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IFAC PapersOnLine 55-1 (2022) 522–527

Quaternion Constrained Robust Attitude Control Using Barrier Lyapunov Function Based Back-stepping

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Abstract: A novel, robust quaternion constrained attitude control law for rigid bodies is presented in this paper. The controller is formulated using the backstepping philosophy, and Barrier Lyapunov Functions (BLFs) are used to prevent attitude constraint violation. This is done by ensuring the boundedness of BLFs in the closed-loop Lyapunov stability analysis. The analogy between a standard Quadratic Lyapunov Function based attitude control law and the BLF based control law is highlighted, and a systematic procedure to select control constants is detailed. Finally, the BLF based control law is verified by carrying out numerical simulations of the rigid body in the presence of initial attitude errors and bounded disturbance torques. In the first case, the attitude errors are asymptotically driven to zero while strictly satisfying the quaternion constraints. In the presence of bounded disturbance torques, the attitude errors remain bounded within the constraint boundary.

Keywords: Attitude control, Backstepping, Barrier Lyapunov Function, Constraints, Lyapunov stability analysis.

1. INTRODUCTION

The problem of attitude control for rigid bodies finds its applications in various fields. Some of them include two-link robot manipulators for industrial use, UAVs, satellites for space exploration and antennas, laser systems for communication and defense. Maintaining a fixed, set-point attitude is a crucial component of the control requirement in the scenarios mentioned above. However, in a practical setting, unknown disturbance torques, uncertainty in system parameters and initial condition variation will cause the rigid body to lose its orientation accuracy. Both the safety and objective may be compromised when the attitude errors increase beyond a certain threshold. Therefore the goal of an effective attitude control problem is to track a set-point attitude without violating any attitude constraints. The attitude constraints on the rigid body are typically expressed as inequality constraints on the kinematic state variables, viz., Euler angles, Quaternions, Rodriguez parameters etc. Although, a number of feedback control strategies, both linear and nonlinear, have been developed to achieve asymptotic tracking (Slotine et al. (1991); Marquez (2003)), the challenges of incorporating state constraints still remain unanswered.

An output constrained control design for SISO systems was proposed by Tee et al. (2009) for a class of nonlinear systems in strict feedback form. To prevent output constraint violation, the authors use a BLF that was first proposed by Ngo et al. (2005). The BLF grows to infinity when the output error approaches a certain finite limit and therefore by ensuring boundedness of the BLF during closed loop lyapunov analysis, the bounds on the output error is ensured. The work was later extended by Tee and Ge (2009, 2011) to include partial as well as full state constraints using a logarithmic BLF based control architecture. Tee et al. (2011) considered the case of a time varying output constraint and employed a time varying BLF to ensure output constraint satisfaction. A full state constrainted control design for SISO nonlinear systems in pure feedback form was proposed by Liu and Tong (2016). The mean value theorem is employed to transform the system to a strict feedback structure with residual non-affine terms. A BLF based backstepping procedure is introduced to handle the non-affine terms as well as state constraints. An Integral BLF based full state constrained controller for pure feedback type SISO systems was formulated by Tang et al. (2016). The integral BLF is capable of handling unknown system non-linearities as well as time varying bounded disturbances. Recently, a classical dynamic inversion based state constrained control for a class of coupled MIMO nonlinear systems was proposed by Sachan and Padhi (2018).

Note that in the aforementioned works, the output/state inequality constraints are assumed to be linear. But in the case of many practical applications, such as the rigid body attitude control problem, the state constraints appear as nonlinear inequality constraints. Deriving motivation from the above works, the goal of this paper is to develop a BLF based nonlinear feedback controller for a rigid body that ensures asymptotic stability of closed-loop attitude tracking without violating attitude constraints. The controller will be formulated using the backstepping philosophy. The performance of the controller is analyzed in the presence

⋆ The work was supported by a research grant (Grant No: ISTC-19-0004) from the Space Technology Cell at the Indian Institute of Science, Bangalore, India.

10.1016/j.ifacol.2022.04.086

of bounded unknown disturbances. Finally, the superiority of the proposed controller has been demonstrated by comparing with an existing backstepping control strategy designed using standard Quadratic Lyapunov Functions (QLFs).

2. RIGID BODY DYNAMICS

In this paper, quaternions $\boldsymbol{q} = [q_1\ q_2\ q_3\ q_4]^T \in \mathbb{R}^4$ will be used as the kinematic variables of interest. The angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the rigid body imparts a change in the quaternions as

$$\dot{\boldsymbol{q}} = \boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{\omega} \tag{1}$$

Here q_1, q_2, q_3 represent the components of the vector portion of the quaternion and q_4 represents the scalar part. The quaternions satisfy the constraint $q_1^2+q_2^2+q_3^2+q_4^2 = 1$. The term $\boldsymbol{g}\{\boldsymbol{q}\}$ is

$$\boldsymbol{g}\{\boldsymbol{q}\} = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

Upon the application of an external torque, the angular velocity of the rigid body is governed by the following equation

$$\dot{\boldsymbol{\omega}} = \boldsymbol{h}\{\boldsymbol{\omega}\} + \boldsymbol{u} \tag{2}$$

where $\boldsymbol{h}\{\boldsymbol{\omega}\}$ is given as $J^{-1}(J\boldsymbol{\omega} \times \boldsymbol{\omega})$. J denotes the moment of inertia of the rigid body and $\boldsymbol{u} \in \mathbb{R}^3$ is the angular acceleration. Consider the rigid body frame $\mathcal{B}$ oriented with respect to a desired attitude frame $\mathcal{D}$. The direction cosine matrix which relates the orientation between the two frames is

$$\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{bmatrix} = \begin{bmatrix} 1-2q_2^2-2q_3^2 & 2\left(q_1q_2-q_3q_4\right) & 2\left(q_1q_3+q_2q_4\right) \\ 2\left(q_1q_2+q_3q_4\right) & 1-2q_3^2-2q_1^2 & 2\left(q_2q_3-q_1q_4\right) \\ 2\left(q_3q_1-q_2q_4\right) & 2\left(q_2q_3+q_1q_4\right) & 1-2q_1^2-2q_2^2 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

The attitude errors between the two frames are defined as the angles between the axis $\hat{b}_i$ and $\hat{d}_i$, $i = 1,2,3$ respectively and are represented in Fig. 1. At any time during the control, the angle between $\hat{b}_i$ and $\hat{d}_i$ axes are enforced to be less than a maximum angle $\bar{\delta}_i$. The attitude errors are transformed into quaternion inequality constraints using the orientation between the $\mathcal{B}$ and $\mathcal{D}$ frames as

$$\begin{aligned} &\hat{d}_1 \cdot \hat{b}_1 = \cos\left(\delta_1\right) = 1-2q_2^2-2q_3^2 \\ &\delta_1 = \cos^{-1}(1-2q_2^2-2q_3^2) < \bar{\delta}_1 \\ &q_{23} = q_2^2+q_3^2 < \bar{e}_1,\ \bar{e}_1 = \frac{1-\cos\left(\bar{\delta}_1\right)}{2} \end{aligned} \tag{3}$$

$\bar{\delta}_1$ indicates the maximum attitude error between $\hat{b}_1$ and the corresponding desired axis $\hat{d}_1$ respectively. Similarly, the remaining attitude errors written in terms of quaternion constraints are

$$q_{13} = q_1^2+q_3^2 < \bar{e}_2 \tag{4}$$

$$q_{12} = q_1^2+q_2^2 < \bar{e}_3 \tag{5}$$

3. SYNTHESIS OF QUATERNION CONSTRAINED ROBUST CONTROL

In this section, a nonlinear feedback controller is designed to align the body frame to the desired attitude frame, i.e., to achieve a desired quaternion $\boldsymbol{q^*} = [0\ 0\ 0\ 1]^T$. The output variables are chosen as $\boldsymbol{y} = [q_{23}\ q_{13}\ q_{12}\ q_4]^T$. The objective of the controller is to asymptotically drive the quaternions $\boldsymbol{q}$ to their desired value $\boldsymbol{q}^*$ and simultaneously enforce quaternion inequality constraints in Eq. (3), (4) and (5). It is easily seen that that driving the output $\boldsymbol{y}$ to $\boldsymbol{y}^* = [0\ 0\ 0\ 1]^T$ uniquely implies $\boldsymbol{q} \to \boldsymbol{q}^*$. During the derivation of the controller, the following assumptions and lemma are taken into account.

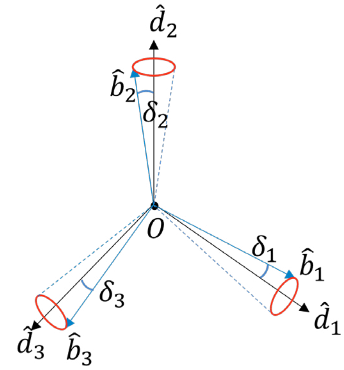


Fig. 1. Rigid body orientation in desired frame.

Assumption 1. All the states viz., the quaternions and angular velocities are available for feedback control design.

Assumption 2. The attitude pointing errors at the initial time satisfy the quaternion constraints $(\delta_i(t_0) < \bar{\delta}_i)$.

Assumption 3. The unknown disturbance torques are assumed to be bounded, i.e., $\|\boldsymbol{d}\| \leq \bar{d}$

Lemma 1. Tee et al. (2009). For any positive constant k_a, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R}_+ : 0 \leq z_1 < k_a\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system $\dot{\eta} = h(t,\eta)$ where $\eta := [w, z_1]^T \in \mathcal{N}$, and $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in z_1, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose there exists functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains such that $V_1(z_1) \to \infty$ as $z_1 \to k_a$. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0)$ belongs to the set $z_1 \in (0, k_a)$. If the inequality holds: $\dot{V} = \frac{\partial V}{\partial \eta} h \leq 0$. Then $z_1(t)$ remains in the open set $z_1 \leq (0, k_a) \forall t \in [0, \infty]$.

Following the principle of backstepping design, the nonlinear feedback controller is designed as

Step 1: Let the components of output error be defined as $e_1 = q_{23} - q_{23}^*$, $e_2 = q_{13} - q_{13}^*$, $e_3 = q_{12} - q_{12}^*$ and $e_4 = q_4 - q_4^*$. Let $\boldsymbol{z}_q = \boldsymbol{q} - \boldsymbol{q}^*$ and $\boldsymbol{z}_\omega = \boldsymbol{\omega} - \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is a stabilizing function to be designed for the quaternion kinematics. In order to simultaneously achieve asymptotic tracking of desired attitude and to enforce output error constraints, a combination of three logarithmic BLF's and one QLF is chosen as the Lyapunov function.

$$\begin{aligned} L_1 = \frac{1}{2}k_{l_1}\log\left(\frac{\bar{e}_1}{\bar{e}_1 - e_1}\right) + \frac{1}{2}k_{l_2}\log\left(\frac{\bar{e}_2}{\bar{e}_2 - e_2}\right) \\ + \frac{1}{2}k_{l_3}\log\left(\frac{\bar{e}_3}{\bar{e}_3 - e_3}\right) + \frac{1}{2}k_{l_4}e_4^2 \end{aligned} \tag{6}$$

BLF's are specifically chosen for the first three components of output errors that require a strict enforcement of constraints. k_{l_i}, $i = 1,2,3,4$ denote positive constants and $\log(\cdot)$ denotes the natural logarithm. The Lyapunov funciton L_1 is positive definite and C^1 continuous when $e_i < \bar{e}_i,\ i = 1, 2, 3$. The derivative of L_1 is given by

$$\dot{L}_1 = \frac{1}{2}k_{l_1}\frac{\dot{e}_1}{(\bar{e}_1 - e_1)} + \frac{1}{2}k_{l_2}\frac{\dot{e}_2}{(\bar{e}_2 - e_2)} + \frac{1}{2}k_{l_3}\frac{\dot{e}_3}{(\bar{e}_3 - e_3)} + k_{l_4}e_4\dot{e}_4$$

Expanding the derivatives of the components of output error leads to

$$\dot{L}_1 = k_{l_1}\frac{q_2\dot{q}_2 + q_3\dot{q}_3}{(\bar{e}_1 - e_1)} + k_{l_2}\frac{q_1\dot{q}_1 + q_3\dot{q}_3}{(\bar{e}_2 - e_2)} + k_{l_3}\frac{q_1\dot{q}_1 + q_2\dot{q}_2}{(\bar{e}_3 - e_3)} + k_{l_4}\left(q_4 - 1\right)\dot{q}_4$$

The above equation is rearranged and written in terms of quaternion error and its derivative as

$$\dot{L}_1 = \boldsymbol{z}_q^T K_{L_1}\dot{\boldsymbol{z}}_q \tag{7}$$

where the diagonal elements of the positive definite *diagonal matrix* K_{L_1} are

$$\begin{aligned}
[K_{L_1}]_{1,1} =& \frac{k_{l_3}}{\bar{e}_3 - e_3} + \frac{k_{l_2}}{\bar{e}_2 - e_2}\\
[K_{L_1}]_{2,2} =& \frac{k_{l_1}}{\bar{e}_1 - e_1} + \frac{k_{l_3}}{\bar{e}_3 - e_3}\\
[K_{L_1}]_{3,3} =& \frac{k_{l_1}}{\bar{e}_1 - e_1} + \frac{k_{l_2}}{\bar{e}_2 - e_2}\\
[K_{L_1}]_{4,4} =& k_{l_4}
\end{aligned}$$

Utilizing the definitions of terms $\boldsymbol{z}_q$ and $\boldsymbol{z}_\omega$ and the derivative of the quaternion from Eq. (1) leads to

$$\dot{L}_1 = \boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{\xi} + \boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{z}_\omega$$

Designing the stabilizing function $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = -\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q \tag{8}$$

where K_q is a diagonal matrix which takes the form

$$\begin{aligned}
[K_q]_{1,1} =& \frac{k_{q_3}}{\bar{e}_3 - e_3} + \frac{k_{q_2}}{\bar{e}_2 - e_2}\\
[K_q]_{2,2} =& \frac{k_{q_1}}{\bar{e}_1 - e_1} + \frac{k_{q_3}}{\bar{e}_3 - e_3}\\
[K_q]_{3,3} =& \frac{k_{q_1}}{\bar{e}_1 - e_1} + \frac{k_{q_2}}{\bar{e}_2 - e_2}\\
[K_q]_{4,4} =& k_{q_4}
\end{aligned}$$

$k_{q_i},\ i = 1, 2, 3, 4$ denote positive constants. Substituting Eq. (8) results in

$$\dot{L}_1 = -\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q + \boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{z}_\omega \tag{9}$$

The coupling term $\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{z}_\omega$ is canceled in the subsequent step.

Step 2: In this step, a control $\boldsymbol{u}$ that enables the spacecraft to track the desired attitude $\boldsymbol{q}^*$ is designed. This is done by augmenting a Lyapunov function L_1 as

$$L_2 = L_1 + \frac{1}{2}\boldsymbol{z}_\omega^T K_{L_2}\boldsymbol{z}_\omega \tag{10}$$

K_{L_2} is a positive definite diagonal matrix. The time derivative of L_2 is given by

$$\dot{L}_2 = -\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q + \left(\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\} + \dot{\boldsymbol{z}}_\omega^T K_{L_2}\right)\boldsymbol{z}_\omega$$

Upon further simplification, the above equation reduces to

$$\dot{L}_2 = -\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q + \left(\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\} + \left(\boldsymbol{h}\{\boldsymbol{\omega}\} + \boldsymbol{u} - \dot{\boldsymbol{\xi}}\right)^T K_{L_2}\right)\boldsymbol{z}_\omega \tag{11}$$

The control law $\boldsymbol{u}$ is designed as

$$\boldsymbol{u} = -K_{L_2}^{-1}\boldsymbol{g}^T\{\boldsymbol{q}\}K_{L_1}\boldsymbol{z}_q - K_{L_2}^{-1}K_\omega\boldsymbol{z}_\omega - \boldsymbol{h}\{\boldsymbol{\omega}\} + \dot{\boldsymbol{\xi}} \tag{12}$$

where K_ω is a positive definite diagonal matrix. Substituting Eq. (12) into Eq. (11) yields

$$\dot{L}_2 = -\boldsymbol{z}_q^T K_{L_1}\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q - \boldsymbol{z}_\omega^T K_\omega\boldsymbol{z}_\omega \tag{13}$$

By choosing the matrix $K_{L_1} = \mu K_q,\ \mu > 0$ and $\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}$ being positive-semi definite, Eq. (13) leads to

$$\dot{L}_2 = -\left(\boldsymbol{z}_q^T K_q\boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q\right)\mu - \boldsymbol{z}_\omega^T K_\omega\boldsymbol{z}_\omega \le 0 \tag{14}$$

The closed loop error dynamics involving $\boldsymbol{z}_q$ and $\boldsymbol{z}_\omega$ can be summarized as

$$\dot{\boldsymbol{z}}_q = \boldsymbol{g}\left(\boldsymbol{q}\right)\boldsymbol{z}_\omega - \boldsymbol{g}\left(\boldsymbol{q}\right)\boldsymbol{g}^T\left(\boldsymbol{q}\right)K_q\boldsymbol{z}_q \tag{15}$$

$$\dot{\boldsymbol{z}}_\omega = -K_{L_2}^{-1}\boldsymbol{g}^T\left(\boldsymbol{q}\right)K_{L_1}\boldsymbol{z}_{\boldsymbol{q}} - K_{L_2}^{-1}K_\omega\boldsymbol{z}_\omega \tag{16}$$

Theorem 1. Consider the closed loop error dynamics shown in Eqs. (15) and (16), stabilizing function $\boldsymbol{\xi}$ in Eq. (8) and control law $\boldsymbol{u}$ in Eq. (12). Under the assumptions defined previously, the following properties hold.

(1) The quaternion error converges to zero asymptotically. The output errors $e_i(t),\ i = 1, 2, 3, 4$ remain in the compact set defined by

$$\chi_{e_i} := \{e_i \in \mathbb{R} : e_i \le D_{e_i}\}, i = 1, 2, 3$$
$$D_{e_i} = \bar{e}_i\left(1 - e^{-\frac{2L_2(t_0)}{k_{l_i}}}\right)$$
$$\chi_{e_4} := \{e_4 \in \mathbb{R} : e_4 \le \sqrt{\frac{2L_2\left(t_0\right)}{k_{l_4}}}\}$$

where L_2 is the Lyapunov function candidate defined in Eq. (10).

(2) The output components $q_{23}(t), q_{13}(t)$ and $q_{12}(t)$ do not violate the constraints. This implies the attitude errors strictly satisfy $\delta_i < \bar{\delta}_i,\ i = 1, 2, 3$ and all the closed loop signals are bounded.

(3) Without disturbance, the output errors $e_i,\ i = 1, 2, 3, 4$, $\boldsymbol{z}_q$ and $\boldsymbol{z}_\omega$ converge to zero asymptotically. In the presence of a bounded disturbance $\|\boldsymbol{d}\| \le \bar{d}$ in Eq. (2), with an appropriate selection of constants, errors $\boldsymbol{z}_q$ and $\boldsymbol{z}_\omega$ remain bounded and the output tracking errors $e_i,\ i = 1, 2, 3$ stay within the constraints.

Proof 1. (1) From Eq. (14), Lyapunov stability theorem Marquez (2003) ensures the stability and boundedness of L_2 along the error dynamics in Eq. 15. Moreover, it is straightforward to show that $\dot{L}_2 = 0$ implies $(\boldsymbol{z}_q, \boldsymbol{z}_\omega) = (\boldsymbol{0}, \boldsymbol{0})$. Using Krasovskii-LaSalle's theorem Marquez (2003), it implies $\boldsymbol{q} \to \boldsymbol{q}^*$ as $t \to \infty$. According to Lemma 1, the output errors e_i are ensured to satisfy $e_i < \bar{e}_i,\ i = 1, 2, 3\ \forall t > 0$. It follows that $L_2\left(t\right) \le L_2\left(t_0\right)$. This leads to

$$\frac{1}{2}k_{l_i}\log\left(\frac{\bar{e}_i}{\bar{e}_i - e_i(t)}\right) \le L_2\left(t_0\right), i = 1, 2, 3$$

Simplification of the above equation leads to the inequality constraint

$$e_i\left(t\right) \le \bar{e}_i\left(1 - e^{-\frac{2L_2(t_0)}{k_{l_i}}}\right),\ i = 1, 2, 3 \tag{17}$$

Likewise, from Eq. (13)

$$\frac{1}{2}k_{l_4}e_4^2(t) \le L_2(t_0)$$

This leads to the inequality constraint

$$e_4(t) \le \sqrt{\frac{2L_2(t_0)}{k_{l_4}}} \quad (18)$$

(2) Since $q_{23}(t) = e_1(t) + q_{23}^*$, and $e_1(t) \le D_{e_i} < \bar{e}_i$, with $q_{23}^* = 0$ it follows that $q_{23}(t) < \bar{e}_i$ which is the constraint requirement in Eq. (3). A similar analysis is followed for $q_{13}(t)$ and $q_{12}(t)$, respectively. It is clear that the boundedness of $e_i(t),\ i = 1,2,3,4$ implies that the quaternions $q_i(t), i = 1,2,3,4$ remain bounded. Together with the boundedness of $\boldsymbol{z}_\omega(t)$, leads to the fact that the stabilizing function $\boldsymbol{\xi}(t)$ defined in Eq. (8) is bounded. Boundedness of $\boldsymbol{\xi}(t)$ and $\boldsymbol{z}_\omega(t)$ implies that the angular velocity of the spacecraft $\boldsymbol{\omega}(t)$ remains bounded since $\boldsymbol{\omega}(t) = \boldsymbol{z}_\omega(t) + \boldsymbol{\xi}(t)$. Finally the control signal $\boldsymbol{u}$ defined in Eq. (12) made up of the bounded signals as defined above also remains bounded.

(3) In the presence of bounded disturbances the Lyapunov derivative in Eq. (13) gets modified as

$$\dot{L}_2 = -(\boldsymbol{z}_q^T K_q \boldsymbol{g}\{\boldsymbol{q}\}\boldsymbol{g}^T\{\boldsymbol{q}\}K_q\boldsymbol{z}_q)\mu - \boldsymbol{z}_\omega^T K_\omega \boldsymbol{z}_\omega + \boldsymbol{z}_\omega^T K_{L_2}\boldsymbol{d}$$

Using Rayleigh Inequality in Marquez (2003) and Young's inequality in Peajcariaac and Tong (1992), the equation is further simplified as

$$\dot{L}_2 \le -\mu\lambda_{\min}(G(\boldsymbol{q}))\|K_q z_q\|^2 - \lambda_{\min}(K_\omega)\|z_\omega\|^2 + \frac{\sigma}{2}\|\boldsymbol{z}_\omega\|^2 + \frac{1}{2\sigma}\boldsymbol{d}^T K_{L_2}^T K_{L_2}\boldsymbol{d},\ \sigma > 0 \quad (19)$$

$\lambda(\cdot)$ indicates the Eigenvalue of a matrix and $G(\boldsymbol{q})$ corresponds to matrix $\boldsymbol{g}(\boldsymbol{q})\boldsymbol{g}^T(\boldsymbol{q})$. Further using the identity in Ren et al. (2010), within the constraint $0 < e_i < \bar{e}_i,\ i = 1,2,3$

$$\log \bar{e}_i/(\bar{e}_i - e_i) < e_i/(\bar{e}_i - e_i)\ ,\ i = 1,2,3 \quad (20)$$

Equation (19) is further simplified as

$$\dot{L}_2 \le -\lambda_{\min}(G(\boldsymbol{q}))\bar{k}_q L_1 - \frac{2\lambda_{\min}(K_\omega) - \sigma}{2\lambda_{\max}(K_{L_2})}\frac{\boldsymbol{z}_\omega^T K_{L_2}\boldsymbol{z}_\omega}{2} + \frac{\lambda_{max}(K_{L_2}^T K_{L_2})\bar{d}^2}{2\sigma} \quad (21)$$

Here $\bar{k}_q$ indicates the minimum diagonal value of the matrix K_q. The above equation is written as

$$\dot{L}_2 \le -aL_2 + b \quad (22)$$

where

$$a = min\{\lambda_{\min}(G(\boldsymbol{q}))\bar{k}_q, \frac{2\lambda_{\min}(K_\omega) - \sigma}{2\lambda_{\max}(K_{L_2})}\}$$

$$b = \frac{\lambda_{\max}\left(K_{L_2}^T K_{L_2}\right)\bar{d}^2}{2\sigma},\ \sigma > 0$$

For any $\sigma > 0$, there exists an appropriate choice of constant K_ω which results in $a, b > 0$. This leads to boundedness of L_2 (Tang et al. (2016); Liu and Tong (2016)), which implies the output errors $e_i < \bar{e}_i,\ i = 1,2,3$.

4. RESULTS AND DISCUSSION

4.1 Analogy to Traditional Backstepping Integrator

A traditional backstepping controller enables asymptotic tracking of the desired attitude $\boldsymbol{q}^*$. However, it is not capable of enforcing attitude error constraints. The controller uses the following Quadratic Lyapunov Functions (QLFs) during the formulation

$$\tilde{L}_1 = \boldsymbol{z}_q^T \tilde{K}_{L_1}\boldsymbol{z}_q \quad (23)$$

$$\tilde{L}_2 = \tilde{L}_1 + \frac{1}{2}\boldsymbol{z}_\omega^T \tilde{K}_{L_2}\boldsymbol{z}_\omega \quad (24)$$

$\tilde{K}_{L_1}$ and $\tilde{K}_{L_2}$ are positive definite diagonal matrices. The selection of the stabilizing function $\tilde{\boldsymbol{\xi}}$ as well as the controller $\tilde{\boldsymbol{u}}$ follows the same philosophy used in section 3. The resulting QLF based stabilizing function and controller are given as

$$\tilde{\boldsymbol{\xi}} = -\boldsymbol{g}^T\{\boldsymbol{q}\}\tilde{K}_q\boldsymbol{z}_q \quad (25)$$

$$\tilde{\boldsymbol{u}} = -\tilde{K}_{L_2}^{-1}\boldsymbol{g}^T\{\boldsymbol{q}\}\tilde{K}_{L_1}\boldsymbol{z}_q - \tilde{K}_{L_2}^{-1}\tilde{K}_\omega\boldsymbol{z}_\omega - \boldsymbol{h}\{\boldsymbol{\omega}\} + \dot{\tilde{\boldsymbol{\xi}}} \quad (26)$$

$\tilde{K}_q$ and $\tilde{K}_\omega$ are chosen as positive definite diagonal matrices. The BLF and QLF based attitude control laws have multiple matrices that are to be selected as design parameters. In this work, a systematic procedure is developed to construct the matrices. Let the positive definite matrices take the form

$$\tilde{K}_{V_1} = \tilde{\alpha}I_{4\times4},\ \tilde{K}_{V_2} = \tilde{\beta}I_{3\times3},\ \tilde{K}_q = \tilde{k}_q I_{4\times4},\ \tilde{K}_\omega = \tilde{k}_\omega I_{3\times3}$$

Consider a small angular rotation from a given spacecraft attitude $\boldsymbol{q}$ to the desired pointing direction $\boldsymbol{q}^*$. Under this assumption, the vector part of the quaternion and the angular velocity is written in terms of Euler axis $\hat{e}$ and rotation angle θ_e as $[q_1, q_2, q_3]^T = \frac{\theta_e}{2}\hat{e}$ and $\boldsymbol{\omega} = \dot{\theta}_e\hat{e}$, respectively. The control law in Eq. (26) is modified as

$$\tilde{\boldsymbol{u}} = -\frac{\tilde{k}_q\dot{\theta}_e}{4}\hat{e} - \boldsymbol{h}\{\boldsymbol{\omega}\} - \frac{\tilde{\alpha}\theta_e}{4\tilde{\beta}}\hat{e} - \frac{\tilde{k}_\omega\dot{\theta}_e}{\tilde{\beta}}\hat{e} - \frac{\tilde{k}_\omega\tilde{k}_q\theta_e}{4\tilde{\beta}}\hat{e} \quad (27)$$

Upon substituting the simplified control law into Eq. (2) results in a closed loop dynamics which has a second-order LTI form

$$\left(\ddot{\theta}_e + \left(\frac{\tilde{k}_\omega}{\tilde{\beta}} + \frac{\tilde{k}_q}{4}\right)\dot{\theta}_e + \left(\frac{\tilde{k}_\omega\tilde{k}_q}{4\tilde{\beta}} + \frac{\tilde{\alpha}}{4\tilde{\beta}}\right)\theta_e\right)\hat{e} = 0 \quad (28)$$

Upon substituting $\tilde{\beta} = 1$ and $\tilde{\alpha} = \mu\tilde{k}_q$, the damping constant ζ and natural frequency ω_n of the system that are related to settling time $t_s = \frac{4}{\zeta\omega_n}$ and are used to obtain the remaining constants $\tilde{k}_q$ and $\tilde{k}_\omega$. Once the constants for the QLF based control law are calculated, the controller constants for the BLF based controller are subsequently selected as

$$k_{q_i} = \frac{\tilde{k}_q\bar{e}_i}{2}\text{for } i = 1,2,3,\ k_{q_4} = \tilde{k}_q \quad (29)$$

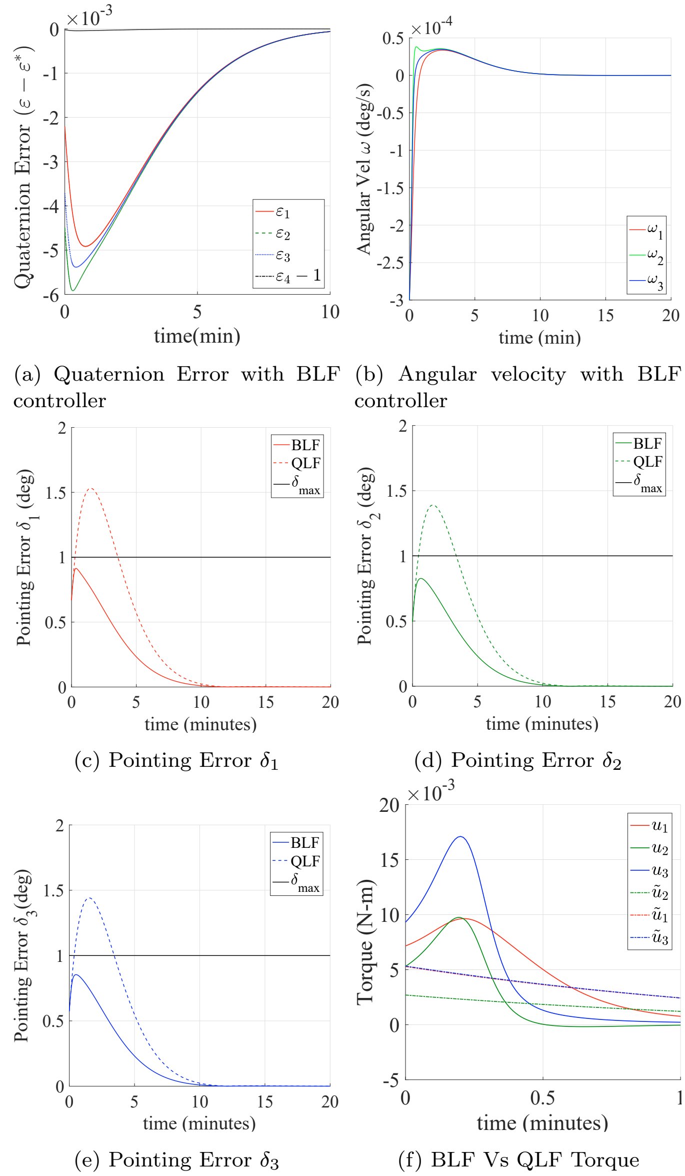
$$K_\omega = \tilde{k}_\omega I_{3\times3},\ K_{V_1} = \mu K_q,\ K_{v_2} = I_{3\times3} \quad (30)$$

Note that the selection of constants for the BLF based controller are done in a way that when $e_i,\ i = 1,2,3$ are set to zero, the controller in Eq. (12) becomes equivalent to Eq. (26) i.e., the QLF based controller.

4.2 Simulations with Initial Condition Variation

Consider a rigid body with principal moment of inertia values $I_{11} = 1000\ kg - m^2, I_{22} = 500\ kg - m^2, I_{33} = 1000\ kg - m^2$ and product of inertias $I_{12} = -5\ kg - m^2, I_{13} = -20\ kg - m^2, I_{23} = -30\ kg - m^2$. In an ideal scenario, the body frame $\mathcal{B}$ must be perfectly aligned to the desired attitude frame $\mathcal{D}$. But due to various external disturbances, the attitude accuracy may be lost, resulting in an error in the ideal attitude states. A BLF based attitude control law in Eq. (12) will be used to drive the attitude errors back to zero, while simultaneously preventing attitude constraint violation. In the following scenario, an attitude pointing error accuracy of $\bar{\delta}_i = 1^{\circ}$ is considered, i.e., during the control, the angle between $\hat{b}_i$ and $\hat{d}_i$, $i = 1, 2, 3$ is constrained to be less than 1°. Multiple attitude states are generated by individually perturbing $\boldsymbol{q}^*$ and $\boldsymbol{\omega}^*$ such that the initial conditions lie within the constraint boundary. Using a settling time of 10 minutes, damping constant value 0.8 and μ value 0.005, the necessary control constants are calculated using the equations in Section (4.1). Application of the BLF based control law for a particular initial condition results in an attitude error profile shown in Fig. 2. Using the same values of settling time and damping constant, a QLF based control law results in constraint violation in attitude errors along all the three rigid body axes. It is observed that the magnitude of control torques generated by the BLF based controller is larger that its counterpart. This is due to the fact that the control gains are dependent on the attitude error and hence increases as the attitude errors reach close to the constraint value. Figure 3 indicates the application of the BLF and QLF based control laws for multiple initial conditions. It is observed that using the BLF based control law, the pointing errors asymptotically go to zero without violating the pointing constraint.

In the next scenario, the performance of both the control laws in the presence of a bounded disturbance torque of $10^{-5}\ N - m$ is studied. In section 3 it was proved that under a bounded external disturbance, the BLF based controller ensures the attitude errors to lie within the constraint. The same is observed in Fig. 4. On the other hand, the attitude errors using the QLF based controller settle at a value beyond the constraint.



(a) Quaternion Error with BLF controller (b) Angular velocity with BLF controller

(c) Pointing Error δ_1 (d) Pointing Error δ_2

(e) Pointing Error δ_3 (f) BLF Vs QLF Torque

Fig. 2. Comparison of evolution of attitude errors using BLF and QLF control laws

5. CONCLUSION

In this paper, a BLF based attitude control law is designed for a rigid body using backstepping philosophy. The controller ensures an asymptotic reduction in the output error while simultaneously satisfying the attitude error constraints. Violation of the constraint is avoided by ensuring the boundedness of the BLFs in the closed loop Lyapunov analysis. The BLF based controller outperforms the QLF based traditional backstepping controller by ensuring the constraints are strictly satisfied in the presence of initial attitude errors as well as bounded disturbance torques.

REFERENCES

Liu, Y.J. and Tong, S. (2016). Barrier lyapunov functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints. *Automatica*, 64, 70–75.

Marquez, H.J. (2003). *Nonlinear control systems: analysis and design*, volume 1. Wiley-Interscience Hoboken, NJ.

Ngo, K.B., Mahony, R., and Jiang, Z.P. (2005). Integrator backstepping using barrier functions for systems with multiple state constraints. In *Proceedings of the 44th IEEE Conference on Decision and Control*, 8306–8312. IEEE.

Peajcariaac, J.E. and Tong, Y.L. (1992). *Convex functions, partial orderings, and statistical applications*. Academic Press.

Ren, B., Ge, S.S., Tee, K.P., and Lee, T.H. (2010). Adaptive neural control for output feedback nonlinear systems using a barrier lyapunov function. *IEEE Transactions on Neural Networks*, 21(8), 1339–1345.

Sachan, K. and Padhi, R. (2018). Barrier lyapunov function based state-constrained control for a class of nonlinear systems. *IFAC-PapersOnLine*, 51(1), 7–12.

Slotine, J.J.E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.

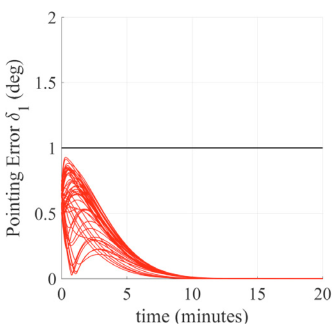
Tang, Z.L., Ge, S.S., Tee, K.P., and He, W. (2016). Robust adaptive neural tracking control for a class of perturbed uncertain nonlinear systems with state constraints. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 46(12), 1618–1629.

Tee, K.P. and Ge, S.S. (2009). Control of nonlinear systems with full state constraint using a barrier lyapunov function. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, 8618–8623. IEEE.

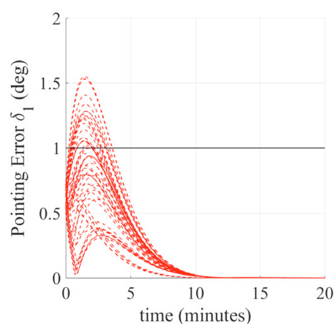
Tee, K.P. and Ge, S.S. (2011). Control of nonlinear systems with partial state constraints using a barrier lyapunov function. *International Journal of Control*, 84(12), 2008–2023.

Tee, K.P., Ge, S.S., and Tay, E.H. (2009). Barrier lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, 45(4), 918–927.

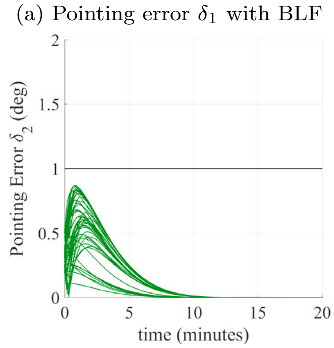
Tee, K.P., Ren, B., and Ge, S.S. (2011). Control of nonlinear systems with time-varying output constraints. *Automatica*, 47(11), 2511–2516.



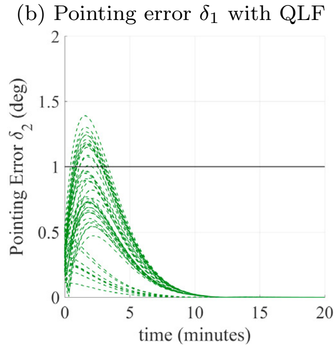
(a) Pointing error δ_1 with BLF



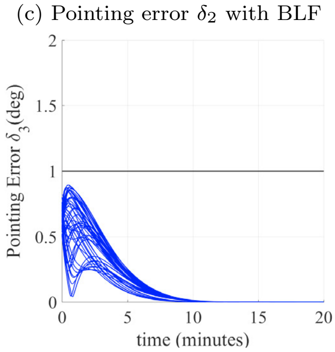
(b) Pointing error δ_1 with QLF



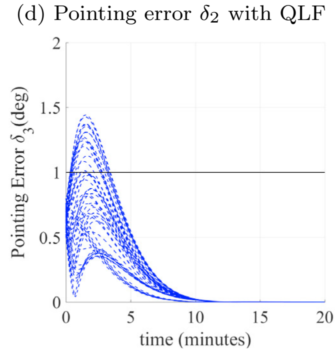
(c) Pointing error δ_2 with BLF



(d) Pointing error δ_2 with QLF

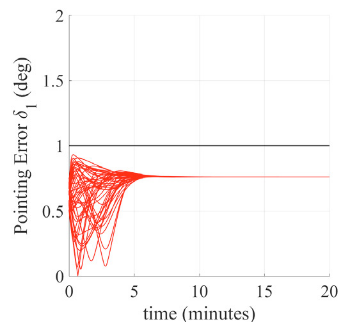


(e) Pointing error δ_3 with BLF

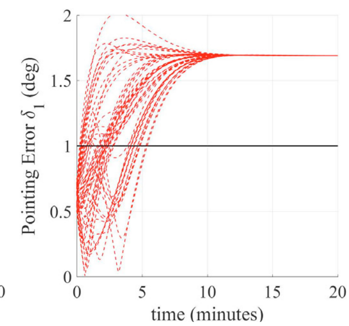


(f) Pointing error δ_3 with QLF

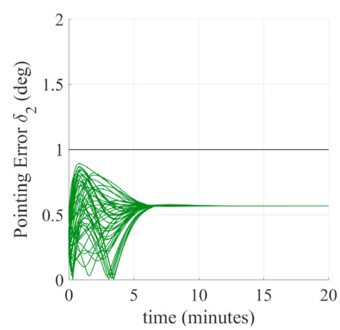
Fig. 3. Comparison of attitude errors without disturbance



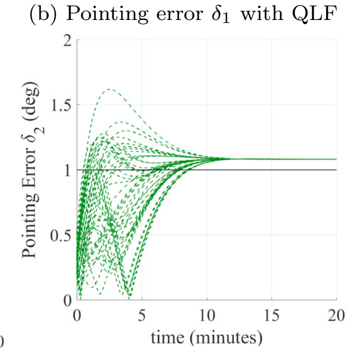
(a) Pointing error δ_1 with BLF



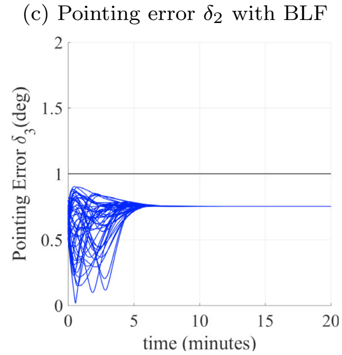
(b) Pointing error δ_1 with QLF



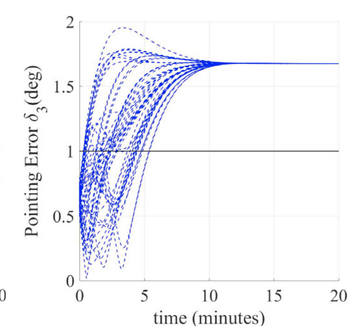
(c) Pointing error δ_2 with BLF



(d) Pointing error δ_2 with QLF controller



(e) Pointing error δ_3 with BLF



(f) Pointing error δ_3 with QLF

Fig. 4. Comparison of attitude errors with bounded disturbance