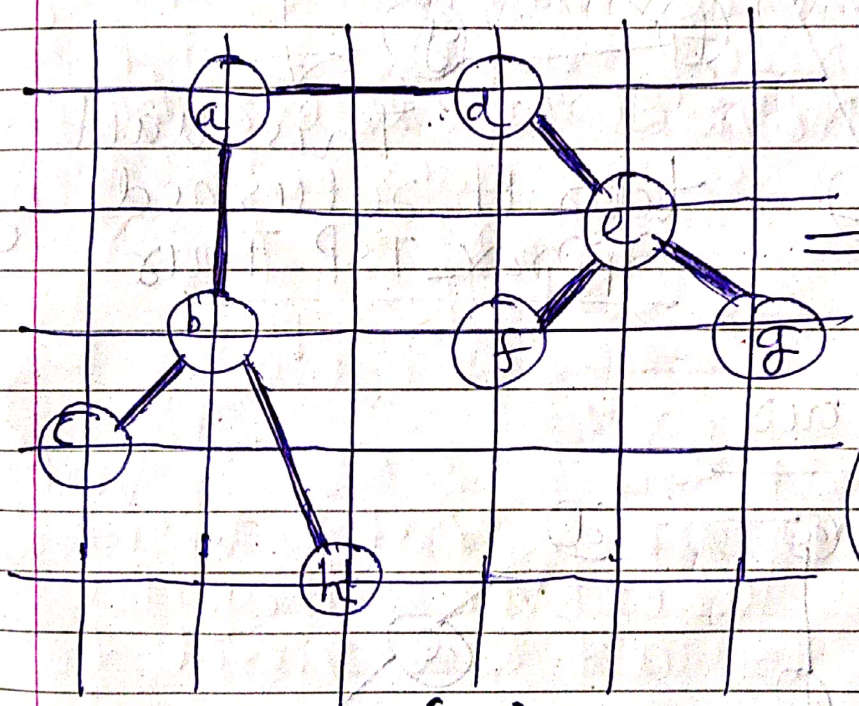
$$c(u,w)\leq c(u,v)+c(v,w)$$

→ If the vertices of the graph are points in the plane and the cost of traveling between two vertices is the ordinary euclidean distance between them, then the triangular inequality is satisfied.



→ Complete Graph
→ Distance between them is Euclidean distance between points

↓ Minimum Spanning Tree



MST=(T)

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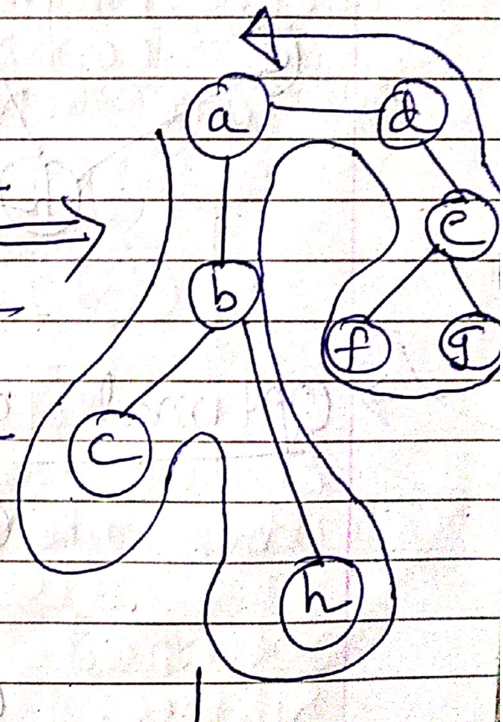

Walk on T

Preorder walk on T

a b c b h b a d e f e g e d a

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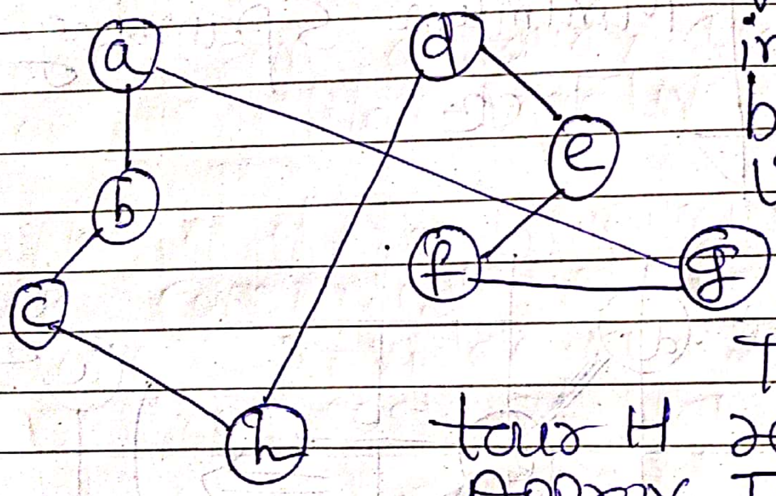
→ Apply Pre-Order Walk over Min Spanning Tree T

→ We will get Pre-Order Walk W as
a b c b h b a d e f e g e d a

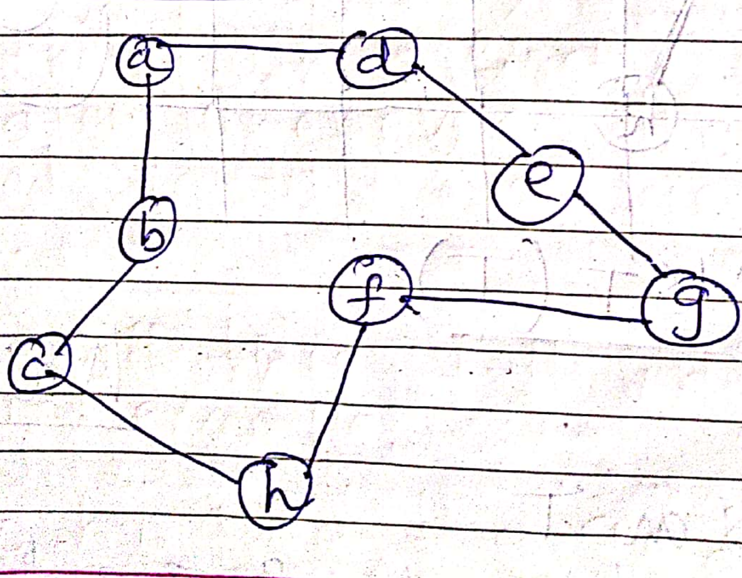
Remove Repeated Vertices

→ a b c h d e f g a (A Tour obtained by visiting the vertices in the order given by the pre-order walk)



This walk is tour H returned by Approx_TSP_Tour

→ Optimal Tour



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APPROX_TSP_TOUR (G, C)
{
1. Select a vertex $r \in G.V$ to be a root Vertex

2. Compute a Minimum Spanning Tree T for G from root r using

MST_PRIM(G, C, r)

3. Let H be a list of vertices, ordered according to when they are first visited in a preorder tree walk of T.

4. return the Hamiltonian cycle H
}

▷ MST_PRIM(G, C, r) is having $O(|V|^2)$ Running time i.e. polytime

i.e. Above algo. can run in polynomial time.

**Theorem:** Approx_TSP Tour is a polynomial-time 2-approximation algorithm for the TSP problem with the Triangle Inequality

**Proof:** We have already seen that Approx_TSP Tour runs in polynomial time.

Let $H^*$ denote optimal Tour for the given set of vertices

We obtain spanning Tree by deleting any edge from a tour, and each edge cost is Non-Negative.

$\therefore$ Weight of the Minimum Spanning Tree T computed in Line 2 of Approx_TSP Tour provides a lower bound on the cost of an optimal tour.

$\therefore \boxed{C(T) \leq C(H^*)} \longrightarrow (1)$

A full walk of T lists the vertices when they are first visited and also whenever they are returned to after a visit to a subtree.

Let us call this full walk W.
In our example,

a b c b h b a d e f e g e d a

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→ Since the full Walk traverses every edge of T exactly twice, we have

$$\boxed{C(W) = 2\,C(T)} \longrightarrow (2)$$

Now from (1) and (2)

$$C(W) = 2\,C(T)$$

$$\leq 2\,C(H^*)$$

$$\therefore \boxed{C(W) \leq 2\,C(H^*)} \longrightarrow (3)$$

→ Full Walk W is generally not a Tour, since it visits some vertices more than once.

→ By Triangle Inequality, we can delete a visit to any vertex from W and the cost doesn't increase

→ If we delete vertex v from W between visits to u and w, the resulting Ordering specifies going directly from u to w.

→ By repeatedly applying this operation, we can remove from W all but the first visit to each vertex.

∴ a b c h d e f g

This Ordering is same as that obtained by Pre-Order walk of the Tree T.

→ Let H be the cycle corresponding to this preorder walk (after deletion)

→ It is a hamiltonian cycle, since every vertex is visited exactly once

→ Since H is obtained by deleting vertices from walk W

$$\boxed{c(H) \leq c(W)} \longrightarrow (4)$$

→ Combining (3) and (4), we will get

$$c(H) \leq c(W) \leq 2c(H^*)$$

$$\Rightarrow c(H) \leq 2c(H^*)$$

$$\Rightarrow \boxed{\frac{c(H)}{c(H^*)} \leq 2}$$

i.e. Approx. Ratio is $\boxed{2}$

Which Completes the Proof

Note: This is based on Triangular Inequality

i.e.

$$d(u,v) \leq \underbrace{d(u,w) + d(w,v)}_{\text{via}}$$

direct

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★ We will show that Without the Triangle Inequality, a polynomial-time approx. algorithm with a constant approximation ratio does not exist unless P=NP.

Th: If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time algorithm with approximation ratio $\rho$ for the general TSP problem

Proof: The proof is by Contradiction.

→ Suppose to the contrary that for some number $\rho \geq 1$, there is a Polynomial time approx. algorithm A with approx. ratio $\rho$.

→ Without Loss of Generality, we assume that $\rho$ is an integer (by rounding if required)

→ We shall then show how to use A to solve instances of the Hamiltonian cycle problem in polynomial time.

We know that Hamiltonian cycle problem is NP-Complete and if we can solve it in polynomial time, then P=NP.

Let $G = \langle V, E \rangle$ be an instance of the Hamiltonian cycle problem.

→ We wish to determine efficiently whether G contains a Hamiltonian cycle by making use of the hypothesized approximation algo A.

→ We turn G into an instance of TSP as follows.

Let $G' = (V, E')$ be the complete graph on V; that is,

$$E' = \{(u, v) : u, v \in V \text{ and } u \neq v\}$$

Assign an integer cost to each edge in $E'$ as follows:

$$c(u,v) = \begin{cases} 1 & , \text{if } (u,v) \in E \\ S|V| + 1 & , \text{otherwise} \end{cases}$$

Number of Vertices

We can create representation of $G'$ and c from a representation of G in time polynomial in $|V|$ and $|E|$

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→ Now Consider the TSP problem $(G', C)$.

→ If the original graph $G$ has a Hamiltonian Cycle $H$, then cost function assigns to each edge of $H$ a cost of 1.

$\therefore$ $(G', C)$ contains a tour of cost $|V|$.

→ If $G$ doesn't contain a Hamiltonian Cycle, then any tour of $G'$ must use some edge not in $E$.

→ Any Tour not using edge in $E$ has a cost of atleast

$$\underbrace{\rho|V|+1}_{\text{One edge not in } E \text{ has cost}} + \underbrace{(|V|-1)}_{(|V|-1) \text{ edges}} * \underbrace{1}_{\text{cost } 1}$$

**Note**: Remember Tour (Hamiltonian) has exactly $|V|$ edges.

$\therefore \rho|V| \cancel{+1} + |V| \cancel{-1}$

$= \rho|V| + |V|$

$> \rho|V|$

Because edges not in $G$ are so costly, there is a gap of atleast $\rho|V|$ between the cost of a tour that is a hamiltonian cycle in $G$ (cost $|V|$) and the cost of any other Tour (cost at least $\rho|V|+|V|$).

∴ The cost of a Tour that is not Hamiltonian cycle in $G$ is atleast a factor $\rho+1$ greater than the cost of a Tour that is a hamiltonian cycle in $G$.

$$\text{i.e. } \underset{\text{atleast}}{\Sigma} \quad \frac{\rho|V|+|V|}{|V|} = \frac{|V|(\rho+1)}{|V|}$$

$$= \rho+1$$

→ Now suppose we apply approx. algo. A to the TSP problem $(G', c)$.

→ Because A is guaranteed to return a tour of cost no more than $\rho$ times the cost of an optimal tour, if $G$ contains a

Hamiltonian Cycle, then A must return it.

→ If G has no hamiltonian cycle, then A returns a tour of cost more than $\rho|V|$.

∴ We can solve the Hamiltonian cycle problem by using A in polynomial time. ∎