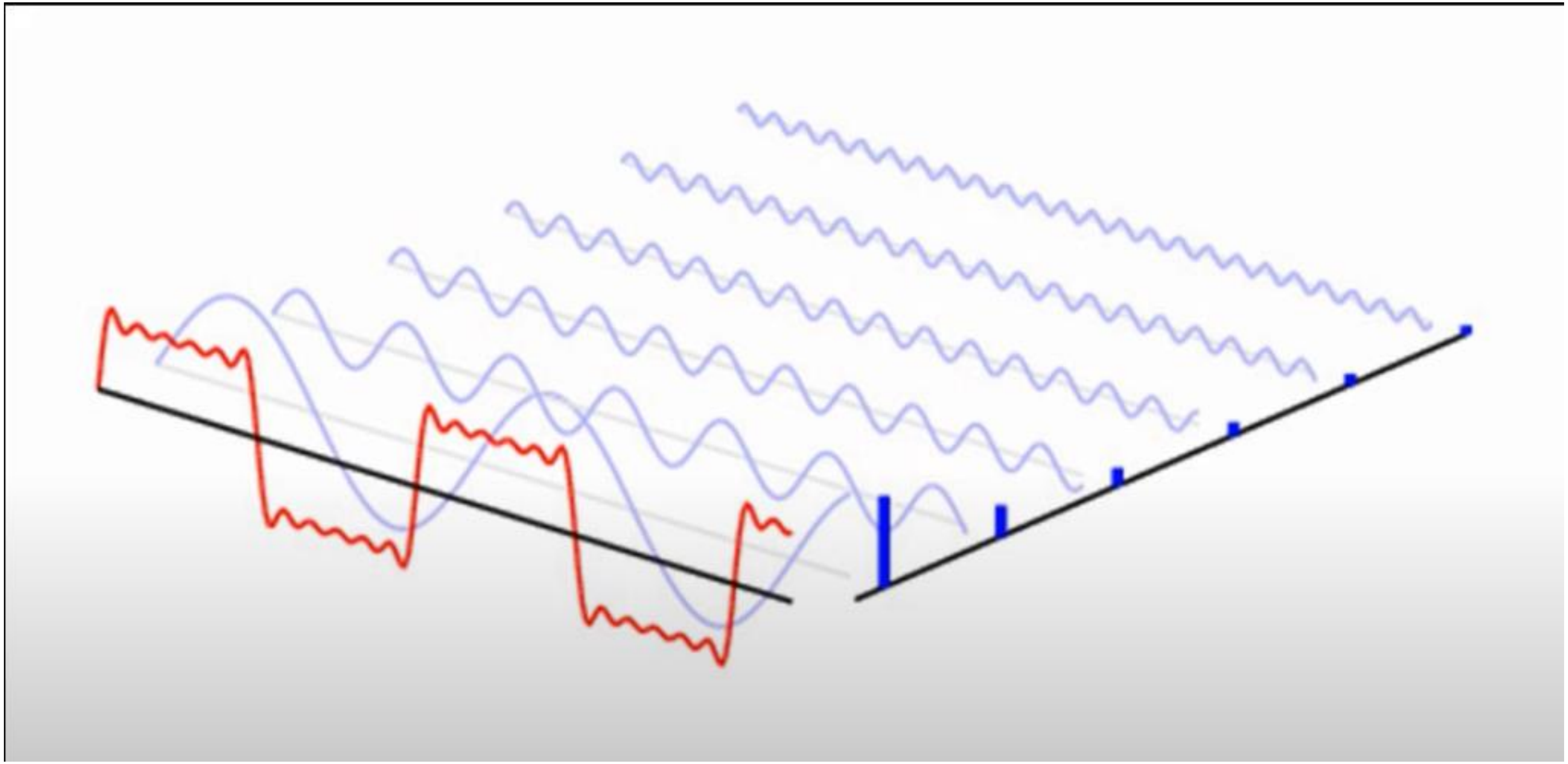
Filtering in Frequency Domain

Lecture 1 (Introduction and Maths Primer)

Prepared by: Neha Fotedar

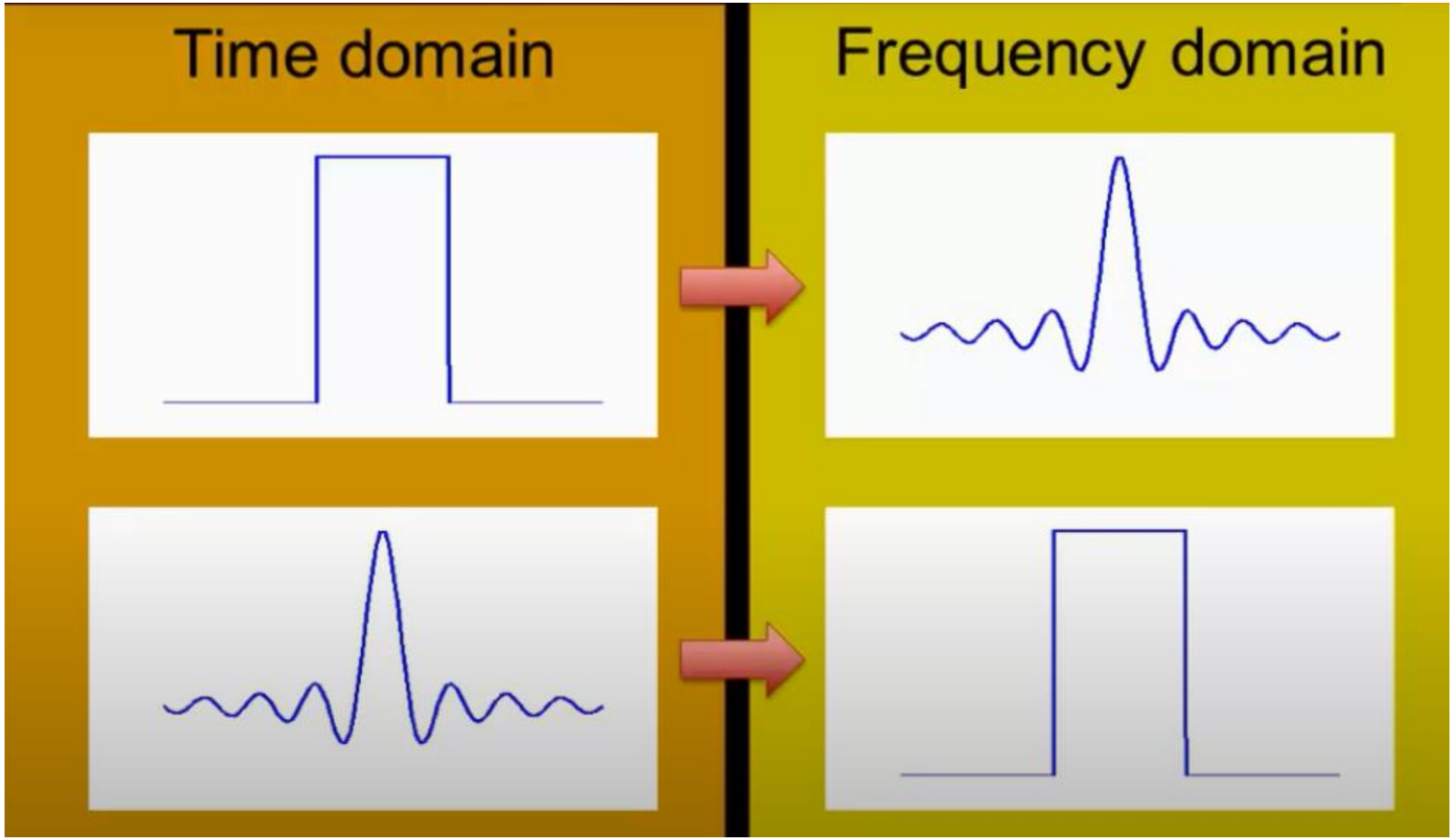
Why we switch to Frequency Domain?

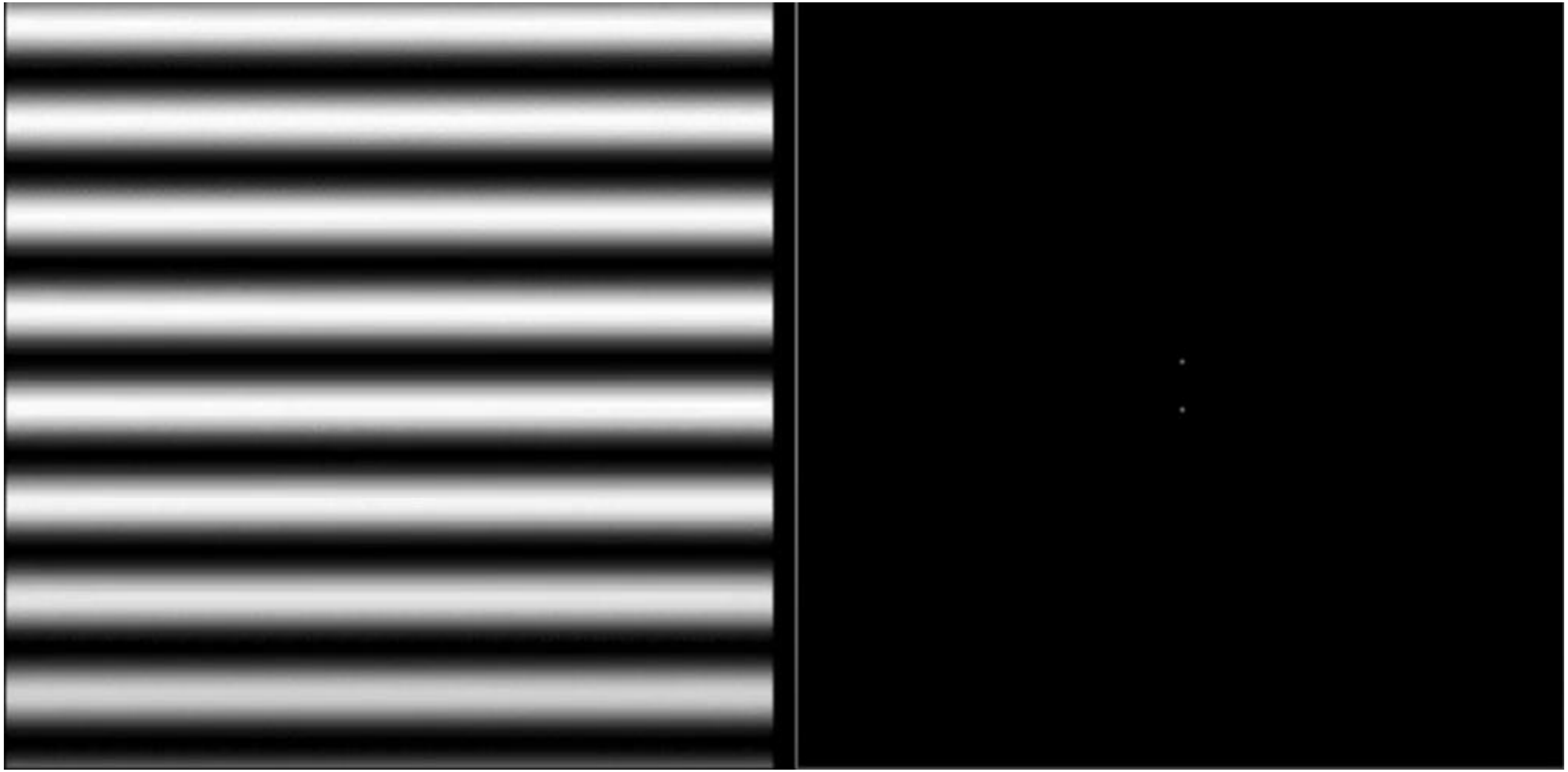
Fourier Transform

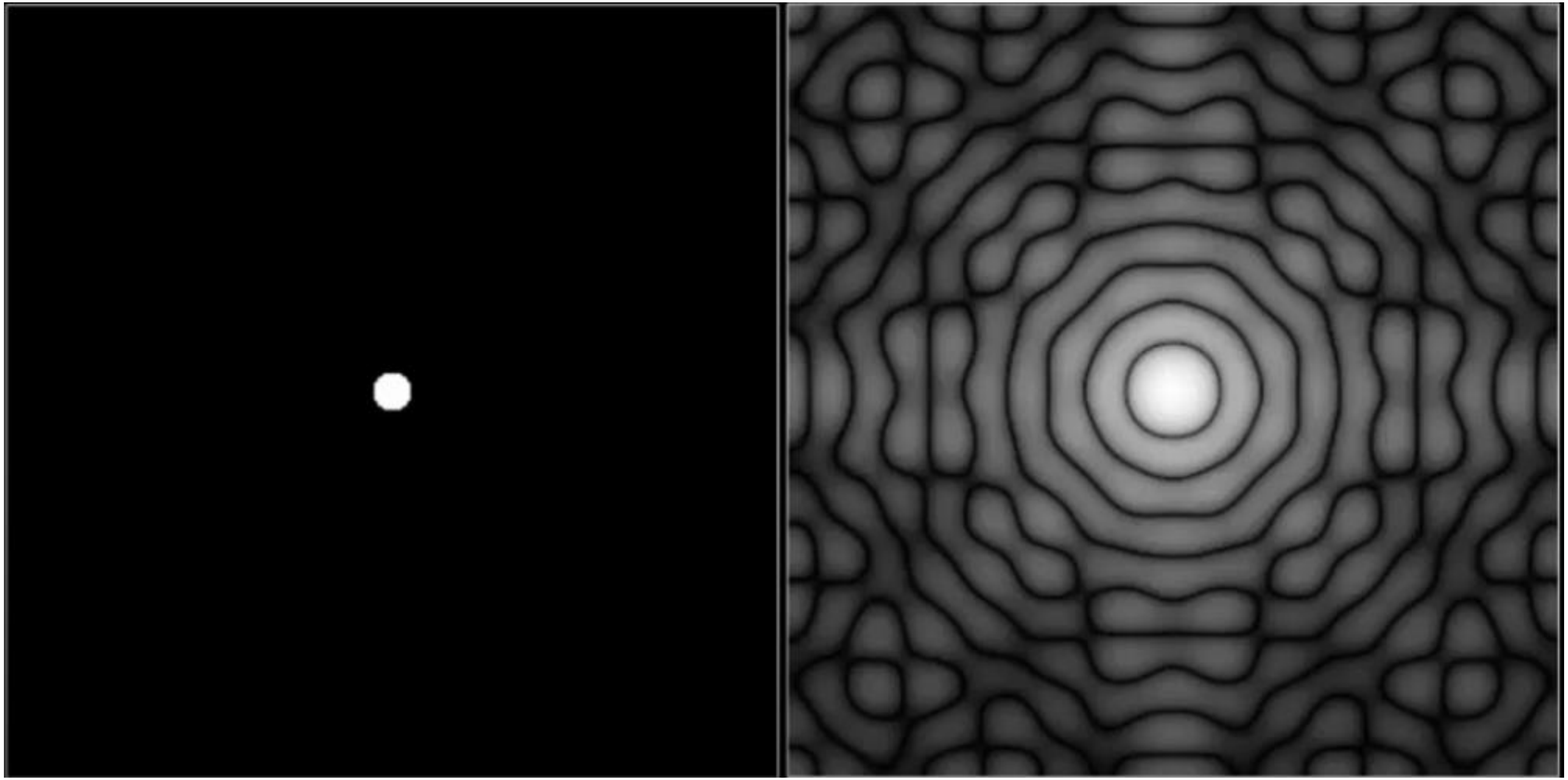


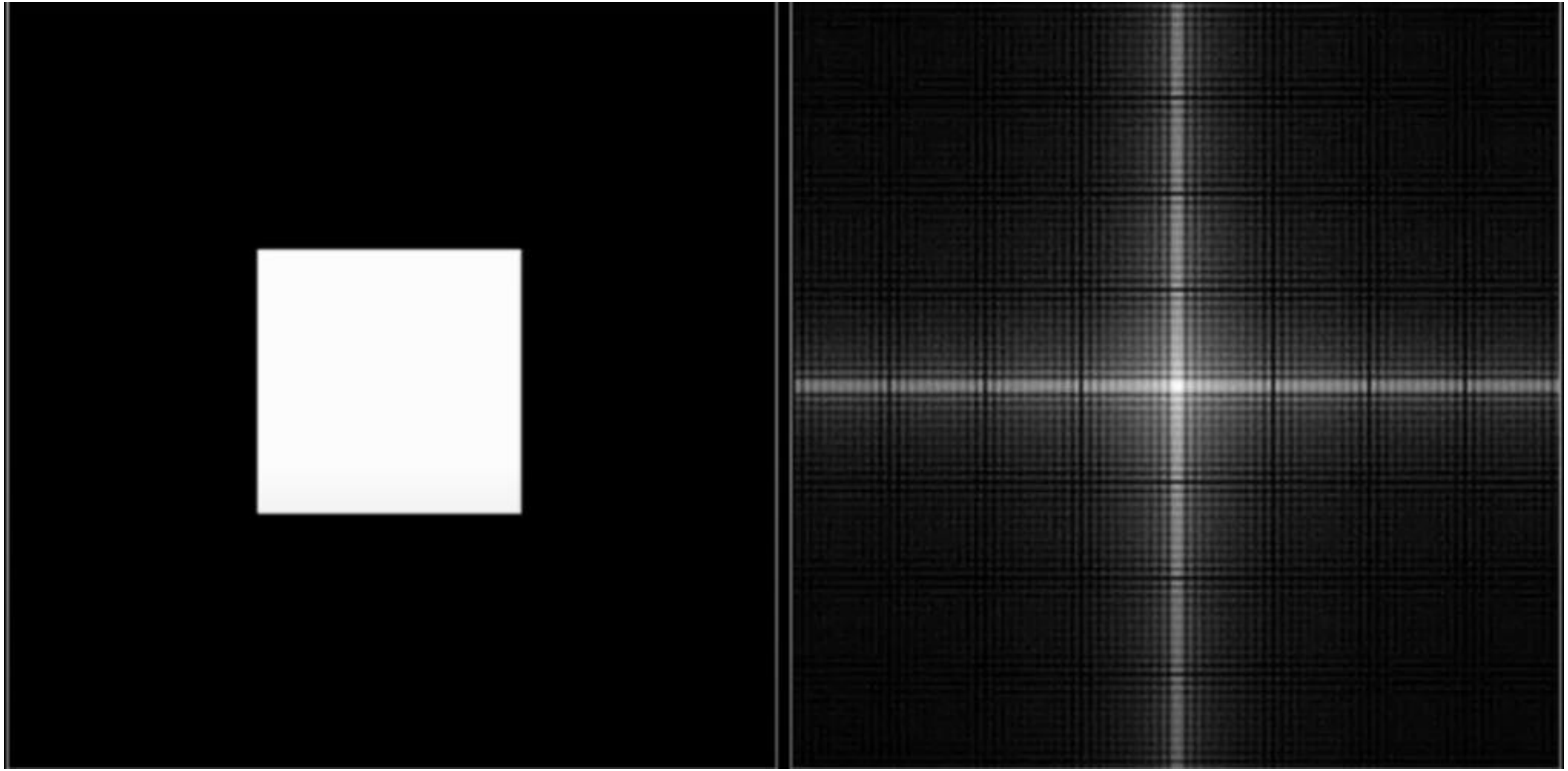
Applications

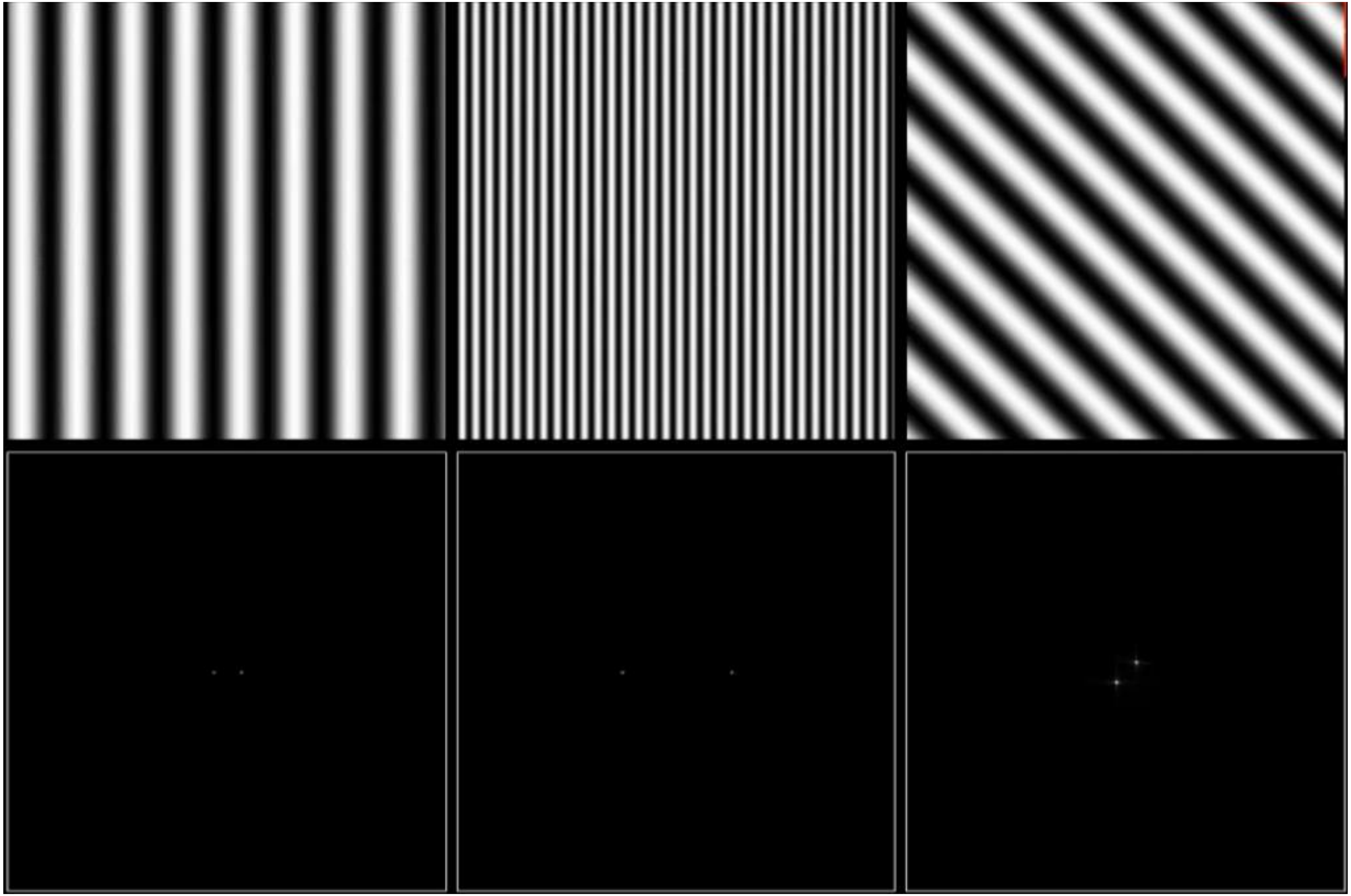
- Reading Text – Captcha
- Transferring books to electronic copies
- Number plate recognition
- Automating the input of handwritten forms


Time domain
Frequency domain









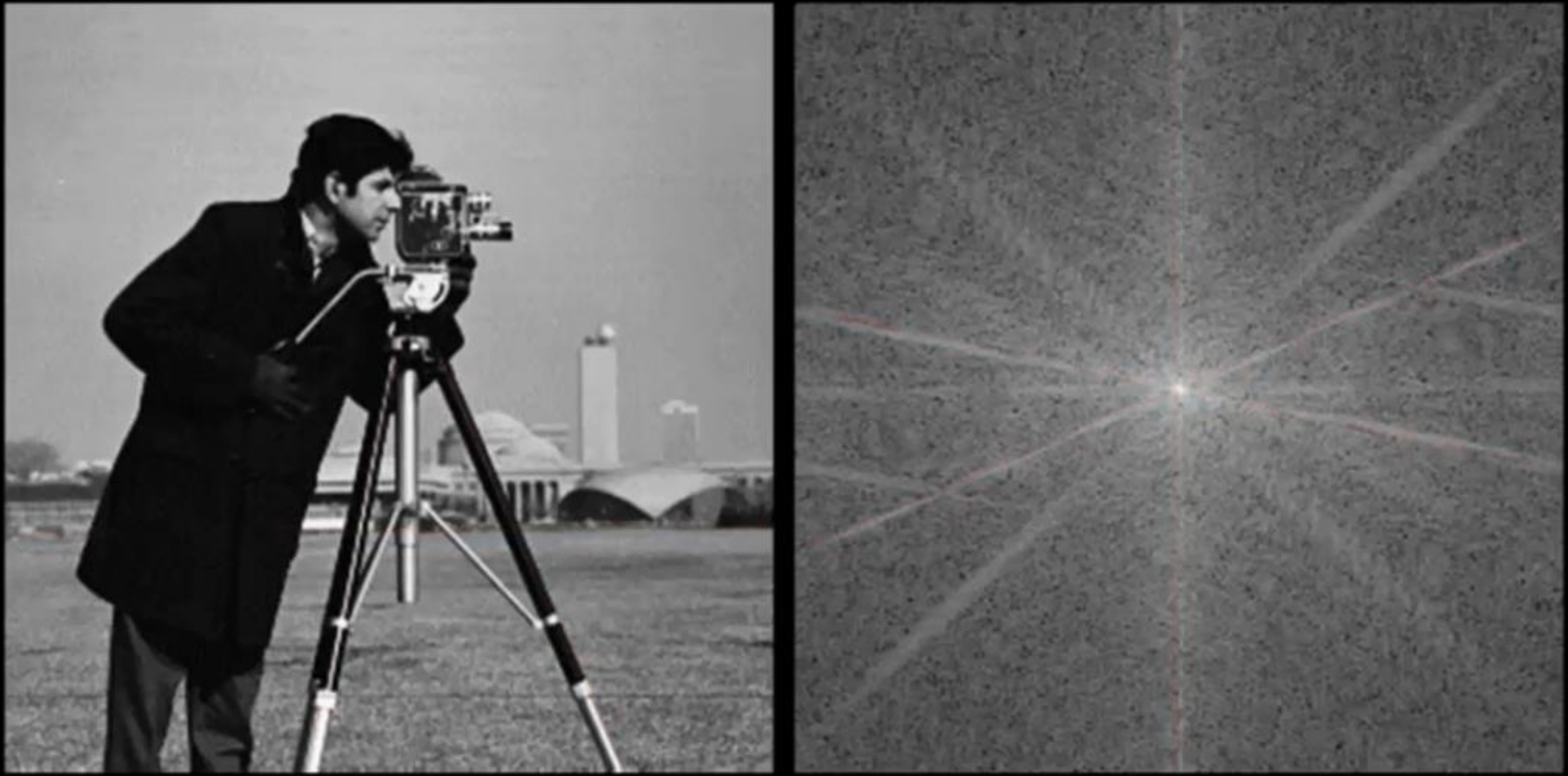
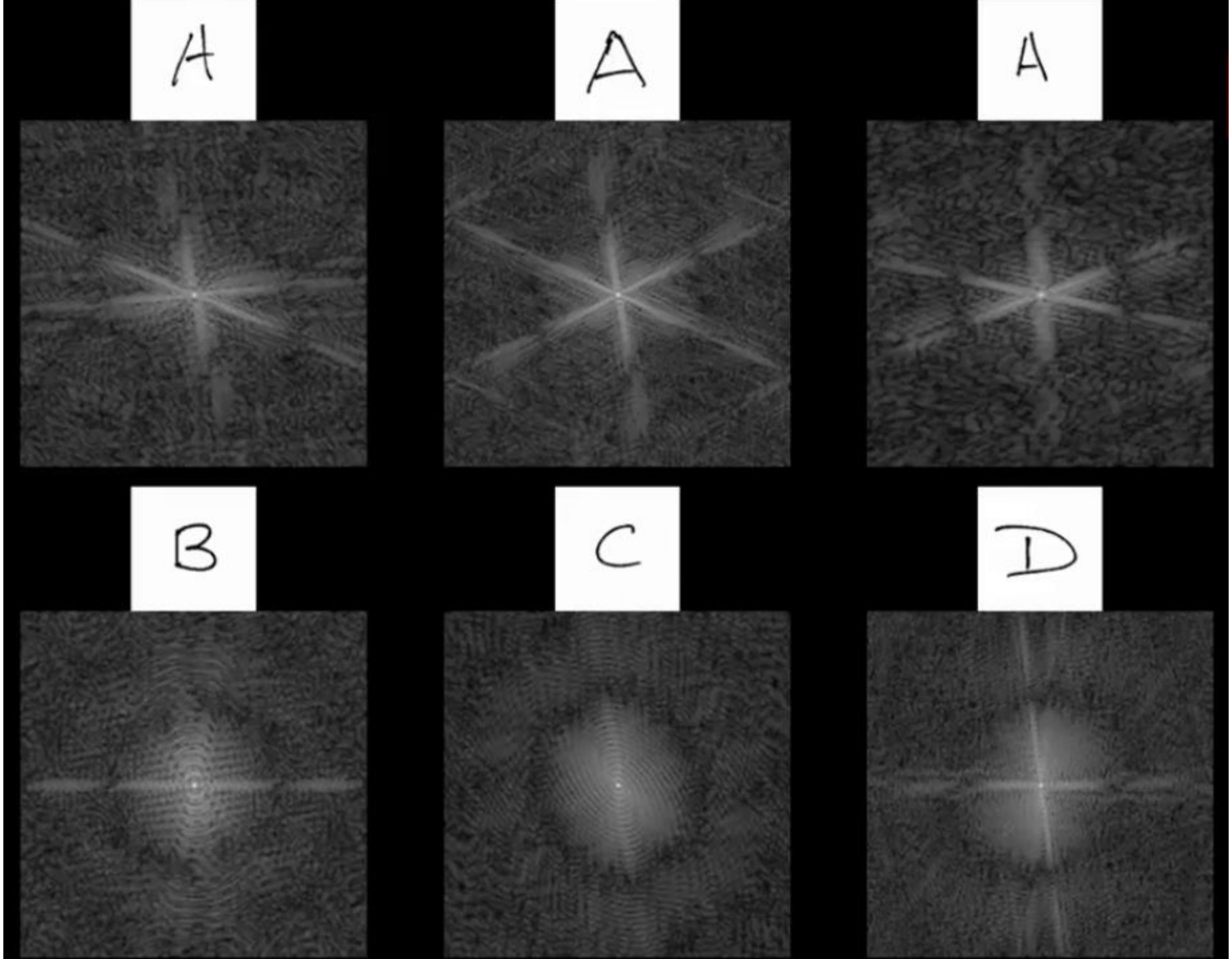
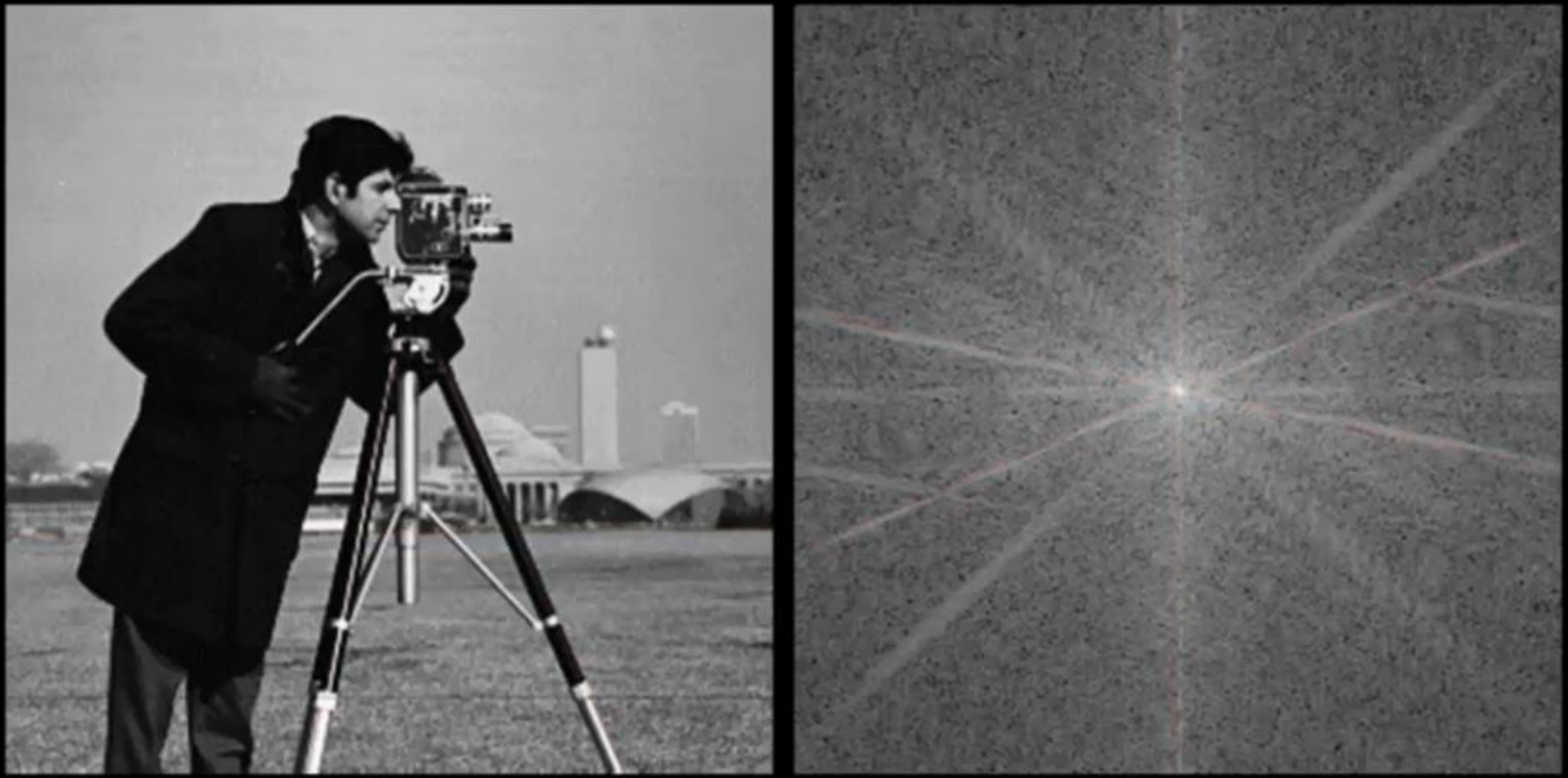


Image: 'cameraman'

https://au.mathworks.com/help/images/examples/deblurring-images-using-the-blind-deconvolution-algorithm.html


A
A
A
B
C
D



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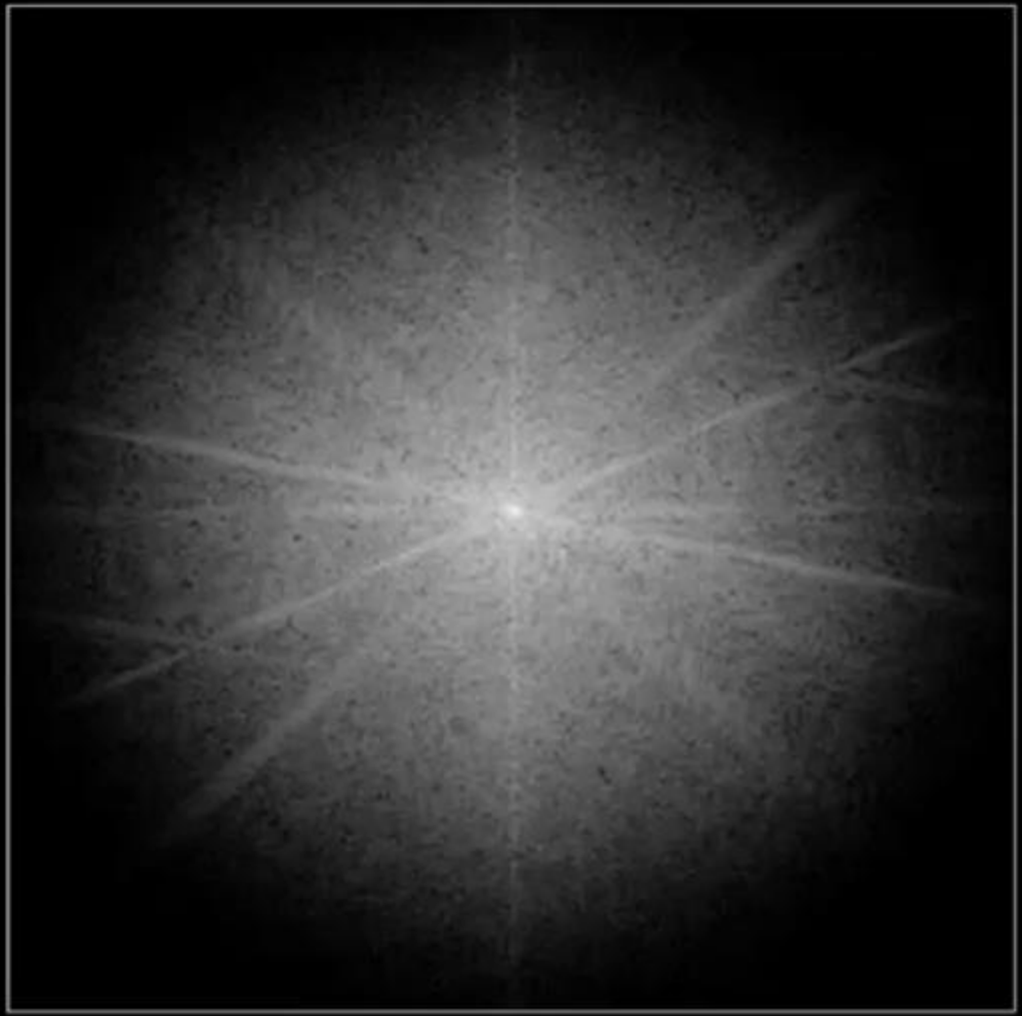

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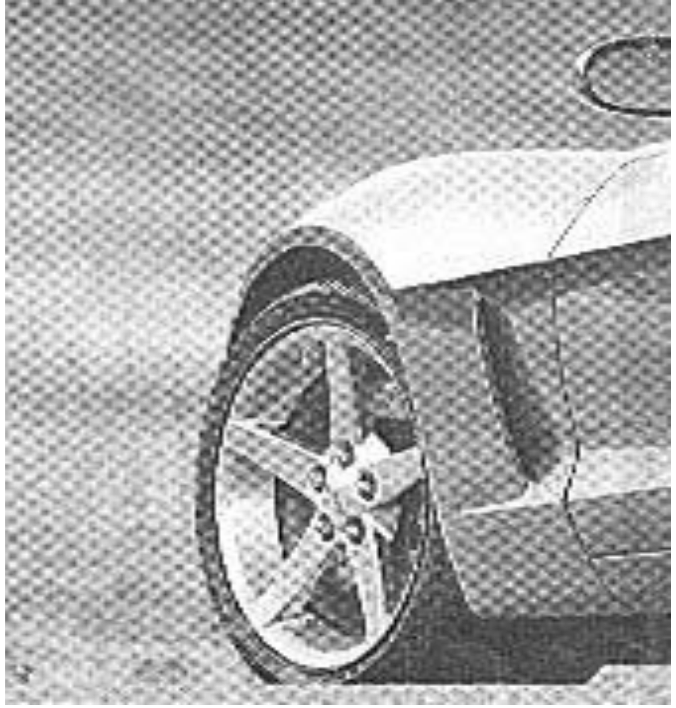
a b
c d

FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern.
(b) Spectrum.
(c) Butterworth notch reject filter multiplied by the Fourier transform.
(d) Filtered image.



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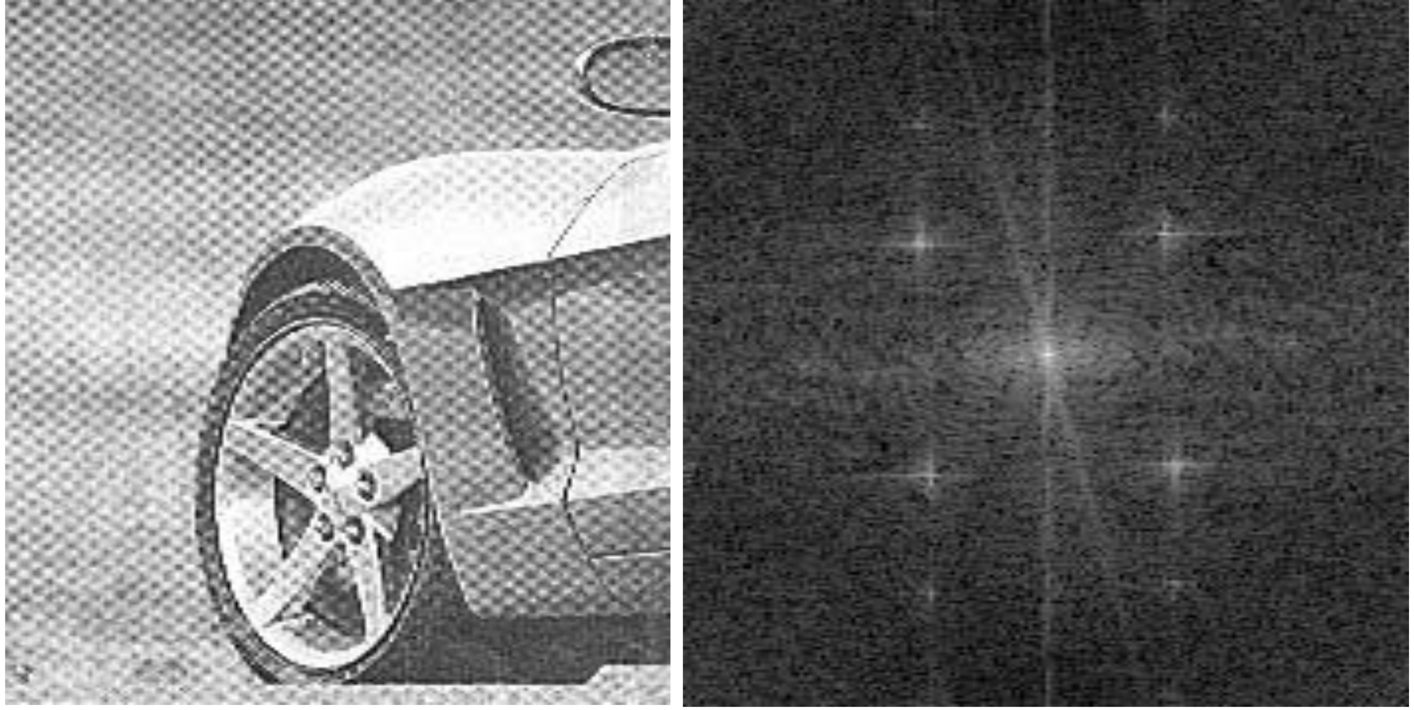

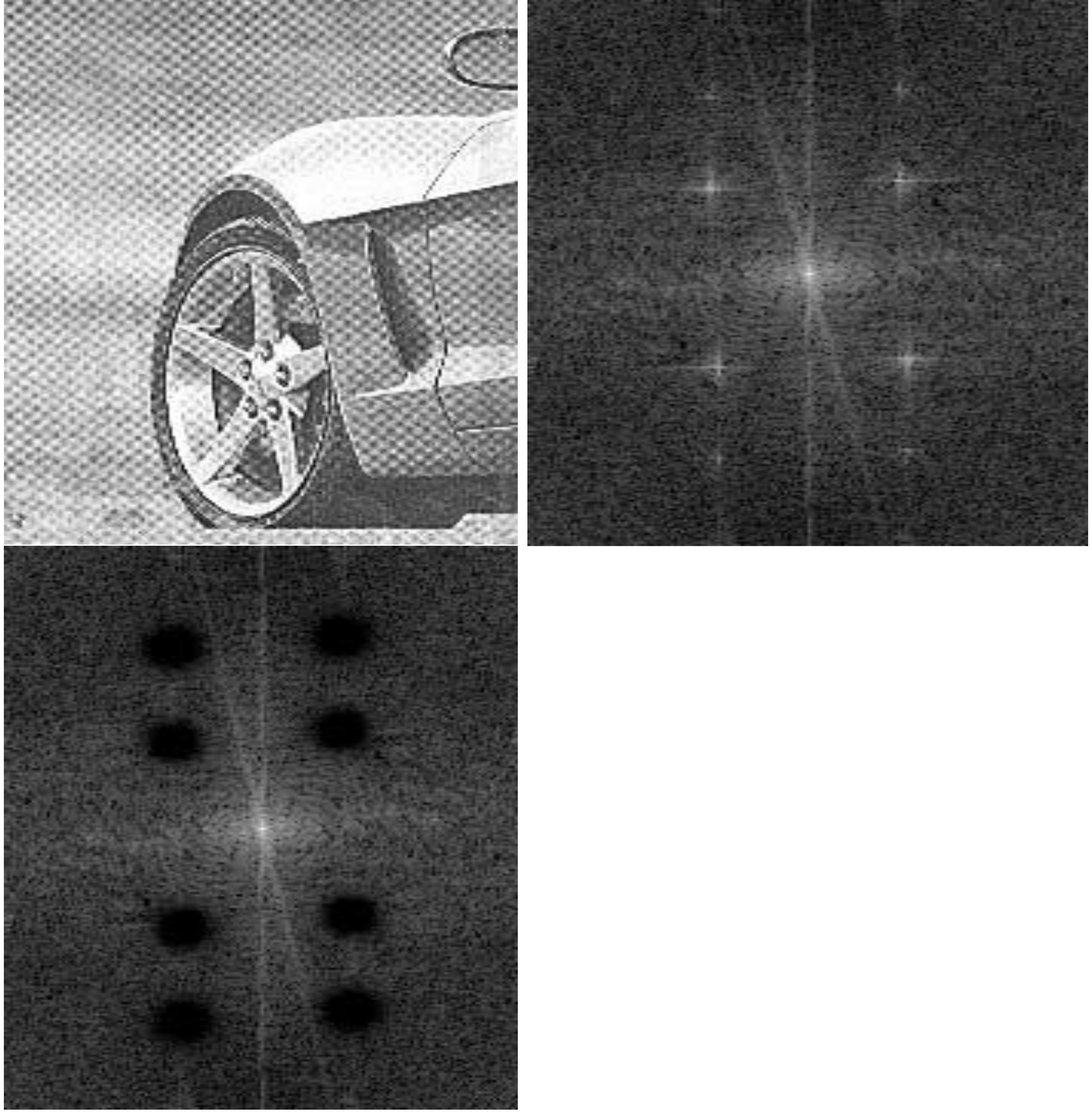

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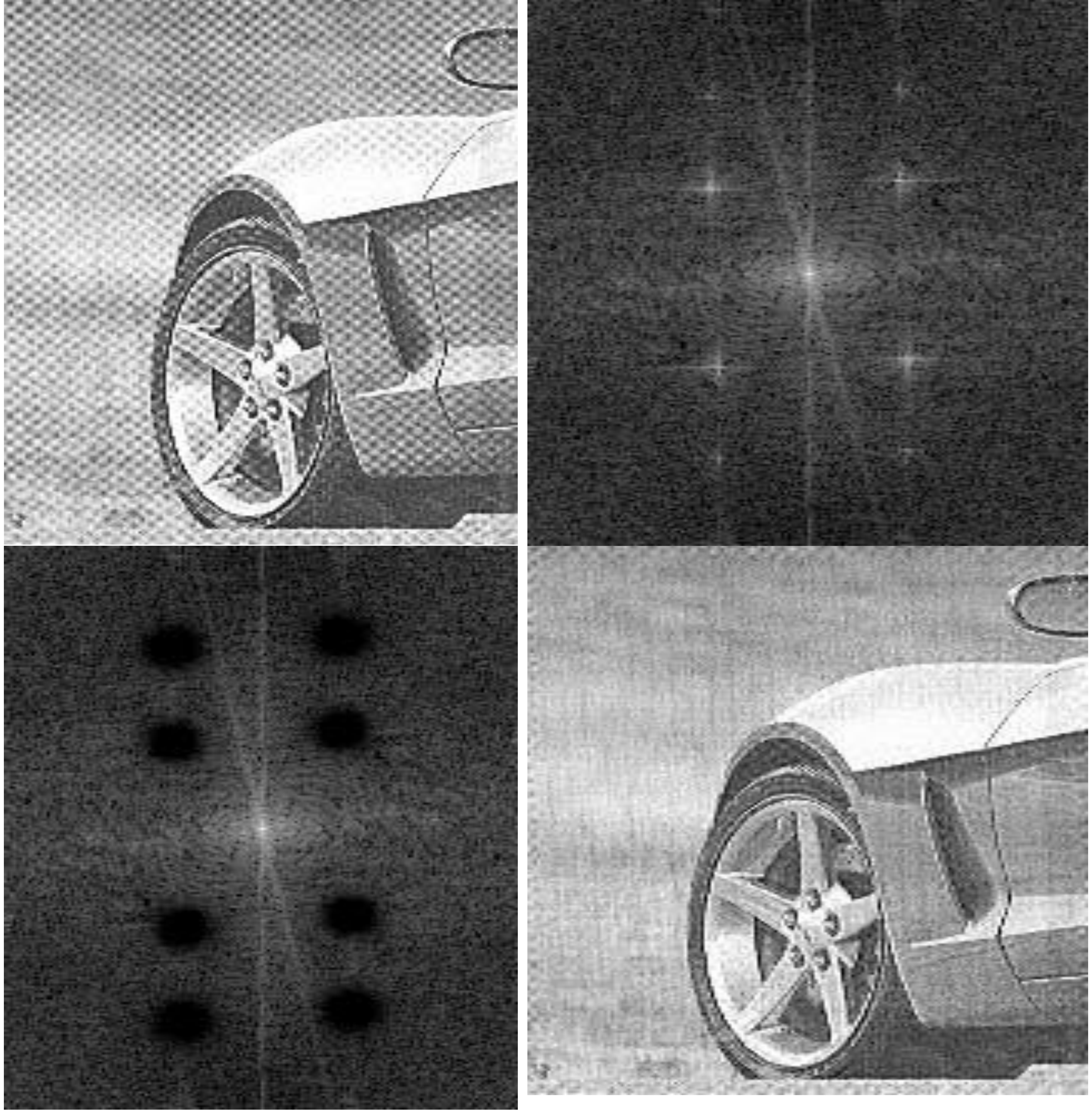

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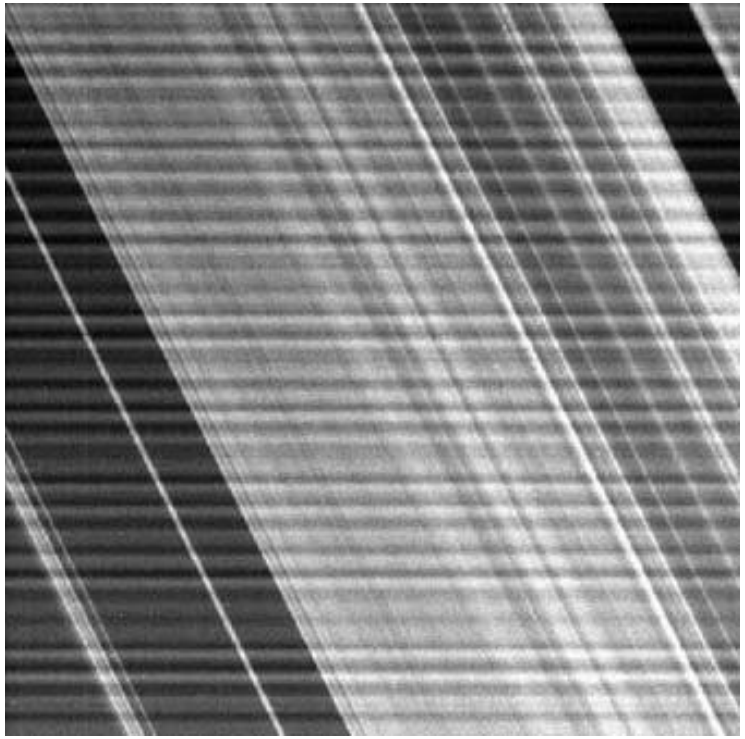
FIGURE 4.65
(a) 674×674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

a	b
c	d



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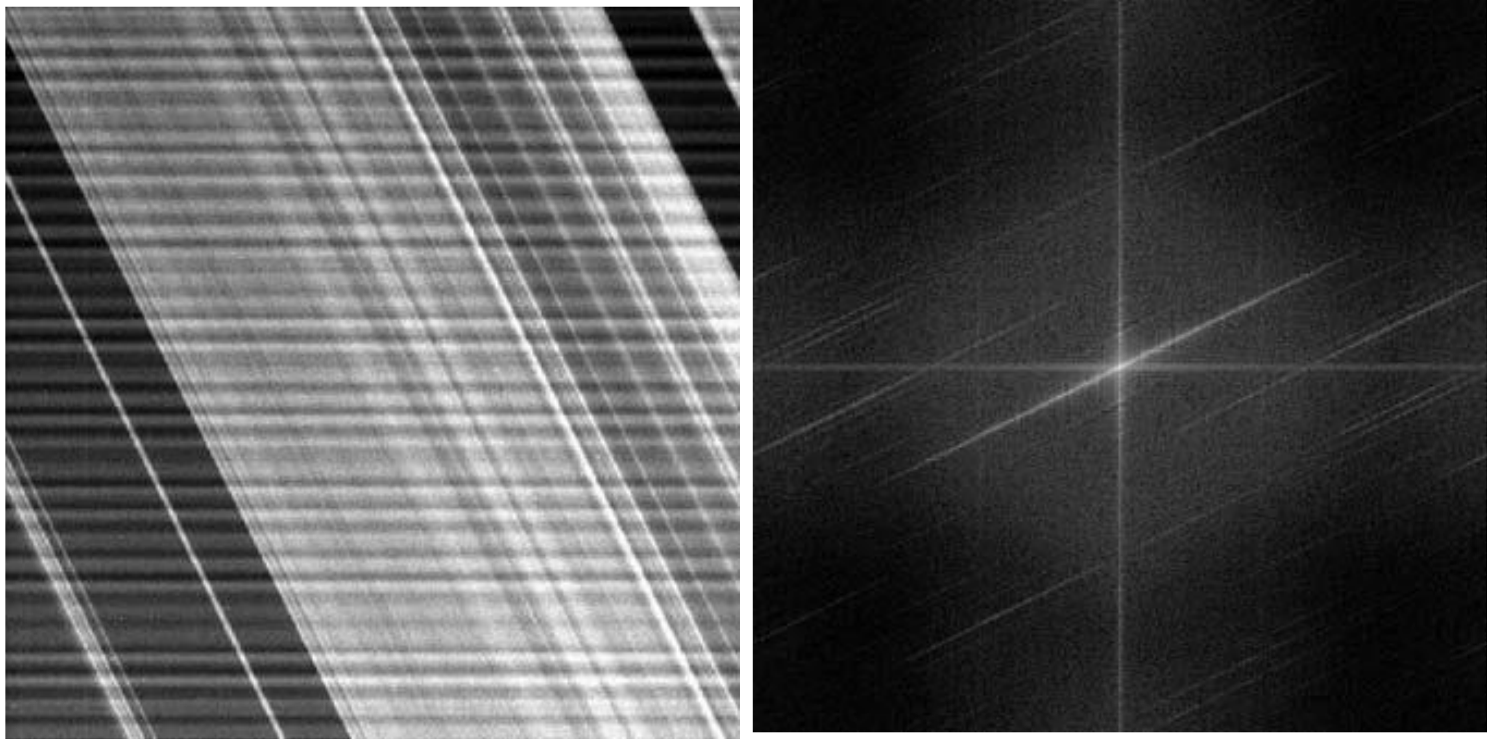

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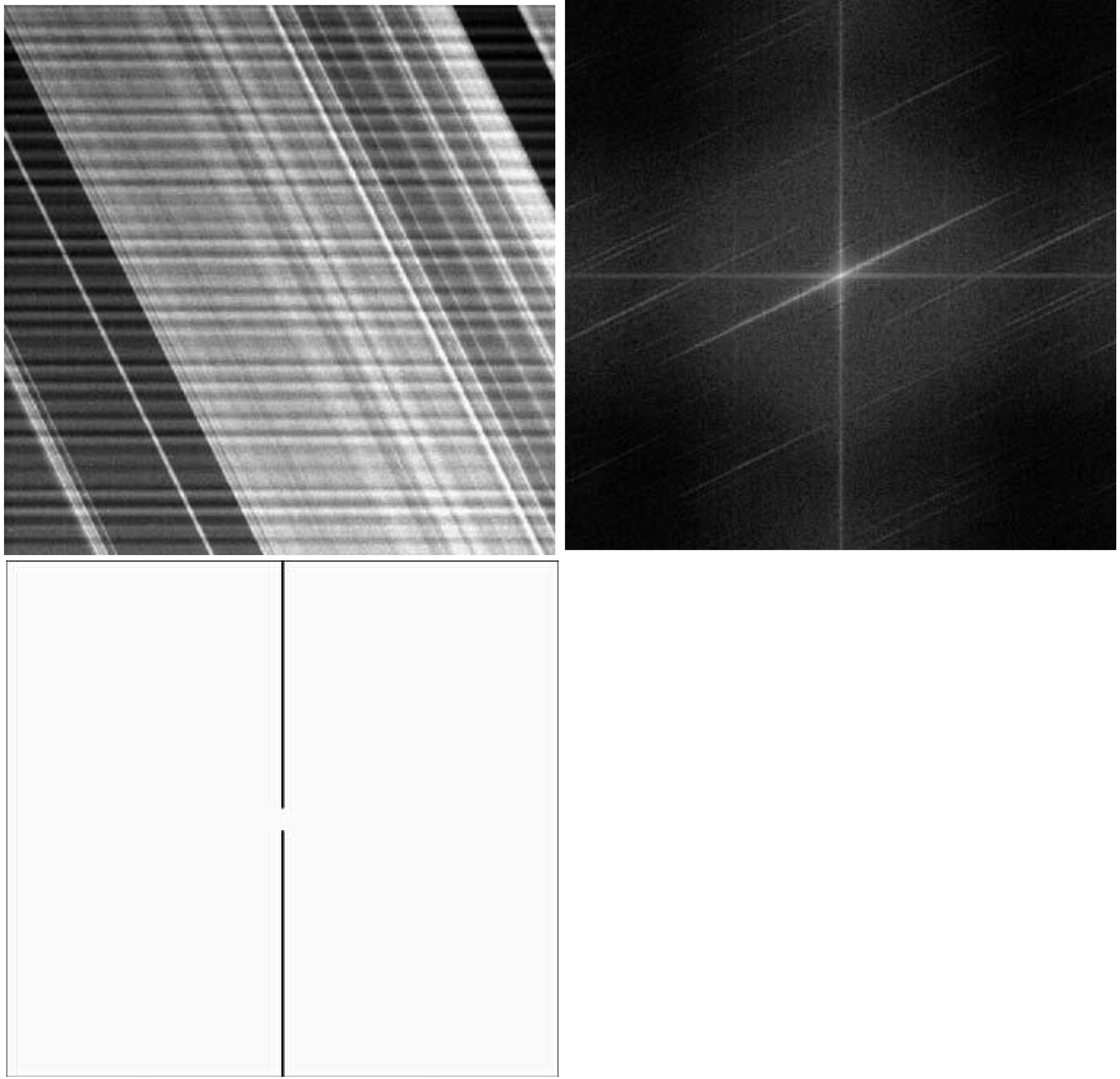

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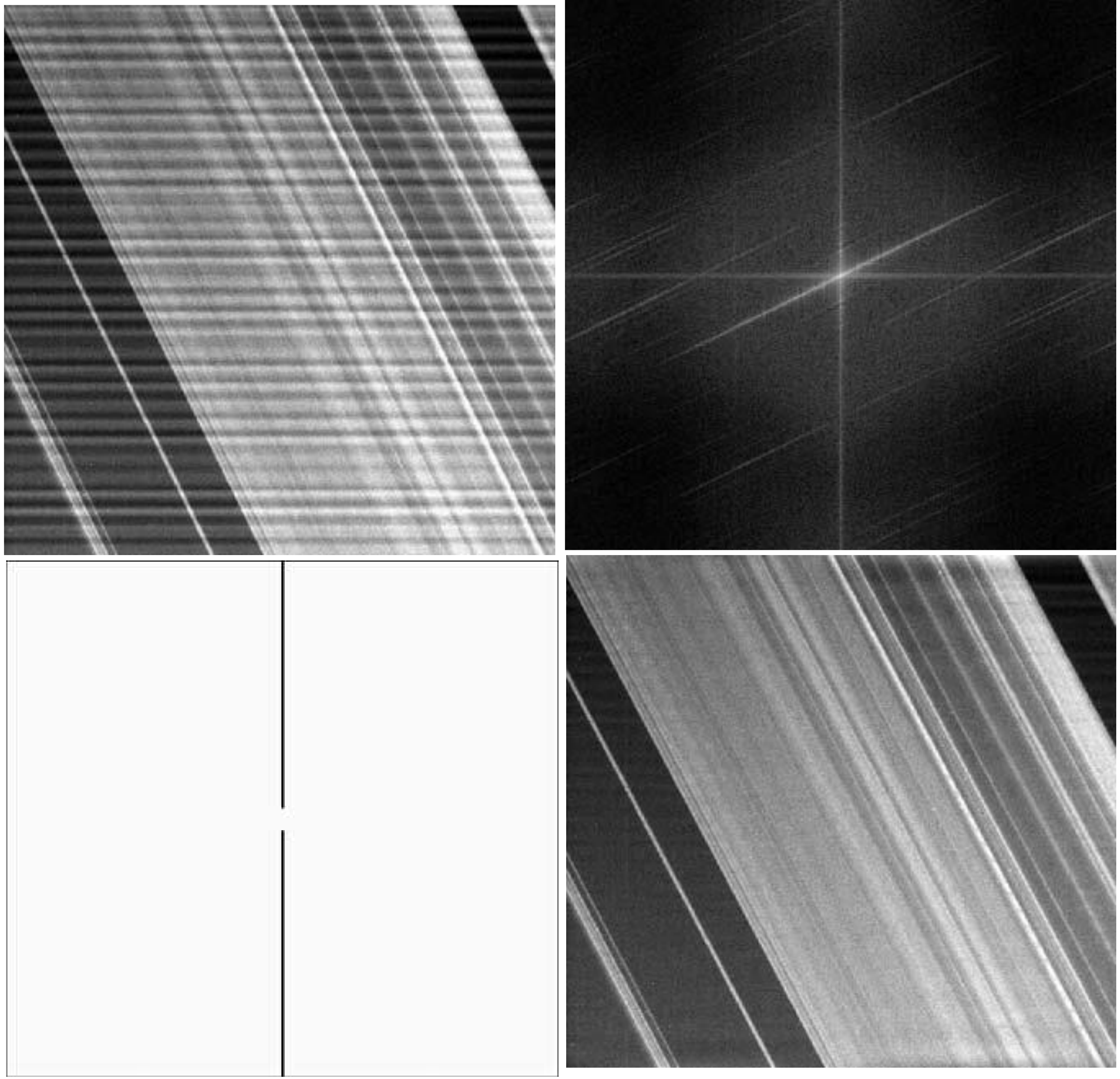

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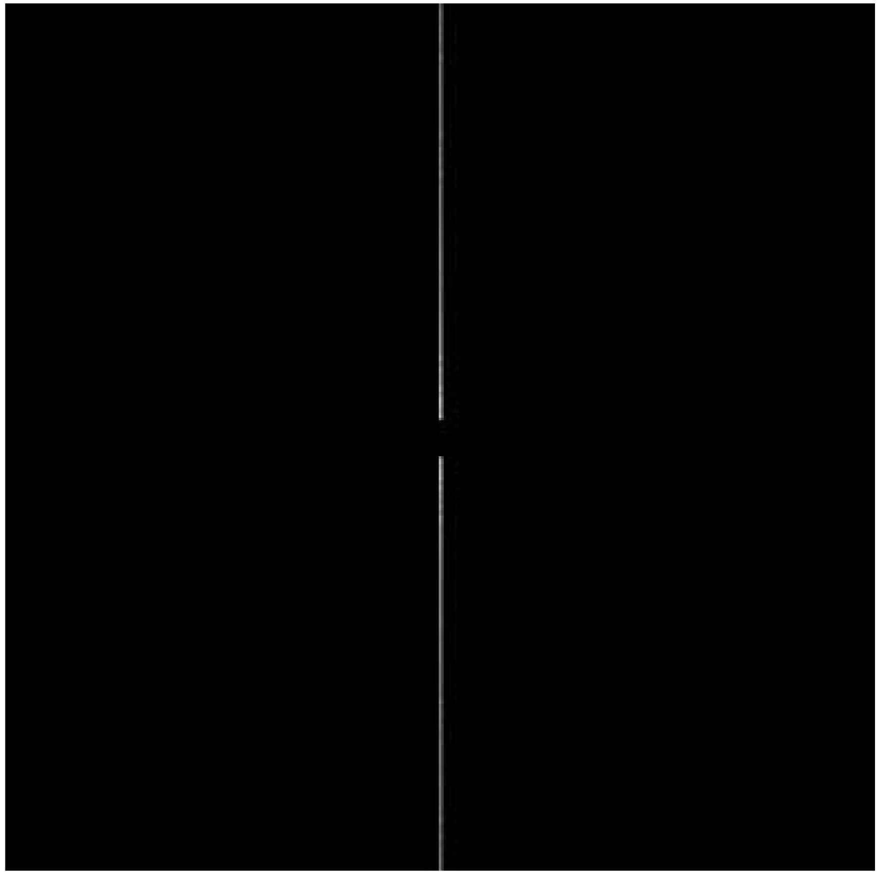
(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).



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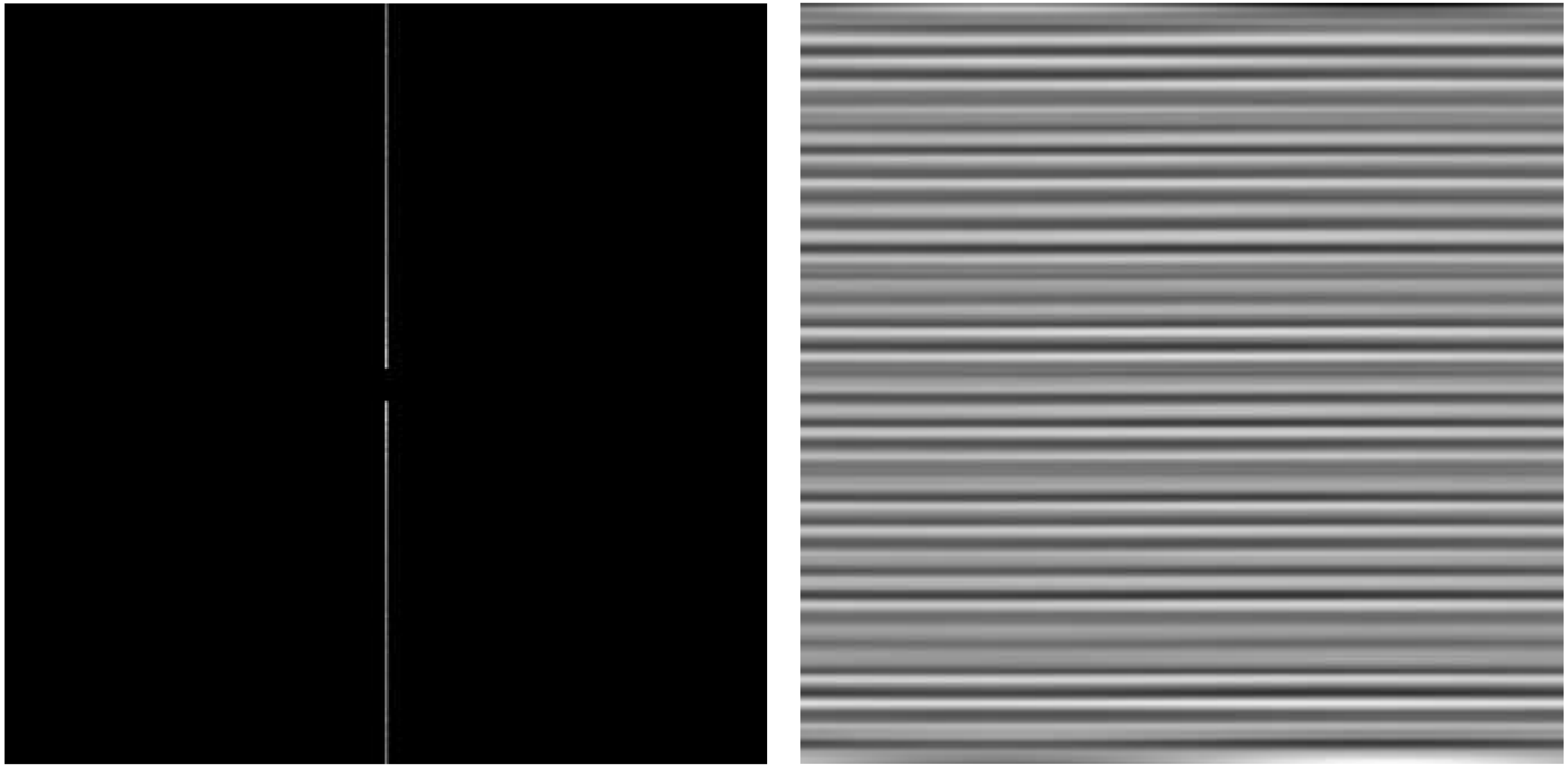

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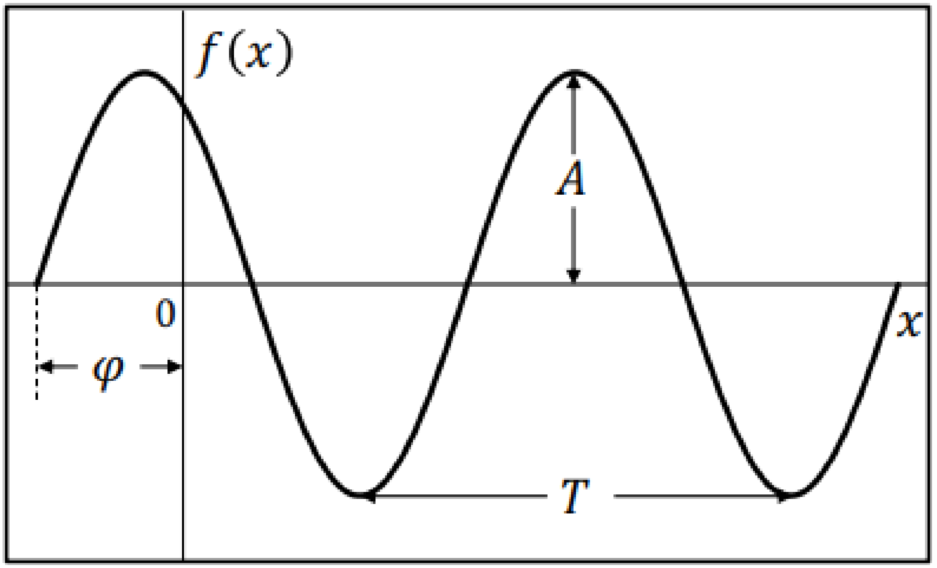
Jean Baptiste Joseph Fourier

(1768-1830)

Any Periodic Function can be rewritten as a Weighted Sum of Infinite Sinusoids of Different Frequencies.

Sinusoid

$$f(x) = A\sin(2\pi ux + \varphi)$$



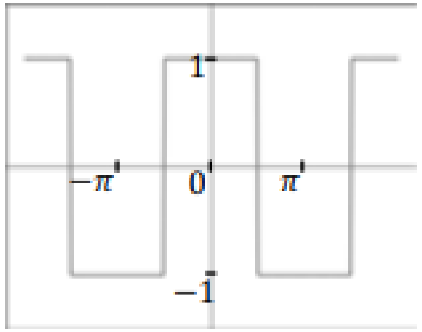
A: Amplitude

T: Period

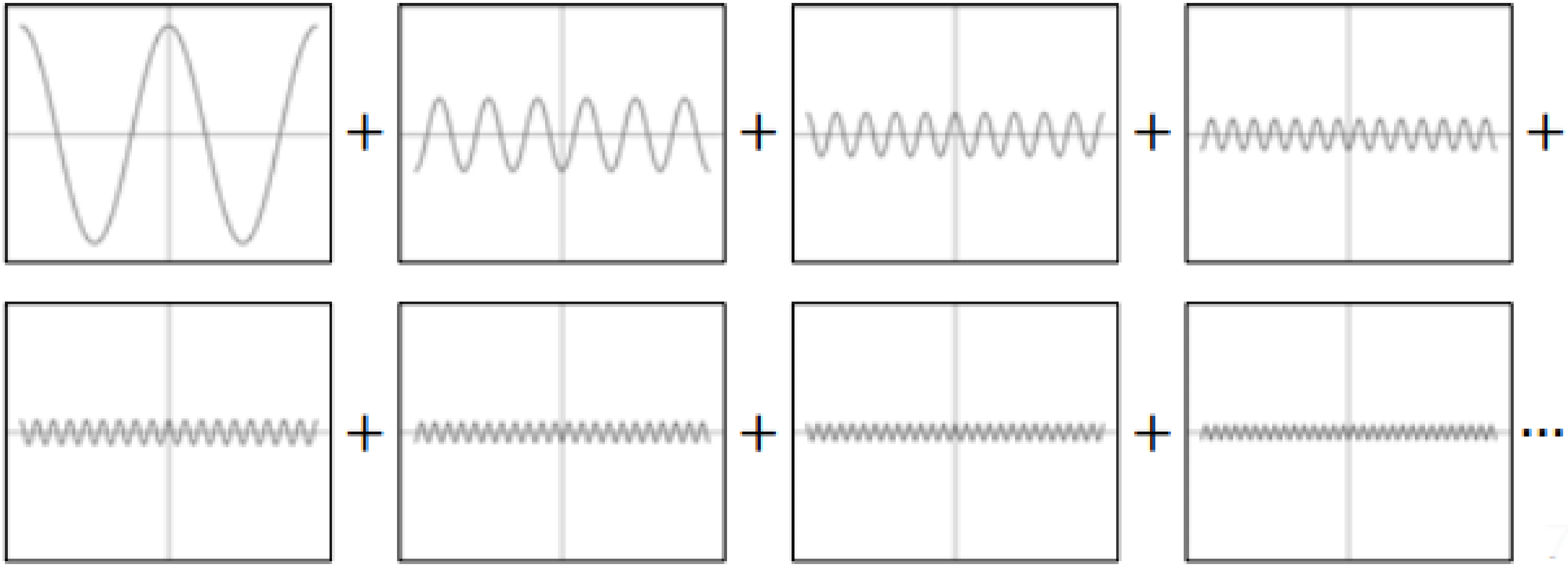
φ: Phase

u: Frequency $(1/T)$

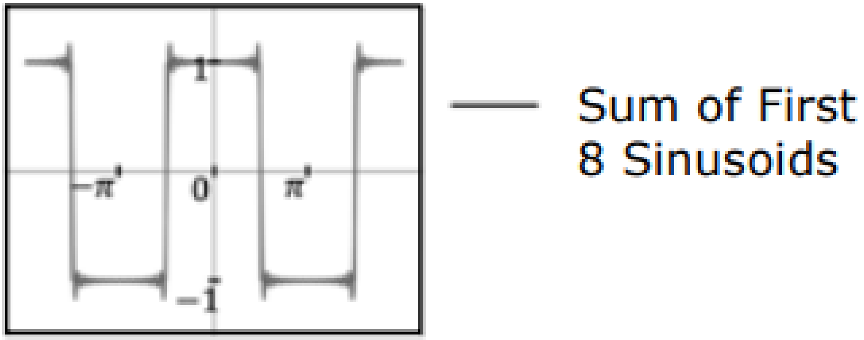
Fourier Series

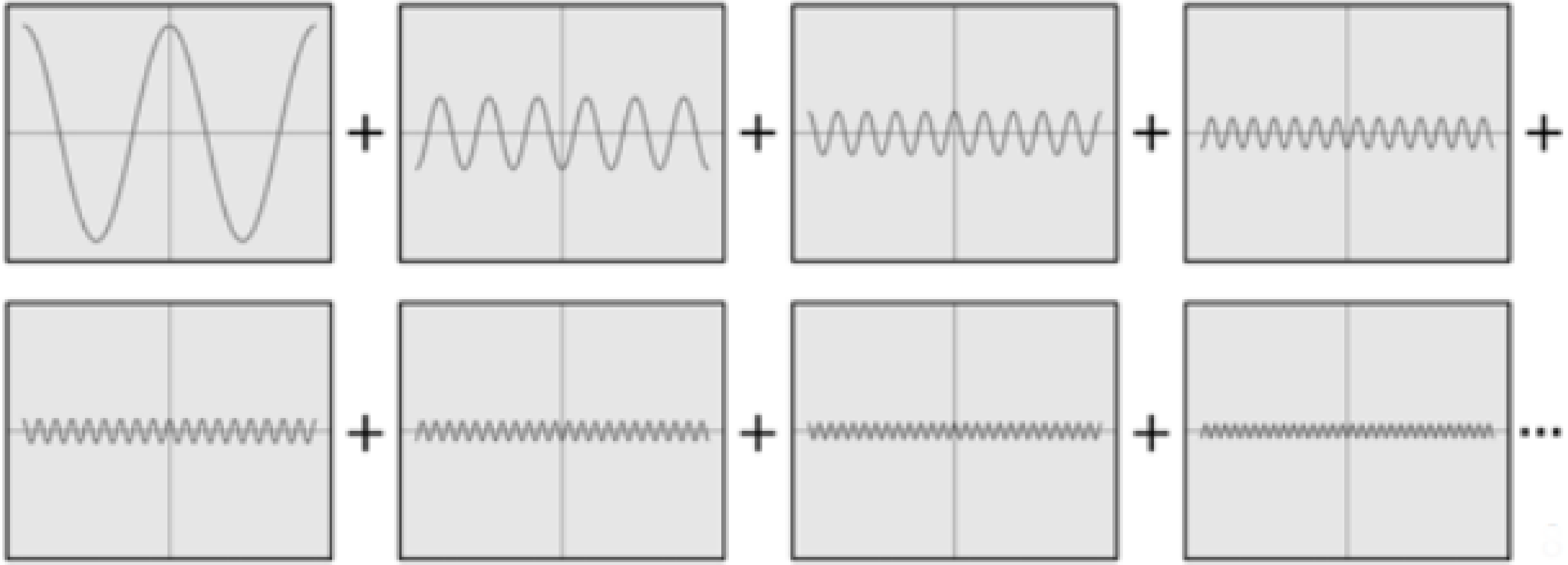


Square Wave
(Period 2π)

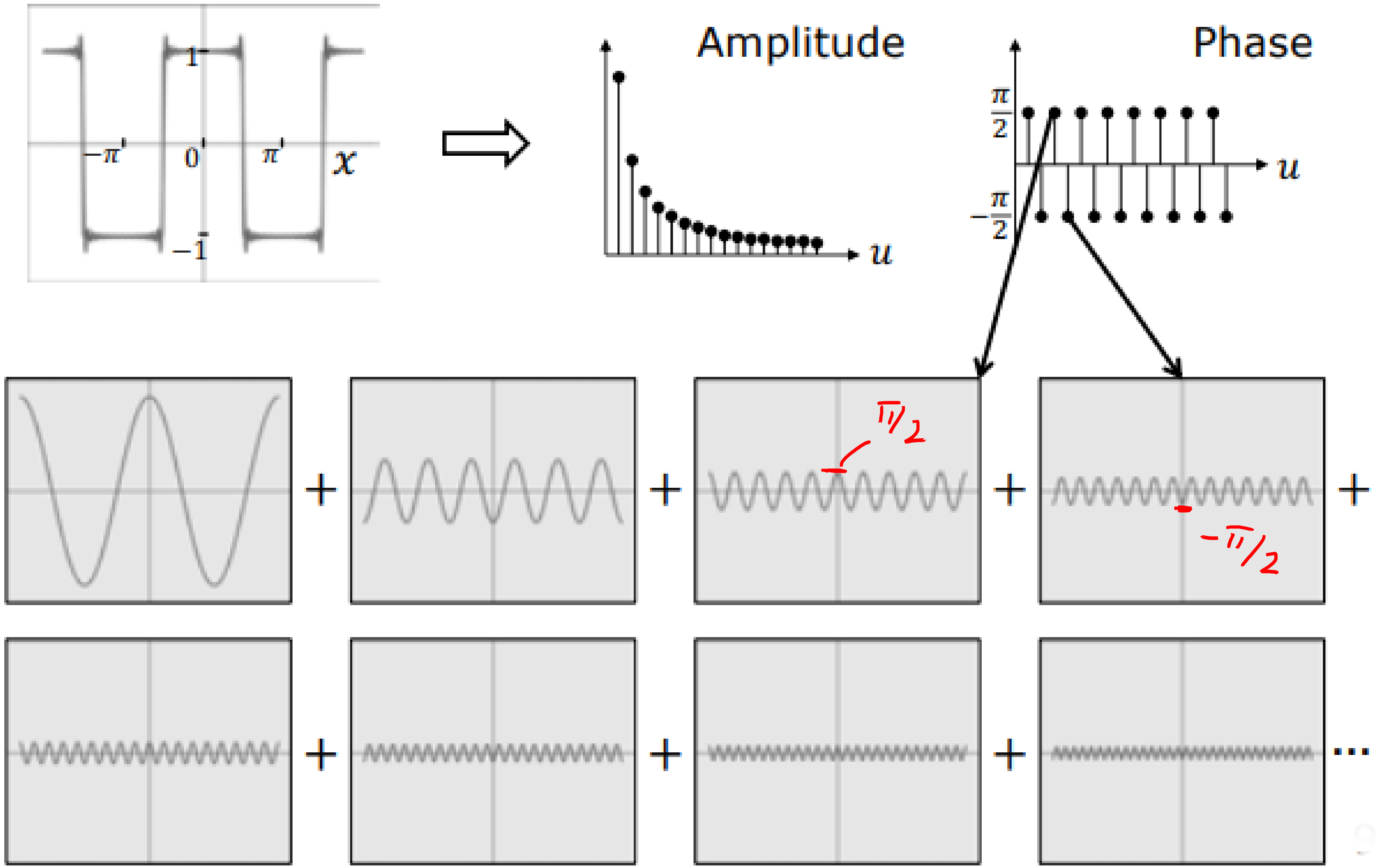


Fourier Series

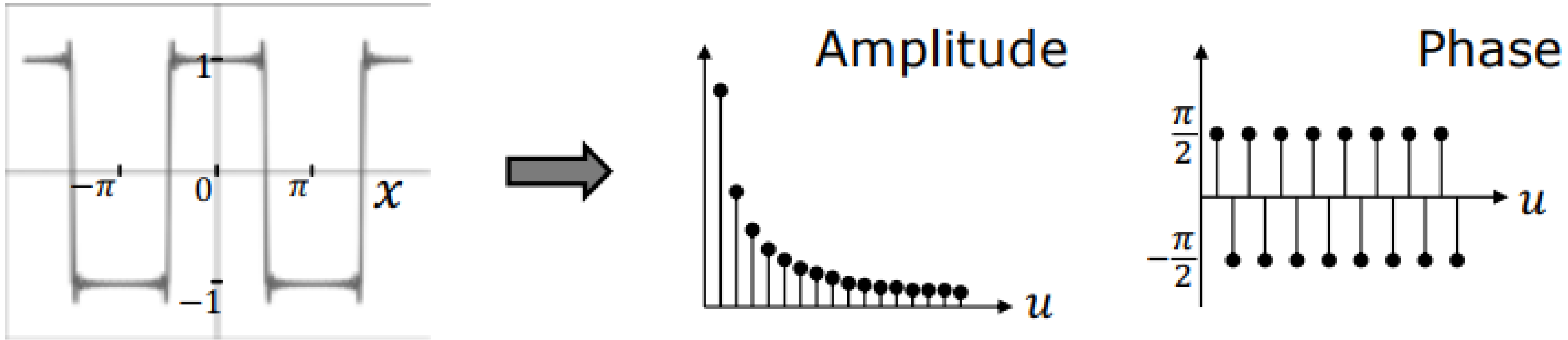




Frequency Representation of Signal



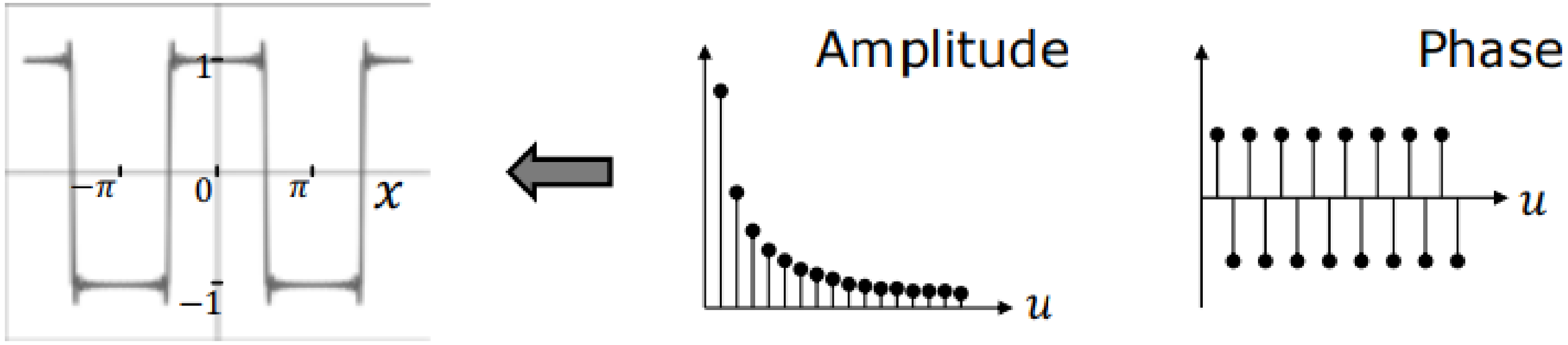
Fourier Transform (FT)



Represents a signal $f(x)$ in terms of Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longrightarrow \boxed{\mathrm{FT}} \longrightarrow F(u)$$

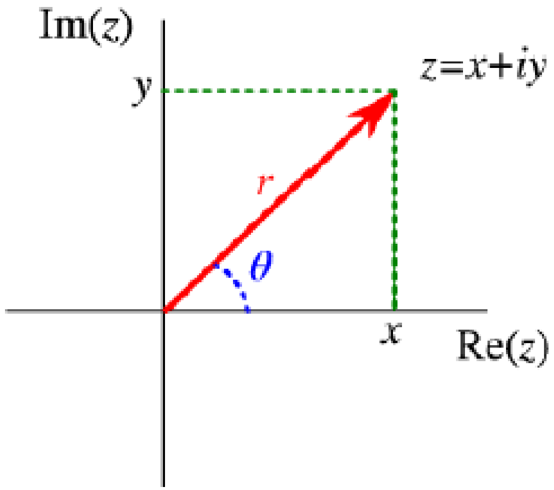
Inverse Fourier Transform (IFT)



Computes the signal $f(x)$ from the Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longleftarrow \boxed{\mathrm{IFT}} \longleftarrow F(u)$$

Maths Primer(Preliminary Concept)



- A complex number C, is defined as

$$C = R + jI$$

$$C^* = R - jI$$

- Sometimes, it is useful to represent complex numbers in polar coordinates

$$C = |C|(\cos\theta + j\sin\theta) \qquad |C| = \sqrt{(R^2 + I^2)} \qquad \theta = \arctan(I/R)$$

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad C = |C|e^{j\theta}$$

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where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Impulses

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- And is constrained also to satisfy the identity

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$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Sifting Property of Impulses

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Unit Discrete Impulse

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- Unit discrete impulse located at x = 0,

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- More generally using a discrete impulse located at $x = x_0$

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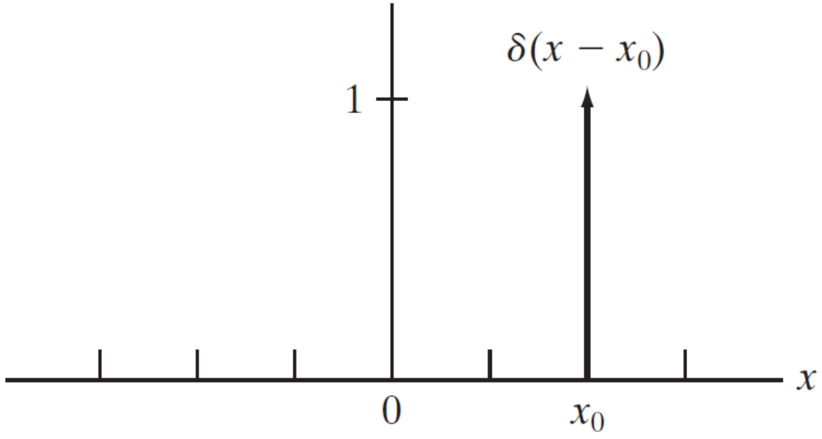

FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Impulse Train

Impulse Train

- Sum of infinitely many periodic impulses ΔT units apart:

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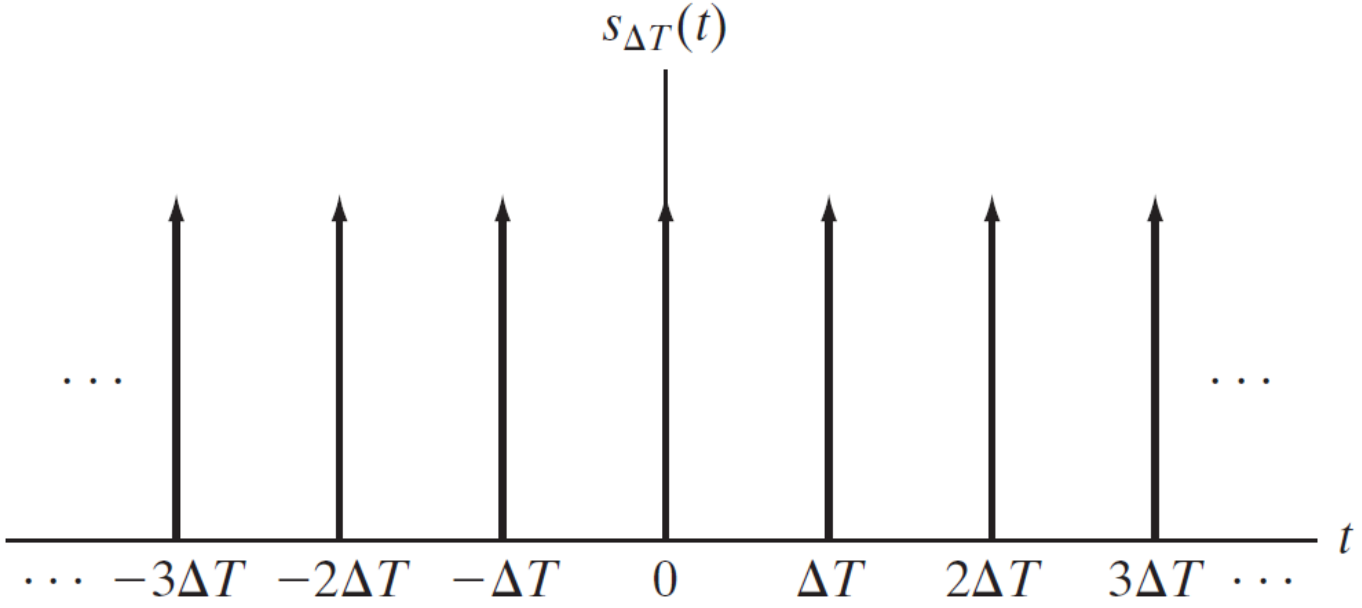

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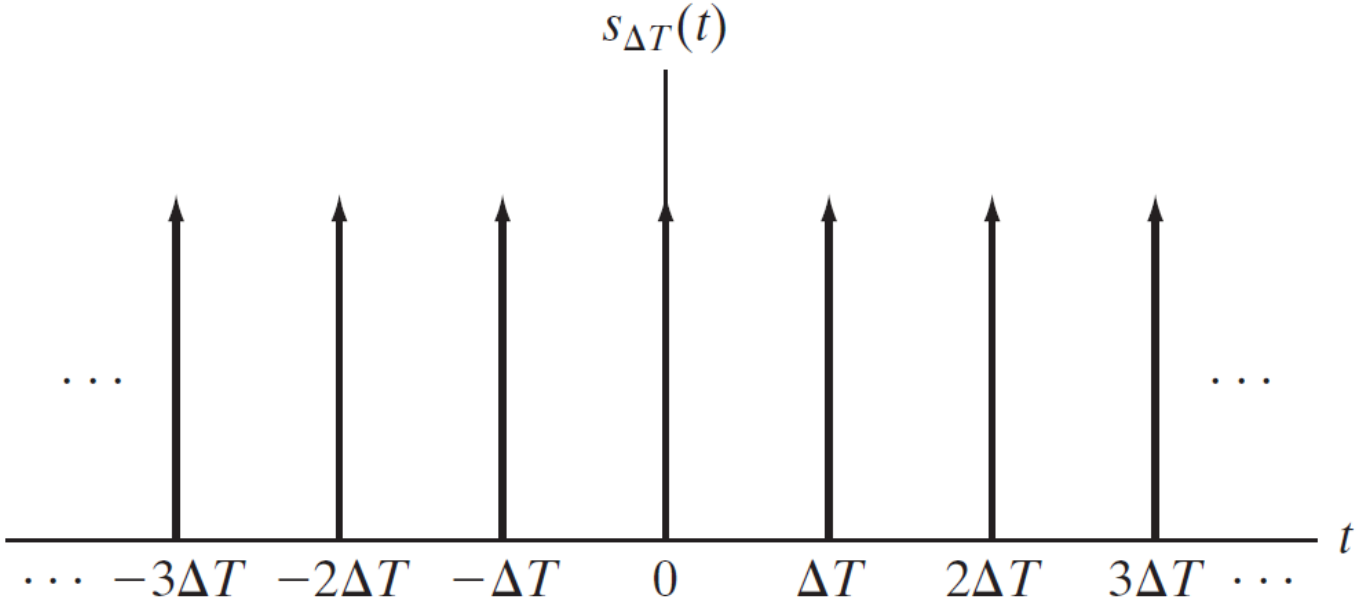

FIGURE 4.3 An impulse train.

Fourier Transform of functions of one continuous variable

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- Fourier transform of a continuous function $f(t)$ of a continuous variable t :

- Inverse Fourier Transform

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$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

- Inverse Fourier Transform

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- Fourier transform of a continuous function $f(t)$ of a continuous variable t :

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\left[\cos(2\pi\mu t) - j\sin(2\pi\mu t)\right] dt$$

- Inverse Fourier Transform

$$\Im^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}\, d\mu$$

Example

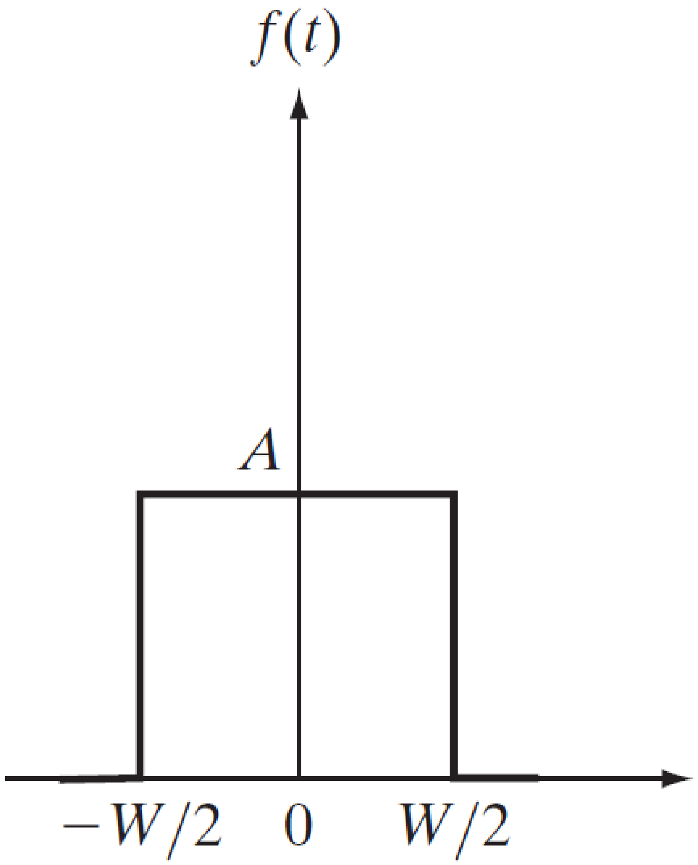
- Example 1: Find Fourier Transform of function shown in figure below

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\Big[e^{-j2\pi\mu t}\Big]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\Big[e^{-j\pi\mu W} - e^{j\pi\mu W}\Big]$$

$$= \frac{A}{j2\pi\mu}\Big[e^{j\pi\mu W} - e^{-j\pi\mu W}\Big]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \qquad \sin\theta = (e^{j\theta} - e^{-j\theta})/2j.$$



- The result in the last step of the preceding expression is known as *sinc* function:

- Where sinc(0) = 1 and sinc(m) = 0 for all other integer values of m.

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$$\operatorname{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

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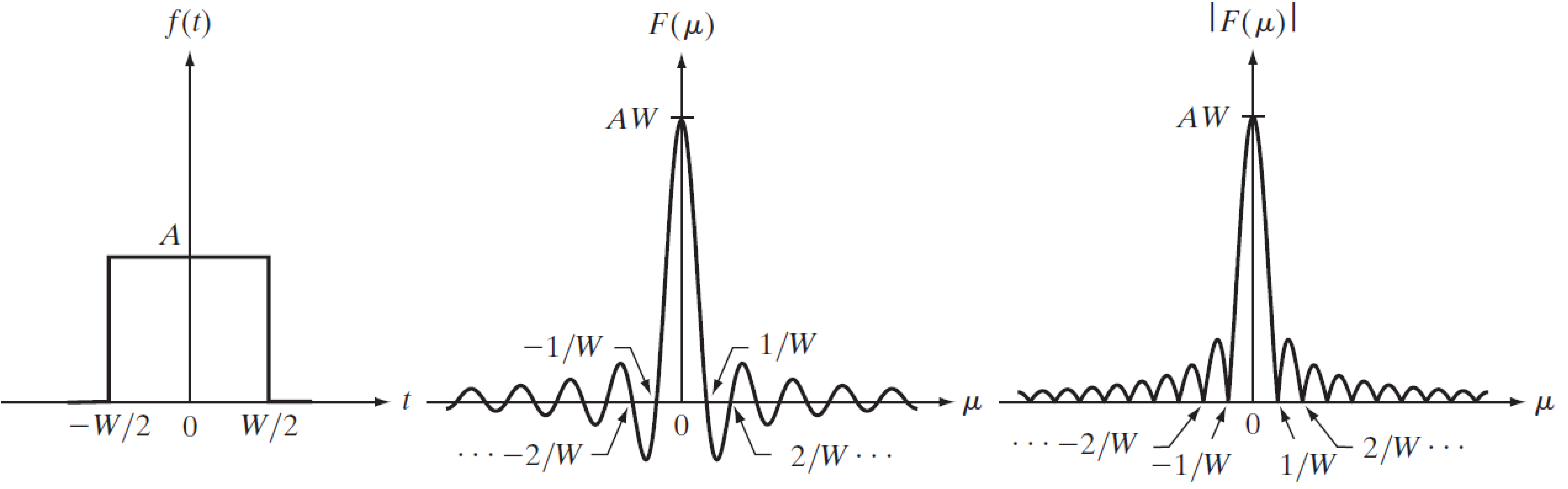

a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

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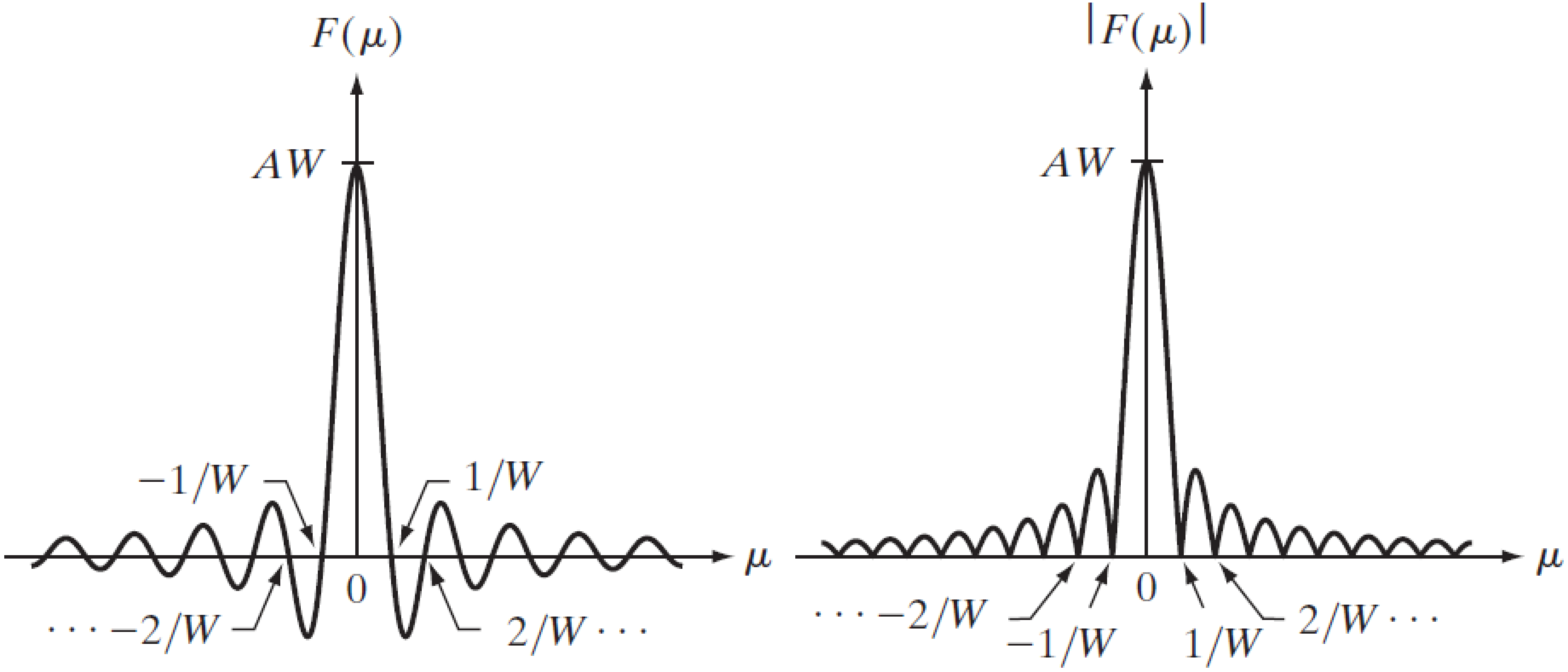

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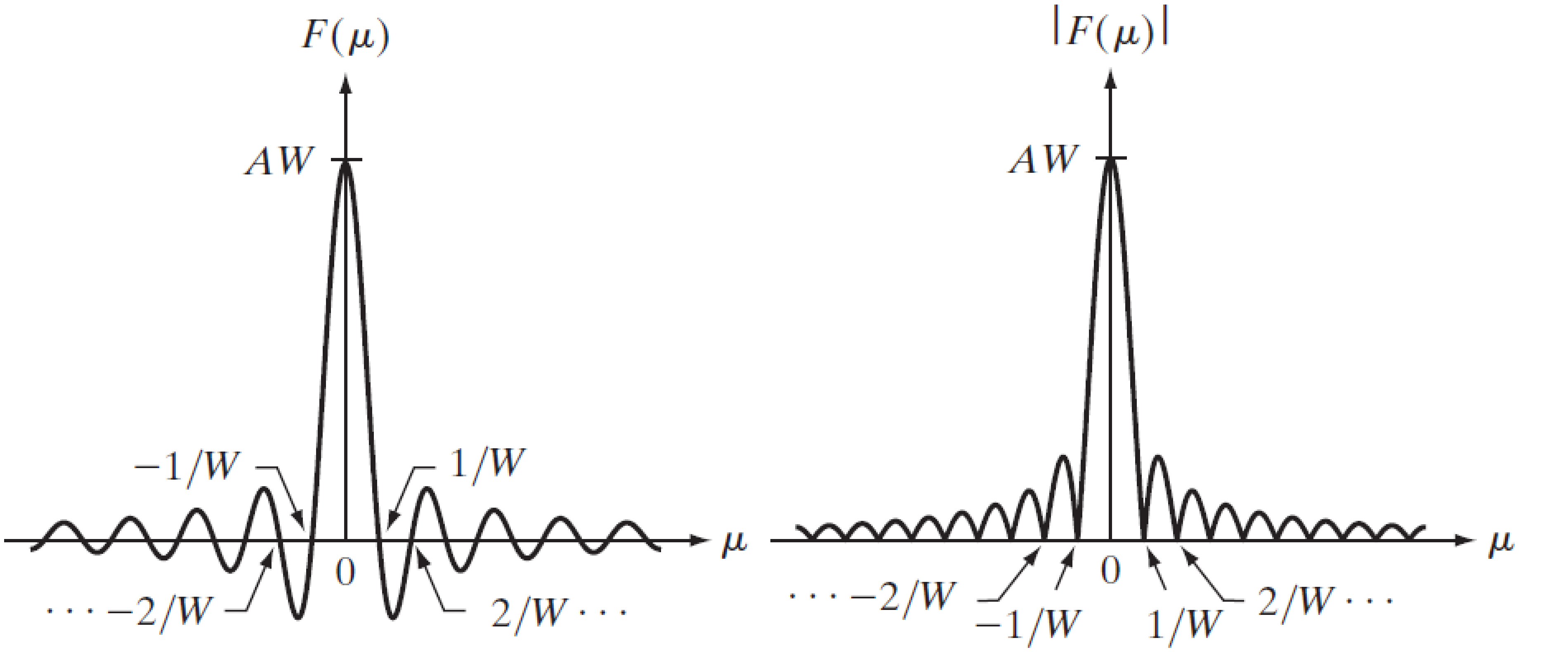
The locations of the zeros of both F(μ) and |F(μ)| are *inversely* proportional to the width, W, of the "box" function



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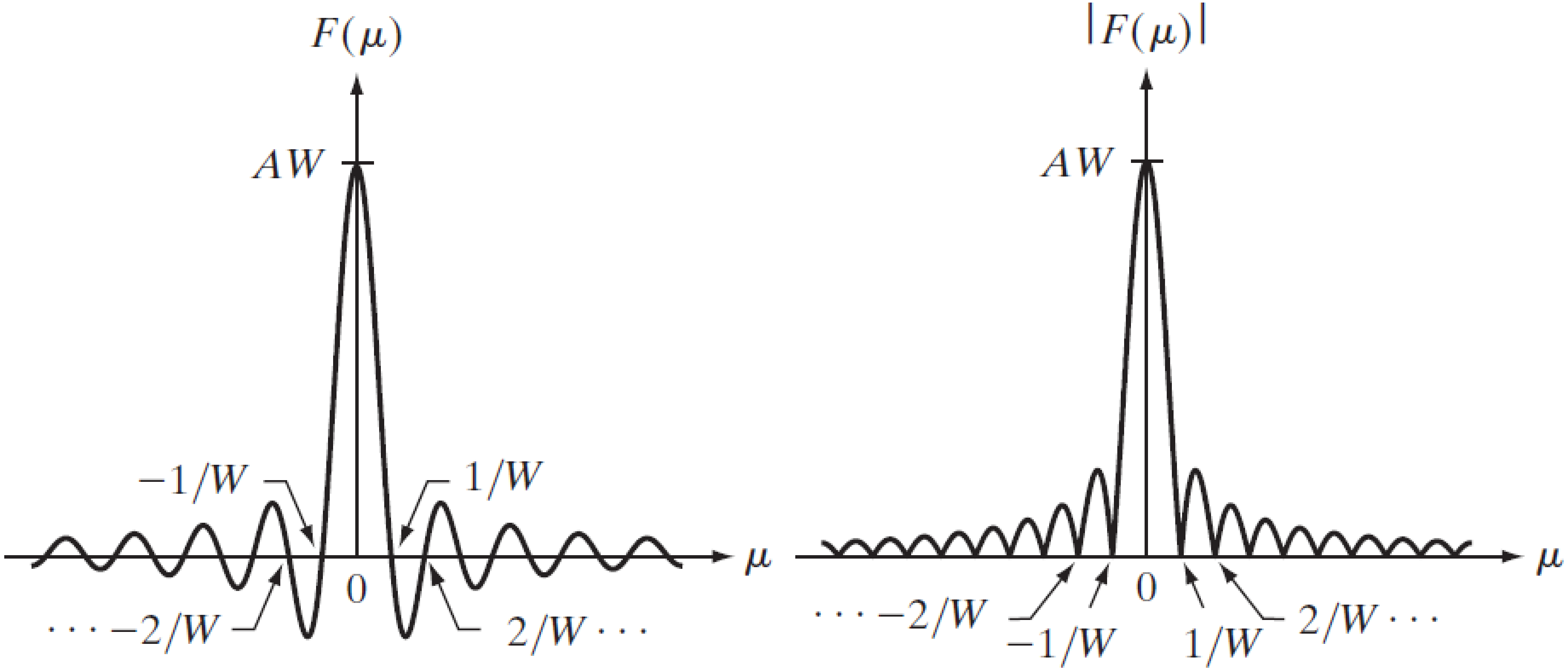

a b c

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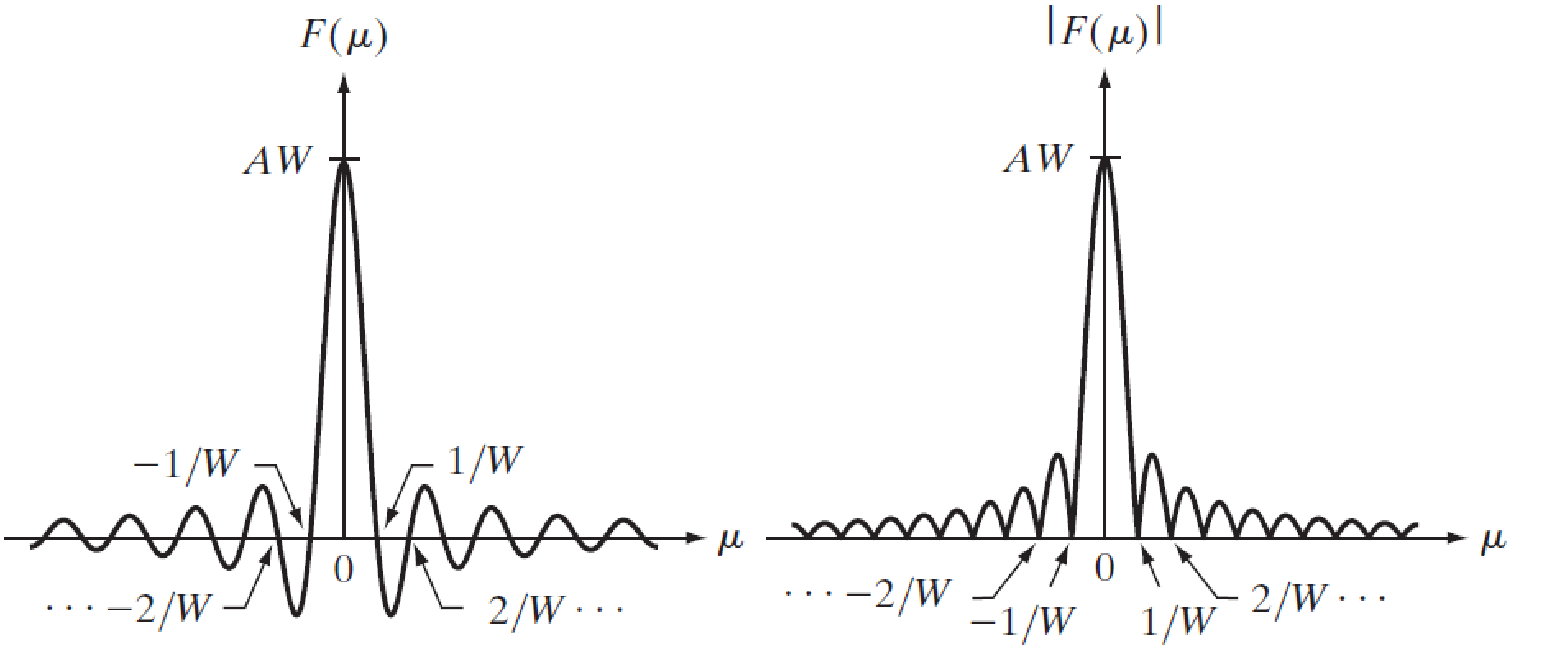
The function extends to infinity for both positive and negative values of μ



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Sifting property of impulse function

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$$F(\mu) = \delta(\mu - \mu_0)$$

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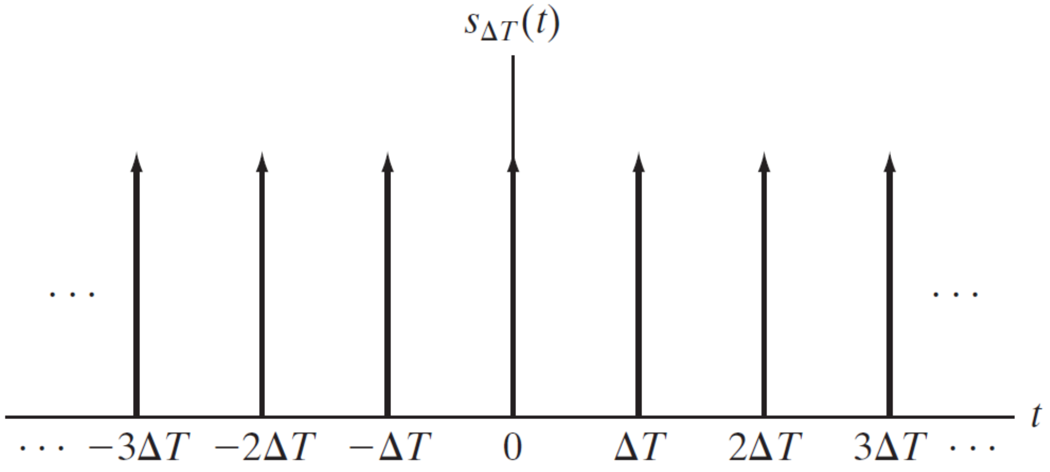

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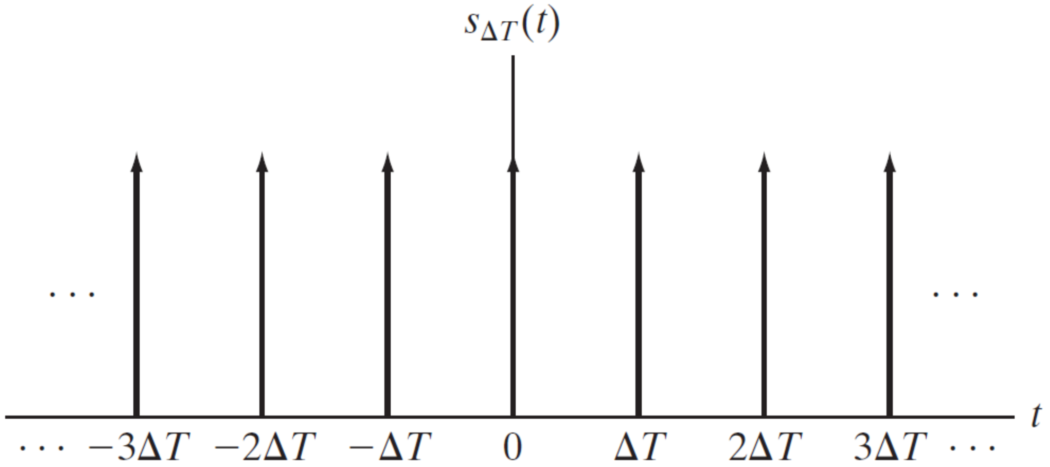

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The Fourier series expansion then becomes

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s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{\,j\frac{2\pi n}{\Delta T}t}
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This fundamental result tells us that Fourier transform of an impulse train with period ΔT is also an impulse train, whose period is 1/ ΔT