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On the Second Lyapunov Method for Quaternionic Differential Equations

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Abstract
In this paper we mainly study the stability of quaternion-valued differential equations. We first establish the second Lyapunov method over quaternion field and then we use this theory to study the stability in both autonomous systems and periodic non-autonomous systems cases.

Keyword Quaternion differential equation · The second Lyapunov method · Stability · Periodic solution

Mathematics Subject Classification 34D08 · 20G20 · 34K23 · 37C60

1 Introduction

The theory of stability for real or complex differential equations has been deeply studied from many different points of view, while because of the non-commutativity of quaternions, the study on quaternion-valued differential equations (QDEs) becomes more difficult and sophisticated. At present the study on quaternion-valued equations has gained considerable attention because of its widely applications in many fields, such as quantum and fluid mechanics,see e.g. [1–6], spatial kinematic modelling and attitude dynamics, see e.g. [7,8], etc.

In the analysis of stability properties of nonlinear systems it is very often useful to employ what is now called the second Lyapunov method or the method of Lyapunov functions(also called V-function), this method was originally developed for studying the stability of fixed points for autonomous or non-autonomous differential equations. It was then extended from fixed points to sets, from differential equations to dynamical systems, it was the most widely used technique to analyze the stability of various dynamic systems, including differential equations, hybrid or stochastic differential

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equations, and many other equations, and the Lyapunov function can also be used in the study of automatic regulation system, optimal control and limit cycle.

In 2003, De Leo and Ducati [9] prove the existence and uniqueness for quaternionic intial value problems. Then Campos and Mawhin [10] have initiated a study of the T-periodic solutions of quaternion-valued first order differential equations. Later, Wilczyński [11,12] continued this study and proved the existence of at least two periodic solutions of quaternionic Riccati equation and the existence of at least one periodic solutions of the quaternionic polynomial equations. Gasull [13] demonstrated the dynamics of real four-dimensional polynomial differential equations of the forms $\dot{q} = aq^k\overline{q}^m$ or $\dot{q} = q^k aq^m$ and proved the existence of periodic orbits, homoclinic loops, invariant tori. And Zhang [14] is devoted to the global structure of the quaternion autonomous homogeneous Bernoulli equations [15]. Studied the polynomial differential equations over the quaternions [16]. Gave and proved the Cauchy matrix and liouville formula for homogeneous and nonhomogeneous quaternion impulsive dynamic equations and obtained the existence, uniqueness and expression of their solutions. By decomposing the quaternion-valued neural networks into four real-valued system and combine with the real-valued Lyapunov function method, Liu et al. [17] studied the μ-stability and power-stability for the considered quaternion-valued neural networks [18]. Presented an algorithm to evaluate the fundamental matrix and gave a method to construct the fundamental matrix when the linear system has multiple eigenvalues. And the basic theory of linear QDEs such as the Wronskian and the algebraic structure of solutions was studied by Kou and Xia [19]. After that, [20] studied the Floquet theory for QDEs with periodic coefficients and the stability of quaternionic periodic differential equations.

To the best of our knowledge, there is no paper is trying to study the stability of QDEs by the second Lyapunov method, in view of this and its wide range of applications, we go one step further and develop the second Lyapunov method for QDEs. In this paper we mainly study the stability of autonomous system and periodic non-autonomous systems, we establish the second Lyapunov method for quaternion-valued differential equations, in what follows Lyapunov stability, asymptotic stability and instability theorems are obtained, and we prove the first approximation method of stability, then we use this method to analysis the stability of autonomous systems and periodic non-autonomous systems. As a natural generalization of ODEs and due to the non-commutativity of quaternions, we may confront various challenges when studying QDEs: First, a quaternion matrix has exactly n (right) eigenvalues, but the set of all eigenvectors corresponding to a non-real eigenvalue is not a module; Second, the definition of determinant for quaternion matrices are more complicated and the Cramer's rules for quaternion matrix equations is more strict; Third, as for Quaternion-valued functions, their analytical properties are not very good, they can not compare numerical values, and they do not have monotony. All these leads to the difficulties of the research, therefore we need to revise the definition of V function. Different from general Lyapunov equation, we should give more severe and rigorous conditions for the solvability of the Lyapunov equation on quaternion filed.

The remaining part of the paper proceeds as follows. In Sect. 2, we collect some basic facts and theorems concerning quaternions and quaternionic matrices. Section 3 is devoted to the second Lyapunov method and the stability of autonomous systems. In

Sect. 4, we study the stability of periodic solution of period non-autonomous systems. At last, we give some examples in Sect. 5.

2 Preliminaries

Quaternions are simple hypercomplex, which were first described by Hamilton in 1843 [21], The algebra of quaternions is often denoted by

$$\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where i,j,k satisfying the multiplication table formed by

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The addition of $q, p \in \mathbb{H}$ is given by

$$q + p = (q_0 + p_0) + (q_1 + p_1)i + (q_2 + p_2)j + (q_3 + p_3)k$$

and their multiplication is defined by expending the product

$$\begin{aligned} qp &= (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) \\ &= (q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3) + (q_1 p_0 + q_0 p_1 - q_3 p_2 + q_2 p_3)i \\ &\quad + (q_2 p_0 + q_3 p_1 + q_0 b_2 - q_1 p_3)j + (q_3 p_0 - q_2 p_1 + q_1 p_2 + q_0 p_3)k. \end{aligned}$$

We can define the inner product, and the norm, respectively, by

$$\begin{aligned} \langle q, p \rangle &= q_0 p_0 + q_1 p_1 + q_2 p_2 + q_3 p_3, \\ \|q\| &= (\langle q, q \rangle)^{1/2} = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}. \end{aligned}$$

For every quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$, we introduce the *scalar or real part*

$$\mathcal{R}(q) = q_0,$$

as well as the *vector or imaginary part* by

$$\mathcal{V}(q) = q_1 i + q_2 j + q_3 k.$$

If $q = \mathcal{V}(q)$, we call it *pure imaginary quaternion.* In general the multiplication of two quaternions p and q is not commutative but its projection on the real part satisfies, i.e.

$$\mathcal{R}(pq) = \mathcal{R}(qp).$$

However this is not true for three or more quaternions, for example

$$\mathcal{R}(ijk) = -1 \neq 1 = \mathcal{R}(jik),$$

but we can prove that for any $r \in \mathbb{H}$

$$\mathcal{R}(pqr) = \mathcal{R}(qrp) = \mathcal{R}(rpq),$$

i.e. only cyclic permutations are satisfied. The *conjugate* is defined by

$$\overline{q} = \mathcal{R}(q) - \mathcal{V}(q) = q_0 - q_1 i - q_2 j - q_3 k,$$

if $q \neq 0$, then $q^{-1} = \dfrac{\overline{q}}{\|q\|^2}$.

Let $\mathbb{H}^{n\times n}$ denote the collection of all $n \times n$ matrices with quaternion entries and $\mathbb{H}^n$ denote the collection of all $n \times 1$ quaternion column vectors.

For any $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{H}^n$, the norm of A and α are

$$\|A\| = \sum_{i,j=1}^{n} \|a_{ij}\|, \qquad \|\alpha\| = \sum_{j=1}^{n} \|\alpha_j\|.$$

The exponential of $A \in \mathbb{H}^{n\times n}$ is defined by

$$\exp(A) = \sum_{i=1}^{\infty} \frac{A^n}{n!} = E + \frac{A}{1!} + \frac{A^2}{2!} + \cdots .$$

Just like complex matrices, we denote $\overline{A}$, the *conjugate* of A, A^T the *transpose* of A, and $A^* = (\overline{A})^T$, the *conjugate transpose* of A.

We say A is *unitary* if $AA^* = A^*A = I$, where I is the identity matrix; and is *invertible* if $AB = BA = I$ for some $B \in \mathbb{H}^{n\times n}$. A list of facts follows immediately: $(AB)^* = B^*A^*$ and $(A^{-1})^* = (A^*)^{-1}$.

Now we introduce the definition of quaternion matrix determinant and some useful results which can be found in [22,23].

Definition 2.1 [22,23] The ith row determinat of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined by

$$rdet_i A = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \cdots a_{i_{k_1+l_1} i} \cdots a_{i_{k_r} i_{k_r+1}} \cdots a_{i_{k_r+l_r} i_{k_r}}$$

for all $i = 1, 2, \ldots, n$, where r is the rank of A, S_n is the symmetric group on the set $I_n = \{1, 2, \ldots, n\}$, the elements of the permutation σ are indices of each monomial. The left-ordered cycle notation of the permutation σ is written as follows,

$$\sigma = (i i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1} i)(i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r}).$$

The index i opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_1} < i_{k_2} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

Definition 2.2 [22,23] The determinant of corresponding Hermitian matrix of $A \in \mathbb{H}^{n\times n}$ is called its double determinant, i.e.

$$ddet A = det A^*A = det AA^* = rdet(A^*A).$$

By this definition, we have the following Cramer's rule for quaternion matrix equations.

Theorem 2.1 [22,23] *Suppose $AX = B$ is a right matrix equation where $A, B \in \mathbb{H}^{n\times n}$ are given, $X \in \mathbb{H}^{n\times n}$ is unknown, if $ddet A \neq 0$, then this equation has a unique solution.*

The following result can be found in e.g. [20,24–26].

Theorem 2.2 [20,24–26] *Let $A \in \mathbb{H}^{n\times n}$ then the following statements hold.*

(i) A has exactly n (right) eigenvalues which are complex numbers with non-negative imaginary parts. These eigenvalues are called standard eigenvalues of A.

(ii) If A is in triangular form, then every diagonal element is a (right)eigenvalue of A.

(iii) There exists a invertible matrix P, such that $P^{-1}AP = J \in \mathbb{C}^{n\times n}$ is a triangular matrix.

(iv) $C \in \mathbb{H}^{n\times n}$ is invertible, then there exists $B \in \mathbb{H}^{n\times n}$, such that $e^B = C$.

(v) If $A \in \mathbb{H}^{n\times n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the standard eigenvalues of A repeated according to their multiplicity, then $e^{\widetilde{\lambda_1}}, e^{\widetilde{\lambda_2}}, \ldots, e^{\widetilde{\lambda_n}}$ are the standard eigenvalues of e^A, where $\widetilde{\lambda_j}, (j = 1, 2, \cdots, n)$ is defined by

$$\widetilde{\lambda_j} = \begin{cases} \lambda_j, & \textit{if } e^{\lambda_j} \textit{ has nonnegative imaginary part;} \\ \overline{\lambda_j}, & \textit{otherwise.} \end{cases}$$

3 Stability of Autonomous Systems

We know Lyapunov had proposed two methods to study the stability of nonlinear systems for ordinary differential equations, . The first method (indirect method) is to use the series expansion method to find the general solution and then judge the stability. The second method (direct method, V-function method) does not need to solve the special solution or general solution of the differential equation, but seeks some special auxiliary functions and uses it to judge the stability. In this section, we are devoted to studying the second Lyapunov method and the stability of quaternion autonomous system

$$\dot{q} = f(q), \tag{3.1}$$

where $q \in \mathbb{H}^n$, $f(q)$ is a $\mathbb{H}^n$-valued function.

3.1 The Second Lyapunov Second Method for Quaternion Autonomous Systems

For the second Lyapunov method of quaternion autonomous system, the special auxiliary quaternion functions $V(q)$ are very important, and we start with several definitions.

Definition 3.1 Let $\Omega \subset \mathbb{H}^n$, $V : \mathbb{H}^n \to \mathbb{R}$ is continuous and $V(0) = 0$, if for any $q \in \Omega \backslash \{0\}$, $V(q) > 0(< 0)$, then V(q) is called definite positive function (definite negative function). Similarly, if for any $q \in \Omega$, $V(q) \geq 0(\leq 0)$, then $V(q)$ is called semi-definite positive function (semi-definite negative function).

Quaternion quadratic form is the simplest and most commonly used V-function, moreover, some complicated V-functions are also obtained on the basis of quaternion quadratic form.

Definition 3.2 [24] Let $A \in \mathbb{H}^{n\times n}$, $A^* = A$ and $q \in \mathbb{H}^n$, we say that $f(q) = q^*Aq$ is a *quaternion quadratic form*. Moever if for any $q \in \mathbb{H}^n \backslash \{0\}$, we have $f(q) = q^*Aq > 0(< 0)$, then we say $f(q)$ is a *positive definite quadratic form*(*negative definite quadratic form*). If for any $q \in \mathbb{H}^n$, we have $f(q) = q^*Aq \geq 0(\leq 0)$, then we say $f(q)$ is a *positive semi-definite quadratic form* (*negative semi-definite quadratic form*).

The definition of stability for QDEs is as follows.

Definition 3.3 We say the solution $\Phi(t) \in \mathbb{H}^n$ with initial value $\Phi(t_0) = q_0$ of quaternion syetem (3.1) is stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ (δ is generally related to ε and t_0) such that when $\|q - q_0\| < \delta$, $\|\Phi(t, t_0, q) - \Phi(t, t_0, q_0)\| < \varepsilon$ holds for all t $\geq t_0$. $\Phi(t)$ is called asymptotically stable if the solution $\Phi(t, t_0, q_0)$ is stable and there exists δ_0 such that when $\|q - q_0\| < \delta_0$,

$$\lim_{t\to\infty} \|\Phi(t, t_0, q) - \Phi(t, t_0, q_0)\| = 0.$$

Moreover, when we apply the second Lyapunov method, we should not only consider the function $V(q)$, but also the derivative of V to t.

Definition 3.4 Suppose that $q = q(t)$ is the solution of quaternion autonomous system (3.1) Take it into $V = V(q)$ and take the derivative of t

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} = \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \frac{\mathrm{d}q_k}{\mathrm{d}t}. \tag{3.2}$$

We call (3.2) the total derivative of V to t along the system (3.1). When there is no misunderstanding, we abbreviate it as dV/dt.

Let $f = (f_1, f_2, \ldots, f_n)^T$, from Eqs. (3.1) and (3.2) we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \cdot f_k, \tag{3.3}$$

or write it in the form of inner product

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \langle \nabla V, f \rangle,$$

where

$$\nabla V = (\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \dots, \frac{\partial V}{\partial q_n})^T$$

is called the gradient of V. From equation (3.3) we know dV/dt can be obtained without knowing the solution of system (3.1). Because of this characteristic, in the second Lyapunov method, it is not necessary to solve the differential equation, but directly judge the stability of the zero solution according to the sign property of the special V function and its total derivative to t.

Now we have the stable, asymptotically stable and unstable theorem for the second Lyapunov method for quaternion autonomous systems.

Theorem 3.1 (Stable and asymptotically stable theorem) *Let $\Omega \subset \mathbb{H}^n$ be the neighborhood of origin,*

(1) if there exists a definite positive (definite negative) quaternion function $V(q) \in \Omega$ and the total derivative of V to t along the system (3.1) is semi-definite negative (semi-definite positive), then the zero solution of system (3.1) is stable;

(2) if there exists a definite positive (definite negative) quaternion function $V(q) \in \Omega$ and the total derivative of V to t along the system (3.1) is definite negative (definite positive), then the zero solution of system (3.1) is asymptotically stable.

Proof (1) We prove that V is definite positive and dV/dt is semi-definite negative, the other case can be proved analogously.

Let $\varepsilon > 0$ is small enough such that the sufficient small neighborhood of the origin $\Omega_\varepsilon = \{q : \|q\| < \varepsilon\}$ is all contained in Ω, and let

$$c = \min_{q \in \partial\Omega_\varepsilon} V(q).$$

Since V is definite positive function, so $c > 0$, $V(q)$ is continuous and $V(0) = 0$, then there exists $0 < \delta < \varepsilon$ such that when $\|q\| < \delta, \quad 0 \le V(q) < c$. We denote this region by $\Omega_\delta = \{q : \|q\| < \delta\}$, and dV/dt is semi-definite negative, so the solution of function V along system (3.1) does not increase, therefore when $t = t_0$, any solution $q = q(t)$ which has the initial value $q(t_0) = q_0 \in \Omega$ satisfies

$$V(q(t)) - V(q(t_0)) = \int_{t_0}^{t} \frac{\mathrm{d}V(q(t))}{\mathrm{d}t} dt \le 0,$$

i.e.

$$V(q(t)) \le V(q_0) < c.$$

Since c is the minimum value of V in $\partial\Omega_\varepsilon$, then $q = q(t)$ won't arrive at $\partial\Omega_\varepsilon$ when $t > t_0$ This means that it won't leave Ω_ε. Thus we have that $\|q(t)\| < \varepsilon$, and the zero solution of system (3.1) is stable.

Analogously, if V is definite negative and dV/dt is semi-definite positive, we can get the same result.

(2) Since dV/dt is definite negative, the zero solution of system (3.1) is obviously stable. We take δ determined in the proof (1) as δ_0, i.e. take $\delta_0 = \delta$, hence when $\|q\| < \delta_0$, $\|q(t)\| < \eta$ holds for all $t \geq t_0$ where $\eta > \varepsilon$. In order to prove $\lim_{t\to\infty} \|q(t)\| = 0$, we first need to prove

$$\lim_{t\to\infty} V(q(t)) = 0$$

Since dV/dt is a definite negative function, $V(q(t))$ is decreasing with respect to t, so there is a limit

$$\lim_{t\to\infty} V(q(t)) = c.$$

If $c \neq 0$, then for any $t \geq t_0$ we have $V(q(t)) > c$. And because $V(q)$ is continuous, definite positive and $V(0) = 0$, therefore there exists a $\lambda > 0$ such that for any $t \geq t_0$ we have $\|q(t)\| > \lambda$. If we choose

$$m = \sup_{\lambda \leq \|q\| \leq \eta} \frac{\mathrm{d}V(q(t))}{\mathrm{d}t},$$

since dV/dt is definite negative, hence $m < 0$, it follows from that

$$V(q(t)) - V(q_0) = \int_{t_0}^{t} \frac{\mathrm{d}V(q(t))}{\mathrm{d}t} dt \leq m(t - t_0).$$

i.e.

$$V(q(t)) \leq V(q_0) + m(t - t_0).$$

When t increases continuously, the above formula makes $V(q(t))$ become negative, this is in contradiction with the fact that $V(q(t))$ is definite positive. Consequently c must be zero, then $\lim_{t\to\infty} V(q(t)) = 0$ holds.

Now we prove $\lim_{t\to\infty} \|q(t)\| = 0$ by using reduction to absurdity. Since the zero solution is stable, $q(t)$ is bounded, thus there exists a sequence $\{t_k\}$ $(k = 1, 2, \ldots)$ when $k \to \infty$, $t_k \to \infty$ such that

$$\lim_{k\to\infty} \|q(t_k)\| = \|q^*\| \neq 0.$$

So according to the fact that $V(q)$ is continuous and definite positive, we have

$$\lim_{k\to\infty} V(q(t_k)) = V(q^*) \neq 0,$$

this is contradict with $\lim_{t\to\infty} V(q(t)) = 0$, hence $\lim_{t\to\infty} \|q(t)\| = 0$ holds. This implies the zero solution of system (3.1) is asymptotically stable. □

Besides, we have the following instability theorem.

Theorem 3.2 (Unstable theorem) *Let $\Omega \subset \mathbb{H}^n$ be the neighborhood of origin,*

(1) if there exists a continuous differentiable quaternion function $V(q) \in \Omega$, $V(0) = 0$, and in any neighborhood of origin, $V(q)$ can always take positive values (negative values), i.e. $V(q)$ is not semi-definite negative (semi-definite positive), and the total derivative of V to t along the system (3.1) is definite positive (definite negative), then the zero solution of system (3.1) is unstable;

(2) if there exists a continuous differentiable quaternion function $V(q) \in \Omega$, $V(0) = 0$, and in any neighborhood of origin, $V(q)$ can always take positive values (negative values), i.e. $V(q)$ is not semi-definite negative (semi-definite positive), and the total derivative of V to t along the system (3.1) can be written as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \mu V(q) + U(q),$$

where μ is positive constant, $U(q) = 0$ or is a semi-definite positive (semi-definite negative), then the zero solution of system (3.1) is unstable.

Proof (1) Take $\varepsilon > 0$ is small enough such that the sufficient small neighborhood of the origin $\Omega_\varepsilon = \{q : \|q\| < \varepsilon\}$ is all contained in Ω. Let dV/dt be positive in Ω_ε, from assumption of this theorem (1), we know that no matter how small $\delta > 0(\delta < \varepsilon)$ would be, we can always find q_0 such that $0 < \|q_0\| < \delta$, $V(q_0) > 0$.

We need to prove the solution $q = q(t)$ with initial value $q(t_0) = q_0$ at some time $t > t_0$ is not in Ω_ε, therefore the zero solution of system (3.1) is unstable.

We prove it by reduction to absurdity. Suppose that the solution $q = q(t)$ is always in Ω_ε when $t > t_0$. Since dV/dt is positive in Ω_ε, so $V(q(t))$ increases with respect to t, i.e. for $t > t_0$ we have

$$V(q(t)) \geq V(q_0) > 0,$$

then there exists $\lambda > 0$, such that when $t > t_0$

$$\lambda \leq \|q(t)\| \leq \varepsilon.$$

If we choose

$$a = \inf_{\lambda \leq \|q\| \leq \varepsilon} \frac{\mathrm{d}V(q(t))}{\mathrm{d}t},$$

since dV/dt is positive, hence $a > 0$, it follows from that

$$V(q(t)) - V(q_0) = \int_{t_0}^{t} \frac{\mathrm{d}V(q(t))}{\mathrm{d}t} dt \geq a(t - t_0),$$

i.e.

$$V(q(t)) \geq V(q_0) + a(t - t_0).$$

When t goes to infinity, the right hand of above formula goes to infinity, hence $V(q(t))$ goes to infinity; But $V(q)$ is continuous in Ω_ε, so it is bounded, this is contradict. Therefore the zero solution of system (3.1) is unstable, this has completed the proof. □

(2) We prove that $U(q)$ is semi-definite positive and $V(q)$ can always take positive values (other cases can be proved analogously).

Similar to (1), consider the solution $q = q(t)$ with initial value $q(t_0) = q_0$, we need to prove at some time $t > t_0$, $q(t)$ is not in Ω_ε.

We prove it by reduction to absurdity. Suppose that the solution $q = q(t)$ is always in Ω_ε when $t > t_0$, since $U(q) \geq 0$ in Ω_ε, so from this theorem (2) we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} \geq \mu V(q).$$

Also because $V(q_0) > 0, \quad \mu > 0$, then for $t > t_0$, $V(q(t)) > V(q_0) > 0$. Integrate above formula from t_0 to t, we have

$$V(q(t)) \geq V(q_0)e^{\mu(t-t_0)}.$$

Hence when t goes to infinity, $V(q(t))$ goes to infinity,. This is contradict with that $V(q)$ is bounded in Ω_ε. From this we can know that in some time $t > t_0$, $q(t)$ is not in Ω_ε, therefore the zero solution of system (3.1) is unstable.

We can employ these theorems to judge the stability of quaternion autonomous systems and we do not need to solve the special solution or general solution of the quaternion differential equation, but to find the special auxiliary function V and use it to judge the stability. This method is called the second Lyapunov method (or direct method, V-function method). When using the second Lyapunov method, the key is to construct appropriate V-function to meet the requirements of corresponding theorems. The function we used in the Lyapunov second method usually called Lyapunov function.

3.2 The Existence of Lyapunov Functions for Quaternion Autonomous System

Since many Lyapunov functions of nonlinear system can be obtained from linear systems, hence we consider Lyapunov functions of linear quaternion autonomous system

$$\dot{q} = Aq, \tag{3.4}$$

where $A = [a_{ij}] \in \mathbb{H}^{n\times n}$ and $q \in \mathbb{H}^n$.

Let V take the form of quaternion quadratic function $V(q) = q^*Bq$, and B is a quaternion self-conjugate matrix, the derivative of the solution of V along the system (3.4) to t is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = q^*Cq \triangleq W(q), \tag{3.5}$$

where C is given by equation

$$A^*B + BA = C. \tag{3.6}$$

Obviously, C is also a self-conjugate matrix. Equation (3.6) gives the relationship between matrix B and C, it is called *Lyapunov equation on quaternion field*. We want to determine the matrix B according to the given matrix C such that V is the required Lyapunov function.

First, we need the solvability condition of equation (3.6).

Lemma 3.1 *Suppose that constant quaternion A has n eigenvalues which are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with non-negative imaginary parts, for any given self-conjugate matrix C, equation (3.6) has a unique self-conjugate matrix solution B if and only if*

$$\overline{\lambda_i} + \lambda_j \neq 0, \quad (i, j = 1, 2, \cdots, n) \tag{3.7}$$

Proof From Theorem 2.2 we know there exists a non-singular matrix $P \in \mathbb{H}^{n\times n}$ such that

$$P^{-1}AP = K = [k_{ij}] \in \mathbb{C}^{n\times n} \tag{3.8}$$

is Jordan canonical, and the element on its main diagonal is the eigenvalue of A. Without losing generality, suppose they can be ordered from top to bottom as $\lambda_1, \lambda_2, \ldots, \lambda_n$. From Eqs. (3.6) and (3.8) we have

$$K^*P^*BP + P^*BPK = P^*CP. \tag{3.9}$$

Let $P^*BP = D$, $P^*CP = Q$, then equation (3.9) becomes

$$K^*D + DK = Q.$$

Since B and C are self-conjugate, then D and Q are also self-conjugate, therefore we can denote

$$K = \begin{pmatrix} \lambda_1 & k_2 & & \\ & \lambda_2 & k_3 & \\ & & \ddots & k_n \\ & & & \lambda_n \end{pmatrix}, \quad K^* = \begin{pmatrix} \overline{\lambda_1} & & & \\ k_2 & \overline{\lambda_2} & & \\ & & \ddots & \\ & & k_n & \overline{\lambda_n} \end{pmatrix},$$

$$D=\begin{pmatrix} d_{11} & d_{12}\cdots & d_{1n}\\ \overline{d_{12}} & \ddots\ \ \cdots & d_{2n}\\ \vdots & \cdots\ \ \ddots & \vdots\\ \overline{d_{1n}} & \overline{d_{2n}}\ \ldots & d_{nn}\end{pmatrix},\quad Q=\begin{pmatrix} q_{11} & q_{12}\cdots & q_{1n}\\ \overline{q_{12}} & \ddots\ \ \cdots & q_{2n}\\ \vdots & \cdots\ \ \ddots & \vdots\\ \overline{q_{1n}} & \overline{q_{2n}}\ \ldots & q_{nn}\end{pmatrix},$$

where $k_i=0$ or 1. It can be written in the form of the following equations

$$\begin{cases} k_j d_{1j-1}+(\overline{\lambda_1}+\lambda_j)d_{1j}=q_{1j},\\ (\overline{\lambda_i}+\lambda_j)d_{ij}+k_i d_{i-1j}+k_j d_{ij-1}=q_{ij},\end{cases}\tag{3.10}$$

where $i=2,\ldots,n$; $j=1,2,\ldots,n$. Here we denote $k_1=d_{i0}=0$, because the coefficient matrix M of equation (3.10) is lower triangular, and its double determinant is $|M|=ddet M=rdet M^*M$. We notice that K is in $\mathbb{C}^{n\times n}$ and $\lambda_i(i=1,2,\ldots,n)$ is also in $\mathbb{C}$, so is the coefficient matrix M. And when M is a complex matrix we have $rdet M^*M=rdet M^*\cdot rdet M$, therefore

$$|M|=[2^n Re\lambda_1 Re\lambda_2\cdots\lambda_n\prod_{i<j}(\overline{\lambda_i}+\lambda_j)]^2=[\prod_{i\le j}(\overline{\lambda_i}+\lambda_j)]^2.$$

According to Theorem 2.1 the Cramer's rule of quaternion equations, then (3.10) has a unique solution if and only if $ddet M\neq 0$, i.e.

$$\prod_{i\le j}(\overline{\lambda_i}+\lambda_j)\neq 0.$$

So equation (3.7) holds. Since D and Q correspond to B, C, respectively, then equation (3.6) has a unique self-conjugate matrix solution B if and only if (3.7) holds. □

Theorem 3.3 *Suppose that all standard eigenvalues of matrix A of system (3.4) have negative real parts, and for any given definite negative quaternion quadratic form $W(q)=q^*Cq$, there exists a unique quaternion quadratic form $V(q)=q^*Bq$ such that*

$$\frac{\mathrm{d}V}{\mathrm{d}t}=W(q)$$

and V(q) is definite positive.

Proof Since all standard eigenvalues of matrix A of system (3.4) have negative real parts, so (3.7) holds. Thus there exists a unique quaternion quadratic form $V(q)=q^*Bq$ such that

$$\frac{\mathrm{d}V}{\mathrm{d}t}=W(q).$$

Let's prove $V(q)=q^*Bq$ is definite positive by using reduction to absurdity.

Suppose that $V(q) = q^*Bq$ is not definite positive, therefore there exists $q_0 \neq 0$ in the neighborhood of the origin such that $V(q_0) \leq 0$. Consider the solution $q(t)$ of system (3.4) with this initial value q_0, since $V(q) = q^*Bq$ is definite negative, along this solution we have

$$\frac{\mathrm{d}V(q(t))}{\mathrm{d}t} = q^*(t)Cq(t) < 0.$$

From this we can infer that

$$V(q(t)) < V(q(t_1)) < V(q(t_0)) \leq 0, \quad t > t_1 > t_0.$$

On the other hand, since all eigenvalues of matrix A of system (3.4) have negative real parts, then the solution $q(t)$ should tend to 0 when t tends to infinity, therefore we have $\lim_{t\to\infty} V(q(t)) = 0$, which is contradict to above equation. This completes the proof. □

3.3 Stability of Periodic Solution for Quaternion Autonomous Systems

We consider the stability of periodic solution $\psi(t)$ of system (3.1) with T as its period, i.e.

$$\psi(t + T) = \psi(t). \tag{3.11}$$

In order to study the behavior of the solution of equation (3.1) near $\psi(T)$, we introduce a new unknown function p and let

$$q = \psi(t) + p, \tag{3.12}$$

then

$$\dot{p} = \dot{q} - \dot{\psi}(t).$$

Suppose that the right side of equation (3.1) has a second-order continuous partial derivative with respect to the component of q. Notice that $\psi(t)$ is a solution of equation (3.1), we have:

$$\begin{aligned} \dot{p} &= \dot{q} - \dot{\psi}(t) \\ &= f(q) - f(\psi(t)) \\ &= f(\psi(t) + p) - f(\psi(t)), \end{aligned} \tag{3.13}$$

it equals to

$$\dot{p}_i = \sum_j \frac{\partial f_i(\psi(t))}{\partial q_j} p_j + r_i(t, p)(i = 1, 2, \cdots, n), \tag{3.14}$$

where

$$r_i(t,p)=f_i(\psi(t)+p)-f_i(\psi(t))-Df_i(\psi(t))p_i$$

is a second order infinitely small quantity. Linearize this equation, we obtain the linearized system(or first approximation system)

$$\dot{p}=A(t)p, \tag{3.15}$$

where

$$A(t)=[a_{ij}(t)]_{n\times n}=\left[\frac{\partial f_i(\psi(t))}{\partial q_j}\right]_{n\times n}.$$

Since we have assumed that $\psi(t)$ is periodic with period T, Under this assumption, we have $A(t+T)=A(t)$, i.e. system (3.15) is a system of periodic coefficient equations with period T.

Then we need the definition of characteristic multiplier.

Definition 3.5 Let $p=\Phi(t)$ be the arbitrary solution of equation (3.15) where $A(t)$ is periodic quaternionic matrix with period T, if there always exists a constant quaternionic matrix C(non-singular) such that

$$\Phi(t+T)=\Phi(t)C,$$

then C is called basic matrix of the solution $p=\Phi(t)$, the standard eigenvalues of matrix C are called characteristic multipliers of (3.15).

Theorem 3.4 *System (3.15) has at least one characteristic multiplier* $\rho=1$.

Proof Take $q=\psi(t)$ into equation (3.1) we get

$$\dot{\psi}(t)=f(\psi(t)),$$

then take the derivative of t, we get the equation for the ith component as follows

$$\frac{\mathrm{d}\dot{\psi}_i(t)}{\mathrm{d}t}=\sum_{j=1}^{n}\frac{\partial f_i(\psi(t))}{\partial q_j}\dot{\psi}_j(t),\quad (i=1,2,\cdots,n)$$

i.e.

$$\frac{\mathrm{d}\dot{\psi}(t)}{\mathrm{d}t}=A(t)\dot{\psi}(t).$$

This implies that $\dot{\psi}(t)$ is a periodic solution of system (3.1), then since (3.15) has a T-periodic solution if and only if there is a characteristic multiplier $\rho=1$ (see Corollary 4.10 in [20]). Therefore system (3.15) has at least one characteristic multiplier $\rho=1$.

We say a solution $q = q(t)$ is nontrivial if it is not a constant quaternion vector, we point out that for quaternion autonomous systems, there is no asymptotically stable nontrivial periodic solution. □

Theorem 3.5 *If q(t) is a nontrivial periodic solution of quaternion autonomous system (3.1), then it can not be asymptotically stable in the sense of Lyapunov.*

Proof We prove it by reduction to absurdity. If $q = \psi(t)$ is asymptotically stable in the sense of Lyapunov, then there exists $\eta(t_0) > 0$ such that when $\|q_0 - \psi(t_0)\| < \eta(t_0)$, any solution $q = q(t)$ which determined by the initial value (t_0, q_0) all satisfies

$$\lim_{t\to\infty} \|q(t) - \psi(t)\| = 0.$$

Specially, we can choose δ (δ is small enough) such that

$$0 < r \triangleq \|\psi(t_0) - \psi(t_0 + \delta)\| < \eta(t_0).$$

Since the system is autonomous, the solution determined by the initial value of $(t_0, \psi(t_0 + \delta))$ is $q = \psi(t + \delta)$. Therefore we have

$$\lim_{t\to\infty} \|\psi(t + \delta) - \psi(t)\| = 0. \tag{3.16}$$

Meanwhile because the period of $\psi(t)$ is T, let $t_n = t_0 + nT$, then for any positive integer n we have

$$\|\psi(t_n + \delta) - \psi(t_n)\| = \|\psi(t_0 + \delta) - \psi(t_0)\| = r,$$

which is contradict with equation (3.16), it shows that $q = \psi(t)$ can't be asymptotically stable in the sense of Lyapunov. This has completed the proof. □

But for quaternion non-autonomous system, we can prove that it has periodic solution which can be asymptotically stable in the sense of Lyapunov.

4 Stability of Non-Autonomous Systems

In this section, we discuss the stability of periodic solution for periodic quaternion non-autonomous systems, we consider the system of differential equations the right-hand side of differential equations depends on the independent variable t

$$\dot{q} = f(t, q), \tag{4.1}$$

where $q \in \mathbb{H}^n$, $f(t, q)$ is a $\mathbb{H}^n$-valued function.

Consider the periodic function with T as the period of time t on the right side of equation (4.1), i.e.

$$f(t + T, q) = f(t, q).$$

We study the stability of periodic solution $\psi(t)$ with T as its period, i.e. $\psi(t)$ satisfies (3.11). With the same arguments as above quaternion autonomous systems, if we take (3.12) into (4.1), then we can get

$$\dot{p} = A(t)p + r(t, p), \tag{4.2}$$

and the linearized system is

$$\dot{p} = A(t)p, \tag{4.3}$$

where

$$A(t) = [a_{ij}(t)]_{n\times n} = \left[\frac{\partial f_i(t, \psi(t))}{\partial q_j}\right]_{n\times n}.$$

Since we have assumed that $f(t, p)$ and $\psi(t)$ are both periodic with period T, Under these assumptions, equation (4.3) is also a system of periodic quaternion coefficients equations with period T.

In order to study the stability of periodic quaternion non-autonomous systems, we begin some important lemmas.

Lemma 4.1 (First approxiamtion theory) *Let $p \in \mathbb{H}^n$ and*

$$\dot{p} = Bp + R(t, p), \tag{4.4}$$

where $B = [b_{ij}]$ is a constant quaternion matrix and all standard eigenvalues of B has negative real parts. For $c > 0$, the remainder $R(t, p)$ has a definition for $t > t_0$, $\|p\| < c$ and satisfies the estimation formula

$$\|R(t, p)\| \le r\|p\|^2,$$

where $r > 0$ is a constant, then the zero solution of equation (4.4) is asymptotically stable.

Proof Take the definite negative quaternion quadratic form $W(p) = -\|p\|^2$, according to theorem 3.3 we know for the linear approximate system

$$\dot{p} = Bp, \tag{4.5}$$

there exists definite positive quaternion quadratic form

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.5)} = \langle \nabla V, Bp\rangle = -\|p\|^2.$$

We choose V from above equation as the Lyapunov function of system (4.4), then

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.4)} &= \langle \nabla V, Bp + R(t,p)\rangle \\ &= -\|p\|^2 + \langle \nabla V, R(t,p)\rangle \\ &= W(p) + G(t,p),\end{aligned} \tag{4.6}$$

where $G(t,p) \triangleq \langle \nabla V, R(t,p)\rangle$ and

$$\frac{\|G(t,p)\|}{\|p\|^2} \leq \frac{\|\nabla V\| \cdot \|R(t,p)\|}{\|p\|^2} = \frac{\|\nabla V\|}{\|p\|} \cdot \frac{\|R(t,p)\|}{\|p\|} \triangleq a.$$

Since V is a quadratic form, so $\|\nabla V\|$ and $\|p\|$ are of the same order. In addition $\|R(t,p)\| \leq r\|p\|^2 (r > 0)$, thus when $\|p\|$ tends to zero, a tends to zero. It follows that $\|G(t,p)\| \leq a\|p\|^2$ and a can be infinitely small when $\|p\|$ is small enough.

From (4.6) we get that as long as the neighborhood of the origin is small enough, it can make the symbols of dV/dt and $W(p)$ be the same, therefore, it is definite negative, because $V(p)$ is definite positive, by theorem 3.1 (2), we know the zero solution of system (4.4) is is asymptotically stable. This has completed the proof. □

Definition 4.1 Equation (4.3) is said to be equivalent to

$$\dot{q} = B(t)q, \tag{4.7}$$

if there exists a linear transformation $q = S(t)p$, which transforms equation (4.3) into equation (4.7), and $B(t)$ and $S(t)$ are also periodic quaternion matrixs with the same period T.

Lemma 4.2 *Equation (4.3) is equivalent to (4.7) if and only if these two equations exists solutions $p = \Phi(t)$ and $q = \Psi(t)$ with the same basic matrix.*

Proof First we assume that equation (4.3) is equivalent to (4.7), let $p = \Phi(t)$ is an arbitrary solution of equation (4.3) with basic matrix C, so $q = \Psi(t) = S(t)\Phi(t)$ is the solution of equation (4.7), what's more, we have

$$\Psi(t+T) = S(t+T)\Phi(t+T) = S(t)\Phi(t+T) = S(t)\Phi(t)C = \Psi(t)C.$$

Hence, the basic matrix of solution $p = \Phi(t)$ is also the basic matrix of the solution $q = \Psi(t)$.

Conversely, we suppose that Eqs. (4.3) and (4.7) has the solution $p = \Phi(t)$ and $q = \Psi(t)$ respectively, they have the same basic matrix C. Therefore we get

$$\Phi(t+T) = \Phi(t)C, \quad \Psi(t+T) = \Psi(t)C,$$

since C is non-singular, it equals to

$$\Psi(t+T)\Phi^{-1}(t+T) = \Psi(t)\Phi^{-1}(t),$$

hence

$$S(t) = \Psi(t)\Phi^{-1}(t) \tag{4.8}$$

has the period T and the equation $\Psi(t) = S(t)\Phi(t)$ holds. Because solutions $\Phi(t)$ and $\Psi(t)$ uniquely determine their respective equations , from (4.8) we infer that equation (4.7) is obtained from equation (4.3) by the linear transformation $q = S(t)p$. This has completed the proof. □

Lemma 4.3 *Every periodic quaternion coefficient equation (4.3) is equivalent to a constant quaternion coefficient equation*

$$\dot{q} = Bq. \tag{4.9}$$

Proof Let C be the basic matrix of the solution (4.3), then there exists B such that $e^{TB} = C$. Since $q = e^{tB}$ is the solution of equation (4.9), if we consider B as a periodic matrix with period T, then the basic matrix of solution $q = e^{tB}$ is $C = e^{TB}$, i.e.

$$e^{(t+T)B} = e^{tB}e^{TB} = e^{tB}C,$$

this is because of the fact that if $A, B \in \mathbb{H}^{n\times n}$ are commutable, then $e^A e^B = e^{A+B}$. Since the basic matrix of the solution under consideration is exactly the same, therefore the two equations are equivalent. □

Now, we have the following significant theorems.

Theorem 4.1 *Let $f(t, q)$ of system (4.1) is a periodic quaternion function and its period is T with respect to t, $\psi(t)$ is its periodic solution which also has period T, if the all characteristic multipliers of equation (4.3) have modulus less than 1, then the solution $\psi(t)$ is asymptotically stable.*

Proof According to Definition 4.1 and Lemma 4.3, we know there exists transformation $q = T(t)p$, where $T(t)$ has period T, under this transformation, equation (4.3) becomes $\dot{q} = Bq$, e^{tB} is the solution of this equation and e^{TB} is its basic matrix. This means that e^{TB} is also the basic matrix of equation (4.3).

According to the hypothesis of theorem, the modulus of all standard eigenvalues of e^{TB} is less than 1 and the standard eigenvalues of matrix e^{TB} has the form $e^{T\widetilde{\lambda_i}}$ (λ_i is the standard eigenvalues of matrix B), so $|e^{T\widetilde{\lambda}}| < 1$. From this we know that all standard eigenvalues of B have negative real parts.

Apply this transformation $q = T(t)p$ to equation (4.2), then it becomes the form (4.4), since all standard eigenvalues of B have negative real parts, thus by Lemma 4.1, the solution is asymptotically stable. This has completed the proof. □

Since autonomous systems' periodic solutions can't be asymptotically stable in the sense of Lyapunov, but as a special case of non-autonomous systems, if we weaken the conditions of Theorem 4.1 and as a corollary of Theorem 4.1, we have the following stable theorem for quaternion autonomous systems.

Theorem 4.2 *Suppose that $q=\psi(t)$ is the nontrivial periodic solution of system (3.1) with period T it, if the characteristic multipliers of system (3.15) all have modulus not larger than 1 and the algebraic multiplicity equals to 1 of each characteristic multipliers with modulus one, then the solution is stable under the sense of Lyapunov.*

5 Examples

In this section, we give some examples which have used the second Lyapunov method to judge their stability.

Example 5.1 Consider the system $\dot{q}=Aq$, where

$$A=\begin{pmatrix} -1+j+\frac{k}{10} & i+k \\ j+\frac{k}{10} & -1+i+k \end{pmatrix}$$

We can choose $V(q_1,q_2)=\|q_1-q_2\|^2$, thus V is a semi-definite positive function, and

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= (\dot{q_1}-\dot{q_2})\overline{q_1-q_2}+(q_1-q_2)\overline{\dot{q_1}-\dot{q_2}} \\
&= \left[\left(-1+j+\frac{k}{10}\right)q_1+(i+k)q_2-\left(\left(j+\frac{k}{10}\right)q_1+(-1+i+k)q_2\right)\right](\overline{q_1-q_2}) \\
&\quad +(q_1-q_2)\left[\overline{q_1}\left(-1-j-\frac{k}{10}\right)-\overline{q_2}(-i-k)-(\overline{q_1}\left(-j-\frac{k}{10}\right)\right. \\
&\quad \left.+\overline{q_2}(-1-i-k))\right] \\
&= [(-q_1+q_2]\overline{(q_1-q_2)}+(q_1-q_2)[-\overline{q_1}+\overline{q_2}] \\
&= -2\|q_1\|^2-2\|q_2\|^2+q_1\overline{q_2}+q_2\overline{q_1}+q_2\overline{q_1}+q_1\overline{q_2} \\
&= -2\|q_1\|^2-2\|q_2\|^2+2\mathcal{R}(q_1\overline{q_2})+2\mathcal{R}(q_2\overline{q_1}) \\
&= -2\|q_1-q_2\|^2 \\
&\le 0.
\end{aligned}$$

Therefore dV/dt is a semi-definite negative function, according to Theorem 3.1, this system is stable.

Example 5.2 Consider the system $\dot{q}=Aq$,where

$$A=\begin{pmatrix} -1+i & 0 \\ 0 & -2+j \end{pmatrix}$$

We can choose $V(q_1, q_2) = \|q_1\|^2 + \|q_2\|^2$, obviously V is a definite positive function, and

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \dot{q}_1\overline{q_1} + q_1\dot{\overline{q_1}} + \dot{q}_2\overline{q_2} + q_2\dot{\overline{q_2}} \\ &= (-1+i)q_1\overline{q_1} + q_1[\overline{q_1}(-1-i)] + (-2+j)q_2\overline{q_2} + q_2[\overline{q_2}(-2-j)] \\ &= -2\|q_1\|^2 - 4\|q_2\|^2.\end{aligned}$$

Therefore dV/dt is a definite negative function, then according to Theorem 3.1, the solution of this system is asymptotically stable.

In fact, we can calculate that the standard eigenvalues of A is $\lambda_1 = -1 + i, \lambda_2 = -2 + j$. To find the eigenvector of $\lambda_1 = -1 + i$, we consider the following equation

$$Aq = q\lambda_1,$$

i.e.

$$\begin{cases} (-1+i)q_1 = q_1(-1+i) \\ (-2+j)q_2 = q_2(-1+i) \end{cases} \tag{5.1}$$

Thus,we can take one eigenvector as

$$v_1 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To find the eigenvector of $\lambda_2 = -2 + j$,we consider the following equation

$$Aq = q\lambda_2,$$

i.e

$$\begin{cases} (-1+i)q_1 = q_1(-2+j) \\ (-2+j)q_2 = q_2(-2+j) \end{cases}$$

thus we can take one eigenvector as

$$v_2 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since

$$ddet(v_1, v_2) = ddet\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = rdet\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \neq 0.$$

Therefore the eigenvectors v_1 and v_2 are linearly independent, hence the fundamental matrix is

$$M(t) = (v_1 e^{\lambda_1 t}, v_2 e^{\lambda_2}) = \begin{pmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-2+j)t} \end{pmatrix}.$$

Obviously $\lim_{t\to\infty} ||M(t)|| = 0$, from this we know the solution is asymptotically stable.

Example 5.3 Consider the system $\dot{q} = Aq$,where

$$A = \begin{pmatrix} -2+i-j & -i+j \\ 1+i+k & -3-k \end{pmatrix}$$

We can choose $V(q_1, q_2) = \|q_1\|^2 + \|q_2\|^2$, thus V is a definite positive function, and

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \dot{q_1}\overline{q_1} + q_1\dot{\overline{q_1}} + \dot{q_2}\overline{q_2} + q_2\dot{\overline{q_2}} \\
&= [(-2+i-j)q_1 + (-i+j)q_2]\overline{q_1} + q_1[\overline{q_1}(-2-i+j) + \overline{q_2}(i-j)] \\
&\quad + [(1+i+k)q_1 + (-3-k)q_2]\overline{q_2} + q_2[\overline{q_1}(1-i-k) + \overline{q_2}(-3+k)] \\
&= (-2+i-j)\|q_1\|^2 + (-i+j)q_2\overline{q_1} + \|q_1\|^2(-2-i+j) + q_1\overline{q_2}(i-j) \\
&\quad + (1+i+k)q_1\overline{q_2} + (-3-k)\|q_2\|^2 + q_2\overline{q_1}(1-i-k) + \|q_2\|^2(-3+k) \\
&= -4\|q_1\|^2 - 6\|q_2\|^2 + 2\mathcal{R}(q_1\overline{q_2}(i-j))) + 2\mathcal{R}(1+i+k)q_1\overline{q_2} \\
&= -[(q_1 + (i-j)q_2)(\overline{q_1} - \overline{q_2}(i-j))] \\
&\quad - [(q_1 - \overline{(1+i+k)}q_2)(\overline{q_1} - \overline{q_2}(1+i+k))] - 2\|q_1\|^2 - \|q_2\|^2 \\
&= -(\|q_1 + (i-j)q_2\|^2 + \|q_1 - (1-i-k)q_2\|^2 + 2\|q_1\|^2 + \|q_2\|^2) < 0.
\end{aligned}$$

Therefore according to Theorem 3.1, the solution is asymptotically stable.

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