A Geometric Algebra Differential Flatness based approach to the control of a Subactuated aircraft

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Abstract— **This is the general notes for proving differential flatness of the quad rotorcraft using Geometric Algebra.**

I. Introduction

Notation and Terminology

In GA, only real numbers are associated with "scalar" quantities which are denoted here using lowercase italic font i.e., $b, k, \ell, x, \alpha, \iota, \theta, \tau \in \mathbb{R}$. Traditional (polar) vectors are associated with directed lengths (here in Euclidean 3-space) and denoted using lowercase upright bold font i.e., $\mathbf{b}, \mathbf{e}, \mathbf{u}, \mathbf{w}, \mathbf{x}, \mathbf{v}, \mathbf{f} \in \mathbb{L}^3$. Traditional axial vectors are closely related with directed areas (bivectors) in Euclidean 3-space and denoted using uppercase upright bold font i.e., $\mathbf{B}, \mathbf{E}, \boldsymbol{\Upsilon}, \boldsymbol{\Omega} \in \mathbb{B}^3$. Trivectors in 3-space are pseudoscalars that can be represented as scaled versions of a unit directed volume denoted with $\mathbf{I}$, as $c\mathbf{I} \in \mathbb{T}^3$. Finally, a general multivector in Euclidean 3-space $A, B, M, R \in \mathbb{G}^3$ is the direct sum of the scalars (signed line segments), vectors (directed line segments), bivectors (directed areas), and trivectors (directed volumes) in Euclidean 3-space

$$\mathbb{G}^3 = \mathbb{R} \oplus \mathbb{L}^3 \oplus \mathbb{B}^3 \oplus \mathbb{T}^3 \tag{1}$$

and can be represented in terms of an orthonormal basis

$$\{1,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_3\mathbf{e}_1,\ \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\} \tag{2}$$

as

$$\begin{aligned} M = {} & (m_0)(1) && \text{(scalar)} \\ & + m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3 && \text{(vector)} \\ & + m_4\mathbf{e}_1\mathbf{e}_2 + m_5\mathbf{e}_2\mathbf{e}_3 + m_6\mathbf{e}_3\mathbf{e}_1 && \text{(bivector)} \\ & + m_7\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. && \text{(trivector)} \end{aligned} \tag{3}$$

One exception to the notational rules provided above is that unit bivectors $\mathbf{j} \in \{\mathbf{B} \in \mathbb{B}^3 : \|\mathbf{B}\| = 1\}$ are denoted with *lowercase* upright bold font $\mathbf{j}$ because they have the property

$$\mathbf{j}^2 = -1 \tag{4}$$

and furthermore, they play a role similar to that of the traditional imaginary number (as a generator of rotations). Finally, a rotor is a unit norm multivector used to represent a rotation. In Euclidean 3-space, a rotor R is formed as the sum of a scalar and bivector $R \in \{M \in \mathbb{R} \oplus \mathbb{B}^3 : \|M\| = 1\}$ and is isomorphic to the traditional unit quaternion. The key benefits of rotors over quaternions are that rotations maintain clear geometric interpretability and that the rotor belongs to the same space, $\mathbb{G}^3$, as all other elements in our dynamic model. The reader interested in GA is referred to Macdonald's works [1]–[3].

Related Work

Main Contributions

Organization of the Manuscript

II. Problem Statement

The actuators of a quad rotorcraft are four motors, with a designated propeller. Two motors spin counter-clockwise, while the other two spin clockwise. Four thrusts are available for control; one for each motor. The torques generated by the thrusts are used to manipulate the orientation and position of the system. A popular convention is to place the inertial and body coordinate frame in a North-East-Down (NED) configuration. The inertial frame is taken to be the origin of the system. The body frame is placed at the center of mass of the platform. Traditionally, to perform stabilization and tracking, the following states are required: the coordinates corresponding to the position, velocity, orientation, and the angular velocity of the platform. All the states are with respect to the inertial frame. The main goal of this work is to prove the dynamics of the quad rotorcraft are deferentially flat.

III. Dynamic Model

The derived illustration of the quad rotorcraft's dynamic model is shown in Fig. 1. The inertial frame is represented using the set of orthonormal vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The frame is oriented in a NED configuration, where $\mathbf{e}_1$ points along the North, $\mathbf{e}_2$ points along the East, and $\mathbf{e}_3$ points in the Down direction. The body frame is represented using another set

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of orthonormal vectors, $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. When the frames are perfectly aligned, the body frame is in a NED orientation. The dynamics of the quad rotorcraft are separated into two subsystems. The first subsystem is the rotational dynamics. The second subsystem is the translational dynamics.

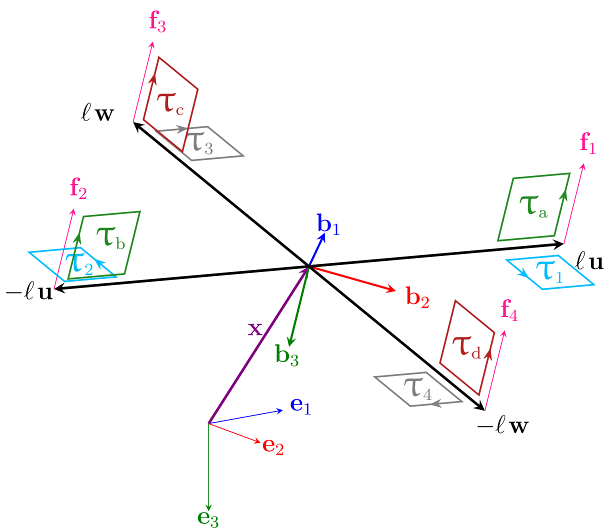

Fig. 1: Illustration of the GA-based quad rotorcraft dynamic model. The body frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a rotated and translated copy of the inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The rotation is represented with rotor R (not illustrated) and the translation is represented by the vector $\mathbf{x}$. The quad rotorcraft arms are represented as a superposition of the body frame vectors $\pm\ell\mathbf{u} = \pm\ell(\mathbf{b}_1 + \mathbf{b}_2)/\sqrt{2}$ and $\pm\ell\mathbf{w} = \pm\ell(\mathbf{b}_1 - \mathbf{b}_2)/\sqrt{2}$, where ℓ is the arm length. The forces generated by the motors are related to the body frame by $\mathrm{f}_1 = -f_1\mathbf{b}_3$, $\mathrm{f}_2 = -f_2\mathbf{b}_3$, $\mathrm{f}_3 = -f_3\mathbf{b}_3$, and $\mathrm{f}_4 = -f_4\mathbf{b}_3$, where f_i are scalars. The torques generated by the motors are related to the body frame by $\tau_1 = -\alpha_1\mathbf{b}_1 \wedge \mathbf{b}_2$, $\tau_2 = -\alpha_2\mathbf{b}_1 \wedge \mathbf{b}_2$, $\tau_3 = \alpha_3\mathbf{b}_1 \wedge \mathbf{b}_2$, $\tau_4 = \alpha_4\mathbf{b}_1 \wedge \mathbf{b}_2$, $\tau_\mathrm{a} = \ell\mathbf{u} \wedge \mathrm{f}_1$, $\tau_\mathrm{b} = -\ell\mathbf{u} \wedge \mathrm{f}_2$, $\tau_\mathrm{c} = \ell\mathbf{w} \wedge \mathrm{f}_3$, and $\tau_\mathrm{d} = -\ell\mathbf{w} \wedge \mathrm{f}_4$, where α_i are scalars.

The equation of motion for the translation movement of the quad rotorcraft system is

$$m\dot{\mathbf{v}} = mg\mathbf{e}_3 - u_h\mathbf{b}_3, \tag{5}$$

where m is the mass of the platform, g is the acceleration due to gravity, $\mathbf{v}$ is the linear velocity vector, and u_h is the control input from the altitude controller. The equation of motion for the rotational movement of the system is given by

$$\dot{\boldsymbol{\Omega}}_\mathrm{b} = \mathcal{I}^{-1}\Big(R^\dagger \boldsymbol{\tau} R + \big[\boldsymbol{\Omega}_\mathrm{b}, \mathcal{I}(\boldsymbol{\Omega}_\mathrm{b})\big]\Big), \tag{6}$$

where $\boldsymbol{\Omega}_\mathrm{b}$ is the angular velocity bivector, $\mathcal{I}$ is the inertial tensor, R is rotor describing the orientation, and $\boldsymbol{\tau}$ is the input torque bivector.

IV. Differential Flatness

The choice of flat outputs will be the vector representing the translational position, $\mathbf{x}$ and

A. Translation position, velocity, and acceleration

The translational states of the system are easily recognized as flat through the following relations

$$\mathbf{x} = \mathbf{x}, \tag{7}$$

$$\mathbf{v} = \dot{\mathbf{x}}, \tag{8}$$

$$\dot{\mathbf{v}} = \ddot{\mathbf{x}}. \tag{9}$$

As can be seen, the translational states can all be represented using the flat output $\mathbf{x}$ and its derivatives. The translational states are flat.

B. Thrust input

The altitude control input to the system is show to be flat by utilizing equation (5),

$$m\dot{\mathbf{v}} = mg\mathbf{e}_3 - u_h\mathbf{b}_3.$$

The $\mathbf{b}_3$ basis vector is a rotated copy of the $\mathbf{e}_3$ inertial vector, so equation (5) can be rewritten as

$$m\dot{\mathbf{v}} = mg\mathbf{e}_3 - u_h R\mathbf{e}_3 R^\dagger.$$

The rotated basis vector can be isolated to result in

$$\frac{-(m\dot{\mathbf{v}} - mg\mathbf{e}_3)}{u_h} = R\mathbf{e}_3 R^\dagger.$$

Squaring both sides of the above equation results in

$$\frac{(-(m\dot{\mathbf{v}} - mg\mathbf{e}_3))^2}{u_h^2} = (R\mathbf{e}_3 R^\dagger)(R\mathbf{e}_3 R^\dagger).$$

Using the fact that $RR^\dagger = 1$ and $\mathbf{e}_3\mathbf{e}_3 = 1$, the above equation reduces to

$$\frac{(-(m\dot{\mathbf{v}} - mg\mathbf{e}_3))^2}{u_h^2} = 1.$$

Solving for the altitude control input, u_h, yields the following

$$u_h = m\sqrt{(\dot{\mathbf{v}} - g\mathbf{e}_3)^2}. \tag{10}$$

The above representation is considered flat, it can be placed into a form that is more familiar. Call the argument of the square root, $\mathbf{T}$. Then the expanded form of $\mathbf{T}$ is

$$\mathbf{T} = (\dot{\mathbf{v}} - g\mathbf{e}_3) = \ddot{x}_1\mathbf{e}_1 + \ddot{x}_2\mathbf{e}_2 + (\ddot{x}_3 - g)\mathbf{e}_3.$$

Then, the square of the vector $\mathbf{T}$, is

$$\mathbf{T}^2 = \ddot{x}_1{}^2 + \ddot{x}_2{}^2 + (\ddot{x}_3 - g)^2,$$

which is the extent of the vector. Thus the square root of the extent results in the norm of $\mathbf{T}$. With this, equation (10), can be rewritten as

$$u_h = m\sqrt{(\dot{\mathbf{v}} - g\mathbf{e}_3)^2} = m\|\mathbf{T}\|. \tag{11}$$

Both equations (9) and (11), show that the altitude control is flat.

C. Body basis vectors

The first of the basis vectors to be proven as flat is the $\mathbf{b}_3$ basis vector. The $\mathbf{b}_3$ basis vector must align with the thrust vector of the quad rotorcraft. Using equation (5), $\mathbf{b}_3$ can be expressed as

$$\mathbf{b}_3 = \frac{mg\mathbf{e}_3 - m\dot{\mathbf{v}}}{u_h} \tag{12}$$

The above expression for $\mathbf{b}_3$ is flat. To establish that $\mathbf{b}_3$ is along the thrust vector, recall that $u_h = m\|\mathbf{T}\|$, and the definition of $\mathbf{T}$. Equation (12) can then be written as

$$\mathbf{b}_3 = \frac{-m(\dot{\mathbf{v}} - g\mathbf{e}_3)}{m\|\mathbf{T}\|} = -\frac{\mathbf{T}}{\|\mathbf{T}\|} \tag{13}$$

The $\mathbf{b}_3$ vector is along $\mathbf{T}$, but in the opposite direction, which matches the selected NED basis.

The $\mathbf{b}_1$ basis vector is one of the flat outputs that will represent the system. It is chosen by the user and is called $\mathbf{b}_1^d$. $\mathbf{b}_1$ must be orthogonal to $\mathbf{b}_3$, to create the body basis frame. To ensure $\mathbf{b}_1$ meets this criteria, it must be projected onto the plane that is perpendicular to $\mathbf{b}_3$. This is easily handled in GA by the use of the dual operation. The dual of $\mathbf{b}_3$ is

$$\mathbf{b}_3^* = \mathbf{b}_3\mathbf{I}^{-1}.$$

Then the projection of $\mathbf{b}_1^d$ onto the plane perpendicular to $\mathbf{b}_3$ is simply,

$$\mathbf{b}_1 = \mathsf{P}_{\mathbf{b}_3^*}(\mathbf{b}_1^d) = (\mathbf{b}_1^d \cdot \mathbf{b}_3^*)(\mathbf{b}_3^*)^{-1}. \tag{14}$$

The last basis vector will be perpendicular to the plane spanned by $\mathbf{b}_1$ and $\mathbf{b}_3$. Once again the dual offers this vector easily,

$$\mathbf{b}_2 = (\mathbf{b}_3 \wedge \mathbf{b}_1)^* = (\mathbf{b}_3 \wedge \mathbf{b}_1)\mathbf{I}^{-1}. \tag{15}$$

D. Orientation

The orientation of the quad rotorcraft is represented using the Rotor, R. The body basis frame and the inertial frame of the system are related by the following,

$$\mathbf{b}_k = R\mathbf{e}_k R^\dagger. \tag{16}$$

The rotor responsible for rotating the inertial frame to the body frame is given by,

$$R = \beta(1 + \mathbf{b}_1\mathbf{e}^1 + \mathbf{b}_2\mathbf{e}^2 + \mathbf{b}_3\mathbf{e}^3), \tag{17}$$

where β is a scalar to ensure the magnitude of the rotor is equal to unity, and $\mathbf{e}^k$ is the reciprocal frame of the inertial frame.

The derivative of R is given by the following expression

$$\dot{R} = \beta(1 + \dot{\mathbf{b}}_1\mathbf{e}^1 + \dot{\mathbf{b}}_2\mathbf{e}^2 + \dot{\mathbf{b}}_3\mathbf{e}^3). \tag{18}$$

The second derivative of R is given as

$$\ddot{R} = \beta(1 + \ddot{\mathbf{b}}_1\mathbf{e}^1 + \ddot{\mathbf{b}}_2\mathbf{e}^2 + \ddot{\mathbf{b}}_3\mathbf{e}^3). \tag{19}$$

E. Angular velocity and acceleration

The angular velocity bivector is given by the following rotor equation,

$$\mathbf{\Omega} = -2R\dot{R}. \tag{20}$$

The angular velocity bivector can be rotated into the body frame to give the body frame angular velocity

$$\mathbf{\Omega}_b = R^\dagger \mathbf{\Omega} R. \tag{21}$$

The angular acceleration bivector is given the the following rotor equation,

$$\dot{\mathbf{\Omega}} = -2(\ddot{R}R^\dagger + \dot{R}\dot{R}^\dagger). \tag{22}$$

The body frame angular acceleration is given as the following,

$$\dot{\mathbf{\Omega}}_b = R^\dagger \dot{\mathbf{\Omega}} R \tag{23}$$

F. Torque Input

The control input to the attitude is a Torque bivector, $\boldsymbol{\tau}$. The equation of motion for the attitude, equation 6 can be arranged to solve for the input torque,

$$\boldsymbol{\tau} = R\Big[\mathcal{I}\Big(\dot{\mathbf{\Omega}}_b\Big) - \big[\mathbf{\Omega}, \mathcal{I}(\mathbf{\Omega}_b)\big]\Big)\Big]R^\dagger \tag{24}$$

V. Controller Design

With the dynamics of the quad rotorcraft proven to be flat, controllers can now be developed to leverage flatness. This section specifies the design choices made in designing a position controller for the quad rotorcraft.

A. Errors

Given the quad rotorcraft's current position, $\mathbf{x}$, and a desired position, $\mathbf{x_d}$, the error in the quad rotorcraft's position is

$$\mathbf{e_x} = \mathbf{x} - \mathbf{x}_d. \tag{25}$$

Similarly, given the quad rotorcraft's velocity, $\mathbf{v}$ and a desired velocity, $\mathbf{v_d}$, the error in the quad rotorcraft's velocity is

$$\mathbf{e_v} = \mathbf{v} - \mathbf{v}_d. \tag{26}$$

The error of the

B. Positional Controller

References

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