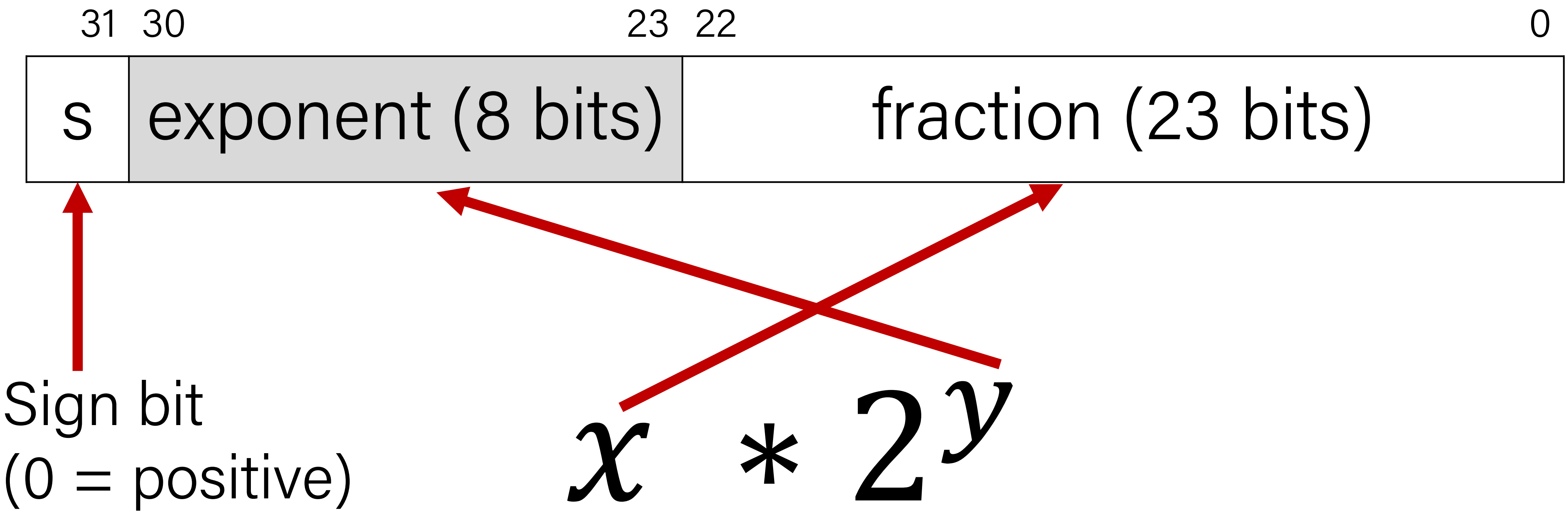
IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$x * 2^{y}$$

With this format, 32-bit floats represent numbers in the range ~1.2 x10^{-38} to ~3.4 x10^{38}! Is every number between those representable? **No**.

IEEE Single Precision Floating Point



Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	?
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	?

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	127
11111101	126
11111100	125
…	…
00000011	-124
00000010	-125
00000001	-126
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

- The exponent is **not** represented in two's complement.
- Instead, exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive). This makes bit-level comparison fast.
- **Actual value = binary value – 127 ("bias")**

`11111110`	`254 - 127 = 127`
`11111101`	`253 - 127 = 126`
…	…
`00000010`	`2 - 127 = -125`
`00000001`	`1 - 127 = -126`

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x * 2^{y}$$

- We could just encode whatever x is in the fraction field. But there's a trick we can use to make the most out of the bits we have.

An Interesting Observation

In Base 10:

$42.4 \times 10^5 = 4.24 \times 10^6$

$324.5 \times 10^5 = 3.245 \times 10^7$

$0.624 \times 10^5 = 6.24 \times 10^4$

We tend to adjust the exponent until we get down to one place to the left of the decimal point.

In Base 2:

$10.1 \times 2^5 = 1.01 \times 2^6$

$1011.1 \times 2^5 = 1.0111 \times 2^8$

$0.110 \times 2^5 = 1.10 \times 2^4$

Observation: in base 2, this means there is always a 1 to the left of the decimal point!

Representing Zero

The float representation of zero is all zeros (with any value for the sign bit)

Sign	Exponent	Fraction
`any`	`All zeros`	`All zeros`

- This means there are two representations for zero! ☹

Representing Small Numbers

If the exponent is all zeros, we switch into "denormalized" mode.

Sign	Exponent	Fraction
`any`	`All zeros`	`Any`

- We now treat the exponent as -126, and the fraction as *without* the leading 1.
- This allows us to represent the smallest numbers as precisely as possible.

Representing Exceptional Values

If the exponent is all ones, and the fraction is all zeros, we have +- infinity.

Sign	Exponent	Fraction
`any`	`All ones`	`All zeros`

- The sign bit indicates whether it is positive or negative infinity.
- Floats have built-in handling of over/underflow!
  - Infinity + anything = infinity
  - Negative infinity + negative anything = negative infinity
  - Etc.

Representing Exceptional Values

If the exponent is all ones, and the fraction is nonzero, we have
Not a Number (NaN)

Sign	Exponent						Fraction
any	1	…	…	…	…	1	Any nonzero

- NaN results from computations that produce an invalid mathematical result.
  - Sqrt(negative)
  - Infinity / infinity
  - Infinity + -infinity
  - Etc.

Number Ranges

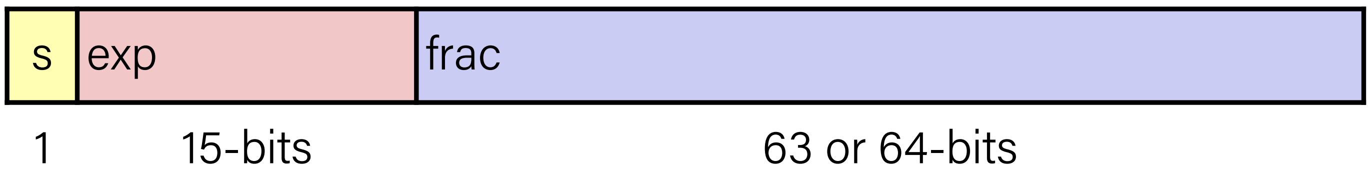
- 32-bit integer (type **int**):
  › -2,147,483,648 to 2147483647
  › Every integer in that range can be represented
- 64-bit integer (type **long**):
  › −9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
- 32-bit floating point (type **float**):
  – ~1.2 x10^{-38} to ~3.4 x10^{38}
  – Not all numbers in the range can be represented (not even all integers in the range can be represented!)
  – Gaps can get quite large! (larger the exponent, larger the gap between successive fraction values)
- 64-bit floating point (type **double**):
  – ~2.2 x10^{-308} to ~1.8 x10^{308}

Precision options

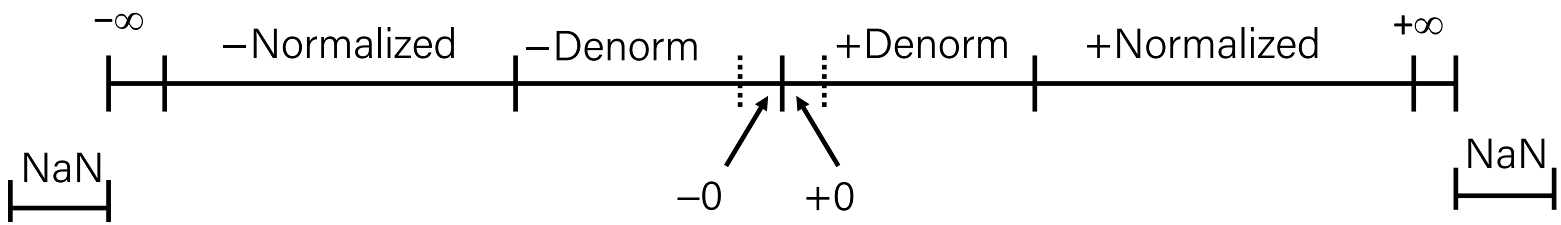
- Single precision: 32 bits

- Double precision: 64 bits

- Extended precision: 80 bits (Intel only)



Visualization: Floating Point Encodings



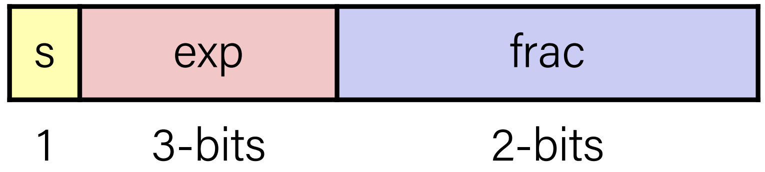
Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$
n: E = Exp – Bias
d: E = 1 – Bias

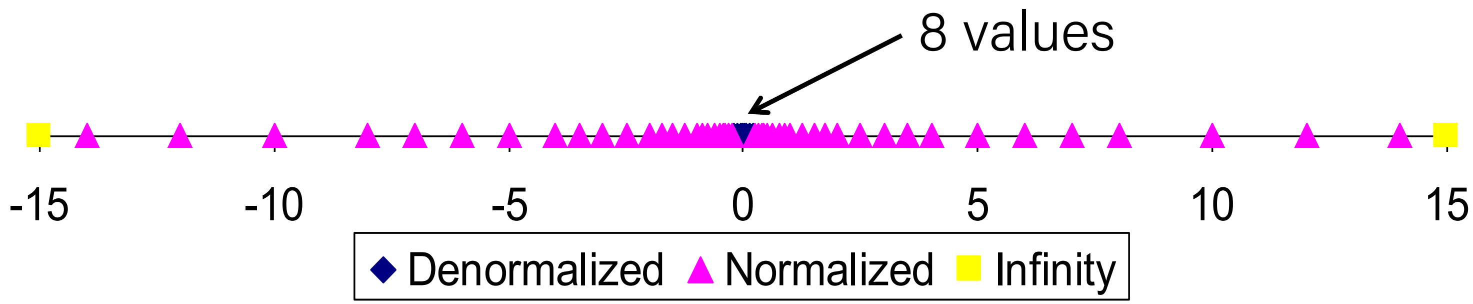
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$



- Notice how the distribution gets denser toward zero.



Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $-0 = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011
1e20:	0	11000001	01011010111100011101100

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
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float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Floating point arithmetic is not associative. The order of operations matters!

- The first line loses precision when first adding 3.14 and 1e20, as we have seen.
- The second line first evaluates 1e20 – 1e20 = 0, and then adds 3.14

Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither **d** nor **f** is NaN

```
• x == (int)(float) x                          False
• x == (int)(double) x                         True
• f == (float)(double) f                       True
• d == (float) d                               False
• f == -(-f);                                  True
• 2/3 == 2/3.0                                 False
• d < 0.0      ⇒    ((d*2) < 0.0)              True (OF?)
• d > f        ⇒    -f > -d                    True
• d * d >= 0.0                                 True (OF?)
• (d+f)-d == f                                 False
```