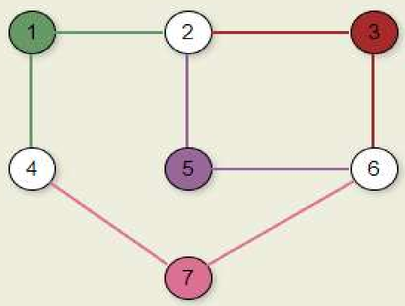
Advanced Algorithms

NP-Hard & NP-Complete Problems

Independent Set

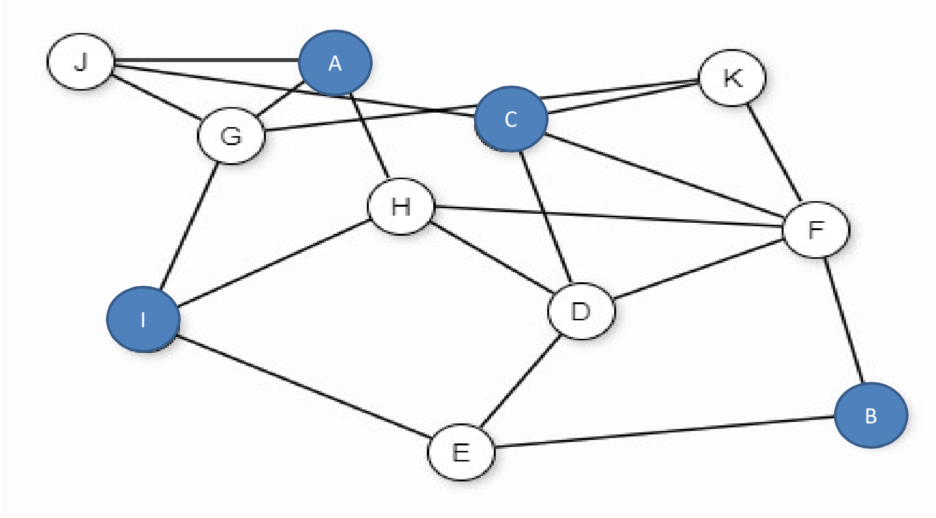
An Independent Set of a graph is a set of vertices such that no two of them are connected i.e. there exists no edge between any two vertices of an Independent Set.

The largest possible Independent Set of a graph is called the "Maximum Independent Set".



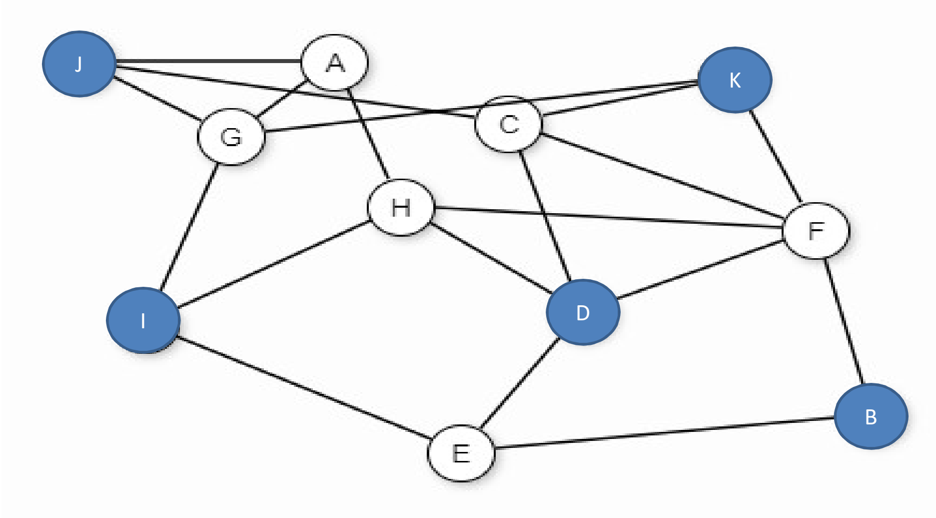
Here, set {1,3,5,7} forms an independent set

Independent Set



Here, set {A,B,C,I} forms an independent set

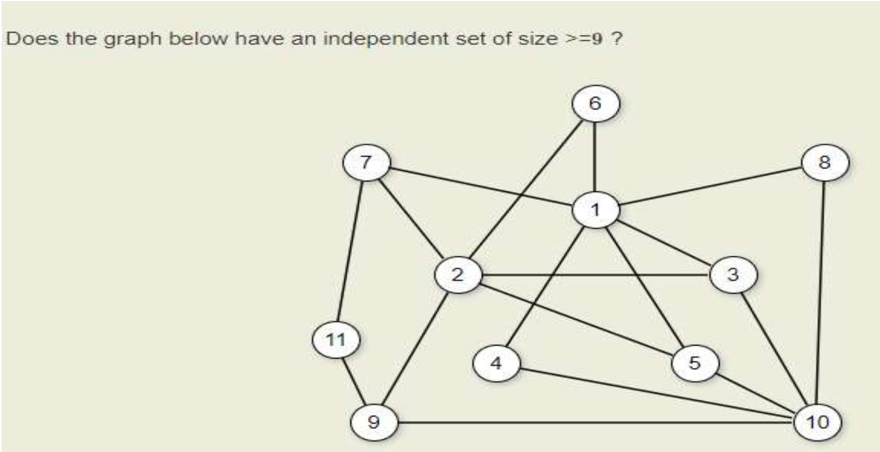
Independent Set



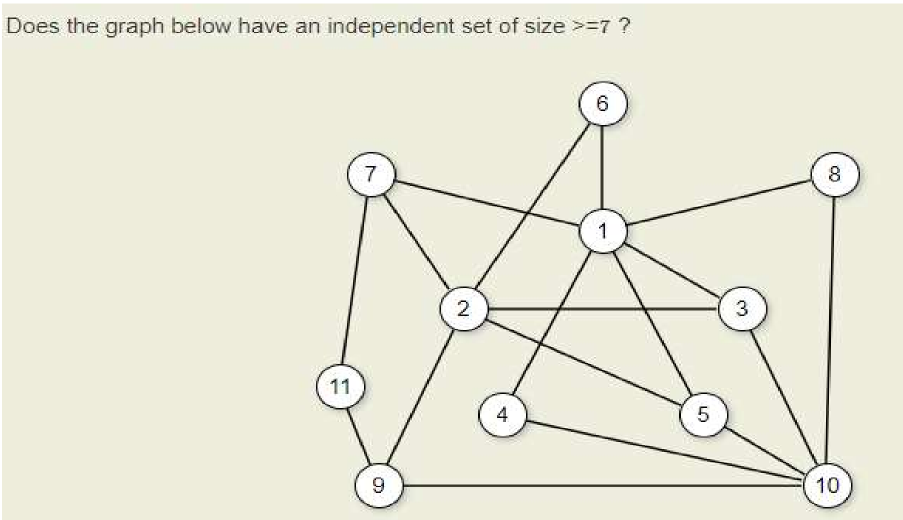
Here, set {B, D, I, J, K} forms MAXIMUM independent set

Independent Set Problem

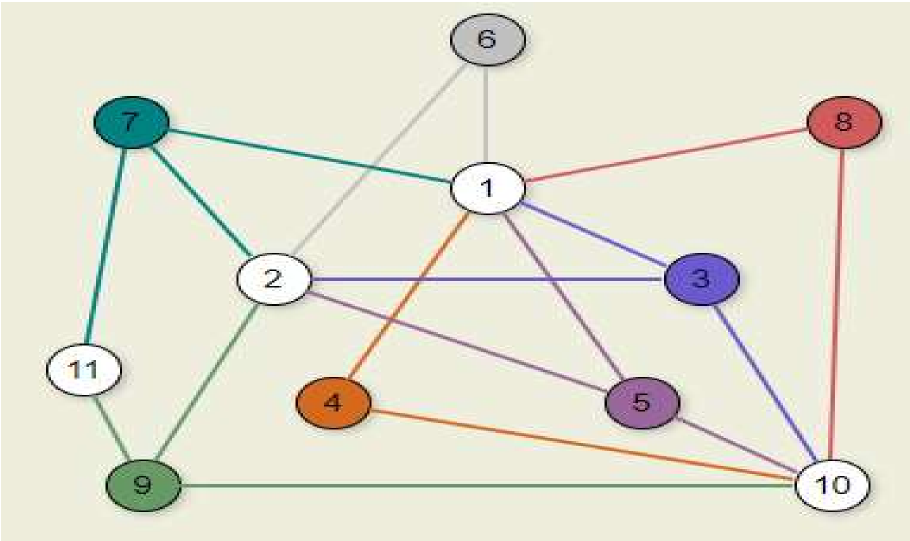
Given a graph G=(V,E), find the Maximum Independent Set in G.
OR
Given a graph G=(V,E) and a number k, does G contain an independent Set of size >=k ?



Independent Set Problem



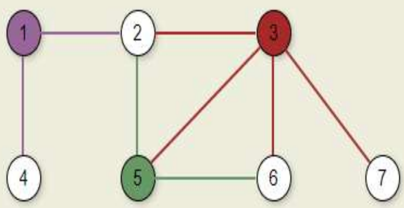
Independent Set Problem



Vertex Cover Problem

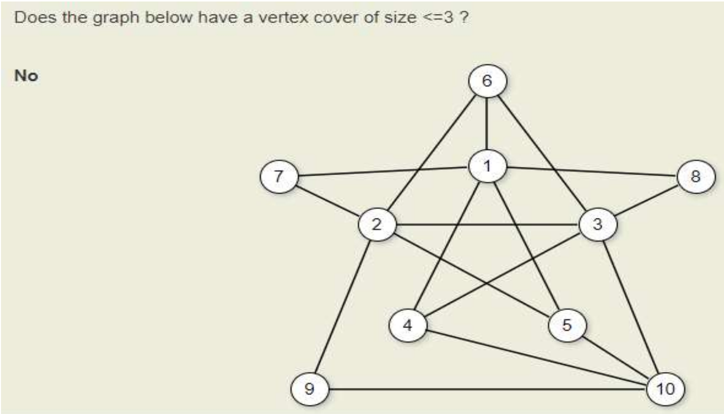
A Vertex Cover of a graph is a set of vertices such that any edge of the graph is incident on at least one vertex of the set.

The smallest possible Vertex Cover of a graph is called the "Minimum Vertex cover".



Vertex Cover Problem

Given a graph G=(V,E), find the Minimum Vertex Cover in G.
OR
Given a graph G=(V,E) and a number k, does G contain vertex cover of size <=k ?



Vertex Cover Problem

Given a graph G=(V,E), find the Minimum Vertex Cover in G.
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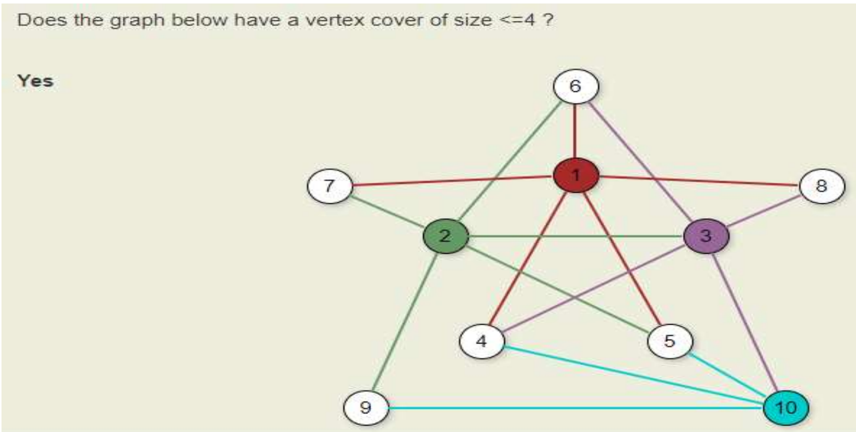
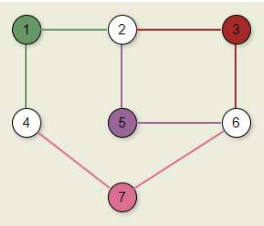


Reduction of Independent Set to Vertex Cover



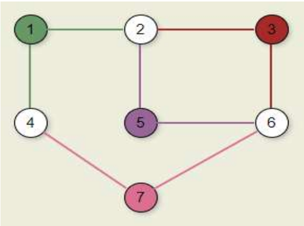
S={1,3,5,7}
Here, S is an independent set.
There is no edge e=(u,v) in G, such that u, v ϵ S

Hence, for any edge e=(u,v) , atleast one of u, v must lie in V-S

For edge(1,2), '2' lies in V-S . V-S={2}
For edge(1,4), '4' lies in V-S V-S={2,4}
For edge(2,3), Do nothing
For edge(2,5), Do nothing
For edge(3,6), '6' lies in V-S. V-S={2,4,6}
For edge(5,6), Do nothing
For edge(4,7), Do nothing
For edge(7,6), Do nothing

Hence, {2,4,6} is a Vertex Cover

Reduction of Independent Set to Vertex Cover



S={1,3,5,7}
Here, S is an independent set.

Let's assume that {2,4,6} is a Vertex Cover

Consider any two nodes u and v is S. If they were joined by an edge e, then neither end of e would lie in V-S, contradicting our assumption that V-S is a vertex cover.

It follows that no two nodes in S are joined by an edge, so S is an Independent set.

Independent Set $\leq_R$ Vertex Cover

Proof:

If we have a black box to solve Vertex Cover, then we can decide whether G has an independent set of size at least k, by asking the black box whether G has a vertex cover of size at most n-k

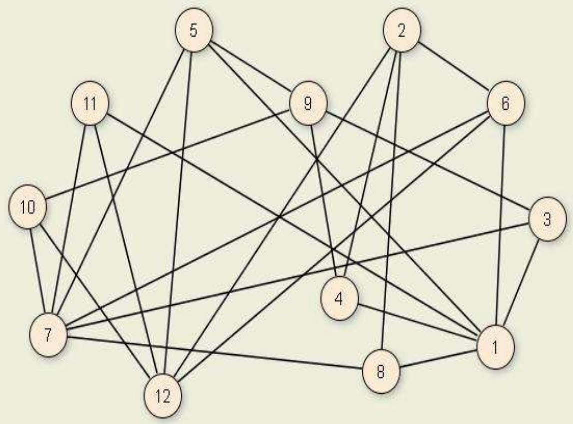
Vertex Cover $\leq_R$ Independent Set

Proof:

If we have a black box to solve Independent Set, then we can decide whether G has a vertex cover of size at most k, by asking the black box whether G has an independent set of size at least n-k

EXAMPLE 2

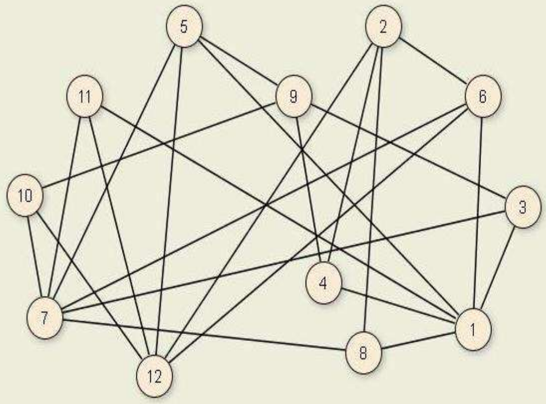
Does this 12-node graph have an Independent Set of size >= 9 ?



EXAMPLE 2

Does this 12-node graph have an Vertex Cover of size <= 3 ?

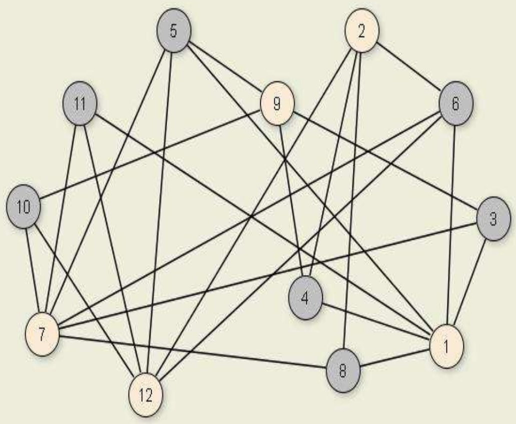
No



EXAMPLE 2

Does this 12-node graph have an Independent Set of size >= 7 ?

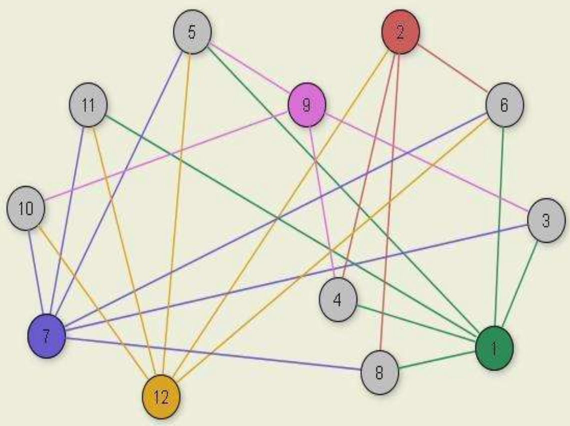
Yes



EXAMPLE 2

Does this 12-node graph have an Vertex Cover of size <= 5 ?

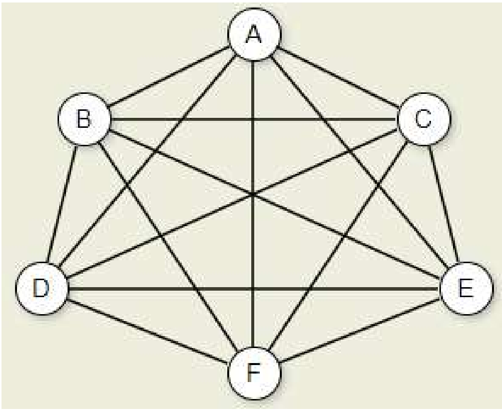
Yes



Max Clique Problem

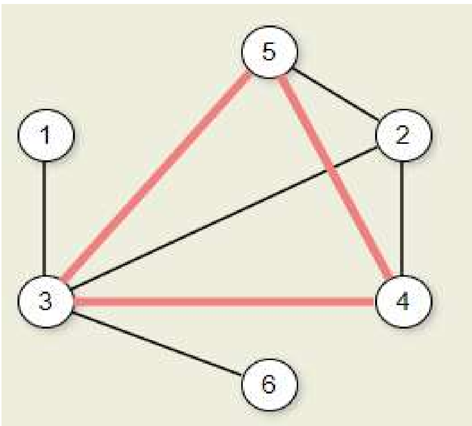
A Clique is a complete graph .

Each node is connected to every other nodes by atleast one edge



Max Clique Problem

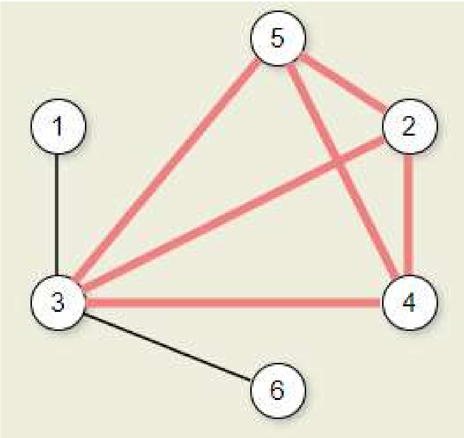
If in a graph G, there exists a complete subgraph of k nodes, then G is said to contain k-clique.



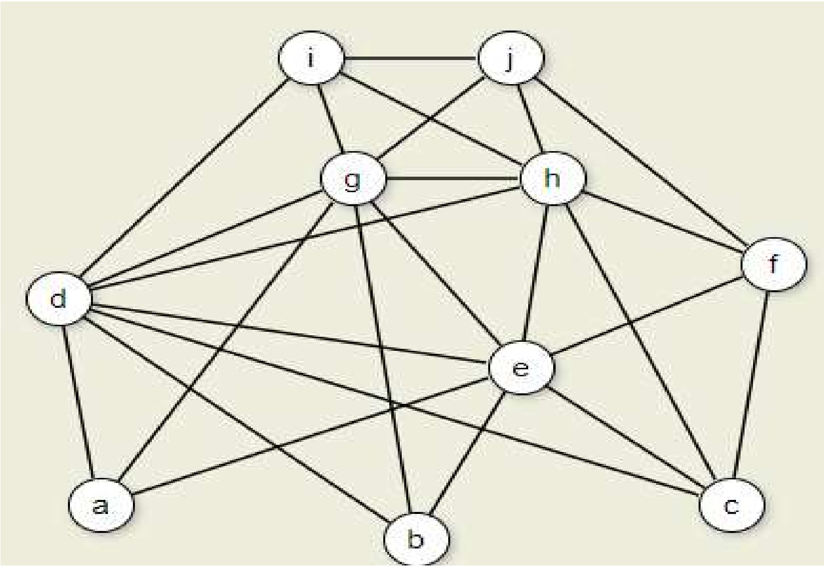
Max Clique Problem

If in a graph G, there exists a complete subgraph of k nodes, then G is said to contain k-clique.

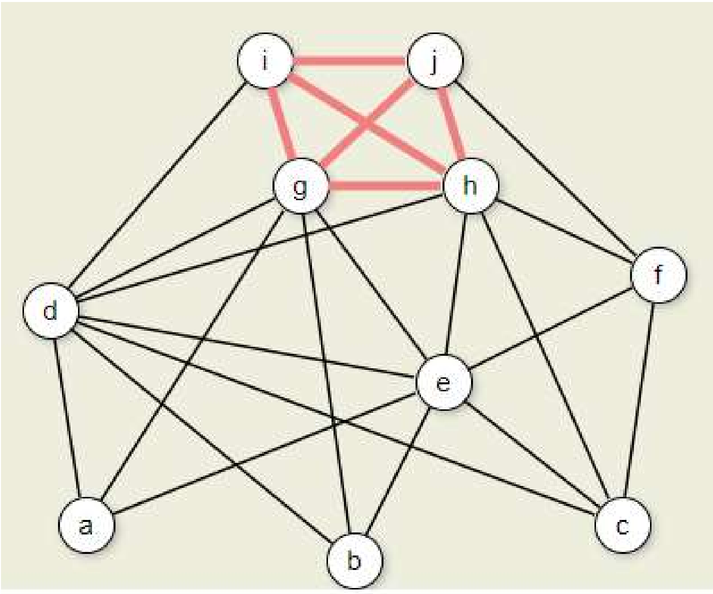
Clique with largest number of vertices in a graph G is called **Maximum Clique** in G



Find the Maximum Clique



Maximum Clique of size 4



Maximum Clique $\leq_R$ Independent Set

Reduction of Clique to Independent Set

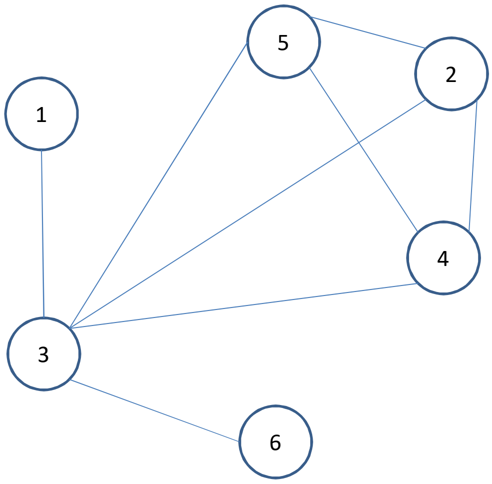
To reduce a Clique Problem to an Independent Set problem for a given graph $G = (V, E)$, construct a complimentary graph $G' = (V', E')$ such that

1. $V = V'$, that is the compliment graph will have the same vertices as the original graph

2. E' is the compliment of E that is G' has all the edges that is **not** present in G

Note: Construction of the complimentary graph can be done in polynomial time

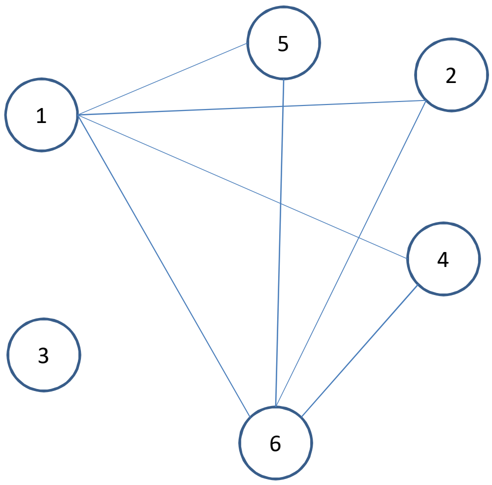
Maximum Clique $\leq_R$ Independent Set



Graph G

Maximum Clique $\leq_R$ Independent Set

Graph G'



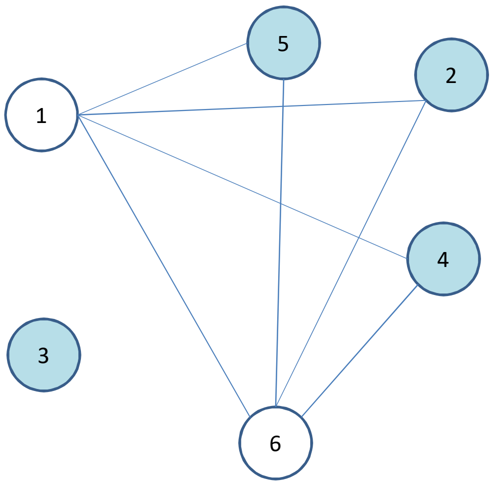
Maximum Clique $\leq_R$ Independent Set

Clique problem reduced to Independent Set

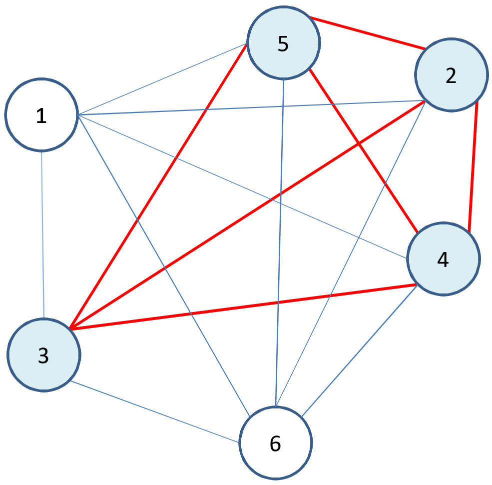
1. **If there is an independent set of size k in the complement graph** G', it implies no two vertices share an edge in G' which further implies all of those vertices share an edge with all others in G forming a clique. that is **there exists a clique of size k in** G

Maximum Clique $\leq_R$ Independent Set

Graph G'



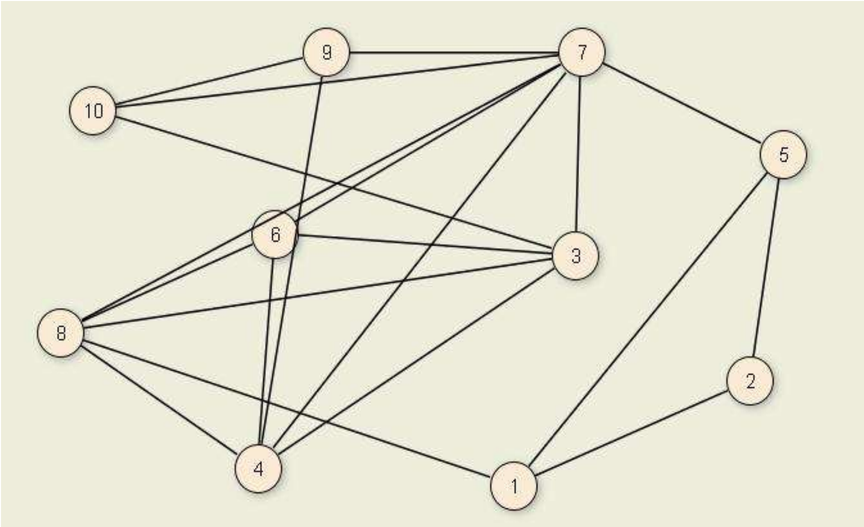
Maximum Clique $\leq_R$ Independent Set



Maximum Clique $\leq_R$ Independent Set

2. **If there is a clique of size** k **in the graph** G, it implies all vertices share an edge with all others in G which further implies no two of these vertices share an edge in G' forming an Independent Set. that is **there exists an independent set of size k in** G'

EXAMPLE 2



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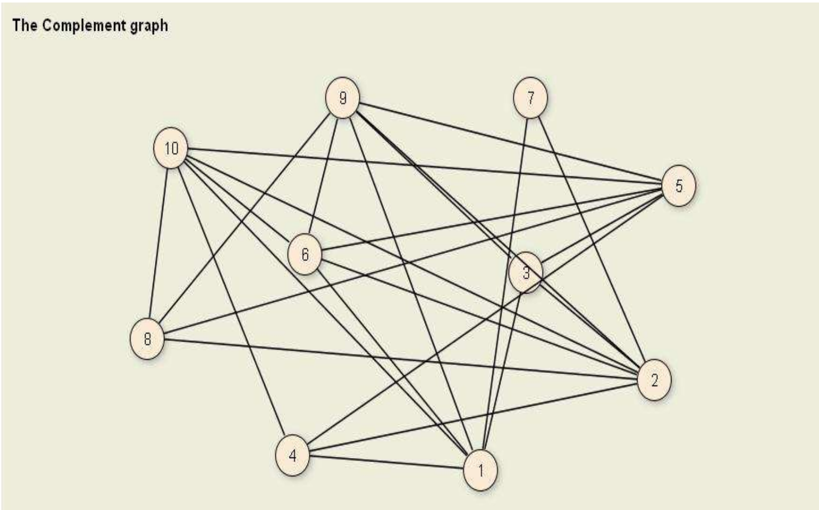
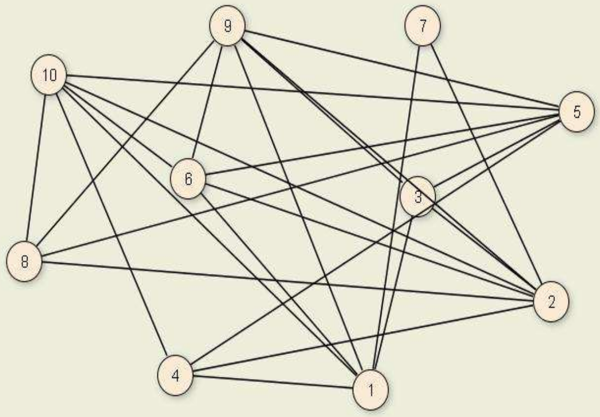


EXAMPLE 2

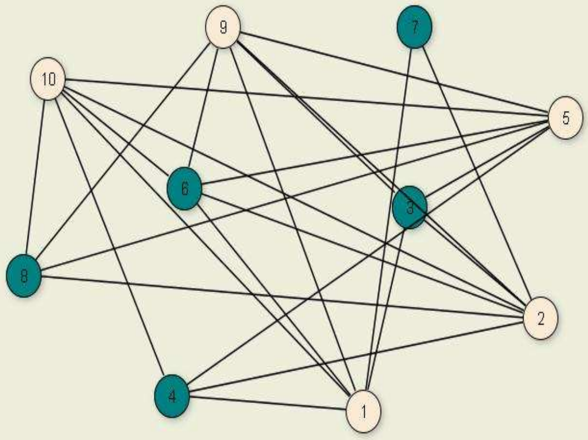
Does G' below have an independent set of size 5?



EXAMPLE 2

Does G' below have an independent set of size 5?

YES



EXAMPLE 2