Theory of Automata & Formal Languages

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Outline

- Properties of Context-Free Languages

    - ❖ Closure, Union and Concatenation of Context-Free Languages

    - ❖ Intersection of Context-Free Languages

    - ❖ Intersection of Context-Free Language with Regular Language

Recall

- If L is a Context Free Language, then L* is also a Context-Free Language
- L={a}

  $S \rightarrow a$
- $L^* = \{a^k \mid k >= 0\}$

  $S \rightarrow aS \mid \varepsilon$

Recall

- If L_{21} and L_{22} are Context-Free Languages then L_{21} U L_{22} is also a Context Free Language
- $L_{21}=\{a^i b^j \mid i=j\}$

  $S_{21} \rightarrow aS_{21}b \mid \varepsilon$
- $L_{22}=\{b^j c^k \mid j=k\}$

  $S_{21} \rightarrow bS_{21}c \mid \varepsilon$
- $L_{21} U L_{22}= \{a^i b^j c^k \mid i=j \text{ or } j=k\}$

  $S \rightarrow S_{21} \mid S_{22}$

Recall

- If L_{11} and L_{12} are Context-Free Languages then $L_{11}L_{12}$ is also a Context Free Language
- $L_{11}=\{a^i b^j | i=j\}$

  $S_{11} \rightarrow aS_{11}b | \varepsilon$
- $L_{12}=\{c^k | \ k >=0\}$

  $S_{12} \rightarrow cS_{12} \ | \ \varepsilon$

$L_{11}L_{12} = \{a^i b^j c^k \ | \ i=j \ , k >=0\} = \{a^i b^j c^k \ | \ i=j \ \}$

$S_1 \rightarrow S_{11}S_{12}$

Theorem

- Intersection of two Context-Free Languages is not always a Context-Free Language

Using a suitable example, we will prove the above statement.

Theorem

- Intersection of two Context-Free Languages is not always a Context-Free Language

$L_1 = \{a^i b^j c^k \mid i=j\}$

$L_2 = \{a^i b^j c^k \mid j=k\}$

$L = L_1 \cap L_2 = \{a^i b^j c^k \mid i=j\} \cap \{a^i b^j c^k \mid j=k\}$

$= \{a^i b^j c^k \mid i=j \text{ and } j=k\}$

We can write CFGs for both L1 and L2 and also construct PDA individually for both languages.

But, We can not construct PDA which can compare/match 'b's with both 'a' and 'c' at once.

One More Example

Intersection of two Context-Free Languages is not always a Context-Free Language

Proof. • $L_1 = \{a^i b^i c^j \mid i, j \geq 0\}$ is a CFL

– Generated by a grammar with rules $S \to XY$; $X \to aXb|\epsilon$; $Y \to cY|\epsilon$.

• $L_2 = \{a^i b^j c^j \mid i, j \geq 0\}$ is a CFL.

– Generated by a grammar with rules $S \to XY$; $X \to aX|\epsilon$; $Y \to bYc|\epsilon$.

• But $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$ is not a CFL.

Here, it is a Context-Sensitive Language

What is the intersection of given three Context- Free Languages ?

$$L_1 = \{a^i b^j c^k \mid i \le j\}$$

$$L_2 = \{a^i b^j c^k \mid j \le k\}$$

$$L_3 = \{a^i b^j c^k \mid k \le i\}$$

Example-1

- There are also some examples for which intersection of two CFLs is also a CFL.

- $L_1=\{a^n b^m \mid n >= m\}$

  Strings: a, ab, aab, aaab, aabb, aaabb
- $L_2=\{a^n b^m \mid m >= n\}$

  Strings : b, ab, abb, aabb, abbb, aabbb

$L_1 \cap L_2 = \{a^n b^m \mid n = m\}$

Example-2

- There are also some examples for which intersection of two CFLs is also a CFL.

- $L_1=\{x \in \{a,b\}^* \mid x \text{ is a Palindrome}\}$

- $L_2=\{x \in \{a,b\}^* \mid x \text{ is an odd length Palindrome}\}$

$L_1 \cap L_2 = \{x \in \{a,b\}^* \mid x \text{ is an odd length Palindrome}\}$

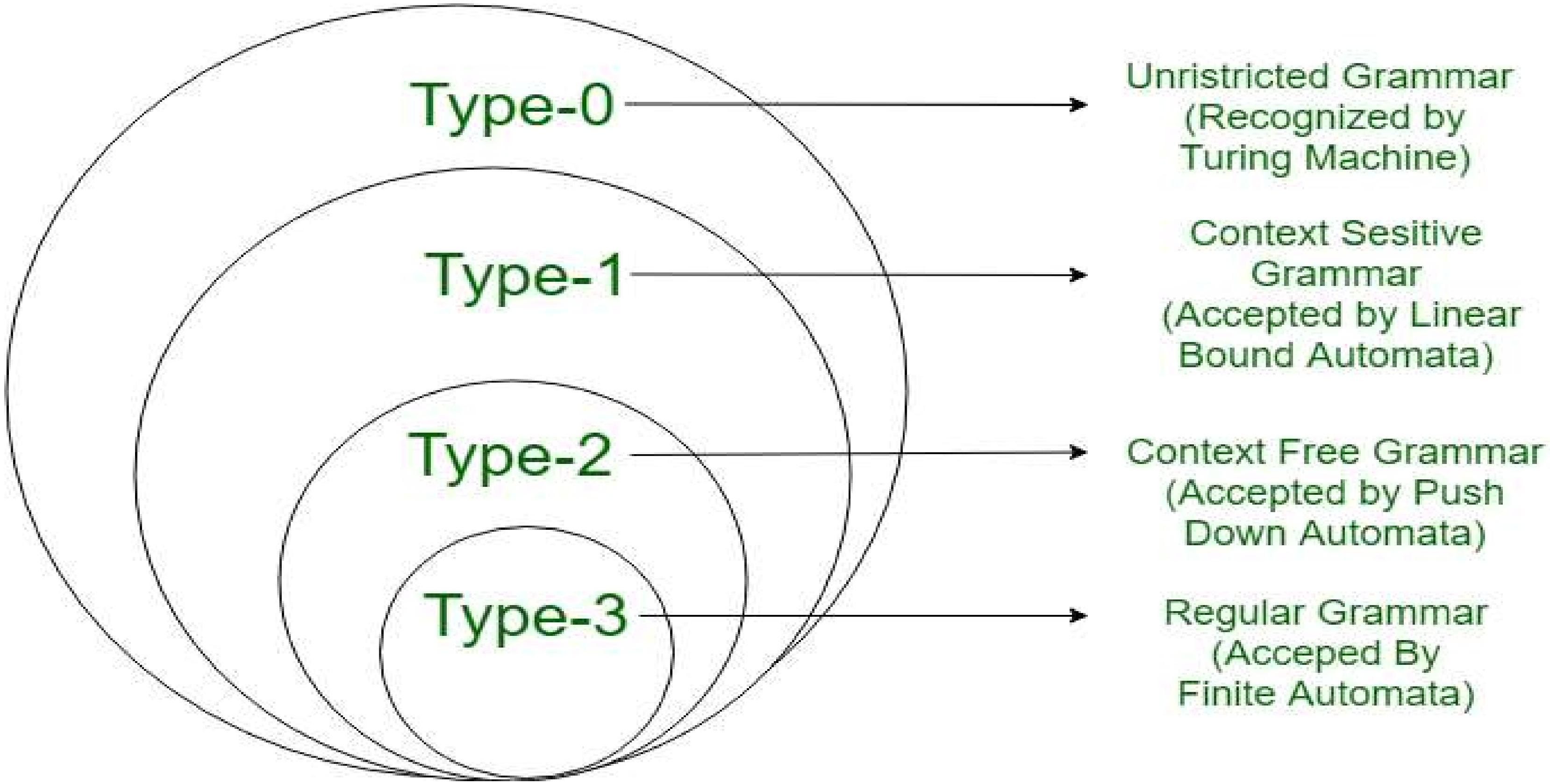
Intersection
of
Context-Free Language
with
Regular Language

Remember

- Intersection of CFL with a Regular Language is always a CFL
- $L_1=\{a^n b^m | n>=0, m>=0\}$

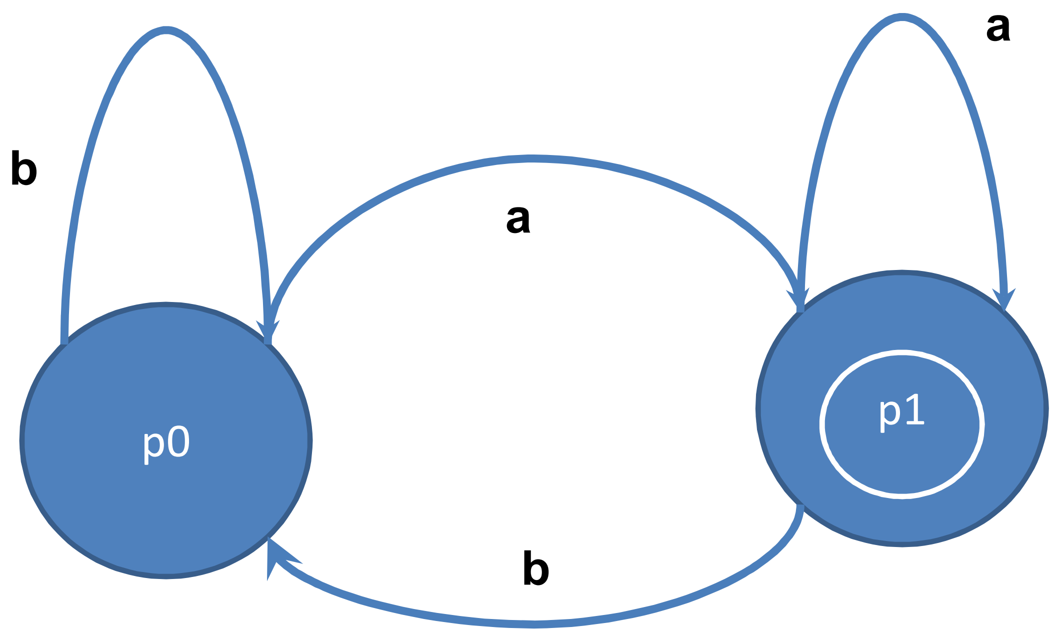
  Here, L_1 is representing Regular Expression a^*b^*
- $L_2=\{a^n b^n \mid n>=0\}$ is a Context-Free Language
- $L_1 \cap L_2$ is clearly L_2 which is a CFL
- Here, $L_1 \cap L_2 = L_2$ is just a coincidence

Chomsky Hierarchy



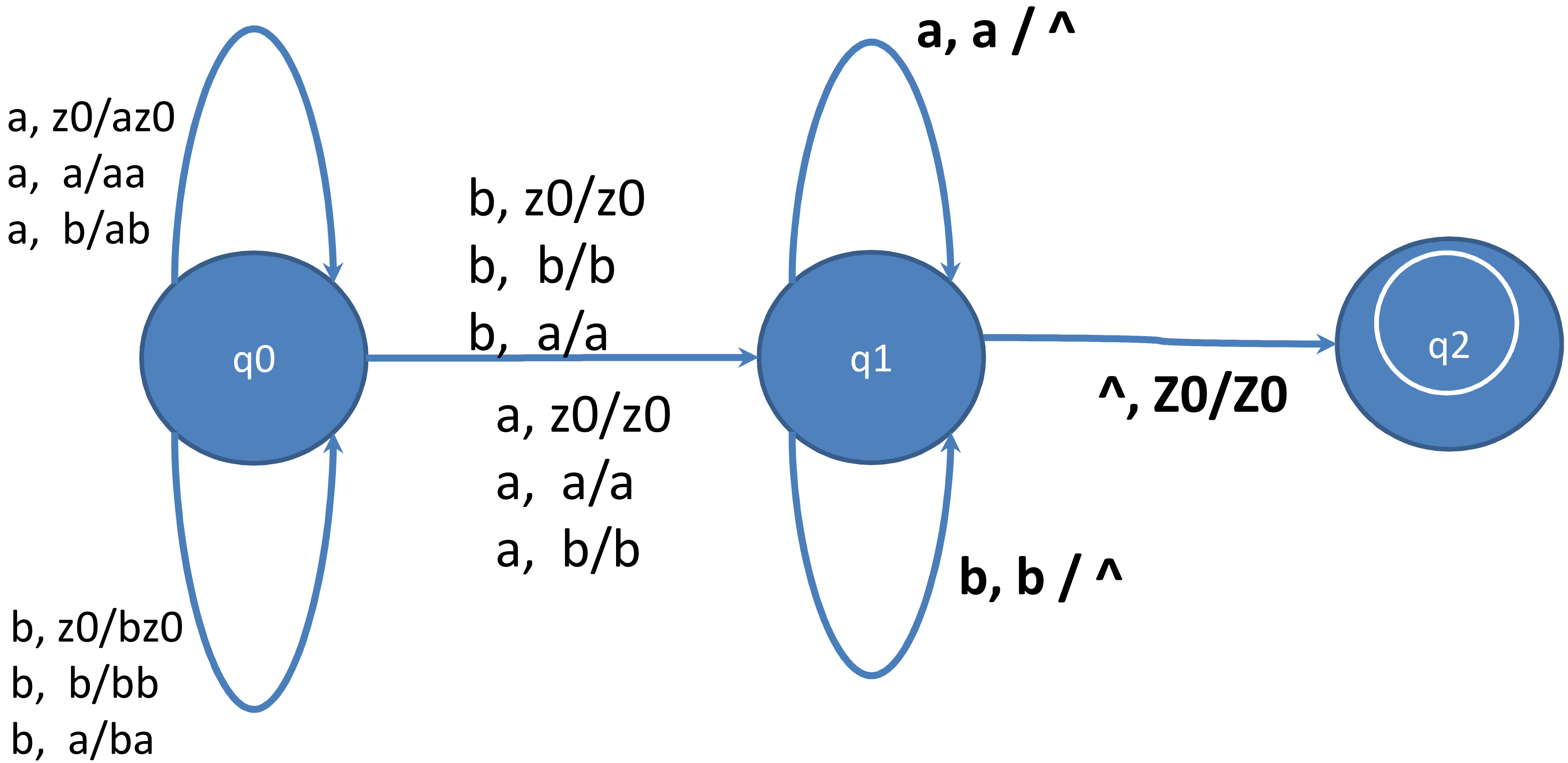
All Regular Languages are also Context-Free Languages

Deterministic Finite Automata

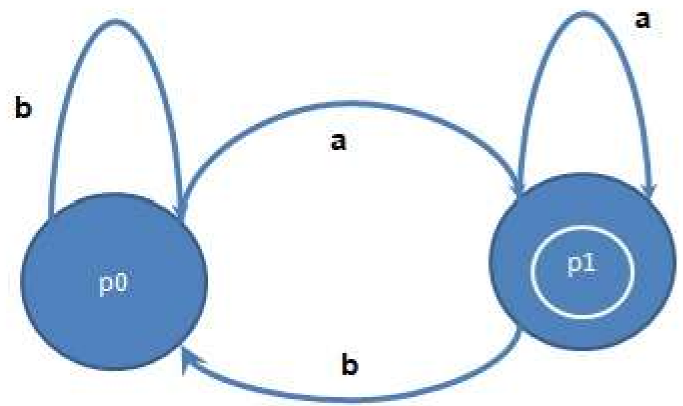


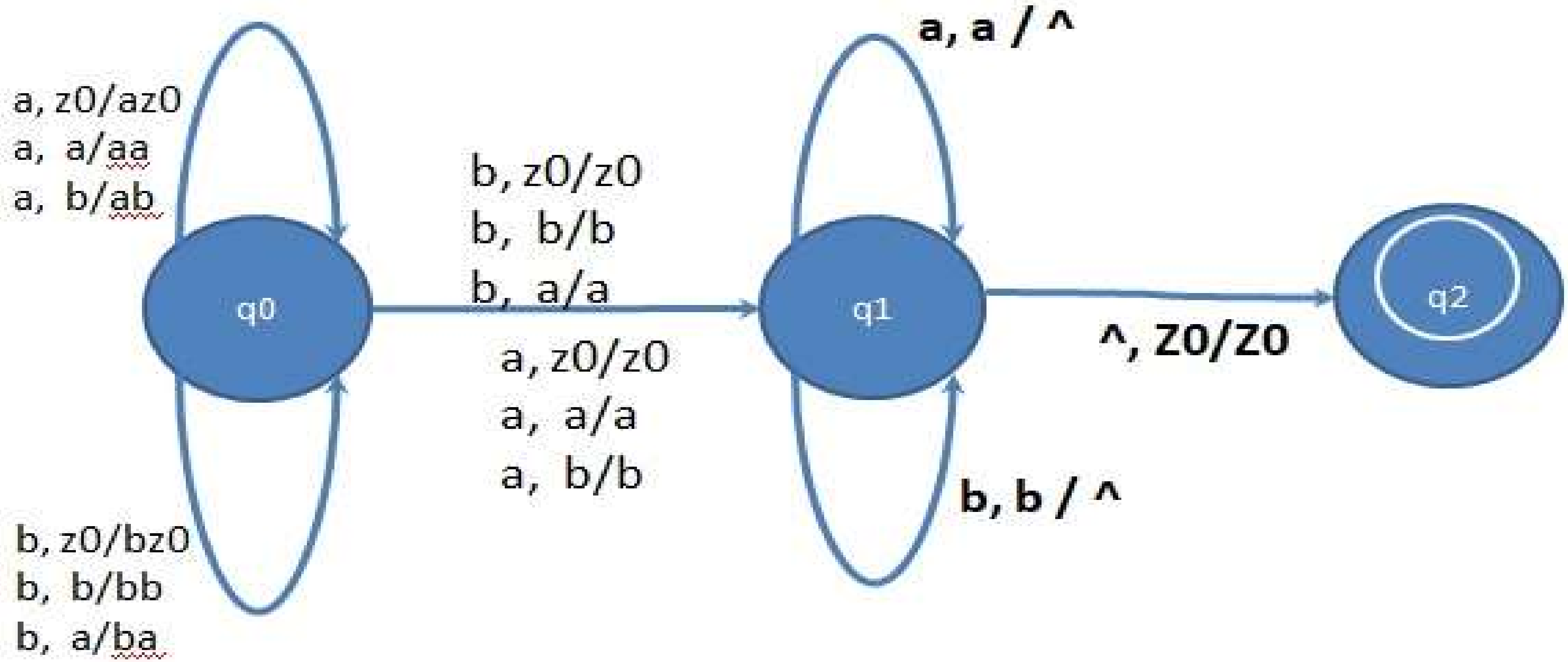
L ={x ϵ {a , b}* | x ends with a}

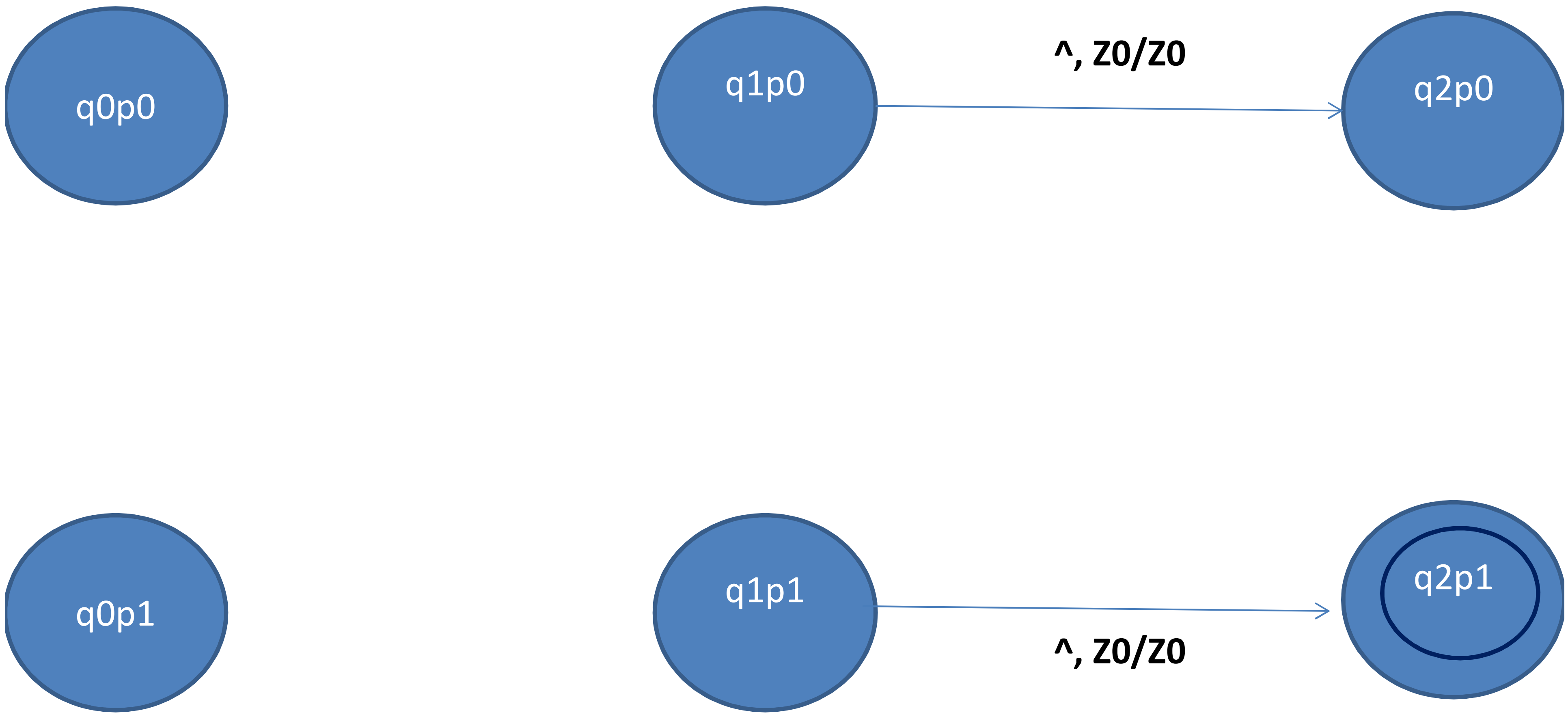
Push Down Automata

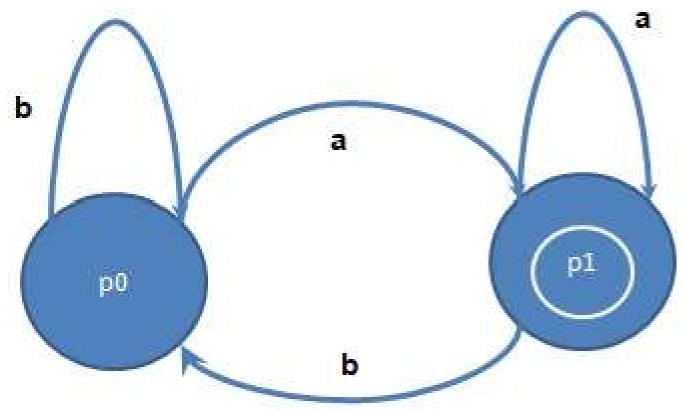


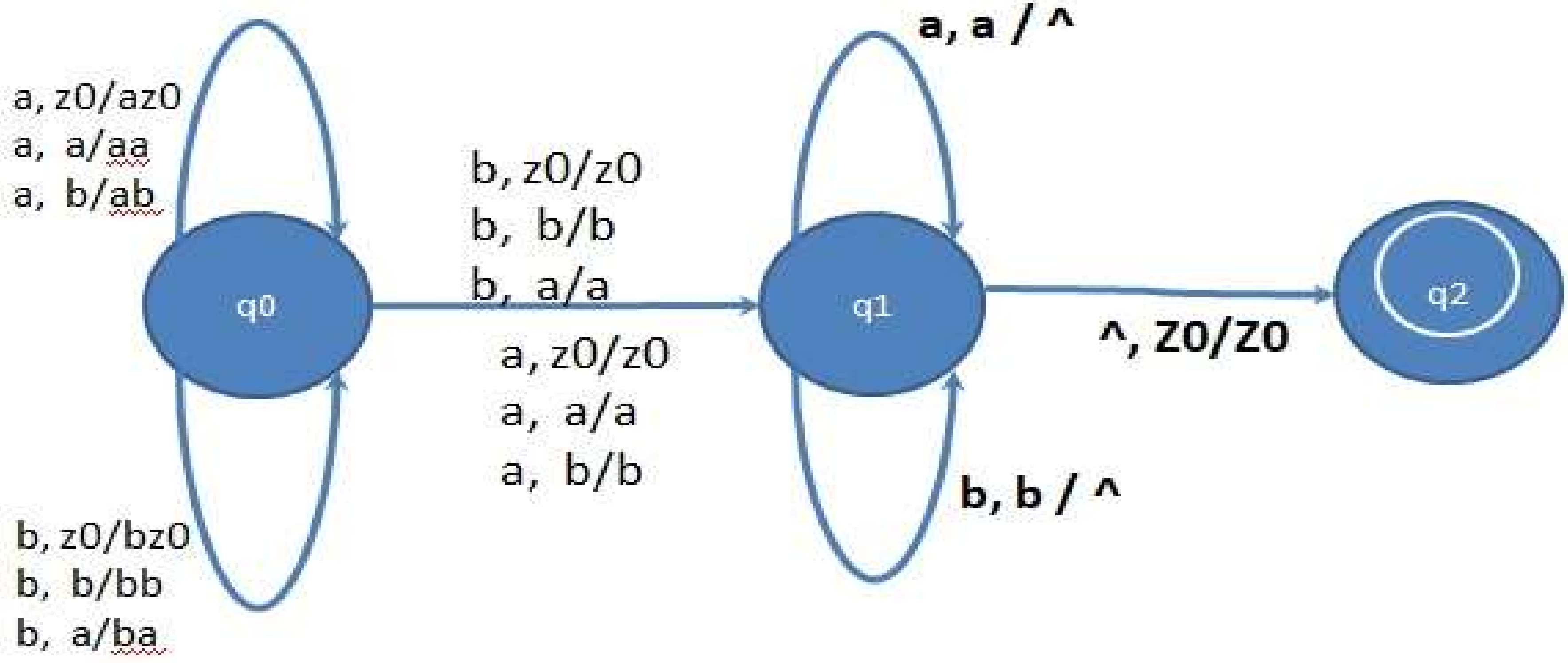
L={x ϵ {a , b }* | x is an odd Length Palindrome }

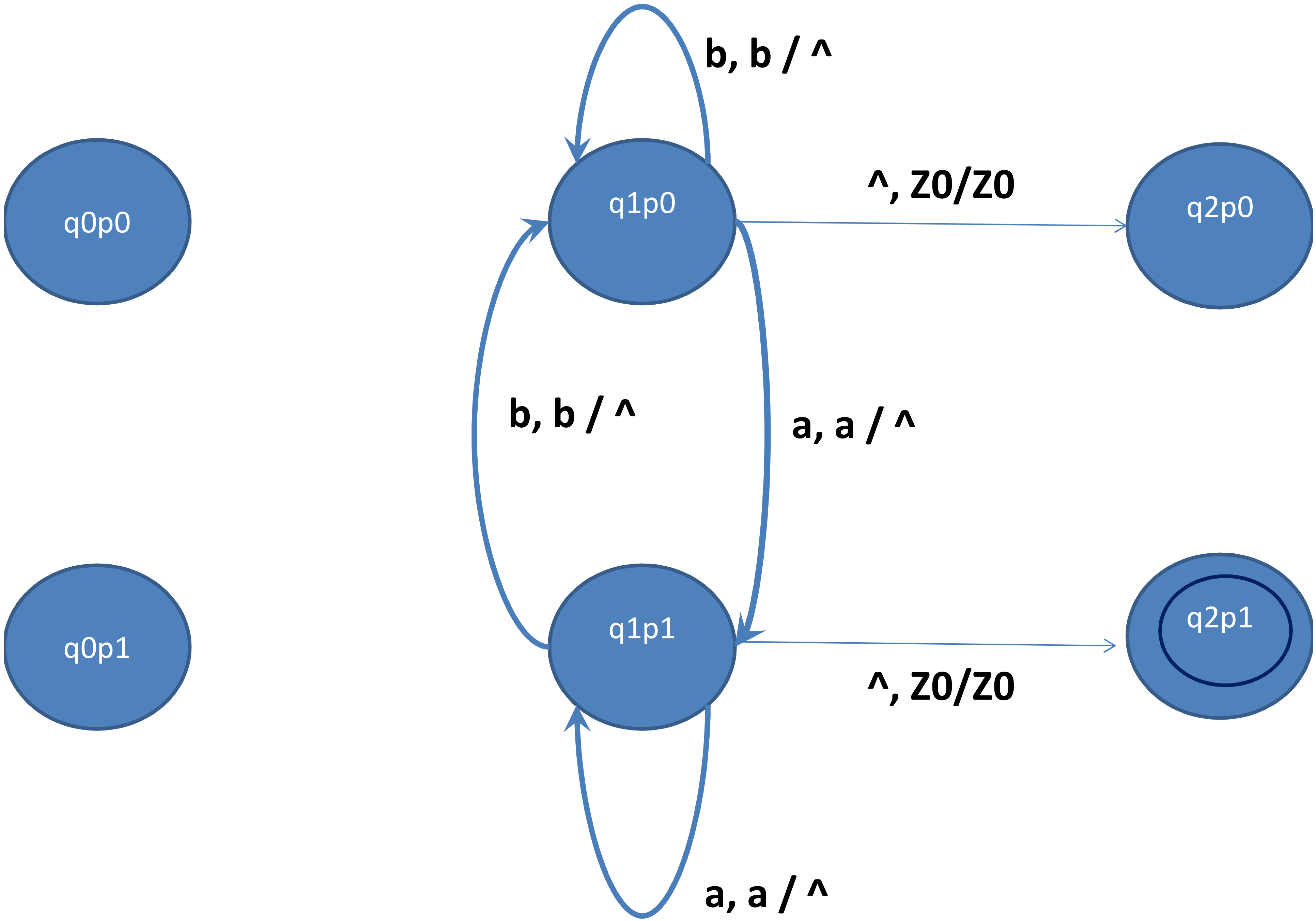

b
a
a
p0
p1
b


a, a / ^
a, z0/az0
a, a/aa
a, b/ab
b, z0/z0
b, b/b
b, a/a
q0
q1
q2
^, Z0/Z0
a, z0/z0
a, a/a
a, b/b
b, b / ^
b, z0/bz0
b, b/bb
b, a/ba

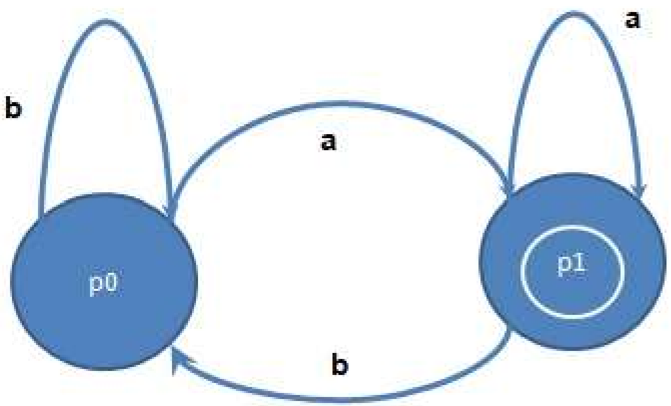

q0p0
q1p0
^, Z0/Z0
q2p0
q0p1
q1p1
^, Z0/Z0
q2p1


b
a
a
p0
p1
b

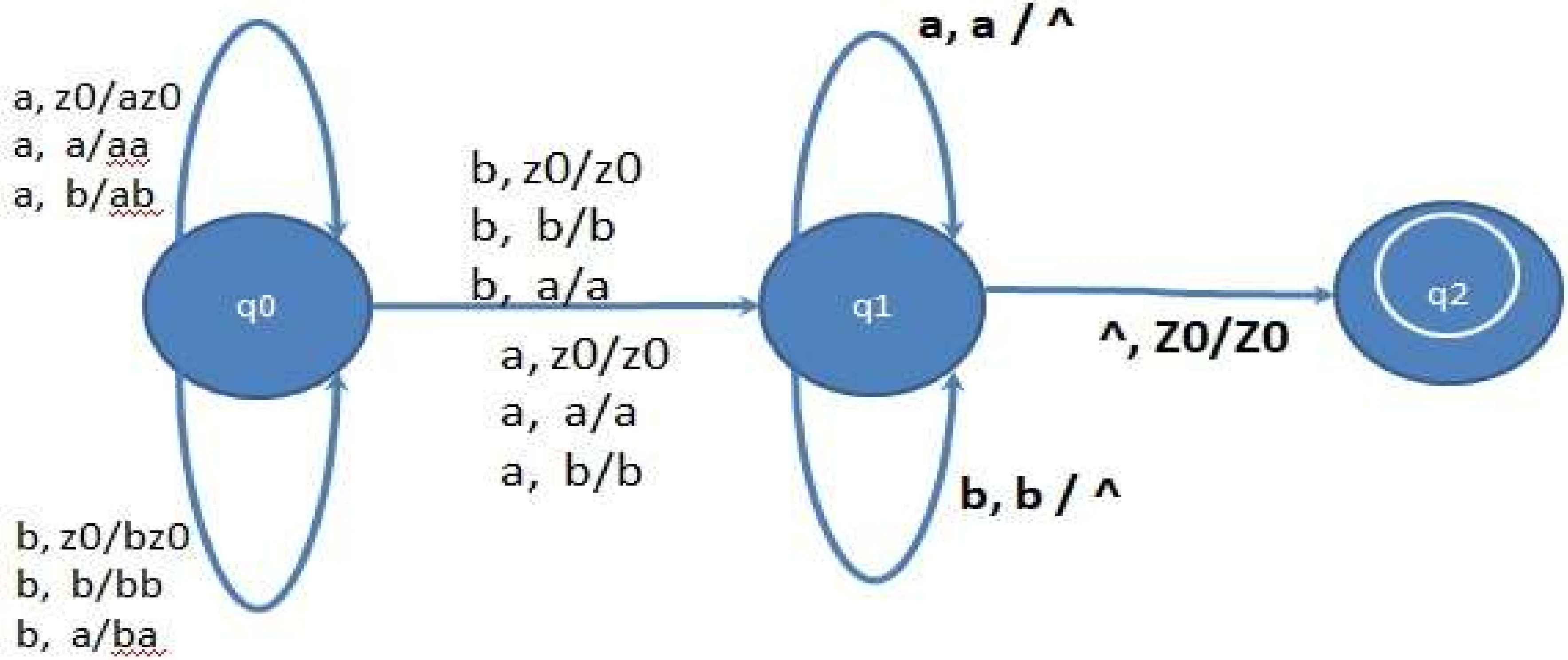

a, z0/az0
a, a/aa
a, b/ab
b, z0/z0
b, b/b
b, a/a
a, a / ^
q0
q1
q2
a, z0/z0
a, a/a
a, b/b
^, Z0/Z0
b, b / ^
b, z0/bz0
b, b/bb
b, a/ba



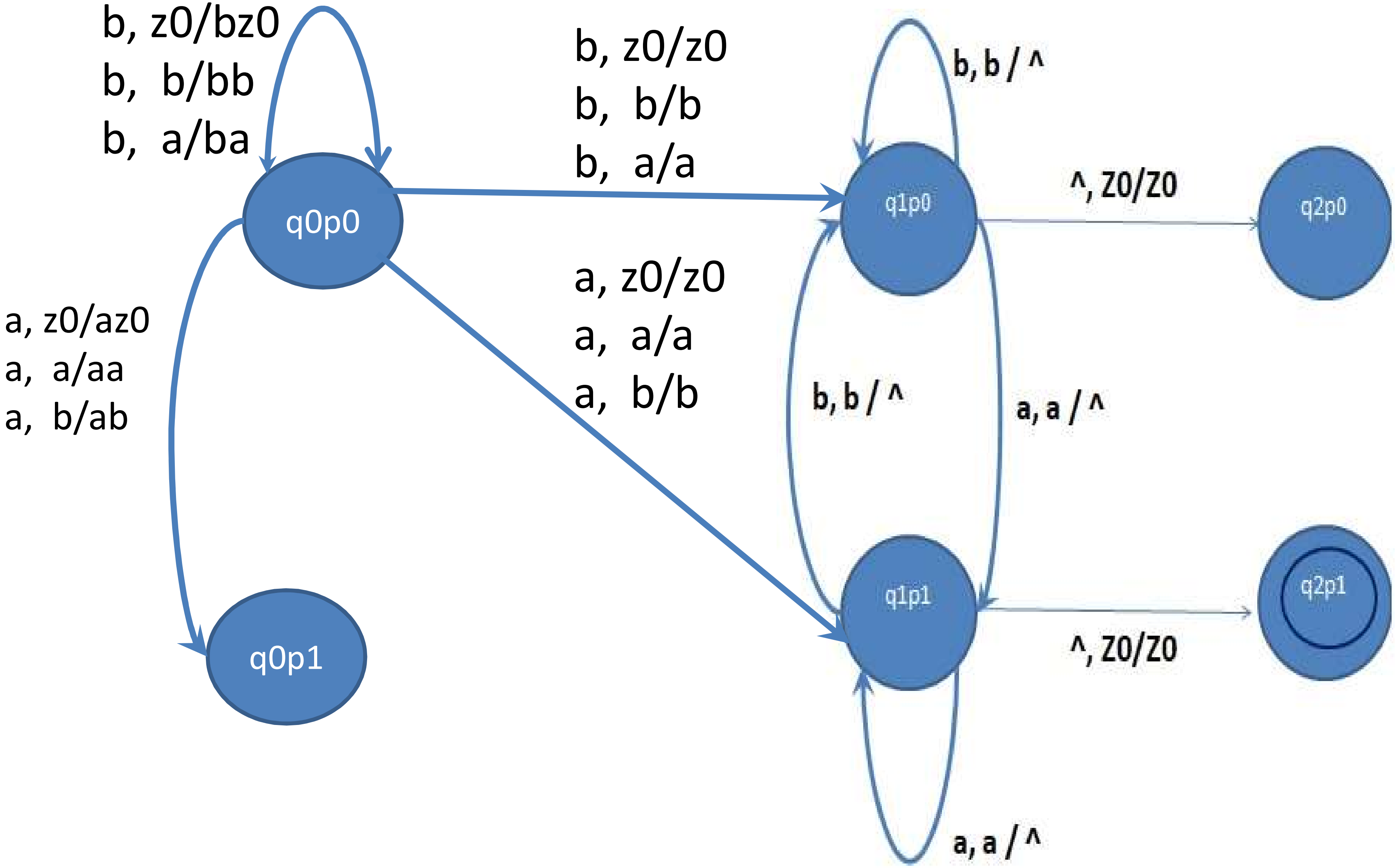
q0p0
q1p0
q2p0
q0p1
q1p1
q2p1
b, b / ^
^, Z0/Z0
b, b / ^
a, a / ^
^, Z0/Z0
a, a / ^



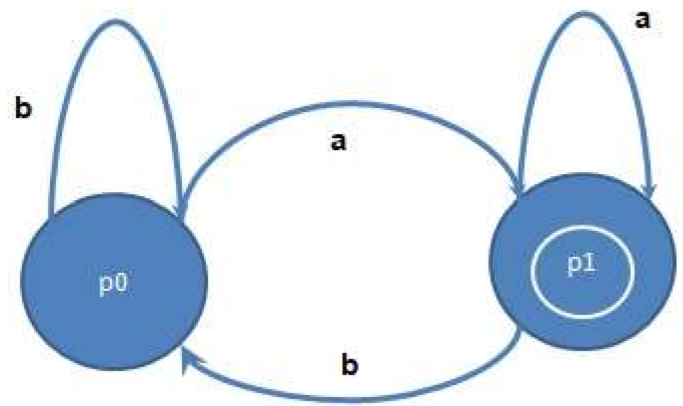
b
a
a
p0
p1
b

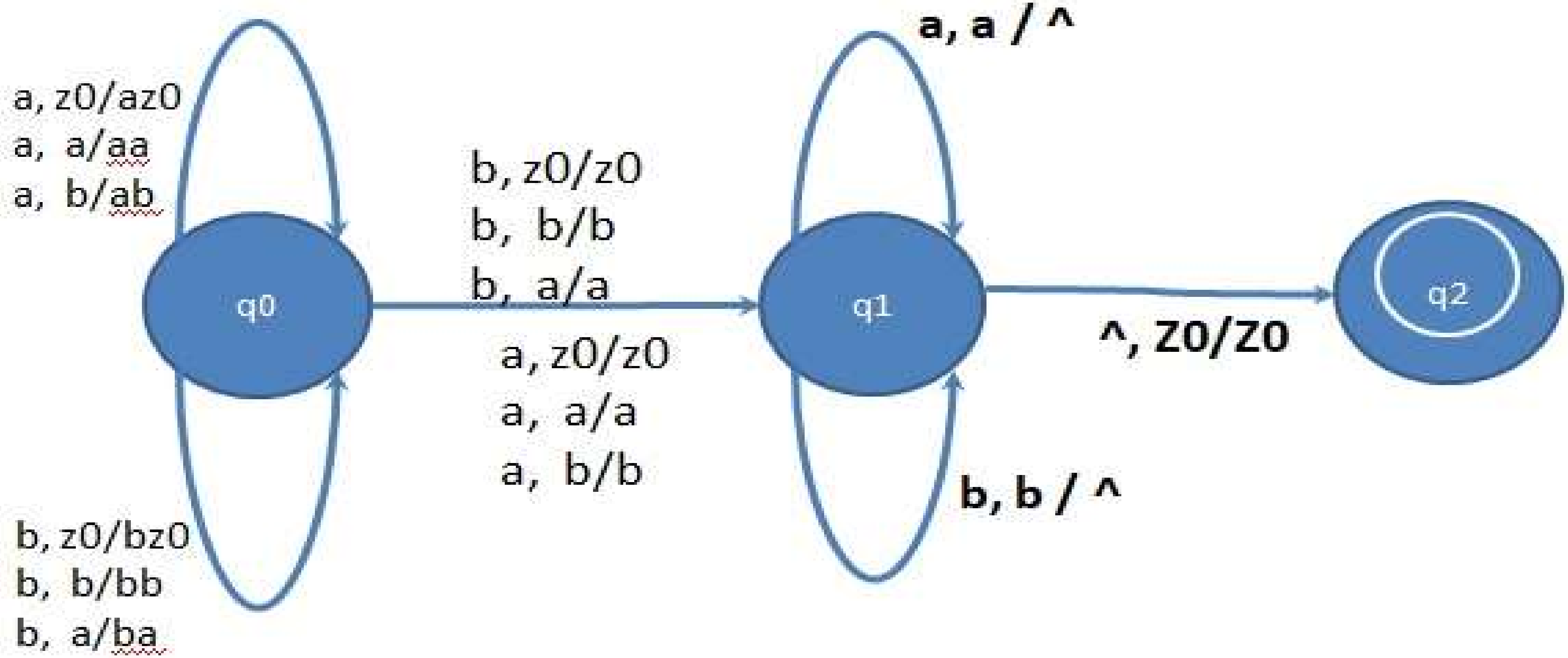


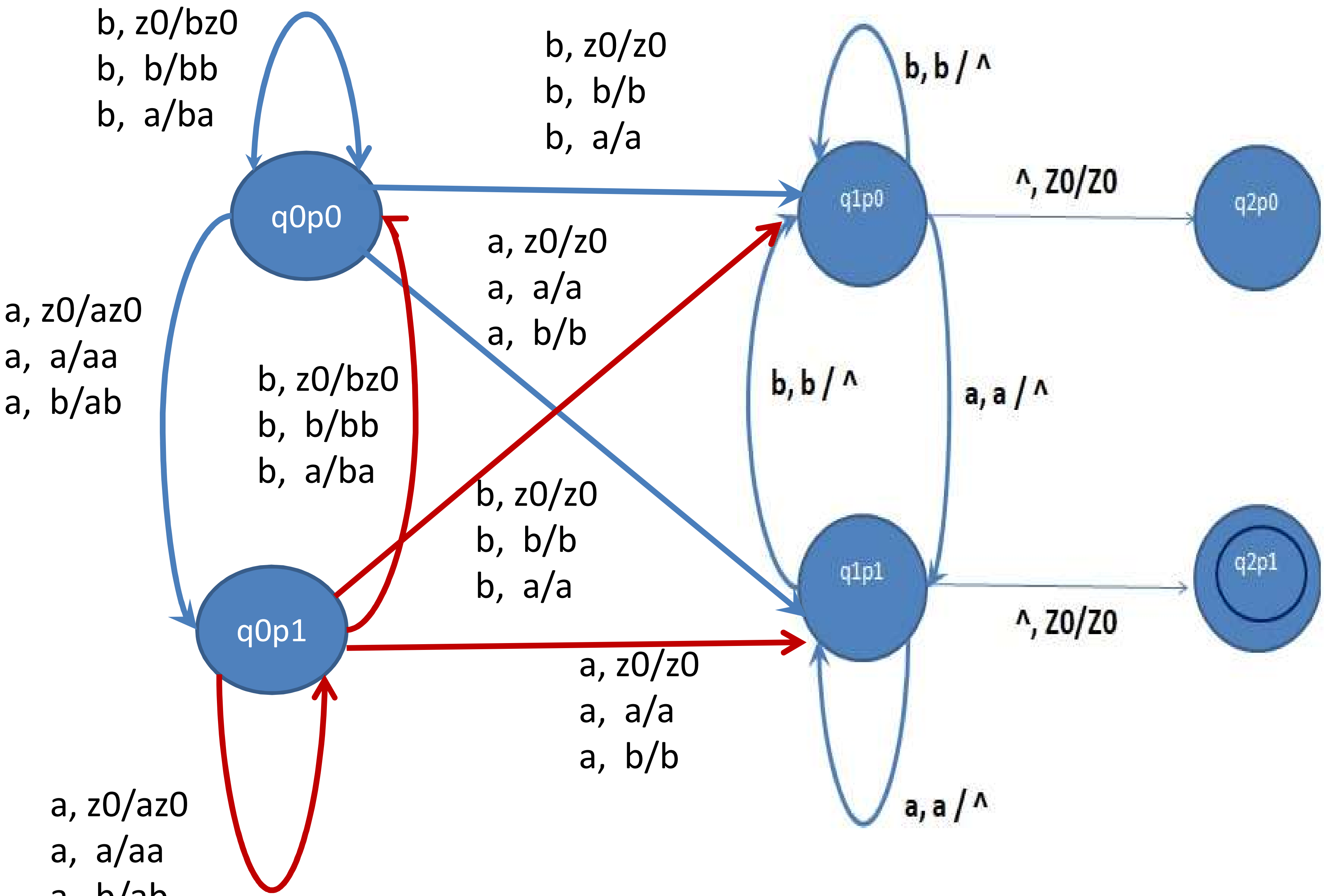
a, z0/az0
a, a/aa
a, b/ab
q0
b, z0/bz0
b, b/bb
b, a/ba
b, z0/z0
b, b/b
b, a/a
a, z0/z0
a, a/a
a, b/b
a, a / ^
q1
b, b / ^
^, Z0/Z0
q2



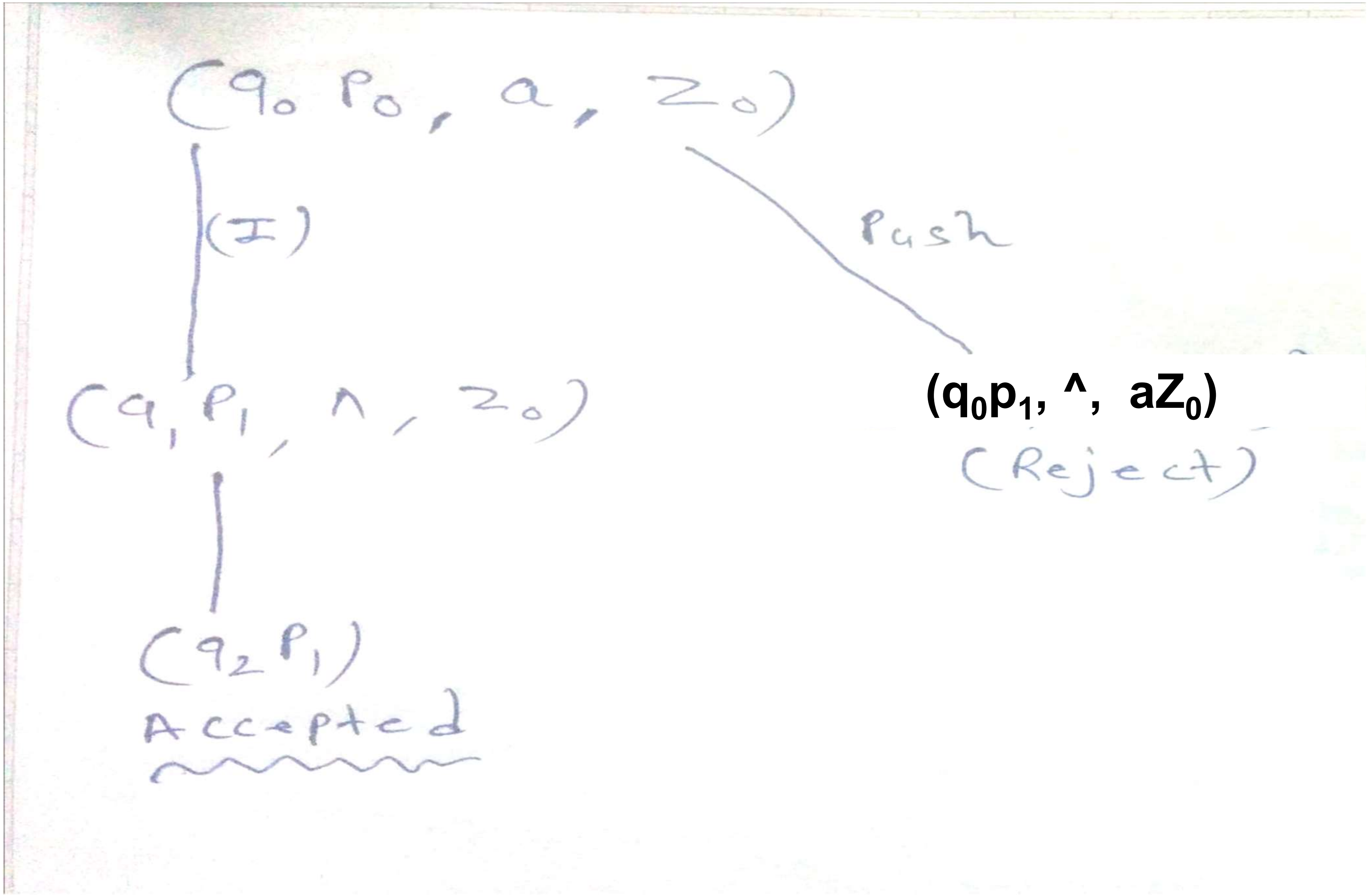
b, z0/bz0
b, b/bb
b, a/ba
q0p0
b, z0/z0
b, b/b
b, a/a
b, b / ^
q1p0
^, Z0/Z0
q2p0
a, z0/az0
a, a/aa
a, b/ab
a, z0/z0
a, a/a
a, b/b
b, b / ^
a, a / ^
q0p1
q1p1
q2p1
^, Z0/Z0
a, a / ^


b
a
a
b
p0
p1


a, z0/az0
a, a/aa
a, b/ab
b, z0/bz0
b, b/bb
b, a/ba
b, z0/z0
b, b/b
b, a/a
a, z0/z0
a, a/a
a, b/b
a, a / ^
b, b / ^
^, Z0/Z0
q0
q1
q2


b, z0/bz0
b, b/bb
b, a/ba
b, z0/z0
b, b/b
b, a/a
b, b / ^
q0p0
q1p0
^, Z0/Z0
q2p0
a, z0/z0
a, a/a
a, b/b
a, z0/az0
a, a/aa
a, b/ab
b, z0/bz0
b, b/bb
b, a/ba
b, b / ^
a, a / ^
b, z0/z0
b, b/b
b, a/a
q1p1
q2p1
q0p1
^, Z0/Z0
a, z0/z0
a, a/a
a, b/b
a, z0/az0
a, a/aa
a, b/ab
a, a / ^

Validation of Diagram



If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is a CFL.

Accepted	Rejected
a	b
aba	aa, ab, ba, bb
aaa	abb, aab, abba, abb,baa,bbb
ababa	
aaaaa	

Here, $L_1 \cap L_2$ is a Context-Free Language but it is not same as L_1

Intersection of CFL and RL

- Here, Language accepted is

  L ={x ϵ {a, b}* | x is an odd length palindrome with the first and last symbol always a}

Context-Free Grammar :

$S \rightarrow a$

$S \rightarrow a\ S_1 a$

$S_1 \rightarrow a\ S_1\ a \mid b\ S_1\ b$

$S_1 \rightarrow a \mid b$

If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is a CFL.

Sketch of Proof Let $M_1 = (Q_1, \Sigma, \Gamma, q_1, Z_0, A_1, \delta_1)$ be a PDA accepting L_1 and $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ an FA accepting L_2. Then we define the PDA $M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ as follows.

$$Q = Q_1 \times Q_2 \qquad q_0 = (q_1, q_2) \qquad A = A_1 \times A_2$$

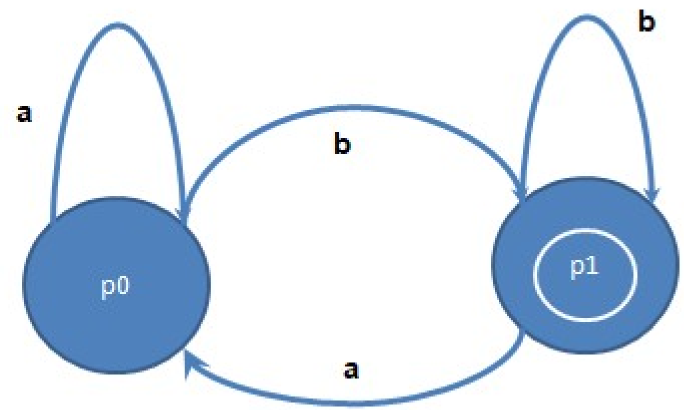
For $p \in Q_1$, $q \in Q_2$, and $Z \in \Gamma$,

1. For every $\sigma \in \Sigma$, $\delta((p, q), \sigma, Z)$ is the set of pairs $((p', q'), \alpha)$ for which $(p', \alpha) \in \delta_1(p, \sigma, Z)$ and $\delta_2(q, \sigma) = q'$.
2. $\delta((p, q), \Lambda, Z)$ is the set of pairs $((p', q), \alpha)$ for which $(p', \alpha) \in \delta_1(p, \Lambda, Z)$.

Practice Problem

Find intersection of PDA and FA given below

Move Number	State	Input	Stack Symbol	Move(s)
1	q_0	a	Z_0	(q_1, aZ_0)
2	q_1	a	a	(q_1, aa)
3	q_1	b	a	(q_2, Λ)
4	q_2	b	a	(q_2, Λ)
5	q_2	Λ	Z_0	(q_3, Z_0)
	(all other combinations)			none



We have discussed.....

- Properties of Context-Free Languages

    - ❖ Closure, Union and Concatenation of Context-Free Languages

    - ❖ Intersection of Context-Free Languages

    - ❖ Intersection of Context-Free Language with Regular Language