COMP 341 Intro to AI
Constraint Satisfaction Problems

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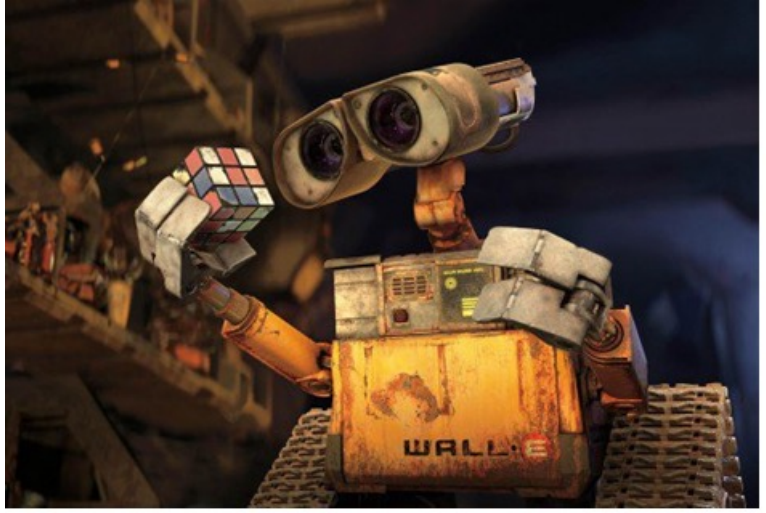
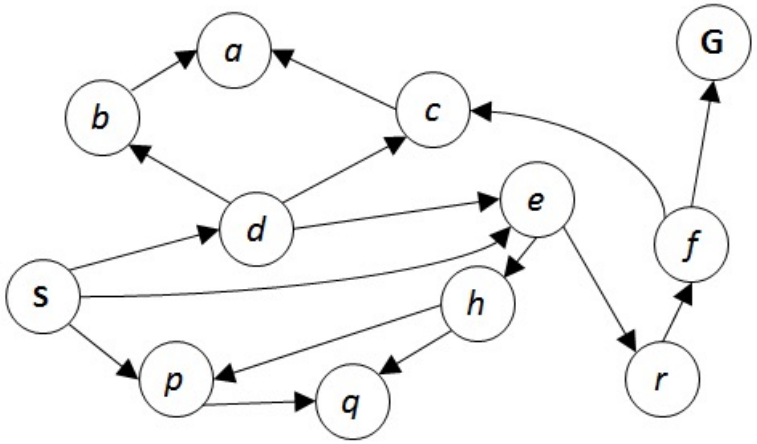
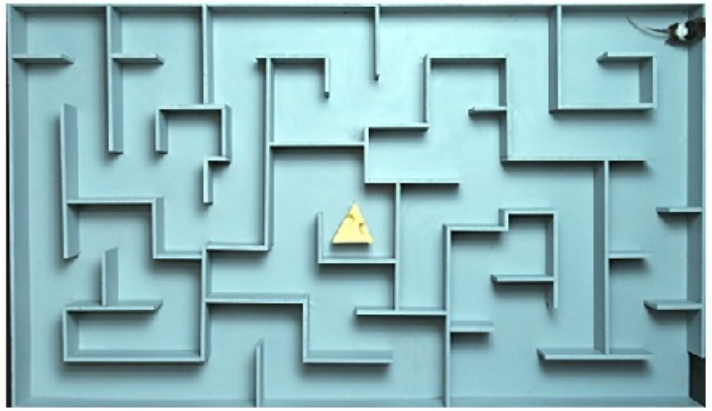
Today

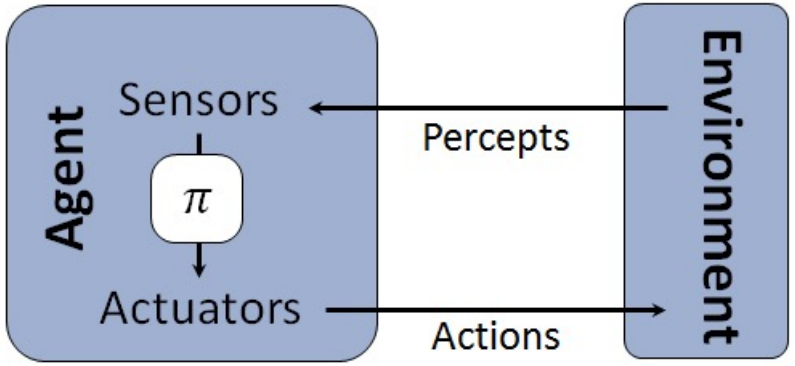
- Recap

- Constraint Satisfaction Problems
  - Definitions, Examples
  - Backtracking Search
  - CSP Search Heuristics

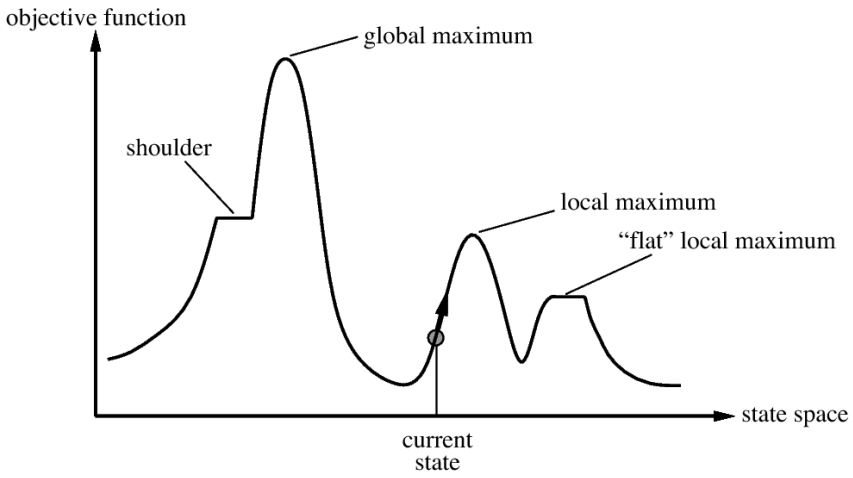


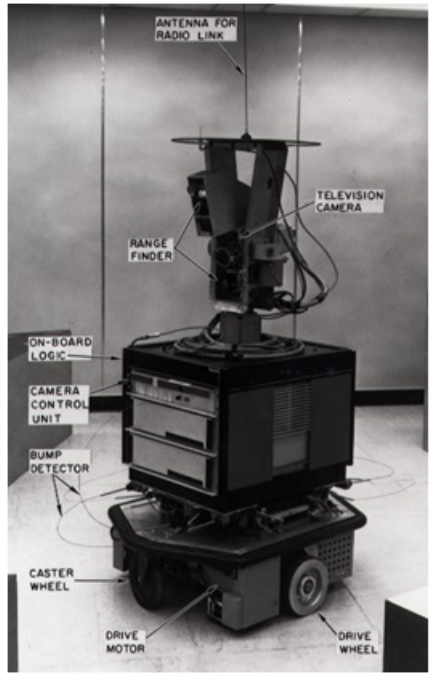






Previously on Intro to AI





Search

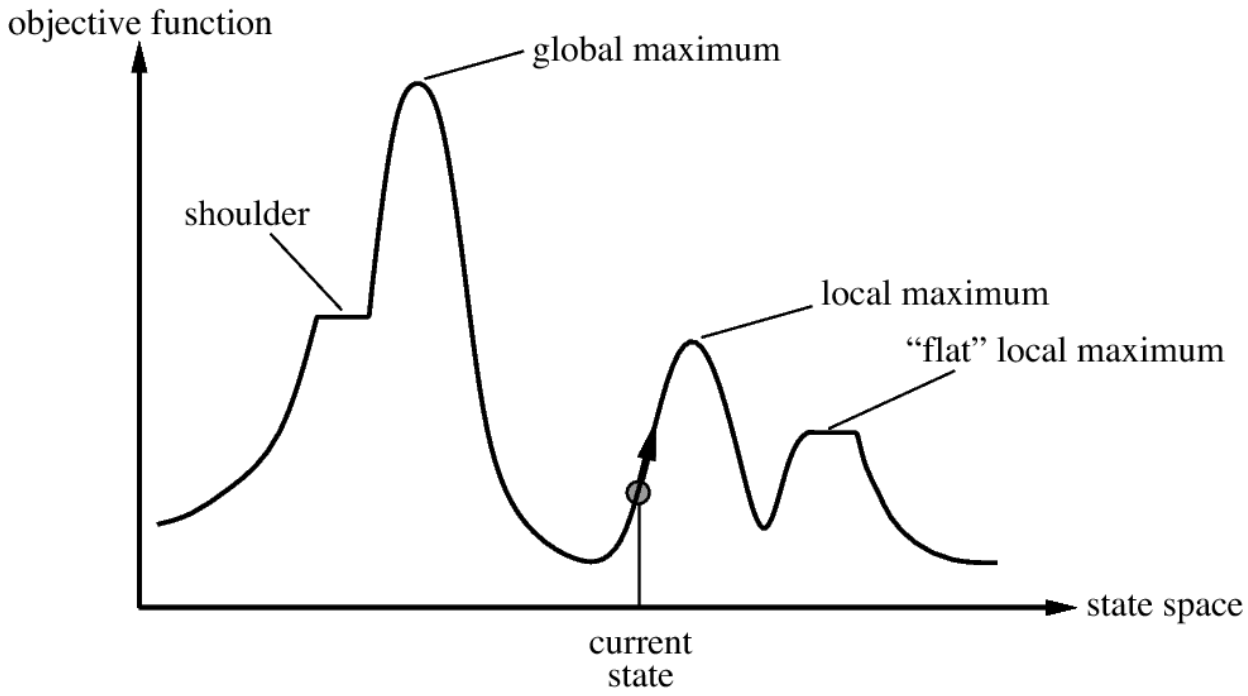
- Uninformed
  - DFS, BFS, UCS
  - No domain knowledge
- Informed
  - Greedy, A*
  - Heuristics based on domain/problem knowledge
- Solution is a path to goal

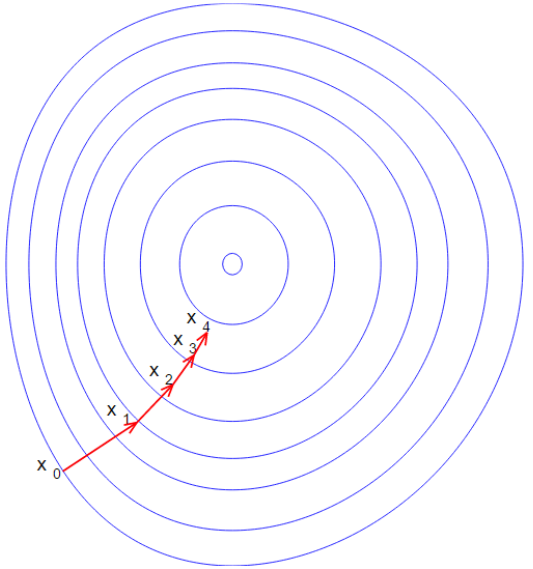
Local Search

- Solution is important, not the path!

- Hill Climbing
- Simulated Annealing
- Local Beam Search
- Genetic Algorithms

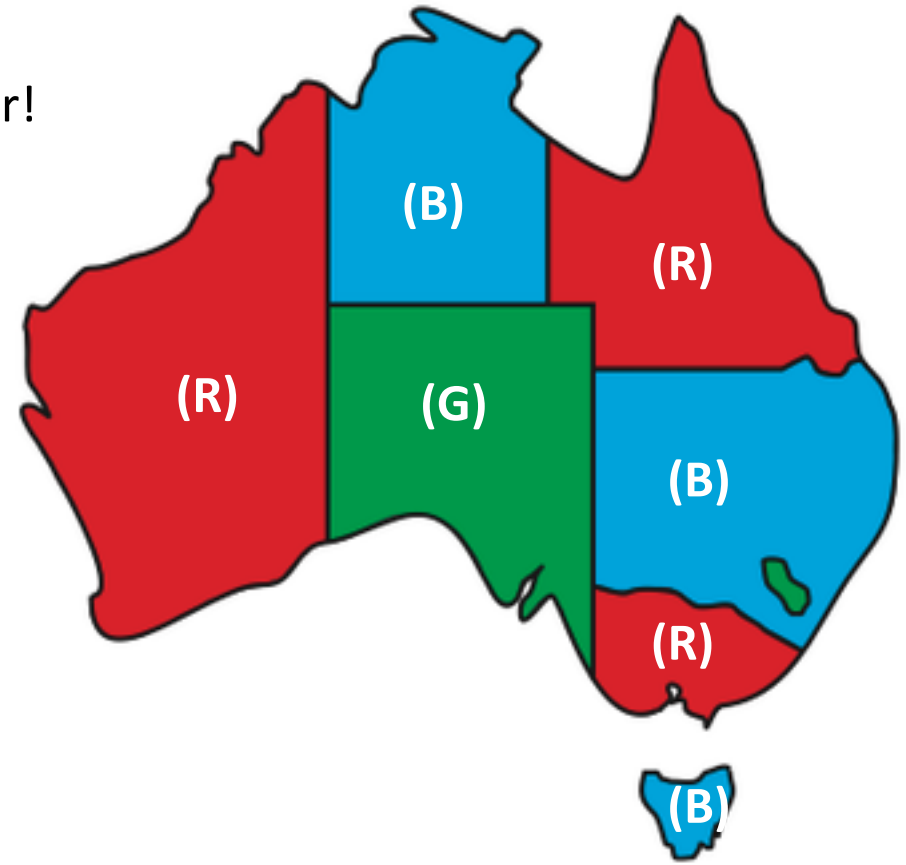
- Gradient Descent





Constraint Satisfaction Problems

No neighbors with the same color!



Search So Far

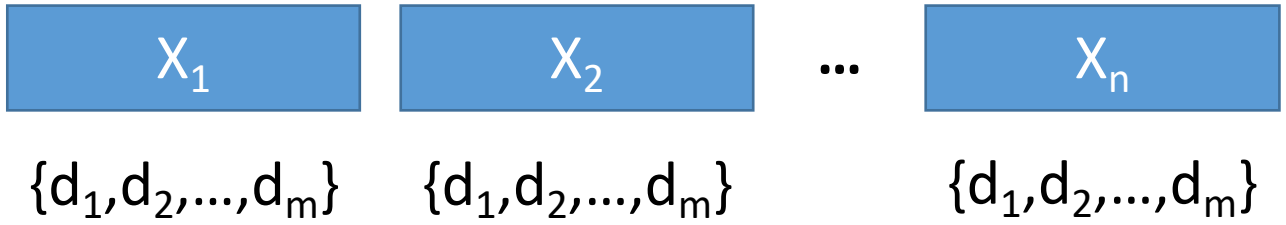
- Classical Search:
  - Solution is path to a goal state
- Local Search:
  - Solution is the goal state itself
- CSPs?
  - Goal matters
  - States and goal test have specific structure!
  - Allows for general heuristics

Constraint Satisfaction Problems

- Standard Search
  - State is a black box data structure
  - Goal test: Can be any Boolean function of states
  - Successor Function: Can be anything that returns valid states
  - Heuristic function: Can be anything that maps states to a non-negative scalar
- CSPs
  - State is defined by variables X_i with values from domain D_i.
    - Map Coloring Example: Variables are the color of each Australian state and the domain is the set of allowable colors
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
    - Map Coloring Example: All states are colored and neighboring states do not have the same color
- This structure allows useful general-purpose algorithms with more power than standard search algorithms

CSPs

- State is defined by variables X_i with values from domain D_i



Domains of variables can be different!

- Goal test is a set of constraints specifying allowable combinations of values for subsets of variables. E.g.

$$\sum_{i \in A} X_i == k \qquad X_i \neq X_j \text{ for } i \neq j, i \in A, j \in A$$

Real Life Example (!) - Carpool

- Ahmet, Elif, Mehmet, Zeynep want to carpool to Bolu
  - Variables are A,E,M,Z
- There are only 2 cars
  - Domains = $\{C_1, C_2\}$
- The cars belong to Ahmet and Zeynep
  - Consraints: $A = C_1$ and $Z = C_2$
- Ahmet and Elif do not like each other
  - Constraint: $A \neq E$
- Mehmet has a crush on Zeynep
  - Constraint: $M = Z$
- A solution
  - $A=C_1$, $E=C_2$, $M= C_2$, $Z=C_2$

Side Note: They should just sit Elif in the front and Ahmet in the back and take 1 car

But the problem does not model that!

Small Warning

- The domains and variables were flipped during the lecture
- You can flip the domains and variables if you allow for “set assignments”.
- However;
  - It is less intuitive
  - Unassigned variables, and variables with an empty set assignment (e.g. an car that has not been assigned yet vs an empty car) must be distinct
  - This would cause issues with the subsequent algorithms

Solving CSPs

- Each state of the problem is a possible assignment to some or all the variables

- **Legal Assignment**: no violations
- **Complete Assignment**: every variable assigned

Map Coloring

- Color the map such that no two neighbors have the same color

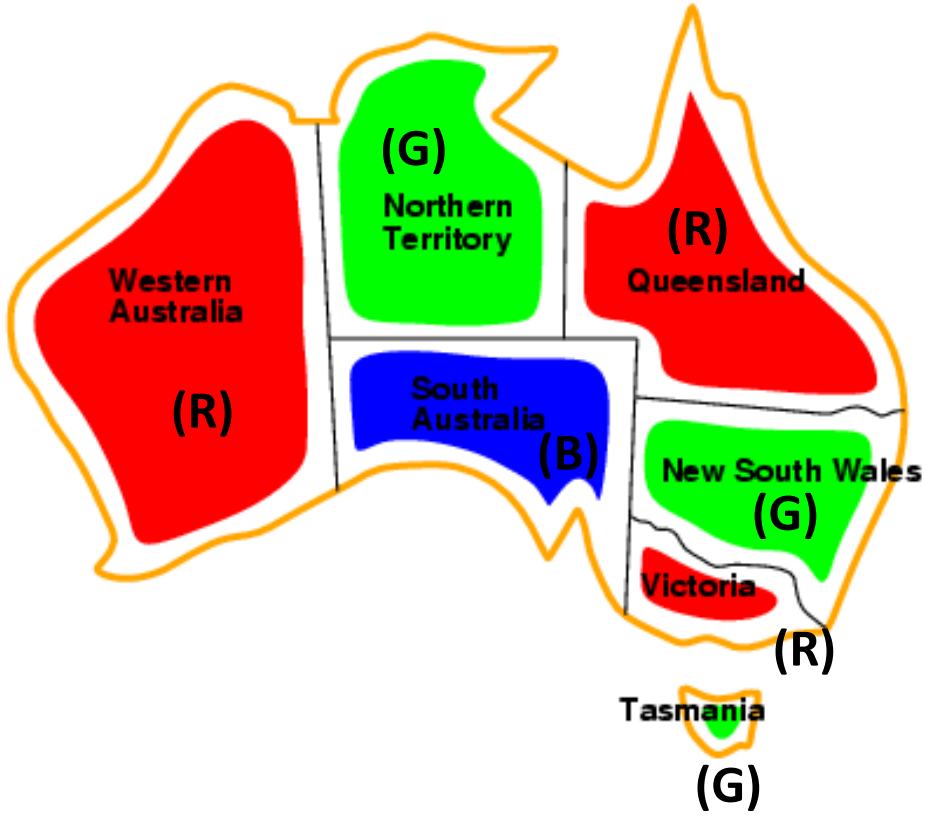
- Variables:
  - *WA, NT, Q, NSW, V, SA, T*
- Domains:
  - D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
  - Implicit: *WA* ≠ *NT*
  - Explicit: (*WA, NT*) ∈ {(red, green), (red, blue), (green, red), (green, blue), . . .}

Map Coloring

- Color the map such that no two neighbors have the same color

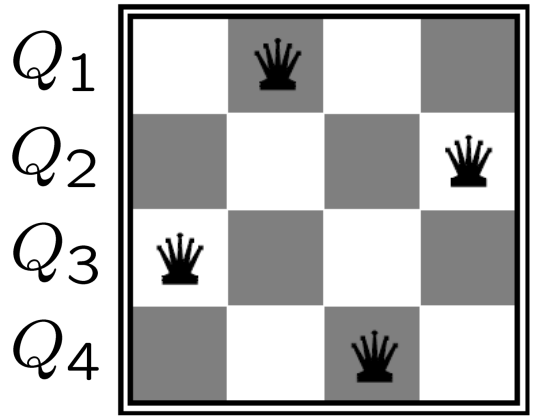
- Solutions are assignments satisfying all constraints, e.g.,

{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green}



Example: N-Queens

- Variables: Q_k
- Domains: {1,2,…,N}
- Constraints:



Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

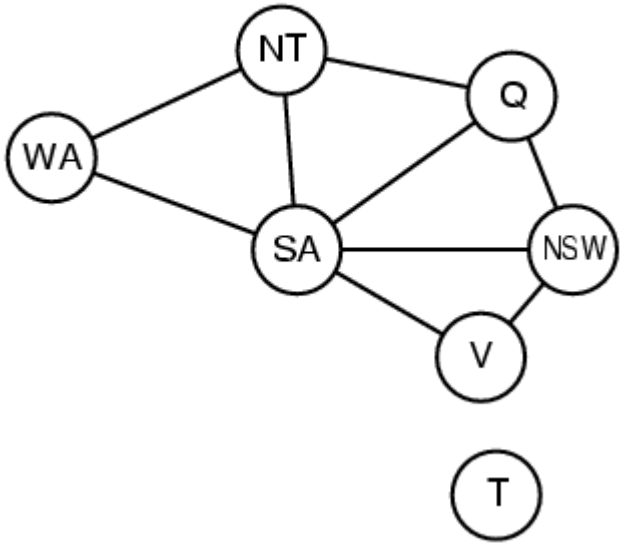
Explicit: $(Q_1, Q_2) \in \{(1,3),(1,4),\ldots\}$

$\ldots$

Constraint Graph

- Binary CSP: each constraint relates (at most) two variables

- **Binary Constraint Graph** is a data structure we use to represent the problem
  - Nodes are variables
  - Arcs show which variables are constrained



Example: Cryptarithmetic

- Variables:

$$F \; T \; U \; W \; R \; O \; X_1 \; X_2 \; X_3$$

- Domains:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

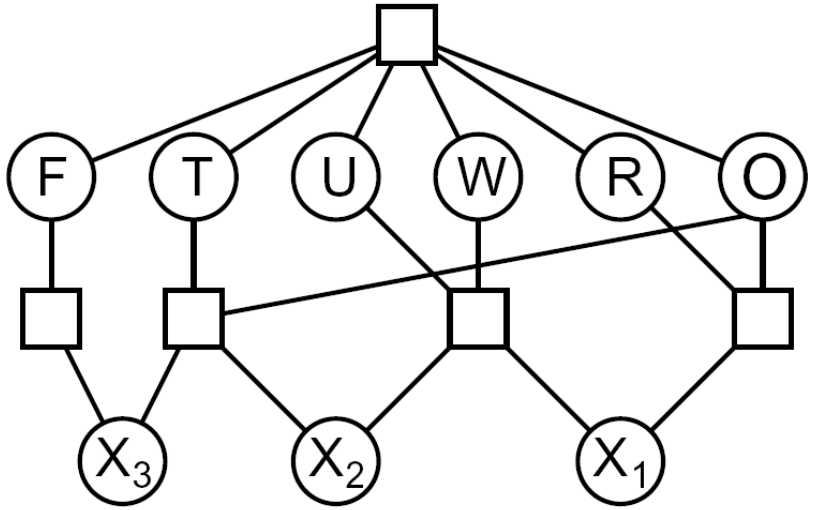
- Constraints:

$$\text{alldiff}(F, T, U, W, R, O)$$

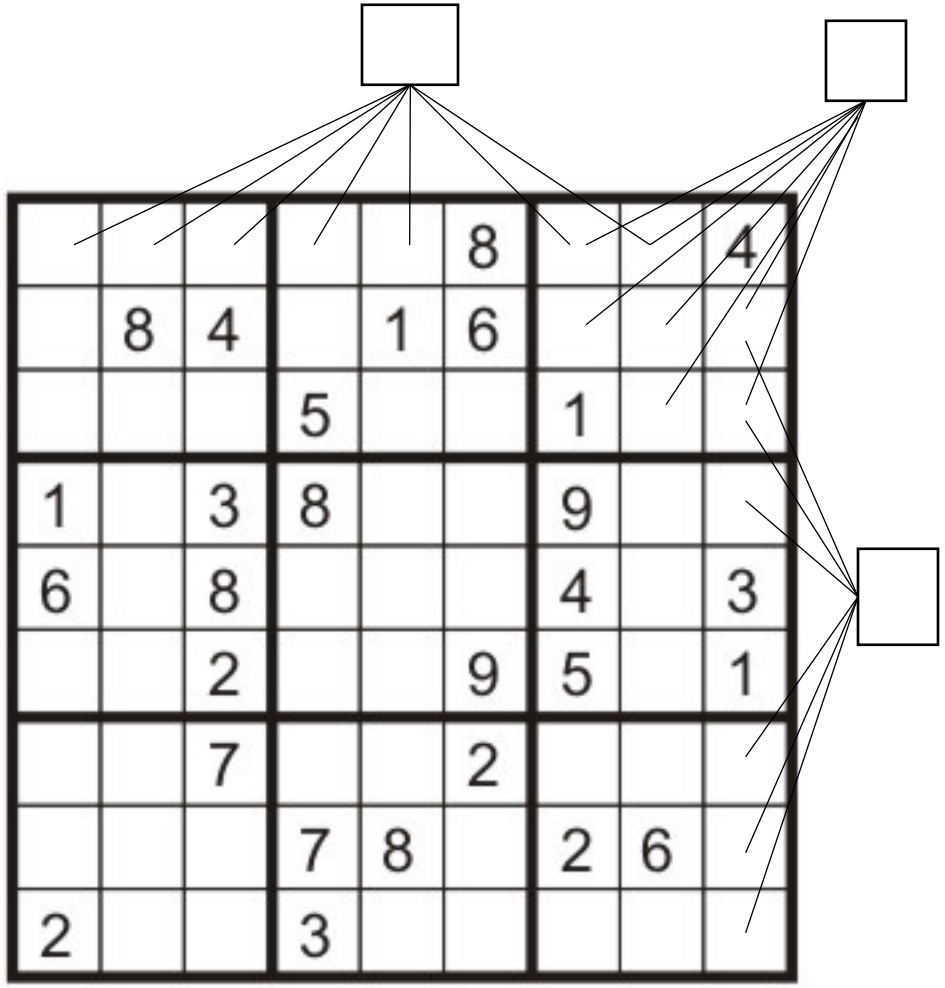
$$O + O = R + 10 \cdot X_1$$

$$\cdots$$

```
   T W O
+  T W O
-------
F O U R
```



Most Famous CSP - Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

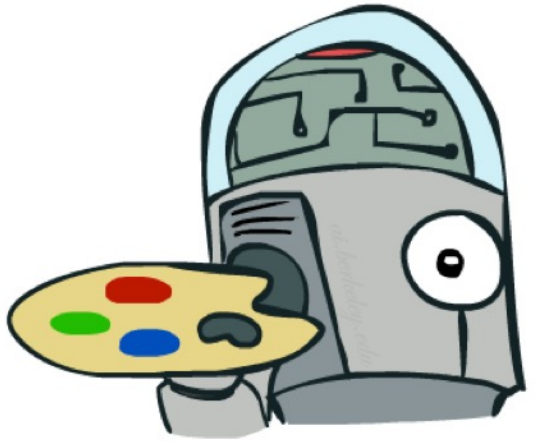
Other Real-World Examples

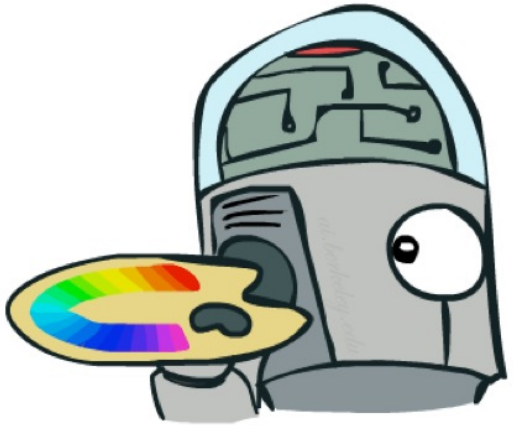
- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floor Planning
- …

- Many real-world problems involve real-valued variables…

Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Need constraint language: *Job1 + 5 < Job2*
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by Linear Programming methods (ever heard of the Simplex Method?)



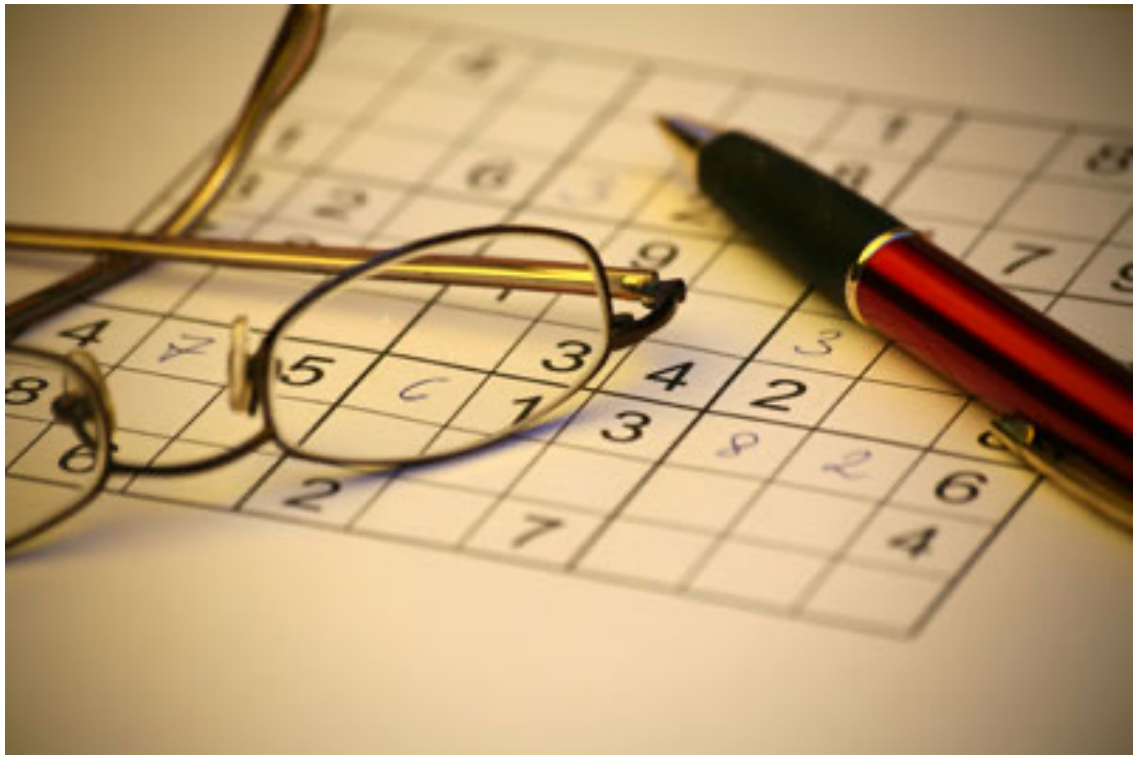


Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains)
    e.g.: *SA ≠ green*
  - Binary constraints involve pairs of variables,
    e.g.: *SA ≠ WA*
  - Higher-order constraints involve 3 or more variables:
    e.g.: cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives rise to constrained optimization problems
  - We might come back to this later on in the course

Solving CSPs



Search Formulation

Local Search Formulation

Search Formulation for CSPs

- Initial State: {}
- Successor(): assign a value (consistent with constraints) to an unassigned variable
- Goal Test(): All variables are assigned and all constraints are satisfied
- Failure: No legal assignment to do

- This is the **same** for all CSPs!
- Path is irrelevant
- Every solution appears at depth ***n*** with ***n*** variables
  - DFS anyone
- Complexity (*n* vars, *d* values)
  - Branch factor: *(n-l)d* at depth *l*
  - $n!d^n$ leaves!

Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are **commutative**, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which **do not conflict** previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for $n \approx 25$

Detour: Recursive DFS

```
function RECURSIVE-DFS(problem) returns a solution, or failure
  return RECURSIVE-DFS_(MAKE-NODE(problem.INITIAL-STATE), problem)

function RECURSIVE-DFS_(node,problem) returns a solution, or failure
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  for each action in problem.ACTIONS(node.STATE) do
    child ← CHILD-NODE(problem, node, action)
    result ← RECURSIVE-DFS_(child, problem)
    if result != failure then return result
  return failure
```

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
    return BACKTRACK({ }, csp)

function BACKTRACK(assignment , csp) returns a solution, or failure
    if assignment is complete then return assignment
    var ←SELECT-UNASSIGNED-VARIABLE(csp)
    for each value in ORDER-DOMAIN-VALUES(var , assignment , csp) do
        if value is consistent with assignment then
            add {var = value} to assignment
            inferences ←INFERENCE(csp, var , value)
            if inferences ≠ failure then
                add inferences to assignment
                result ←BACKTRACK(assignment , csp)
                if result ≠ failure then
                    return result
        remove {var = value} and inferences from assignment
    return failure
```

select a variable

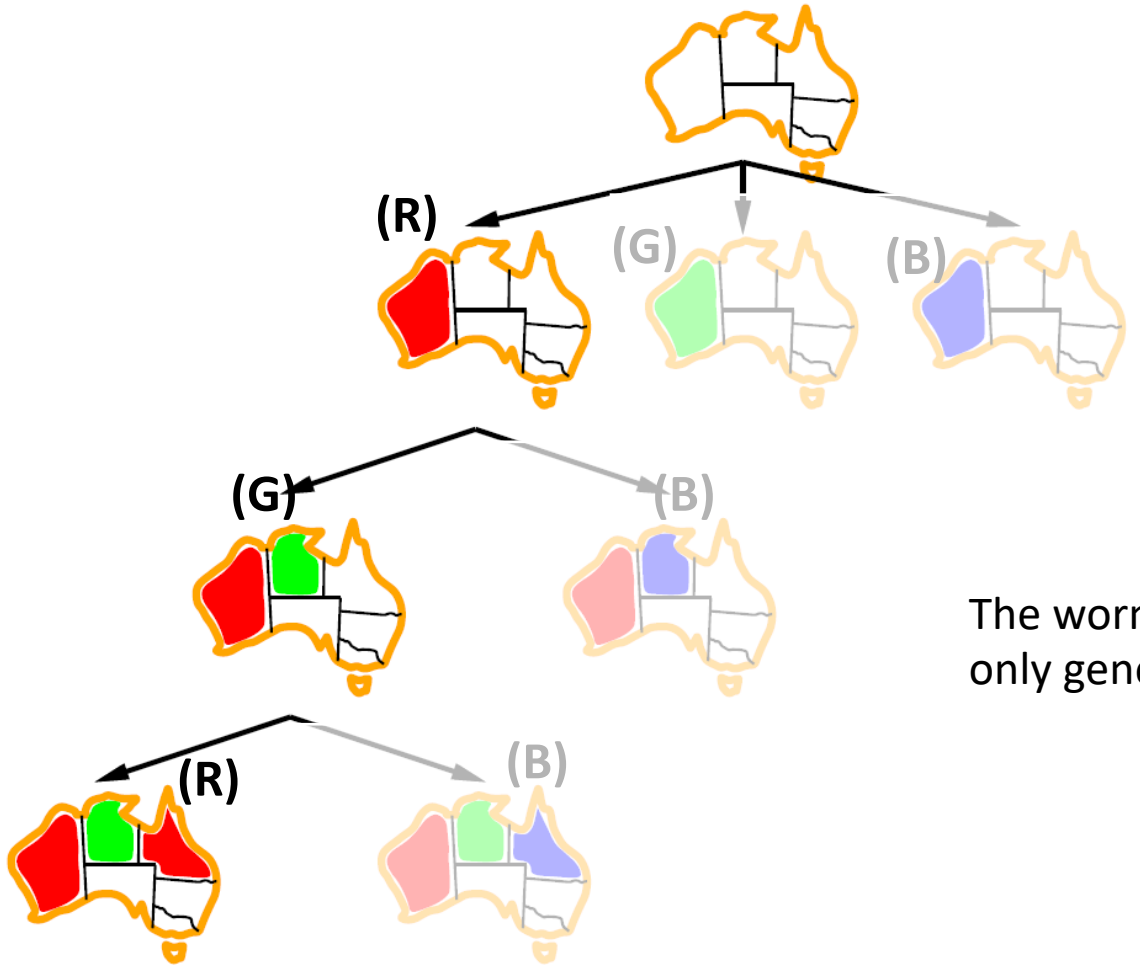
Find a value consistent with constraints

Recurse to assign another

only keeps a single representation of the assigned state!

If no consistent assignment exists, return failure, which causes another value to be tried

Backtracking Example



The worn-out states are only generated if needed

CSPs Recap

- Variables (e.g. Australian States)
- Domains of Variables (e.g. map colors)
- Assign values to variables from the corresponding domains (WA = red)
- Constraints (WA ≠ SA)

- Solution Method: Backtracking Search
  - DFS
  - One variable at a time
  - Check constraints along the way

Improvements

- Backtracking: DFS + variable ordering + constraint checking
- Uninformed: Add heuristics to improve

- **General Purpose Heuristics** thanks to the structure of CSPs
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: What inference can be made to detect failures early ?
- Structure: Can we exploit the problem structure?

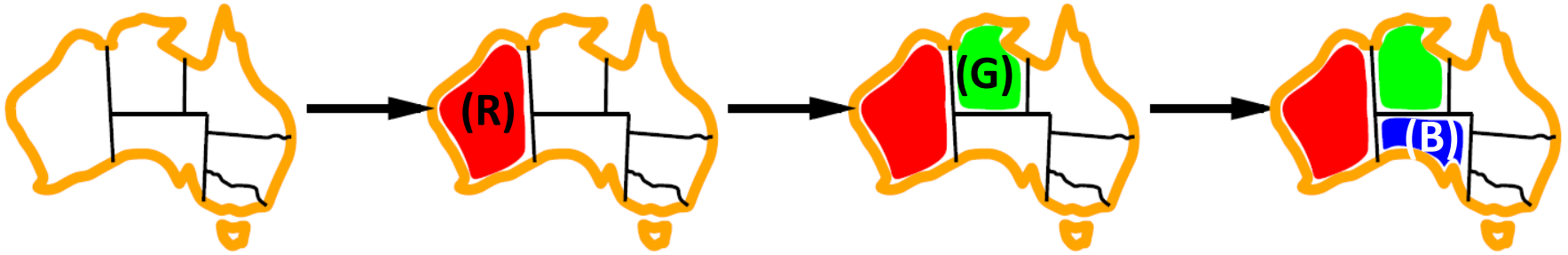
Ordering: What variable to assign next?

- Fixed order
- Random

- Other ideas?
  - Let’s look at the number of constraints per variable!

Minimum Remaining Values (MRV)

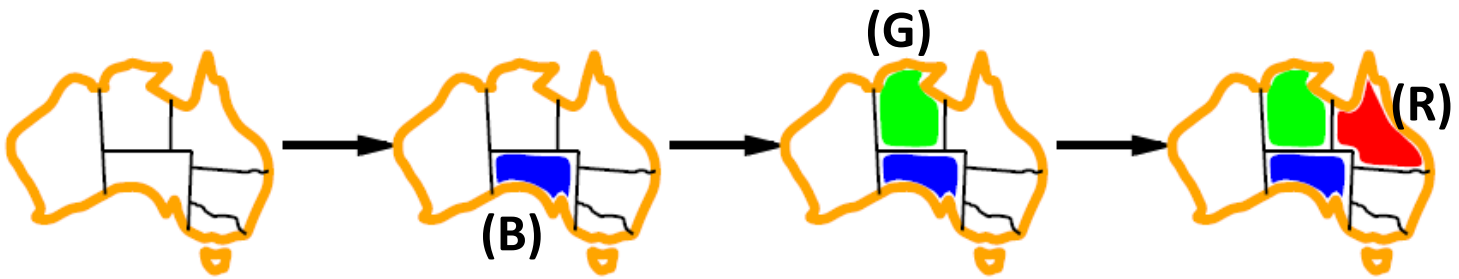
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

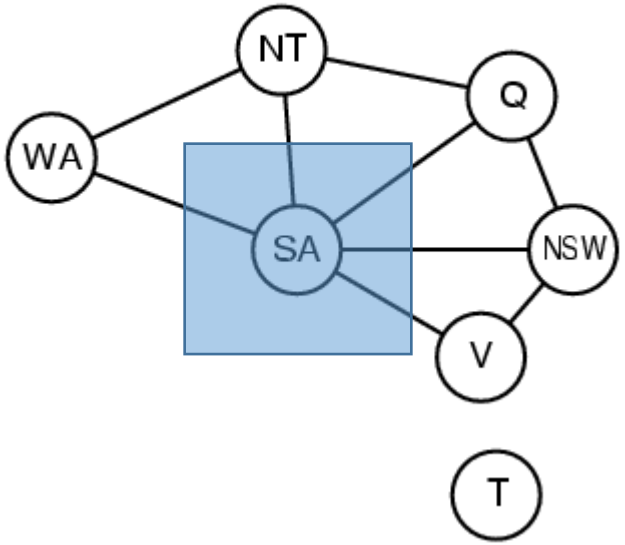


- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering + not running out of options

Degree Heuristic

- Tie breaker among MRV variables
- Choose the variable with most constraints on remaining variables



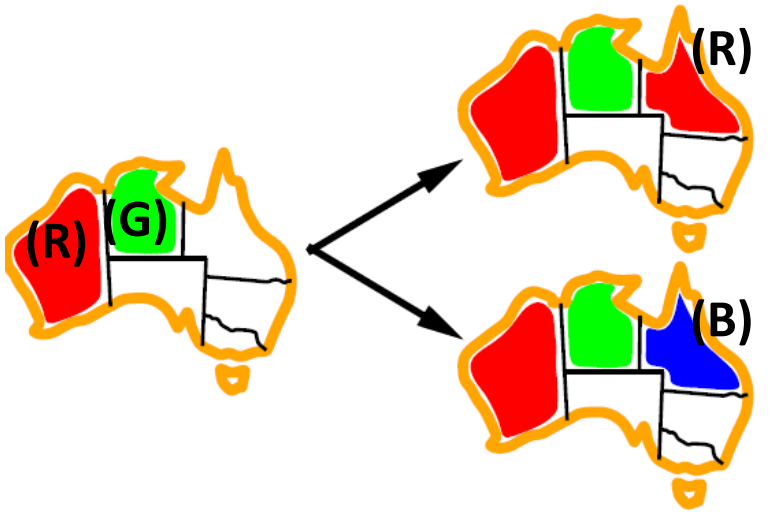


Least Constraining Value

- Which value to assign next?
- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the **least constraining value**
  - I.e., the one that **rules out the fewest values** in the remaining variables
  - Note that it may take some computation
- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible

1 Value for SA



0 Values for SA!

Summary of Ordering

- Which variable to select next and which value to assign to it?
- Detect failures early (MRV + DH)
- Enter the most promising branch (LCV)
- Note that the heuristics do not change the theoretical bounds!

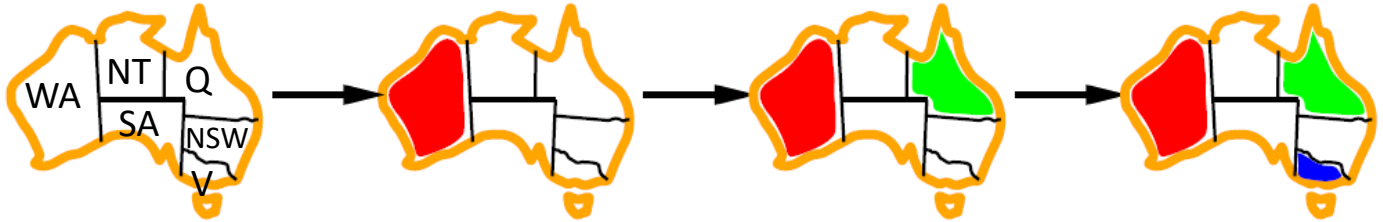
Filtering

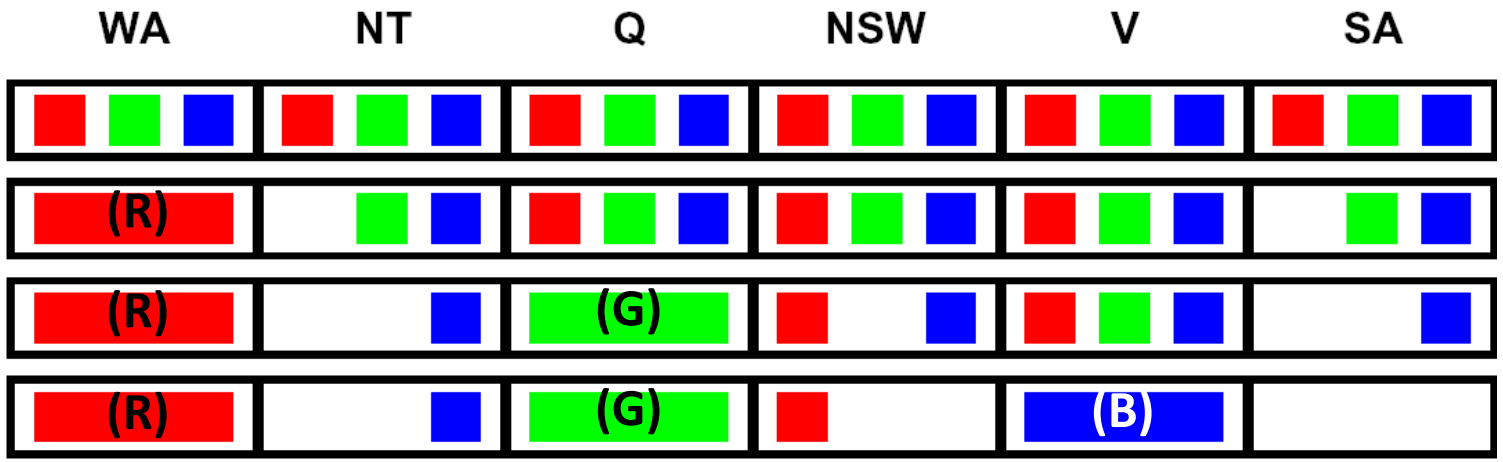
- How to detect failures early?
- Filtering: Keep track of domains for unassigned variables and cross off bad options

- Forward Checking
- Constraint Propagation - Arc consistency

Forward Checking

- Forward checking: Cross off values that violate a constraint when added to the existing assignment
- Backtrack when no assignments left



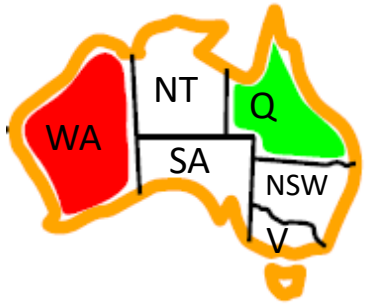


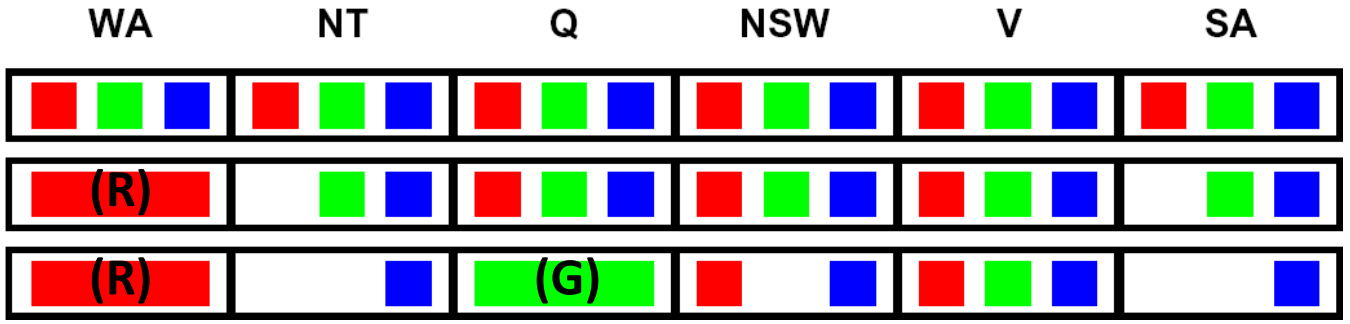
Legend:
Left: Red
Middle: Green
Right: Blue

MRV + Forward Checking: FC can be used to compute what MRV needs!

Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures
- After deleting neighbors, check constraints for all other variables



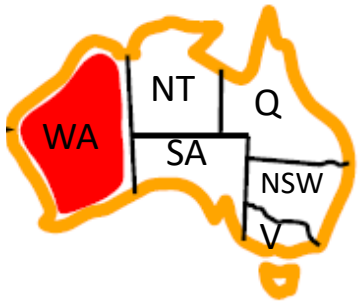


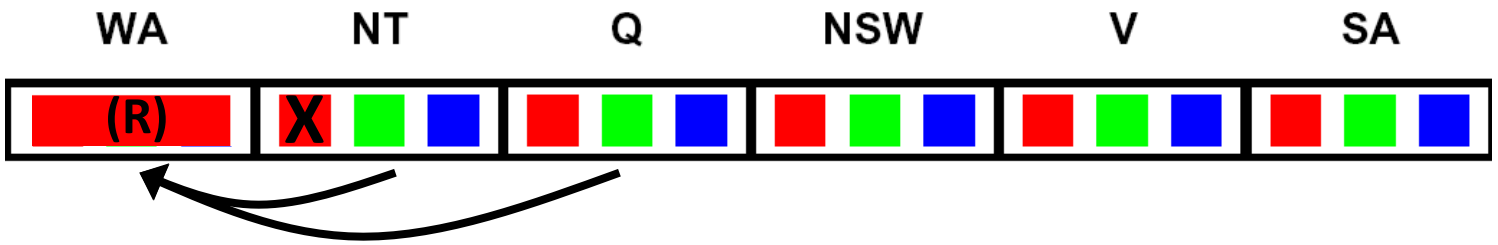
- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Legend:
Left: Red
Middle: Green
Right: Blue

Consistency of A Single Arc

- An arc X $\rightarrow$ Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint



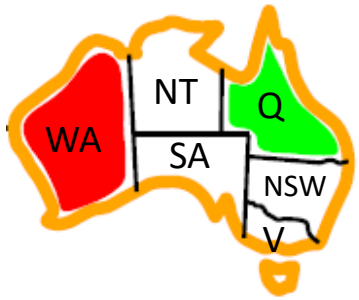


Legend:
Left: Red
Middle: Green
Right: Blue

- Delete from tail!
- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:



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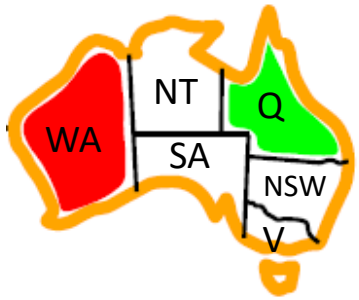

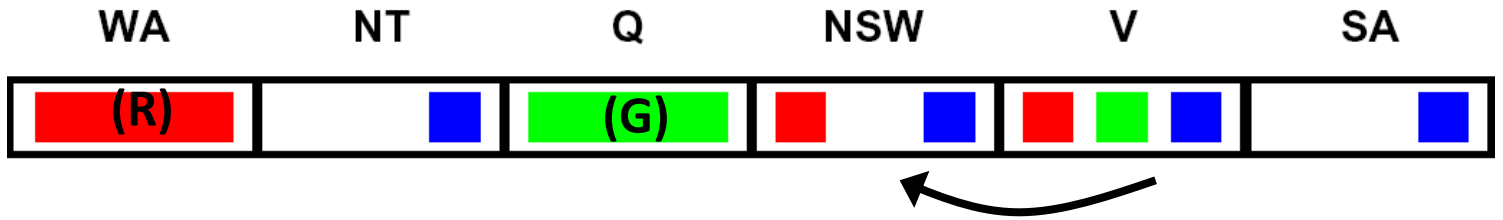

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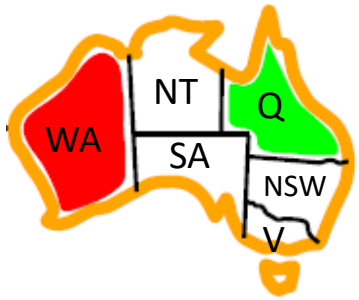

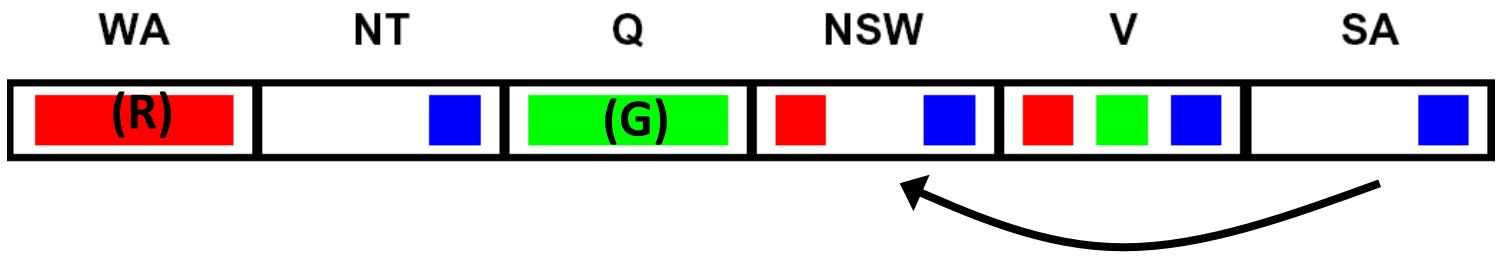

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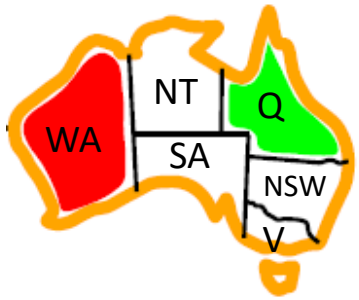

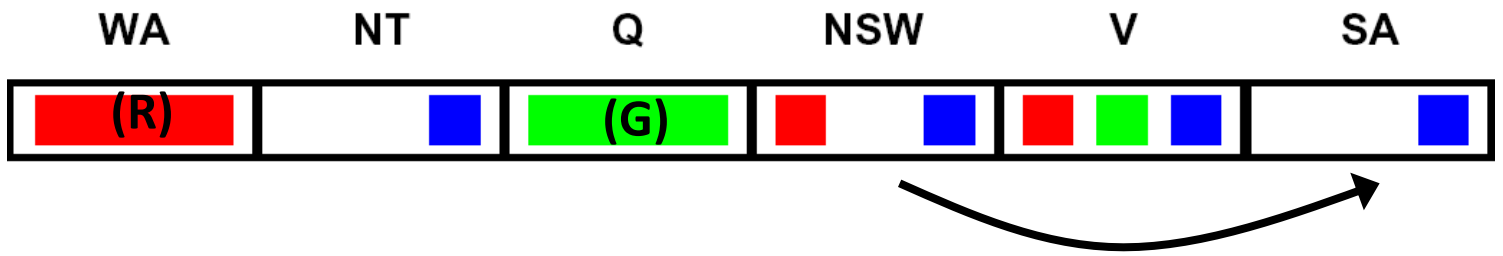

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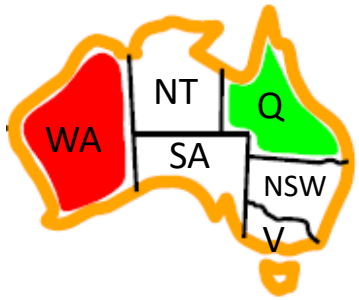

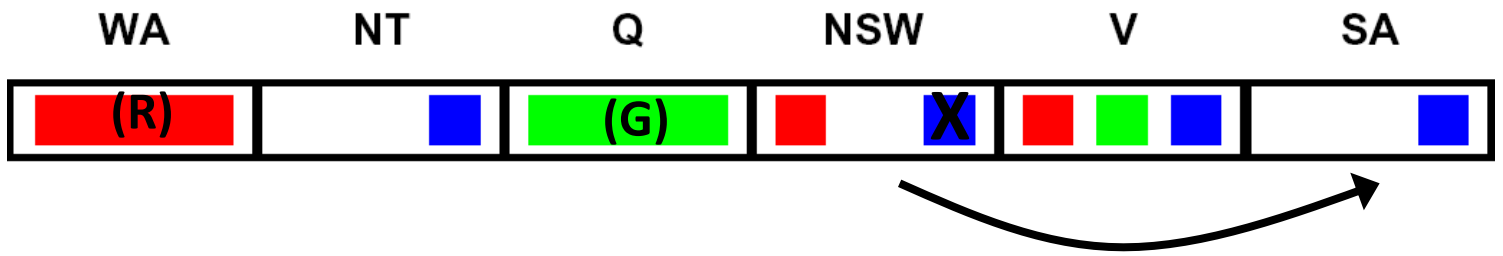

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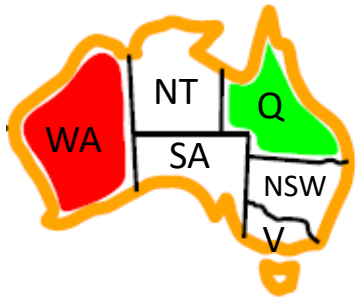

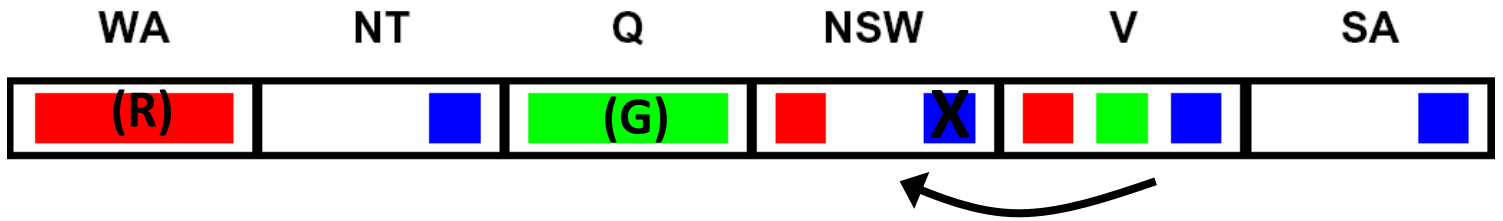

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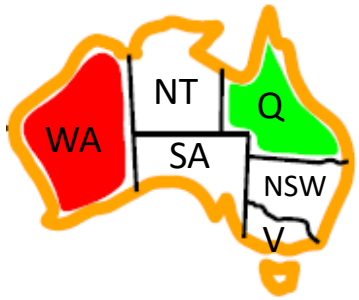

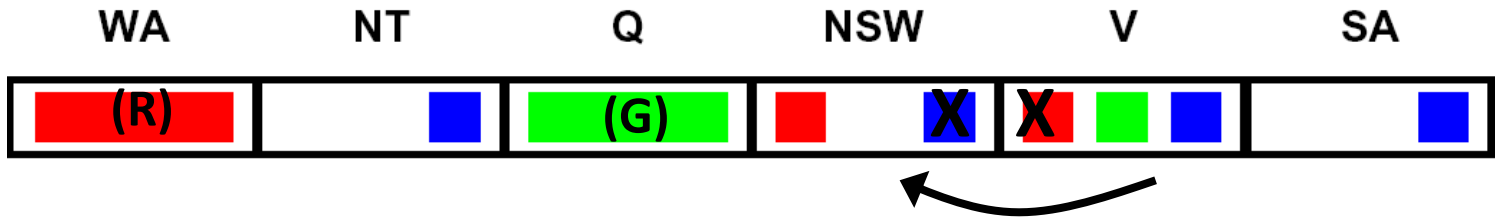

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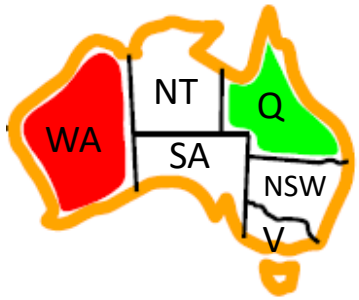

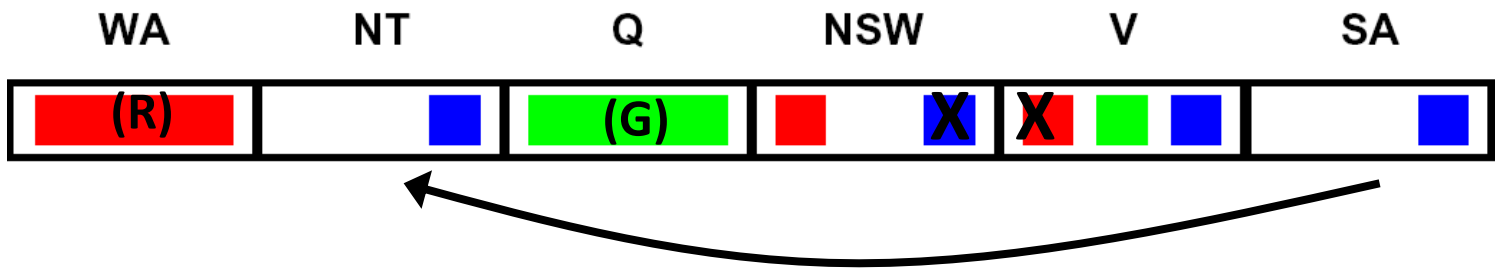

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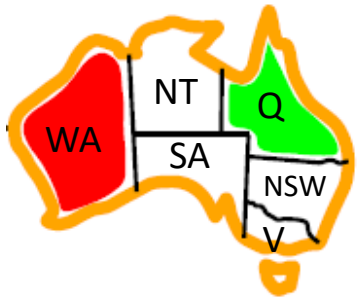

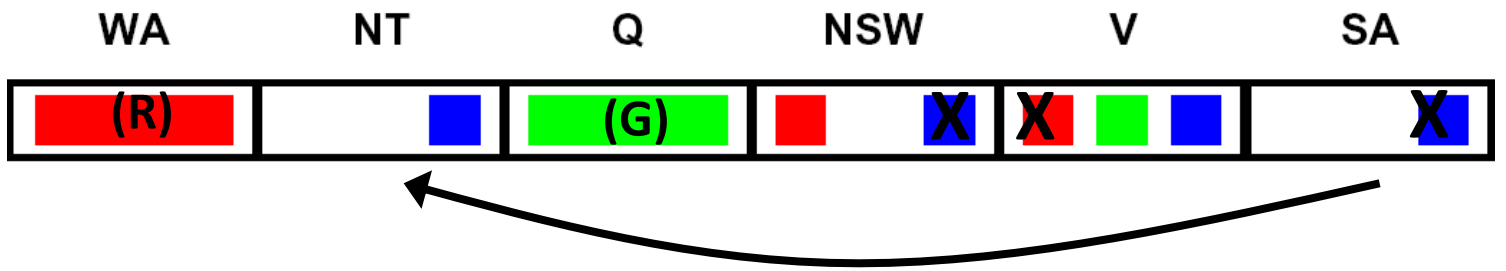

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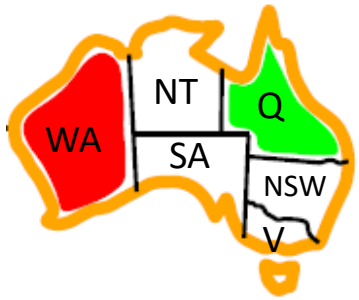
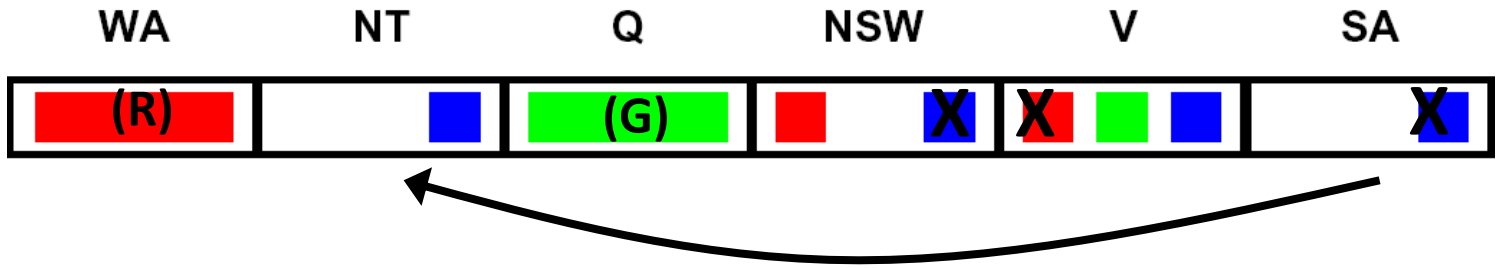




- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as before or after each assignment
- What's the downside of enforcing arc consistency?

Legend:
Left: Red
Middle: Green
Right: Blue

Remember: Delete from the tail!

Enforcing Arc Consistency in a CSP

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components *(X, D, C)*
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*
  while *queue* is not empty **do**
    (X_i, X_j)←REMOVE-FIRST(*queue*)
    if REVISE(*csp*, X_i, X_j) **then**
      if size of D_i = 0 **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - {X_j} **do**
        add (X_k, X_i) to queue
  return *true*

Check consistency and remove value if necessary

Add all the neighbors to the queue if something is removed

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised ←*false*
  for each *x* **in** D_i **do**
    if no value *y* in D_j allows (*x*,*y*) to satisfy the constraint between X_i and X_j **then**
      delete *x* from D_i
      revised ←*true*
  return *revised*

Delete from tail to enforce consistency

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard

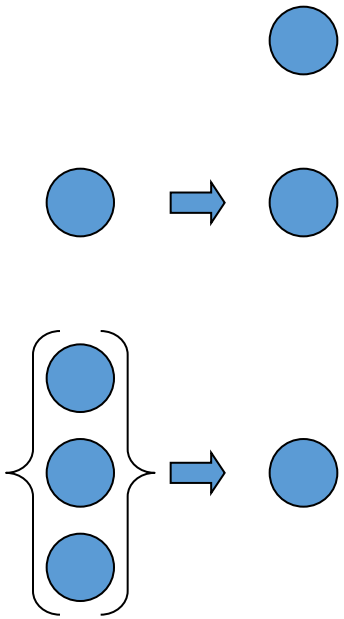
Arc Consistency and Forward Checking

- FC is essentially an arc consistency check for a single node!
  - Head is the assigned node, tails are its neighbors
- If you ran AC, no need to run FC

- FC is faster per value-assignment
- AC can catch failures earlier

K-Consistency

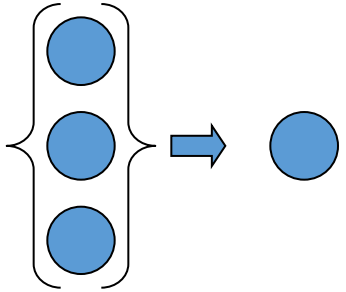
- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each value in a single node's domain must meet that **node's unary constraints**
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one node can **be extended to the other**
  - K-Consistency: For each k nodes, any consistent assignment to k-1 nodes can be **extended to the k^{th} node**.



- Higher k more expensive to compute
- (k=2 is called arc consistency, k=3 is called path consistency)

K-Consistency

- K-Consistency: For each k nodes, any consistent assignment to k-1 nodes can be **extended to the kth node**.
  - You must look at every consistent k − 1 assignment and every other variable (not in the selected k-1 set) and show that the extension exist
  - A single counter example is enough to rule-out consistency
  - (We are not deleting anything from the domains while checking consistency. Constraint propagation algorithms do this to enforce consistency)

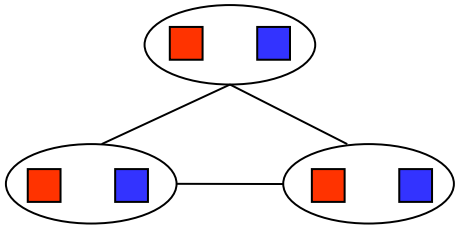


x
$D_x = \{1,2,3,4,5\}$
$x < 3$

Is this 1-consistent? No!

x, y
$D_{x,y} = \{1,2,3,4,5\}$
$x > 0, \quad x + y < 3$

Is this 1-consistent? Yes!
Is this 2-consistent? No!



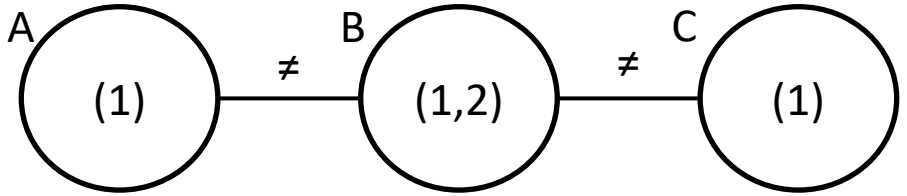
Is this 1-consistent? Yes!
Is this 2-consistent? Yes!
Is this 3-consistent? No!

K-Consistency

- K-Consistency: For each k nodes, any <u>consistent assignment to k-1 nodes</u> can be **extended to the k^{th} node**.
  - You must look at every consistent k − 1 assignment and every other variable (not in the selected k-1 set) and show that the extension exist
  - A single counter example is enough to rule-out consistency
  - We are not deleting anything from the domains

x, y
$D_{x,y} = \{1,2,3,4,5\}$
$x > 3, x = y$

Is this 1-consistent? No!
Is this 2-consistent? Yes!



Is this 1-consistent? Yes!
Is this 2-consistent? No!
Is this 3-consistent? Yes!

B=1 is 1-consistent, however, it cannot be extended!

(A=1,B=2), (A=1,C=1),(B=2,C=1) are all consistent and they can be extended to (A=1,B=2,C=1)

Strong K-Consistency

- Strong k-consistency: also k-1, k-2, … 1 consistent
- If a CSP is n-consistent, we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency

Summary of Filtering

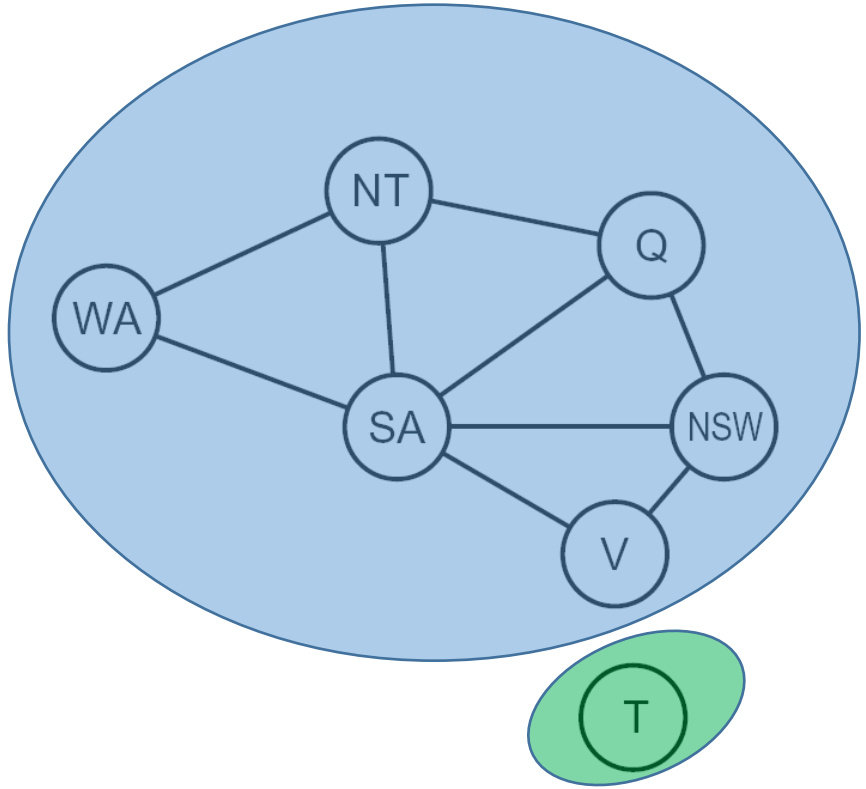
- FC: Remove values from the domains neighboring nodes
  - Fast to compute
  - Plays well with MRV
  - Does not catch some failures early

- Arc Consistency: Make all arcs consistent after an assignment
  - Keep checking arcs until there is no change
  - Entails FC
  - Earlier failure detection
  - Costly to Compute (especially if there are a lot of constraints)

- K-consistency: Check the consistency based on multiple (k-1) assignments

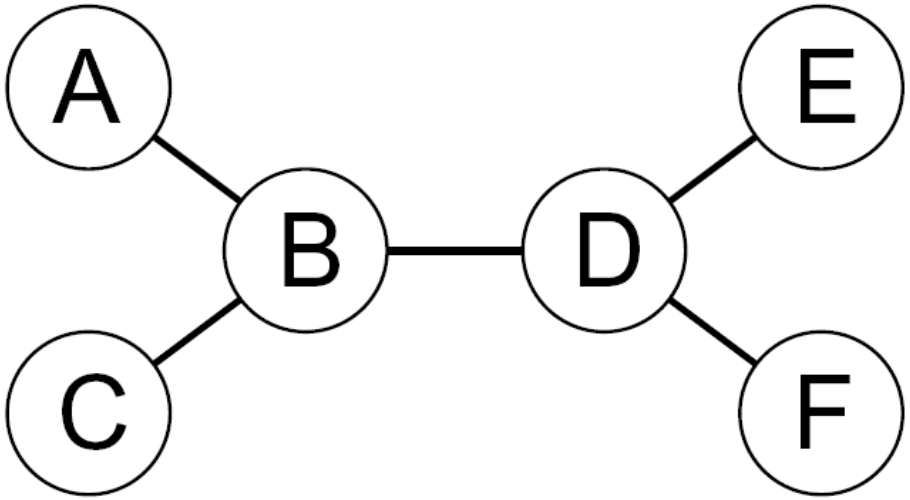
Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



It's rare to find unconnected parts of the graph

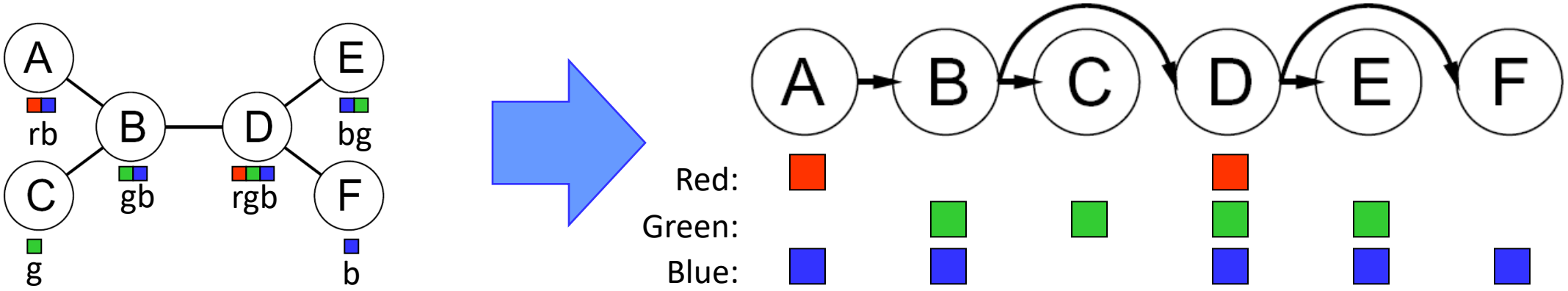
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(nd^2)$ time
- Compare to general CSPs, where worst-case time is $O(d^n)$

Solving Tree-Structured CSPs

- Order: Choose a root variable, order variables so that parents precede children



Constraints: Adjacent nodes have different colors
The domains of the variables are part of the problem definition. Remember that domains of the variables do not have to be the same. In this case, they are given like this. For example, the problem only allows green for C

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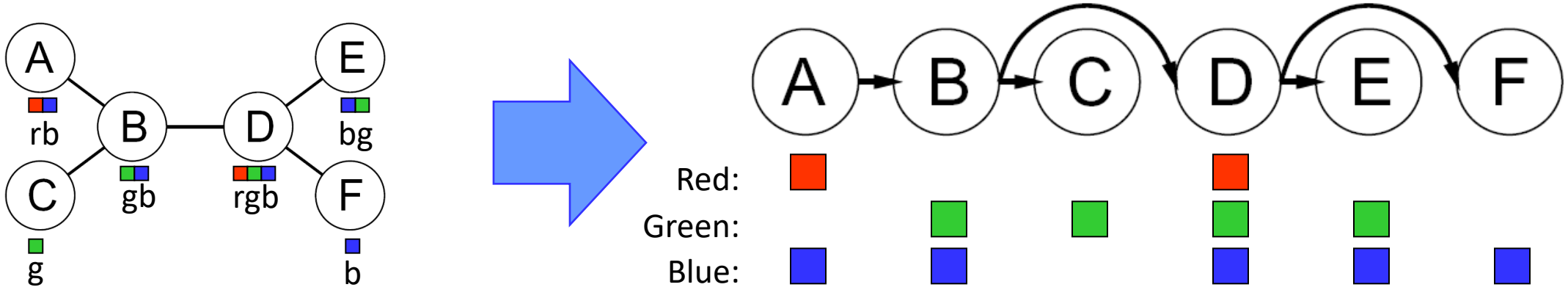

- For $i = n : 2$, apply Revise(Parent(X_i),X_i) – Remove Backward
- For $i = 1 : n$, assign X_i consistently with Parent(X_i) - Assign forward

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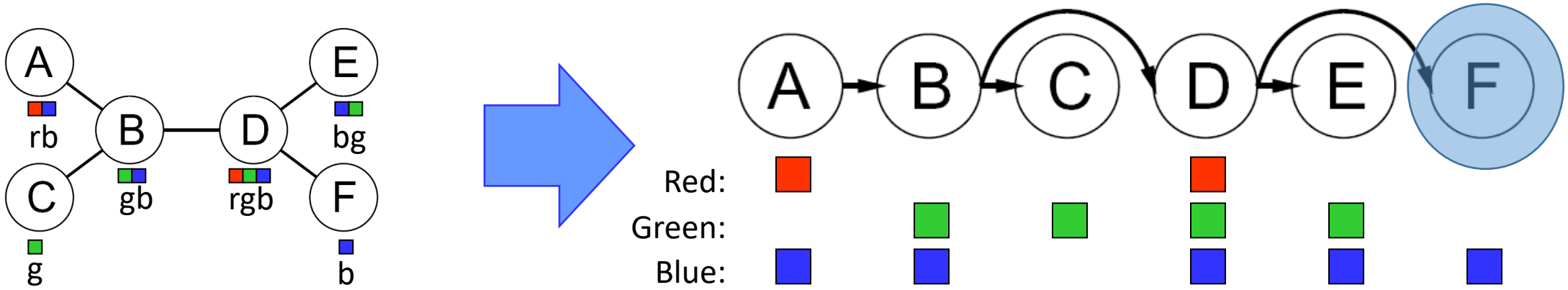

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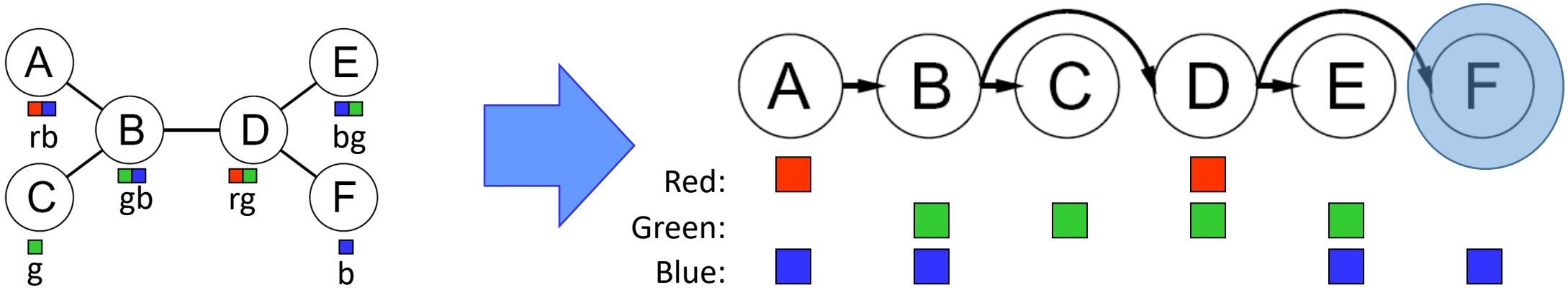

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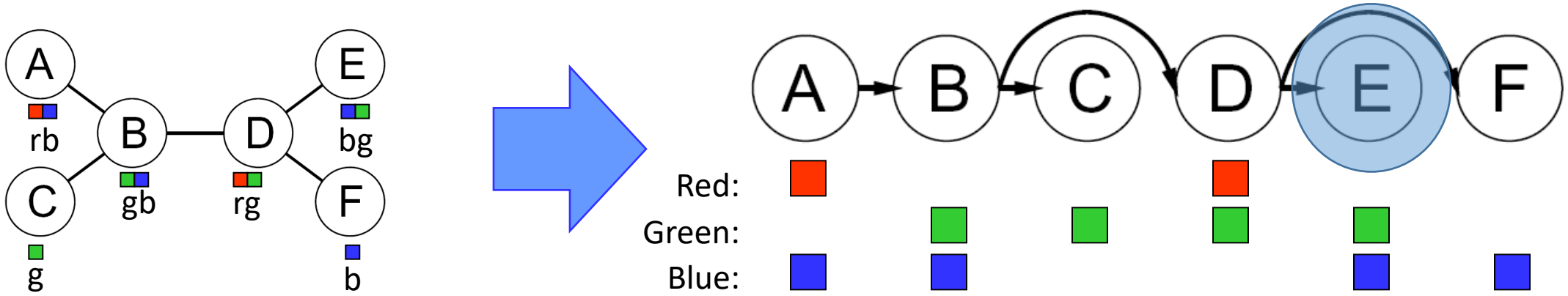

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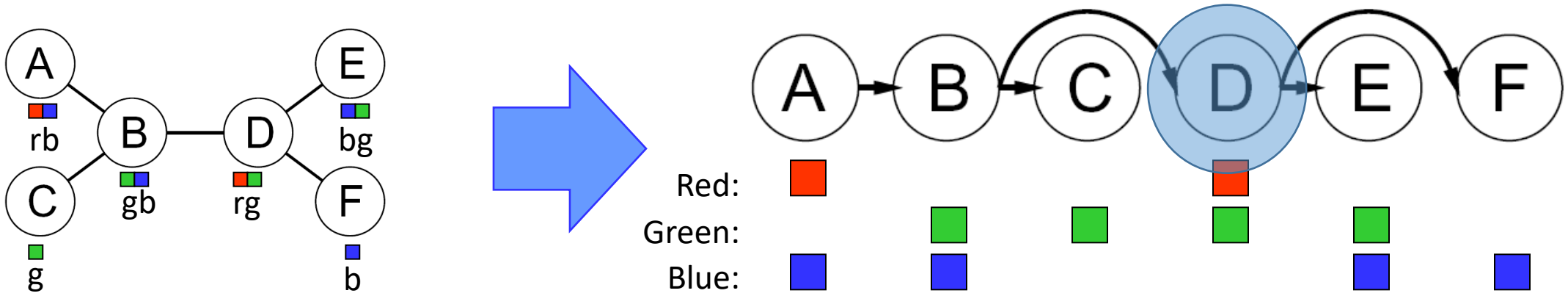

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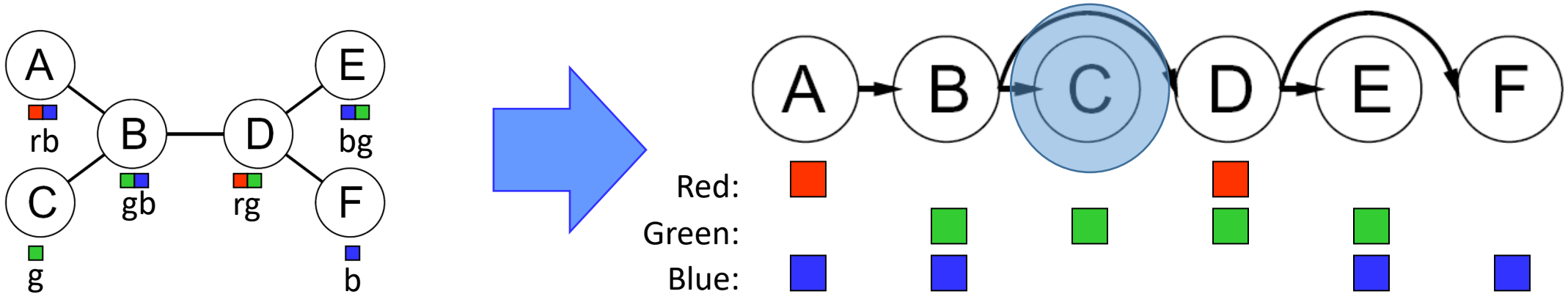

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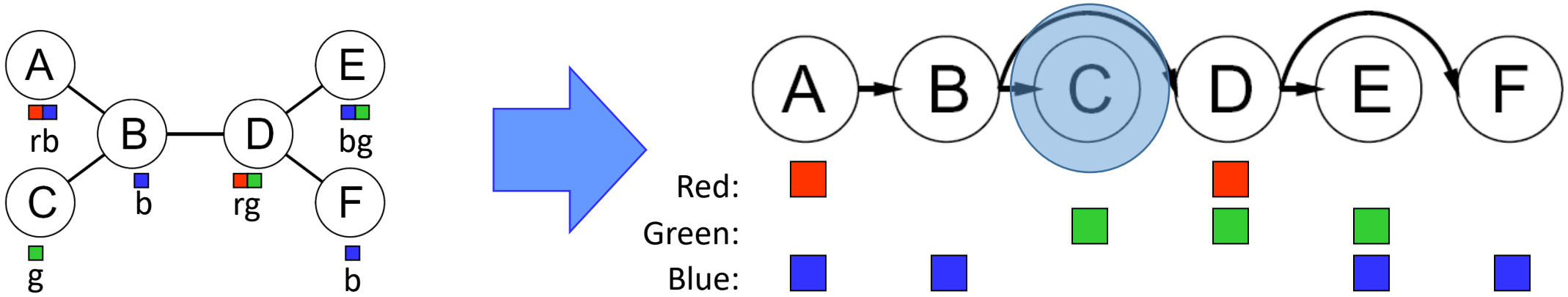

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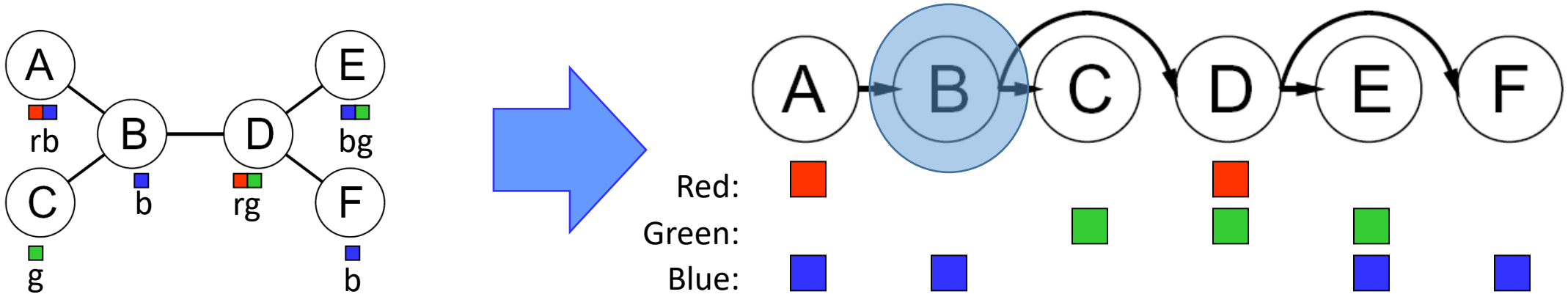

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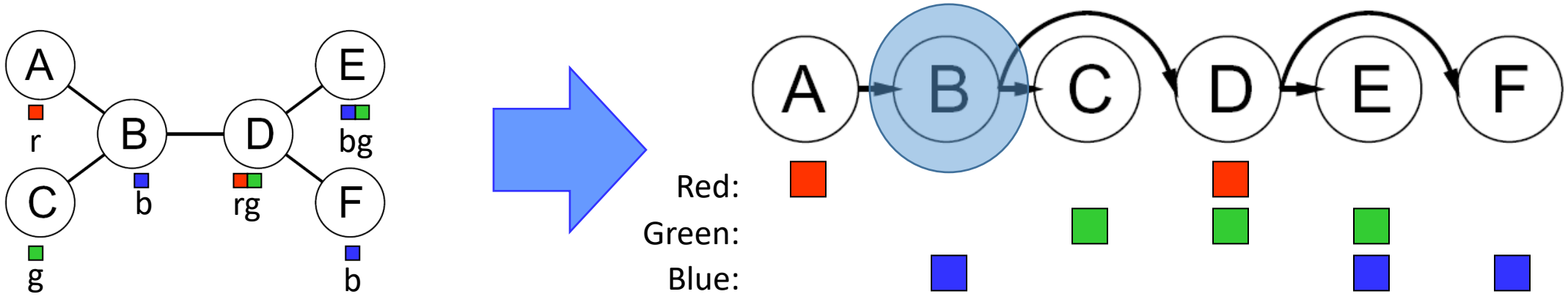

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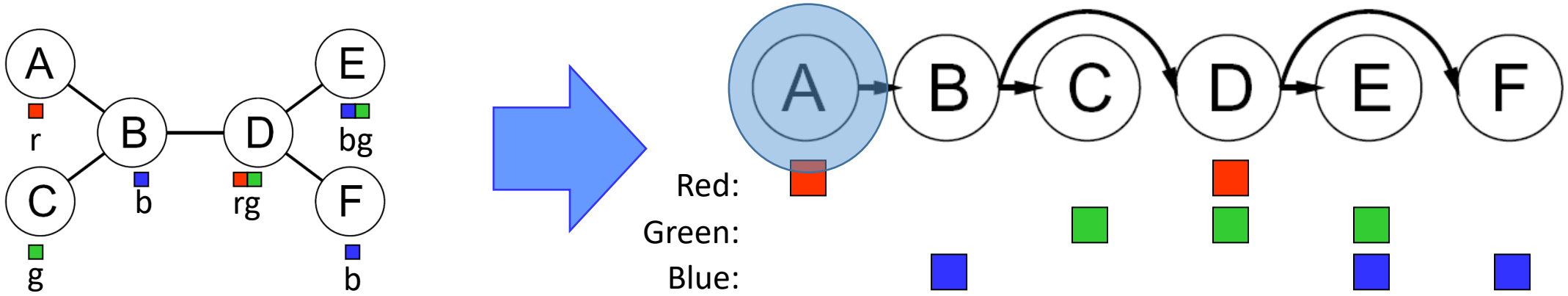

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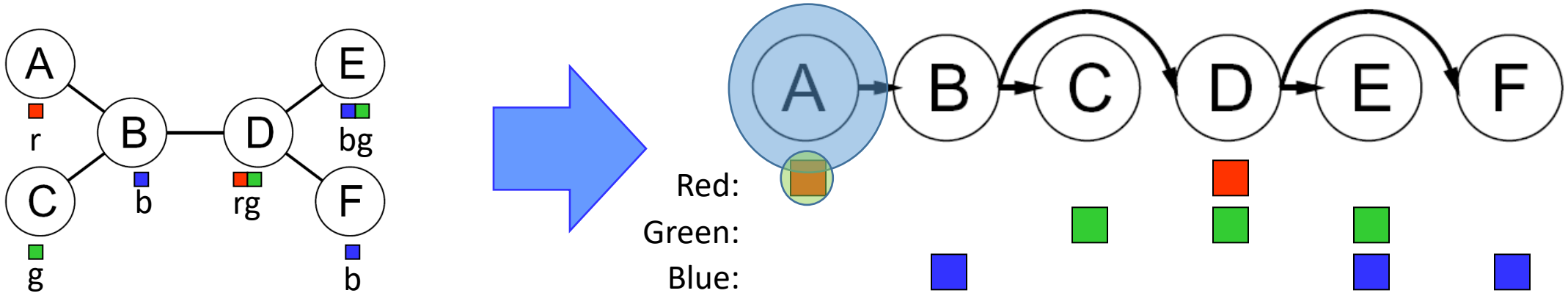

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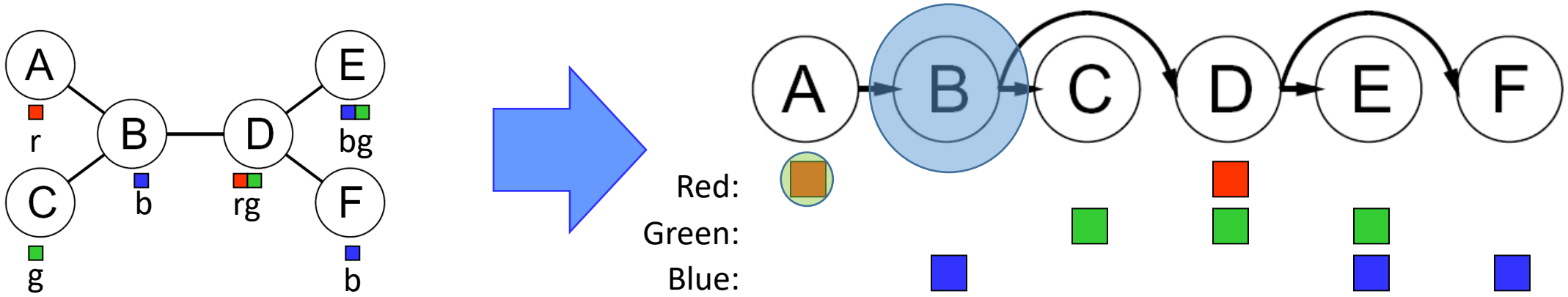

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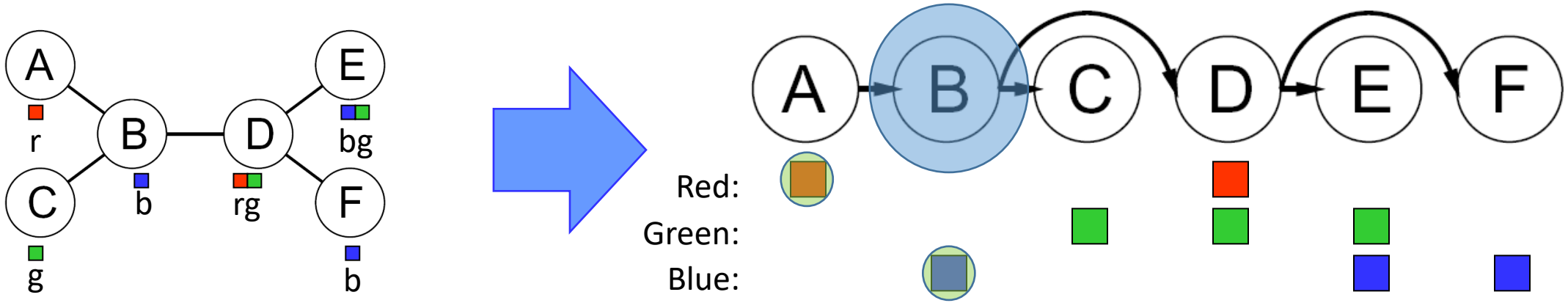

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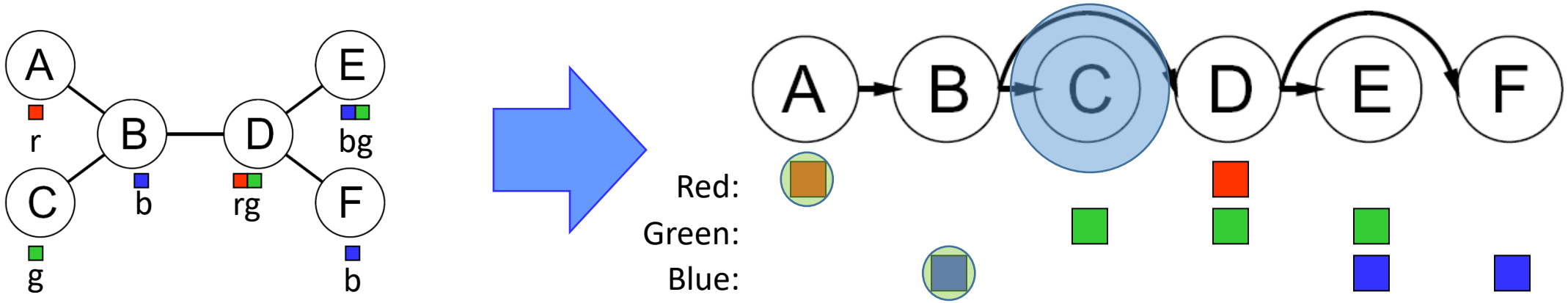

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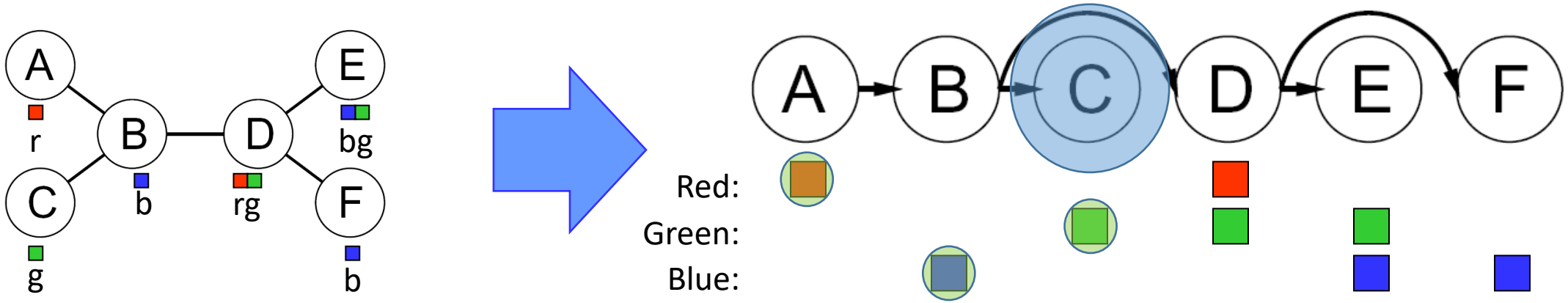

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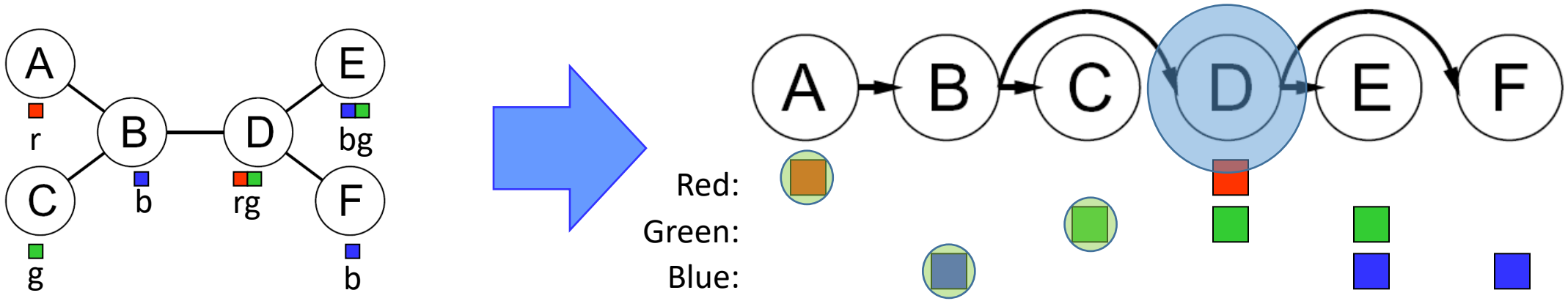

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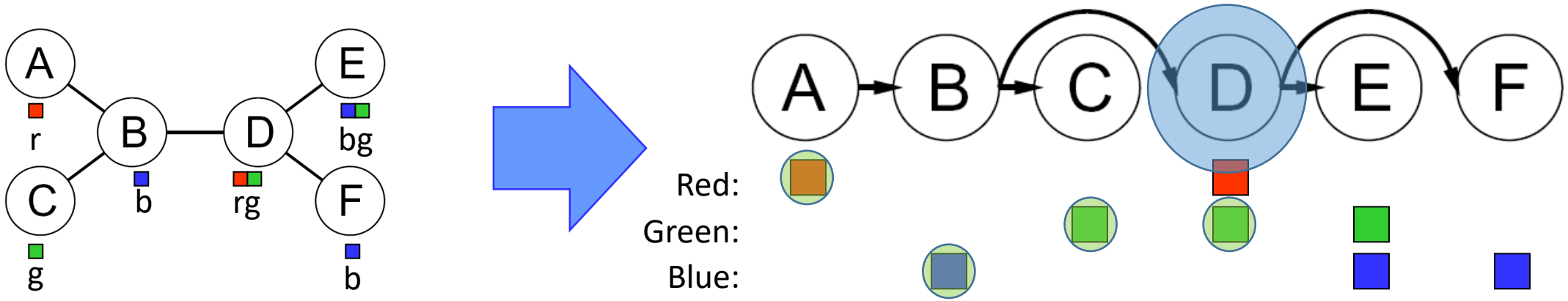

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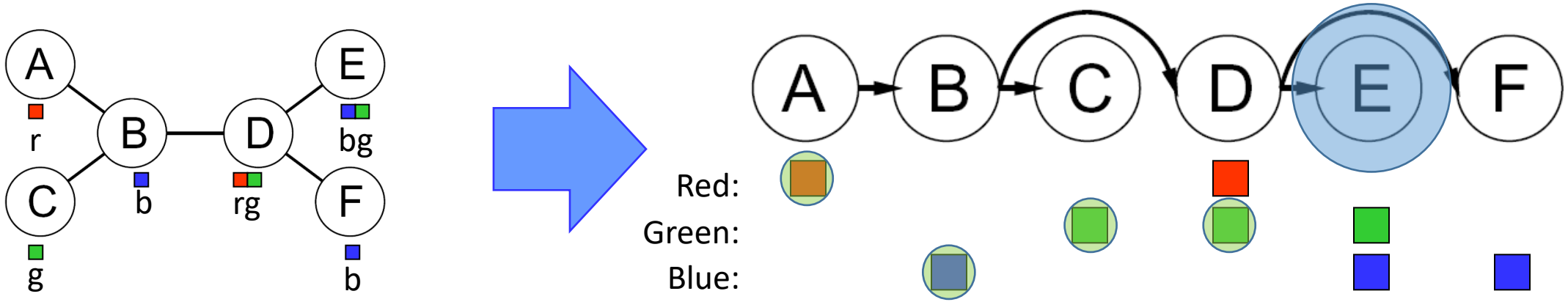

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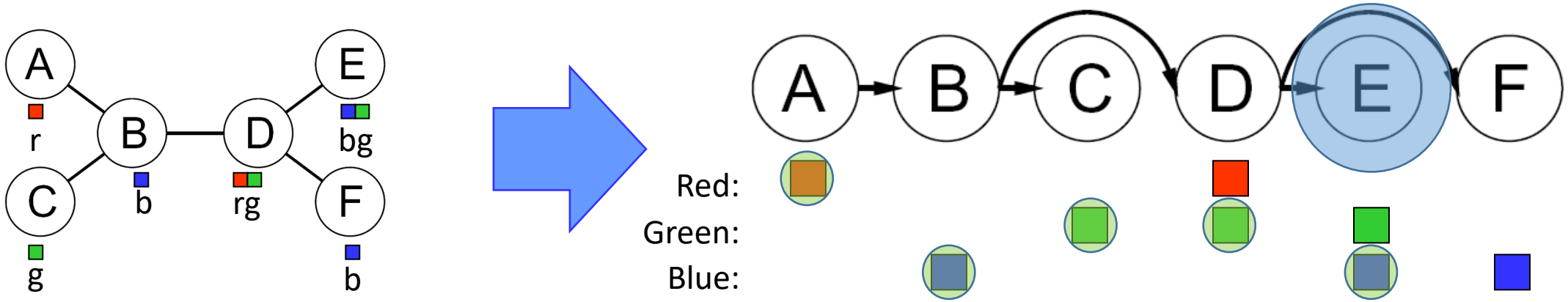

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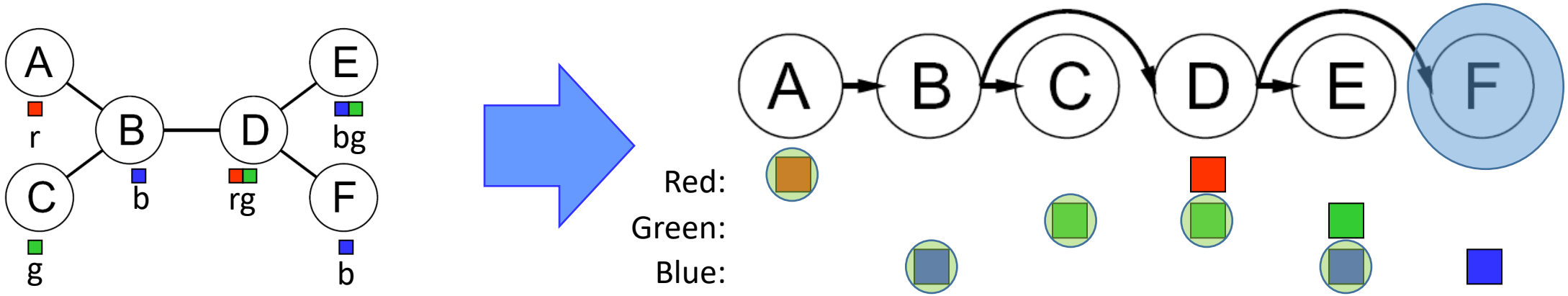

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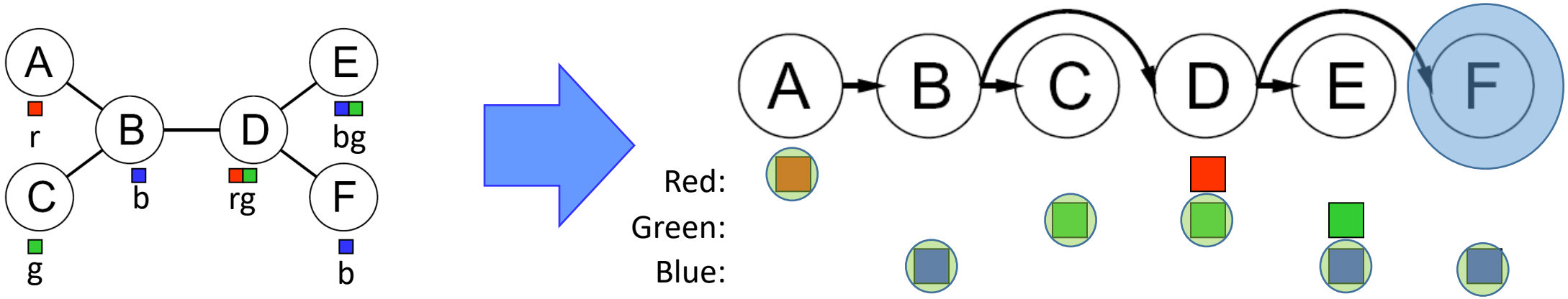

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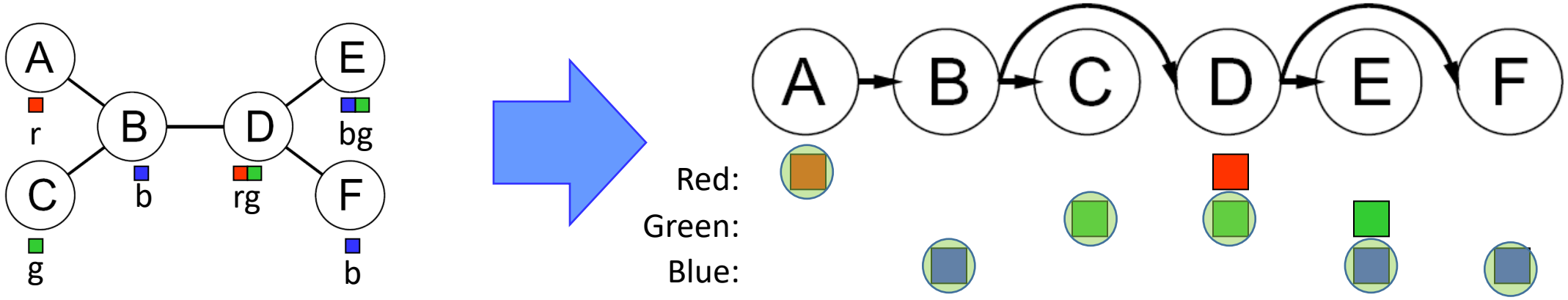

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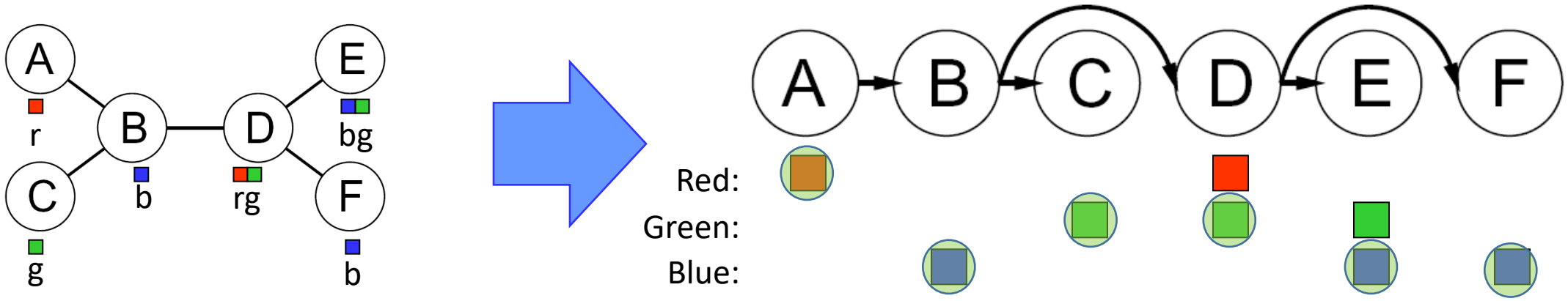

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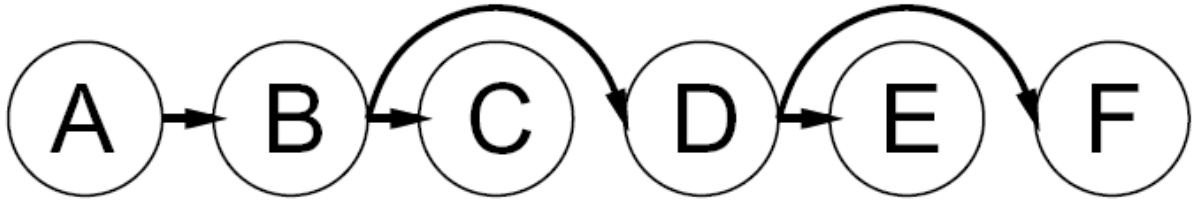
- Runtime: $O(nd^2)$ (why?)

Solving tree Structured CSPs

function TREE-CSP-SOLVER(*csp*) **returns** a solution, or failure
  inputs: *csp*, a CSP with components *X, D, C*
  n←number of variables in *X*
  assignment ←an empty assignment
  root ←any variable in *X*
  X ←TOPOLOGICALSORT(*X*, *root*)
  for *i* = *n* **down to** 2 **do**
    Revise(Parent(X_i),X_i)
    if it cannot be made consistent **then return** *failure*
  for i = 1 **to** n **do**
    assignment [X_i]←any consistent value from D_i
    if there is no consistent value **then return** *failure*
  return *assignment*

Just run BFS since X is a tree!

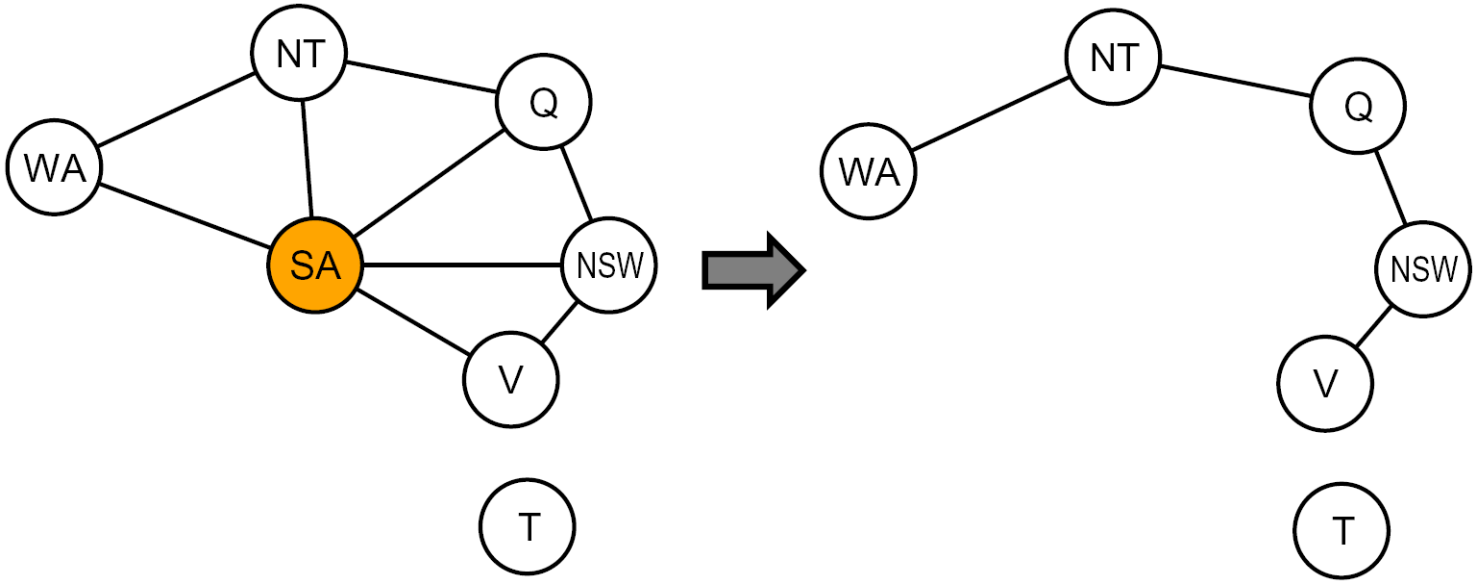
Tree-Structured CSPs



- **Claim 1**: After backward pass, all root-to-leaf arcs are consistent
- **Proof**: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)
- **Claim 2**: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- **Proof**: Induction on position

- Why doesn't this algorithm work with cycles in the constraint graph?

Nearly Tree-Structured CSPs



- **Conditioning**: instantiate a variable, prune its neighbors' domains
- **Cutset conditioning**: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size *c* gives runtime $O((d^c) (n-c) d^2)$, very fast for small *c*

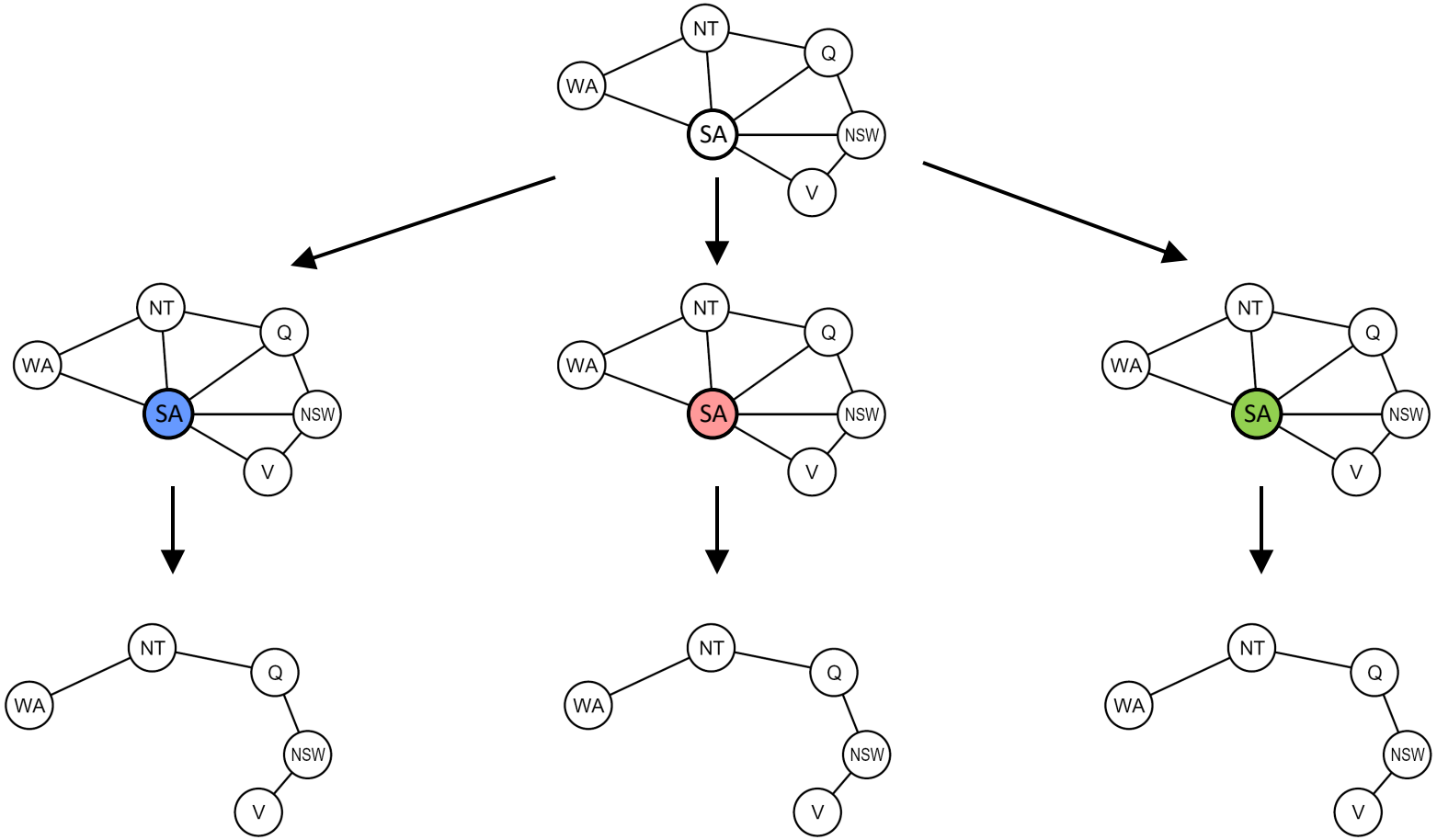
Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

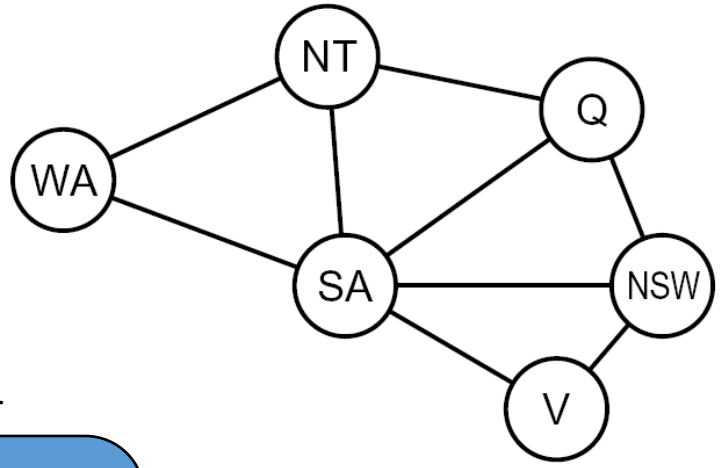
Compute residual CSP for each assignment

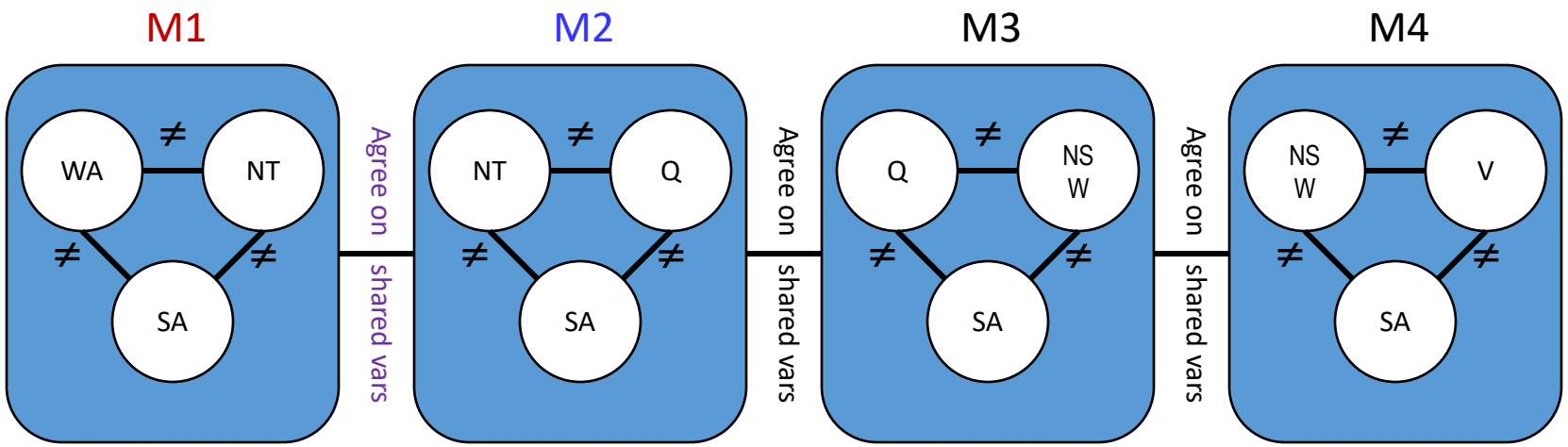
Solve the residual CSPs (tree structured)



Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions





{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

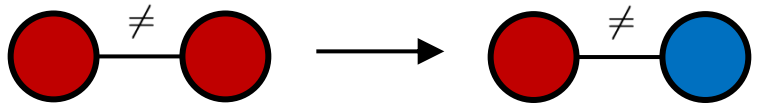
Agree: (M1,M2) ∈
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

Summary of Problem Structure Related Ideas

- Smaller connected components – rare to get lucky
- Tree CSPs – again not common
- Cutset Conditioning – finding minimal-size cutset is NP-complete
  - Efficient methods exist to find cycle-cutsets but not the minimal one
- Tree-Decomposition

What about Local Search for solving CSPs?

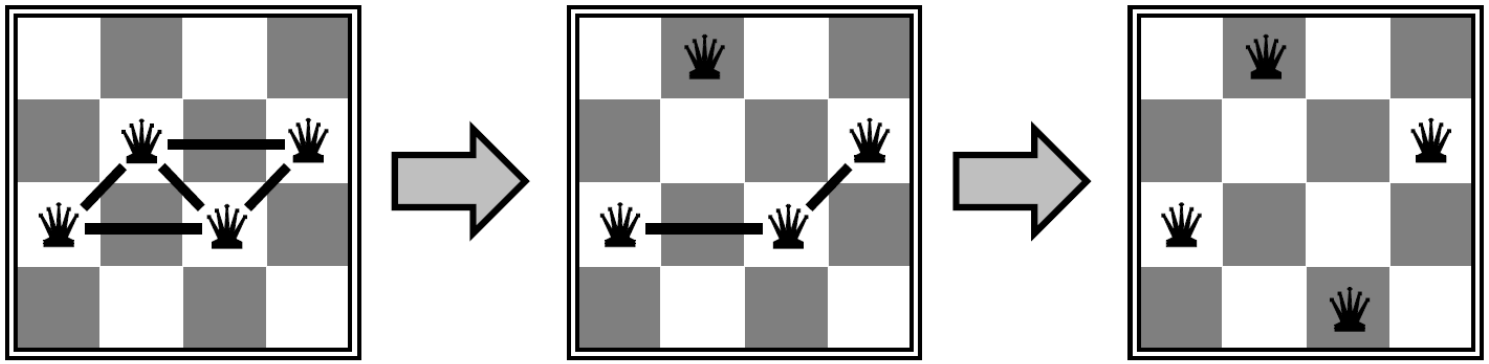
- Remember path does not matter!
- State: All variables assigned
  - Even if in conflict!
- Successors/Neighbors: *reassign* variable values
- Algorithm: While not solved,
  - **Initialize**: Random or greedy
    - Greedy: min number of violated constraints per variable in turn
  - **Variable selection**: randomly select any conflicted variable
  - **Value selection**: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints



Local Search For CSPs – MIN-CONFLICTS

function MIN-CONFLICTS(*csp*, *max_steps*) **returns** a solution or failure
  inputs: *csp*, a constraint satisfaction problem
  max_steps, the number of steps allowed before giving up
  current ←an initial complete assignment for *csp*
  for *i = 1* to *max_steps* **do**
    if *current* is a solution for *csp* **then return** *current*
    var ←a randomly chosen conflicted variable from *csp*.VARIABLES
    value←the value v for var that minimizes CONFLICTS(*var* , *v*, *current* , *csp*)
    set *var* =*value* in *current*
  return failure

Example: 4-Queens

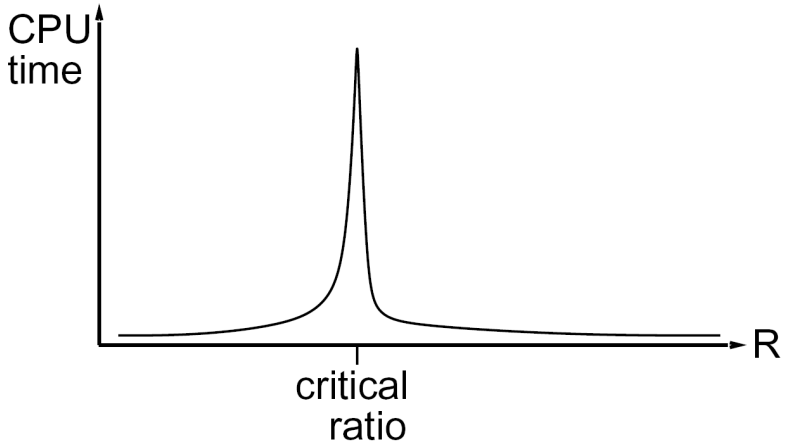


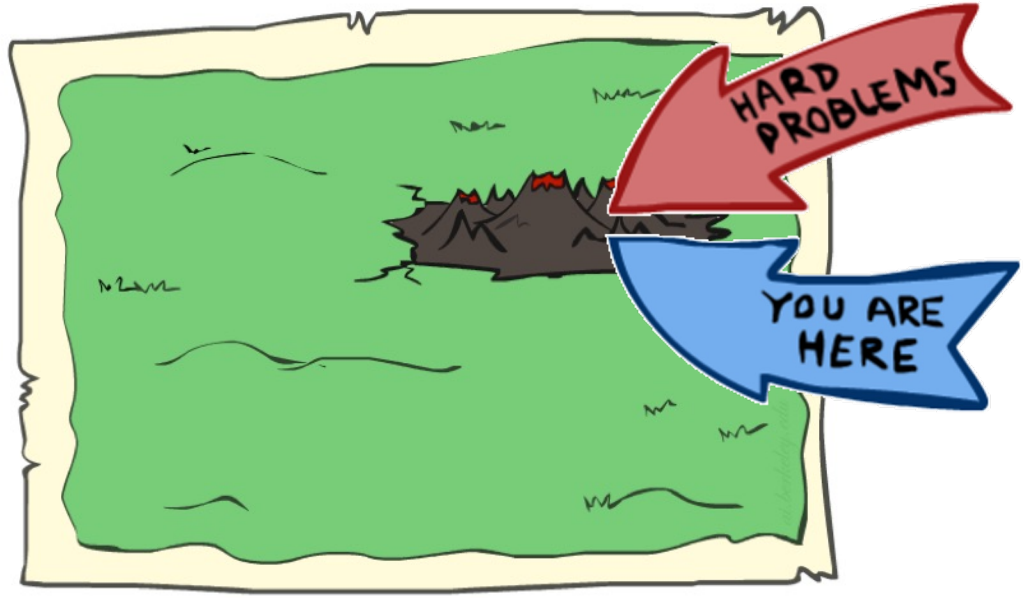
- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$





Related to the “Phase Transition” phenomenon

Summary of CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice