Forward Kinematics using Modified Denavit–Hartenberg Parameters: A Case Study on Our Robot

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Abstract

This report presents a detailed explanation of how the Modified Denavit–Hartenberg (MDH) parameterization method is applied to model the kinematics of our 7-DOF robotic manipulator. It discusses how reference frames are assigned to each joint and link, how the homogeneous transformations are constructed, and how the overall forward kinematics are derived. The goal is to provide a clear understanding of how a robotic arm's geometry can be mathematically represented to describe its motion and position in space. Visual figures and a parameter table are included to support and clarify the explanation.

1 Introduction

The MDH convention provides a structured way to represent each joint and link using four key parameters: the joint angle θ_i, link offset d_i, link length a_i, and twist angle α_i. These parameters allow us to express every link's position and orientation relative to its neighboring link through a homogeneous transformation matrix.

In this project, we applied the MDH convention to our own 7-degree-of-freedom (7-DOF) robotic arm model. By defining coordinate frames for each link, we were able to mathematically describe the entire structure of our robot and compute its forward kinematics. This process not only helps in simulation and motion planning but also ensures accuracy when implementing real-world robotic movements.

2 MDH Parameters for Our Robot

Table 1 shows the MDH parameters we assigned to our robot. Each row defines how one link is positioned and oriented relative to the previous one. These parameters provide a direct connection between the geometric design of the robot and its mathematical model.

The table shows that our robot consists of seven revolute joints, each represented by a variable q_i. The d_i parameters indicate the linear distances between axes, while the a_i parameters represent the link lengths. The twist angle α_i defines how each axis is rotated

Link i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$ (m)	$\boldsymbol{a_i}$ (m)	$\boldsymbol{\alpha_i}$ (rad)
1	q_1	0	0	0
2	q_2	0	0	$-\pi/2$
3	q_3	0.42	0	$+\pi/2$
4	q_4	0	0	$+\pi/2$
5	q_5	0.40	0	$-\pi/2$
6	q_6	0	0	$-\pi/2$
7	q_7	0	0	$+\pi/2$

Table 1: Modified Denavit–Hartenberg (MDH) parameters used for our robot. Each parameter describes one link-to-link transformation.

relative to the previous one. By combining all of these, we can fully describe how each link is connected within the robot.

3 Homogeneous Transformation Matrices

For each link, the homogeneous transformation A_i defines how to move from the coordinate frame of link $i-1$ to link i. This transformation includes both rotation and translation components and is expressed as:

$$A_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

By multiplying all seven transformation matrices, we can find the overall transformation from the base to the end-effector:

$$T_0^7 = A_1A_2A_3A_4A_5A_6A_7$$

This final matrix T_0^7 gives the full position and orientation of the robot's end-effector in space.

4 Forward Kinematics Solved by Hand (Step–by–Step)

We now compute forward kinematics manually by chaining the homogeneous transforms. We denote the local link transform by $g_{i-1,i}$ (same as A_i) and the cumulative/base transform by ${}^0g_i$.

4.1 Per–link transform $g_{i-1,i}$

With the MDH parameters $(\theta_i, d_i, a_i, \alpha_i)$, each link transform is

$$g_{i-1,i} \;=\; R_z(\theta_i)\, T_z(d_i)\, T_x(a_i)\, R_x(\alpha_i) \;=\; \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.2 Cumulative transforms ${}^0g_i$

Start at the base with ${}^0g_0 = I_4$. Then multiply in order:

$${}^0g_1 = g_{0,1}, \quad {}^0g_2 = {}^0g_1\, g_{1,2}, \quad \ldots, \quad {}^0g_7 = {}^0g_6\, g_{6,7}.$$

Write ${}^0g_i$ in block form ${}^0g_i = \begin{bmatrix} {}^0R_i & {}^0o_i \\ 0\;0\;0 & 1 \end{bmatrix}$, where:

- ${}^0R_i \in \mathbb{R}^{3\times 3}$ is the orientation of frame i in the base,
- ${}^0o_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^\top$ is the position of frame i origin in the base.

The end–effector pose equals ${}^0g_7$.

4.3 How to extract x_i, y_i, z_i and $\hat{z}_i$

From ${}^0g_i$,

$${}^0o_i = \left({}^0g_i\right)_{1:3,\,4}, \qquad {}^0\hat{z}_i = \left({}^0R_i\right)\hat{z} = \text{third column of } {}^0R_i.$$

These are often needed for Jacobians and for checking frame assignment.

4.4 Worked multiplication: first two joints

Using our MDH table (with $a_i = 0$ and the listed α_i and d_i),

$$g_{0,1} = \begin{bmatrix} c_1 & -s_1c_{\alpha_1} & s_1s_{\alpha_1} & 0 \\ s_1 & c_1c_{\alpha_1} & -c_1s_{\alpha_1} & 0 \\ 0 & s_{\alpha_1} & c_{\alpha_1} & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad g_{1,2} = \begin{bmatrix} c_2 & -s_2c_{\alpha_2} & s_2s_{\alpha_2} & 0 \\ s_2 & c_2c_{\alpha_2} & -c_2s_{\alpha_2} & 0 \\ 0 & s_{\alpha_2} & c_{\alpha_2} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ and $c_{\alpha_i} = \cos\alpha_i$, $s_{\alpha_i} = \sin\alpha_i$. Then

$${}^0g_2 = g_{0,1}\, g_{1,2} \;=\; \begin{bmatrix} {}^0R_1R_1^2 & {}^0R_1\, p_1^2 + {}^0o_1 \\ 0\;0\;0 & 1 \end{bmatrix},$$

where $p_1^2 = \begin{bmatrix} 0 & 0 & d_2 \end{bmatrix}^\top$. This same pattern repeats for all links.

i	${}^0o_i^x$	${}^0o_i^y$	${}^0o_i^z$	${}^0\hat{z}_i^\top$	note
0	0	0	0	$[\,0\ 0\ 1\,]$	base
1					from ${}^0g_1$
2					from ${}^0g_2$
3					includes $d_3 = 0.42$
4					
5					includes $d_5 = 0.40$
6					
7					end–effector

Table 2: Hand–computed positions and z-axes. Fill from the cumulative transforms ${}^0g_i$.

4.5 Fill–in table for positions and axes

4.6 Numerical example (plug–in angles)

For demonstration, choose the angles

$$\theta = \left[\,0,\ -\tfrac{\pi}{2},\ 0,\ \tfrac{\pi}{2},\ 0,\ 0,\ 0\,\right],$$

with the MDH constants $d_3 = 0.42$ m, $d_5 = 0.40$ m and all $a_i = 0$. Compute $g_{i-1,i}$ for $i = 1 \ldots 7$, then multiply to get ${}^0g_7$. Extract the end–effector position ${}^0o_7 = \left({}^0g_7\right)_{1:3,4} = \begin{bmatrix} x_7 & y_7 & z_7 \end{bmatrix}^\top$ and record it in the table above together with all intermediate ${}^0o_i$. This explicit hand multiplication verifies the FK pipeline used by our controller.

5 Visual Representation and Frame Assignment

To better understand the kinematic modeling of our robot, we created detailed visual diagrams that show how the MDH parameters and frames are assigned.

6 Conclusion

Through the Modified Denavit–Hartenberg (MDH) convention, we were able to model the full kinematic structure of our robot in a systematic and consistent way. By defining the MDH parameters, constructing the homogeneous transformation matrices, and visualizing the coordinate frames, we achieved a complete mathematical description of the robot's motion.

This process allowed us to translate the robot's geometry into equations that can be used for simulation, motion control, and real-time programming. Understanding this connection between theory and physical implementation is crucial in robotics engineering, as it forms the foundation for both forward and inverse kinematic analysis. Overall, this work demonstrates how mathematical modeling transforms into a practical understanding of robotic movement.

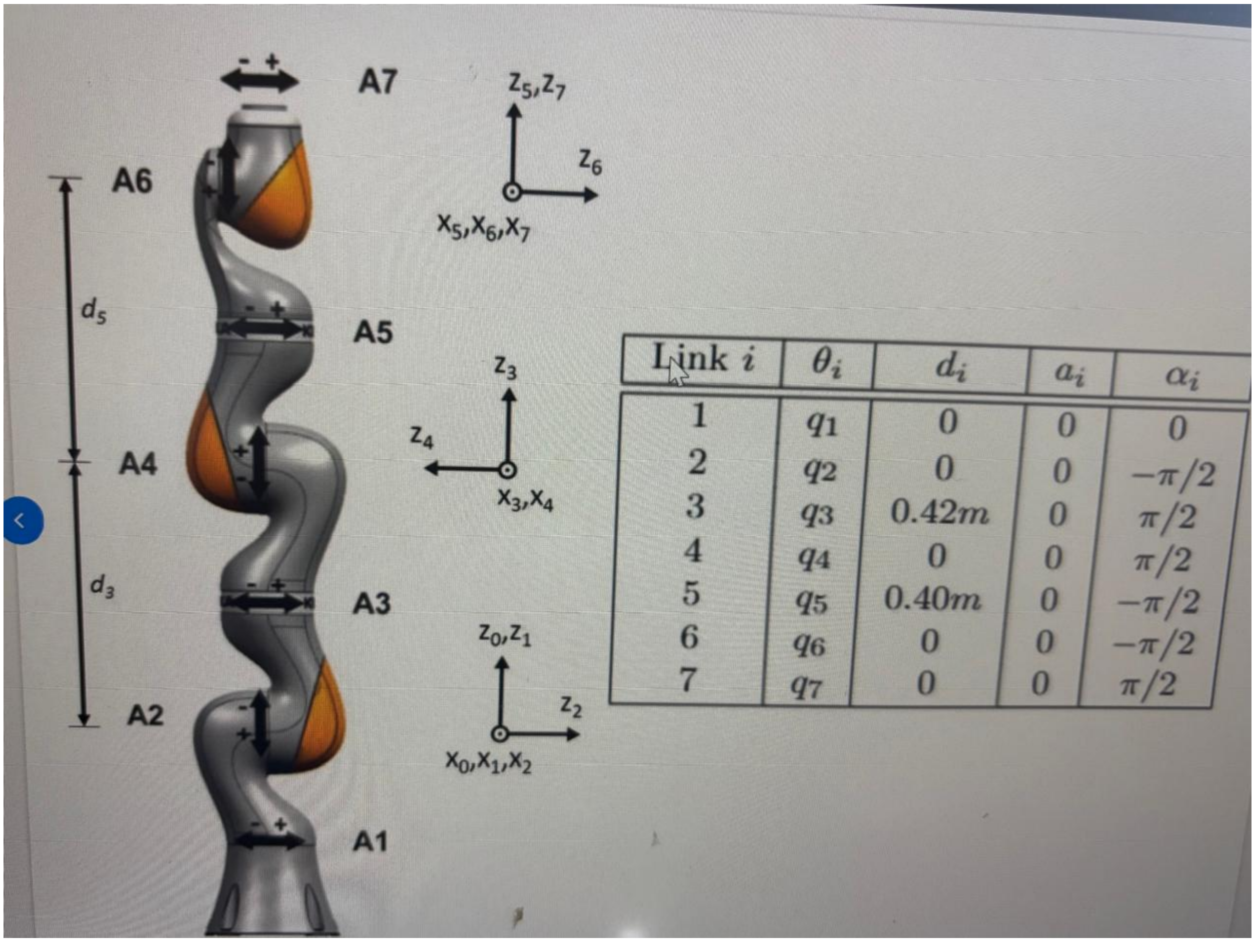

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Figure 1: The diagram above illustrates how the Modified Denavit–Hartenberg (MDH) parameters and coordinate frames are assigned to each link in our robot. Each colored coordinate frame corresponds to one link or joint in the manipulator. The vertical z-axes represent the rotation axes for the revolute joints, while the horizontal x-axes define the directions of the connecting links. The accompanying MDH table lists the parameters θ_i, d_i, a_i, and α_i that mathematically describe the orientation and distance between these coordinate systems. This diagram helps visualize how all seven links are connected in sequence from the base to the end-effector, forming a continuous kinematic chain.

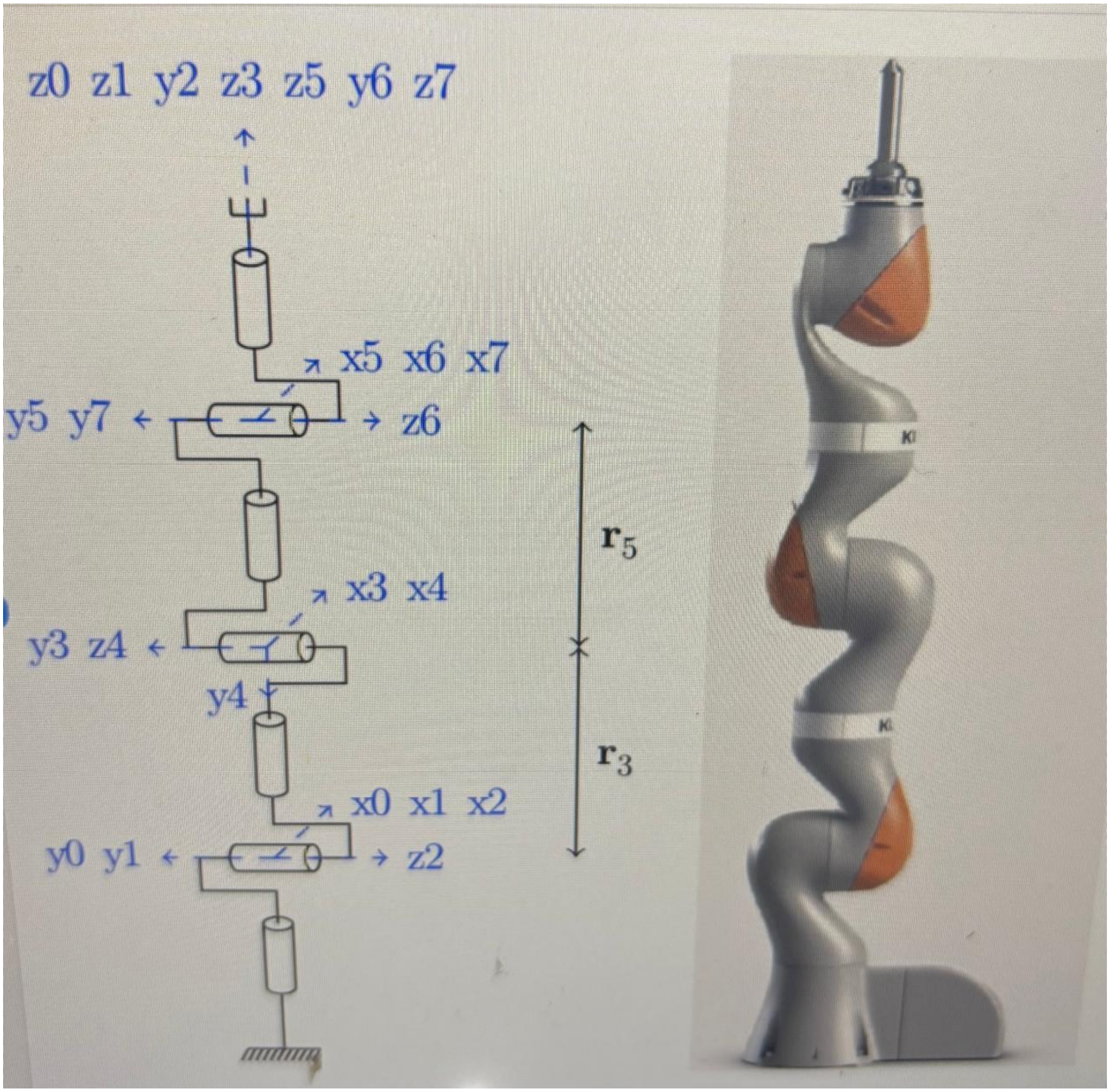

Figure 2: This figure shows both the schematic kinematic diagram (left) and the physical model of our robot (right). On the left, the line drawing displays the sequential arrangement of coordinate frames from the base (z_0, x_0) up to the end-effector (z_7, x_7). Each cylinder represents a revolute joint, and each coordinate frame (x_i, y_i, z_i) defines the orientation of that link using the MDH convention. The z-axes indicate rotation directions, while the x-axes show the connection between links. On the right, the real robot model visually matches the same structure, showing how the theoretical coordinate frames correspond to physical components. The lengths r_3 and r_5 represent the main link distances that define the arm's proportions. This comparison between theory and physical design demonstrates how the mathematical model reflects the real configuration of the manipulator.