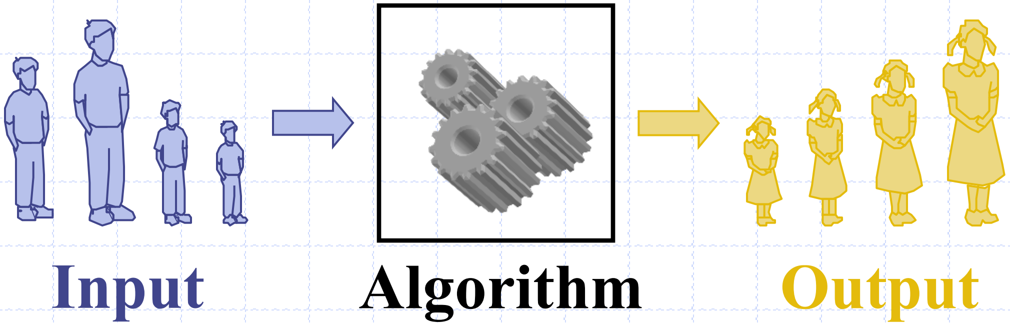
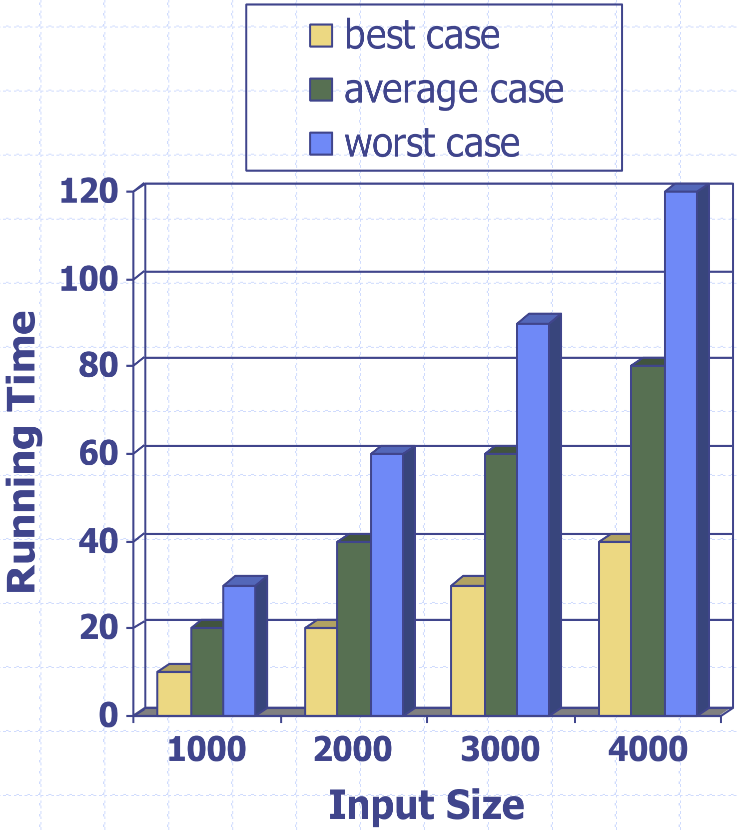
# Analysis of Algorithms



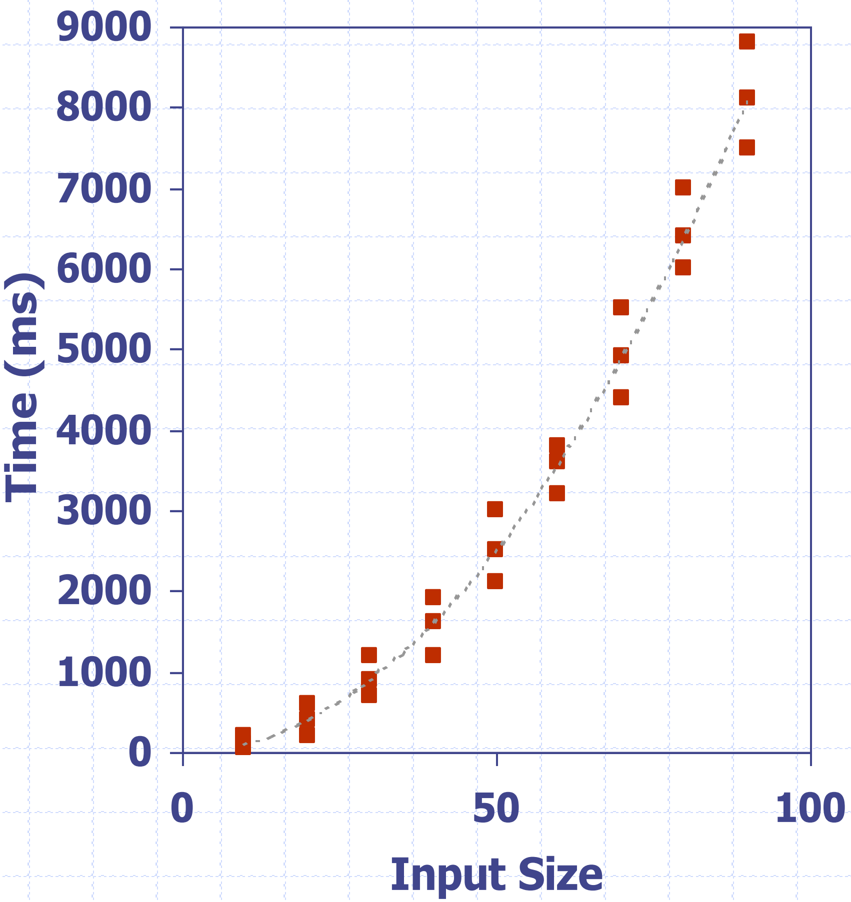
# Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically **grows** with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



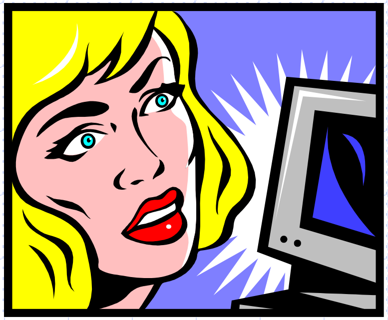
# Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results

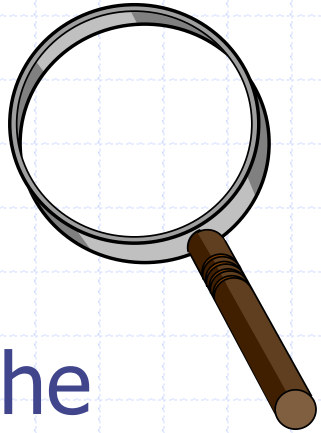


# Limitations of Experiments

- It is necessary to **implement** the algorithm, which may be difficult
- Results may **not** be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the **same** hardware and software environments must be used



# Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm **independent** of the hardware/software environment

# Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

**Algorithm** *arrayMax*($A$, $n$)
  **Input** array $A$ of $n$ integers
  **Output** maximum element of $A$

  $currentMax \leftarrow A[0]$
  **for** $i \leftarrow 1$ **to** $n - 1$ **do**
    **if** $A[i] > currentMax$ **then**
      $currentMax \leftarrow A[i]$
  **return** $currentMax$

# Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  **Algorithm** ***method*** (***arg*** [, ***arg***…])

  **Input** …

  **Output** …
- Method call

  ***var.method*** (***arg*** [, ***arg***…])
- Return value

  **return** ***expression***
- Expressions

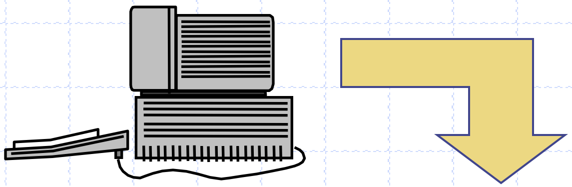
  $\leftarrow$ Assignment (like $=$ in Java)

  $=$ Equality testing (like $==$ in Java)

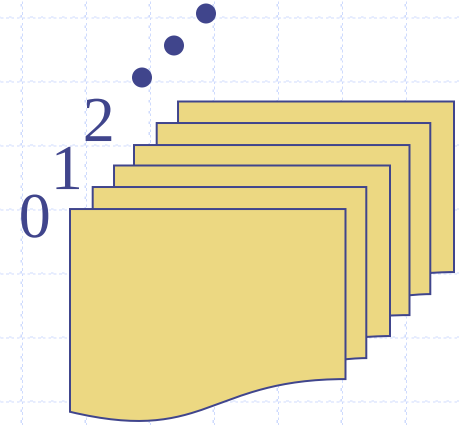
  $n^2$ Superscripts and other mathematical formatting allowed

# The Random Access Machine (RAM) Model

- A **CPU**



- An potentially **unbounded** bank of **memory** cells, each of which can hold an arbitrary number or character




- Memory cells are numbered and accessing any cell in memory takes unit time.

# Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely **independent** from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

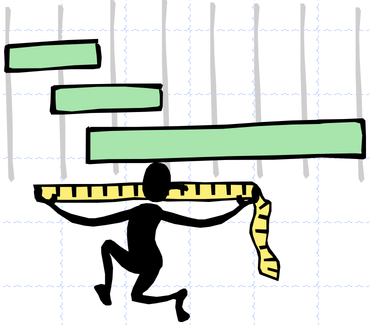
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

# Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

| **Algorithm** *arrayMax*($A$, $n$) | # operations |
| --- | --- |
| $currentMax \leftarrow A[0]$ | $2$ |
| **for** $i \leftarrow 1$ **to** $n - 1$ **do** | $2n$ |
|     **if** $A[i] > currentMax$ **then** | $2(n - 1)$ |
|         $currentMax \leftarrow A[i]$ | $2(n - 1)$ |
| { increment counter $i$ } | $2(n - 1)$ |
| **return** $currentMax$ | $1$ |
| Total | $8n - 2$ |

# Estimating Running Time



- Algorithm ***arrayMax*** executes $8n - 2$ primitive operations in the worst case. Define:

  $a$ = Time taken by the fastest primitive operation

  $b$ = Time taken by the slowest primitive operation

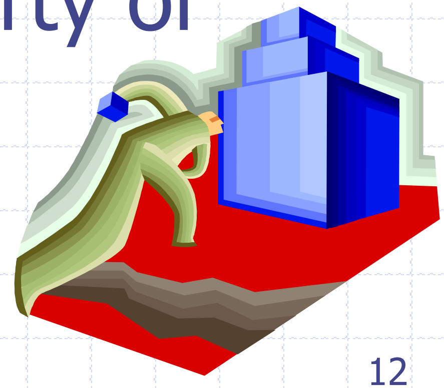
- Let $T(n)$ be worst-case time of ***arrayMax.*** Then

$$a\,(8n - 2) \leq T(n) \leq b(8n - 2)$$

- Hence, the running time $T(n)$ is bounded by two linear functions in the input size *n*

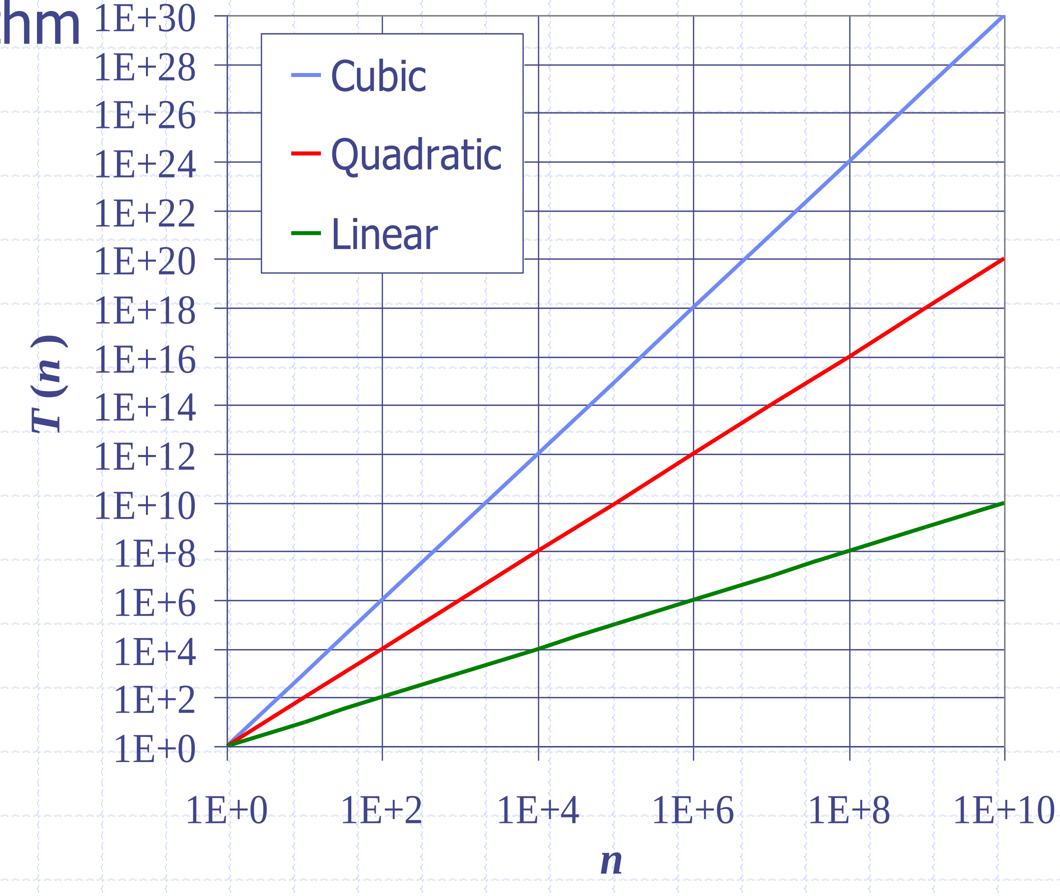
# Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***



# Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ $1$
  - Logarithmic ≈ $\log n$
  - Linear ≈ $n$
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ $n^2$
  - Cubic ≈ $n^3$
  - Exponential ≈ $2^n$
- In a log-log chart, the slope of the line corresponds to the growth rate



# Functions Graphed Using “Normal” Scale

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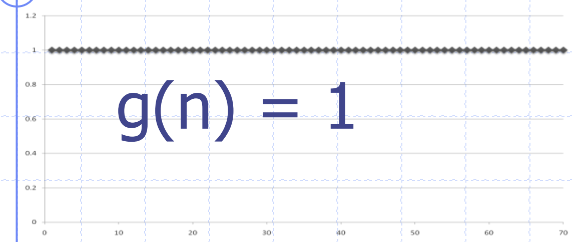


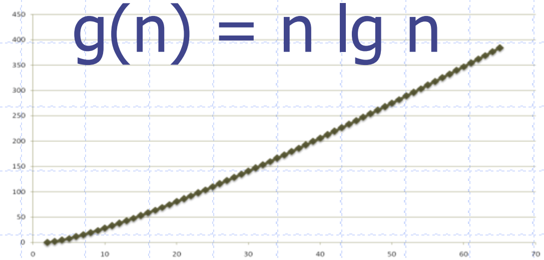


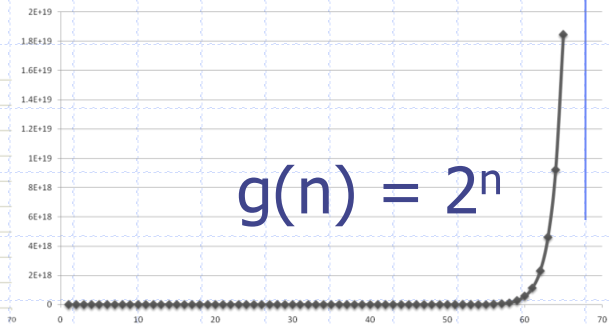


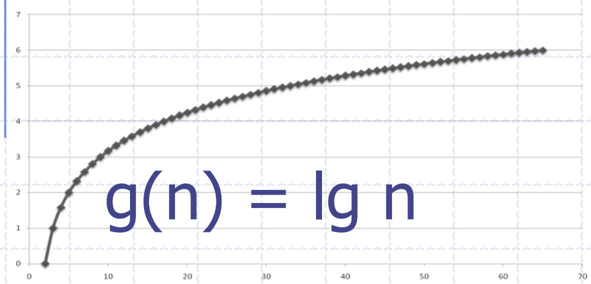


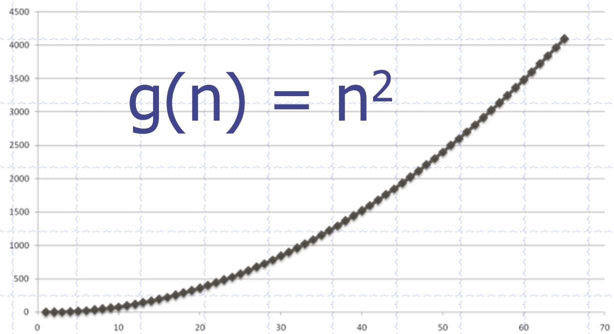


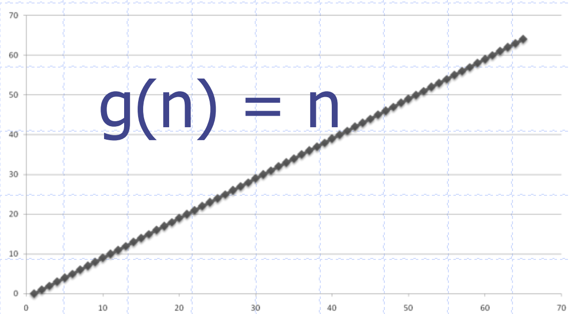


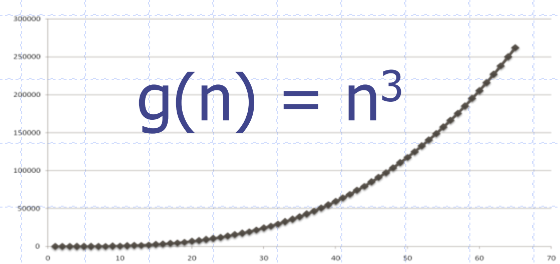

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# Why Growth Rate Matters

| if runtime is... | time for n + 1 | time for 2 n | time for 4 n |
|---|---|---|---|
| c lg n | c lg (n + 1) | c (lg n + 1) | c(lg n + 2) |
| c n | c (n + 1) | 2c n | 4c n |
| c n lg n | ~ c n lg n + c n | 2c n lg n + 2cn | 4c n lg n + 4cn |
| $c\ n^2$ | $\sim c\ n^2 + 2c\ n$ | **$4c\ n^2$** | $16c\ n^2$ |
| $c\ n^3$ | $\sim c\ n^3 + 3c\ n^2$ | $8c\ n^3$ | $64c\ n^3$ |
| $c\ 2^n$ | $c\ 2^{n+1}$ | $c\ 2^{2n}$ | $c\ 2^{4n}$ |

runtime quadruples when problem size doubles

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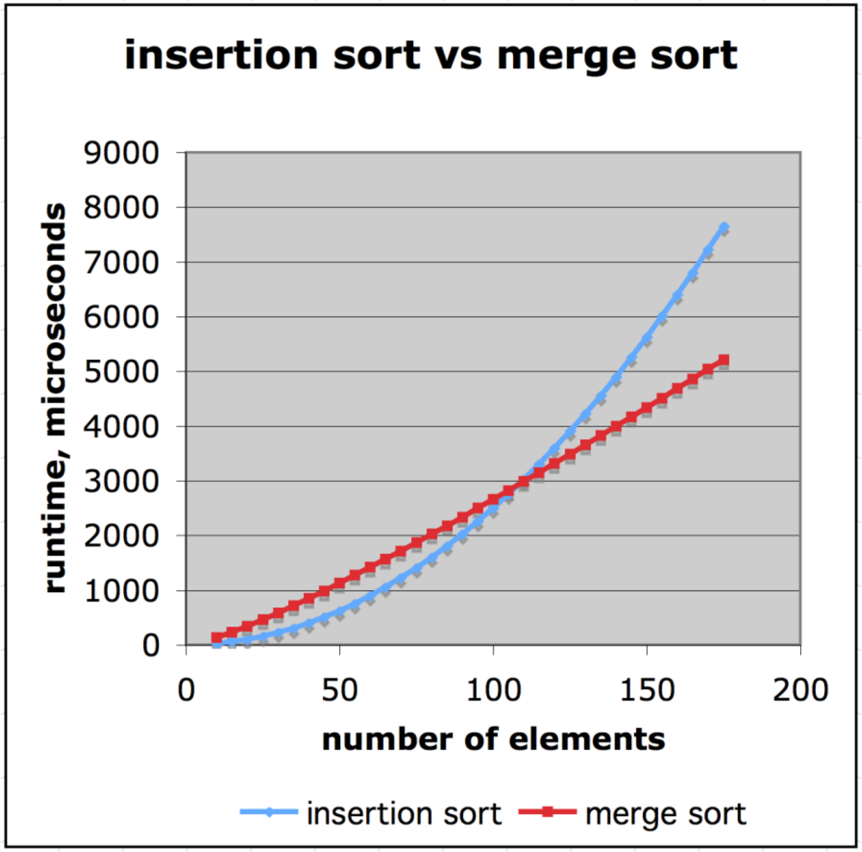
# Comparison of Two Algorithms




insertion sort is

$n^2 / 4$

merge sort is

$2 \, n \lg n$

sort a million items?

insertion sort takes roughly 70 hours

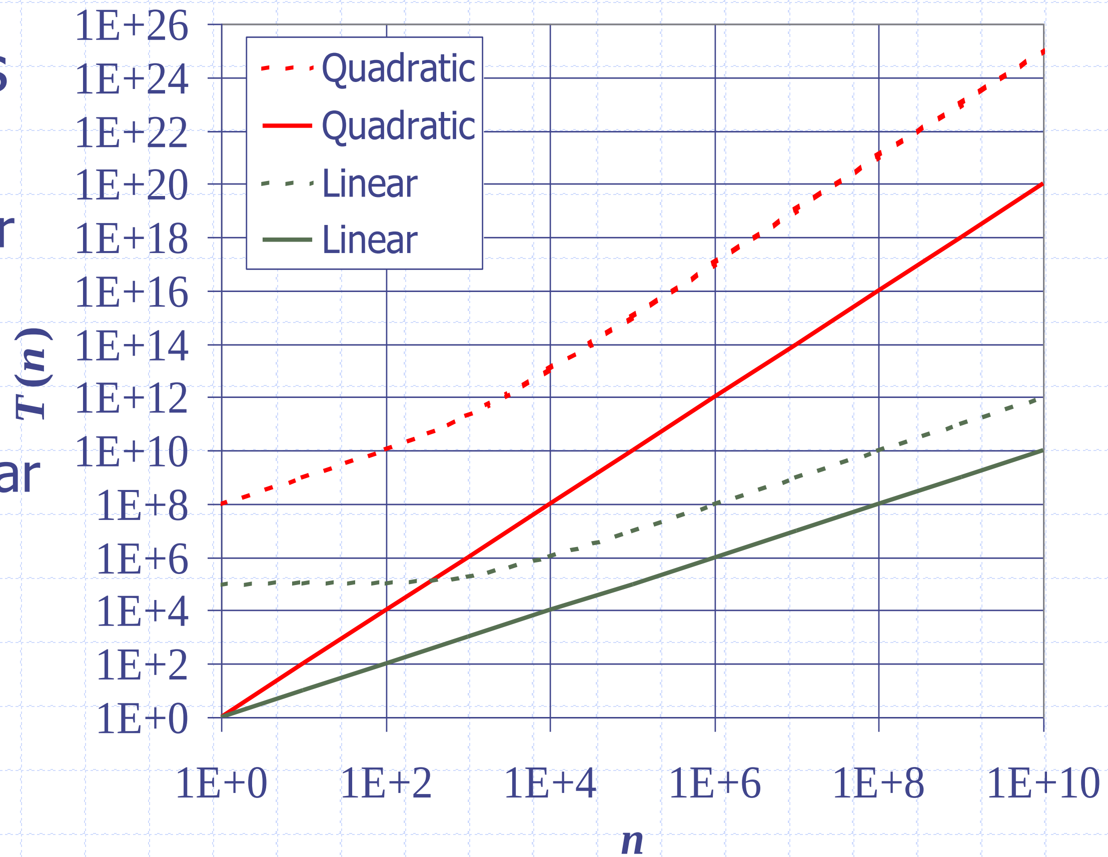
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

# Constant Factors

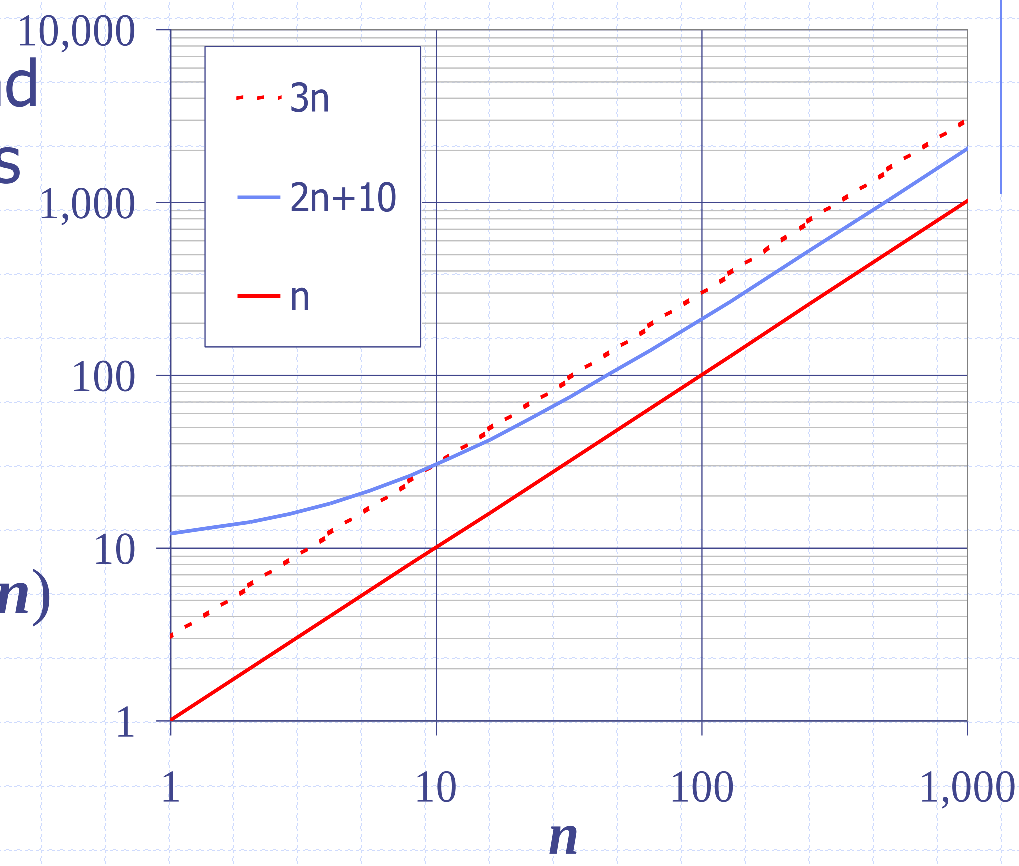
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



# Big-Oh Notation

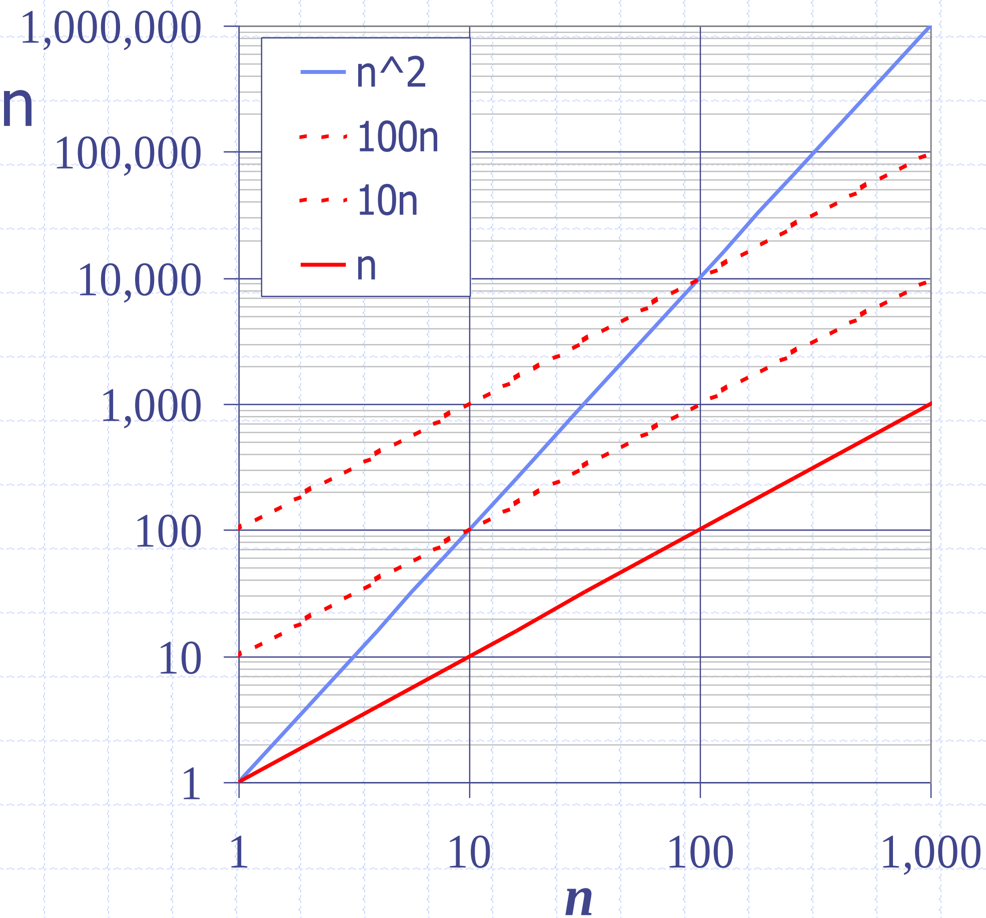
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants $c$ and $n_0$ such that

  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

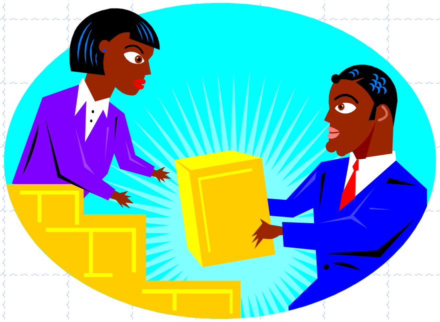


# Big-Oh Example

- Example: the function $n^2$ is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since $c$ must be a constant



# More Big-Oh Examples

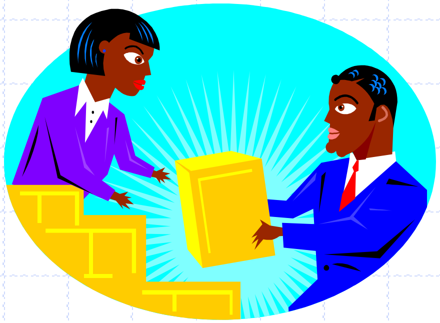


- $7n-2$

- $3n^3 + 20n^2 + 5$

- $3 \log n + 5$

# More Big-Oh Examples



- $7n-2$

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- $3 \log n + 5$

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \bullet \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

# Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

| | $f(n)$ is $O(g(n))$ | $g(n)$ is $O(f(n))$ |
|---|---|---|
| $g(n)$ grows more | Yes | No |
| $f(n)$ grows more | No | Yes |
| Same growth | Yes | Yes |

# Big-Oh Rules

- If is $f(n)$ a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

# Asymptotic Algorithm Analysis

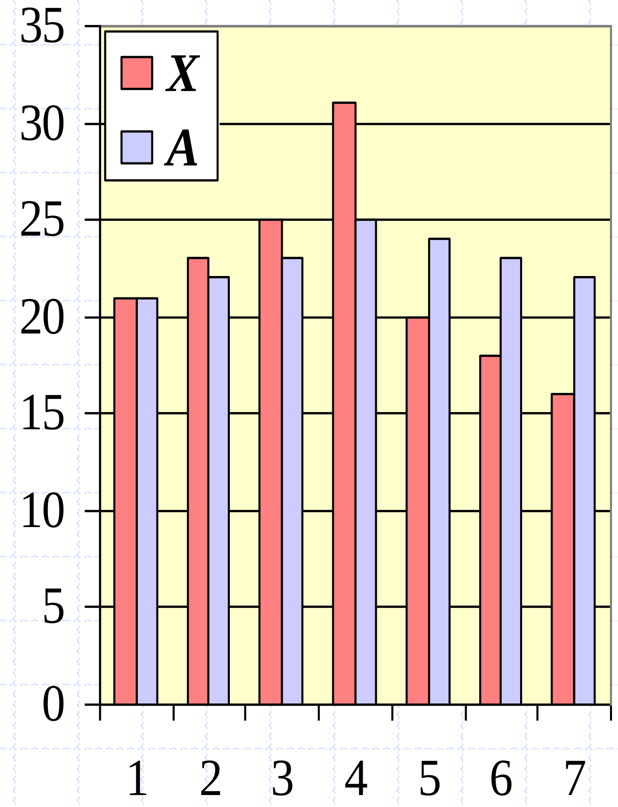
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $8n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

# Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $i$-th prefix average of an array $X$ is average of the first $(i+1)$ elements of $X$:

$$A[i] = (X[0] + X[1] + \dots + X[i])/(i+1)$$

- Computing the array $A$ of prefix averages of another array $X$ has applications to financial analysis



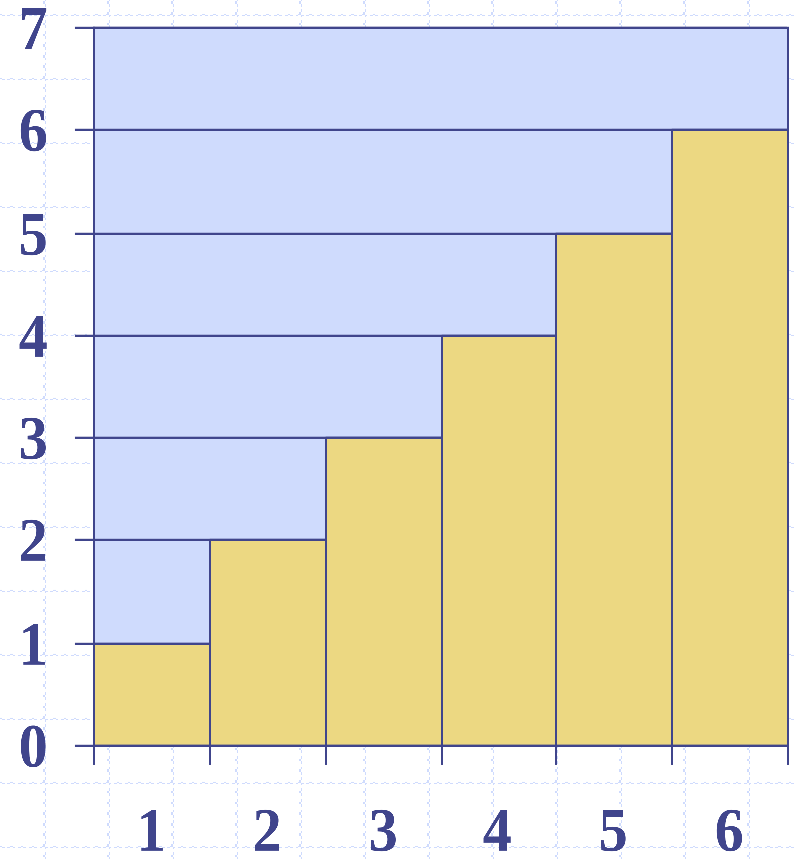
# Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

| | |
|---|---|
| **Algorithm** *prefixAverages1*$(X, n)$ | |
| **Input** array $X$ of $n$ integers | |
| **Output** array $A$ of prefix averages of $X$ | #operations |
| $A \leftarrow$ new array of $n$ integers | $n$ |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $s \leftarrow X[0]$ | $n$ |
| **for** $j \leftarrow 1$ **to** $i$ **do** | $1 + 2 + \ldots + (n - 1)$ |
| $s \leftarrow s + X[j]$ | $1 + 2 + \ldots + (n - 1)$ |
| $A[i] \leftarrow s \,/\, (i + 1)$ | $n$ |
| **return** $A$ | $1$ |

# Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first $n$ integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



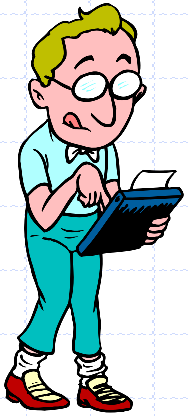
# Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a **running sum**

| | |
|---|---|
| **Algorithm** *prefixAverages2*($X$, $n$) | |
| **Input** array $X$ of $n$ integers | |
| **Output** array $A$ of prefix averages of $X$ | #operations |
| $A \leftarrow$ new array of $n$ integers | $n$ |
| $s \leftarrow 0$ | $1$ |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $\quad s \leftarrow s + X[i]$ | $n$ |
| $\quad A[i] \leftarrow s / (i + 1)$ | $n$ |
| **return** $A$ | $1$ |

- Algorithm *prefixAverages2* runs in $O(n)$ time

# Math you need to Review

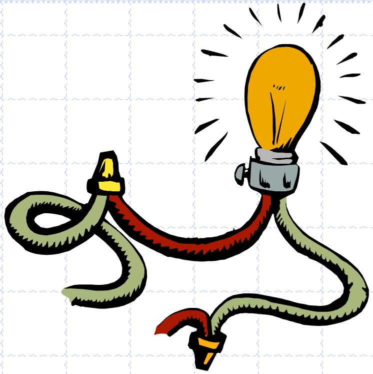


- **Summations**
- **Logarithms and Exponents**
  - **properties of logarithms:**
    - $\log_b(xy) = \log_b x + \log_b y$
    - $\log_b (x/y) = \log_b x - \log_b y$
    - $\log_b xa = a\log_b x$
    - $\log_b a = \log_x a/\log_x b$
  - **properties of exponentials:**
    - $a^{(b+c)} = a^b a^c$
    - $a^{bc} = (a^b)^c$
    - $a^b / a^c = a^{(b-c)}$
    - $b = a^{\log_a b}$
    - $b^c = a^{c*\log_a b}$
- **Proof techniques**
- **Basic probability**

# Relatives of Big-Oh

- **big-Omega**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - $f(n)$ is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$
  - $f(n)$ is $O(g(n))$ and $\Omega(g(n))$
  - $g(n)$ is $O(f(n))$ and $\Omega(f(n))$
  - $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$

# Intuition for Asymptotic Notation



**Big-Oh**

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

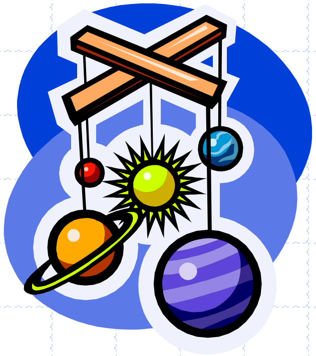
**big-Omega**

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

**big-Theta**

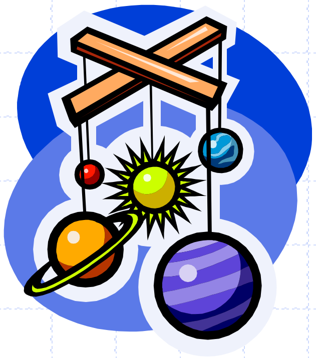
- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

# Example Uses of the Relatives of Big-Oh



- $5n^2$ is $\Omega(n^2)$
- $5n^2$ is $\Omega(n)$
- $5n^2$ is $\Theta(n^2)$

# Example Uses of the Relatives of Big-Oh



- **$5n^2$ is $\Omega(n^2)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Theta(n^2)$**

  $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \leq c \bullet g(n)$ for $n \geq n_0$

  Let $c = 5$ and $n_0 = 1$