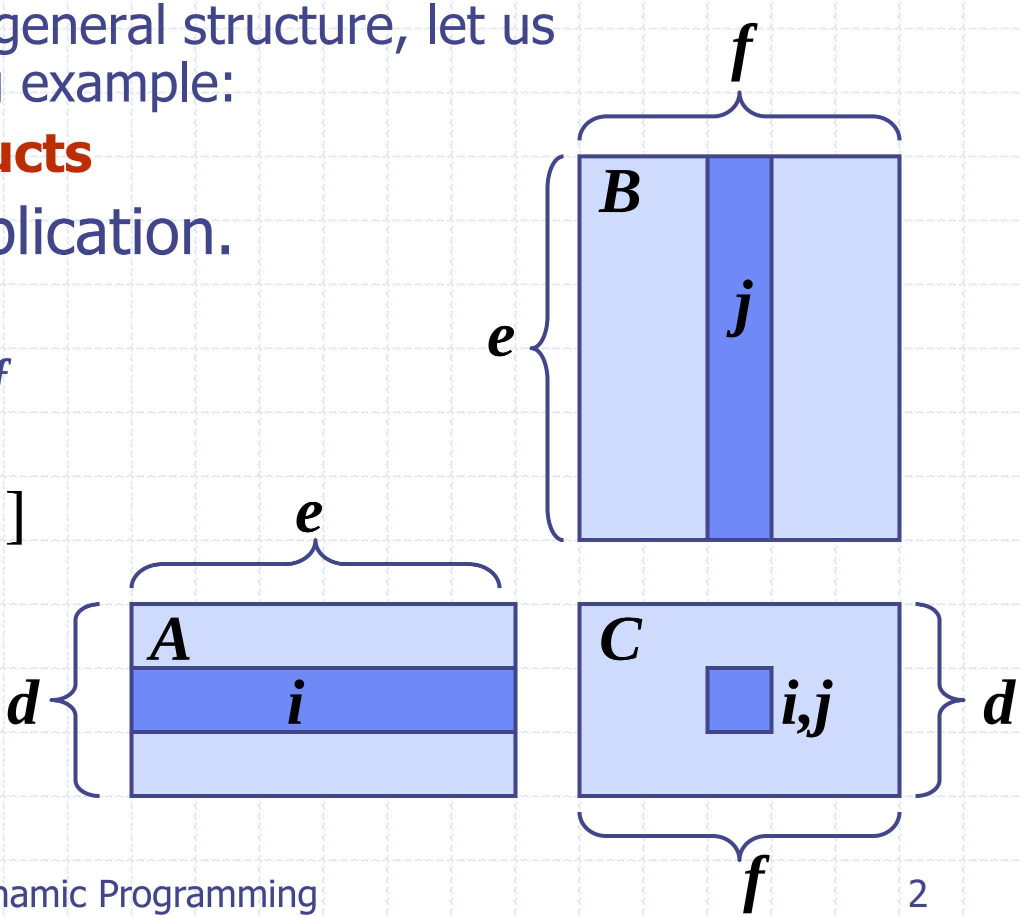
Dynamic Programming

Matrix Chain-Products

- Dynamic Programming is a general algorithm design paradigm.
  - Rather than give the general structure, let us first give a motivating example:
  - **Matrix Chain-Products**
- Review: Matrix Multiplication.
  - $C = A*B$
  - A is $d \times e$ and B is $e \times f$

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$



  - $O(d*e*f)$ time

Matrix Chain-Products

- **Matrix Chain-Product:**
  - Compute $A=A_0*A_1*\ldots*A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to parenthesize?
- Example
  - B is 3×100
  - C is 100×5
  - D is 5×5
  - (B*C)*D takes 1500 + 75 = 1575 steps
  - B*(C*D) takes 1500 + 2500 = 4000 steps

An Enumeration Approach

- **Matrix Chain-Product Alg.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - The number of paranethesizations is equal to the number of binary trees with n nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

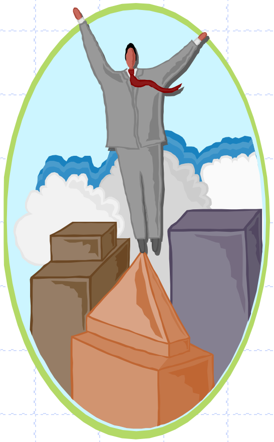
A Greedy Approach

- Idea #1: repeatedly select the product that uses (up) the most operations.
- Counter-example:
  - A is 10 × 5
  - B is 5 × 10
  - C is 10 × 5
  - D is 5 × 10
  - Greedy idea #1 gives (A*B)*(C*D), which takes 500+1000+500 = 2000 ops
  - A*((B*C)*D) takes 500+250+250 = 1000 ops

Another Greedy Approach

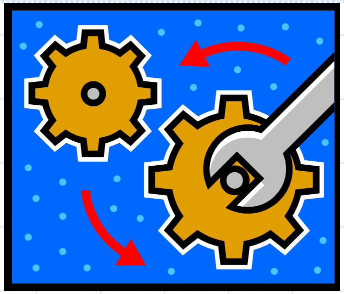
- Idea #2: repeatedly select the product that uses the fewest operations.
- Counter-example:
  - A is 101 × 11
  - B is 11 × 9
  - C is 9 × 100
  - D is 100 × 99
  - Greedy idea #2 gives A*((B*C)*D)), which takes 109989+9900+108900=228789 ops
  - (A*B)*(C*D) takes 9999+89991+89100=189090 ops
- The greedy approach is not giving us the optimal value.

A "Recursive" Approach



- Define **subproblems**:
  - Find the best parenthesization of $A_i*A_{i+1}*\ldots*A_j$.
  - Let $N_{i,j}$ denote the number of operations done by this subproblem.
  - The optimal solution for the whole problem is $N_{0,n-1}$.
- **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiply is at index i: $(A_0*\ldots*A_i)*(A_{i+1}*\ldots*A_{n-1})$.
  - Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.
  - If the global optimum did not have these optimal subproblems, we could define an even better "optimal" solution.

A Characterizing Equation

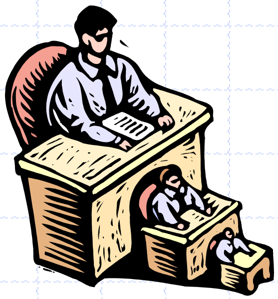


- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- Let us consider all possible places for that final multiply:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- Note that subproblems are not independent--the **subproblems overlap**.

A Dynamic Programming Algorithm



- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $N_{i,i}$'s are easy, so start with them
- Then do length 2,3,… subproblems, and so on.
- The running time is $O(n^3)$

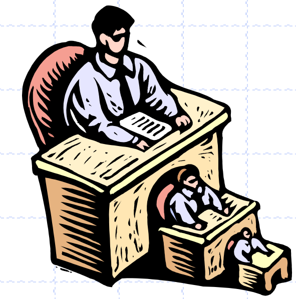
Algorithm *matrixChain*(*S*):

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal paranethization of ***S***

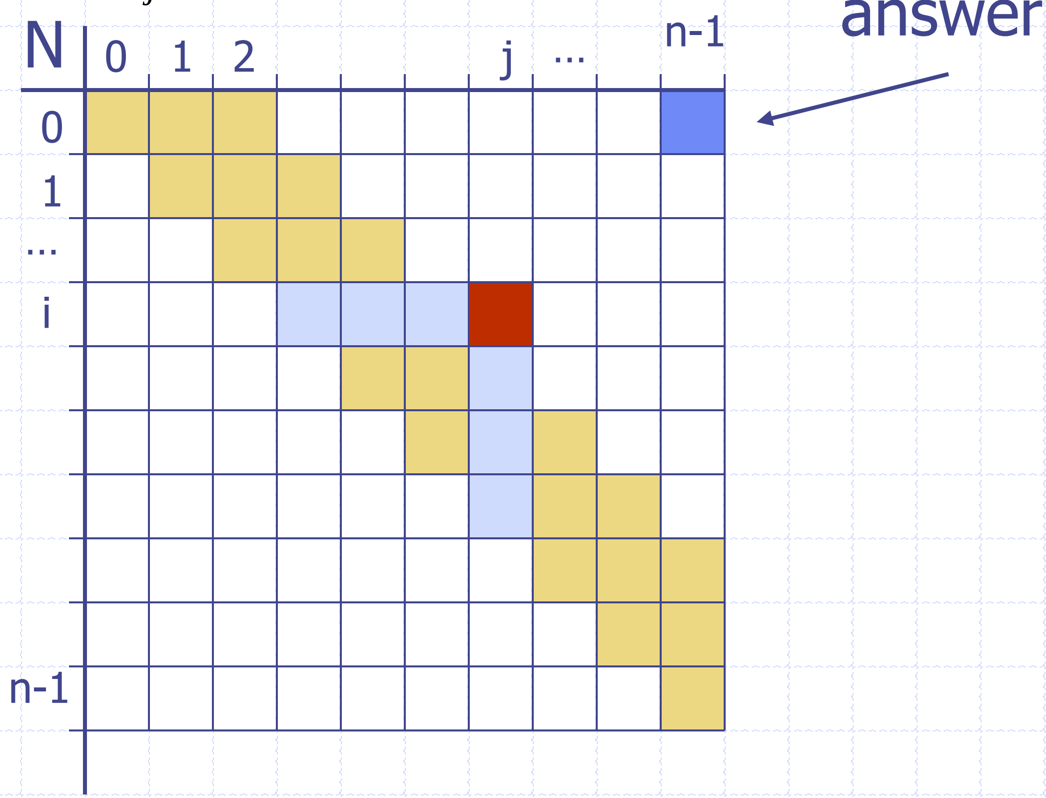
```
for i ← 1 to n-1 do
    N_{i,i} ← 0
for b ← 1 to n-1 do
    for i ← 0 to n-b-1 do
        j ← i+b
        N_{i,j} ← +infinity
        for k ← i to j-1 do
            N_{i,j} ← min{N_{i,j} , N_{i,k} +N_{k+1,j} +d_i d_{k+1} d_{j+1}}
```

A Dynamic Programming Algorithm Visualization

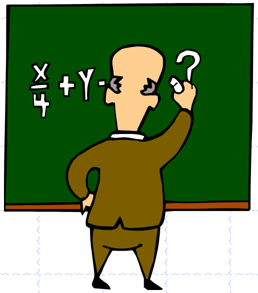


$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from pervious entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering "k" for each N entry



The General Dynamic Programming Technique



- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

Subsequences

- A ***subsequence*** of a character string $x_0x_1x_2\ldots x_{n-1}$ is a string of the form $x_{i_1}x_{i_2}\ldots x_{i_k}$, where $i_j < i_{j+1}$.
- Not the same as substring!
- Example String: ABCDEFGHIJK
  - Subsequence: ACEGJIK
  - Subsequence: DFGHK
  - Not subsequence: DAGH

The Longest Common Subsequence (LCS) Problem

- Given two strings X and Y, the longest common subsequence (LCS) problem is to find a longest subsequence common to both X and Y
- Has applications to DNA similarity testing (alphabet is {A,C,G,T})
- Example: ABCDEFG and XZACKDFWGH have ACDFG as a longest common subsequence

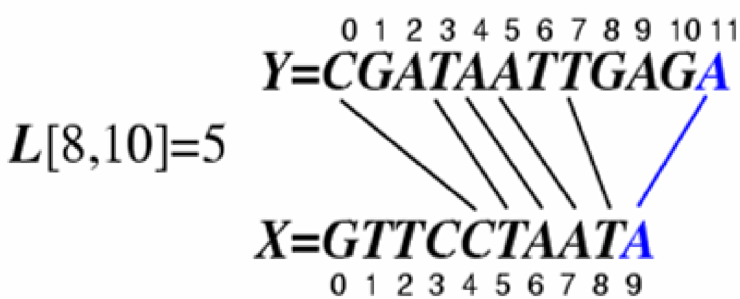
A Poor Approach to the LCS Problem

- A Brute-force solution:
  - Enumerate all subsequences of X
  - Test which ones are also subsequences of Y
  - Pick the longest one.
- Analysis:
  - If X is of length n, then it has 2^n subsequences
  - This is an exponential-time algorithm!

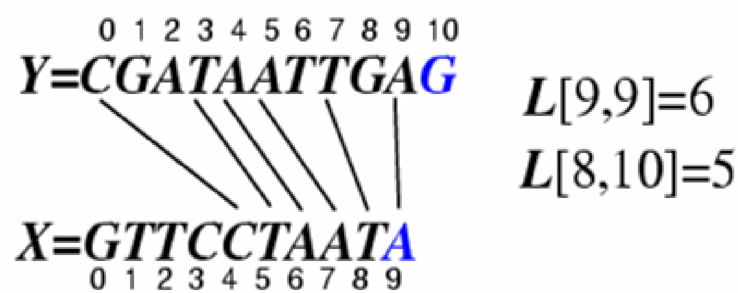
A Dynamic-Programming Approach to the LCS Problem

- Define L[i,j] to be the length of the longest common subsequence of X[0..i] and Y[0..j].
- Allow for -1 as an index, so L[-1,k] = 0 and L[k,-1]=0, to indicate that the null part of X or Y has no match with the other.
- Then we can define L[i,j] in the general case as follows:
  1. If xi=yj, then L[i,j] = L[i-1,j-1] + 1 (we can add this match)
  2. If xi≠yj, then L[i,j] = max{L[i-1,j], L[i,j-1]} (we have no match here)

Case 1:



Case 2:



An LCS Algorithm

Algorithm LCS(X,Y):

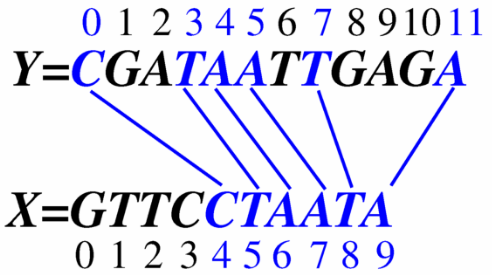
Input: Strings X and Y with n and m elements, respectively

Output: For i = 0,…,n-1, j = 0,...,m-1, the length L[i, j] of a longest string that is a subsequence of both the string X[0..i] = $x_0x_1x_2…x_i$ and the string Y [0.. j] = $y_0y_1y_2…y_j$

for i =1 to n-1 **do**
    L[i,-1] = 0
for j =0 to m-1 **do**
    L[-1,j] = 0
for i =0 to n-1 **do**
    for j =0 to m-1 **do**
        if x_i = y_j **then**
            L[i, j] = L[i-1, j-1] + 1
        else
            L[i, j] = max{L[i-1, j] , L[i, j-1]}
return array L

Visualizing the LCS Algorithm

L	-1	0	1	2	3	4	5	6	7	8	9	10	11
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	2	2	2	2	2	2	2	2	2
2	0	0	1	1	2	2	2	3	3	3	3	3	3
3	0	1	1	1	2	2	2	3	3	3	3	3	3
4	0	1	1	1	2	2	2	3	3	3	3	3	3
5	0	1	1	1	2	2	2	3	4	4	4	4	4
6	0	1	1	2	2	3	3	3	4	4	5	5	5
7	0	1	1	2	2	3	4	4	4	4	5	5	6
8	0	1	1	2	3	3	4	5	5	5	5	5	6
9	0	1	1	2	3	4	4	5	5	5	6	6	6



Analysis of LCS Algorithm

- We have two nested loops
  - The outer one iterates n times
  - The inner one iterates m times
  - A constant amount of work is done inside each iteration of the inner loop
  - Thus, the total running time is $O(nm)$
- Answer is contained in L[n,m] (and the subsequence can be recovered from the L table).