Q.1

a) $f(x)=\frac{1}{2}x^TAx+b^Tx$

$\nabla f(x) = Ax + b$

b) $f(x) = g(h(x))$ $\quad \nabla f(x) = g'(h(x))\,\nabla h(x)$

c) $f(x) = \frac{1}{2}x^TAx + b^Tx$

$\nabla f(x) = Ax + b$ $\quad \nabla^2 f(x) = A + 0 = A$

d) $f(x) = g(a^Tx)$

$\nabla f(x) = g'(a^Tx)\,a$

$\nabla^2 f(x) = g''(a^Tx)\,a\,a^T + 0$

$\cancel{= a^T g''(a^Tx)\,a}$

Q.2 ~~Thursday 29~~

a) $$\underset{2\times 1}{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} z_1 & z_2 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix}}$$

$$Z \quad Z^T = A$$

$\because A = A^T$ ✓ $\quad x^T A x > 0$

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{1\times 2}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} z_1^2 x_1 + z_1 z_2 x_2 \\ z_1 z_2 x_1 + z_2^2 x_2 \end{bmatrix}}$$

$$= z_1^2 x_1^2 + z_1 z_2 x_1 x_2 + z_1 z_2 x_1 x_2 + z_2^2 x_2^2$$

$$= z_1^2 x_1^2 + 2 z_1 z_2 x_1 x_2 + z_2^2 x_2^2 = (z_1 x_1 + z_2 x_2)^2 \geq 0$$

b) $R(A) = R(Z Z^T) = 1$

Q.2

c) $F = (\underset{M\times N}{B}\ \underset{N\times N}{A}\ \underset{N\times M}{B^T})^T = (AB^T)^T B^T = B A^T B^T = B A B^T$

$A^T = A$

$F^T = F$ ✓ $\quad x^T F x \geq 0$ ✓

$x^T B A B^T x \geq 0 = \underbrace{C A C^T}$

Let $C = x^T B$

$\hookrightarrow$ A is PSD

$\therefore C A C^T \geq 0$

$\therefore F$ is PSD

Q.3

a) $A = T \Lambda T^{-1}$, $AT = T\Lambda$

$A(t_0 + t_1 + t_2 \cdots) = (t_0 + t_1 + t_2 + \cdots) \begin{bmatrix} \lambda_0 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \lambda_n \end{bmatrix}$

$At_0 + At_1 + \cdots = t_0 \lambda_0 + t_1 \lambda_1 + \cdots$

$\therefore (t_0, \lambda_0)$ are eigen pairs

Q. 3

$U \to$ orthognal

$UU^T = I$

b) $A = U \Lambda U^T$

$AU = U \Lambda U^T U = U \Lambda$

$A(u_1 + u_2 + \dots) = (u_1 + u_2 + \dots) \Lambda$

$Au_1 + Au_2 + \dots = \lambda_1 u_1 + \lambda u_2 + \dots$

$\therefore (u_1, \lambda_1)$ are eigen pairs

c) A is PSD $\to A = A^T \quad x^T A x \geq 0$

$A x_i = \lambda_i x_i$

$x_i^t A x_i = \lambda_i (x_i^t x_i) = \lambda_i \| \cancel{x}_i \|_2 \to \lambda_i \geq 0$