Filtering in Frequency Domain

(Lecture 2)

Convolution

- Convolution of the two continuous functions, $f(t)$ and $h(t)$, of one continuous variable t.

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

- Fourier Transform of the above equation

$$\Im\left\{f(t) \star h(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi\mu t}\, dt$$

Convolution Theorem: Proof

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau},$$

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\boxed{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,dt\right]}d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau = H(\mu)F(\mu)$$

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau};$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi\mu t}\, dt \qquad \text{put } t = u+\tau,\ dt = du$$

$$= \int_{-\infty}^{\infty} h(u)\, e^{-j2\pi\mu(u+\tau)}\, du$$

$$= \underbrace{\int_{-\infty}^{\infty} h(u)\, e^{-j2\pi\mu u}\, du}_{H(\mu)} \cdot e^{-j2\pi\mu\tau}$$

$$= H(\mu) \cdot e^{-j2\pi\mu\tau}$$

Convolution Theorem

- We have,

$$\Im\left\{f(t)\star h(t)\right\} = H(\mu)F(\mu)$$

- First half of convolution Theorem

$$f(t)\star h(t) \Leftrightarrow H(\mu)F(\mu)$$

- Second half of convolution theorem

$$f(t)h(t) \Leftrightarrow H(\mu)\star F(\mu)$$

Extension to function of two variables
The 2-D Impulse

- The impulse, $\delta\,(t, z)$, of two continuous variables, t and z is defined as

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

- and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1$$

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Sifting property of 2-D Impulse

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t,z)\,dt\,dz = f(0,0)$$

At Origin

Located at $(t_0\ ,\ z_0)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t - t_0, z - z_0)\,dt\,dz = f(t_0, z_0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

Discrete 2-D Impulse

- For discrete variables x and y, the 2-D discrete impulse is defined as

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

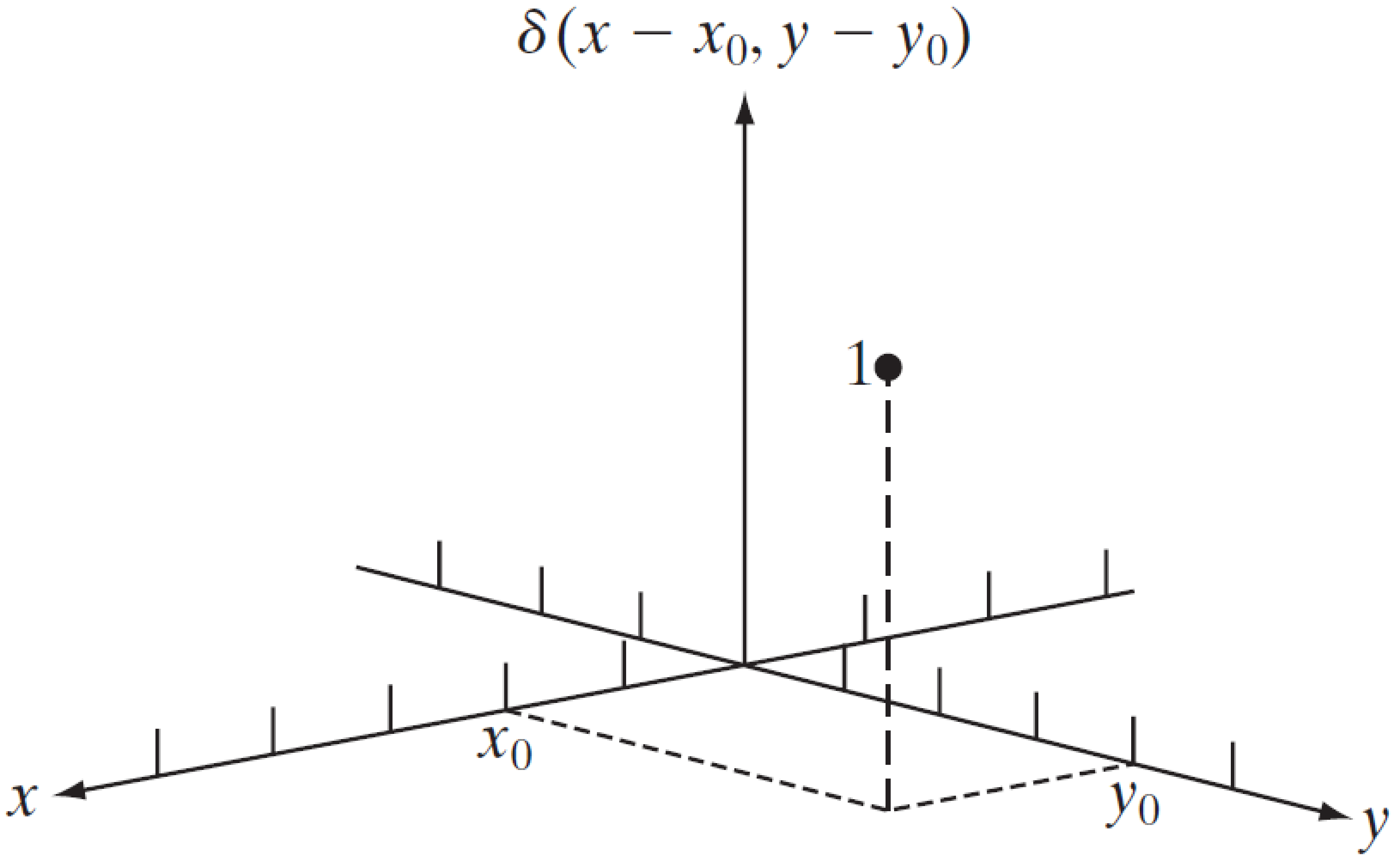
- Sifting property is

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x, y) = f(0, 0)$$

- For impulse located at (x_0 , y_0)

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).



The 2-D Continuous Fourier transform pair

- Let $f\,(t, z)$ be a continuous function of two continuous variables, t and z
- *Fourier Transform of* $\boldsymbol{f\,(t, z)}$

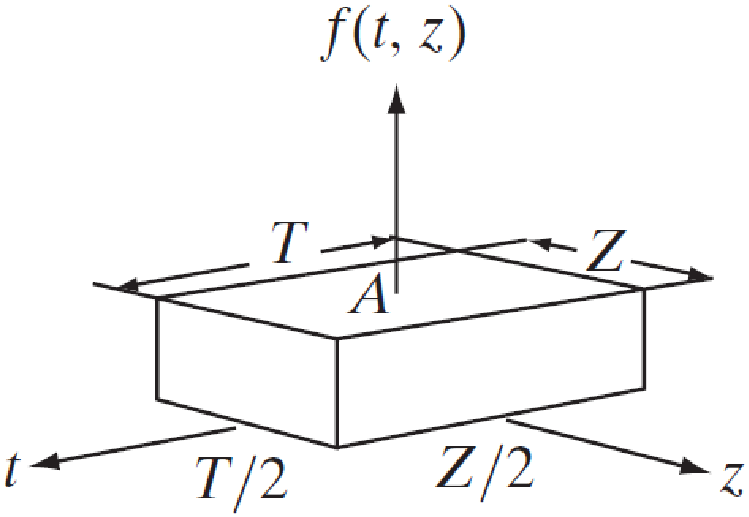
$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

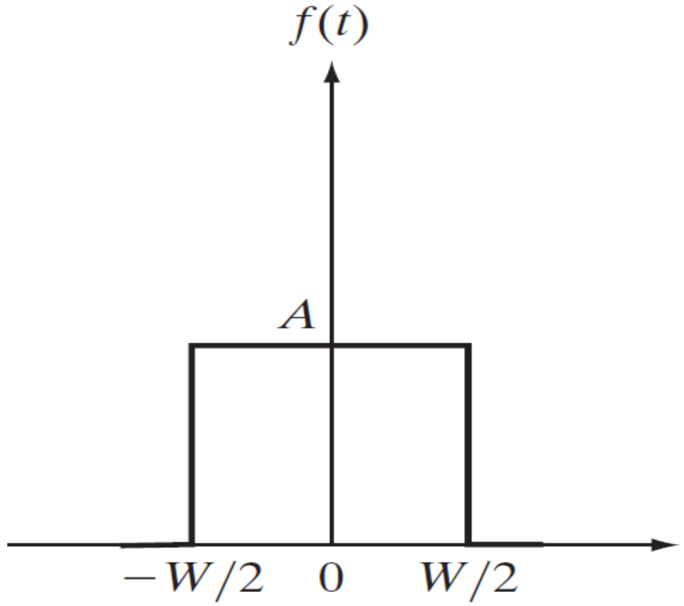
- *Inverse Fourier Transform of* F (μ, v)

$$f(t, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu, \nu)\, e^{j2\pi(\mu t + \nu z)}\, d\mu\, d\nu$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt}$$

- Example 1: Find Fourier Transform of a 2-D function given by





$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

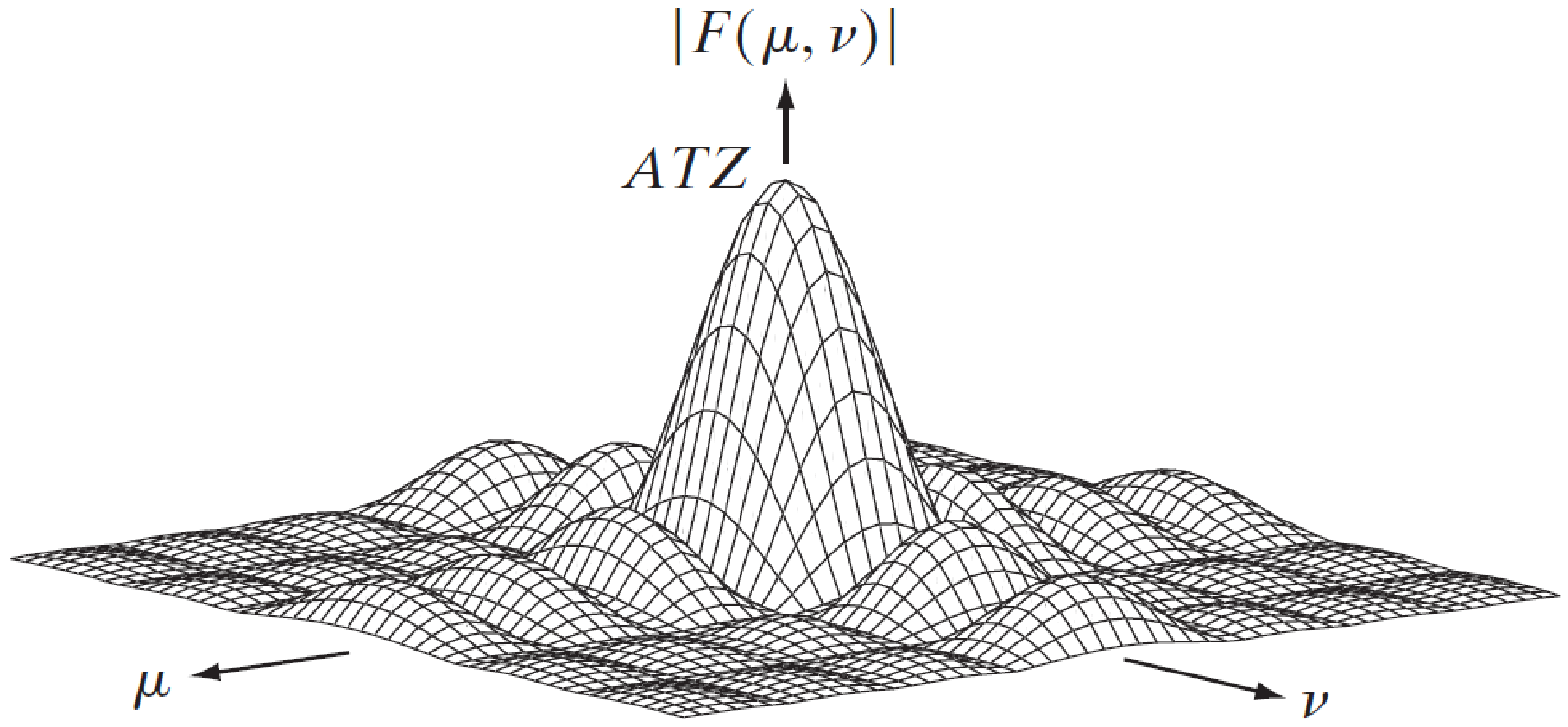
$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

derive properly

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

- The magnitude (spectrum) is given by the expression

$$|F(\mu,\nu)| = ATZ\left|\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right|\left|\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right|$$



Sampling and the Fourier Transform of Sampled functions.

Sampling

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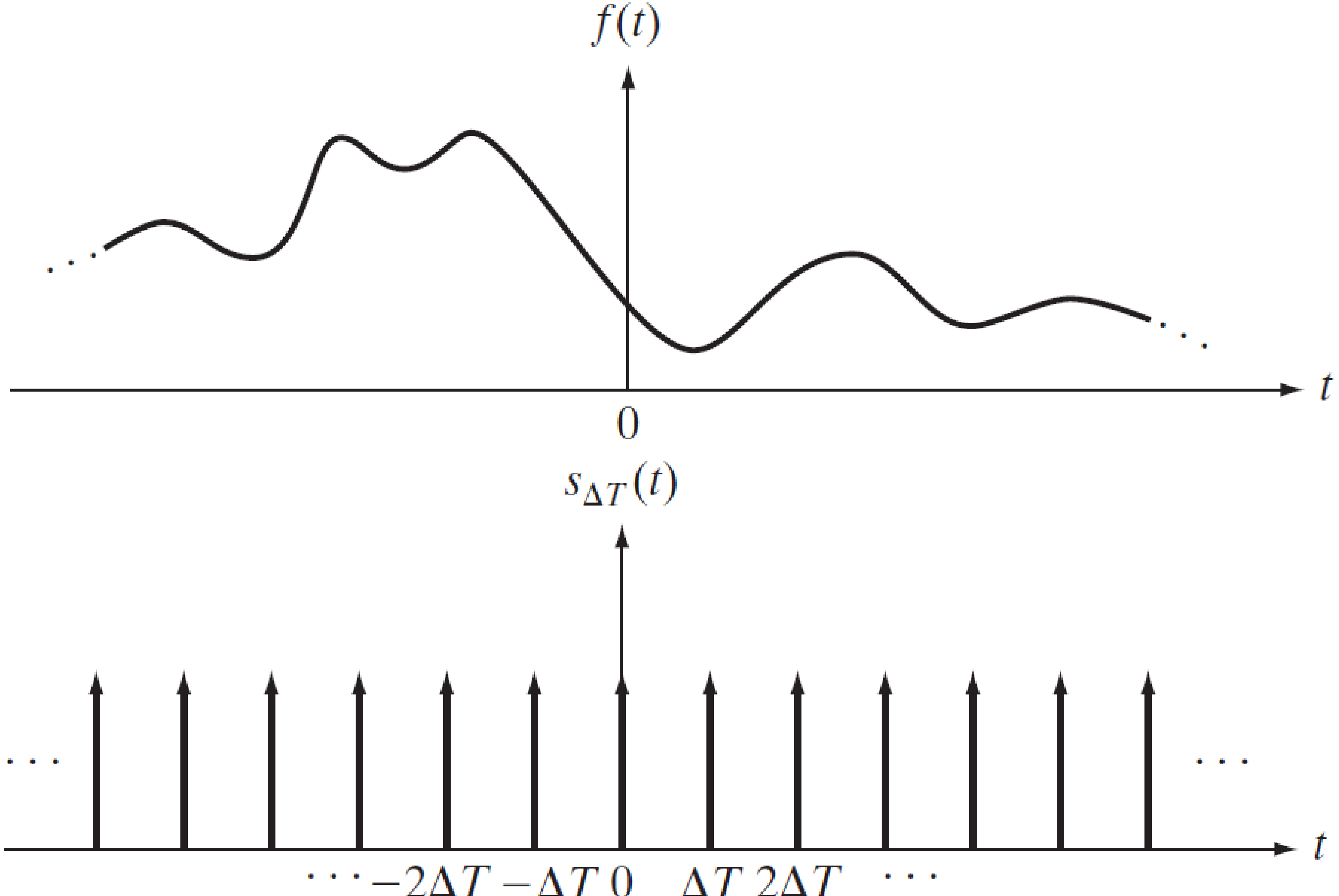
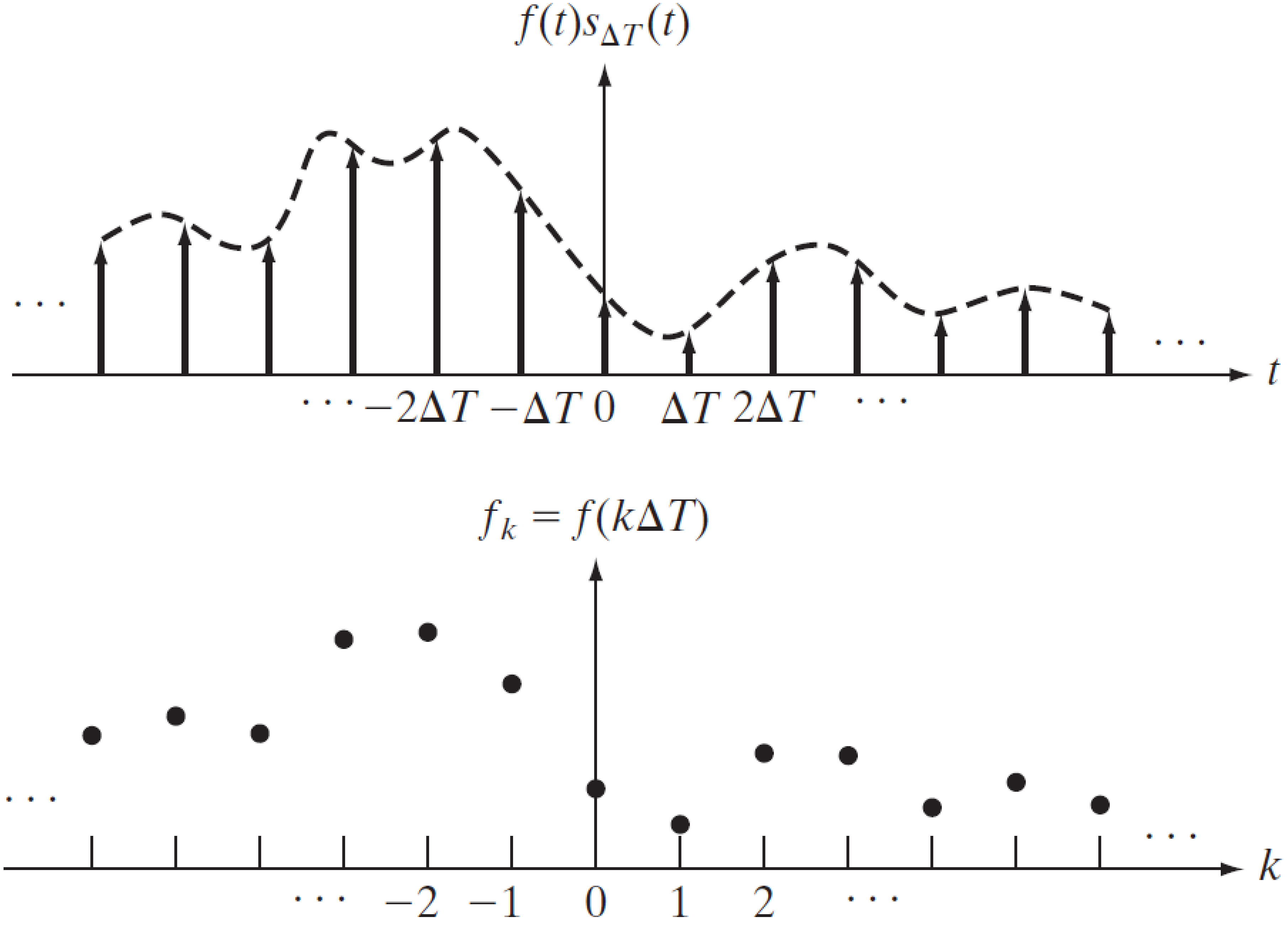




$f(t)s_{\Delta T}(t)$
t
$\cdots -2\Delta T\ -\Delta T\ 0\quad \Delta T\ 2\Delta T \quad \cdots$
$f_k = f(k\Delta T)$
k
$\cdots\ -2\ -1\quad 0\quad 1\quad 2\quad \cdots$

Sampling of 1-D function

- Consider a continuous function, $f(t)$, that we wish to sample at uniform intervals ΔT of the independent variable t.
- One way to model sampling is to multiply $f(t)$ by a *sampling function* equal to a train of impulses ΔT units apart.

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

- Where $\tilde{f}(t)$ denotes the sampled function.
- The value, f_k of an arbitrary sample in the sequence is given by

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt \qquad = f(k\Delta T)$$

The Fourier Transform of Sampled Functions

- Let $F(\mu)$ denote the Fourier transform of a continuous function $f(t)$. As discussed in the previous section, the corresponding sampled, $\tilde{f}(t)$ function, is the product of $f(t)$ and an impulse train.

$$\tilde{F}(\mu) = \Im\{\tilde{f}(t)\}$$

$$= \Im\{f(t)s_{\Delta T}(t)\}$$

$$= F(\mu) \star S(\mu)$$

- We know,

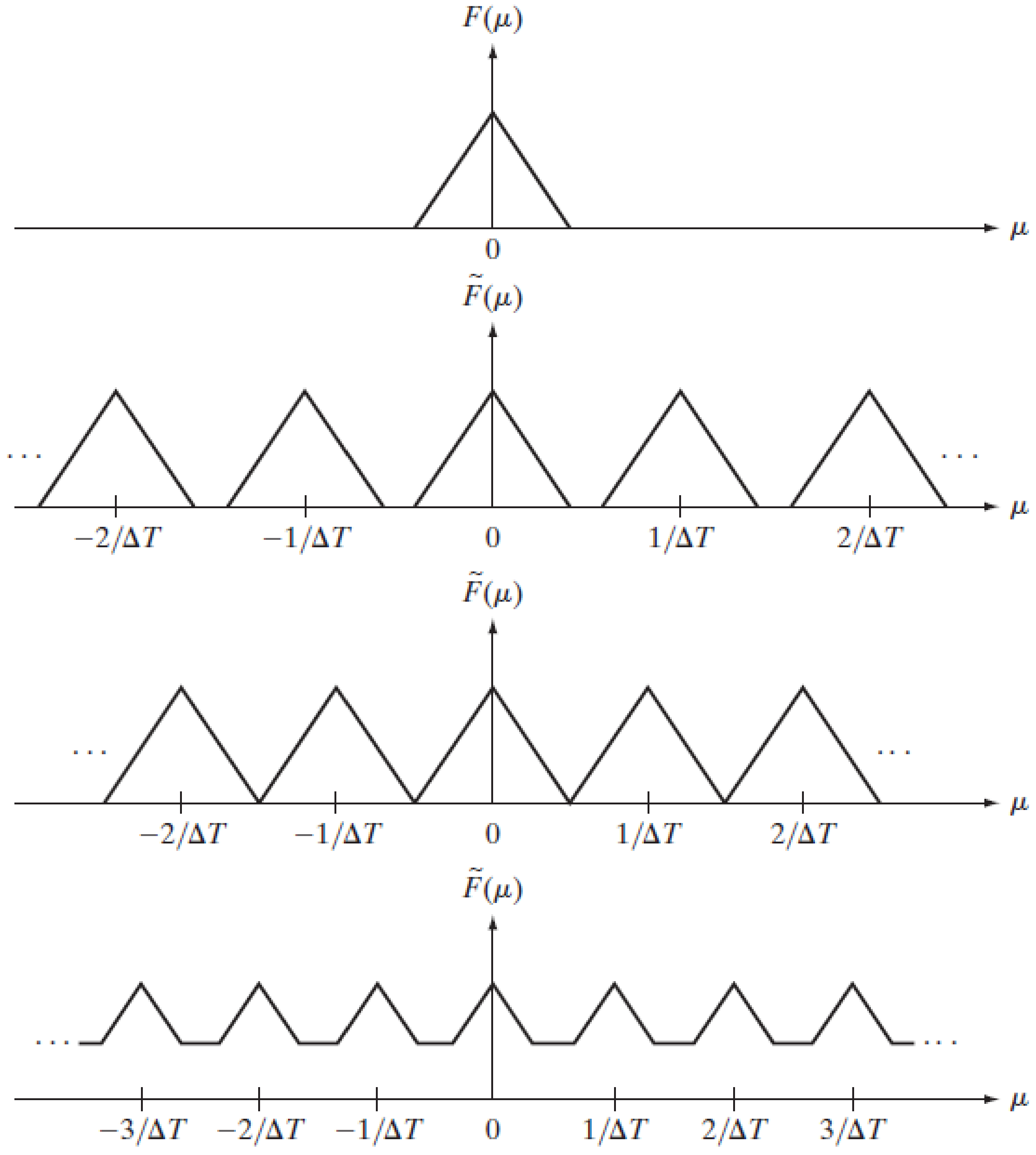
$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

Reference

$$
\begin{aligned}
\widetilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}
$$

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

- The summation shows that the Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$ is an infinite, periodic sequence of copies of $F(\mu)$, the transform of the original continuous function.
- The separation between the copies is determined by the value of $1/\Delta T$.

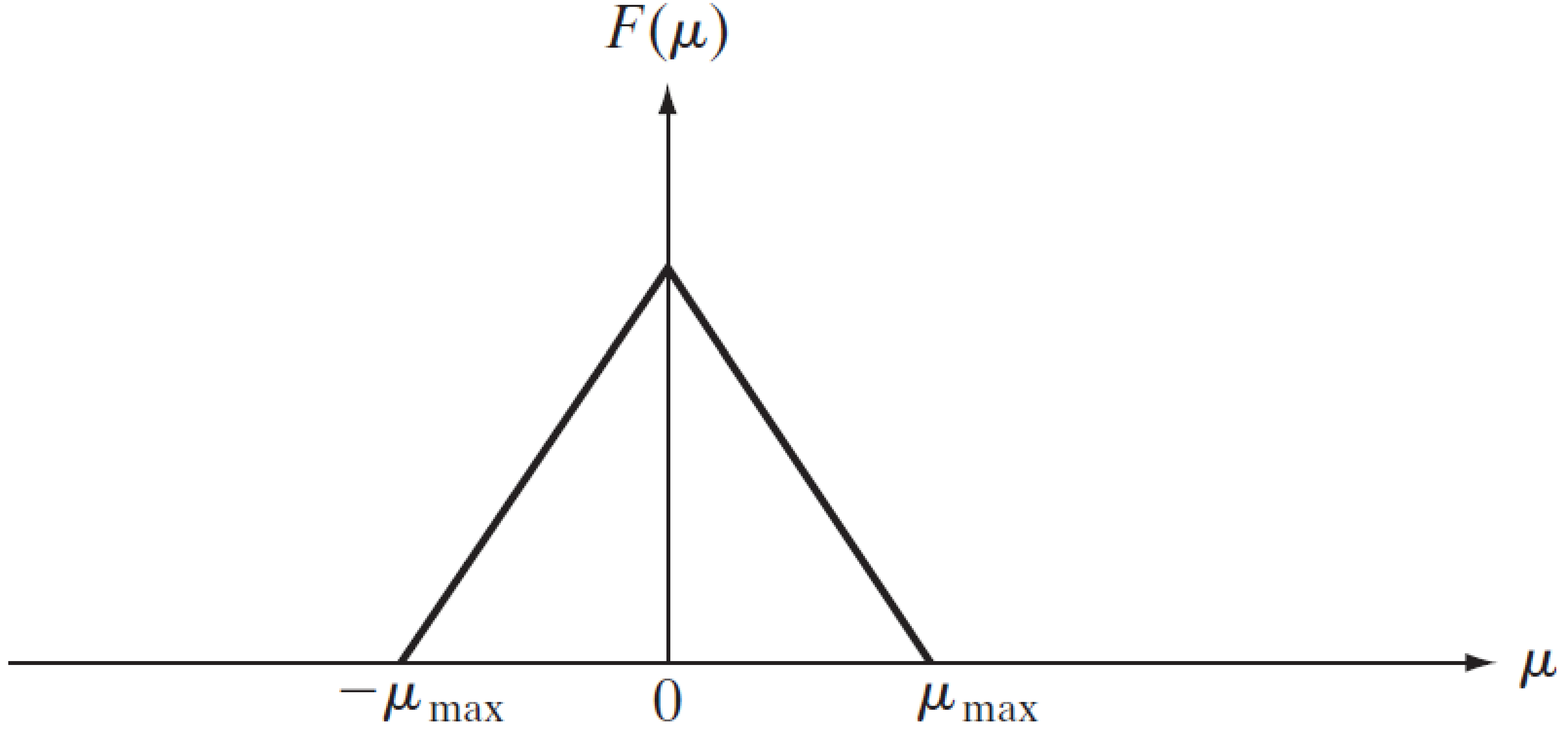


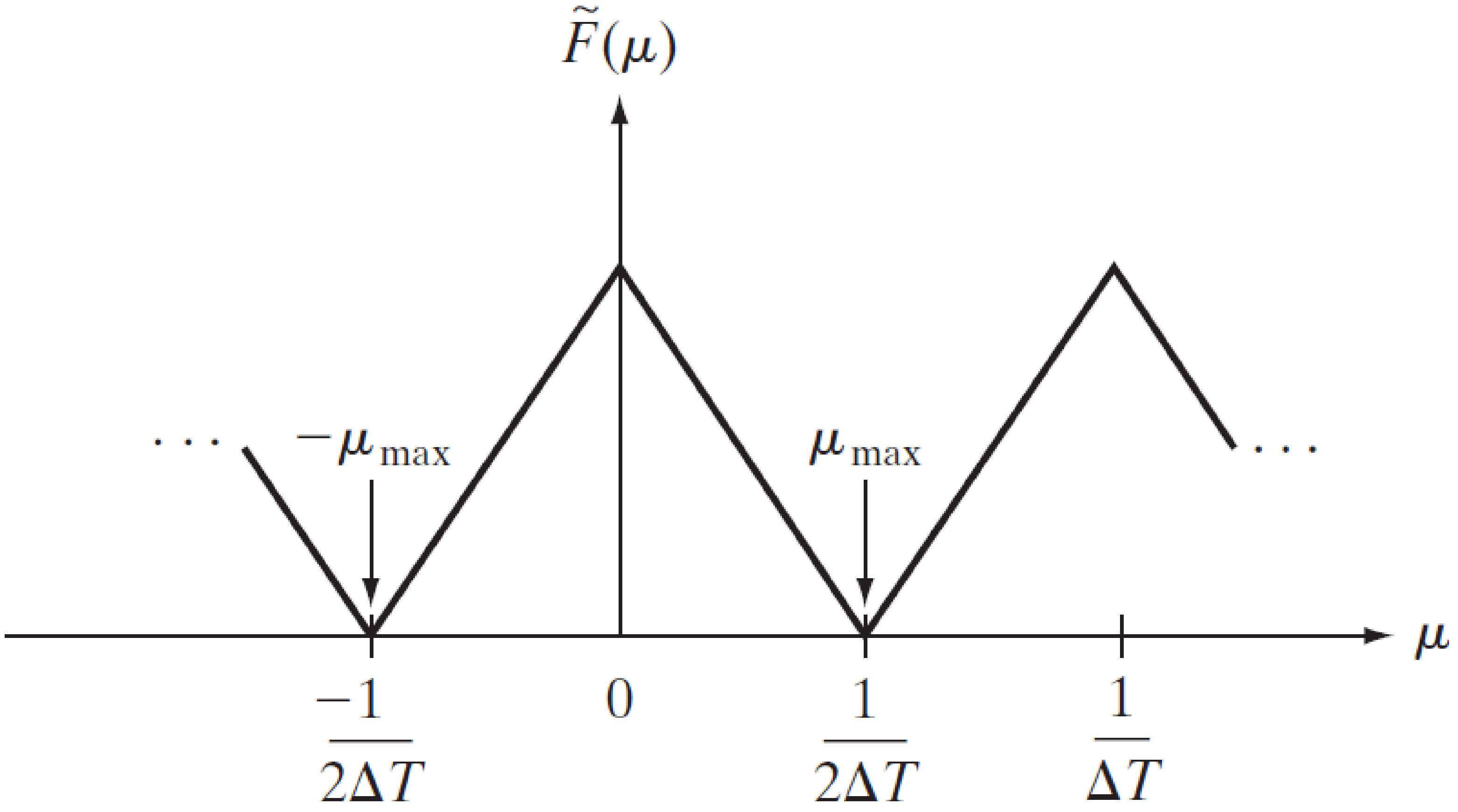
a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

Sampling Process

- Establish the conditions under which a continuous function can be *recovered uniquely* from a set of its samples.
- A function $f(t)$ whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{max}, \mu_{max}]$ about the origin is called a *band-limited function*.





A lower value of $1/\Delta T$ would cause the periods in $\tilde{F}(\mu)$ to merge; a higher value would provide a clean separation between the periods.

Sufficient Separation is guaranteed if

$$\frac{1}{\Delta T} > 2\mu_{max}$$

This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function. This result is known as the *sampling theorem.*

Oversampled

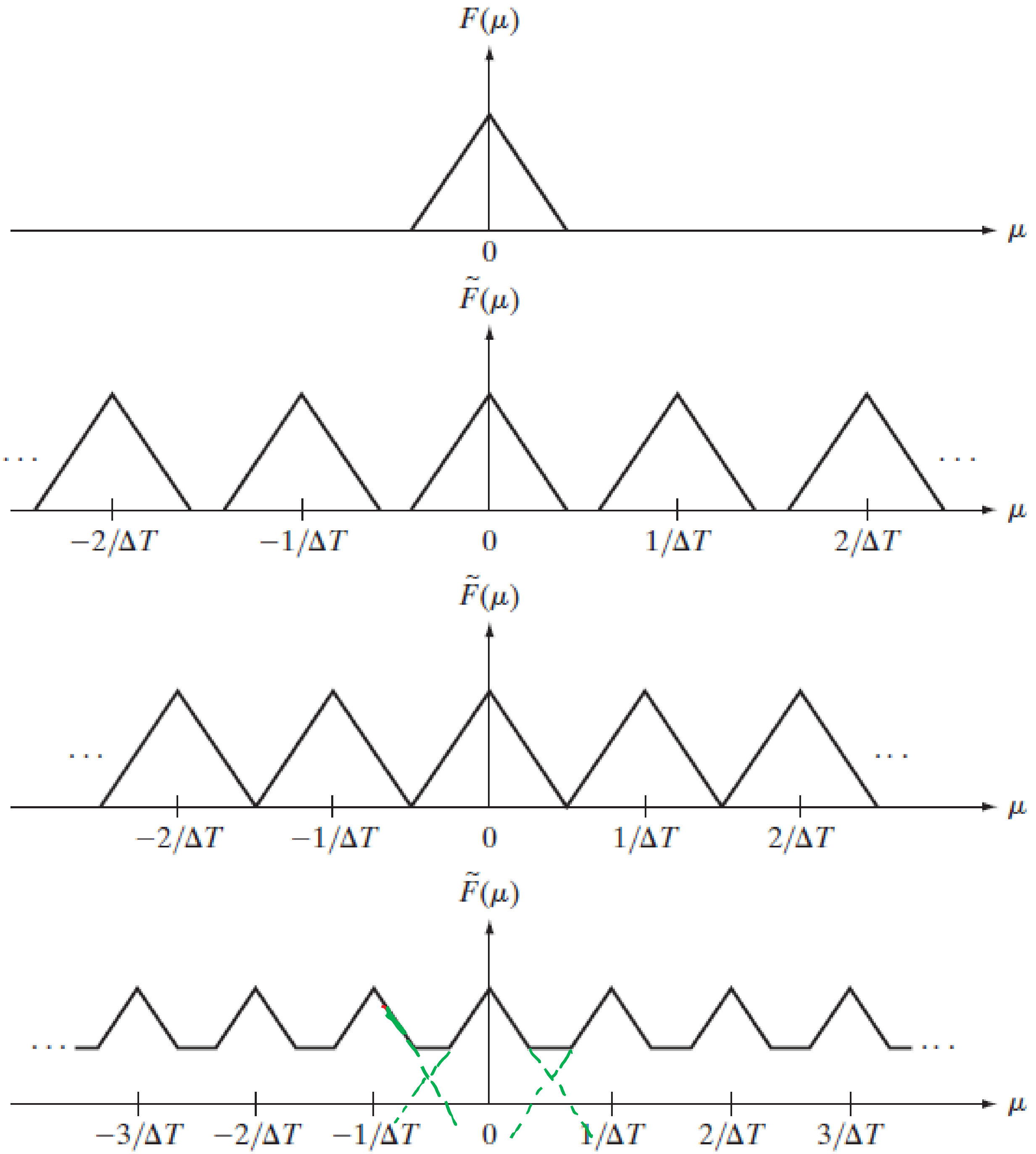
$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled

$$\frac{1}{\Delta T} = 2\mu_{max}$$

Under sampled

$$\frac{1}{\Delta T} < 2\mu_{max}$$



Sampling Theorem

- a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.

Sampling at : $$\frac{1}{\Delta T} = 2\mu_{max}$$ *Nyquist rate*