CS302 – Analysis and Design of Algorithms

Algorithm Analysis

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Introduction

- Algorithm analysis: predicting the resources that the algorithm requires.
- Resources can be:
  - Computational time → most common and important
  - Memory
  - Communication bandwidth
  - Energy consumption
- Computational model:
  - Random Access Machine (RAM) with one processor
  - Execute one operation at time; no concurrent operations
  - Each instruction or data access takes a constant amount of time

Introduction

- Our RAM model have the following instructions:
  - Arithmetic: +, -, %, *, /, floor, ceiling
  - Data movement: load, store, copy
  - Control: conditional and unconditional branch, call, return
  - Bitwise: AND, OR, NOT, XOR, shifting
- RAM model has the following data types:
  - Float, int, char
- RAM model doesn’t have instructions for:
  - Sorting, exponentiation, summation

Introduction

- We cannot compute the running time of an algorithm by implementing it on a programming language and compute the execution time.
  - It depends on the programming language.
  - It depends on the compiler used to compile the program
  - It depends on the libraries used in the program
  - It depends on the architecture and the specifications of your machine
  - It depends on how the user inputs the data

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Analyzing Insertion Sort

- The running time depends on the input size.
  - Larger arrays take longer than smaller arrays.

- Insertion sort depends on the size of the array and how sorted it is.

- The input size has the greatest effect on the performance of the algorithm.
  - Hence, we will describe the running time as a function of the size of its input.
  - To do so, we need to define the terms **running time** and **input size** carefully.

Analyzing Insertion Sort

- The definition of “input size” depends on the application.
  - For sorting an array, the input size means the number of elements in the array.
  - For multiplying two integers, the input size refers to the bit size of the number.
  - For graph-based algorithms, the input size can be the number of edges and vertices.
- The running time is the number of instructions and data accesses executed.
  - Should be independent of any particular computer, but within the framework of the RAM model.
  - We assume that each line in our pseudocode, takes a constant amount of time, c_k, where k is the line number.

Analyzing Insertion Sort

```
INSERTION-SORT(A, n)                                          cost    times
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i - 1].
    j = i - 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j - 1
    A[j + 1] = key
```

Analyzing Insertion Sort

```
INSERTION-SORT(A, n)                                   cost   times
for i = 2 to n                                         c_1    n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

Analyzing Insertion Sort

Note that the loop condition is executed one time more than the loop body – that is when the condition is false

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$\quad key = A[i]$	c_2	$n - 1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.		
$\quad j = i - 1$		
$\quad$ **while** $j > 0$ and $A[j] > key$		
$\quad\quad A[j + 1] = A[j]$		
$\quad\quad j = j - 1$		
$\quad A[j + 1] = key$		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$\quad$ *key* $= A[i]$	c_2	$n-1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$\quad$ $j = i - 1$		
$\quad$ **while** $j > 0$ and $A[j] >$ *key*		
$\quad\quad$ $A[j+1] = A[j]$		
$\quad\quad$ $j = j - 1$		
$\quad$ $A[j+1] =$ *key*		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n - 1$
// Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.	0	$n - 1$
$j = i - 1$	c_4	$n - 1$
while $j > 0$ and $A[j] > key$		
$A[j + 1] = A[j]$		
$j = j - 1$		
$A[j + 1] = key$		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n - 1$
// Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.	0	$n - 1$
$j = i - 1$	c_4	$n - 1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j + 1] = A[j]$		
$j = j - 1$		
$A[j + 1] = key$		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n - 1$
// Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.	0	$n - 1$
$j = i - 1$	c_4	$n - 1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j + 1] = A[j]$	c_6	$\sum_{i=2}^{n} (t_i - 1)$
$j = j - 1$		
$A[j + 1] = key$		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n - 1$
// Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.	0	$n - 1$
$j = i - 1$	c_4	$n - 1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j + 1] = A[j]$	c_6	$\sum_{i=2}^{n} (t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n} (t_i - 1)$
$A[j + 1] = key$		

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n - 1$
// Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n - 1$
$j = i - 1$	c_4	$n - 1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n}(t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n}(t_i - 1)$
$A[j+1] = key$	c_8	$n - 1$

Analyzing Insertion Sort

- The running time of the algorithm, $T(n)$ is

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$\quad key = A[i]$	c_2	$n-1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$\quad j = i-1$	c_4	$n-1$
$\quad$ **while** $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$\quad\quad A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n}(t_i - 1)$
$\quad\quad j = j-1$	c_7	$\sum_{i=2}^{n}(t_i - 1)$
$\quad A[j+1] = key$	c_8	$n-1$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n}(t_i - 1) + c_7 \sum_{i=2}^{n}(t_i - 1) + c_8(n-1) .$$

Analyzing Insertion Sort

- Even for two arrays of the same size, the running time depends on how sorted the array is.

- The best case happens when the array is already sorted.

- In this case, the while loop at line 5 and its body will not execute.
  - The loop header will be executed as condition only - so it is counted as a statement with cost $c_5(n-1)$

Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$\quad key = A[i]$	c_2	$n-1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$\quad j = i - 1$	c_4	$n-1$
$\quad$ **while** $j > 0$ and $A[j] > key$ $\quad c_5(n-1)$	$\cancel{c_5}$	$\cancel{\sum_{i=2}^{n} t_i}$
$\quad\quad \cancel{A[j+1] = A[j]}$	$\cancel{c_6}$	$\cancel{\sum_{i=2}^{n} (t_i - 1)}$
$\quad\quad \cancel{j = j - 1}$	$\cancel{c_7}$	$\cancel{\sum_{i=2}^{n} (t_i - 1)}$
$\quad A[j+1] = key$	c_8	$n-1$

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) .
\end{aligned}
$$

Analyzing Insertion Sort

INSERTION-SORT(A, n)		cost	times
for $i = 2$ **to** n		c_1	n
$\quad key = A[i]$		c_2	$n-1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1:i-1]$.		0	$n-1$
$\quad j = i - 1$		c_4	$n-1$
$\quad$ **while** $j > 0$ and $A[j] > key$	$c_5(n-1)$	~~c_5~~	~~$\sum_{i=2}^{n} t_i$~~
~~$\quad\quad A[j+1] = A[j]$~~		~~c_6~~	~~$\sum_{i=2}^{n} (t_i - 1)$~~
~~$\quad\quad j = j - 1$~~		~~c_7~~	~~$\sum_{i=2}^{n} (t_i - 1)$~~
$\quad A[j+1] = key$		c_8	$n-1$

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= \underbrace{(c_1 + c_2 + c_4 + c_5 + c_8)}_{a} n - \underbrace{(c_2 + c_4 + c_5 + c_8)}_{b} .
\end{aligned}
$$

Analyzing Insertion Sort

INSERTION-SORT(A, n)		*cost*	*times*
for $i = 2$ **to** n		c_1	n
$\quad key = A[i]$		c_2	$n - 1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.		0	$n - 1$
$\quad j = i - 1$		c_4	$n - 1$
$\quad$ **while** $j > 0$ and $A[j] > key$	$c_5(n-1)$	$\cancel{c_5}$	$\cancel{\sum_{i=2}^{n} t_i}$
~~$A[j + 1] = A[j]$~~		$\cancel{c_6}$	$\cancel{\sum_{i=2}^{n} (t_i - 1)}$
~~$j = j - 1$~~		$\cancel{c_7}$	$\cancel{\sum_{i=2}^{n} (t_i - 1)}$
$\quad A[j + 1] = key$		c_8	$n - 1$

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= \underbrace{(c_1 + c_2 + c_4 + c_5 + c_8)n}_{a} - \underbrace{(c_2 + c_4 + c_5 + c_8)}_{b} .
\end{aligned}
$$

The running time is thus a linear function of n.

$an + b$

Analyzing Insertion Sort

- The worst case arises when the array is in reverse sorted order.
  - That is, it starts out in decreasing order.

- The procedure must compare each element $A[i]$ with each element in the entire sorted subarray.
  - The procedure finds that $A[i] > key$ every time in line 5, and the while loop exits only when j reaches 0.
- So, we get the running time expressed as (the same first equation)

$$
\begin{aligned}
T(n) \;=\; & c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) \\
& + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1)\,.
\end{aligned}
$$

Analyzing Insertion Sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1)\,.$$

$$\sum_{i=2}^{n} i = \left(\sum_{i=1}^{n} i\right) - 1 = \frac{n(n+1)}{2} - 1$$

$$\sum_{i=2}^{n} (i-1) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

Analyzing Insertion Sort

- So, the worst-case running time is

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\
&\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\
&= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right) n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right) n \\
&\quad - (c_2 + c_4 + c_5 + c_8)\ .
\end{aligned}
$$

- Thus, the running time is a quadratic function

$$an^2 + bn + c$$

Content

Worst-case and Average-case Analysis

- When analyzing an algorithm, we focus on the worst-case running time. Why?
  1. It gives an upper bound on the running time for any input.
     - If you know it, then you have a guarantee that the algorithm never takes any longer.
  2. For some algorithms, the worst case occurs fairly often.
     - In searching a database, the searching algorithm’s worst case often occurs when the information is not present in the database.
  3. The “average case” is often roughly as bad as the worst case.

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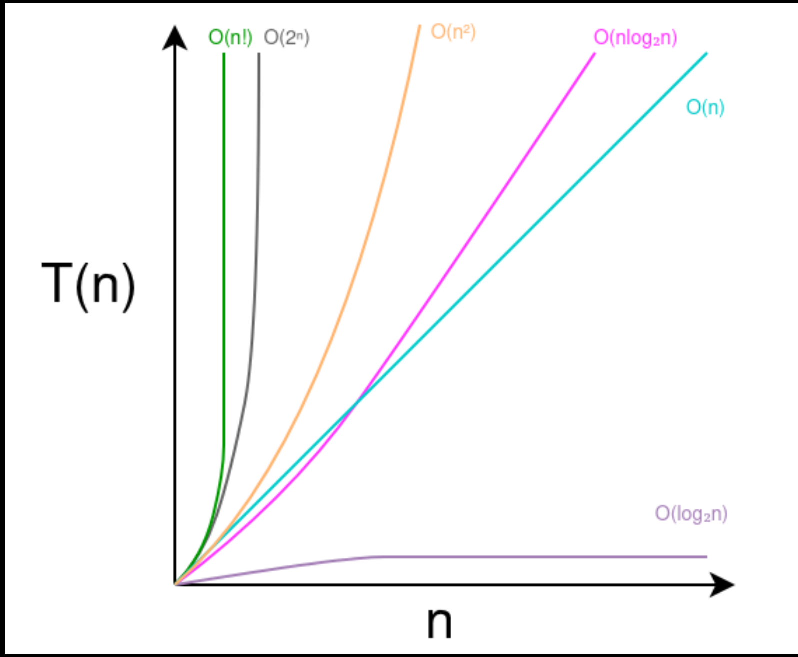
Order of growth

- Order of growth: how the number of computational steps is increasing as the input size n grows.
- To get the order of growth, we ignore constants and low-order terms.
  - For the insertion sort algorithm, the order of growth is n^2.



Order of growth

- For example, suppose that an algorithm implemented on a particular machine takes $n^2/100 + 100n + 17$ microseconds on an input of size n.
  - If you consider some small values for $n = 5, 10, 90$, you find that the $1/100$ coefficient of n^2 is less significant than the 100 coefficient of n.
  - But, when you consider large values for $n = 200$, you will find that n^2 dominates the n.
  - Large data size is what reflects real-life problems.

Order of growth

- When we consider only the dominant variable of high-order of a function, e.g., n^2, we are studying **asymptotic** efficiency.

- **Asymptotic analysis** is a method used to describe the behavior of algorithms or functions as their input size grows towards infinity.

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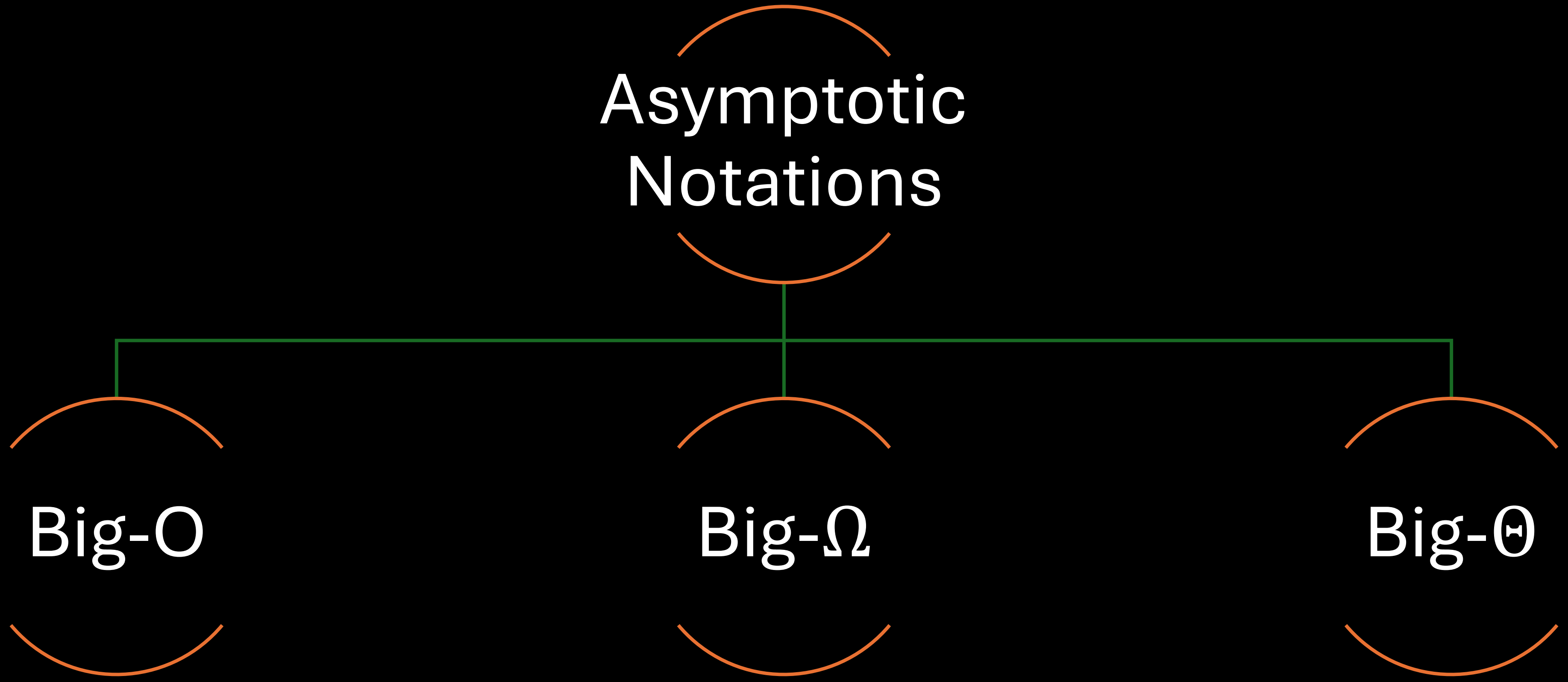
Asymptotic Notations



Asymptotic Notations - O

- O-notation characterizes an **upper bound** on the asymptotic behavior of a function.
  - A function grows no faster than a certain rate, based on the highest-order term.
- For example, the function $7n^3 + 100n^2 - 20n + 6$.
  - Its highest-order term is $7n^3$
  - So, we say that this function's rate of growth is n^3
  - Since the function grows no faster than n^3, we say it is $O(n^3)$
- Can we say it is $O(n^4)$?

Asymptotic Notations - O

- O-notation characterizes an **upper bound** on the asymptotic behavior of a function.
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- For example, the function $7n^3 + 100n^2 - 20n + 6$.
  - Its highest-order term is $7n^3$
  - So, we say that this function's rate of growth is n^3
  - Since the function grows no faster than n^3, we say it is $O(n^3)$
- Can we say it is $O(n^4)$?
  - Yes, because the function still grows no faster than n^4.
  - It is also $O(n^5), O(n^6)$, ...
  - Generally, it's $O(n^c)$ for $c \geq 3$

Asymptotic Notations - Ω

- Ω-notation characterizes a **lower bound** on the asymptotic behavior of a function.
  - A function grows at least as fast as a certain rate, based on the highest-order term.

- The function $7n^3 + 100n^2 - 20n + 6$.
  - It grows at least as fast as $\Omega(n^3)$.
  - It is also $\Omega(n^2), \Omega(n), \ldots$
  - Generally, it is $\Omega(n^c)$ for $c \leq 3$

Asymptotic Notations - Θ

- Θ-notation characterizes a **tight bound** on the asymptotic behavior of a function.
  - A function grows precisely at a certain rate, based on the highest-order term.
- It characterizes the rate of growth of the function to within a constant factor from above and to within a constant factor from below.
  - These two constant factors need not be equal.
- If you show that a function is $O\big(f(n)\big)$ and $\Omega\big(f(n)\big)$, then you have shown it is $\Theta\big(f(n)\big)$
- The function $7n^3 + 100n^2 - 20n + 6$ is $\Theta(n^3)$

Asymptotic Notations

- Analyzing insertion sort running time without evaluating summations.

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

Asymptotic Notations

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

- The procedure has 2 loops:
  - The outer loop runs $n - 1$ times, regardless of the values being sorted.
  - The inner loop depends on the values being sorted.
  - The *while* loop variable j starts at $i - 1$ and decreases by 1 until it either reaches 0 or $A[j] > key$.
  - The while loop might iterate 0 times, $i \ - 1$ times, or anywhere in between.
  - So, the outer loop runs $n - 1$ times, the second loop runs at most $i - 1$times, and because i is at most n. Hence, it is $(n - 1)(n - 1) = \ O(n^2)$

Asymptotic Notations

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

- So, $O(n^2)$ describes an upper-bound for any case of insertion sort.

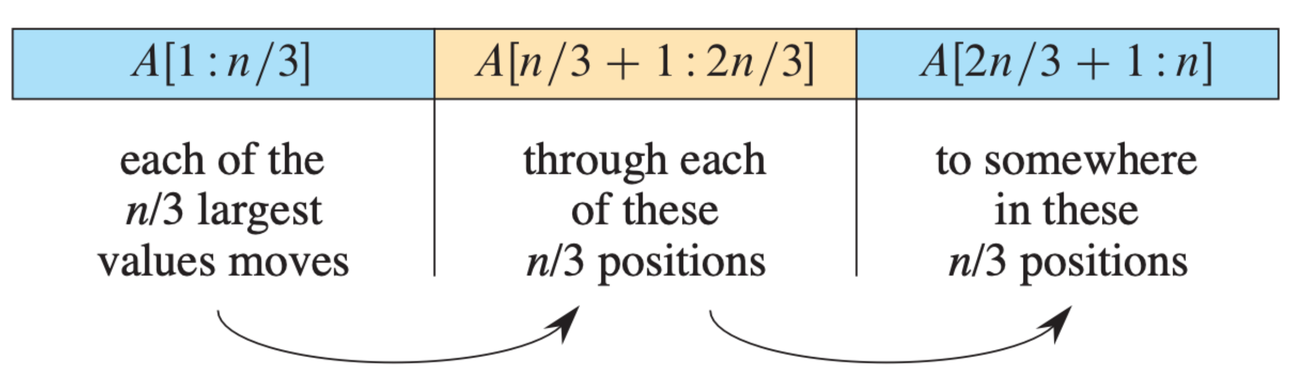
Asymptotic Notations

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

- The lower-bound of the worst-case running time is $\Omega(n^2)$.
- WHY?

Asymptotic Notations

- Assume that we have an array A of size n (multiple of 3), and the $n/3$ largest values are on the positions $A[1:n/3]$



- To sort the array, we will need to move each of the $A[1:n/3]$ to positions $A[2n/3+1:n]$.
  - Thus, the values will pass through the middle $A[n/3+1:2n/3]$ positions.
- Hence, the <u>worst-case lower-bound</u> is $(n/3)(n/3) = \Omega(n^2)$

Asymptotic Notations

- Because we have shown that insertion sort runs in $O(n^2)$ time in all cases.

- And that there is an input that makes it take $\Omega(n^2)$ time.

- We can conclude that the worst-case running time of insertion sort is $\Theta(n^2)$.

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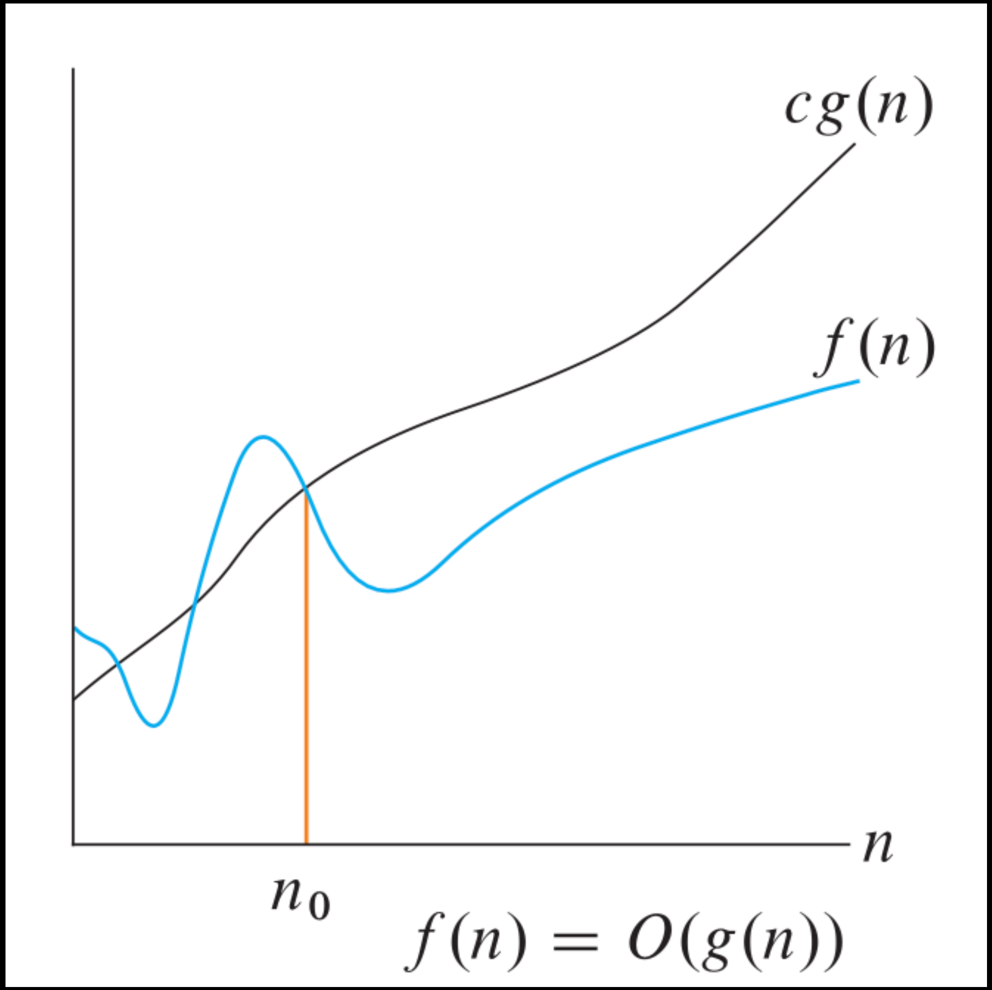
Asymptotic notation: Formal Definitions

- O-notation describes an asymptotic upper bound on a function, to within a constant factor.
- For a given function $g(n)$, we denote by $O\big(g(n)\big)$ the set of functions:

$$O\big(g(n)\big) = \{f(n){:}\ there\ exist\ positive\ constants\ c\ and\ n_0\ such\ that\ 0 \le f(n) \le cg(n)\ for\ all\ n \ge n_0\}$$



Asymptotic notation: Formal Definitions

- The definition of $O(g(n))$ requires that every function $f(n)$ in the set be asymptotically nonnegative
- Consequently, the function $g(n)$ itself must be asymptotically nonnegative, or else the set $O(g(n))$is empty.
- Since $f(n)$ is a function that belongs to the set $O(g(n))$, we should have written it like $f(n) \in O(g(n))$
- But we will adopt the $f(n) = O(g(n))$ notation to indicate the same thing.

Asymptotic notation: Formal Definitions

- Example: show that $4n^2 + 100n + 500 = O(n^2)$.

- To do that, we have to find some positive c and n_0 to make
$$4n^2 + 100n + 500 \leq cn^2$$
- Divide both sides by n^2, we get $4 + 100/n + 500/n^2 \leq c$

- This equation can be satisfied for many choices:
  - $n_0 = 1$ and $c = 604$
  - $n_0 = 10$ and $c = 19$

Asymptotic notation: Formal Definitions

- Example: show that $n^3 - 100n^2$ does not belong to the set $O(n^2)$.

- We express it as $n^3 - 100n^2 \leq cn^2$.

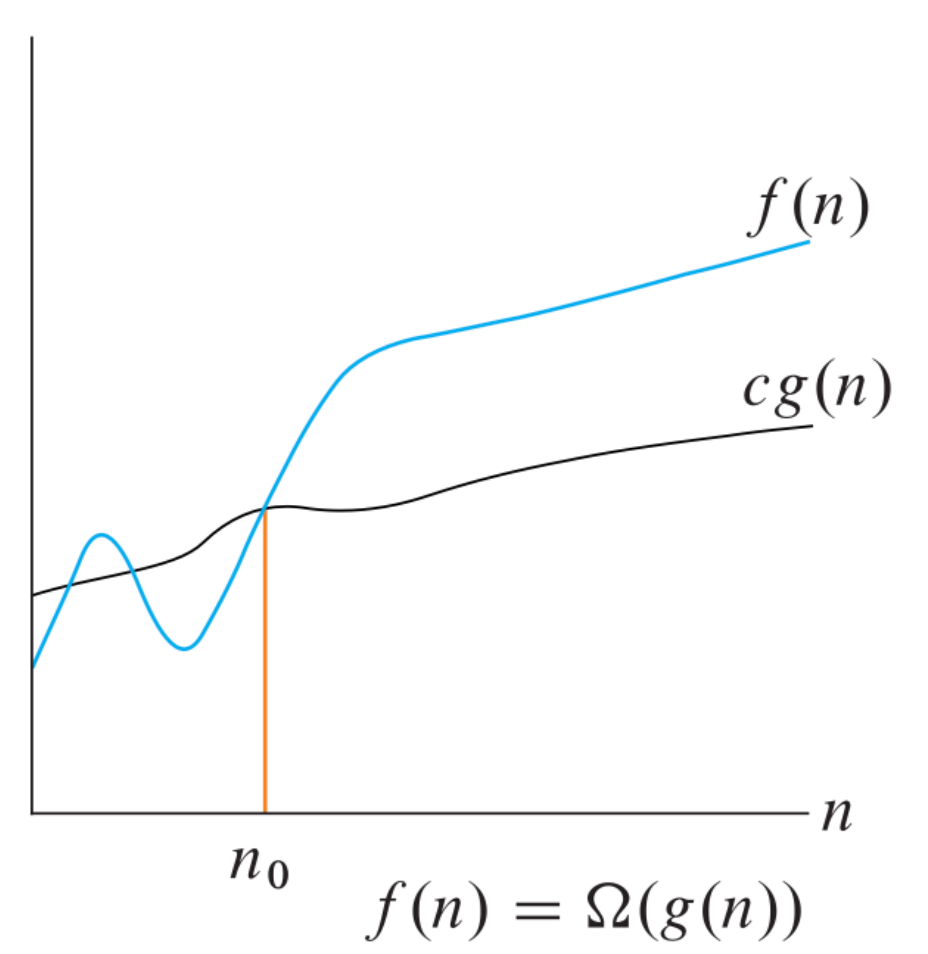
- Divide both sides by n^2, we get $n - 100 \leq c$.

- Regardless of what value we choose for the constant c, this inequality does not hold for any value of n.

Asymptotic notation: Formal Definitions

- Ω-notation provides an asymptotic lower bound.
- For a given function $g(n)$, we denote by $\Omega(g(n))$ the set of functions:

  $\Omega(g(n)) = \{f(n)\colon \textit{there exist positive constants } c \textit{ and } n_0 \textit{ such that } 0 \leq cg(n) \leq f(n) \textit{ for all } n \geq n_0\}$



Asymptotic notation: Formal Definitions

- Example: show that $4n^2 + 100n + 500 = \Omega(n^2)$

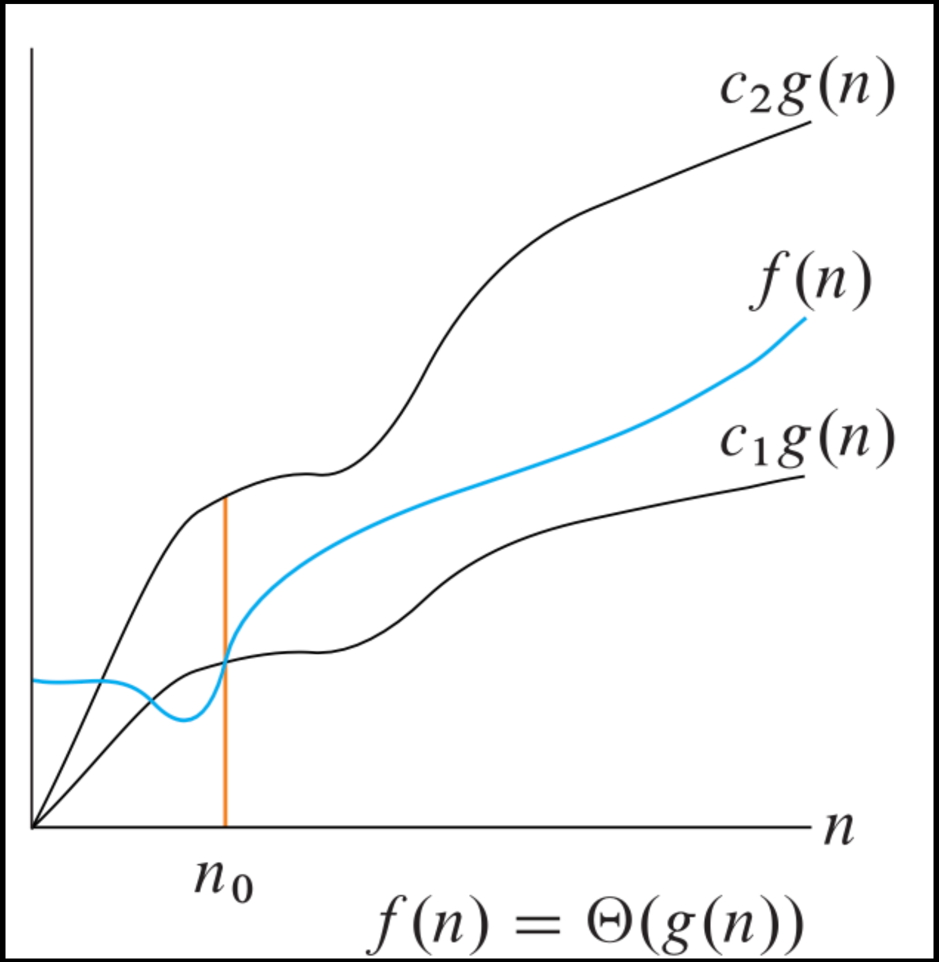
- We need to find positive constants c and n_0 such that $4n^2 + 100n + 500 \geq cn^2$ for all $n \geq n_0$
- Divide both sides by n^2, we get $4 + 100/n + 500/n^2 \geq c$.

- The inequality holds for any positive integer n_0 and $c = 4$.

Asymptotic notation: Formal Definitions

- Θ-notation provides an asymptotic tight bounds.
- For a given function $g(n)$, we denote by $\Theta\big(g(n)\big)$ the set of functions:

  $\Theta\big(g(n)\big) = \{f(n): \textit{there exist positive constants } c_1, c_2 \textit{ and } n_0 \textit{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \textit{ for all } n \ge n_0\}$



Asymptotic notation: Formal Definitions

- Based on the definition of $O-, \Omega-$, and $\Theta -$ notations, we get the following theorem:

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta\big(g(n)\big)$ if and only if $f(n) = O\big(g(n)\big)$ and $f(n) = \Omega\big(g(n)\big)$

Asymptotic Notation and Running Times

- The asymptotic notation we use should be as precise as possible without overstating which running time it applies to.

Asymptotic Notation and Running Times

- Example: for the insertion sort algorithm,
  - We can correctly say that the worst-case running time is $O(n^2), \Omega(n^2)$, and $\Theta(n^2)$
    - The $\Theta(n^2)$ is the most precise one.
  - We can correctly say that the best-case running time is $O(n), \Omega(n)$, and $\Theta(n)$
    - The $\Theta(n)$ is the most precise one.
  - We cannot say that the running time is $\Theta(n^2)$
    - By omitting ”worst-case” we are referring to all the cases, which is not valid.
  - We can correctly say that the running time is $O(n^2)$
    - Because $O(n^2)$ is an upper bound; in all the cases the running time grows no faster than n^2.
  - Likewise, we cannot say that the running time $\Theta(n)$
    - But we can say that its running time is $\Omega(n)$

Asymptotic Notation and Running Times

- Example: for the merge sort algorithm, it runs in $\Theta(n \lg n)$ time in all cases.

- It is correct to say that its running time is $\Theta(n \lg n)$ without specifying ”worst-case”, ”best-case”, or any other case.

Asymptotic Notation and Running Times

- We typically use asymptotic notation to provide the simplest and most precise bounds possible.

- For example, if an algorithm has a running time of $3n^2 + 20n$ in all cases, we use asymptotic notation to write that its running time is $\Theta(n^2)$
- It is correct to say its running time is $O(n^3)$ or $\Theta(3n^2 + 20n)$.
  - Neither of these expressions is useful as $\Theta(n^2)$.
  - $O(n^3)$ is less precise.
  - $\Theta(3n^2 + 20n)$ introduces complexity that obscures the order of growth.

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Asymptotic Notation in Equations and Inequalities

- Although we formally define asymptotic notation in terms of sets, we use the = instead of ∈ within formulas.

- For example, we wrote that $4n^2 + 100n + 500 = O(n^2)$.
- We might also write $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$.

- How do we interpret such formulas?

Asymptotic Notation in Equations and Inequalities

- When the asymptotic notation stands alone on the right-hand side of an equation (or inequality), the equal sign means set membership:
  - $4n^2 + 100n + 500 = O(n^2). \rightarrow 4n^2 + 100n + 500 \in O(n^2)$.

- When asymptotic notation appears in a formula, we interpret it as standing for some anonymous function that we do not care to name.
  - $2n^2 + 3n + 1 = 2n^2 + \Theta(n) \rightarrow 2n^2 + 3n + 1 = 2n^2 + \mathrm{f}(n)$ such that $f(n) \in \Theta(n)$
    - For instance, $f(n) = 3n + 1$, which belongs to $\Theta(n)$

Asymptotic Notation in Equations and Inequalities

- We can chain together a number of such relationships, as in
$$2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$$

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o, ω –notations

- The asymptotic upper bound provided by O-notation may or may not be asymptotically tight.
  - The bound $2n^2 = O(n^2)$ is asymptotically tight.
  - But the bound $2n = O(n^2)$ is not tight.

- We use o-notation to denote an upper bound that is not asymptotically tight.

$$o\big(g(n)\big) = \{f(n)\colon there\ exist\ positve\ constants\ c\ and\ n_0\ such\ that\ 0 \le f(n) < cg(n)\ for\ all\ n \ge n_0\}$$

- For example, $2n = o(n^2)$ but $2n^2 \neq o(n^2)$

o, ω —notations

- Intuitively, in o-notation, the function $f(n)$ becomes insignificant relative to $g(n)$ as n gets large:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

o, ω –notations

- The ω-notation denotes a lower bound that is not asymptotically tight.

$$\omega\big(g(n)\big) = \{f(n): there\ exist\ positive\ constants\ c\ and\ n_0$$
$$such\ that\ 0 \le cg(n) < f(n)\ for\ all\ n \ge n_0\ \}$$

- The relation $f(n) = \omega(n)$ implies that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

- Note that: $f(n) \in \omega\big(g(n)\big)$ if and only if $g(n) \in o\big(f(n)\big)$.
  - For example: $n^2/2 = \omega(n)$ but $n^2/2 \neq \omega(n^2)$

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Comparing Functions

- Asymptotic notations has properties.
- Assume that $f(n)$ and $g(n)$ are asymptotically positive.

Transitivity:

$$
\begin{aligned}
f(n) &= \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \quad \text{imply} \quad f(n) = \Theta(h(n)) , \\
f(n) &= O(g(n)) \text{ and } g(n) = O(h(n)) \quad \text{imply} \quad f(n) = O(h(n)) , \\
f(n) &= \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \quad \text{imply} \quad f(n) = \Omega(h(n)), \\
f(n) &= o(g(n)) \text{ and } g(n) = o(h(n)) \quad \text{imply} \quad f(n) = o(h(n)) , \\
f(n) &= \omega(g(n)) \text{ and } g(n) = \omega(h(n)) \quad \text{imply} \quad f(n) = \omega(h(n)) .
\end{aligned}
$$

Comparing Functions

- Asymptotic notations has properties.
- Assume that $f(n)$ and $g(n)$ are asymptotically positive.

Reflexivity:

$$f(n) = \Theta(f(n)) ,$$
$$f(n) = O(f(n)) ,$$
$$f(n) = \Omega(f(n)).$$

Symmetry:

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)) .$$

Transpose symmetry:

$$f(n) = O(g(n)) \text{ if and only if } g(n) = \Omega(f(n)),$$
$$f(n) = o(g(n)) \text{ if and only if } g(n) = \omega(f(n)) .$$

Comparing Functions

- We can draw an analogy between the asymptotic comparison of two functions f and g and the comparison of two real numbers a and b:

$$\begin{array}{lll} f(n) = O(g(n)) & \text{is like} & a \leq b\ , \\ f(n) = \Omega(g(n)) & \text{is like} & a \geq b\ , \\ f(n) = \Theta(g(n)) & \text{is like} & a = b\ , \\ f(n) = o(g(n)) & \text{is like} & a < b\ , \\ f(n) = \omega(g(n)) & \text{is like} & a > b\ . \end{array}$$

- $f(n)$ is asymptotically smaller than $g(n)$ → $f(n) = o\big(g(n)\big)$
- $f(n)$ is asymptotically larger than $g(n)$ if $f(n) = \omega\big(g(n)\big)$

Comparing Functions

- One property of real numbers, however, does not carry over to asymptotic notation:

  Trichotomy: For any two real numbers a and b, exactly one of the following must hold: $a < b, a = b, or\ a > b$.

- This means that in asymptotic notations, the $<$, $>$, and $=$ can hold at the same time.

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Exercises

3.2-1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$.

Exercises

- **To prove that $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$, we start by recalling the definition of Θ notation**:
  A function $h(n)$ is in $\Theta(k(n))$ if there exist constants $c_1, c_2 > 0$ and n_0 such that $n \geq n_0$, $c_1 k(n) \leq h(n) \leq c_2 k(n)$
- **Establish an upper bound**:

Show that $\max\{f(n), g(n)\} \leq c_2(f(n) + g(n))$ for some constant c_2.

Thus, c_2 can be 1.

So, $\max\{f(n), g(n)\} \leq 1 \cdot (f(n) + g(n))$

Exercises

- **Establish a lower bound:**

Show that $\max\{f(n), g(n)\} \geq c_2\big(f(n) + g(n)\big)$

Thus, c_2 can be $1/2$

So, $\max\{f(n), g(n)\} \geq \frac{1}{2}\big(f(n) + g(n)\big)$

- **Combine the upper and lower bounds:**

$$½\big(f(n) + g(n)\big) \leq \max\{f(n), g(n)\} \leq 1 \cdot \big(f(n) + g(n)\big)$$

For sufficiently large n

- Thus, by the definition of Θ notation, we conclude that: $\max\{f(n)\,, g(n)\} = \Theta\big(f(n) + g(n)\big)$.

Exercises

3.2-2

Explain why the statement, “The running time of algorithm A is at least $O(n^2)$,” is meaningless.

Exercises

- The statement says "at least $O(n^2)$" implies that the running time can be bounded above by a quadratic function.
- However, Big O notation doesn't inherently provide a lower bound. It's about the upper limit on the growth of the function, not the minimum.
- The correct usage is to say either:
  - "The running time of algorithm A is $O(n^2)$ " to indicate that it does not grow faster than a quadratic function.
  - If you want to express a lower bound, you should use Ω notation, e.g., "The running time of algorithm A is $\Omega(n^2)$ " to indicate it grows <u>at least as</u> fast as a quadratic function.

Exercises

3.2-3

Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

Exercises

- $2^{n+1} = 2 \cdot 2^n$ for all $n \geq 0$. So, $2^{n+1} = O(2^n)$, where $c = 2$.

- $2^{2n} = 2^n \cdot 2^n \neq O(2^n)$. There is no n_0 such that $n \geq n\_0$ and there is no c.

Task

3.2-5
Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

3.2-6
Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Task- answer

Suppose the running time is $\Theta(g(n))$. By Theorem 3.1, the running time is $O(g(n))$, which implies that for any input of size $n \geq n_0$ the running time is bounded above by $c_1 g(n)$ for some c_1. This includes the running time on the worst-case input. Theorem 3.1 also imlpies the running time is $\Omega(g(n))$, which implies that for any input of size $n \geq n_0$ the running time is bounded below by $c_2 g(n)$ for some c_2. This includes the running time of the best-case input.

On the other hand, the running time of any input is bounded above by the worst-case running time and bounded below by the best-case running time. If the worst-case and best-case running times are $O(g(n))$ and $\Omega(g(n))$ respectively, then the running time of any input of size n must be $O(g(n))$ and $\Omega(g(n))$. Theorem 3.1 implies that the running time is $\Theta(g(n))$.

Suppose we had some $f(n) \in o(g(n)) \cap \omega(g(n))$. Then, we have

$$0 = \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

a contradiction.