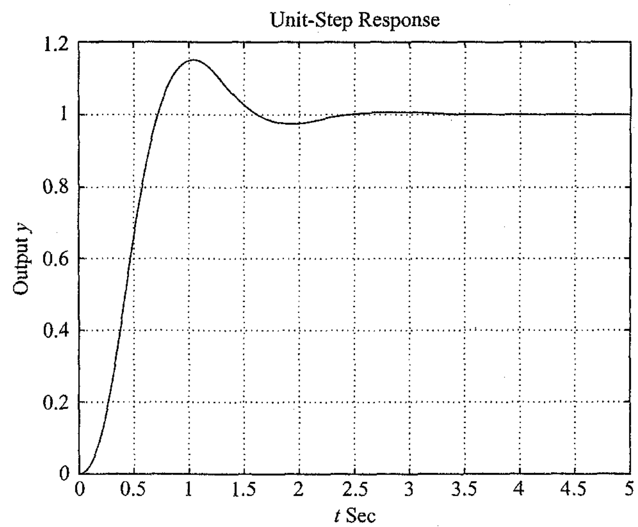

Figure 12–5 Unit-step response curve $y(t)$ versus t for the system designed in Example 12–4.

Note that since

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r$$

we have

$$u(\infty) = -[160 \quad 54 \quad 11]\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} + 160r$$

$$= -[160 \quad 54 \quad 11]\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + 160r = 0$$

At steady state the control signal u becomes zero.

Design of Type 1 Servo System when the Plant Has No Integrator. If the plant has no integrator (type 0 plant), the basic principle of the design of a type 1 servo system is to insert an integrator in the feedforward path between the error comparator and the plant, as shown in Figure 12–6. (The block diagram of Figure 12–6 is a basic form of the type 1 servo system where the plant has no integrator.) From the diagram, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{12–31}$$

$$y = \mathbf{C}\mathbf{x} \tag{12–32}$$

$$u = -\mathbf{K}\mathbf{x} + k_I\xi \tag{12–33}$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x} \tag{12–34}$$

where $\mathbf{x}$ = state vector of the plant (n-vector)

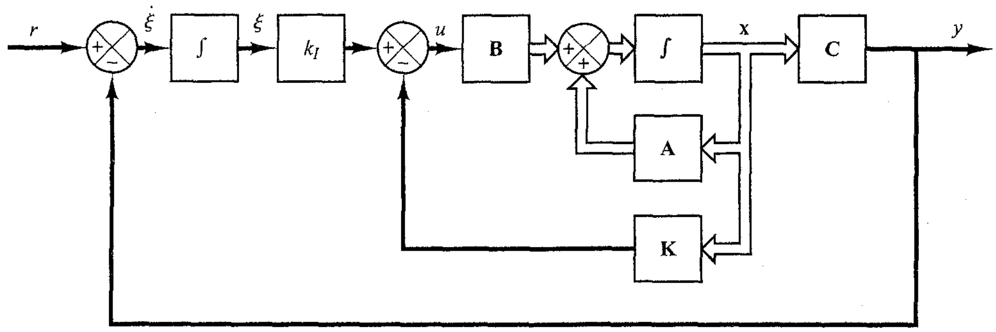

Figure 12–6
Type 1 servo system.

u = control signal (scalar)
y = output signal (scalar)
ξ = output of the integrator (state variable of the system, scalar)
r = reference input signal (step function, scalar)
$\mathbf{A}$ = $n \times n$ constant matrix
$\mathbf{B}$ = $n \times 1$ constant matrix
$\mathbf{C}$ = $1 \times n$ constant matrix

We assume that the plant given by Equation (12–31) is completely state controllable. The transfer function of the plant can be given by

$$G_p(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

To avoid the possibility of the inserted integrator being canceled by the zero at the origin of the plant, we assume that $G_p(s)$ has no zero at the origin.

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by an equation that is a combination of Equations (12–31) and (12–34):

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \tag{12–35}$$

We shall design an asymptotically stable system such that $\mathbf{x}(\infty)$, $\xi(\infty)$, and $u(\infty)$ approach constant values, respectively. Then, at steady state, $\dot{\xi}(t) = 0$, and we get $y(\infty) = r$.

Notice that at steady state we have

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty) \tag{12–36}$$

Noting that $r(t)$ is a step input, we have $r(\infty) = r(t) = r$ (constant) for $t > 0$. By subtracting Equation (12–36) from Equation (12–35), we obtain

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)] \tag{12–37}$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$
$$\xi(t) - \xi(\infty) = \xi_e(t)$$
$$u(t) - u(\infty) = u_e(t)$$

Then Equation (12–37) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t) \tag{12–38}$$

where

$$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t) \tag{12–39}$$

Define a new $(n + 1)$th-order error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Then Equation (12–38) becomes

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \tag{12–40}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

and Equation (12–39) becomes

$$u_e = -\hat{\mathbf{K}}\mathbf{e} \tag{12–41}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \mid -k_I]$$

The state error equation can be obtained by substituting Equation (12–41) into Equation (12–40):

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e} \tag{12–42}$$

If the desired eigenvalues of matrix $\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}$ (that is, the desired closed-loop poles) are specified as $\mu_1, \mu_2, \ldots, \mu_{n+1}$, then the state-feedback gain matrix $\mathbf{K}$ and the integral gain constant k_I can be determined by the pole-placement technique pesented in Section 12–2, provided that the system defined by Equation (12–40) is completely state controllable. Note that if the matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

has rank $n + 1$, then the system defined by Equation (12–40) is completely state controllable. (See Problem **A–12–12.**)

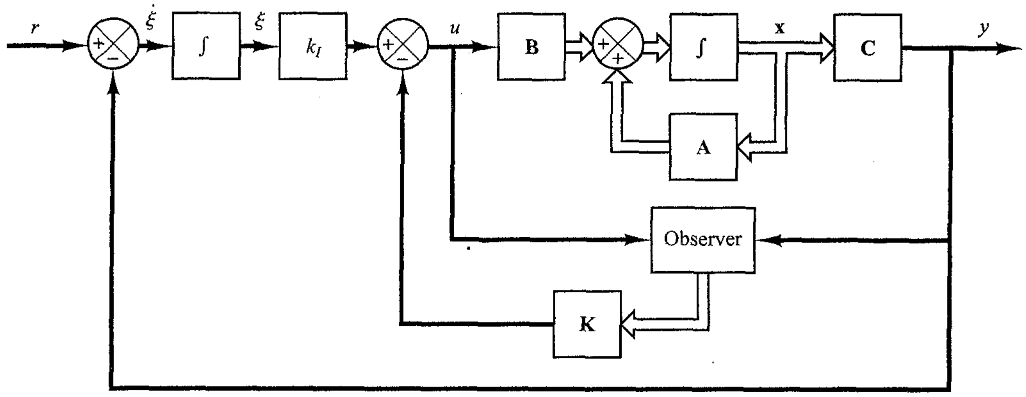

Figure 12–7
Type 1 servo system with state observer.

As is usually the case, not all state variables can be directly measurable. If this is the case, we need to use a state observer. Figure 12–7 shows a block diagram of a type 1 servo system with a state observer. [In the figure, each block with an integral symbol represents an integrator $(1/s)$.] Detailed discussions of state observers are given in Section 12–5.

EXAMPLE 12–5 Consider the inverted-pendulum control system shown in Figure 12–8. In this example, we are concerned only with the motion of the pendulum and motion of the cart in the plane of the page.

It is desired to keep the inverted pendulum upright as much as possible and yet control the position of the cart, for instance, move the cart in a step fashion. To control the position of the cart, we need to build a type 1 servo system. The inverted-pendulum system mounted on a cart does not have an integrator. Therefore, we feed the position signal y (which indicates the position of the cart) back to the input and insert an integrator in the feedforward path, as shown in Figure

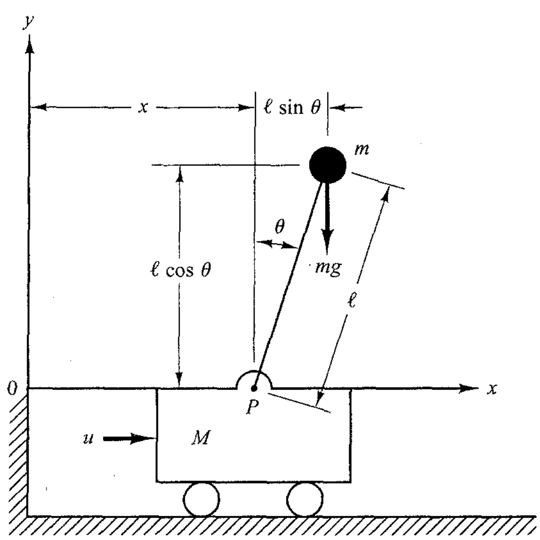

Figure 12–8
Inverted-pendulum control system.

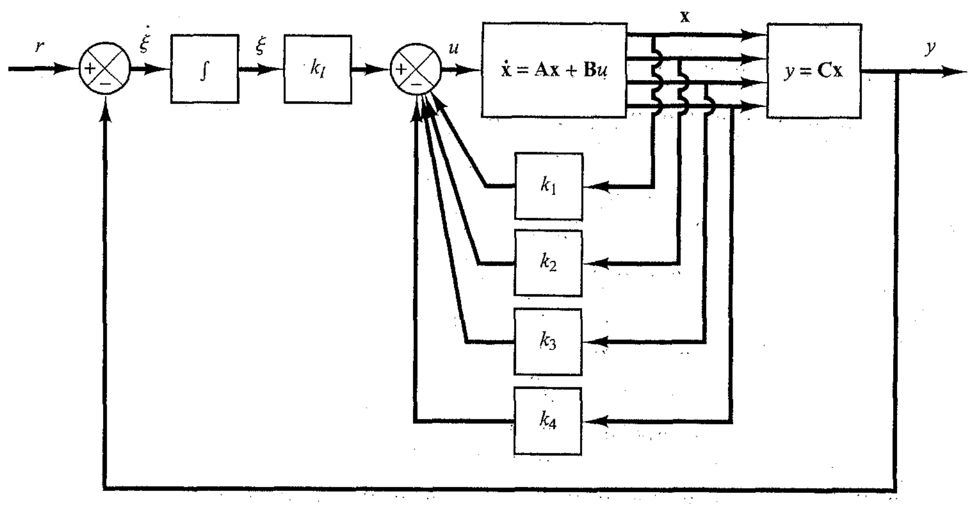

Figure 12–9
Inverted-pendulum control system. (Type 1 servo system when the plant has no integrator.)

12–9. We assume that the pendulum angle θ and the angular velocity $\dot{\theta}$ are small, so that $\sin\theta \doteqdot \theta$, $\cos\theta \doteqdot 1$, and $\theta\dot{\theta}^2 \doteqdot 0$. We also assume that the numerical values for M, m, and l are given as

$$M = 2\text{ kg}, \qquad m = 0.1\text{ kg}, \qquad l = 0.5\text{ m}$$

Referring to Equations (3–59) and (3–60), the equations for the inverted-pendulum control system are

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{12–43}$$

$$M\ddot{x} = u - mg\theta \tag{12–44}$$

When the given numerical values are substituted, Equations (12–43) and (12–44) become

$$\ddot{\theta} = 20.601\theta - u \tag{12–45}$$

$$\ddot{x} = 0.5u - 0.4905\theta \tag{12–46}$$

Let us define the state variables x_1, x_2, x_3, and x_4 as

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$x_3 = x$$

$$x_4 = \dot{x}$$

Then, referring to Equations (12–45) and (12–46) and Figure 12–9 and considering the cart position x as the output of the system, we obtain the equations for the system as follows:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \tag{12–47}$$

$$y = \mathbf{Cx} \tag{12–48}$$

$$u = -\mathbf{Kx} + k_I\xi \tag{12–49}$$

$$\dot{\xi} = r - y = r - \mathbf{Cx} \tag{12–50}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 0 \quad 1 \quad 0]$$

For the type 1 servo system, we have the state error equation as given by Equation (12–40):

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \tag{12–51}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and the control signal is given by Equation (12–41):

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \mid -k_I] = [k_1 \quad k_2 \quad k_3 \quad k_4 \mid -k_I]$$

To obtain a reasonable speed and damping in the response of the designed system (for example, the settling time of approximately 4 ~ 5 sec and the maximum overshoot of 15% ~ 16% in the step response of the cart), let us choose the desired closed-loop poles at $s = \mu_i$ $(i = 1, 2, 3, 4, 5)$, where

$$\mu_1 = -1 + j\sqrt{3}, \qquad \mu_2 = -1 - j\sqrt{3}, \qquad \mu_3 = -5, \qquad \mu_4 = -5, \qquad \mu_5 = -5$$

We shall determine the necessary state-feedback gain matrix by the use of MATLAB.

Before we proceed further, we must examine the rank of matrix **P**, where

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

Matrix **P** is given by

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{12–52}$$

The rank of this matrix can be found to be 5. Therefore, the system defined by Equation (12–51) is completely state controllable, and arbitrary pole placement is possible. MATLAB Program 12–6 produces the state feedback gain matrix $\hat{\mathbf{K}}$.

MATLAB Program 12–6

```
A = [0 1 0 0; 20.601 0 0 0; 0 0 0 1; -0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0];
Ahat = [A zeros(4,1); -C 0];
Bhat = [B;0];
J = [-1+j*sqrt(3) -1-j*sqrt(3) -5 -5 -5];
Khat = acker(Ahat,Bhat,J)

Khat =

 -157.6336 -35.3733 -56.0652 -36.7466 50.9684
```

Thus, we get

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = [-157.6336 \quad -35.3733 \quad -56.0652 \quad -36.7466]$$

and

$$k_I = -50.9684$$

Unit Step-Response Characteristics of the Designed System. Once we determine the feedback gain matrix **K** and the integral gain constant k_1, the step response in the cart position can be obtained by solving the following equation, which is obtained by substituting Equation (12–49) into Equation (12–35):

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_I \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{12–53}$$

The output of the system is $x_3(t)$, or

$$y = [0 \quad 0 \quad 1 \quad 0 \quad 0] \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + [0]r \tag{12–54}$$

Define the state matrix, control matrix, output matrix, and direct transmission matrix of the system given by Equations (12–53) and (12–54) as AA, BB, CC, and DD, respectively. MATLAB Program 12–7 may be used to obtain the step-response curves of the designed system. Notice that, to obtain the unit-step response, we entered the command

```
[y,x,t] = step(AA,BB,CC,DD,1,t)
```

Figure 12–10 shows curves x_1 versus t, x_2 versus t, x_3 (= output y) versus t, x_4 versus t, and x_5 (= ξ) versus t. Notice that $y(t)$ $[= x_3(t)]$ has approximately 15% overshoot and the settling time is approximately 4.5 sec. $\xi(t)$ $[= x_5(t)]$ approaches 1.1. This result can be derived as follows: Since

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{A}\mathbf{x}(\infty) + \mathbf{B}u(\infty)$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} u(\infty)$$

MATLAB Program 12–7

```
%**** The following program is to obtain step response
% of the inverted-pendulum system just designed *****

A = [0 1 0 0;20.601 0 0 0;0 0 0 1;-0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0]
D = [0];
K = [-157.6336 -35.3733 -56.0652 -36.7466];
KI = -50.9684;
AA = [A - B*K B*KI;-C 0];
BB = [0;0;0;0;1];
CC = [C 0];
DD = [0];

%***** To obtain response curves x1 versus t, x2 versus t,
% x3 versus t, x4 versus t, and x5 versus t, separately, enter
% the following command *****

t = 0:0.02:6;
[y,x,t] = step(AA,BB,CC,DD,1,t);

x1 = [1 0 0 0 0]*x';
x2 = [0 1 0 0 0]*x';
x3 = [0 0 1 0 0]*x';
x4 = [0 0 0 1 0]*x';
x5 = [0 0 0 0 1]*x';

subplot(3,2,1); plot(t,x1); grid
title('x1 versus t')
xlabel('t Sec'); ylabel('x1')

subplot(3,2,2); plot(t,x2); grid
title('x2 versus t')
xlabel('t Sec'); ylabel('x2')

subplot(3,2,3); plot(t,x3); grid
title('x3 versus t')
xlabel('t Sec'); ylabel('x3')

subplot(3,2,4); plot(t,x4); grid
title('x4 versus t')
xlabel('t Sec'); ylabel('x4')

subplot(3,2,5); plot(t,x5); grid
title('x5 versus t')
xlabel('t Sec'); ylabel('x5')
```

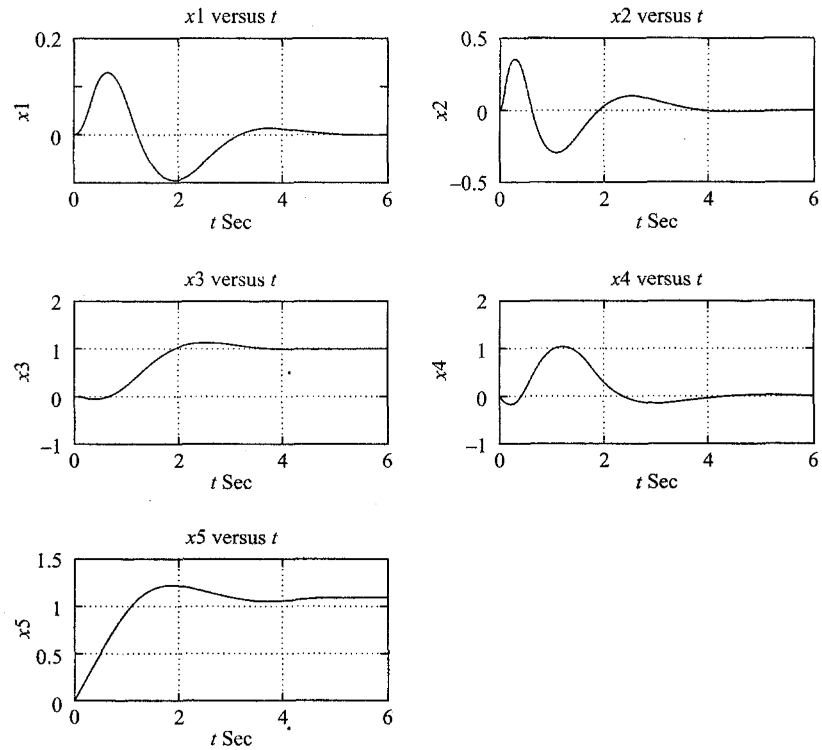

Figure 12–10 Curves x_1 versus t, x_2 versus t, x_3 (= output y) versus t, x_4 versus t, and x_5 (= ξ) versus t.

we get

$$u(\infty) = 0$$

Since $u(\infty) = 0$, we have, from Equation (12–33),

$$u(\infty) = 0 = -\mathbf{K}\mathbf{x}(\infty) + k_1\xi(\infty)$$

and so

$$\xi(\infty) = \frac{1}{k_I}\left[\mathbf{K}\mathbf{x}(\infty)\right] = \frac{1}{k_I}k_3x_3(\infty) = \frac{-56.0652}{-50.9684}r = 1.1r$$

Hence, for $r = 1$, we have

$$\xi(\infty) = 1.1$$

It is noted that, as in any design problem, if the speed and damping are not quite satisfactory, then we must modify the desired characteristic equation and determine a new matrix $\hat{\mathbf{K}}$. Computer simulations must be repeated until a satisfactory result is obtained.

12–5 STATE OBSERVERS

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.