Simple Discretization Methods: Binning

1. Equal-Width (Distance) Partitioning

Formula:

$$W = \frac{B - A}{N}$$

where:

- W = Width of each interval
- A = Minimum value in the dataset
- B = Maximum value in the dataset
- N = Number of bins

Example: Given the dataset:

$$\{1, 3, 7, 10, 15, 18, 22, 25\}$$

Let $N = 3$, $A = 1$, $B = 25$.

Step 1: Compute Interval Width

$$W = \frac{25 - 1}{3} = \frac{24}{3} = 8$$

Step 2: Define Bins

- **Bin 1:** $[1, 9) \rightarrow$ contains $\{1, 3, 7\}$
- **Bin 2:** $[9, 17) \rightarrow$ contains $\{10, 15\}$
- **Bin 3:** $[17, 25] \rightarrow$ contains $\{18, 22, 25\}$

This method is **simple and intuitive**, but **outliers** and **skewed data** can make it less effective.

2. Equal-Depth (Frequency) Partitioning

Concept: Each bin should have **approximately the same number of elements**.

Example: Given the dataset:

$$\{1, 3, 7, 10, 15, 18, 22, 25\}$$

Let $N = 3$. Since there are **8 elements**, each bin should have about $\frac{8}{3} \approx 3$ elements.

Step 1: Sort Data

$$\{1, 3, 7, 10, 15, 18, 22, 25\}$$

Step 2: Assign Bins

- **Bin 1:** $\{1, 3, 7\}$
- **Bin 2:** $\{10, 15, 18\}$
- **Bin 3:** $\{22, 25\}$

This method ensures **equal distribution** of data points in each bin, making it **better for skewed data**.

Bin Processing Methods

1. Bin Means

Concept: Replace all values in a bin with the **mean** (average) of that bin.
Example: Using the same bins from equal-width partitioning:

- **Bin 1:** $\{1, 3, 7\} \rightarrow$ Mean $= \frac{1+3+7}{3} = 3.67 \rightarrow \{3.67, 3.67, 3.67\}$
- **Bin 2:** $\{10, 15\} \rightarrow$ Mean $= \frac{10+15}{2} = 12.5 \rightarrow \{12.5, 12.5\}$
- **Bin 3:** $\{18, 22, 25\} \rightarrow$ Mean $= \frac{18+22+25}{3} = 21.67 \rightarrow \{21.67, 21.67, 21.67\}$

Pros: Reduces variance. **Cons:** May hide important variations in data.

2. Bin Medians

Concept: Replace all values in a bin with the **median** of that bin. The **median** is the middle value when the numbers are sorted. If there is an even number of elements, the median is the average of the two middle values.
Example:

- **Bin 1:** $\{1, 3, 7\} \rightarrow$ Sorted: $\{1, 3, 7\}$, Median $= 3 \rightarrow \{3, 3, 3\}$
- **Bin 2:** $\{10, 15\} \rightarrow$ Sorted: $\{10, 15\}$, Median $= \frac{10+15}{2} = 12.5 \rightarrow \{12.5, 12.5\}$
- **Bin 3:** $\{18, 22, 25\} \rightarrow$ Sorted: $\{18, 22, 25\}$, Median $= 22 \rightarrow \{22, 22, 22\}$

Pros: Handles outliers better than means. **Cons:** May not capture overall distribution well.

3. Bin Boundaries

Concept: Replace each value with the nearest boundary value in the bin.
Example:

- **Bin 1:** $\{1, 3, 7\} \rightarrow$ Boundaries: $1, 7$ - $1 \rightarrow 1$, $3 \rightarrow 1$, $7 \rightarrow 7 \rightarrow \{1, 1, 7\}$
- **Bin 2:** $\{10, 15\} \rightarrow$ Boundaries: $10, 15$ - $10 \rightarrow 10$, $15 \rightarrow 15 \rightarrow \{10, 15\}$
- **Bin 3:** $\{18, 22, 25\} \rightarrow$ Boundaries: $18, 25$ - $18 \rightarrow 18$, $22 \rightarrow 25$, $25 \rightarrow 25 \rightarrow \{18, 25, 25\}$

Pros: Good for preserving extreme values. **Cons:** Can distort middle-range values.

Data Transformation: Normalization

1. Min-Max Normalization

Formula:

$$v' = \frac{v - \min_A}{\max_A - \min_A} \times (\text{new\_max}_A - \text{new\_min}_A) + \text{new\_min}_A$$

where:

- v = Original value
- $\min_A$ = Minimum value of the original dataset
- $\max_A$ = Maximum value of the original dataset
- new_min_A = Minimum value of the new range
- new_max_A = Maximum value of the new range
- v' = Normalized value

Example: Convert age = 30 to a range [0, 1], where min = 10 and max = 80:

$$v' = \frac{30 - 10}{80 - 10} = \frac{20}{70} = \frac{2}{7} \approx 0.2857$$

This transformation scales the value to fall between the new range, in this case, between 0 and 1.

2. Z-Score Normalization

Formula:

$$v' = \frac{v - \text{mean}_A}{\text{stand\_dev}_A}$$

where:

- v = Original value
- mean_A = Mean of the original dataset

$$\text{mean}_A = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- stand_dev_A = Standard deviation of the original dataset

$$\text{Population: } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \text{mean}_A)^2}$$

$$\text{Sample: } s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \text{mean}_A)^2}$$

- v' = Normalized value

The Z-score normalization transforms the data to have a **mean of 0** and a **standard deviation of 1**, making it useful for comparing data from different distributions.

3. Normalization by Decimal Scaling

Formula:

$$v' = \frac{v}{10^j}$$

where j is the smallest integer such that:

$$\text{Max}\left(|v'|\right) < 1$$

This transformation scales the data by dividing each value by a power of 10, where the exponent j is chosen to ensure the maximum absolute value of v' is less than 1.

Five-Number Summary

1. Even-Length Datasets

Formula:

$$\text{Five-number summary} = (\text{Minimum}, Q1, \text{Median (Q2)}, Q3, \text{Maximum})$$

where:

- $Q1 = $ Median of the first half
- $Q2 = $ Overall median (average of middle two values)
- $Q3 = $ Median of the second half

Example 1: Given the dataset: $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$

- **Minimum:** 1
- **Maximum:** 10
- **Median (Q2):** Average of 5th and 6th values $= \frac{5+6}{2} = 5.5$
- **Q1:** Median of $[1, 2, 3, 4, 5] = 3$
- **Q3:** Median of $[6, 7, 8, 9, 10] = 8$

Five-number summary: $(1, 3, 5.5, 8, 10)$

Example 2: Given the dataset: $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$

- **Minimum:** 1
- **Maximum:** 12
- **Median (Q2):** Average of 6th and 7th values $= \frac{6+7}{2} = 6.5$
- **Q1:** Median of $[1, 2, 3, 4, 5, 6] = \frac{3+4}{2} = 3.5$
- **Q3:** Median of $[7, 8, 9, 10, 11, 12] = \frac{9+10}{2} = 9.5$

Five-number summary: $(1, 3.5, 6.5, 9.5, 12)$

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2. Odd-Length Datasets

Example 3: Given the dataset: $[1, 2, 3, 4, 5, 6, 7, 8, 9]$

- **Minimum:** 1
- **Maximum:** 9
- **Median (Q2):** 5th value $= 5$
- **Q1:** Median of $[1, 2, 3, 4] = \frac{2+3}{2} = 2.5$
- **Q3:** Median of $[6, 7, 8, 9] = \frac{7+8}{2} = 7.5$

Five-number summary: $(1, 2.5, 5, 7.5, 9)$

Example 4: Given the dataset: $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]$

- **Minimum:** 1
- **Maximum:** 11
- **Median (Q2):** 6th value $= 6$
- **Q1:** Median of $[1, 2, 3, 4, 5] = 3$
- **Q3:** Median of $[7, 8, 9, 10, 11] = 9$

Five-number summary: $(1, 3, 6, 9, 11)$

Statistical Formulas

1. Median (Q2)

$$Q2 = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} & \text{if } n \text{ is even} \end{cases}$$

2. First Quartile (Q1)

$$p = \frac{n+1}{4}; \quad Q1 = \begin{cases} x_p & \text{if } p \text{ is integer} \\ \frac{x_{\lfloor p \rfloor} + x_{\lfloor p \rfloor + 1}}{2} & \text{otherwise} \end{cases}$$

Example: For $p = 2.3$, interpolate between x_2 and x_3.

3. Third Quartile (Q3)

$$p = \frac{3(n+1)}{4}; \quad Q3 = \begin{cases} x_p & \text{if } p \text{ is integer} \\ \frac{x_{\lfloor p \rfloor} + x_{\lfloor p \rfloor + 1}}{2} & \text{otherwise} \end{cases}$$

Example: For $p = 6.75$, interpolate between x_6 and x_7.

4. Mean

$$\text{Mean} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

5. Midrange

$$\text{Midrange} = \frac{\min(x) + \max(x)}{2}$$

6. Range

$$\text{Range} = \max(x) - \min(x)$$

7. Bin Width

$$\text{Bin Width} = \frac{\text{Range}}{k} \quad (k = \text{number of bins})$$