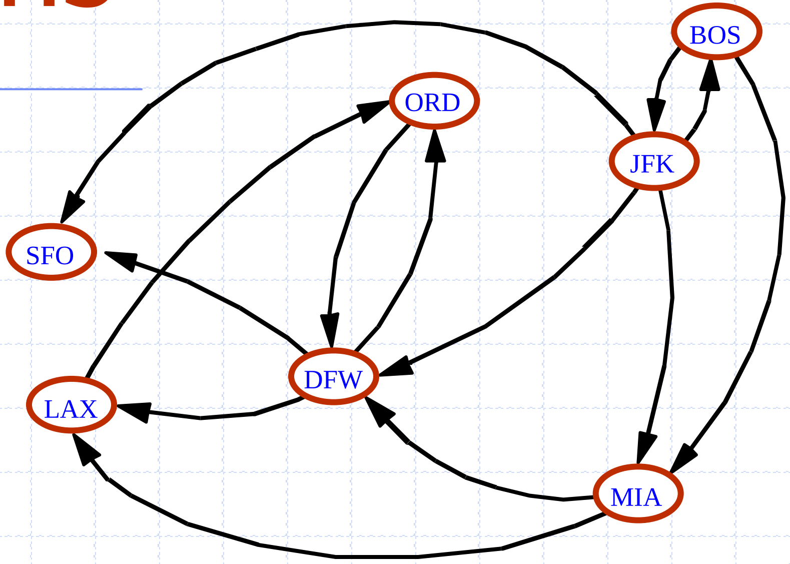
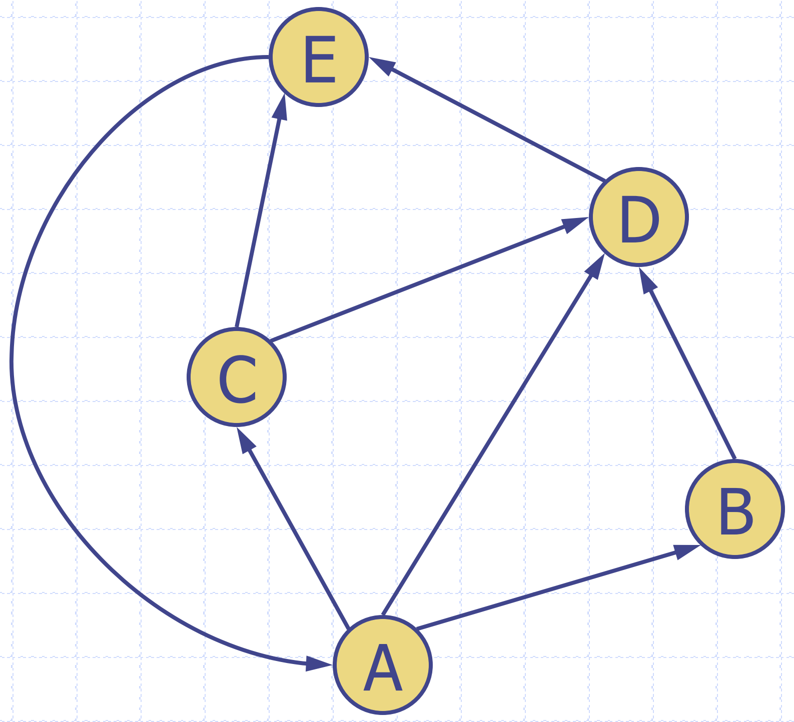
Directed Graphs

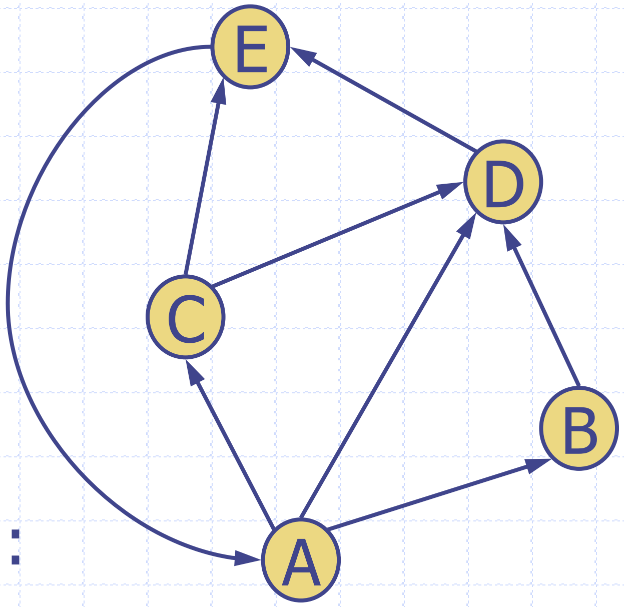


Digraphs

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



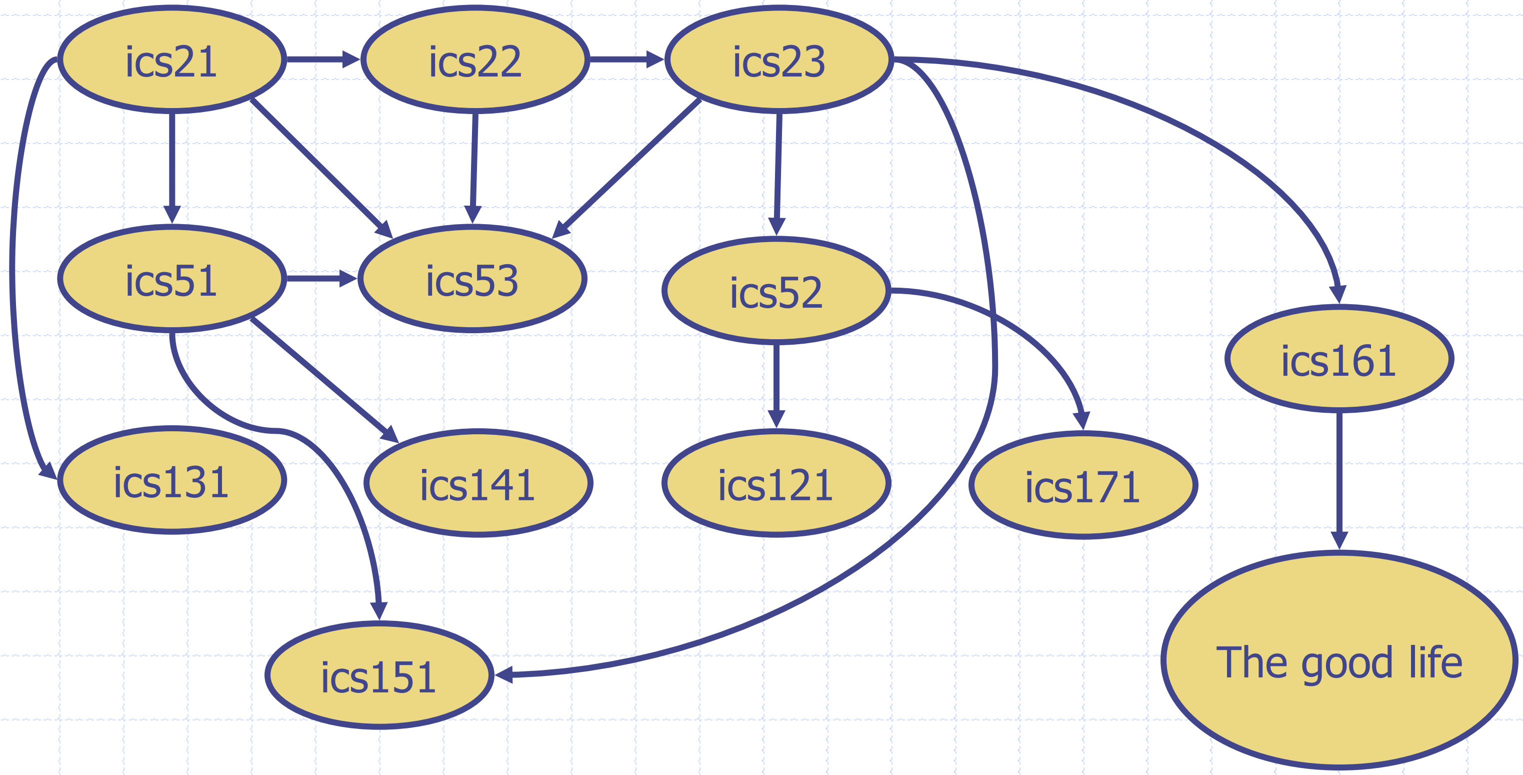
Digraph Properties



- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size

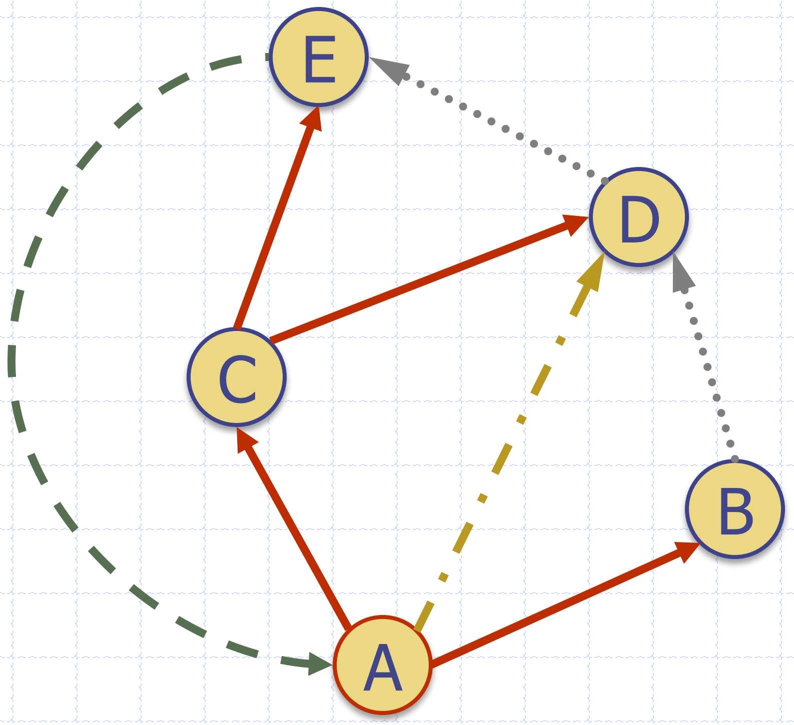
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started



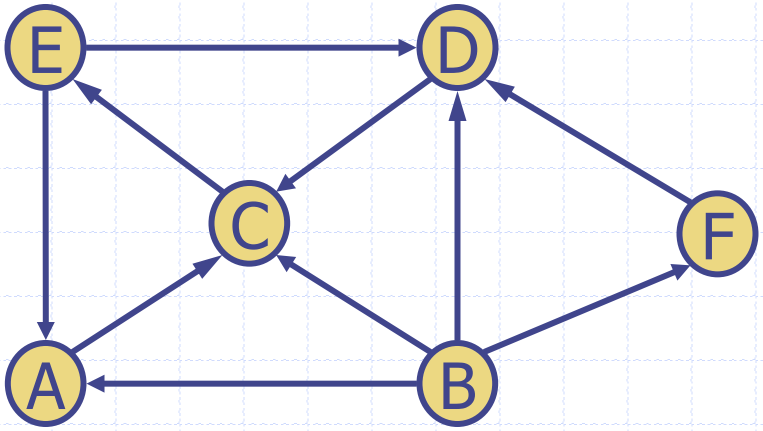
Directed DFS

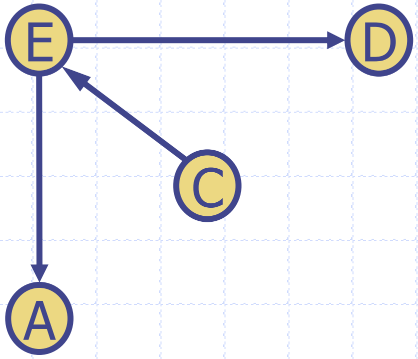
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex *s* determines the vertices reachable from *s*

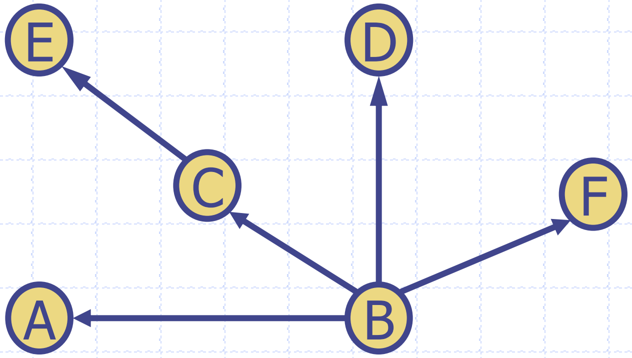


Reachability

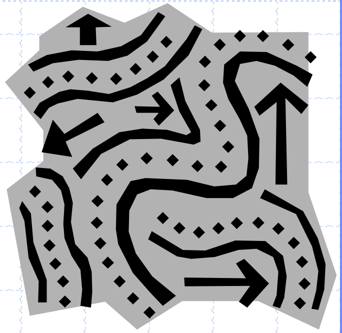
- DFS tree rooted at v: vertices reachable from v via directed paths



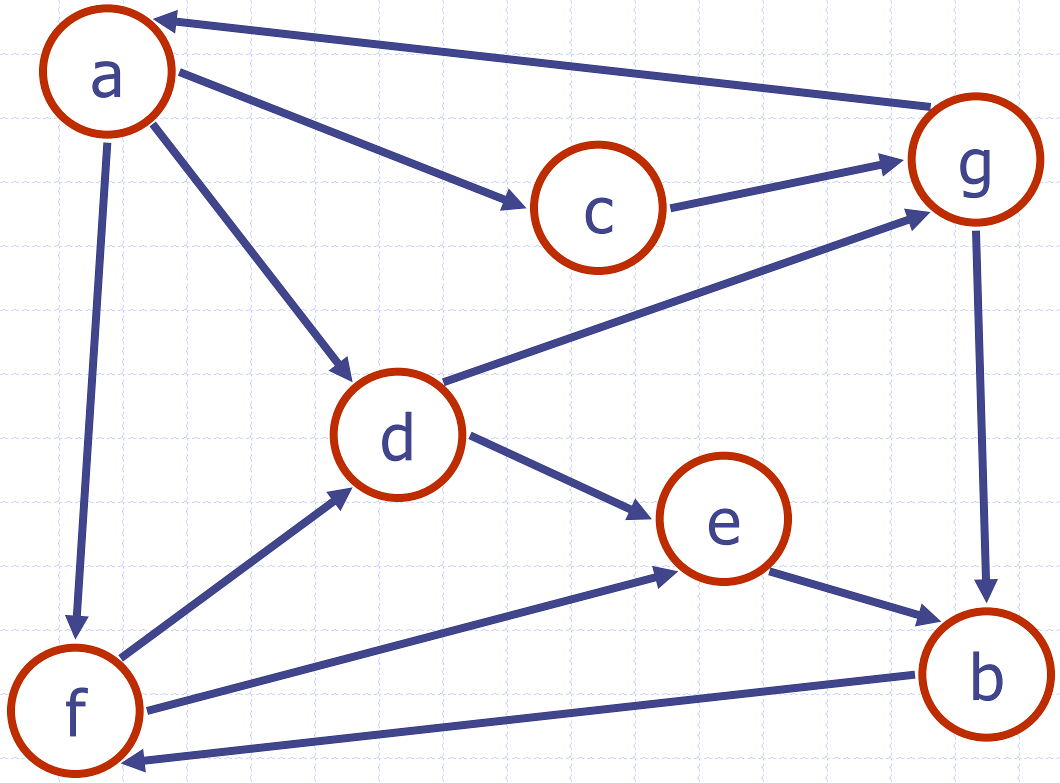




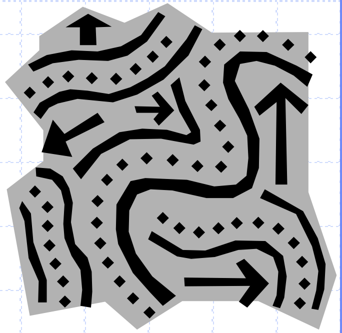
Strong Connectivity



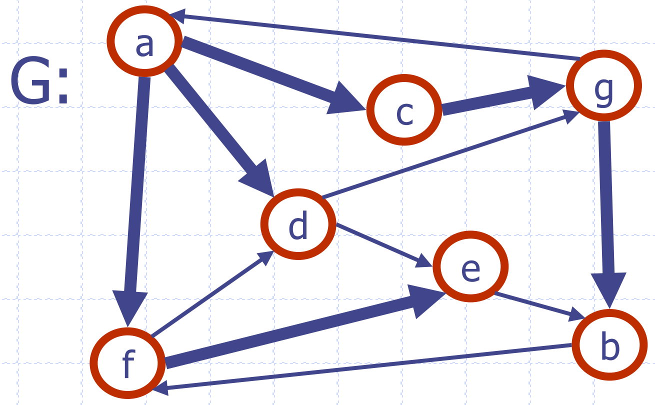
- Each vertex can reach all other vertices

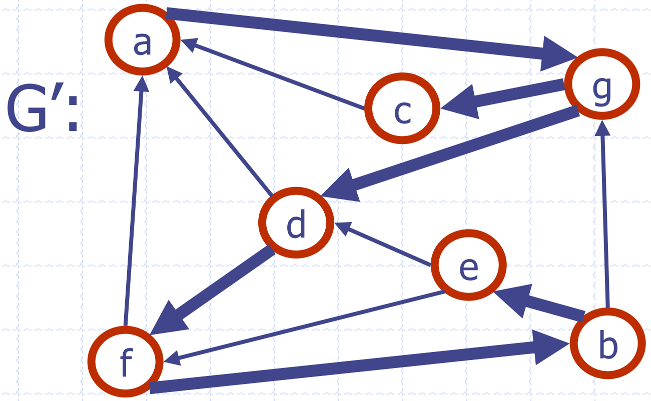


Strong Connectivity Algorithm



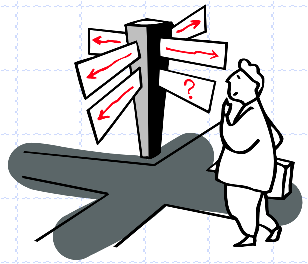
- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: O(n+m)



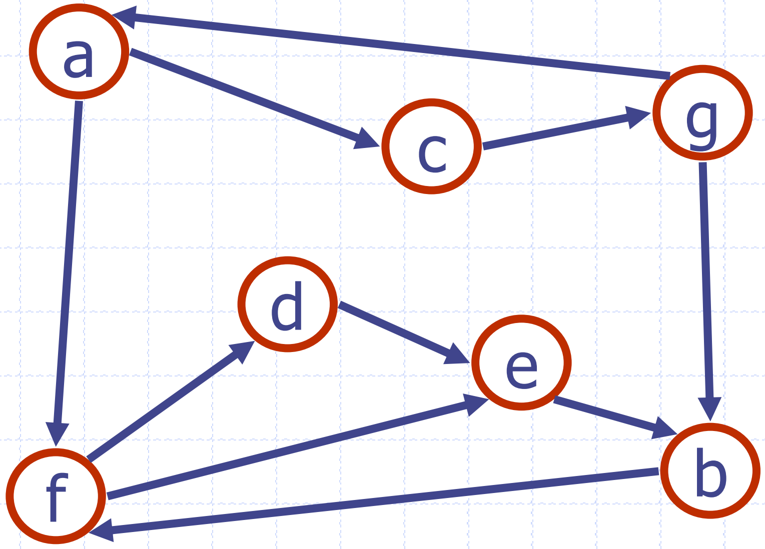


http://www.cs.usfca.edu/~galles/visualization/ConnectedComponent.html

Strongly Connected Components



- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).

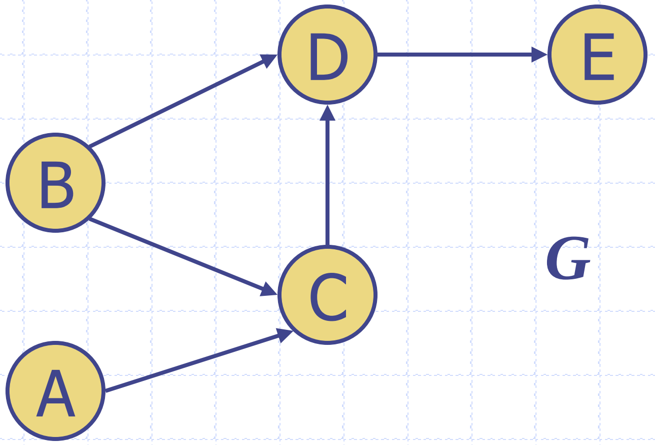


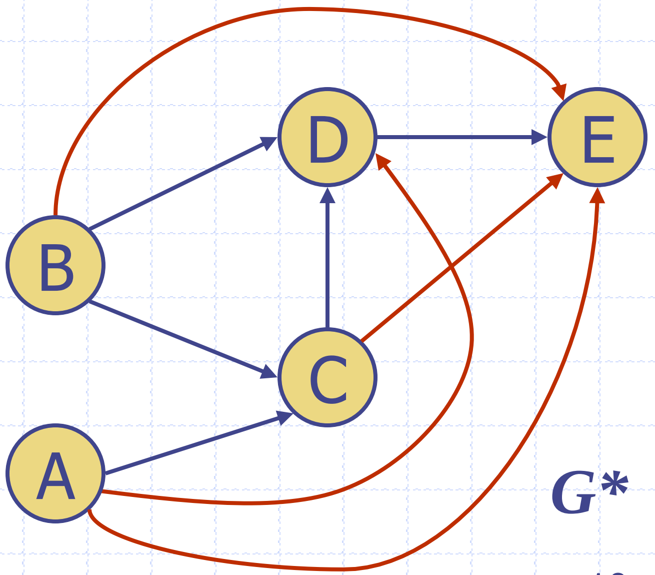
{ a , c , g }

{ f , d , e , b }

Transitive Closure

- Given a digraph ***G***, the transitive closure of ***G*** is the digraph ***G**** such that
  - ***G**** has the same vertices as ***G***
  - if ***G*** has a directed path from ***u*** to ***v*** ($u \neq v$), ***G**** has a directed edge from ***u*** to ***v***
- The transitive closure provides reachability information about a digraph

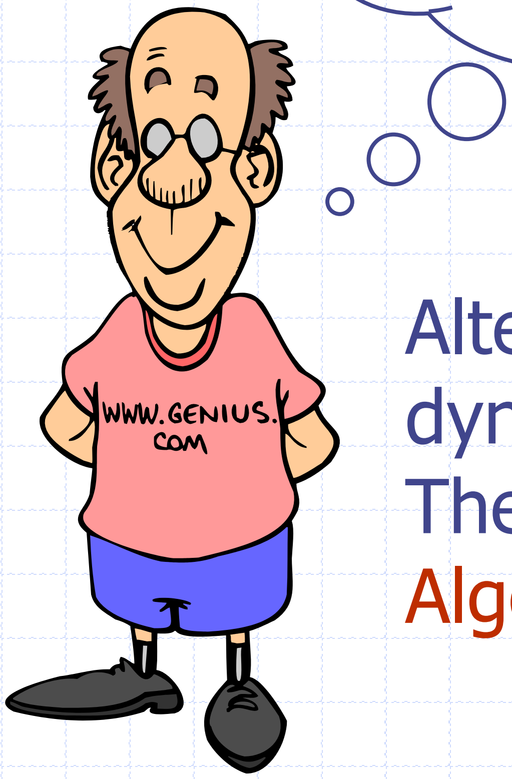




Computing the Transitive Closure

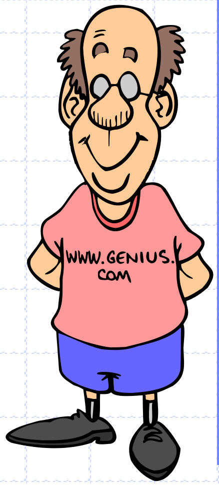
- We can perform DFS starting at each vertex
  - O(n(n+m))

If there's a way to get from A to B and from B to C, then there's a way to get from A to C.

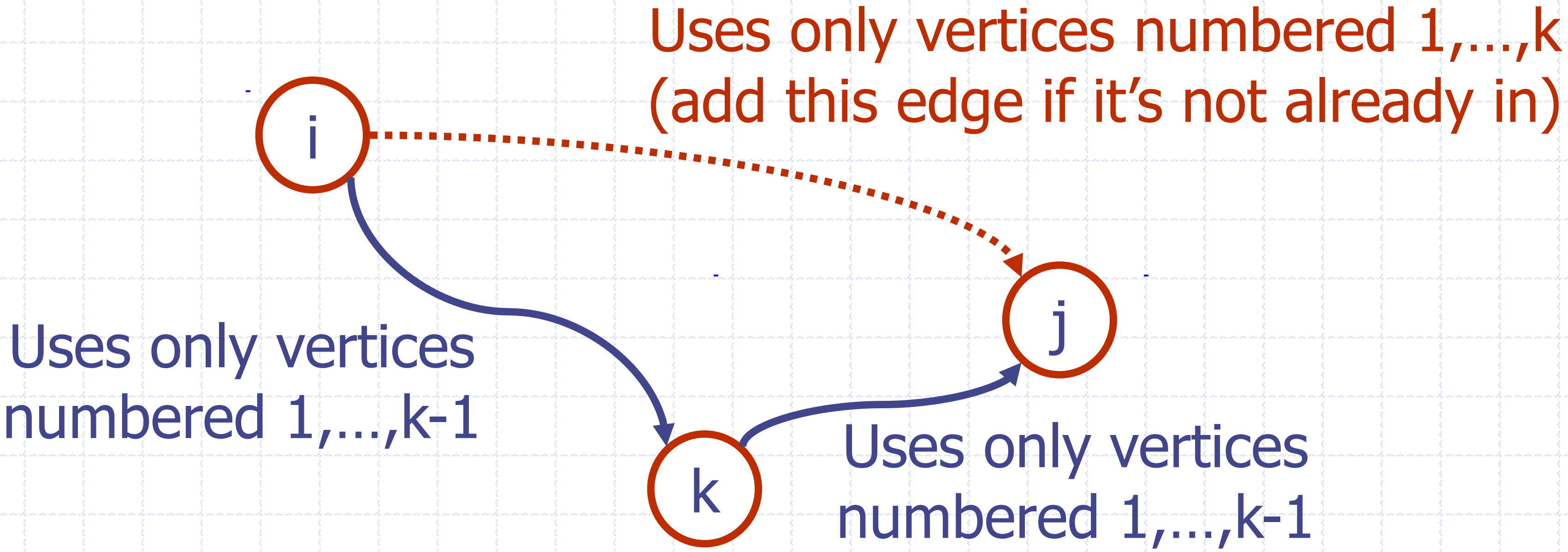


Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

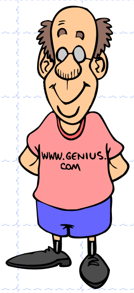
Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:


Uses only vertices numbered 1,…,k
(add this edge if it's not already in)
i
j
k
Uses only vertices numbered 1,…,k-1
Uses only vertices numbered 1,…,k-1

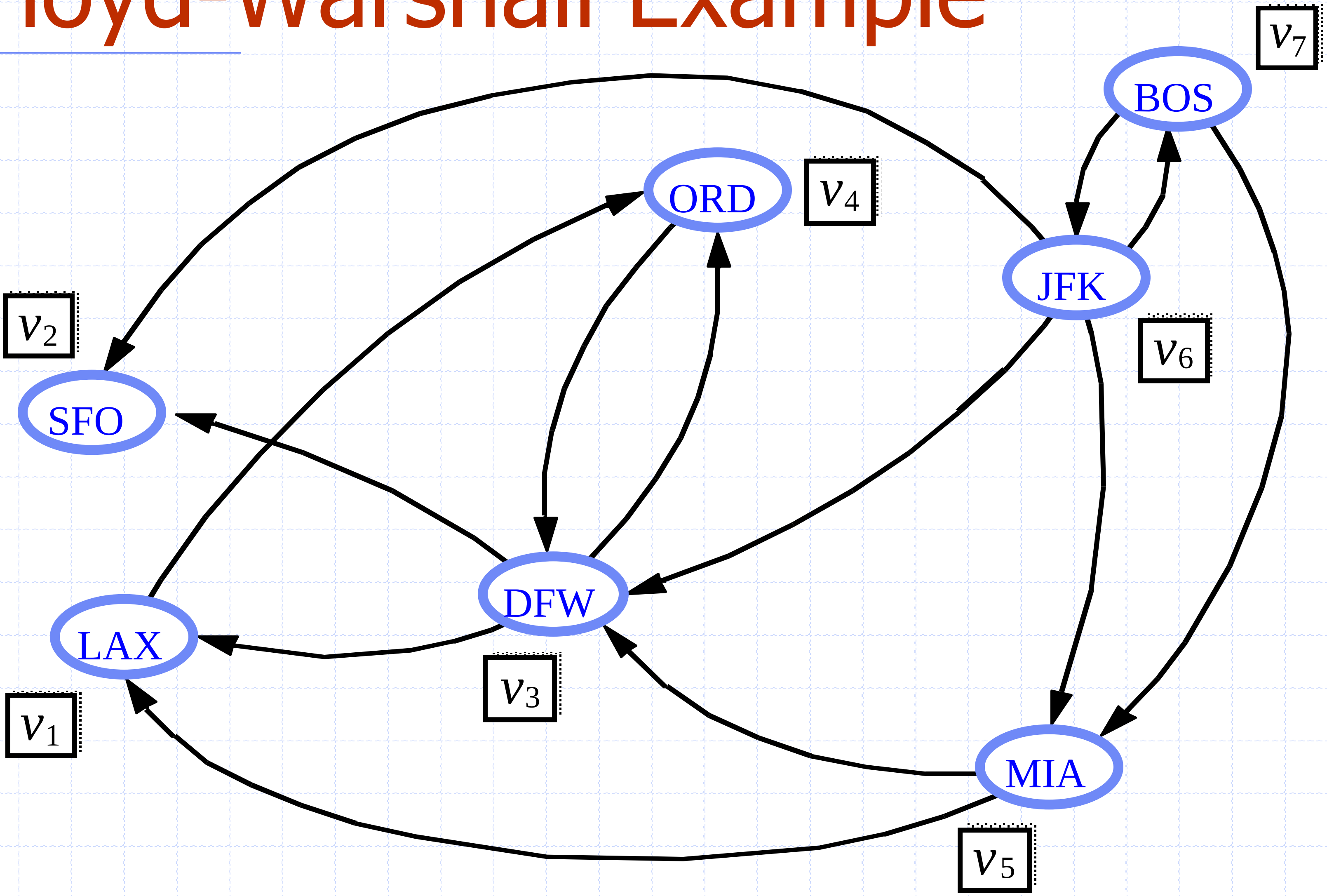
Floyd-Warshall's Algorithm



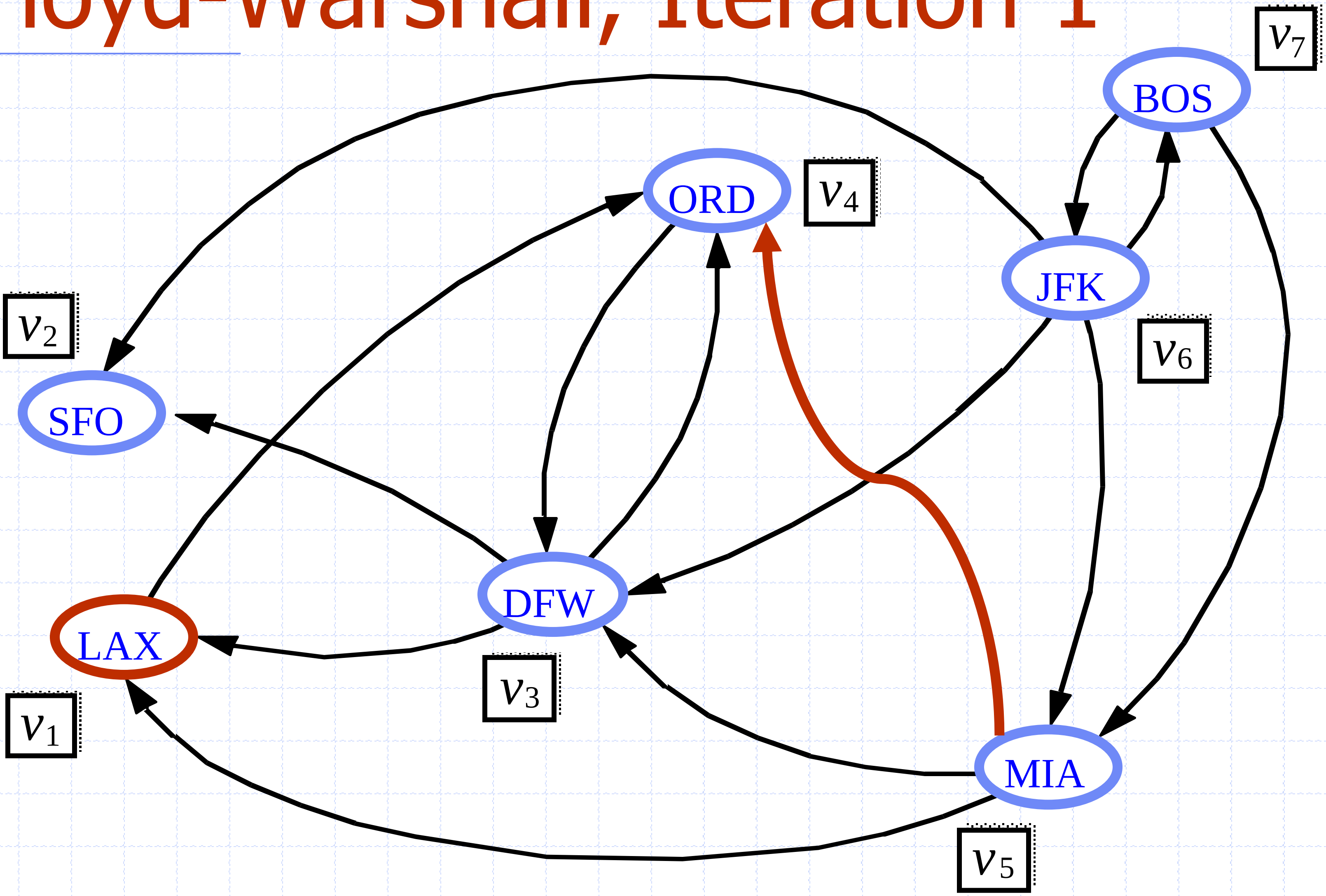
- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

```
Algorithm FloydWarshall(G)
    Input digraph G
    Output transitive closure G* of G
    i ← 1
    for all v ∈ G.vertices()
        denote v as v_i
        i ← i + 1
    G_0 ← G
    for k ← 1 to n do
        G_k ← G_{k−1}
        for i ← 1 to n (i ≠ k) do
            for j ← 1 to n (j ≠ i, k) do
                if G_{k−1}.areAdjacent(v_i, v_k) ∧
                        G_{k−1}.areAdjacent(v_k, v_j)
                    if ¬G_k.areAdjacent(v_i, v_j)
                        G_k.insertDirectedEdge(v_i, v_j , k)
        return G_n
```

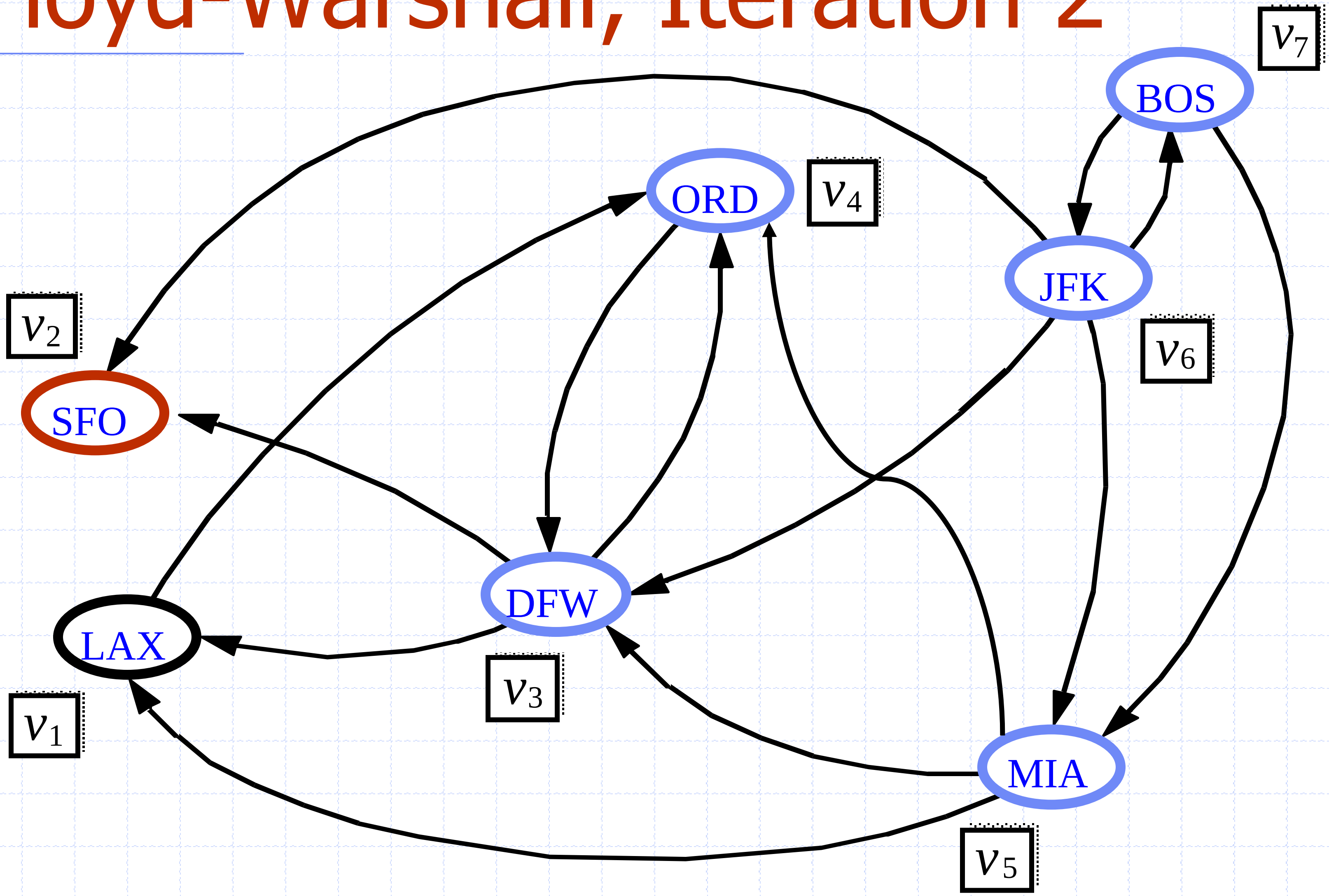
Floyd-Warshall Example



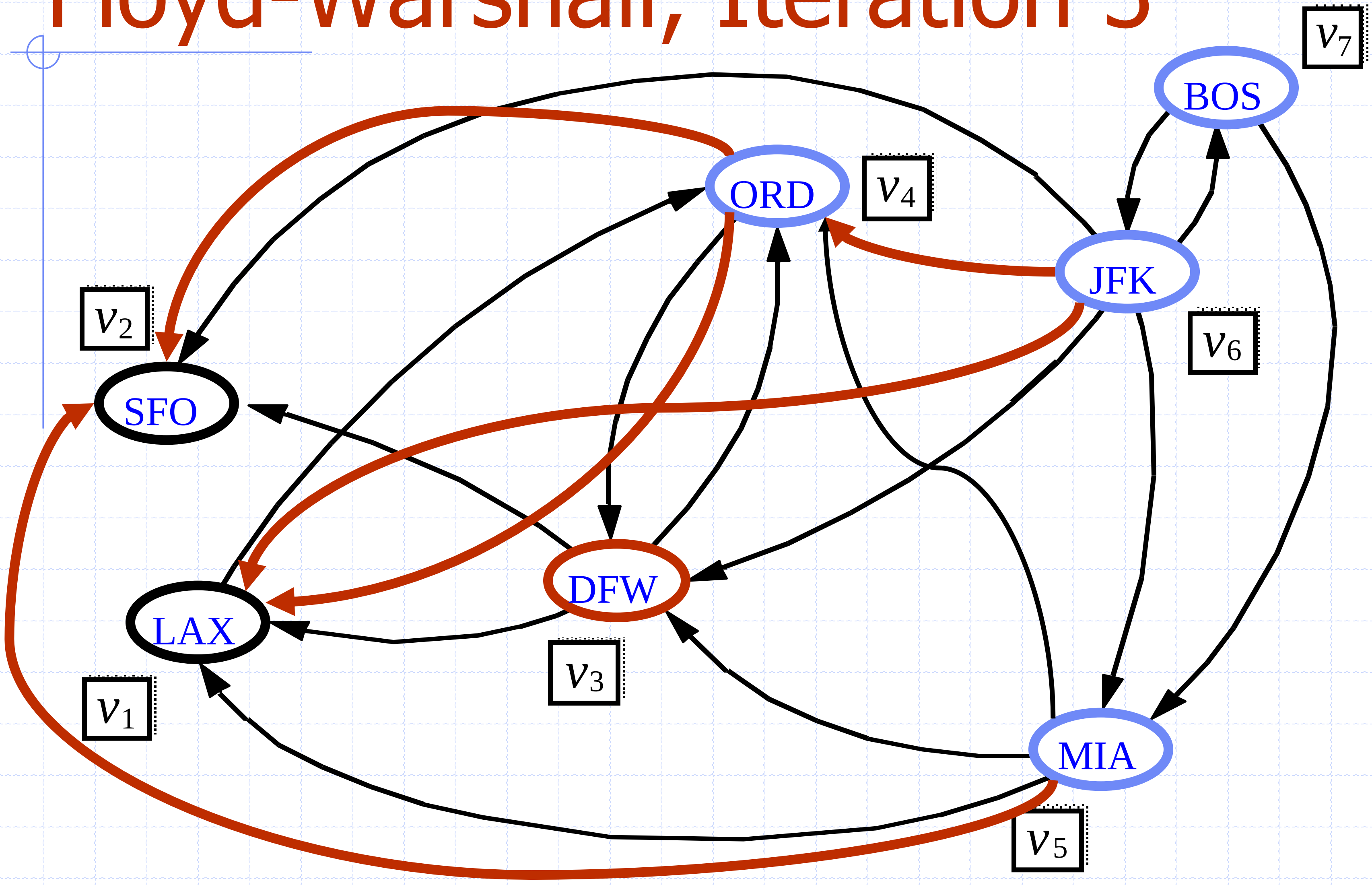
Floyd-Warshall, Iteration 1



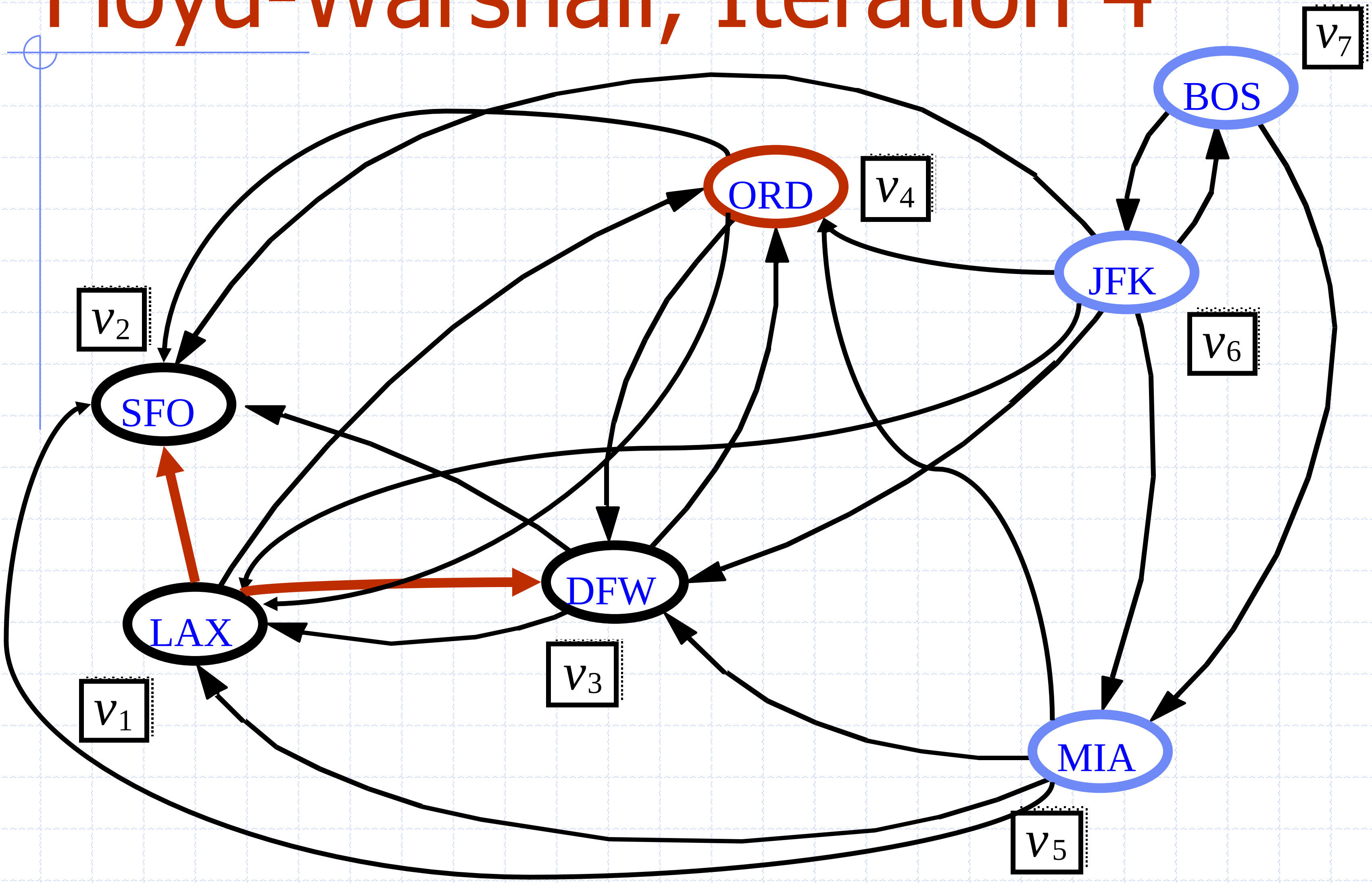
Floyd-Warshall, Iteration 2



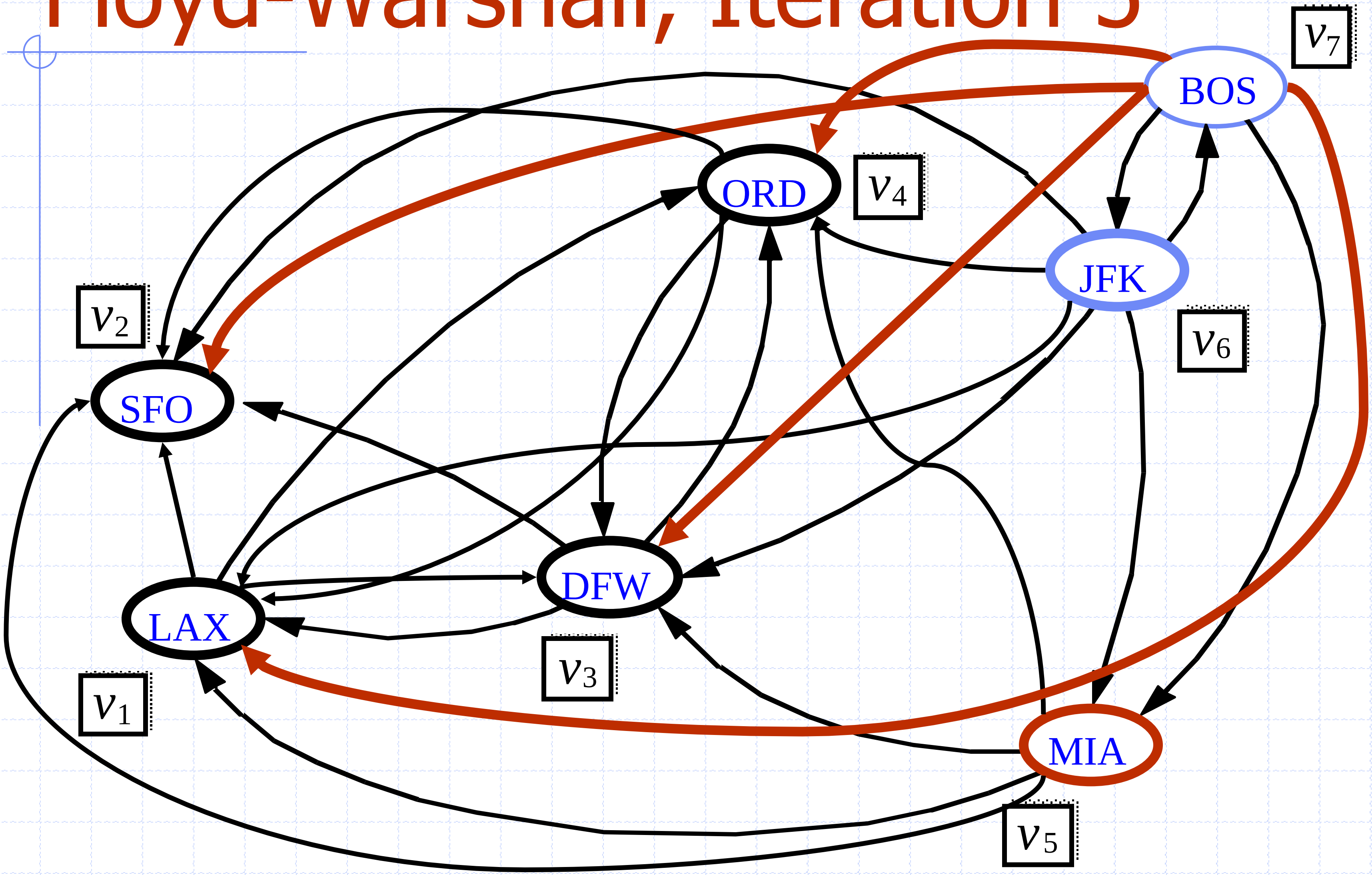
Floyd-Warshall, Iteration 3



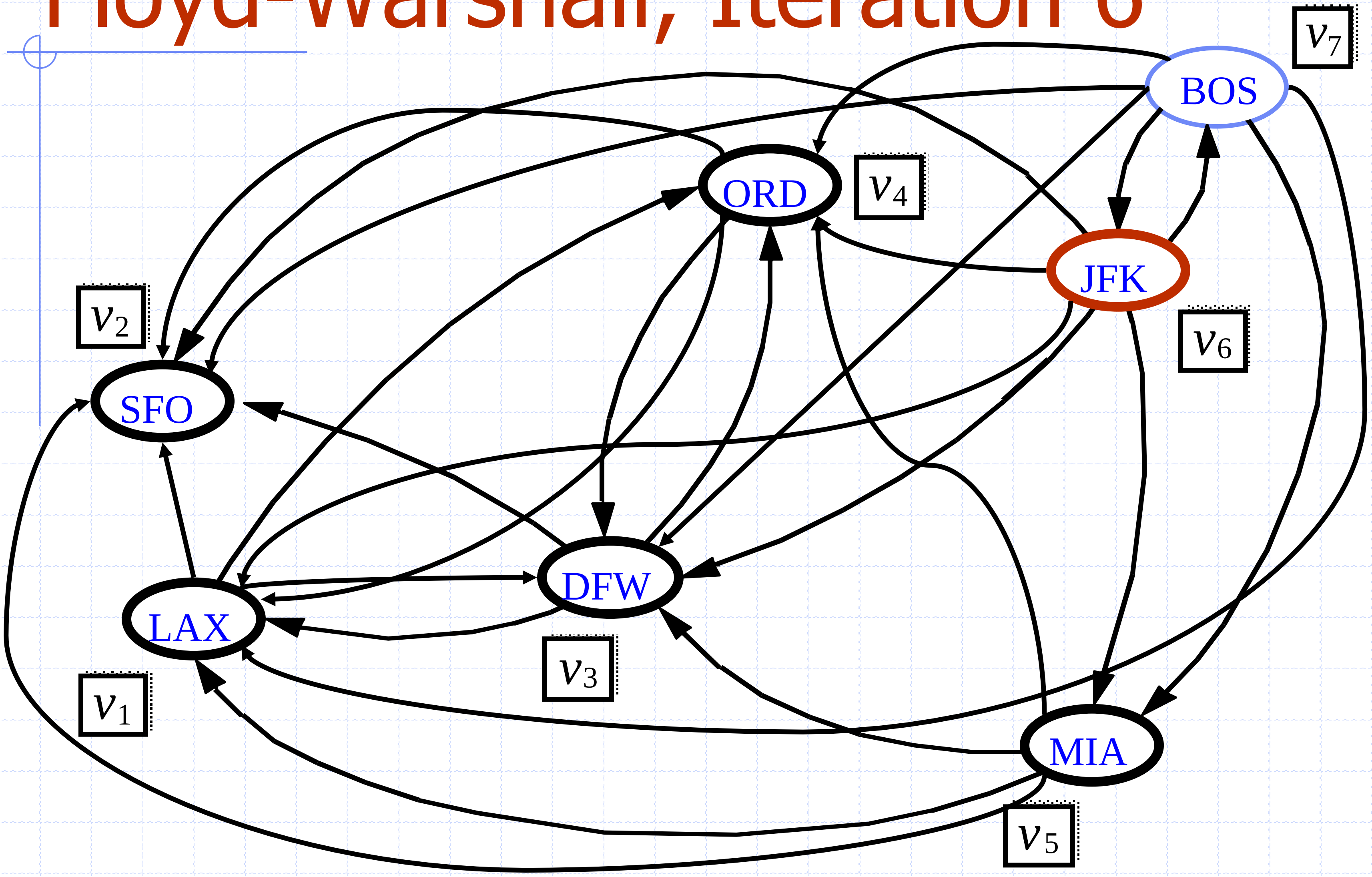
Floyd-Warshall, Iteration 4



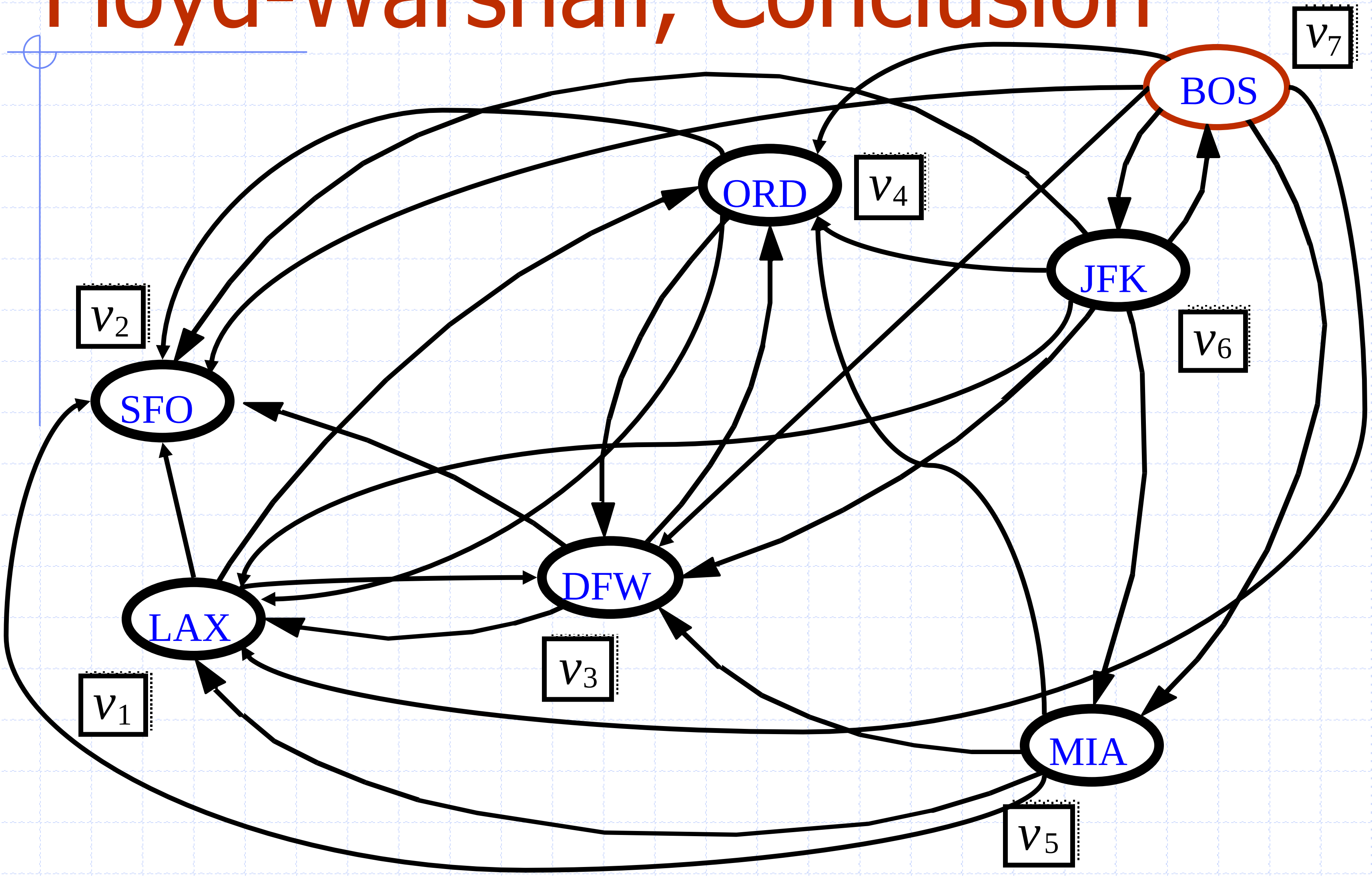
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6



Floyd-Warshall, Conclusion



DAGs and Topological Ordering

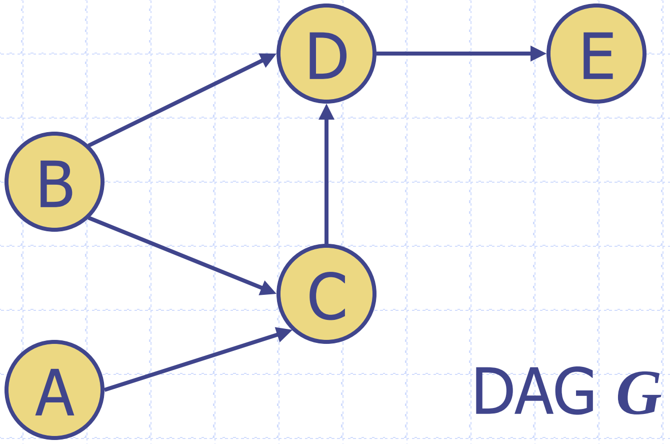
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

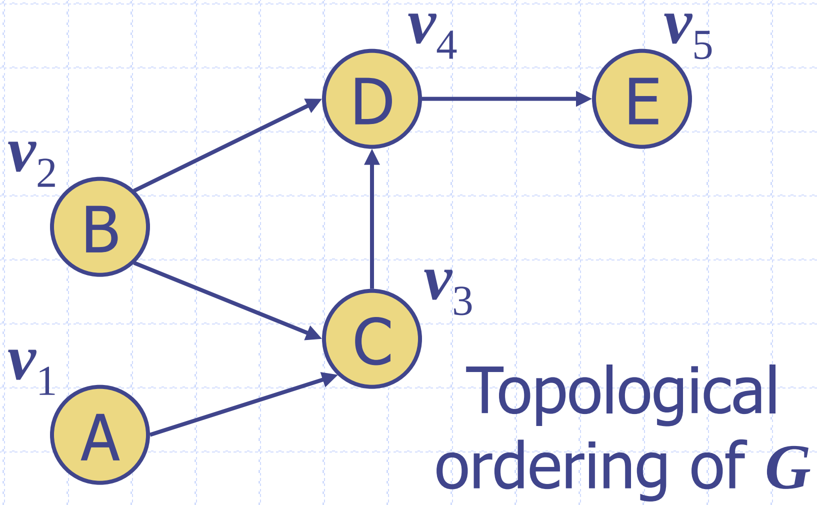
  $$v_1, \ldots, v_n$$

  of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

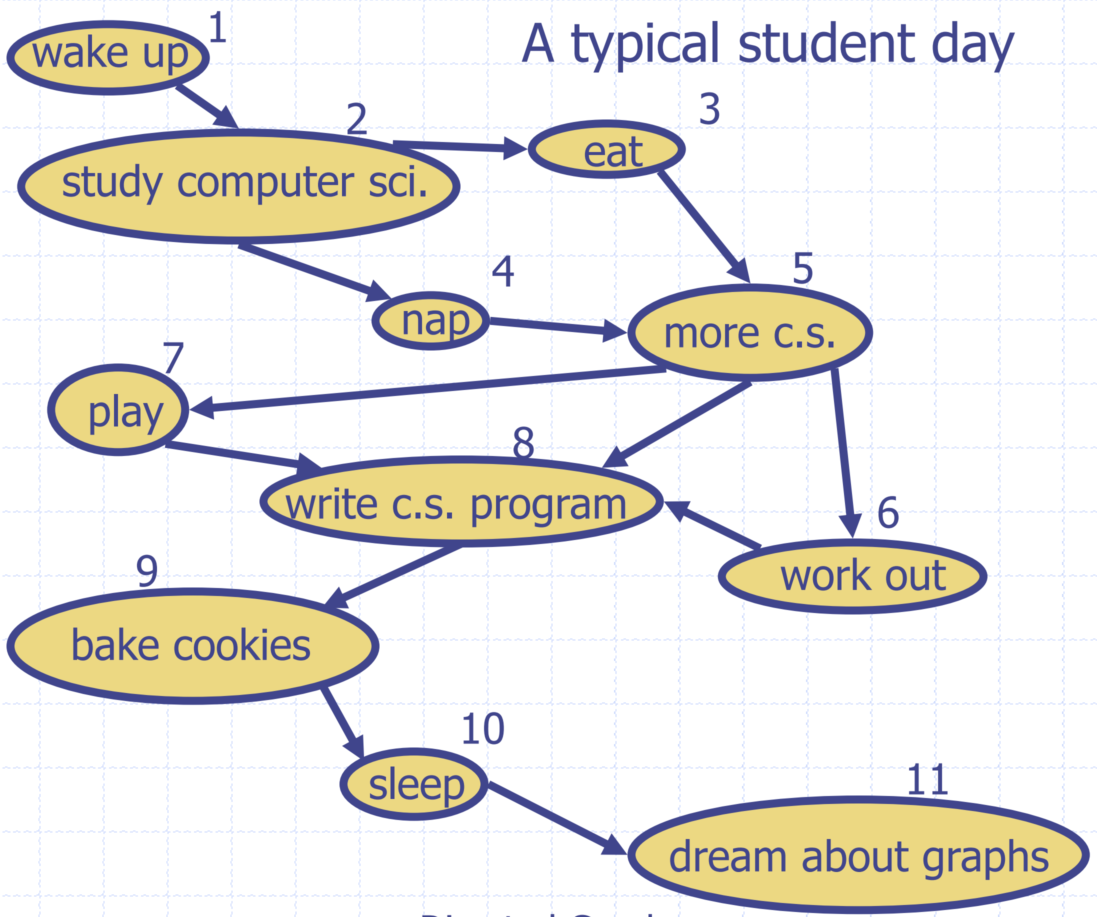
A digraph admits a topological ordering if and only if it is a DAG





Topological Sorting

- Number vertices, so that (u,v) in E implies u < v



Algorithm for Topological Sorting

- Book's algorithm is different: http://www.cs.usfca.edu/~galles/visualization/TopoSortIndegree.html

```
Algorithm TopologicalSort(G)
    H ← G            // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n – 1
        Remove v from H
```

- Running time: $O(n + m)$

http://www.cs.usfca.edu/~galles/visualization/TopoSortDFS.html

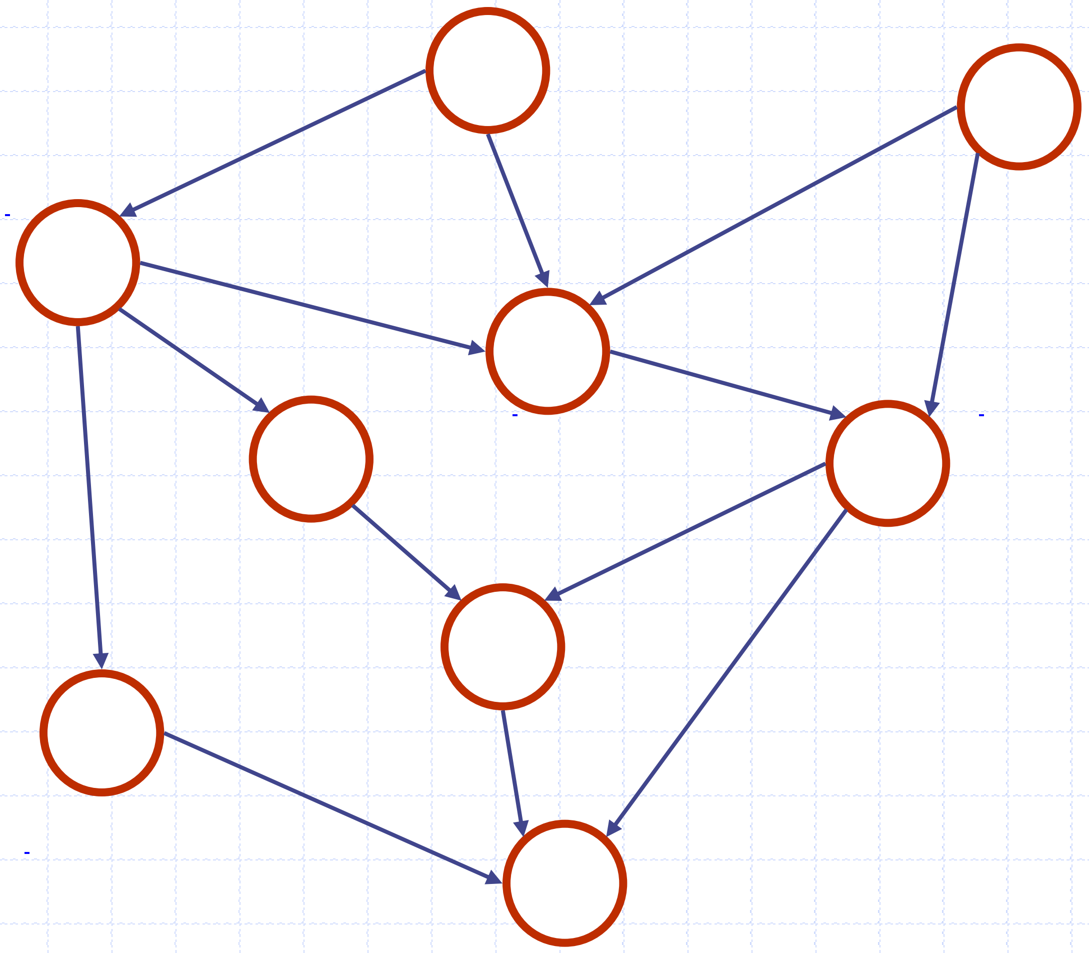
Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

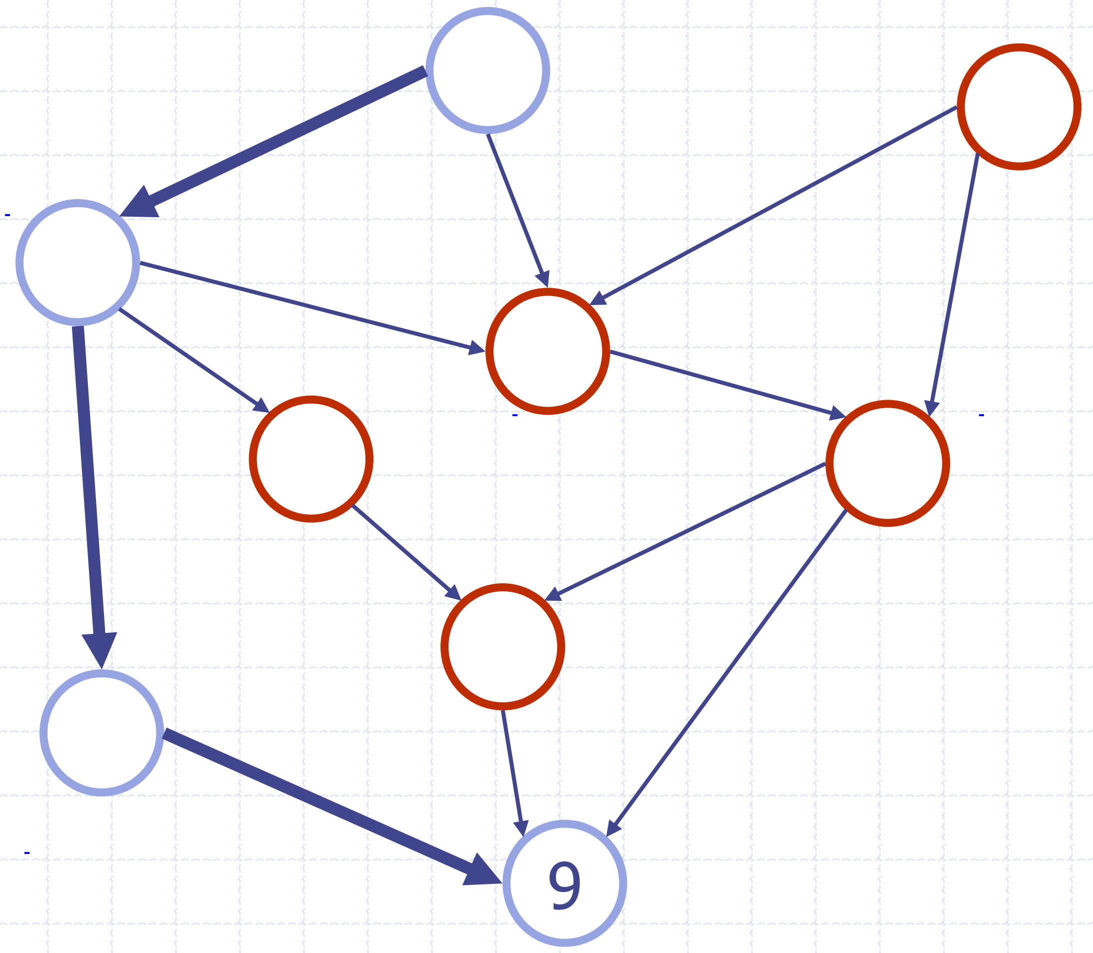
```
Algorithm topologicalDFS(G)
    Input dag G
    Output topological ordering of G
  n ← G.numVertices()
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      topologicalDFS(G, v)
```

```
Algorithm topologicalDFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the vertices of G
    in the connected component of v
  setLabel(v, VISITED)
  for all e ∈ G.outEdges(v)
    { outgoing edges }
    w ← opposite(v,e)
    if getLabel(w) = UNEXPLORED
      { e is a discovery edge }
      topologicalDFS(G, w)
    else
      { e is a forward or cross edge }
  Label v with topological number n
  n ← n - 1
```

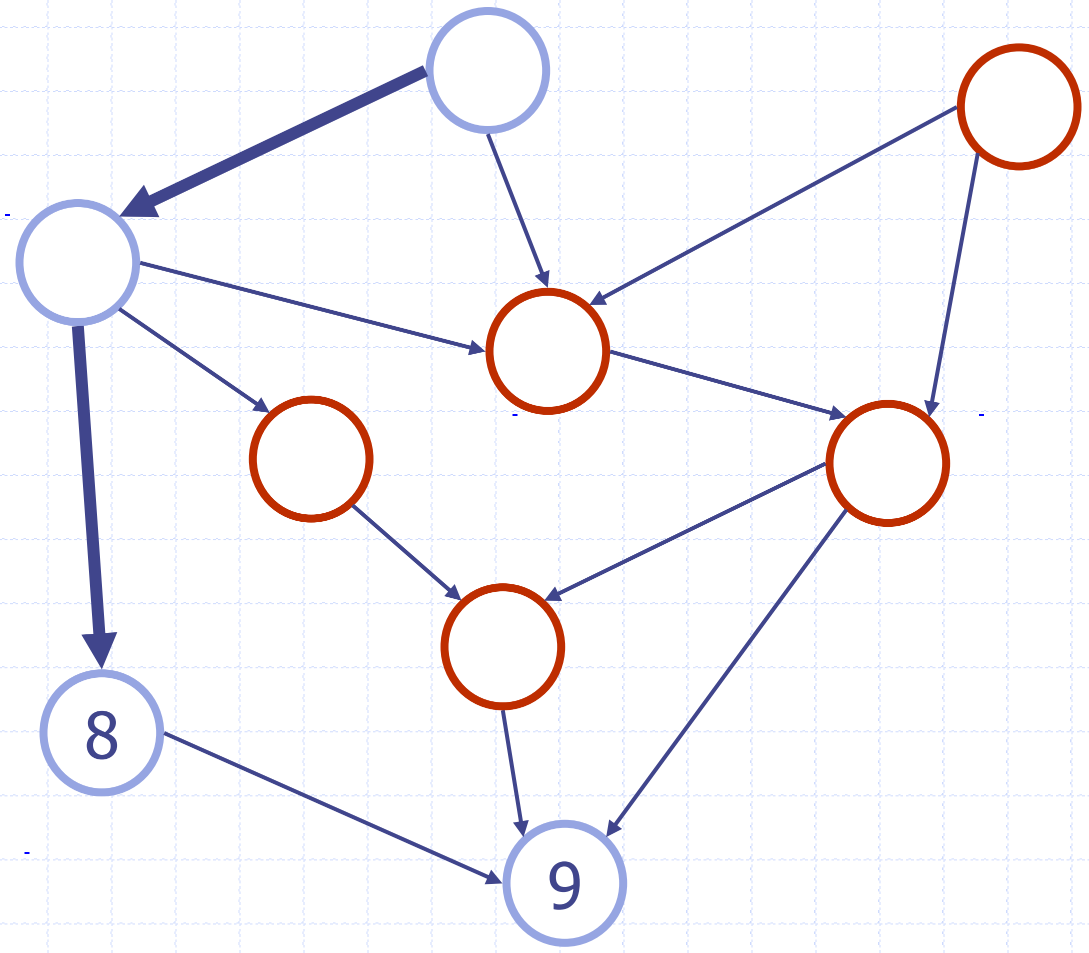
Topological Sorting Example



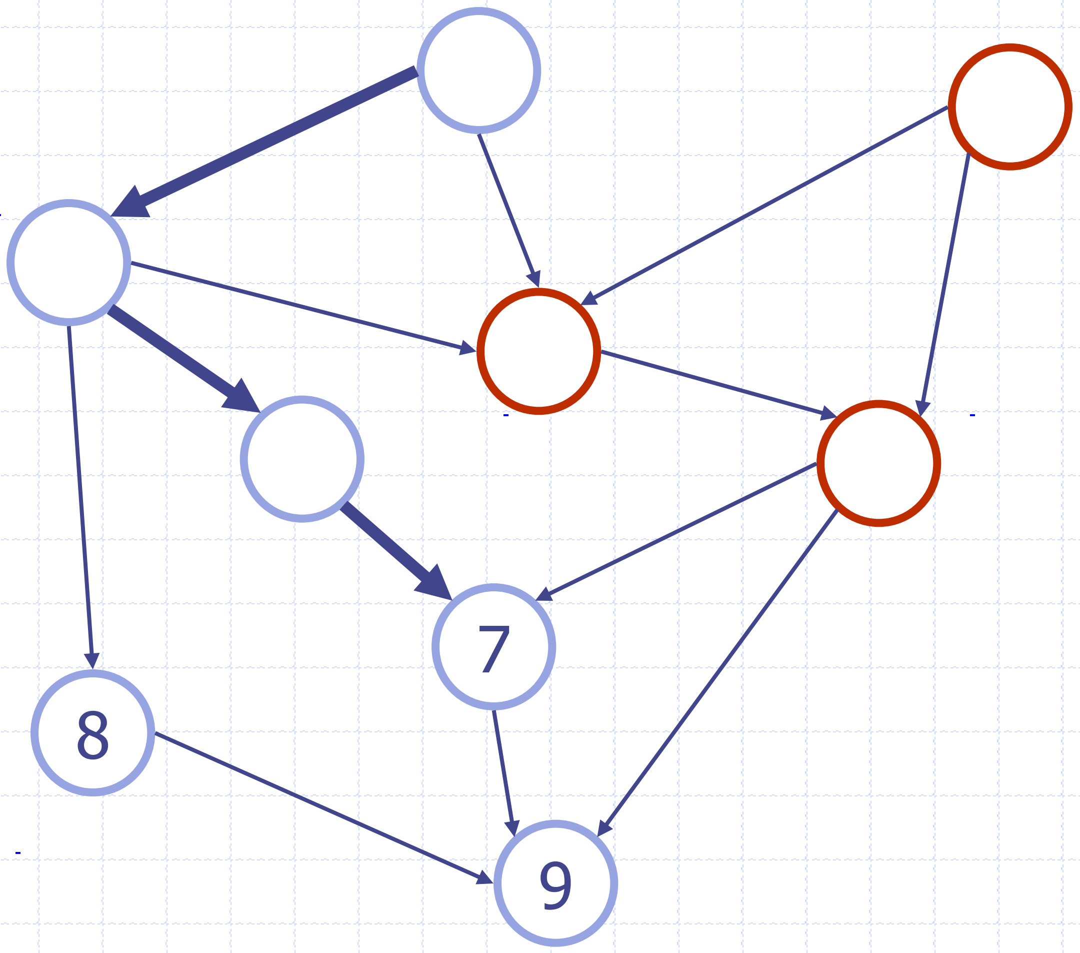
Topological Sorting Example



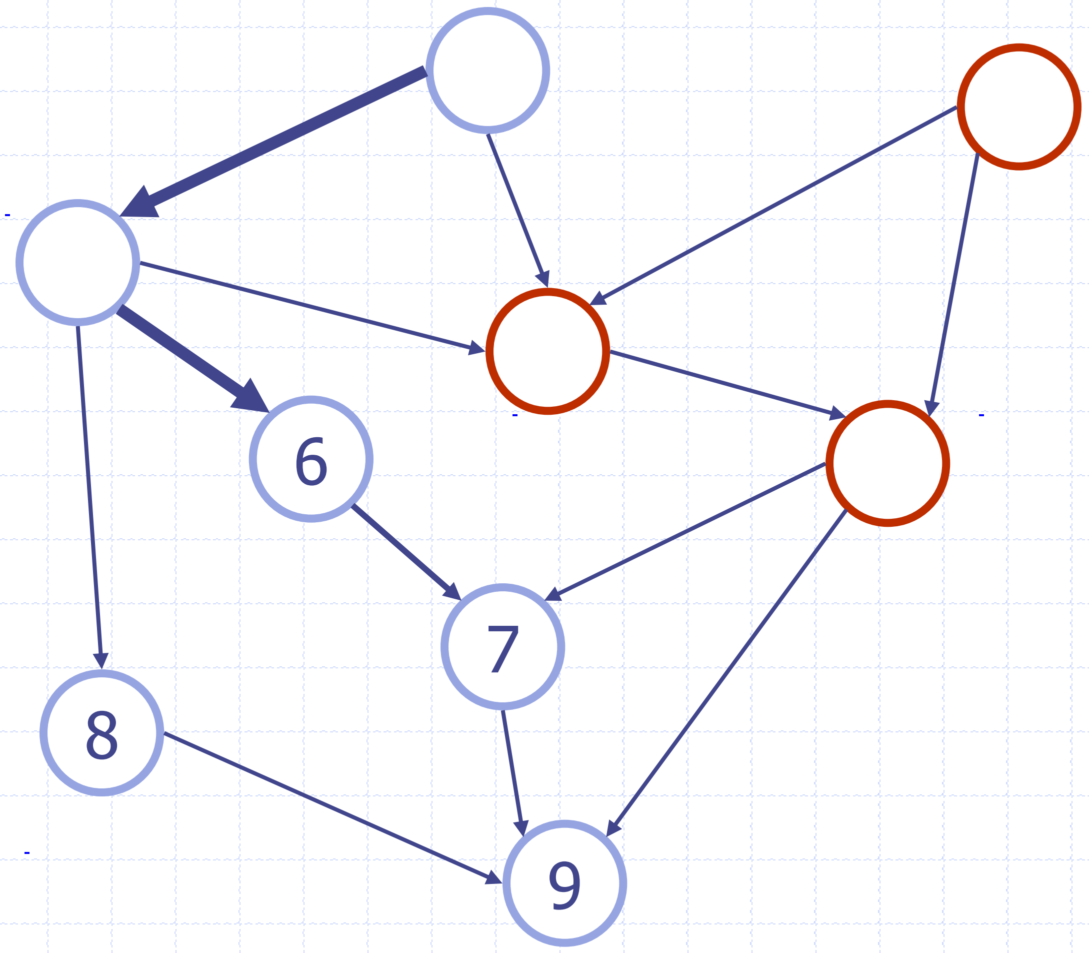
Topological Sorting Example



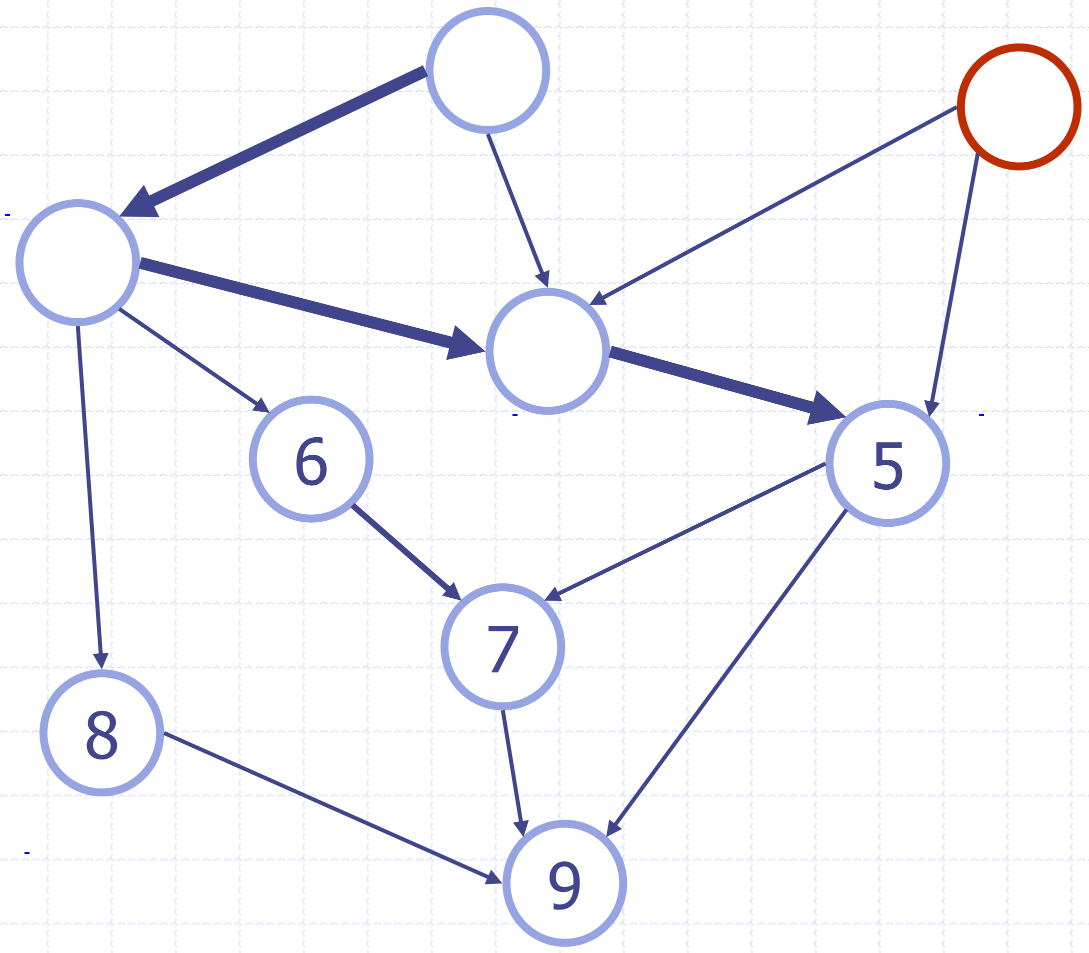
Topological Sorting Example



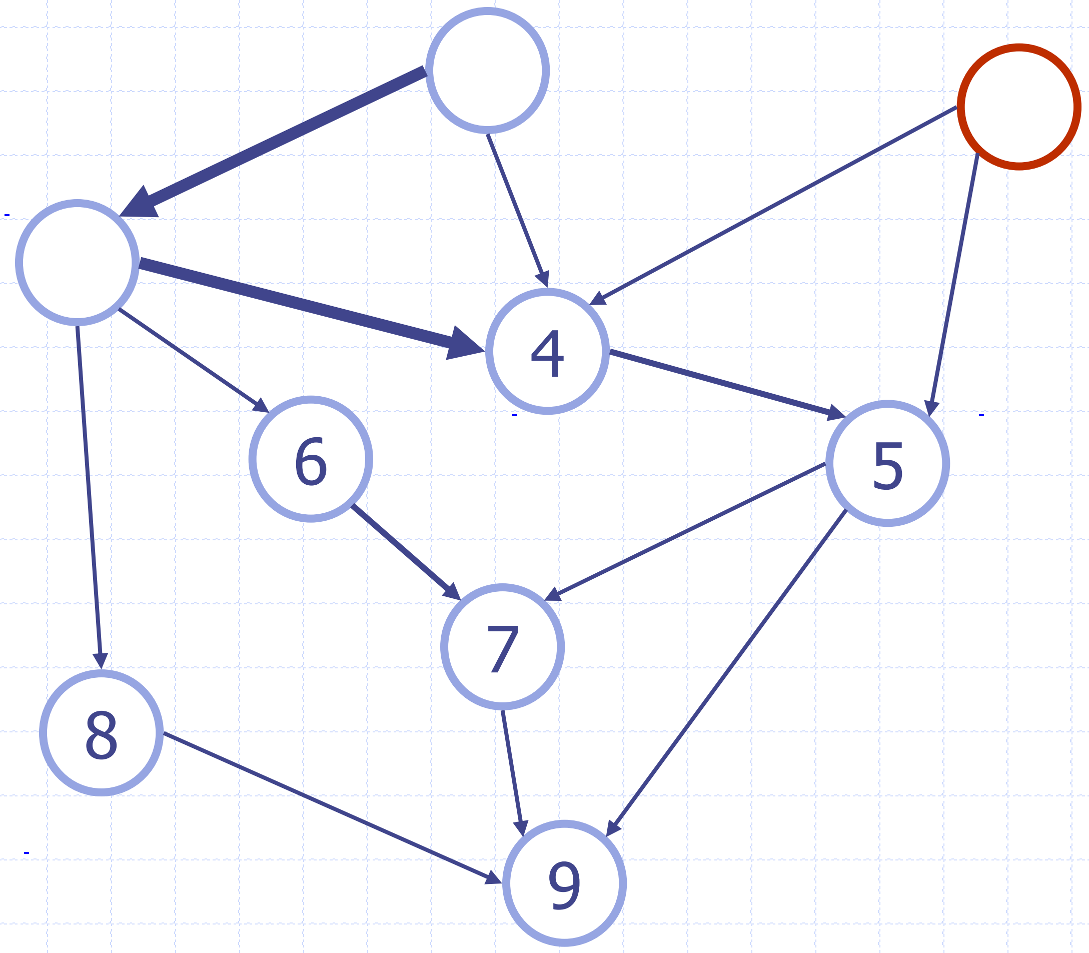
Topological Sorting Example



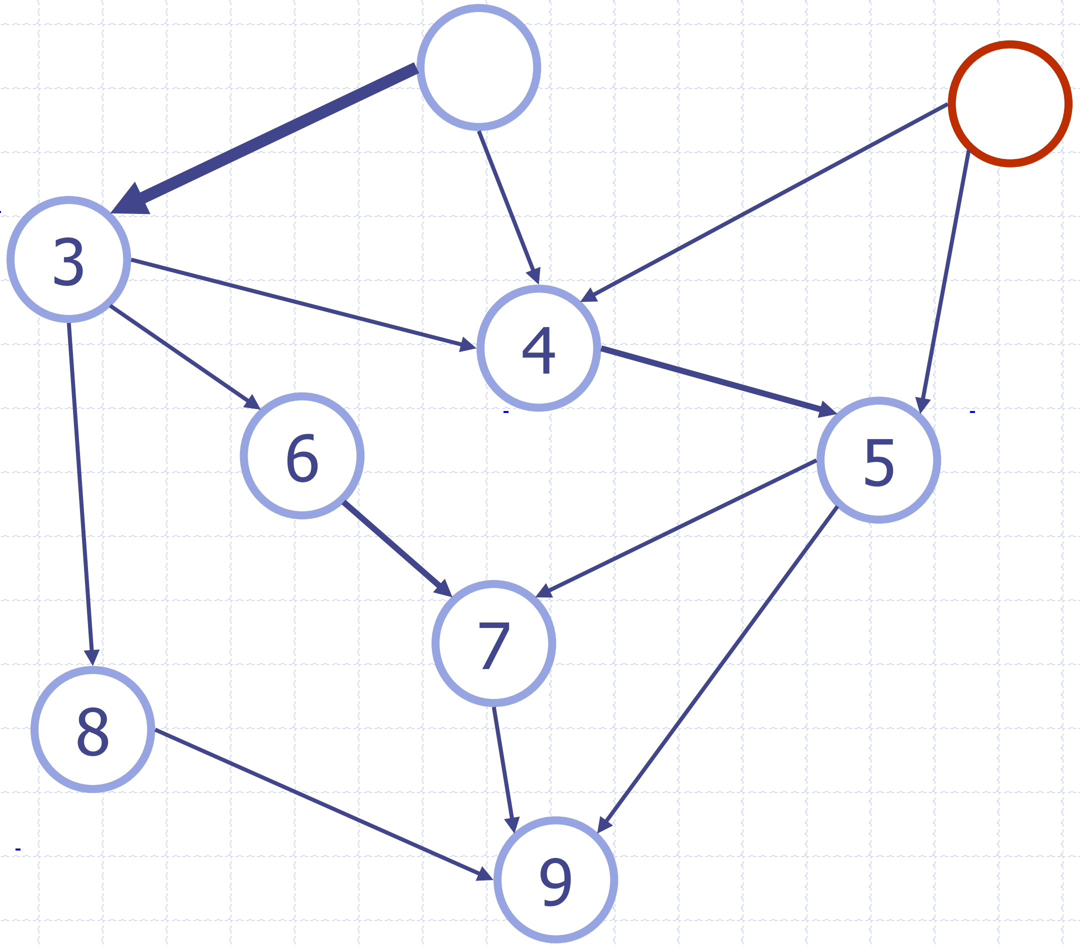
Topological Sorting Example



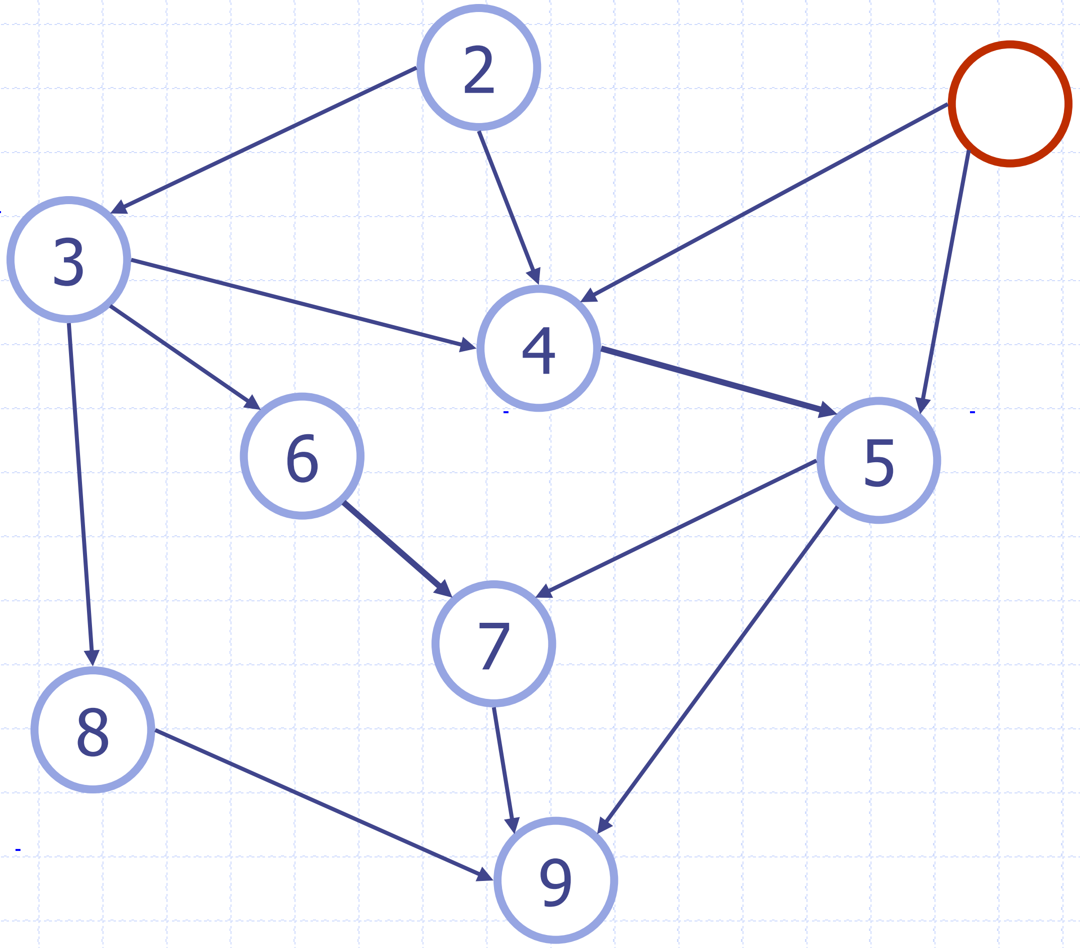
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



Topological Sorting Example