Stabilization and trajectory tracking of a subactuated aircraft based on a Geometric Algebra approach

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Stabilization and trajectory tracking of a subactuated aircraft based on a Geometric Algebra approach

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***Abstract*— A novel approach to the modeling and control of a subactuated aircraft is performed based on Geometric Algebra (GA) principles. The selected platform for analysis is a quad rotorcraft. The derived model leverages objects from GA, such as the rotor, to perform rotations, replacing the need for Euler angles and quaternions. Controllers, which operate exclusively on GA objects, are developed to regulate the altitude, attitude, and translation of the quad rotorcraft. Numerical examples, including way-point navigation and trajectory tracking, illustrate the feasibility of the GA approach.**

I. Introduction

The quad rotorcraft unmanned aircraft system (UAS) dynamics are sub-actuated, i.e., it has fewer control inputs than the number of states to be controlled. Its dynamics are also nonlinear, further complicating controller design [1]. Traditionally, quad rotorcraft controllers have utilized Euler angles to represent and control the orientation of the platform. Euler angles, while intuitive to understand, are subject to singularities limiting the platform from performing complex and aggressive maneuvers [2]. To avoid this design challenge, alternative mathematical objects, such as quaternions, have been proposed. The controllers developed, while effective, are unnecessarily complex and are often non-intuitive. To address the aforementioned issues, this work introduces Geometric Algebra (GA) as the key to developing intuitive, elegant, and effective controllers.

GA may be described as the concept where geometry and algebra can be unified into a linear algebra system that allows the algebraic manipulation of mathematical elements (e.g., vectors, bivector, etc.) in such a way that all elements and operations on those elements have geometric meaning [3]–[6]. The principal argument for the adoption of GA is that it provides a single, simple mathematical framework that eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn [7], [8]. The key innovation of GA is a *true* vector product termed the *geometric product* and denoted by juxtaposition

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} \tag{1}$$

where $\cdot$ is the usual vector dot product (more generally, the left contraction) and $\wedge$ is the wedge product—which is closely related to the usual cross product (at least in three dimensions) [9]. The notation used within this work is presented in the following section.

Notation and Terminology

In GA, only real numbers are associated with "scalar" quantities which are denoted here using lowercase italic font i.e., $b, k, \ell, x, \alpha, \iota, \theta, \tau \in \mathbb{R}$. Traditional (polar) vectors are associated with directed lengths (here in Euclidean 3-space) and denoted using lowercase upright bold font i.e., $\mathbf{b}, \mathbf{e}, \mathbf{u}, \mathbf{w}, \mathbf{x}, \mathbf{v}, \mathbf{f} \in \mathbb{L}^3$. Traditional axial vectors are closely related with directed areas (bivectors) in Euclidean 3-space and denoted using uppercase upright bold font i.e., $\mathbf{B}, \mathbf{E}, \boldsymbol{\Tau}, \boldsymbol{\Omega} \in \mathbb{B}^3$. Trivectors in 3-space are pseudoscalars that can be represented as scaled versions of a unit directed volume denoted with $\mathbf{I}$, as $c\mathbf{I} \in \mathbb{T}^3$. Finally, a general multivector in Euclidean 3-space $A, B, M, R \in \mathbb{G}^3$ is the direct sum of the scalars (signed line segments), vectors (directed line segments), bivectors (directed areas), and trivectors (directed volumes) in Euclidean 3-space

$$\mathbb{G}^3 = \mathbb{R} \oplus \mathbb{L}^3 \oplus \mathbb{B}^3 \oplus \mathbb{T}^3 \tag{2}$$

and can be represented in terms of an orthonormal basis

$$\{1,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_3\mathbf{e}_1,\ \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\} \tag{3}$$

as

$$\begin{aligned} M = (m_0)(1) & \qquad \text{(scalar)} \\ + m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3 & \qquad \text{(vector)} \\ + m_4\mathbf{e}_1\mathbf{e}_2 + m_5\mathbf{e}_2\mathbf{e}_3 + m_6\mathbf{e}_3\mathbf{e}_1 & \qquad \text{(bivector)} \\ + m_7\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. & \qquad \text{(trivector)} \end{aligned} \tag{4}$$

One exception to the notational rules provided above is that unit bivectors $\mathbf{j} \in \{\mathbf{B} \in \mathbb{B}^3 : \|\mathbf{B}\| = 1\}$ are denoted with *lowercase* upright bold font $\mathbf{j}$ because they have the property

$$\mathbf{j}^2 = -1 \tag{5}$$

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and furthermore, they play a role similar to that of the traditional imaginary number (as a generator of rotations). A rotor is a unit norm multivector used to represent a rotation. In Euclidean 3-space, a rotor R is formed as the sum of a scalar and bivector $R \in \{M \in \mathbb{R} \oplus \mathbb{B}^3 : \|M\| = 1\}$ and is isomorphic to the traditional unit quaternion. The benefits of rotors over quaternions are that rotations maintain clear geometric interpretability and that the rotor belongs to the same space, $\mathbb{G}^3$, as all other elements in our dynamic model. For more about GA, refer to Macdonald's works [9]–[11].

Related Work

The dynamics and control of a quad rotorcraft are typically derived using the classical Euler angle approach, see for example [12] and the references therein. Numerous works follow this approach, as in [13], where the tracking of a quad rotorcraft was performed on SE(3). To overcome the limitations encountered in Euler angles-based methodologies, the authors in [14] developed a quaternion-based attitude controller, where a nonlinear P^2-controller is designed to stabilize the attitude of the quad rotorcraft. In [15] the full guidance and control of a quad rotorcraft are conducted using quaternions under numerical simulations. The authors in [16] develop a quaternion-based attitude controller, presenting simulations and a real-time hardware implementation.

These previous works were conducted within classic mathematical modeling frameworks. A few research works that have restructured classical mechanics in a GA sense are [4], [7], [17], [18]. Furthermore, GA has been applied to graphics and computer vision problems in [19]. Within [20]–[23], GA is used for modeling and control of robotic manipulators.

Main Contributions

The first contribution is the development of a six-degrees-of-freedom (6DoF) quad rotorcraft model within the GA framework. The model makes use of the GA rotor object to perform rotations, removing Euler angles and quaternions entirely. With rotors, rotations are simpler to perform mathematically, as well as maintain geometric interpretability. The developed GA model will also be coordinate-free. Secondly, controllers are developed to stabilize the quad rotorcraft and perform trajectory tracking. These controllers operate exclusively on elements from GA, without need of other mathematical frameworks. The behavior of the controllers are then demonstrated through numerical examples.

Organization of the Manuscript

The remainder of this manuscript is structured as follows: Section II rigorously specifies the control problem. Section III conveys the main results of this work. The GA based model is presented, along with a discussion of the methods used to develop controllers for the model. Afterwards, Section IV presents the numerical results of the GA based controllers on the 6DoF GA model. Lastly, Section V analyzes results, and presents future research directions.

II. PROBLEM STATEMENT

The actuators of a quad rotorcraft are four motors, with a designated propeller. Two motors spin counter-clockwise, while the other two spin clockwise. Four thrusts are available for control; one for each motor. Due to the subactuated nature of the quad rotorcraft system, the thrusts from the motors are not only used to control the altitude of the system, the torques generated by the thrusts are used to manipulate the orientation and position of the system. The convention is to place the inertial and body coordinate frame in a North-East-Down (NED) configuration. The inertial frame is taken to be the origin of the system. The body frame is placed at the center of mass of the UAS. To perform the task of stabilization and tracking, the following states are required: the position and velocity of the quad rotorcraft, the orientation of the quad rotorcraft, and the angular velocity of the platform, all of which are with respect to the inertial frame.

The first goal of this work is to completely model the 6DoF dynamics of a quad rotorcraft system in GA. The model will consist exclusively of GA objects and operations. Leveraging GA will allow the model to remain geometrically interpretable, simple to implement, and simple to express. The second goal is to develop controllers to accomplish stabilization and tracking of the GA dynamic model. All the states of the system will be formulated in GA. Control will be conducted completely on the GA states, independent of other mathematical frameworks.

III. MAIN RESULTS

The first contribution is the derivation of the GA-based 6 DoF dynamic model of a quad rotorcraft. The second contribution is the development of GA-based controllers designed to stabilize the UAS and perform trajectory tracking.

A. Dynamic Model

The derived illustration of the quad rotorcraft's dynamic model is shown in Fig. 1. The inertial frame is represented using the set of orthonormal vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The frame is oriented in a NED configuration, where $\mathbf{e}_1$ points along the North, $\mathbf{e}_2$ points along the East, and $\mathbf{e}_3$ points in the Down direction. The body frame is represented using another set of orthonormal vectors, $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. When the frames are perfectly aligned, the body frame is in a NED orientation. The dynamics of the quad rotorcraft are separated into two subsystems. The first subsystem is the rotational dynamics. The second subsystem is the translational dynamics.

1) Rotational Dynamics: The rotational relationship between frames may be expressed as

$$\mathbf{b}_k = R\mathbf{e}_k R^\dagger, \quad k = 1, 2, 3 \tag{6}$$

where $R^\dagger$ denotes the reverse of R and

$$R(t) = \mathrm{e}^{-\mathbf{j}(t)\vartheta(t)/2} \tag{7}$$

is the rotor representing the orientation of the system, (unit bivector) $\mathbf{j}$ is the plane of rotation, and (scalar) ϑ is the angle of rotation in the plane $\mathbf{j}$.

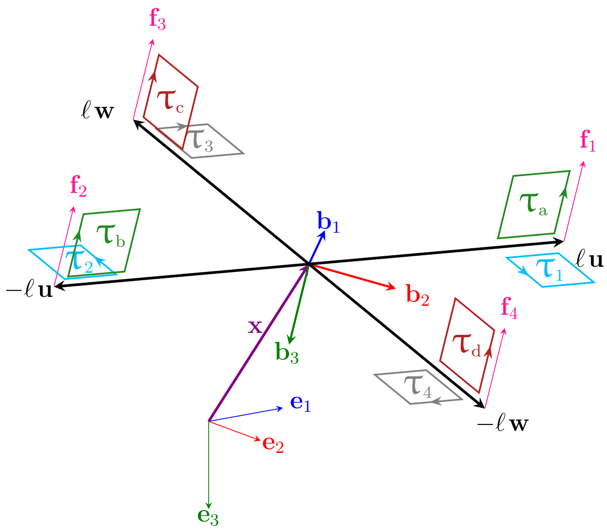

Fig. 1: Illustration of the GA-based quad rotorcraft dynamic model. The body frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a rotated and translated copy of the inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The rotation is represented with rotor R (not illustrated) and the translation is represented by the vector $\mathbf{x}$. The quad rotorcraft arms are represented as a superposition of the body frame vectors $\pm\ell\mathbf{u} = \pm\ell(\mathbf{b}_1 + \mathbf{b}_2)/\sqrt{2}$ and $\pm\ell\mathbf{w} = \pm\ell(\mathbf{b}_1 - \mathbf{b}_2)/\sqrt{2}$, where ℓ is the arm length. The forces generated by the motors are related to the body frame by $\mathbf{f}_1 = -f_1\mathbf{b}_3$, $\mathbf{f}_2 = -f_2\mathbf{b}_3$, $\mathbf{f}_3 = -f_3\mathbf{b}_3$, and $\mathbf{f}_4 = -f_4\mathbf{b}_3$, where f_i are scalars. The torques generated by the motors are related to the body frame by $\mathcal{T}_1 = -\alpha_1\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_2 = -\alpha_2\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_3 = \alpha_3\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_4 = \alpha_4\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_\mathrm{a} = \ell\mathbf{u} \wedge \mathbf{f}_1$, $\mathcal{T}_\mathrm{b} = -\ell\mathbf{u} \wedge \mathbf{f}_2$, $\mathcal{T}_\mathrm{c} = \ell\mathbf{w} \wedge \mathbf{f}_3$, and $\mathcal{T}_\mathrm{d} = -\ell\mathbf{w} \wedge \mathbf{f}_4$, where α_i are scalars.

Remark 1: The relationship between the rotor representing the orientation of the system in terms of Euler angles ϕ, θ, and ψ is [7]

$$R = \exp(-\mathbf{e}_1\mathbf{e}_2\phi/2)\exp(-\mathbf{e}_2\mathbf{e}_3\theta/2)\exp(-\mathbf{e}_1\mathbf{e}_2\psi/2).$$

This relationship is given for the convenience of the reader and remains unused in this work.

The derivative $\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{j}(t)\vartheta(t)\}$ gives the angular velocity bivector

$$\mathbf{\Omega} = \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{j}\vartheta) \tag{8a}$$
$$= -2\dot{R}R^\dagger. \tag{8b}$$

Defining the commutator product denoted by $[\cdot,\cdot]$ as

$$[A, B] \triangleq \tfrac{1}{2}(AB - BA), \tag{9}$$

the rotational equation of motion can be expressed as

$$R\Big(\mathcal{I}(\dot{\mathbf{\Omega}}_\mathrm{b}) - \big[\mathbf{\Omega}_\mathrm{b}, \mathcal{I}(\mathbf{\Omega}_\mathrm{b})\big]\Big)R^\dagger = \mathcal{T} \tag{10}$$

where $\mathcal{T}$ is the total external torque expressed in the inertial frame, $\mathcal{I}(\mathbf{B})$ is the inertia tensor with respect to mass density ρ [7], with

$$\mathcal{I}(\mathbf{B}) = \int \mathrm{d}^3x\rho\mathbf{x} \wedge (\mathbf{x} \cdot \mathbf{B}), \tag{11}$$

and $\mathbf{\Omega}_\mathrm{b}$ is the body angular velocity bivector given by

$$\mathbf{\Omega}_\mathrm{b} = R^\dagger\mathbf{\Omega}R. \tag{12}$$

The total external torque is given by the sum of the torques generated by the motors

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_\mathrm{a} + \mathcal{T}_\mathrm{b} + \mathcal{T}_\mathrm{c} + \mathcal{T}_\mathrm{d}. \tag{13}$$

Torques are related to the body frame by $\mathcal{T}_1 = -\alpha_1\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_2 = -\alpha_2\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_3 = \alpha_3\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_4 = \alpha_4\mathbf{b}_1 \wedge \mathbf{b}_2$, $\mathcal{T}_\mathrm{a} = \ell\mathbf{u} \wedge \mathbf{f}_1$, $\mathcal{T}_\mathrm{b} = -\ell\mathbf{u} \wedge \mathbf{f}_2$, $\mathcal{T}_\mathrm{c} = \ell\mathbf{w} \wedge \mathbf{f}_3$, and $\mathcal{T}_\mathrm{d} = -\ell\mathbf{w} \wedge \mathbf{f}_4$, where α_i are scalars and $\mathbf{f}_i$ are the forces due to the motors. In terms of the body angular velocity $\mathbf{\Omega}_\mathrm{b}$ the time derivative of the rotor R is

$$\dot{R} = -\tfrac{1}{2}\mathbf{\Omega}R = -\tfrac{1}{2}R\mathbf{\Omega}_\mathrm{b} \tag{14}$$

and its reverse

$$\dot{R}^\dagger = -\tfrac{1}{2}R^\dagger\mathbf{\Omega} = -\tfrac{1}{2}\mathbf{\Omega}_\mathrm{b}R^\dagger. \tag{15}$$

Solving for rotational acceleration with respect to the body angular velocity $\dot{\mathbf{\Omega}}_\mathrm{b}$, in the equation of motion yields

$$\mathcal{I}(\dot{\mathbf{\Omega}}_\mathrm{b}) - \big[\mathbf{\Omega}_\mathrm{b}, \mathcal{I}(\mathbf{\Omega}_\mathrm{b})\big] = R^\dagger\mathcal{T}R \tag{16a}$$
$$\mathcal{I}(\dot{\mathbf{\Omega}}_\mathrm{b}) = R^\dagger\mathcal{T}R + \big[\mathbf{\Omega}_\mathrm{b}, \mathcal{I}(\mathbf{\Omega}_\mathrm{b})\big] \tag{16b}$$
$$\dot{\mathbf{\Omega}}_\mathrm{b} = \mathcal{I}^{-1}\Big(R^\dagger\mathcal{T}R + \big[\mathbf{\Omega}_\mathrm{b}, \mathcal{I}(\mathbf{\Omega}_\mathrm{b})\big]\Big). \tag{16c}$$

For symmetric systems with principle planes of rotation $\mathbf{b}_1\mathbf{b}_2$, $\mathbf{b}_2\mathbf{b}_3$, and $\mathbf{b}_3\mathbf{b}_1$ the effect of the inertial tensor $\mathcal{I}(\mathbf{B})$ on an arbitrary bivector $\mathbf{B}$, can be computed by decomposing $\mathbf{B}$ in terms of the principle planes of rotation

$$\mathbf{B} = b_{12}\mathbf{b}_1\mathbf{b}_2 + b_{23}\mathbf{b}_2\mathbf{b}_3 + b_{31}\mathbf{b}_3\mathbf{b}_1 \tag{17}$$

and it follows that

$$\mathcal{I}(\mathbf{B}) = \iota_{12}b_{12}\mathbf{b}_1\mathbf{b}_2 + \iota_{23}b_{23}\mathbf{b}_2\mathbf{b}_3 + \iota_{31}b_{31}\mathbf{b}_3\mathbf{b}_1 \tag{18}$$

and

$$\mathcal{I}^{-1}(\mathbf{B}) = \frac{b_{12}}{\iota_{12}}\mathbf{b}_1\mathbf{b}_2 + \frac{b_{23}}{\iota_{23}}\mathbf{b}_2\mathbf{b}_3 + \frac{b_{31}}{\iota_{31}}\mathbf{b}_3\mathbf{b}_1 \tag{19}$$

where ι_{12}, ι_{23}, and ι_{31} are the principal moments of inertia.

2) Translational Dynamics: The displacement of the body frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ relative to the inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be represented with vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{20}$$

Thus, the velocity of the center of mass of the body is

$$\dot{\mathbf{x}} = \mathbf{v} \tag{21}$$

and the translational equation of motion can be expressed as

$$m\dot{\mathbf{v}} = mg\mathbf{e}_3 + \sum_{i=1}^{4}\mathbf{f}_i \tag{22}$$

where m is the body mass, and g is the gravity acceleration. The forces generated by the motors are related to the body frame by $\mathbf{f}_1 = -f_1\mathbf{b}_3$, $\mathbf{f}_2 = -f_2\mathbf{b}_3$, $\mathbf{f}_3 = -f_3\mathbf{b}_3$, and $\mathbf{f}_4 = -f_4\mathbf{b}_3$, where f_i are scalars.

B. Control Strategies Design

1) Attitude & Heading Control: Let R^{d} represent the *desired* orientation of the body frame

$$\mathbf{b}_k^{\mathrm{d}} = R^{\mathrm{d}} \mathbf{e}_k R^{\mathrm{d}\dagger}, \quad k = 1, 2, 3. \tag{23}$$

To align the desired and actual orientations of the body frame in three dimensions, it is sufficient to align two of the body frame vectors (say, the first two). Thus, we define a rotor

$$R_1 = (\mathbf{b}_1)(\mathbf{b}_1^{\mathrm{d}}) = \mathrm{e}^{\mathbf{j}_1 \theta_1} \tag{24}$$

that aligns $\mathbf{b}_1$ with $\mathbf{b}_1^{\mathrm{d}}$ and a second rotor

$$R_2 = (\mathrm{e}^{-\mathbf{j}_1\theta_1/2} \mathbf{b}_2 \mathrm{e}^{\mathbf{j}_1\theta_1/2})(\mathbf{b}_2^{\mathrm{d}}) = \mathrm{e}^{\mathbf{j}_2 \theta_2} \tag{25}$$

that aligns the rotated $\mathbf{b}_2$ with $\mathbf{b}_2^{\mathrm{d}}$. Then, compose the two rotations R_1 and R_2 as

$$R_{\mathrm{e}} = \mathrm{e}^{-\mathbf{j}_2\theta_2/2} \mathrm{e}^{-\mathbf{j}_1\theta_1/2} = \mathrm{e}^{\mathbf{j}_{\mathrm{e}}\theta_{\mathrm{e}}} \tag{26}$$

Finally, the rotational error bivector is defined as

$$\mathbf{E}_{\mathrm{r}} = -2\mathbf{j}_{\mathrm{e}}\theta_{\mathrm{e}}, \tag{27}$$

noting that it encodes both the (scalar) angle $2\theta_{\mathrm{e}}$ and plane (unit bivector) $\mathbf{j}_{\mathrm{e}}$ of rotational error, as a single quantity.

To control the attitude subsystem of the quad rotorcraft, a (bivector-valued) proportional-integral-derivative (PID) controller is applied with the following control law

$$\boldsymbol{\tau} = k_{\mathrm{p}\tau}\mathbf{E}_{\mathrm{r}} + k_{\mathrm{i}\tau}\int \mathbf{E}_{\mathrm{r}}\,\mathrm{d}t + k_{\mathrm{d}\tau}\frac{\mathrm{d}\mathbf{E}_{\mathrm{r}}}{\mathrm{d}t} \tag{28}$$

where $k_{\mathrm{p}\tau}$, $k_{\mathrm{i}\tau}$, $k_{\mathrm{d}\tau}$ are the PID control gains.

2) Translational Control: Let $\boldsymbol{\xi}$ denote the translational displacement of the body frame in the $\mathbf{e}_1\mathbf{e}_2$ plane

$$\boldsymbol{\xi} = (\mathbf{x}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{x}\cdot\mathbf{e}_2)\mathbf{e}_2 \tag{29}$$

and $\boldsymbol{\xi}^{\mathrm{d}}$ represent the *desired* translational displacement of the body frame in the $\mathbf{e}_1\mathbf{e}_2$ plane. The translational displacement error vector is defined as

$$\mathbf{d}_{\mathrm{t}} = \boldsymbol{\xi} - \boldsymbol{\xi}^{\mathrm{d}} \tag{30}$$

and the translational error bivector as

$$\mathbf{E}_{\mathrm{t}} = -\mathbf{d}_{\mathrm{t}} \wedge \mathbf{e}_3 \tag{31}$$

noting that it encodes both the translational displacement error magnitude $\|\mathbf{E}_{\mathrm{t}}\| = \|\mathbf{d}_{\mathrm{t}}\|$ and plane (unit bivector) of rotation

$$\mathbf{j}_{\mathrm{t}} = \begin{cases} \mathbf{E}_{\mathrm{t}}/\|\mathbf{E}_{\mathrm{t}}\|, & \|\mathbf{E}_{\mathrm{t}}\| \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

required to correct the translation displacement error, as a single quantity.

Due to the subactuated nature of the quad rotorcraft, a change in the orientation of the system is required to perform a linear translation. Let R_{T} be the rotor responsible for performing translational movement. R_{T} can be expressed as

$$R_{\mathrm{T}} = R_{\psi} R_{\mathrm{t}} \tag{33}$$

where, R_{ψ} is the desired heading and R_{t} is the amount of rotation in the $\mathbf{e}_2\mathbf{e}_3$ and $\mathbf{e}_3\mathbf{e}_1$ planes. To form R_{t}, a bivector must be specified. Let $\mathbf{U}_{\mathrm{t}}$ be the bivector corresponding to R_{t}, a (bivector-valued) PID controller with the following control law is applied,

$$\mathbf{U}_{\mathrm{t}} = k_{\mathrm{pt}}\mathbf{E}_{\mathrm{t}} + k_{\mathrm{it}}\int \mathbf{E}_{\mathrm{t}}\,\mathrm{d}t + k_{\mathrm{dt}}\frac{\mathrm{d}\mathbf{E}_{\mathrm{t}}}{\mathrm{d}t} \tag{34}$$

where k_{pt}, k_{it}, k_{dt} are gains of the translational controller. R_{t} is then formed as

$$R_{\mathrm{t}} = \exp(-R^{\dagger}\mathbf{U}_{\mathrm{t}}R/2). \tag{35}$$

R_{T} then serves as the input to the attitude controller.

3) Altitude Control: Let h^{d} represent the *desired* height of the body frame with respect to the inertial frame

$$h = -\mathbf{x}\cdot\mathbf{e}_3. \tag{36}$$

Then, define the altitude error as

$$e_{\mathrm{h}} = h - h^{\mathrm{d}}. \tag{37}$$

To control the altitude subsystem of the quad rotorcraft, a (scalar-valued) PID controller is applied, with the form

$$u_{\mathrm{h}} = k_{\mathrm{ph}}e_{\mathrm{h}} + k_{\mathrm{ih}}\int e_{\mathrm{h}}\,\mathrm{d}t + k_{\mathrm{dh}}\frac{\mathrm{d}e_{\mathrm{h}}}{\mathrm{d}t} = \sum_{i=1}^{4} f_i \tag{38}$$

where k_{ph}, k_{ih}, k_{dh} are the gains of the altitude controller.

IV. Numerical Results

The Julia programming language was utilized to realize the GA model, with the aide of the Grassmann package [24]. The parameters of the quad rotorcraft were heuristically determined as: $m = 1.0$kg, $\ell = 3$m, and $\iota_1 = \iota_2 = \iota_3 = 1.0$kg·m^2. The gains selected for the altitude controller are $k_{\mathrm{ph}} = 15.0$, $k_{\mathrm{ih}} = 1.5$, $k_{\mathrm{dh}} = 9.5$. The gains for the translational controller are: $k_{\mathrm{pt}} = 0.01$, $k_{\mathrm{it}} = 0.0012$, $k_{\mathrm{dt}} = 0.075$. The gains for the attitude controller are: $k_{\mathrm{p}\tau} = 1.0$, $k_{\mathrm{i}\tau} = 0.005$, $k_{\mathrm{d}\tau} = 3.0$. All gains were heuristically tuned until satisfactory performance was achieved.

A. Way-point Navigation

Given specified way-points, $\mathbf{x}_k^{\mathrm{d}}$, where $k \in [1, 2, \ldots, 5]$, the task of the controller is to regulate the UAS's position to $\mathbf{x}_k^{\mathrm{d}}$, while maintaining a constant heading. The initial conditions are $\mathbf{x} = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$, $\mathbf{v} = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$, $R = 1.0$, and $\boldsymbol{\Omega} = 0\mathbf{e}_{12} + 0\mathbf{e}_{23} + 0\mathbf{e}_{31}$. The desired waypoints for this experiment are: $\mathbf{x}_1^{\mathrm{d}} = 5\mathbf{e}_1 + 5\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{x}_2^{\mathrm{d}} = -5\mathbf{e}_1 + 5\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{x}_3^{\mathrm{d}} = -5\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{x}_4^{\mathrm{d}} = 5\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_3$, and $\mathbf{x}_5^{\mathrm{d}} = 5\mathbf{e}_1 + 5\mathbf{e}_2 - \mathbf{e}_3$. The desired waypoints define a square trajectory with a constant altitude.

The desired orientation of the UAS is the unit rotor, meaning that inertial and body frames should be aligned during flight. Simulation results are presented in Figs. 2-5.

Fig. 2 demonstrates that the quad rotorcraft is able to take off from the origin and attain the desired altitude of 1m. The flight path is square, indicating successful tracking of the way-points. The components of $\mathbf{x}$ are plotted in Fig. 3.

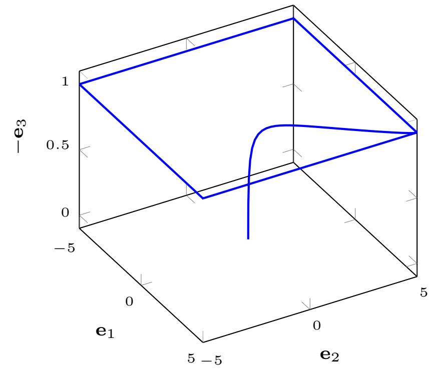

Fig. 2: Position $\mathbf{x}$ of the quad rotorcraft with respect to the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ frame for the way-point tracking flight trajectory.

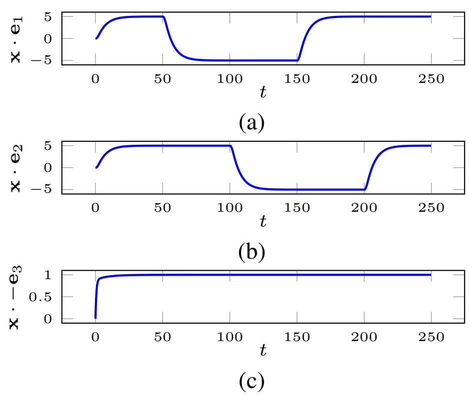

Fig. 3: Position $\mathbf{x}$ in the (a) $\mathbf{e}_1$, (b) $\mathbf{e}_2$, and (c) $\mathbf{e}_3$ directions.

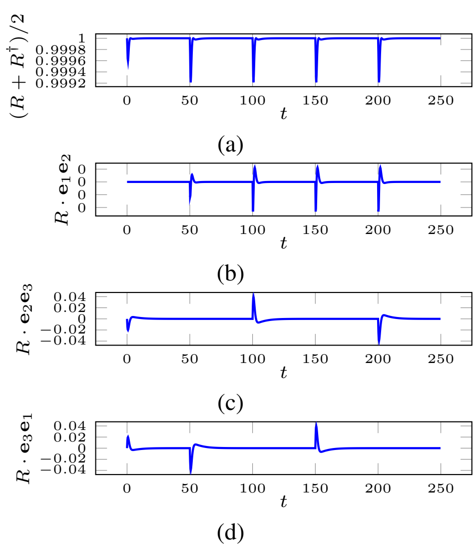

Fig. 4: Rotation of the quad rotorcraft represented by the rotor R with (a) scalar, (b) $\mathbf{e}_1\mathbf{e}_2$, (c) $\mathbf{e}_2\mathbf{e}_3$, and (d) $\mathbf{e}_3\mathbf{e}_1$ components.

To gain further insight into the performance of the attitude/translation controllers, the components of the rotor R are plotted in Fig. 4. Deviation of the scalar component of R from unity is indicative of rotational maneuvers to correct attitude and/or translational error. The mostly zero $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$, and $\mathbf{e}_3\mathbf{e}_1$ components of R, indicate that the body frame is aligned with the inertial frame. The large spikes occurring with period 50 occur when translational movement is required to track the way-points. The error plots in Fig. 5 show that the attitude, altitude, and translation controllers converge to zero, until the way-point changes.

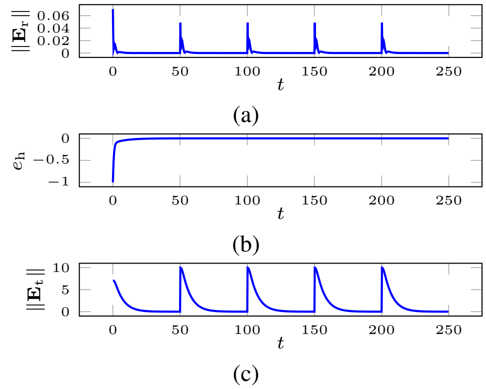

Fig. 5: The quad rotorcraft (a) attitude error magnitude $\|\mathbf{E}_\mathrm{r}\|$, (b) altitude error e_h, and (c) translation error magnitude $\|\mathbf{E}_\mathrm{t}\|$ for the square flight trajectory.

B. Trajectory Tracking

The UAS is tasked to follow a circular trajectory while rotating the heading. The initial conditions are $\mathbf{x} = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$, $\mathbf{v} = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$, $R = 1.0$, $\mathbf{\Omega} = 0\mathbf{e}_{12} + 0\mathbf{e}_{23} + 0\mathbf{e}_{31}$. Results are shown in Figs. 6 through 9.

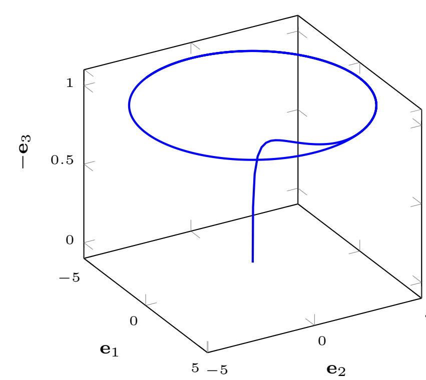

Fig. 6: Position $\mathbf{x}$ of the quad rotorcraft with respect to the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ frame for the circular flight trajectory.

Fig. 6 demonstrates that the UAS is able to take off from the origin and attain the desired altitude of 1m. The flight trajectory is circular, indicating successful tracking. The components of $\mathbf{x}$ are plotted in Fig. 7.

To gain further insight into the performance of the attitude/translation controllers, the components of the rotor R are plotted in Fig. 8. The sinusoidal signals in the scalar and $\mathbf{e}_1\mathbf{e}_2$ (yaw) components of the rotor R are indicative of the heading. Deviation of the $\mathbf{e}_2\mathbf{e}_3$ (roll) and $\mathbf{e}_3\mathbf{e}_1$ (pitch) components of the rotor R are indicative of corrective maneuvers to track the desired trajectory. The error plots in Fig. 9 show that the attitude, altitude, and translation controllers converge to zero.

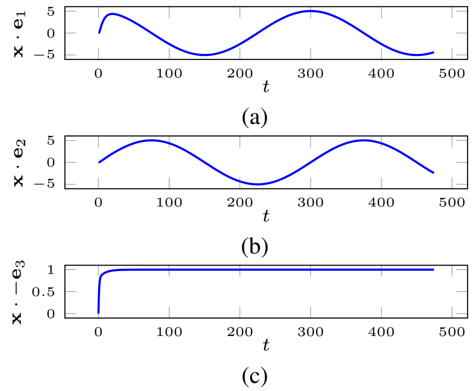

Fig. 7: Position $\mathbf{x}$ of the quad rotorcraft in the (a) $\mathbf{e}_1$, (b) $\mathbf{e}_2$, and (c) $\mathbf{e}_3$ directions.

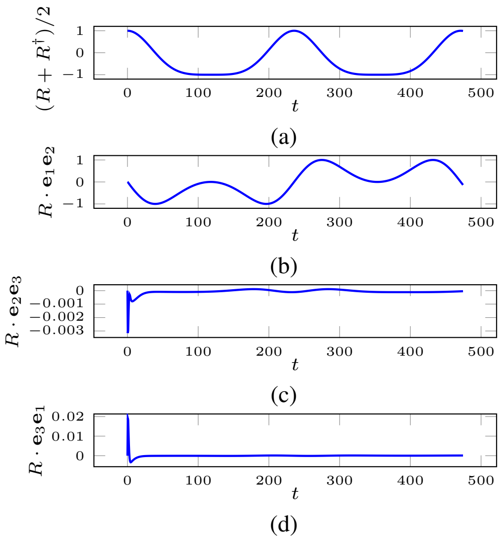

Fig. 8: Rotation of the quad rotorcraft represented by the rotor R with (a) scalar, (b) $\mathbf{e}_1\mathbf{e}_2$, (c) $\mathbf{e}_2\mathbf{e}_3$, and (d) $\mathbf{e}_3\mathbf{e}_1$ components.

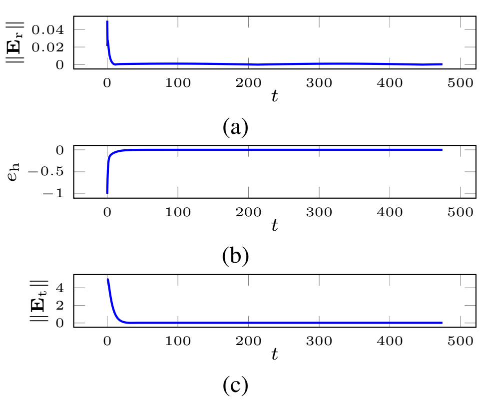

Fig. 9: The quad rotorcraft (a) attitude error magnitude $\|\mathbf{E}_\mathrm{r}\|$, (b) altitude error e_h, and (c) translation error magnitude $\|\mathbf{E}_\mathrm{t}\|$ for the circular flight trajectory.

V. Conclusions

The dynamics of a 6DoF quad rotorcraft system were modeled based on GA principles. Through the use of GA, the quad rotorcraft's orientation is described and manipulated using the GA rotor and the geometric product—thereby replacing the need for other tools such as rotation matrices and quaternions. Novel GA-based controllers were developed to perform stabilization and trajectory tracking tasks. The developed controllers operated strictly on GA objects, independent of other mathematical frameworks.

Numerical simulations demonstrate the ability of the GA controllers to stabilize the quad rotorcraft's altitude, attitude, and position during trajectory tracking tasks.

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