Names: Omar Abdelmotaleb and Benjamin Singleton
Pledge: I pledge my honor that I have abided by the Stevens Honor System. -Omar Abdelmotaleb
Pledge: I pledge my honor that I have abided by the Stevens Honor System. -Benjamin Singleton

Problem 1. (10 points) Show that the following language is decidable by giving a high-level description of a TM that decides the language.

$$\{< M >: M \text{ is a } PDA \text{ and } L(M) \text{ is an infinite language } \}$$

$Prob_1$ = {< M >: M is a PDA and L(M) is an infinite language}

TM T to decide $Prob_1$:

On input < M >

1. Convert M into an equivalent CFG in Chomsky Normal Form.
2. If the equivalent CFG contains the rule A → BC and as it branches off through B and C, if it eventually branches back into the rule A → BC again while adding one or more terminal symbols to the string, then we have a loop and we will accept.
3. If no such loop exists or no terminal symbols are added to the string within a loop, then reject.

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Problem 2. (10 points) Let G be a context-free grammar that generates strings over the alphabet $\Sigma = \{a, b\}$. Show that the problem of determining if G generates a string in a^* is decidable. In other words, show that the following language is decidable:

$$\{< G >: G \text{ } is \text{ } a \text{ } CFG \text{ } over \text{ } \{a, b\} \text{ } and \text{ } a^* \cap L(G) \neq \phi \}$$

$Prob_2 = \{< G >: G$ is a CFG over $\{a,b\}$ and $a^* \cap L(G) \neq \Phi\}$

TM T to decide $Prob_2$:

On input <G>

1. Let H be a CFG for $a^* \cap L(G)$.
2. Construct TM T_1 such that T_1 simulates T.
3. Let T_1 decide if H is empty, where T accepts if H is empty, otherwise it rejects.

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Problem 3. (10 points) Let A be a TM-recognizable language of strings that encode TMs that are deciders. Prove that there is a decidable language which is not decided by any TM in A. (Hint: start with an enumerator for A.)

Let M be the Turing Machine that decides a language L which is not decided by any Turing Machine contained in A. M can be made so that the language it accepts, L, is not accepted by any TM in A. We use an enumerator E to print out each of the TMs in A. Let T_i be the i-th TM in A and let w_i be a string accepted by T_i. We then make the machine M so that when T_i accepts w_i, M would reject w_i; and inversely, when T_i rejects w_i, M would accept w_i. This means that the language L would be accepted by M, but rejected by the given TM T_i. This process would be repeated as E enumerates over all of the string in A. After enumerating over all of A, the machine M would accept the language L which would be rejected by every TM in A. As such, the language L would be decided by M which is outside of A and not decided by any of the TMs within A.

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Problem 4. (10 points) Consider the problem of determining whether a TM M on input w ever attempts to move its head left when its head is on the leftmost tape cell.
a) Formulate this problem as a decision problem for a language, and
b) Show that the language is undecidable.

a) $Prob_4$ = {< M,w >: On input of string w, M attempts to move its head left when its head is on the leftmost tape cell}

b) Proof: We will show that $A_{TM} \leq Prob_4$
Assume that $Prob_4$ is decidable and let R be a TM to decide it.
We will use R to construct a TM S that decides A_{TM}
S: On input < M,w >
   1. Construct TM M_1
   2. Run $R(M_1)$:
      If $R(M_1)$ accepts, then Accept
      If $R(M_1)$ rejects, then Reject

M_1: On input w, mark the leftmost tape cell with a symbol and then shift right one. Begin the program from the second to leftmost cell. If the marked cell is encountered, then the head moves the to the right one cell. If M(w) accepts, then M_1 moves to the leftmost cell and moves left off the tape.

Because this is a contradiction, A_{TM} is undecidable and so is $Prob_4$.