Fuzzy Relations

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Crisp relations

- **Order pairs:**

  Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

  $$A \times B = \{(a, b) | a \in A \; and \; b \in B\}$$

  Note :

  (1) $A \times B \neq B \times A$ (2) $|A \times B| = |A| \times |B|$

  (3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

  A particular mapping so mentioned is called a relation

Crisp relations

Example:

Consider the two crisp sets A and B as given below.

$A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)\}$

Let us define a relation as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x, y) \cup S(x, y) = max(R(x, y), S(x, y))$;
- **Intersection:** $R(x, y) \cap S(x, y) = min(R(x, y), S(x, y))$;
- **Complement:** $\overline{R(x, y)} = 1 - R(x, y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- $R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Find the following
- $R \cup S$
- $R \cap S$
- $\bar{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S \ = \{(x, z) | (x, y) \in \ R \; and \; (y, z) \in \ S \; and \; \forall y \ \in \ Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T \ = \ R \circ S$;

$$T(x, z) \ = \ max\{min\{R(x, y), S(y, z) \; and \; \forall y \ \in \ Y\}\}$$

Composition: Composition

Example : Given $X = \{1, 3, 5\}$; $Y = \{1, 3, 5\}$; $R = \{(x, y) | y = x + 2\}$;

$$S = \{(x, y) | x < y\}$$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1, 3), (3, 5)\}$, $S = \{(1, 3), (1, 5), (3, 5)\}$

$$R = \begin{array}{c} 1 \\ 3 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad S = \begin{array}{c} 1 \\ 3 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using max-min composition

$$R \circ S = \begin{array}{cc} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X1, X2, \ldots, Xn$

- Here, n-tuples $(x1, x2, \ldots, xn)$ may have varying degree of memberships within the relationship.

- The membership values indicate the strength of the relation between the tuples.

Fuzzy relations

Example:

$$X = \{\, typhoid, viral, cold \,\}, Y = \{\, running\, nose, high\, temp, shivering \,\}$$

The fuzzy relation R is defined as

$$R = \begin{array}{c} \\ typhoid \\ viral \\ cold \end{array} \begin{array}{c} \begin{array}{ccc} running\ nose & high\ temperature & shivering \end{array} \\ \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function

given by $\mu_R(x, y) = \mu_{A\text{x}B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$A = \{(a1, 0.2), (a2, 0.7), (a3, 0.4)\}$ and $B = \{(b1, 0.5), (b2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} \boldsymbol{b_1} & \boldsymbol{b_2} \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R\mathrm{U}S}(a,b) = max\{\mu_R(a,b), \mu_S(a,b)\}$
- **Intersection:** $\mu_{R\cap S}(a,b) = min\{\mu_R(a,b), \mu_S(a,b)\}$
- **Complement:** $\mu_{\bar{R}}(a,b) = 1 - \mu_R(a,b)$
- **Composition:** $T = R \circ S$

$$\mu_{R\circ S} = max_{y\in Y}\{min\big(\mu_R(x,y), \mu_S(y,z)\big)\}$$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3), \; Y = (y_1, y_2), \; Z = (z_1, z_2, z_3),$

$$R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \qquad and \qquad S = \begin{array}{c} y_1 \\ y_2 \end{array} \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix}$$

$$R \circ S = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix}$$

$\mu_{R\circ S}\,(x_1, y_1) = max\{min(\mu_R(x_1, y_1), \mu_S(y_1, z_1))\,, min(\mu_R(x_1, y_2)\,, \mu_S(y_2, z_1))\}$

$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = \; 0.5 \; and \; so \; on.$

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, which is stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 1.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using max-min composition

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$