CS302 – Analysis and Design of Algorithms

Red-Black Tree and Complexity Classes

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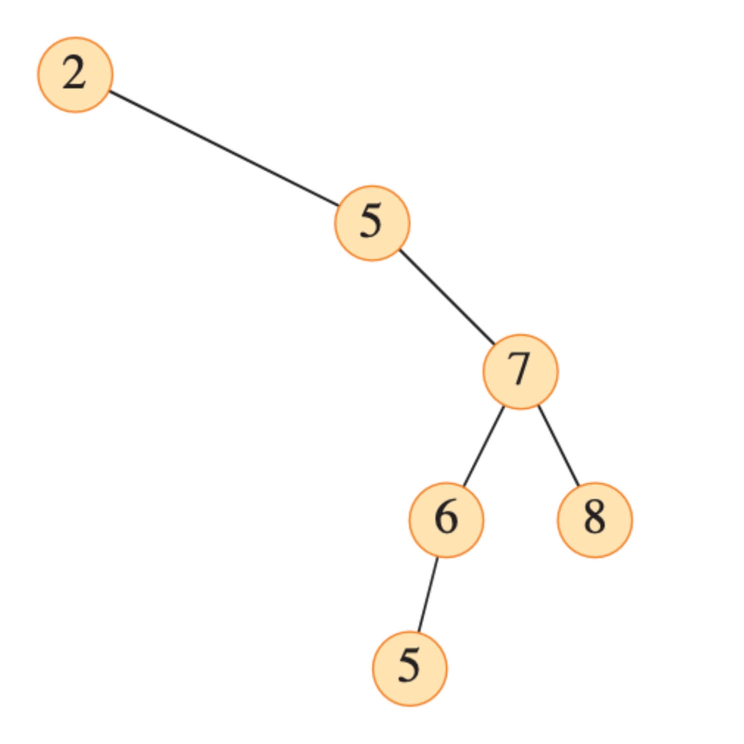
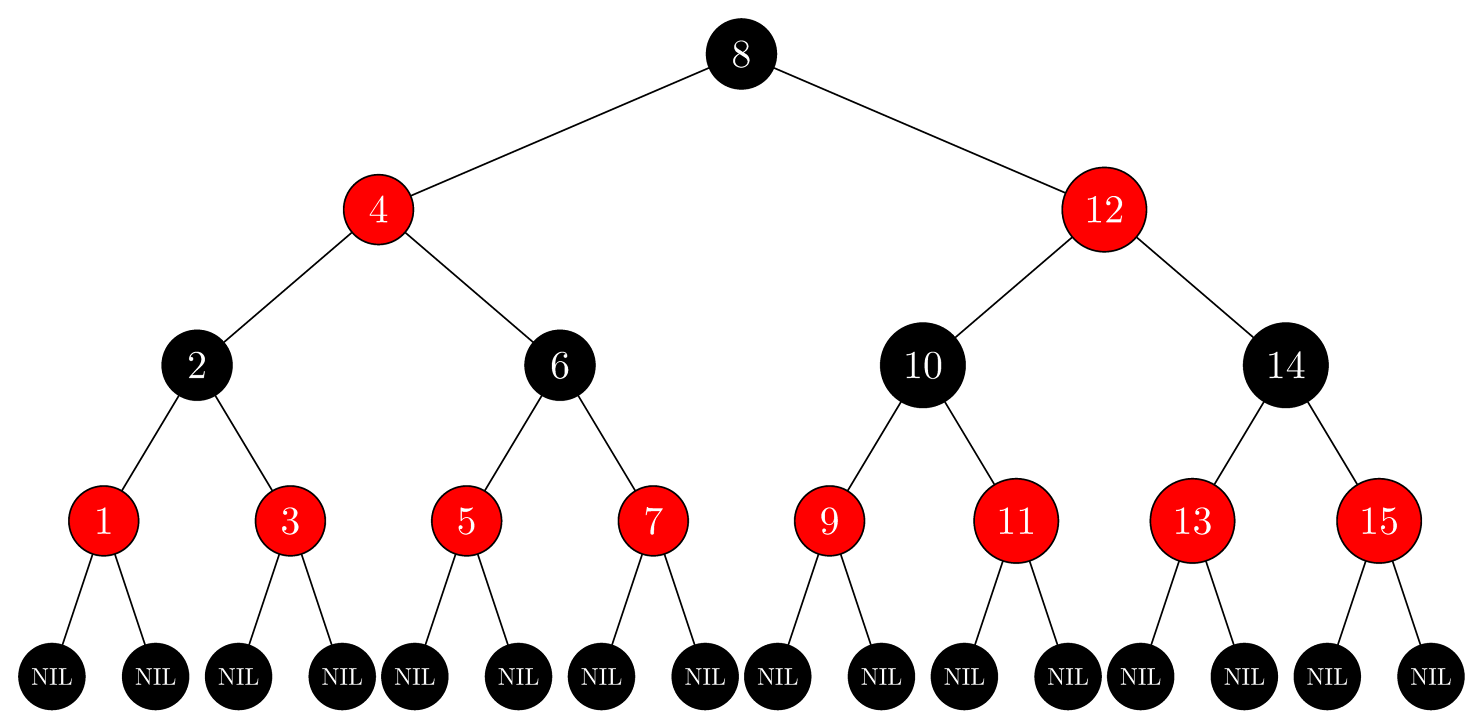
Red-Black Trees

- In BST, operations take $O(h)$ time, where $h = \lg n$ is the height of the tree.
- An unbalanced tree may run no faster than a linked list.



Red-Black Trees

- Red-black trees are binary search tree schemes that are balanced.
- They guarantee $O(\lg n)$ operations in the worst case.



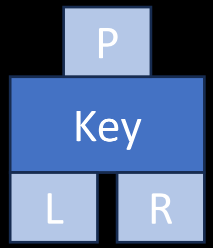
Red-Black Trees

- A red-black tree has an extra bit of storage per node: its color.
  - RED node
  - BLACK node

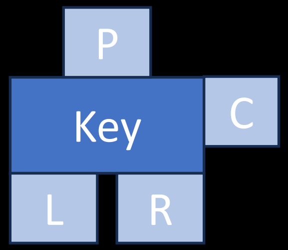
- The height of a red-black tree with n keys is at most $2\lg(n+1) = O(\lg \mathrm{n})$.

Red-Black Trees

- BST node



- Red-black tree node

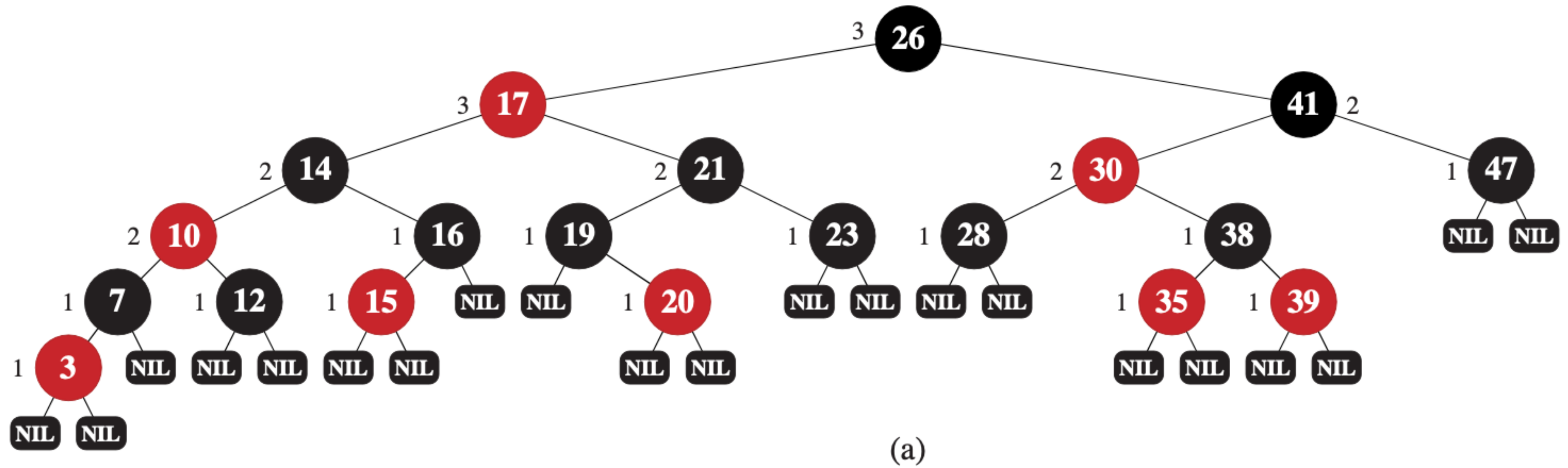


Red-Black Trees

- Properties of RB trees:
  1. Every node is either red or black.
  2. The root is black.
  3. Every leaf (NIL) is black.
  4. If a node is red, then both its children are black.
  5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Red-Black Trees

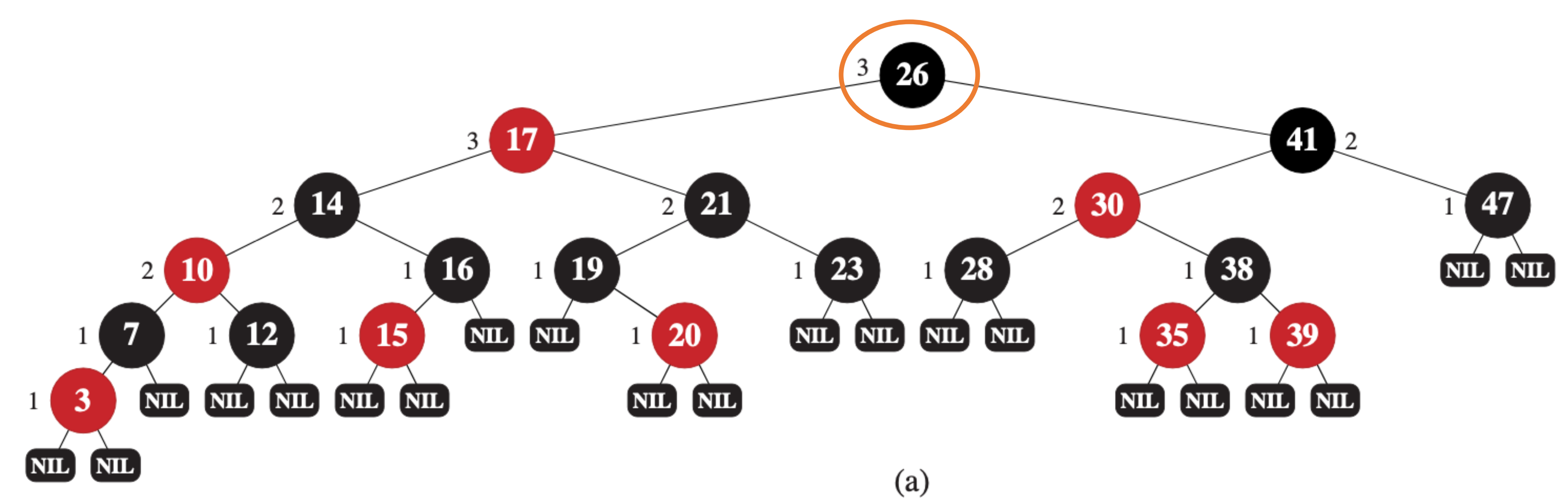
- Example
  - Every node is either red or black.



(a)

Red-Black Trees

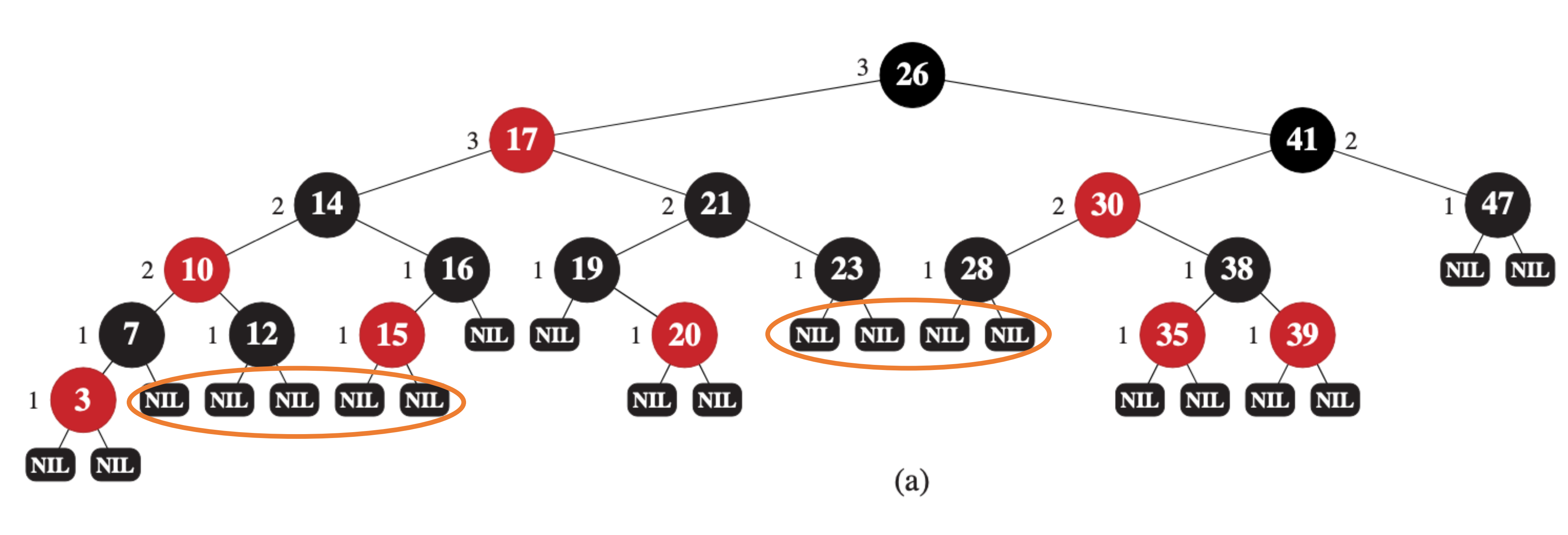
- Example
  - The root is black.



(a)

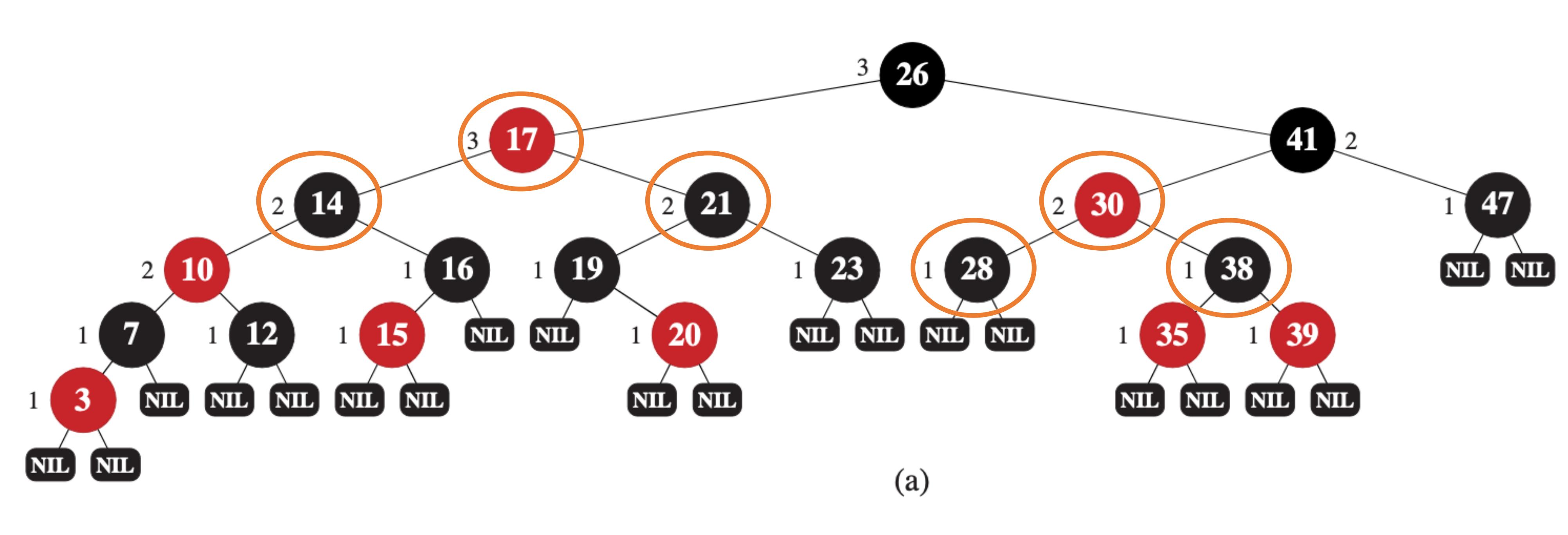
Red-Black Trees

- Example
  - Every leaf (NIL) is black.



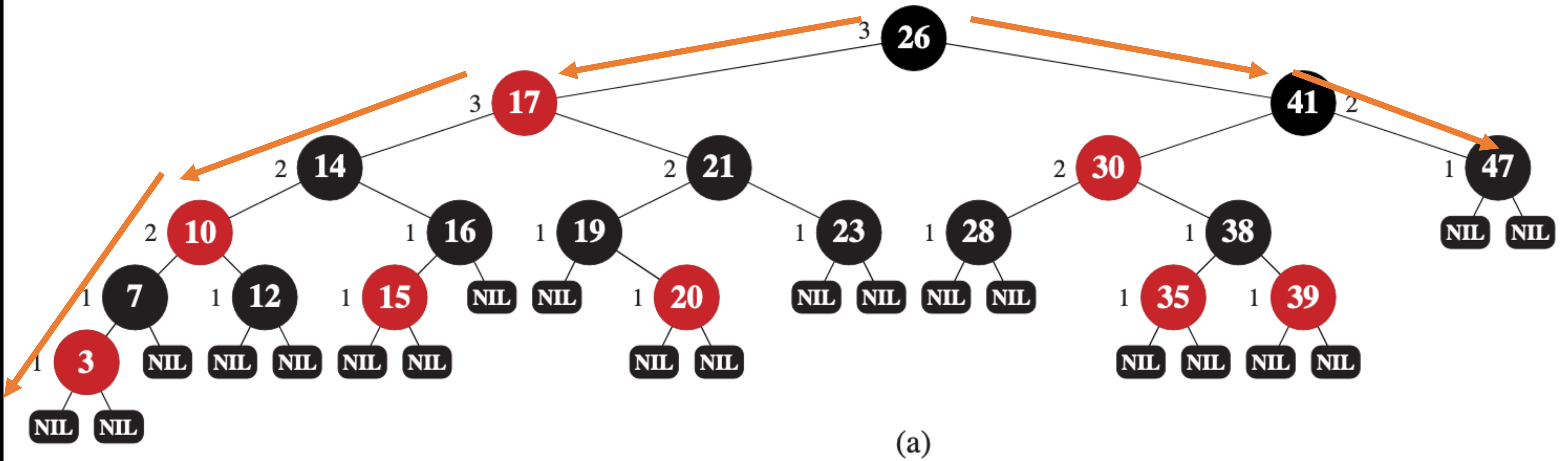
Red-Black Trees

- Example
  - If a node is red, then both its children are black.



Red-Black Trees

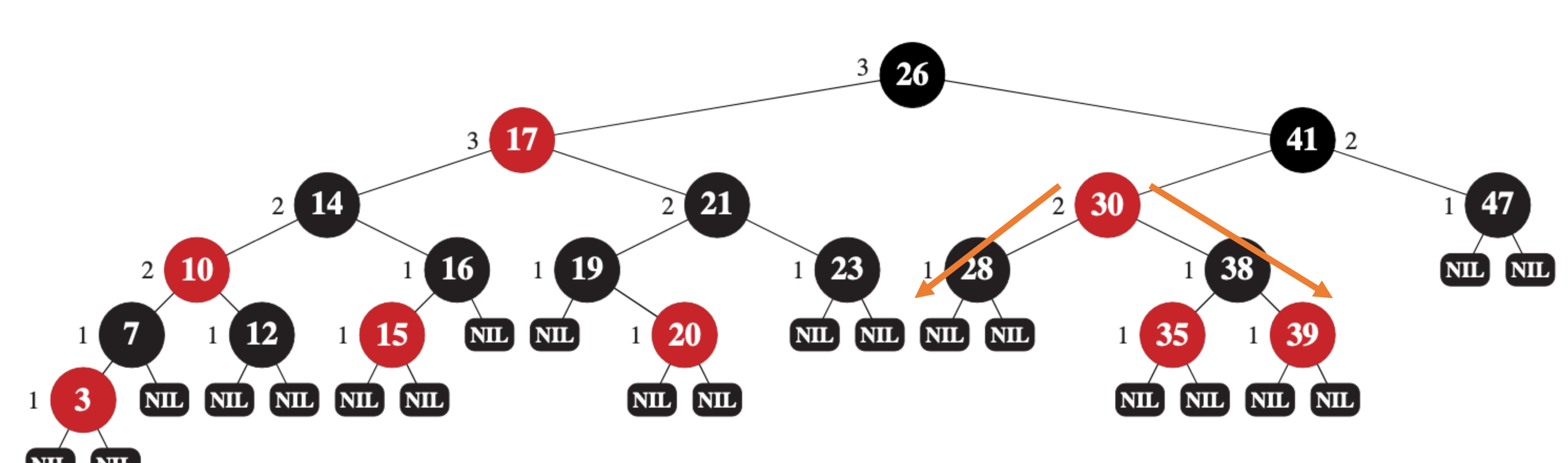
- Example
  - For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.



(a)

Red-Black Trees

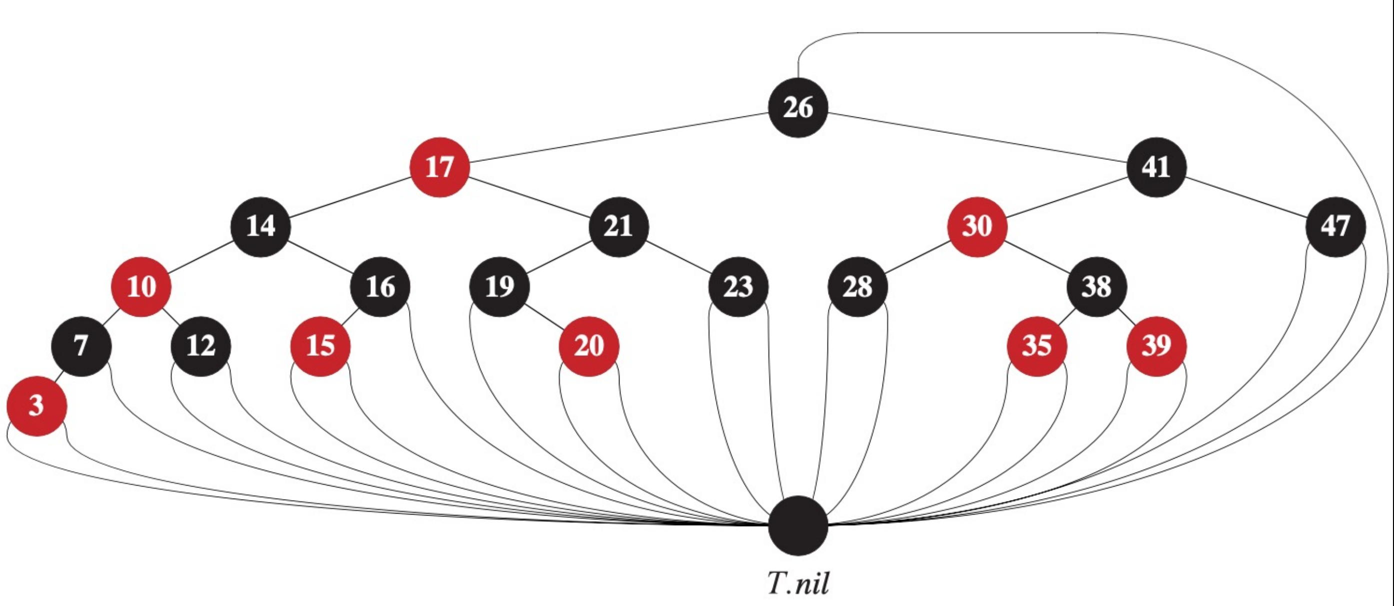
- Example
  - For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.



(a)

Red-Black Trees

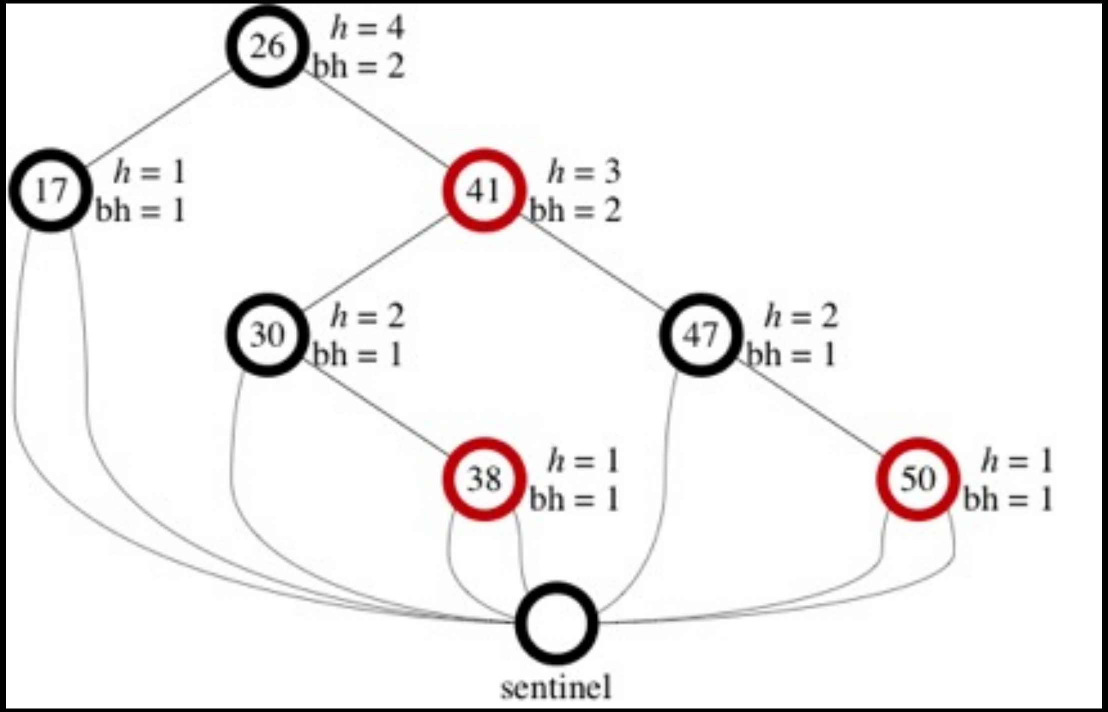
- Since all leaf nodes have their left and right are NILs, assign a sentinel node to all of them.
  - Sentinel node is black→ T.nil
  - The parent of the root point to the sentinel



(b)

Red-Black Trees

- $bh(x)$ refers to the black-height of the node x.
- Black-height is the number of black nodes on any path from node x to a leaf.
    - Node x is not counted



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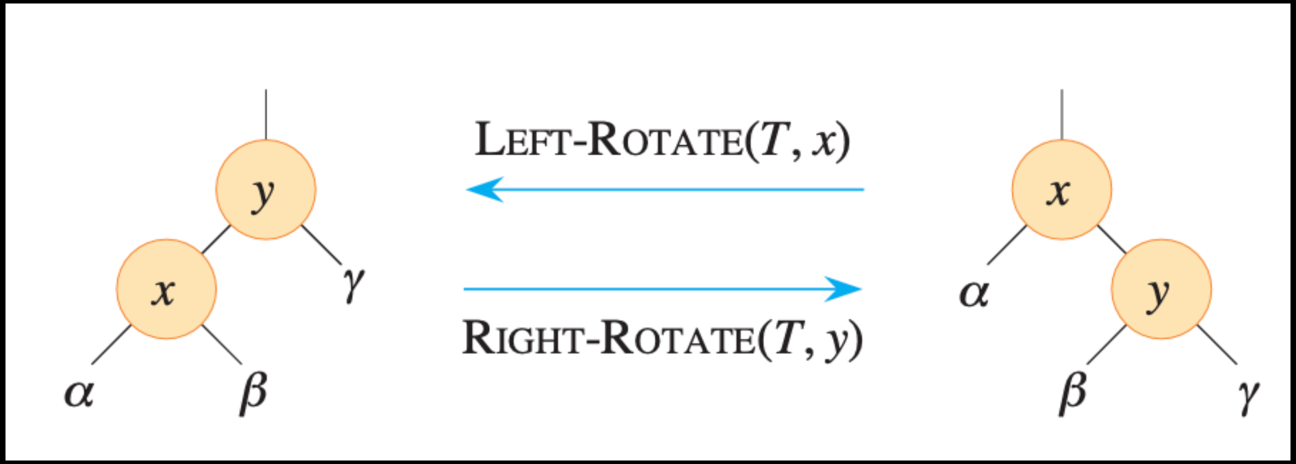
Rotations

- Rotation operation preserves the tree property when inserting or deleting elements.
- There are two rotations: left rotation and right rotation.
  - Both take $O(1)$, because they only change the pointers



Rotations

- Left rotation

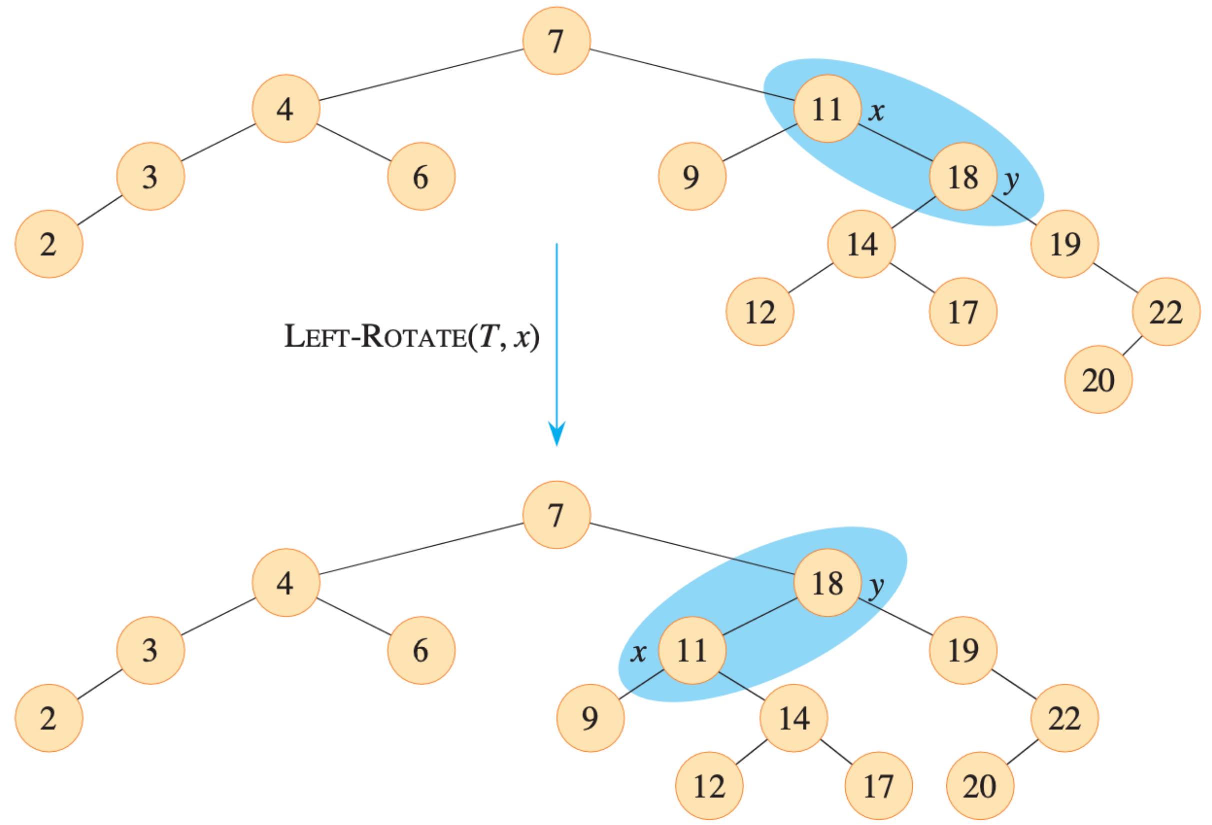
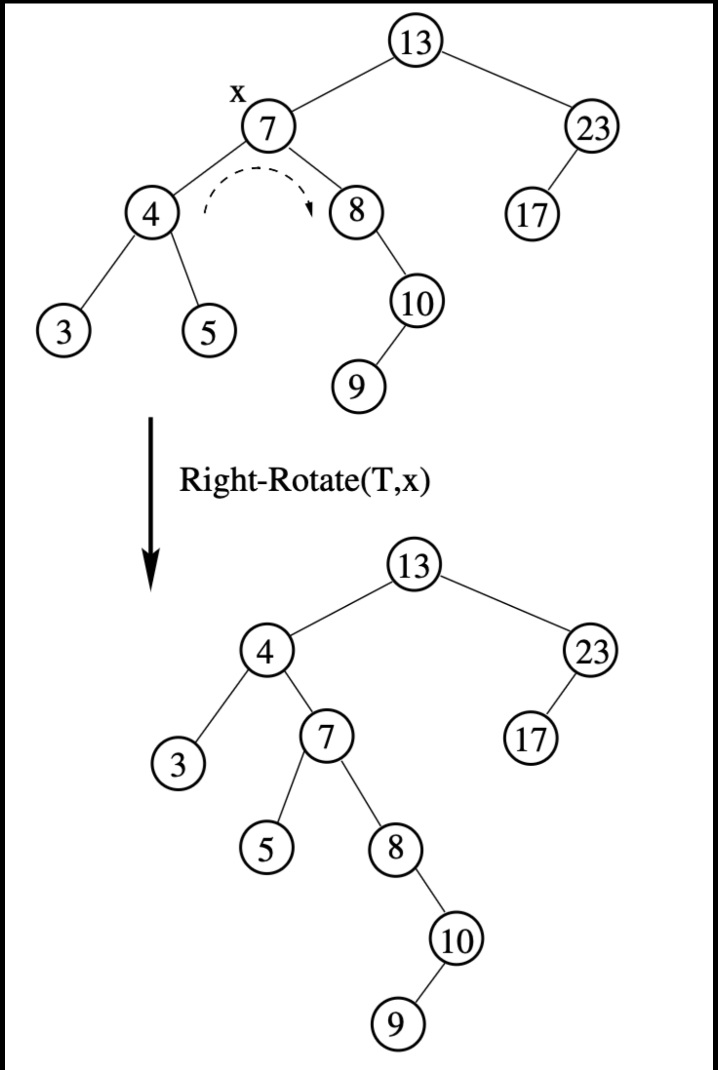


Figure 13.3 An example of how the procedure LEFT-ROTATE(T, x) modifies a binary search tree. Inorder tree walks of the input tree and the modified tree produce the same listing of key values.

Rotations

- Right rotation



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Insertion

- To insert a node z in an RB-tree T:
    1. Set z's color to red
    2. BST-Insert(T, z)
    3. If the parent of z is black
        1. Stop
    4. If the parent of z is red
        1. Fix the tree

Insertion – Fix RB-tree

- While $z.p.color = red$:

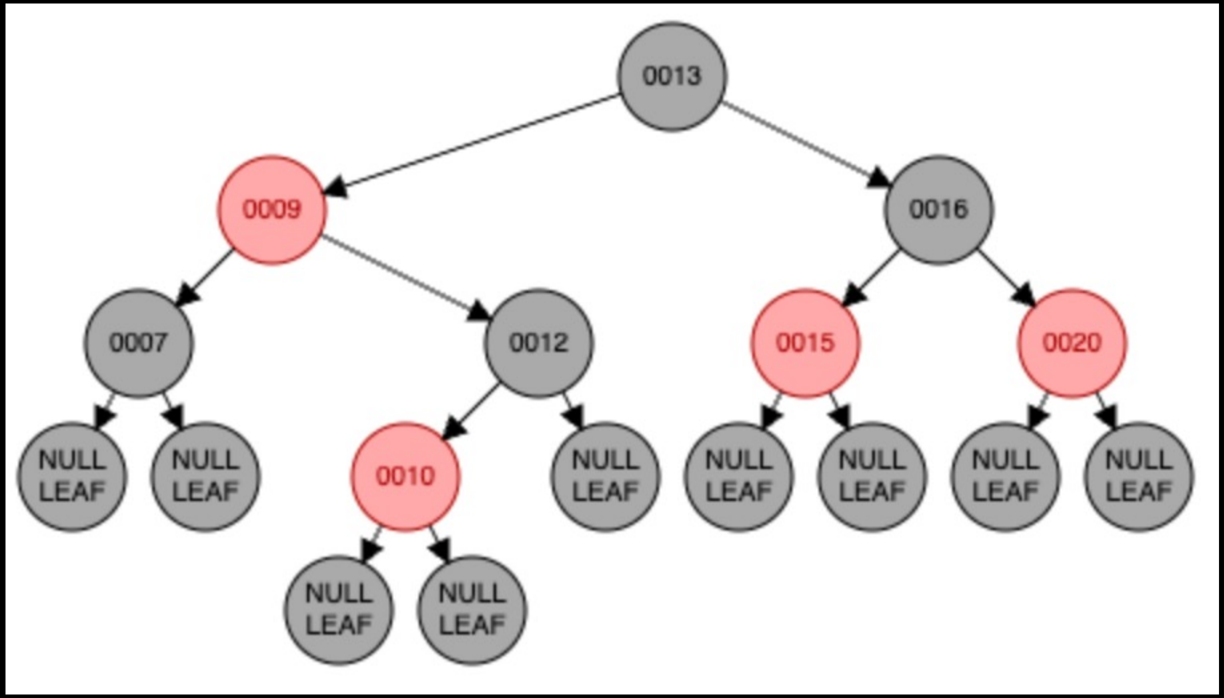
1. If $z.p$ is a left child
    1. y = right parent's sibling (uncle) of z.
    2. If y$.color = red$:
        1. $z.p.color = black$
        2. $y.color = black$
        3. $z.p.p.color = red$
    3. If $y.color = black$
        1. If z is a right child
            1. $z = z.p$
            2. LR(T, z)
        2. $z.p.color = black$
        3. $z.p.p = red$
        4. RR($T, z.p.p$)

2. If $z.p$ is a right child
    1. y = left parent's sibling (uncle) of z.
    2. If $y.color = red$:
        1. $z.p.color = black$
        2. $y.color = black$
        3. $z.p.p.color = red$
        4. $z = z.p.p$
    3. If $y.color = black$:
        1. If z is a left child
            1. $z = z.p$
            2. RR(T, z)
        2. $z.p.color = black$
        3. $z.p.p.color = red$
        4. LR($T, z.p.p$)

$T.root.color = black$

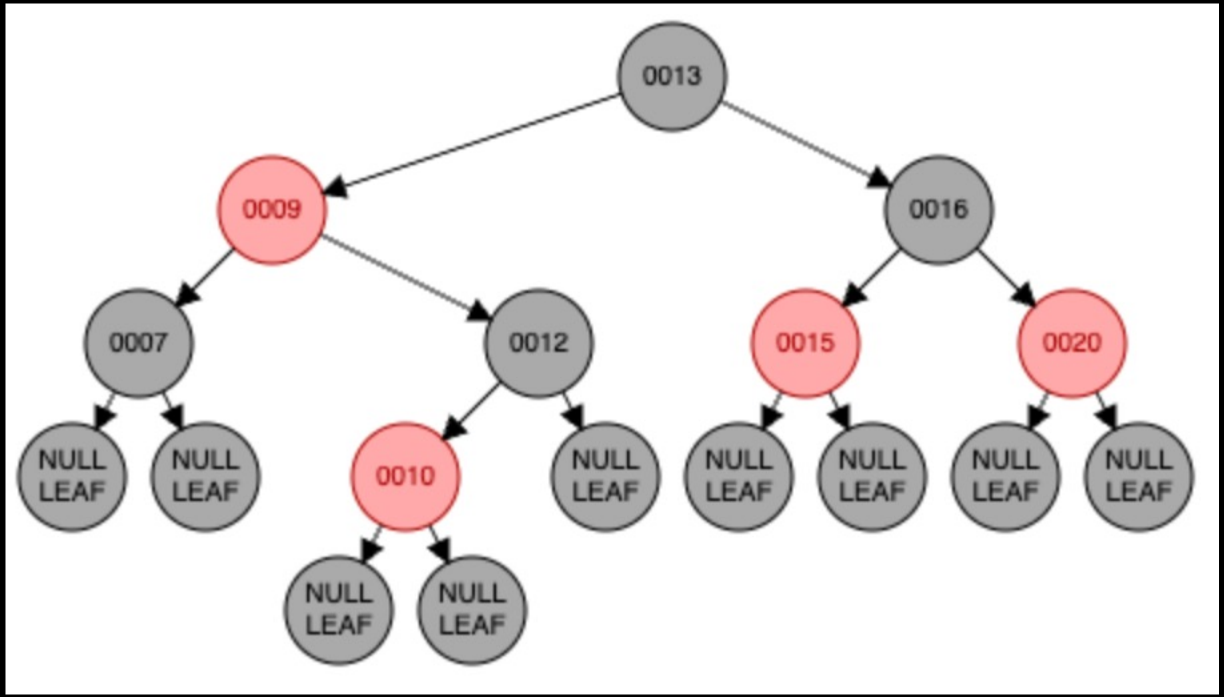
Insertion

- Example: insert 11



Insertion


Inserting 0011
0011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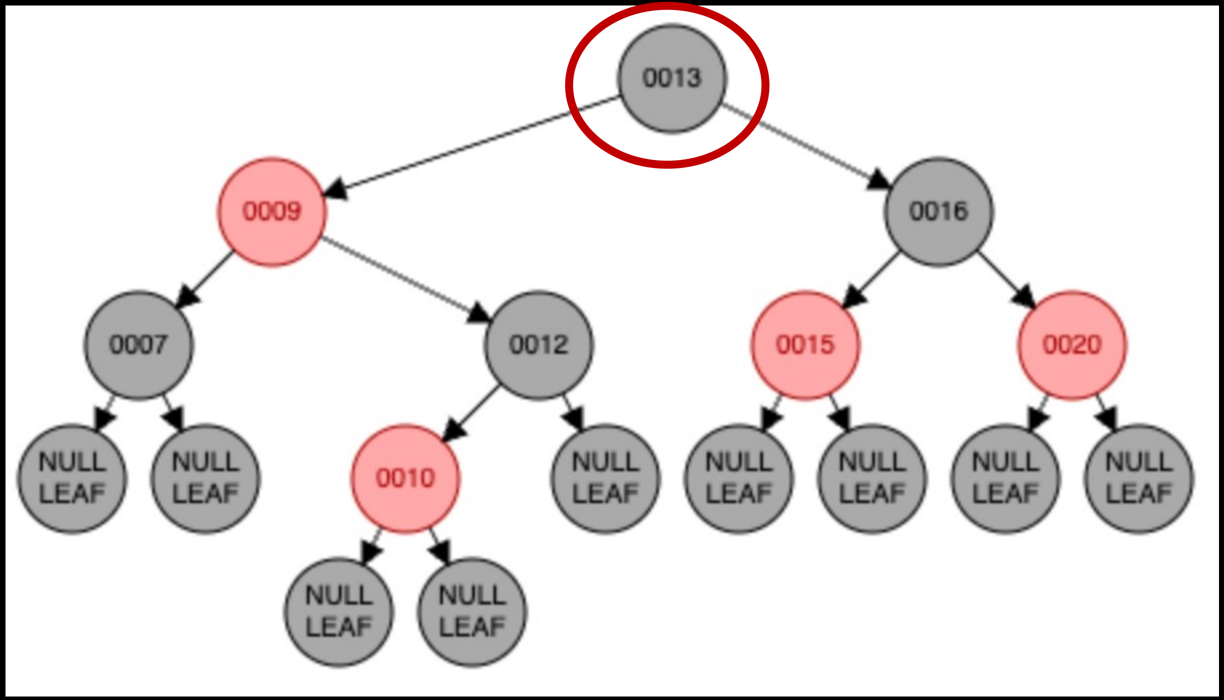
Insertion


Inserting 0011
0011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Insertion

Inserting 0011
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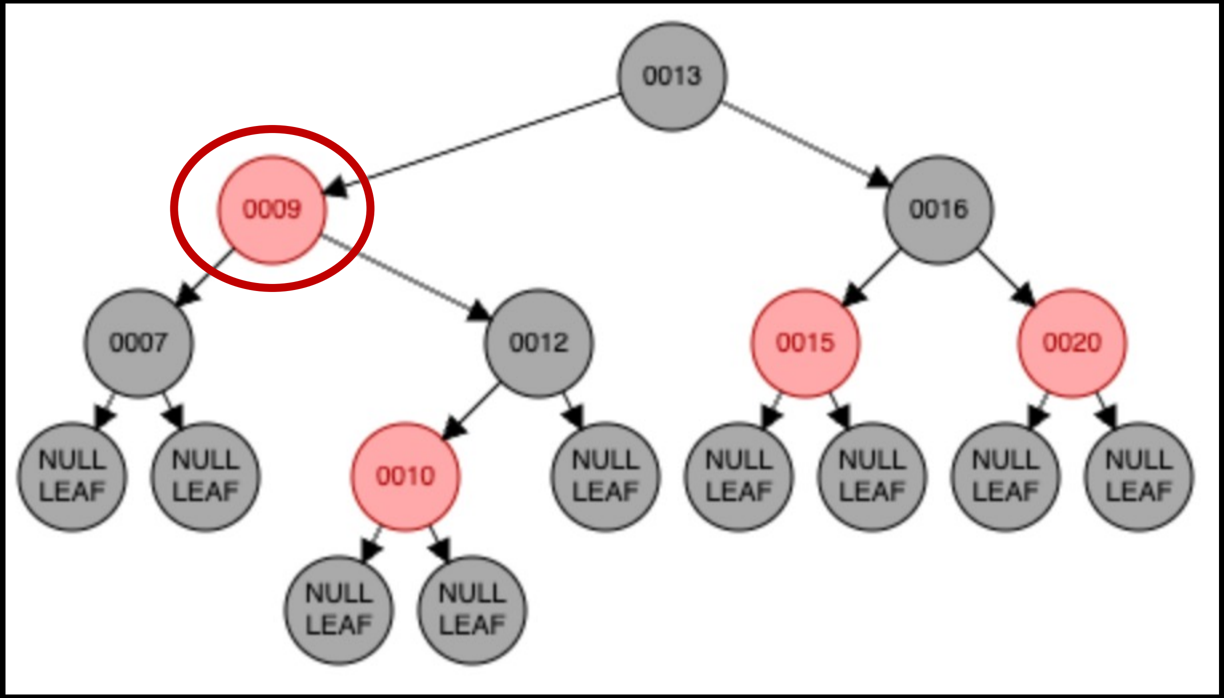

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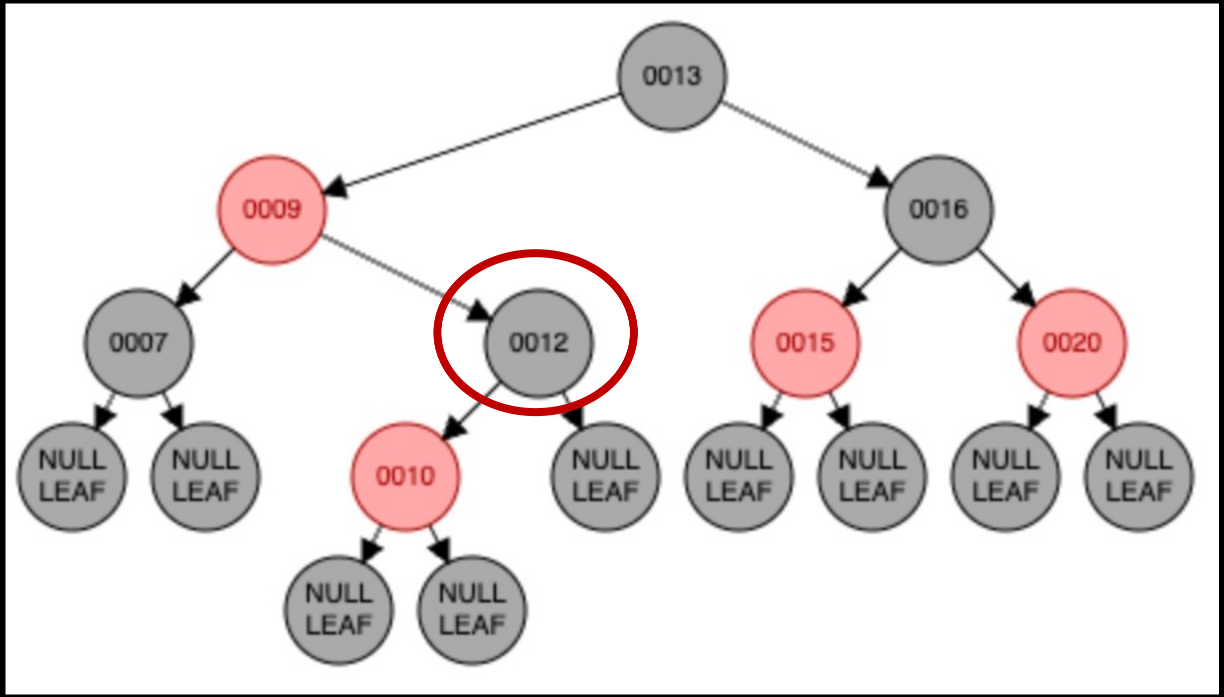
Insertion

Inserting 0011
0011



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Insertion


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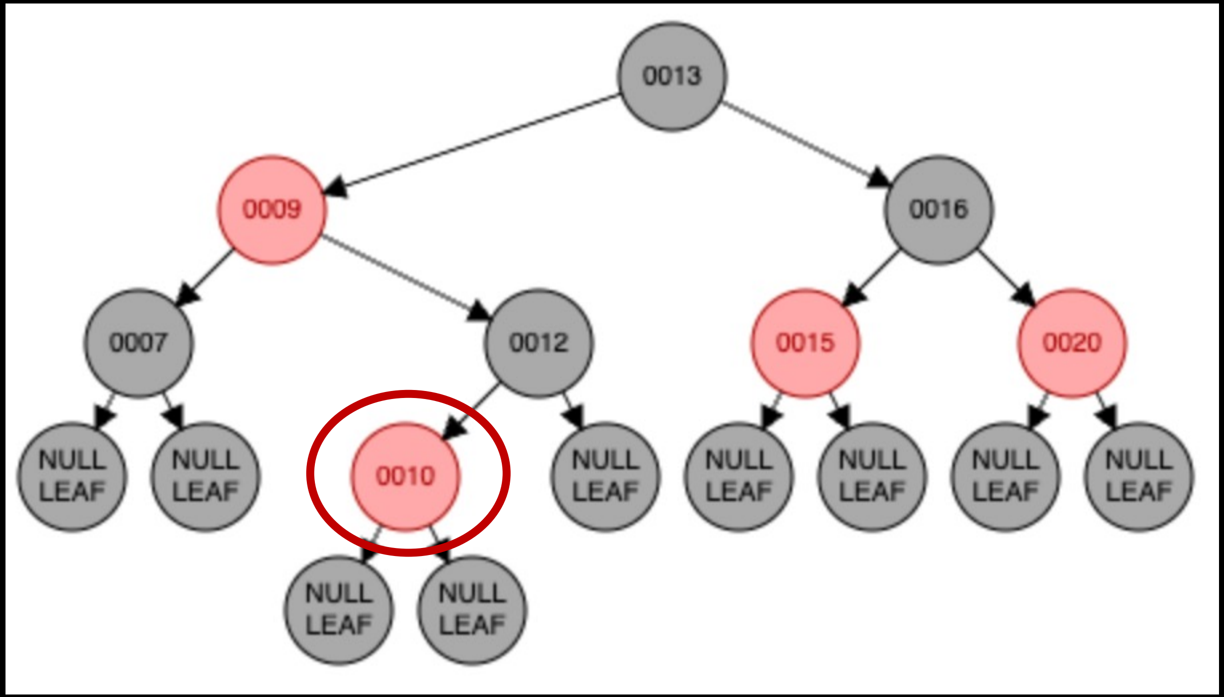
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Insertion


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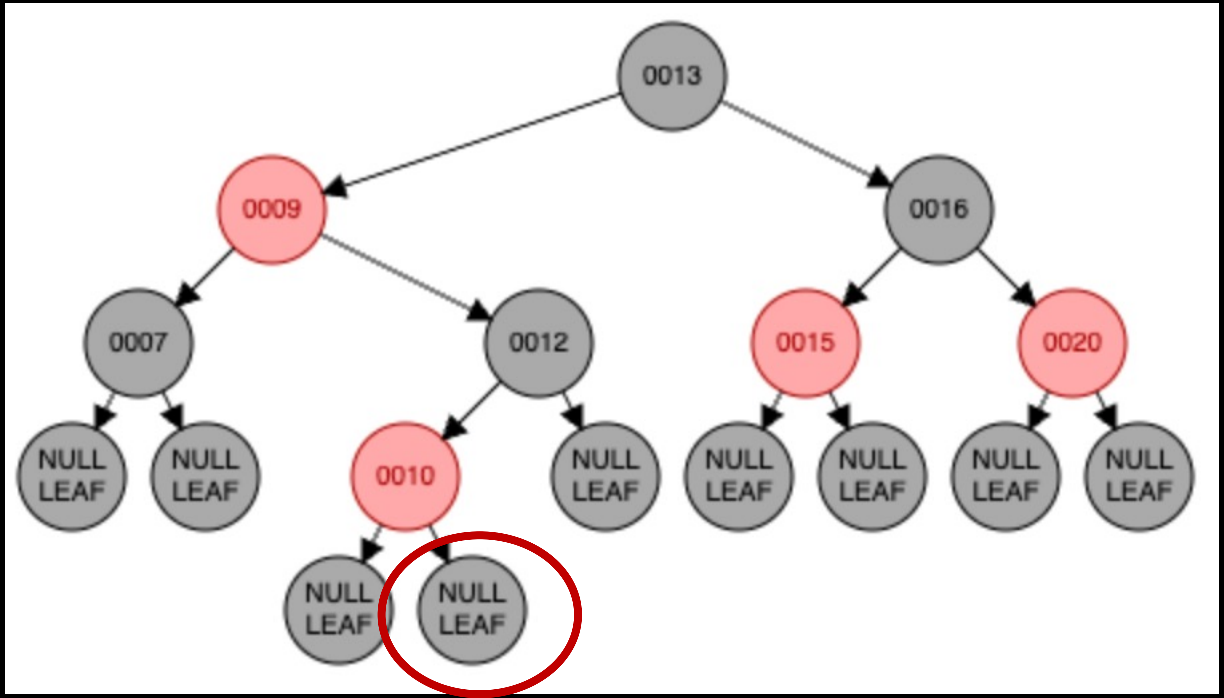
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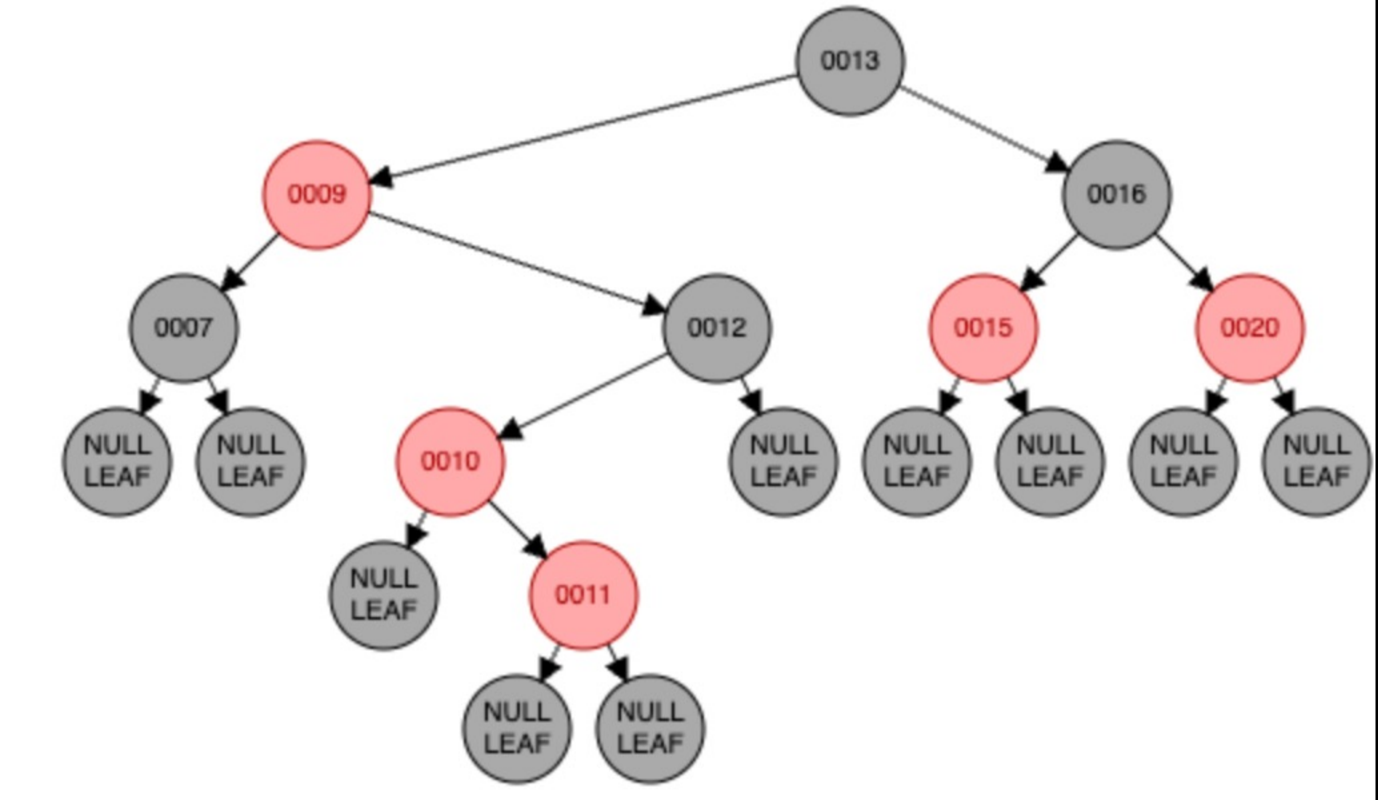
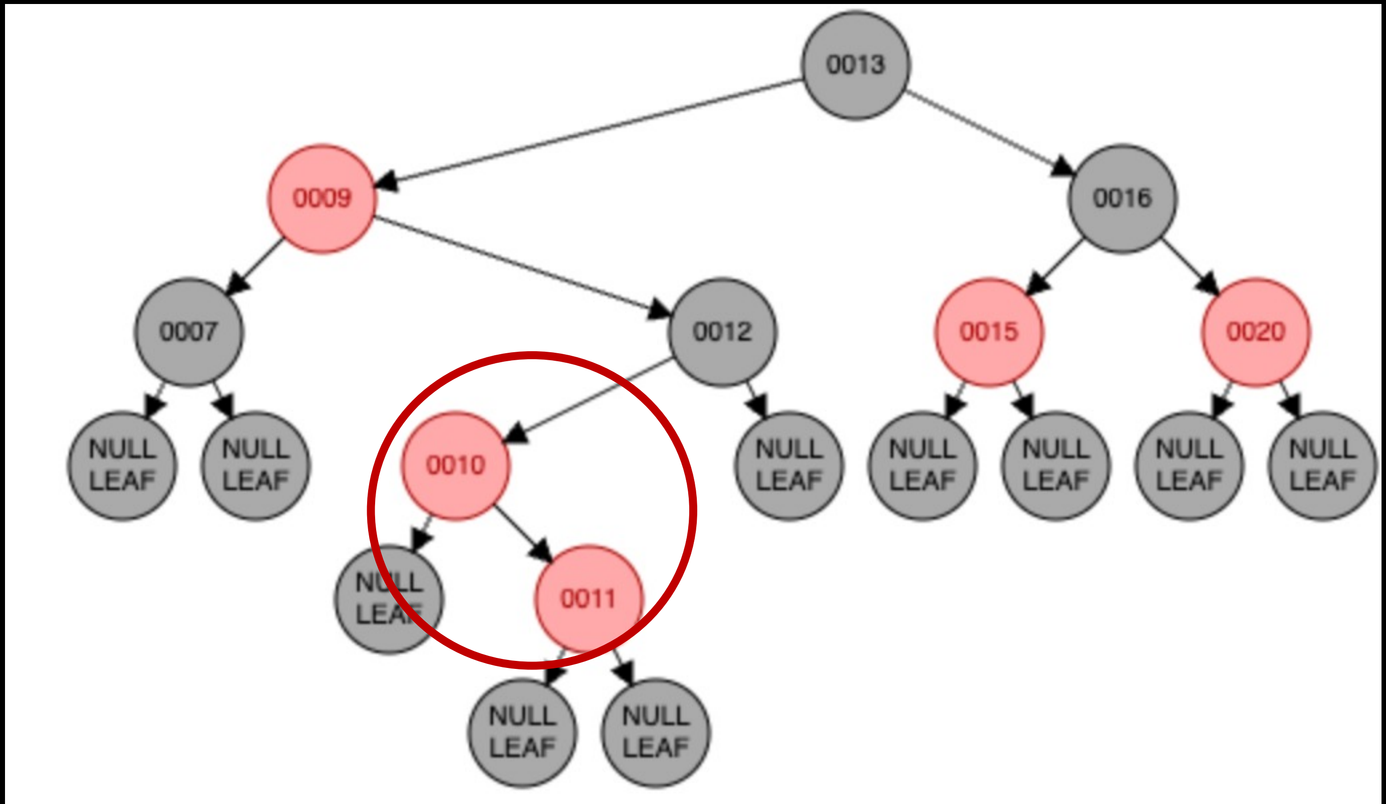
Insertion

- z and parent are both red. z is a right child, parent is a left child



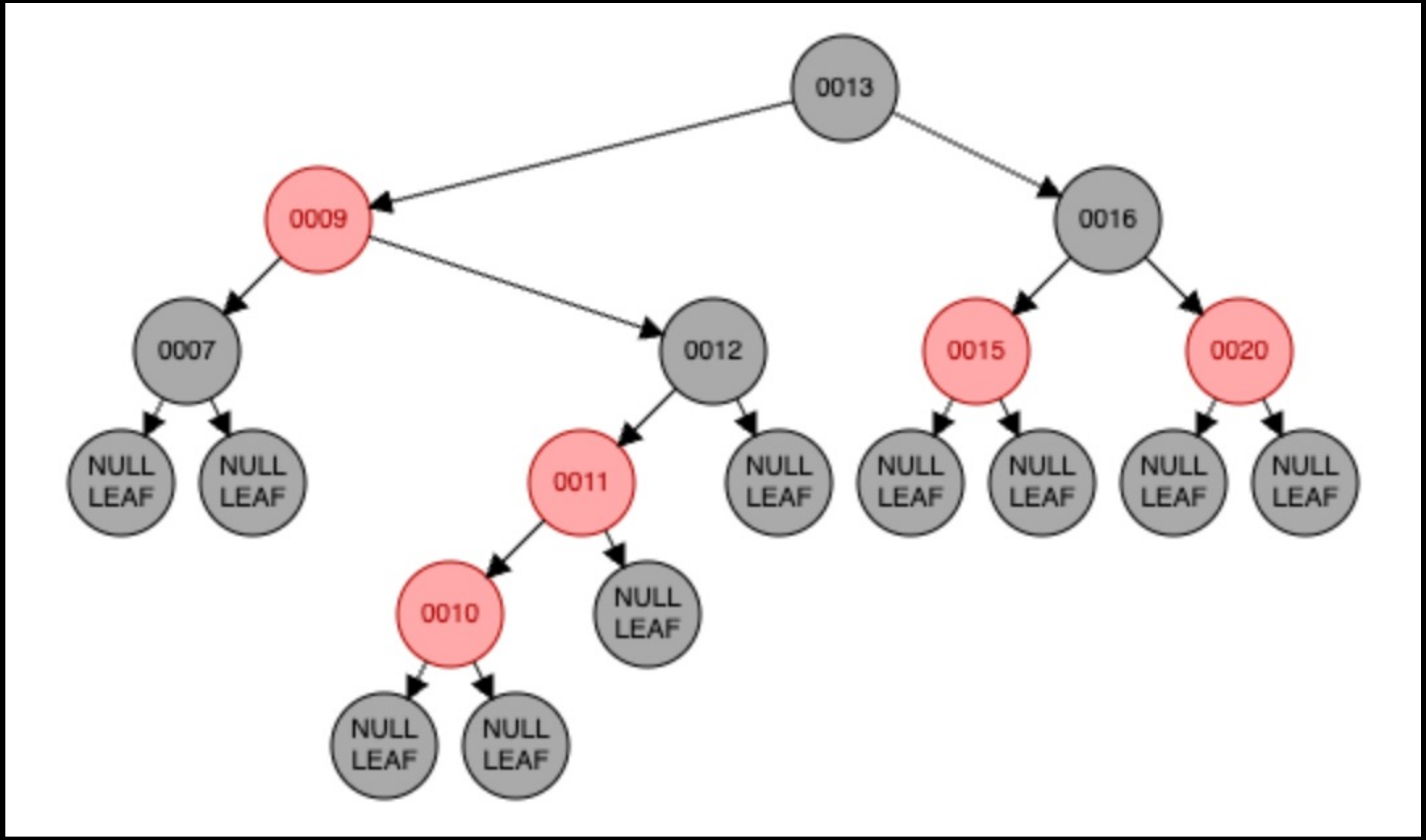
Insertion

- Single left rotation



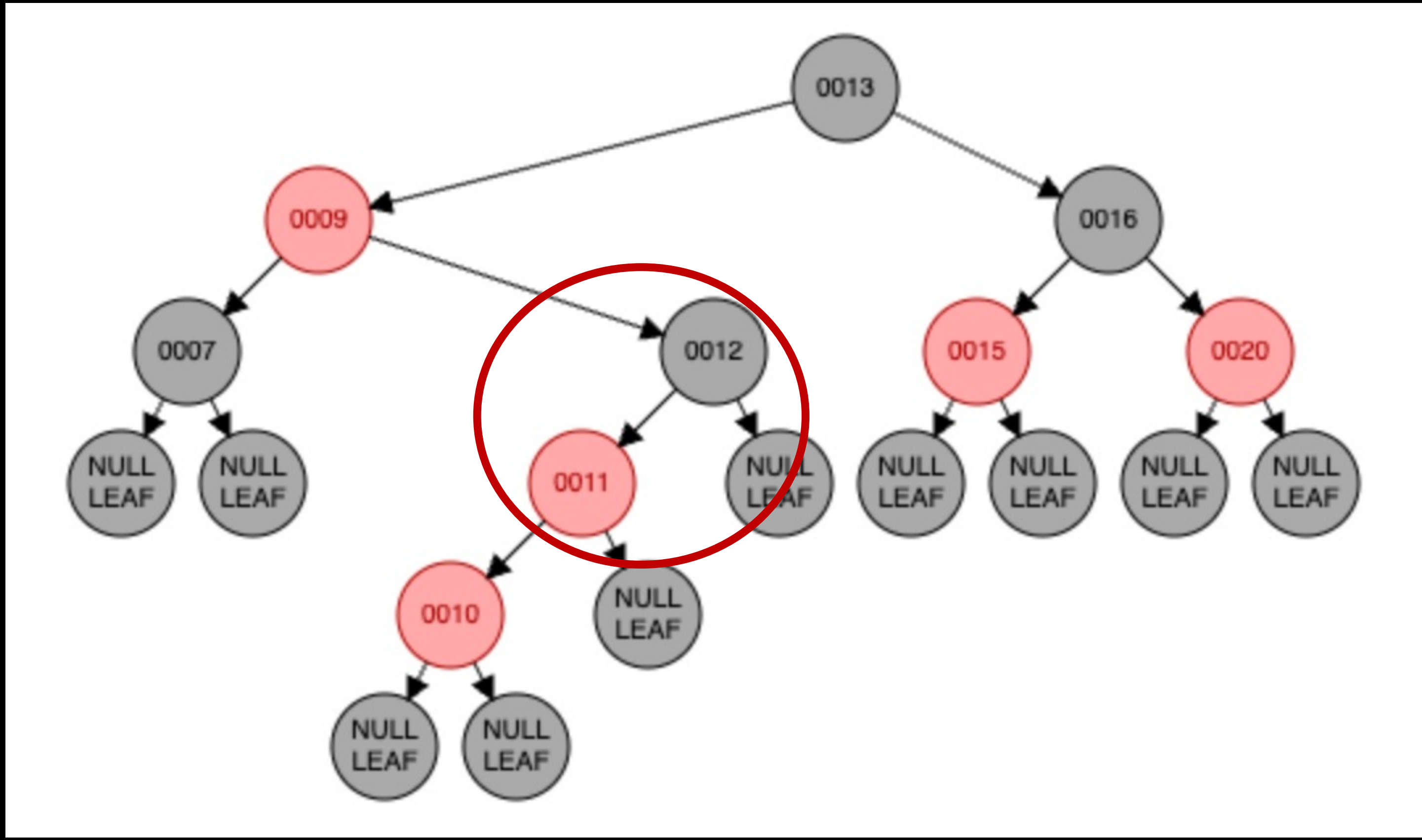
Insertion

- z and parent are red. z is left child, parent is left child



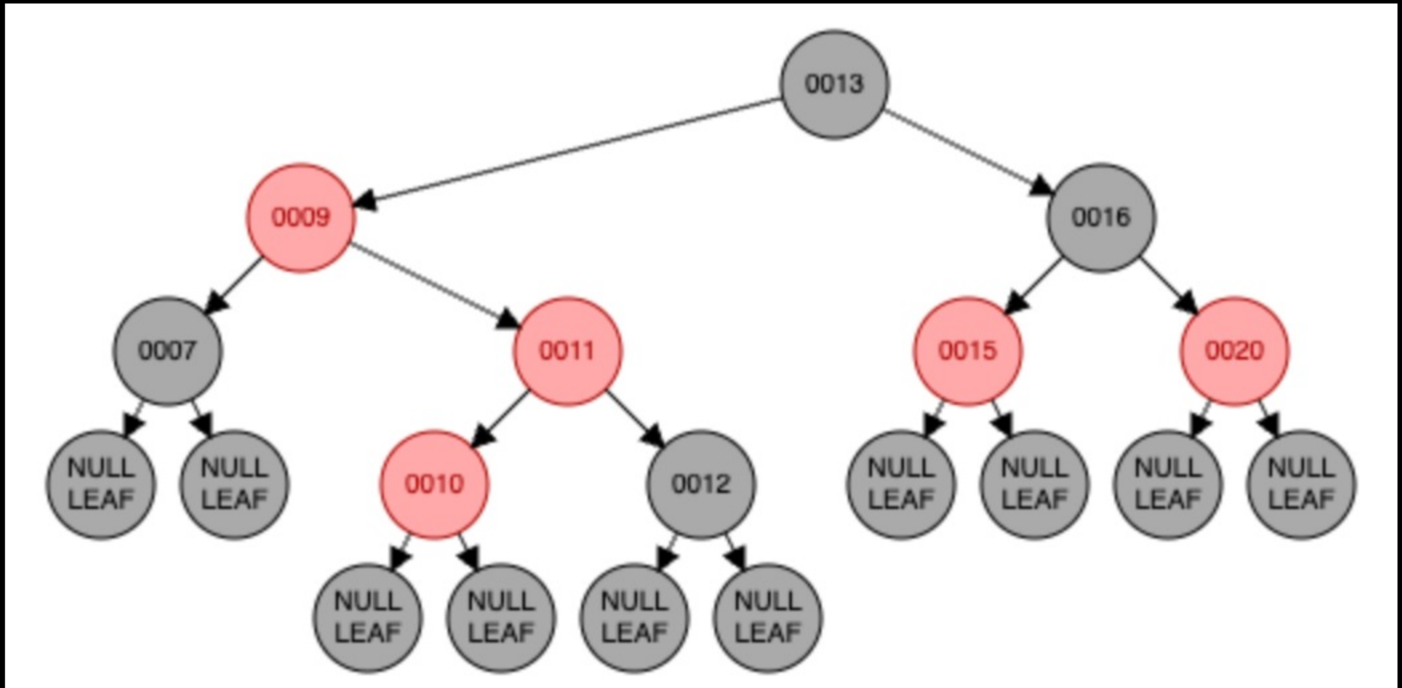
Insertion

- Single right rotation



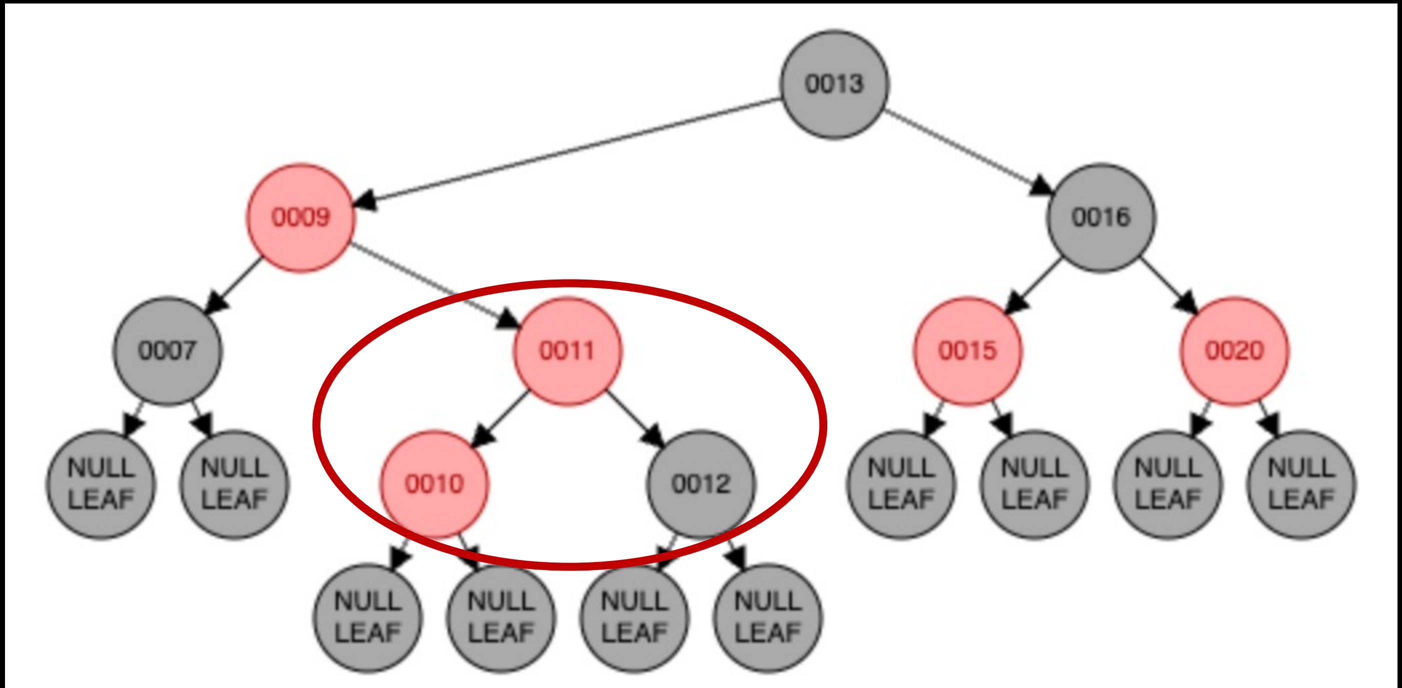
Insertion

- Single right rotation



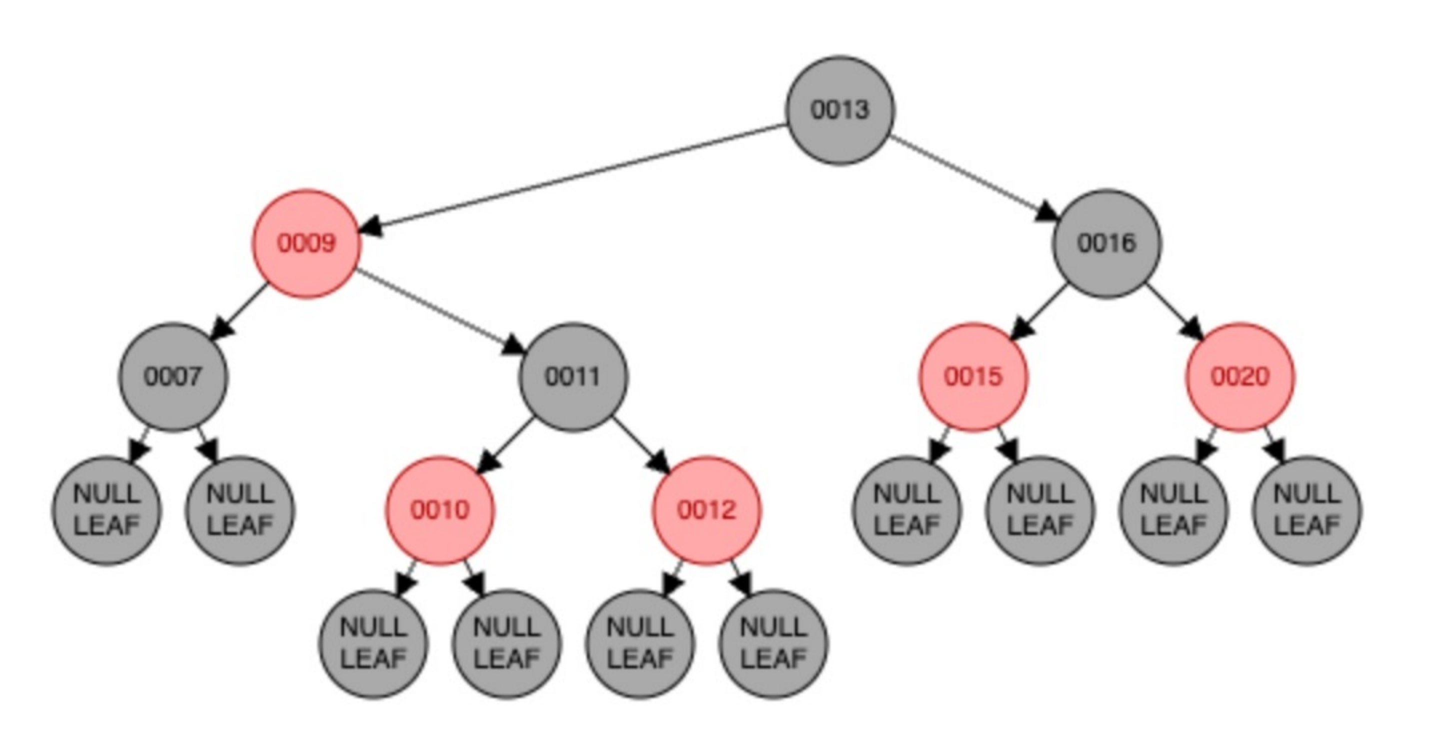
Insertion

- Recolor the nodes



Insertion

- Recolor the nodes



Insertion

- Insert the values [10, 1, 17, 4, 2, 0, 15] in an RB-tree.

Insertion

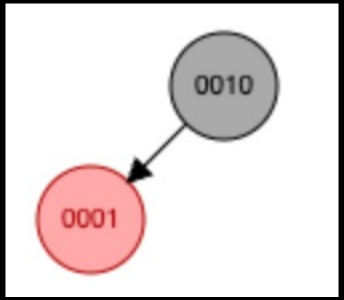
[10, 1, 17, 4, 2, 0, 15] – The tree is empty, so 10 is a black root node

Insertion

[10, 1, 17, 4, 2, 0, 15] – 1 is created as a red node.

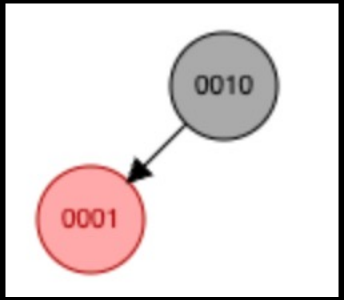
Insertion

[10, 1, 17, 4, 2, 0, 15] – 1 is less than 10, so it's inserted on the left of 10.



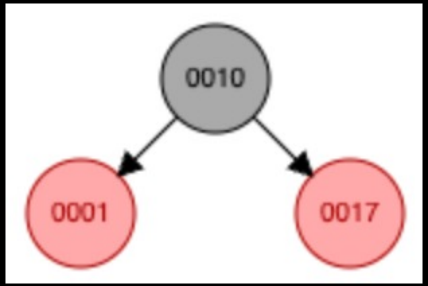
Insertion

[10, 1, 17, 4, 2, 0, 15] – 17 is created as a red node.



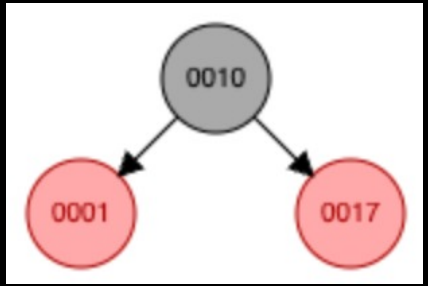
Insertion

[10, 1, 17, 4, 2, 0, 15] – 17 is greater than 10, so it’s inserted on the right.



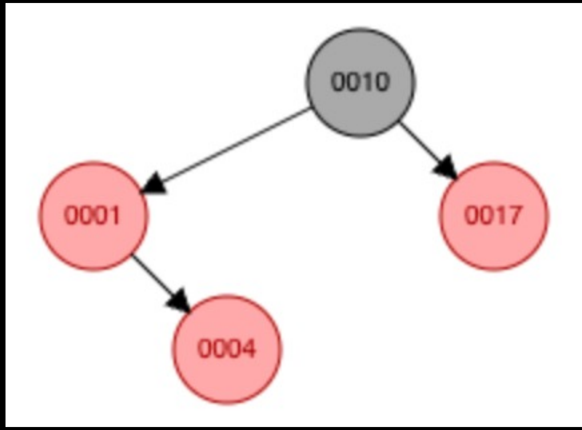
Insertion

[10, 1, 17, 4, 2, 0, 15] – 4 is created as a red node.



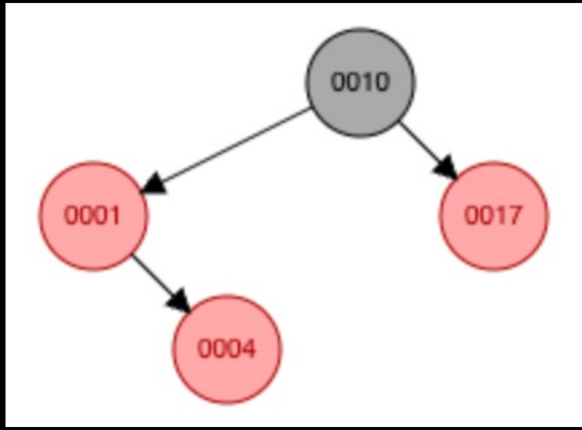
Insertion

[10, 1, 17, 4, 2, 0, 15] – 4 is less than 10 and greater than 1.



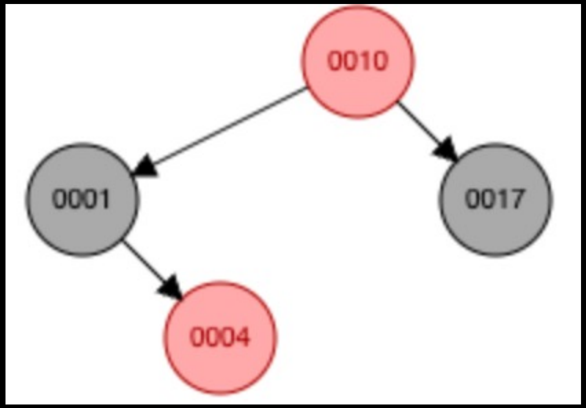
Insertion

[10, 1, 17, 4, 2, 0, 15] – 4 and 1 are both red. Uncle of 4 (17) is red.



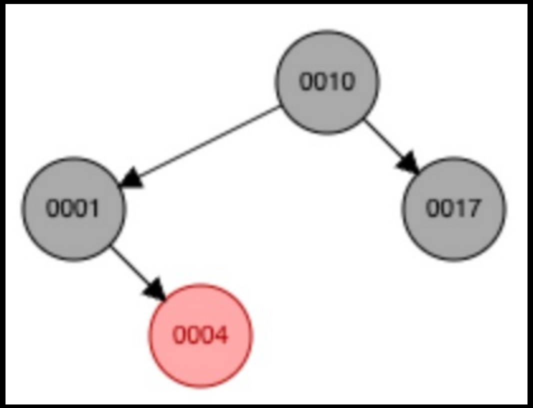
Insertion

[10, 1, 17, 4, 2, 0, 15] – Recolor the nodes from the grandparent.



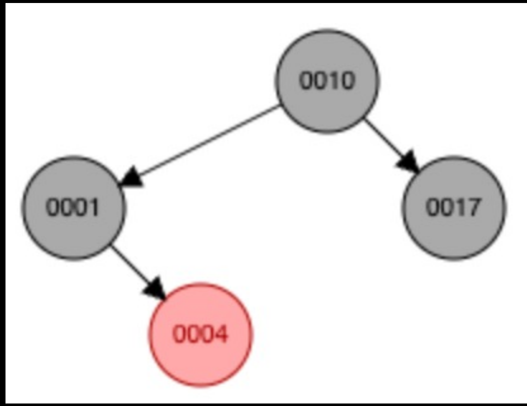
Insertion

[10, 1, 17, 4, 2, 0, 15] – Root is red, set it to black.



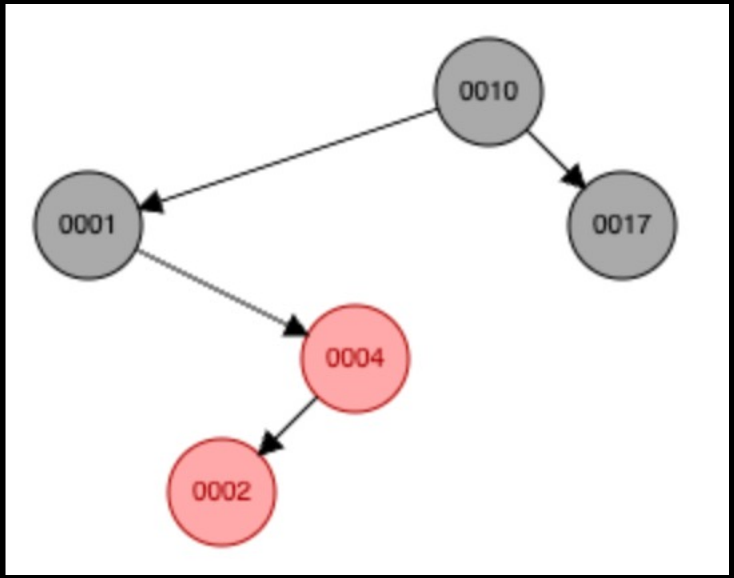
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2 is created as a red node.



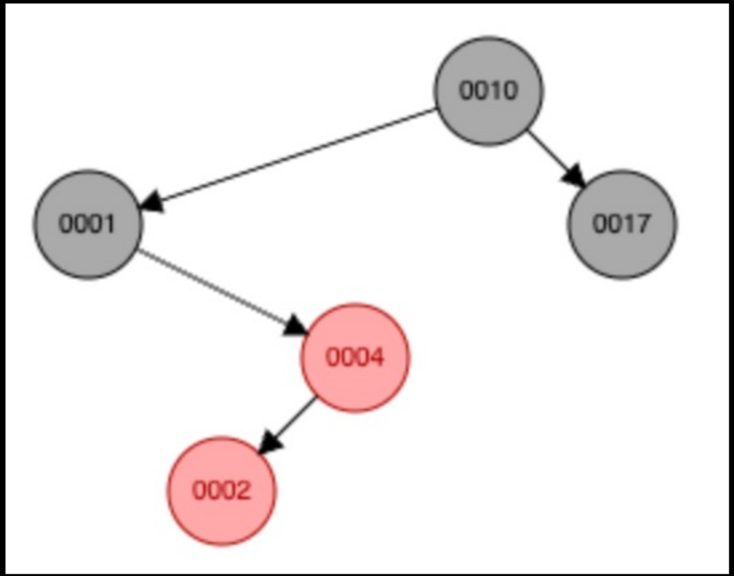
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2<10, 2>1, 2<4.



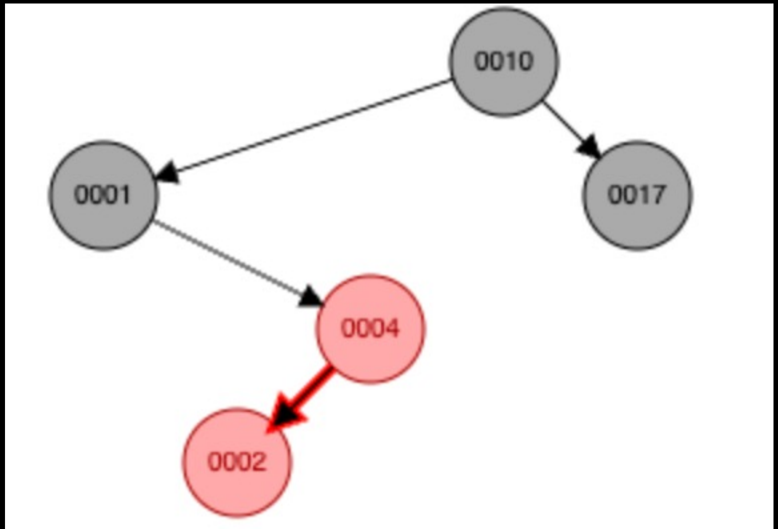
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2 and 4 are both red.



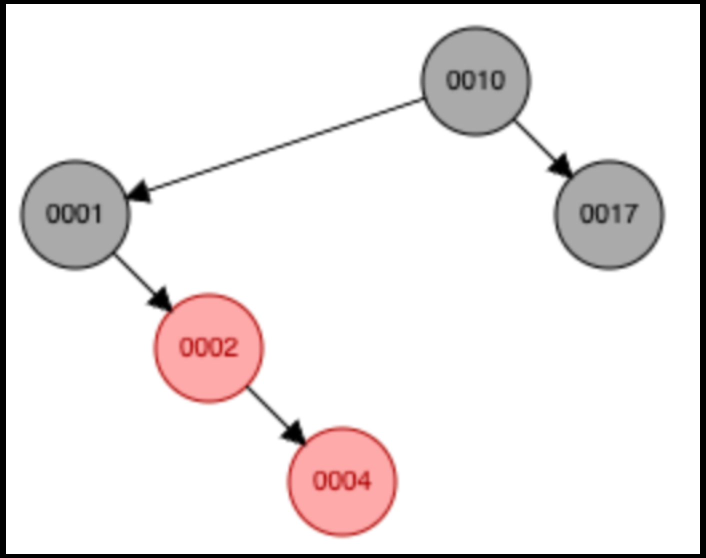
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2 is a left child, 4 is a right child → RR



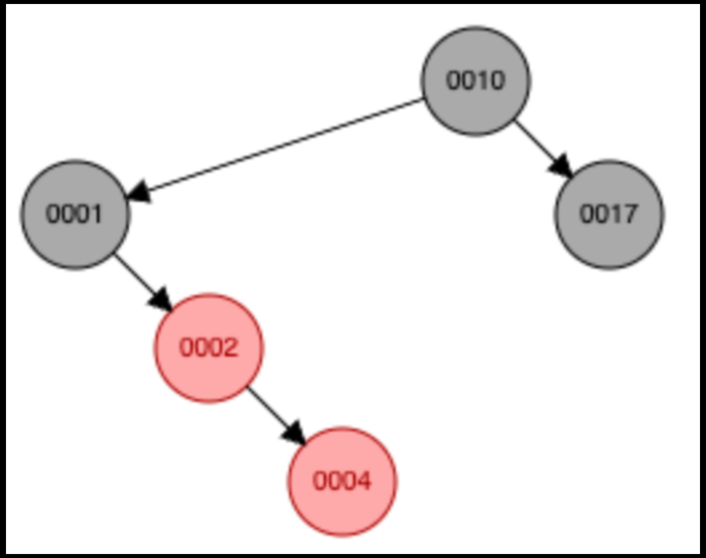
Insertion

[10, 1, 17, 4, 2, 0, 15] – Right rotate



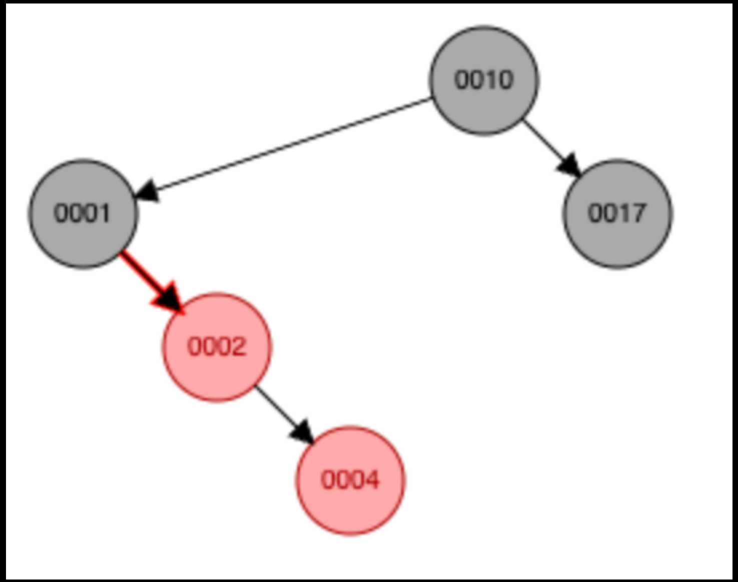
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2 and 4 are red, they are both right child



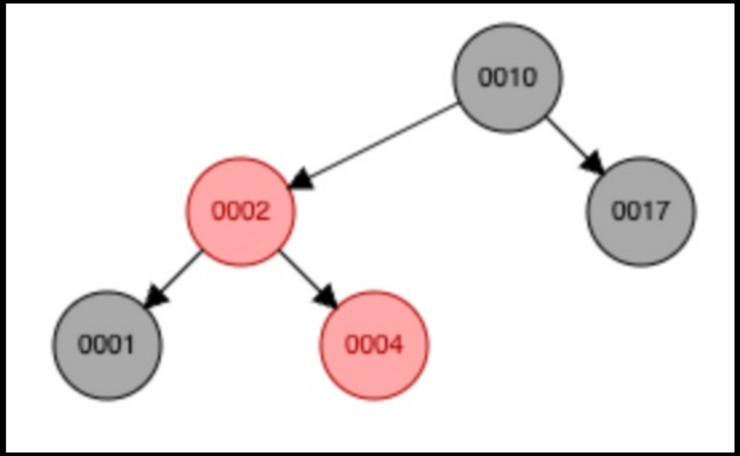
Insertion

[10, 1, 17, 4, 2, 0, 15] – 2 and 4 are red, they are both right child→ LR



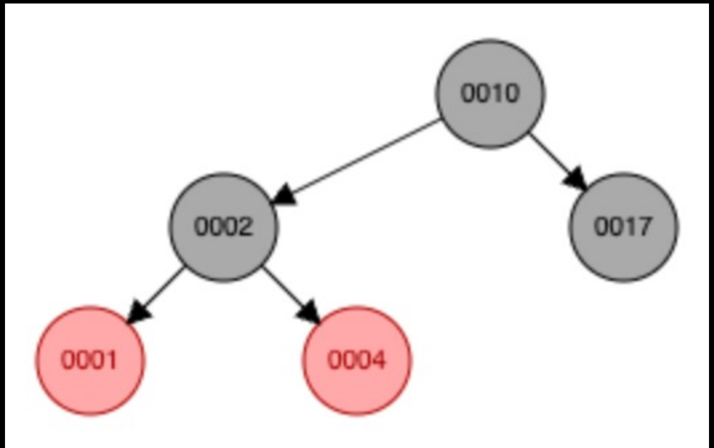
Insertion

[10, 1, 17, 4, 2, 0, 15] – Left rotate



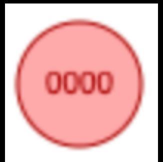
Insertion

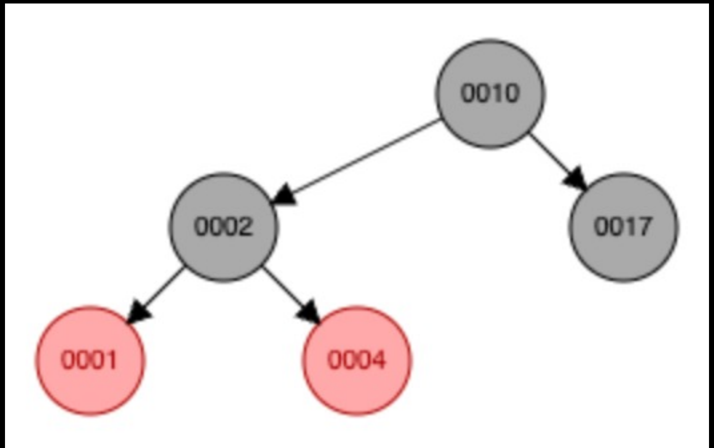
[10, 1, 17, 4, 2, 0, 15] – Recolor the nodes



Insertion

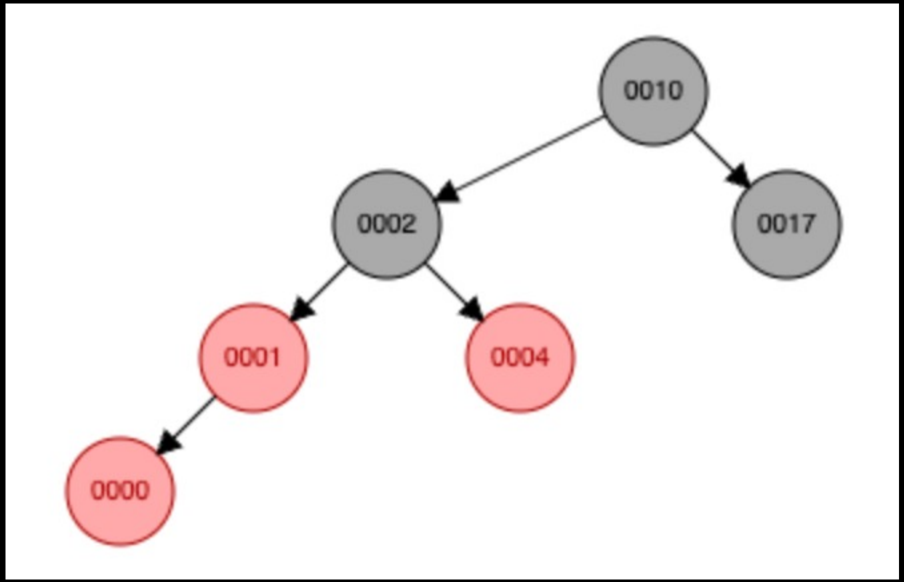
[10, 1, 17, 4, 2, 0, 15] – 0 is created as a red node





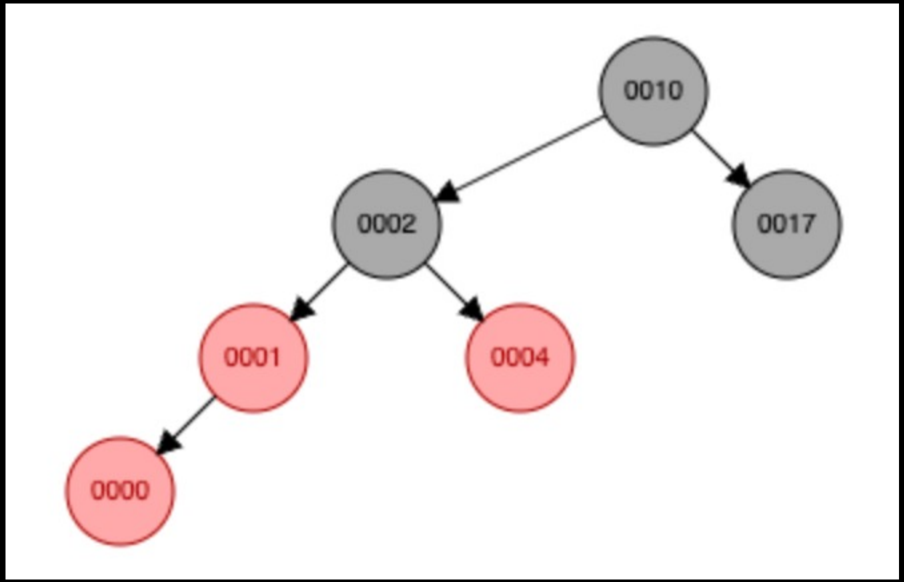
Insertion

[10, 1, 17, 4, 2, 0, 15] – 0<10, 0<2, 0<1



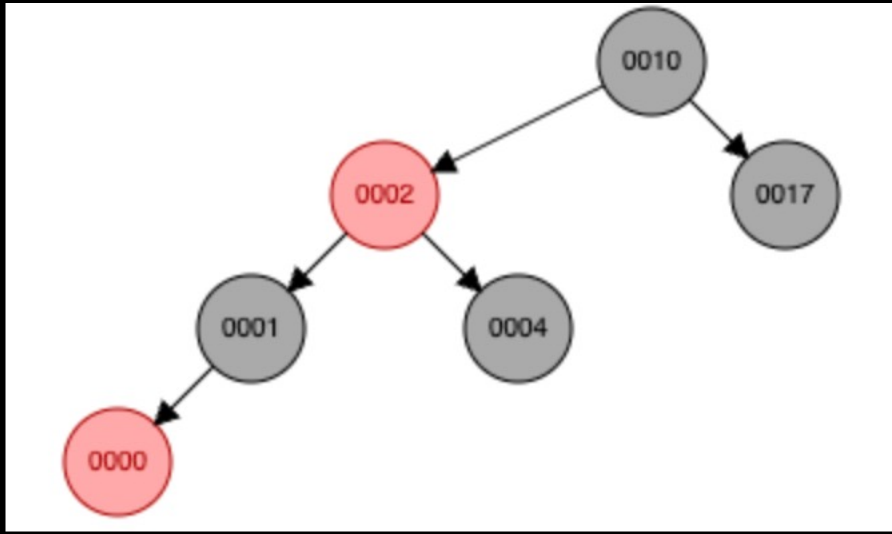
Insertion

[10, 1, 17, 4, 2, 0, 15] – 0 and 1 are both red, uncle (4) is red.



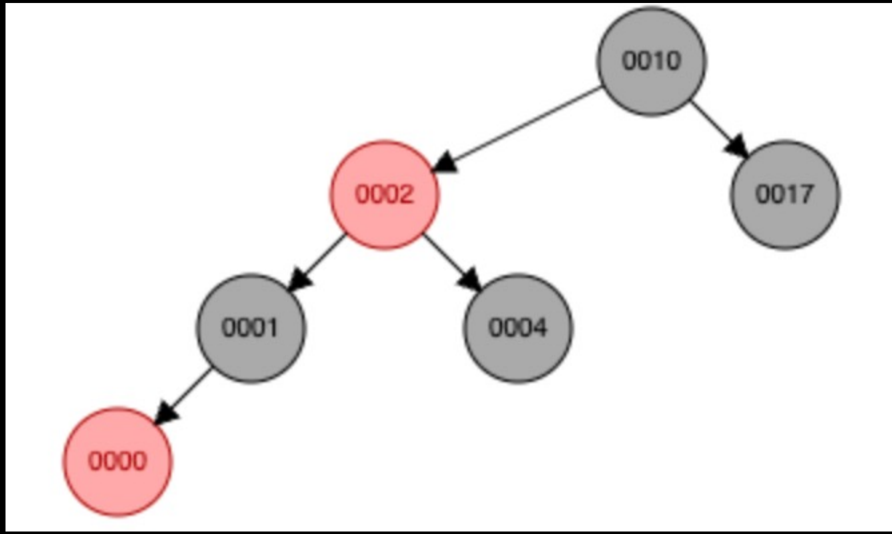
Insertion

[10, 1, 17, 4, 2, 0, 15] – Recolor from the grandparent



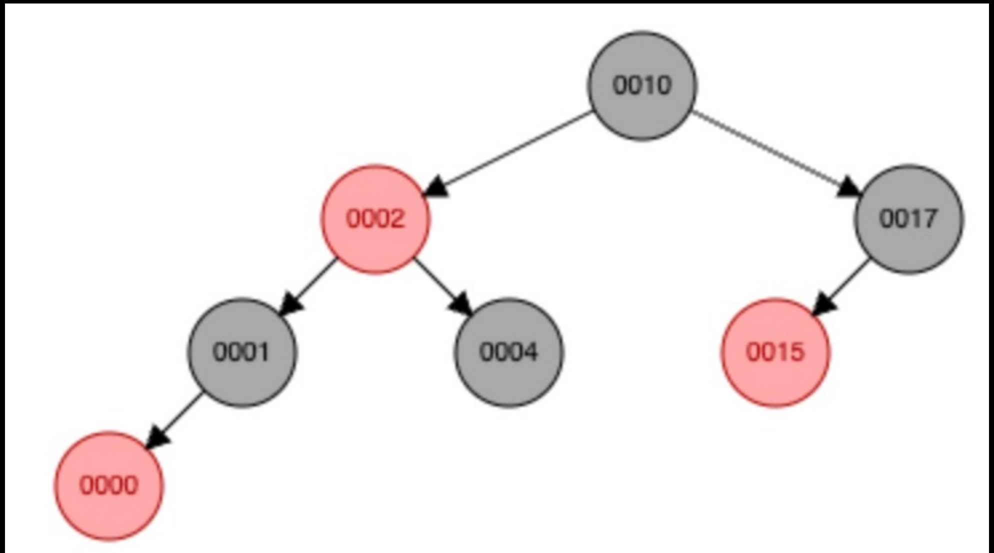
Insertion

[10, 1, 17, 4, 2, 0, 15] – 15 is created as a red node.



Insertion

[10, 1, 17, 4, 2, 0, 15] – 15>10, 15<17.



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Deletion

- RB-delete(T, z)

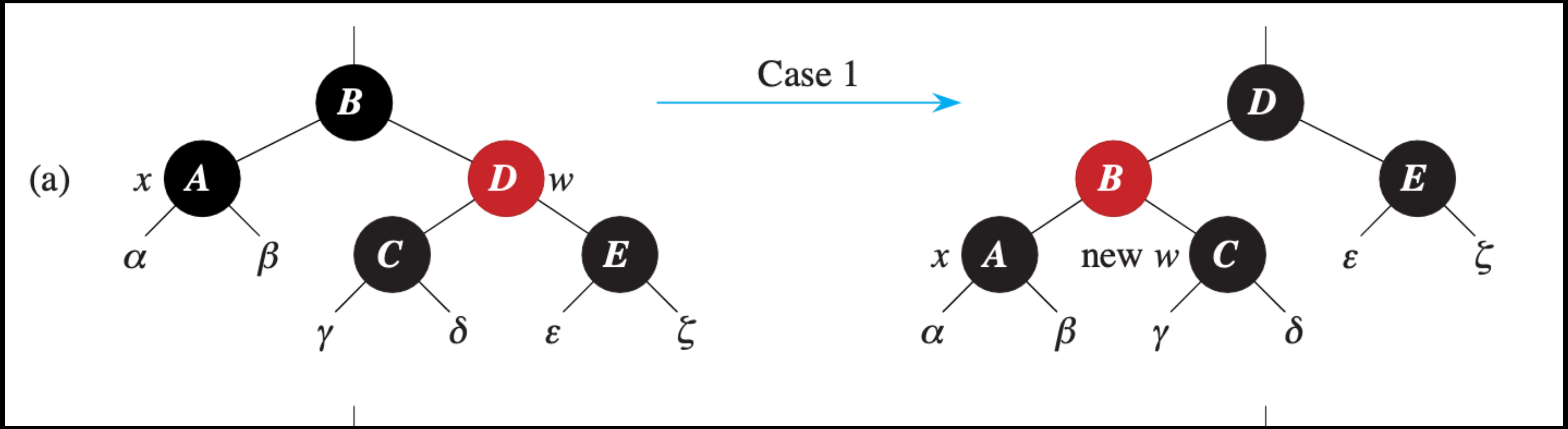
1. $y = z$
2. y-original-color = $y.color$
3. If $z.left = T.nil$:
   1. Replace z by its right child
4. Else If $z.right = T.nil$:
   1. Replace z by its left child

5. Else, $y = Tree - Minimum(z.right)$
   1. y-original-color = $y.color$
   2. $x = y.right$
   3. If $y \neq z.right$
      1. Replace y with its right child
   4. Else, $x.p = y$
   5. Replace z with y
   6. $y.color = z.color$
6. If y-original-color = black
   1. Fix the tree

Deletion

RB-DELETE-FIXUP(T,x)

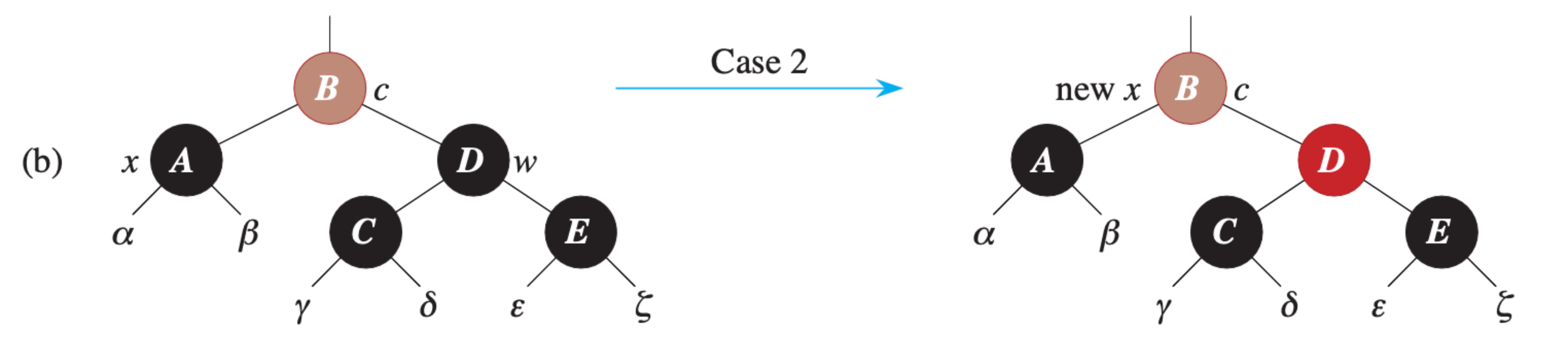
- While $x \neq T.root$ and $x.color = black$
  1. If x is a left child:
     1. $w =$ right sibling of x
     2. If $w.color = red$: → Case 1
        1. $w.color = black$
        2. $x.p.color = red$
        3. LR($T, x.p$)
        4. $w = x.p.right$



Deletion

RB-DELETE-FIXUP(T,x)

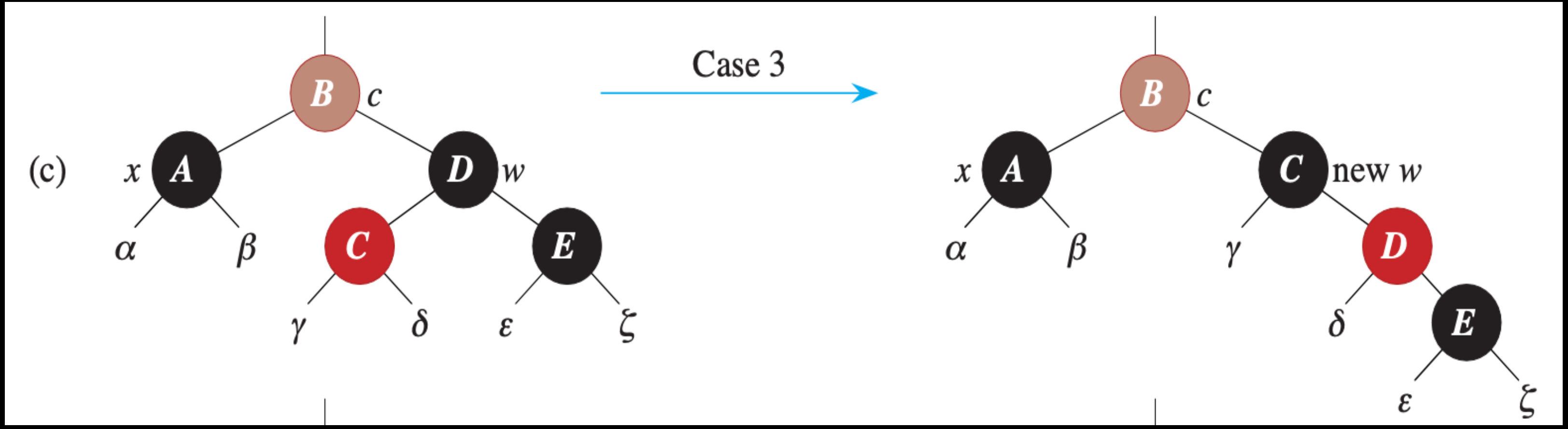
- While $x \neq T.root$ and $x.color = black$
  1. If x is a left child:
     1. $w =$ right sibling of x
     2. If $w.left.color = black$ and $w.right.color = black$: → Case 2
        1. $w.color = red$
        2. $x = x.p$



Deletion

RB-DELETE-FIXUP(T,x)

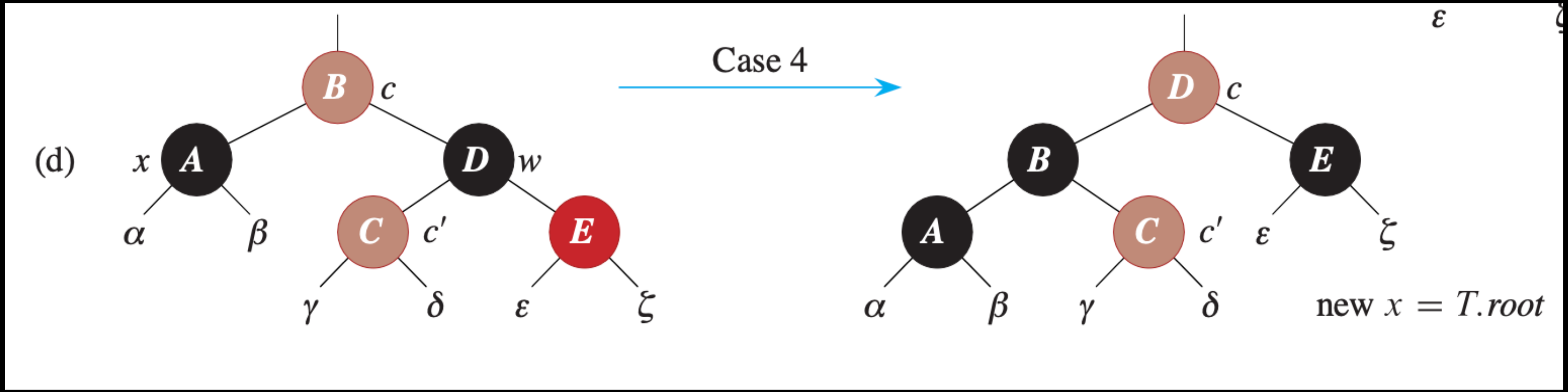
- While $x \neq T.root$ and $x.color = black$
  1. If x is a left child:
     1. $w =$ right sibling of x
     2. Else:
        1. If $w.color = black$: → Case 3
           1. $w.left.color = black$
           2. $w.color = red$
           3. RR(T, w)
           4. $w = x.p.right$



Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  1. If x is a left child:
     1. w = right sibling of x
     2. Else: → Case 4
        1. $w.color = x.p.color$
        2. $x.p.color = black$
        3. $w.right.color = black$
        4. LR($T, x.p$)
        5. $x = T.root$



Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  1. If x is a right child:
     1. $w = x.p.left$ → *Apply the previous 4 cases but exchanging right and left*
     2. If $w.color = red$: → Case 1
        1. $w.color = black$
        2. $x.p.color = red$
        3. RR($T, x.p$)
        4. $w = x.p.left$

Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  1. If x is a right child:
     1. $w = x.p.left$ → *Apply the previous 4 cases but exchanging right and left*
     2. If $w.right.color = black$ and $w.left.color = black$: → Case 2
        1. $w.color = red$
        2. $x = x.p$

Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  1. If x is a right child:
     1. $w = x.p.left$ → *Apply the previous 4 cases but exchanging right and left*
     2. Else:
        1. If $w.color = black$: → Case 3
           1. $w.right.color = black$
           2. $w.color = red$
           3. LR(T, w)
           4. $w = x.p.left$

Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  1. If x is a right child:
     1. $w = x.p.left$ → *Apply the previous 4 cases but exchanging right and left*
     2. Else: → Case 4
        1. $w.color = x.p.color$
        2. $x.p.color = black$
        3. $w.left.color = black$
        4. RR($T, x.p$)
        5. $x = T.root$

Deletion

RB-DELETE-FIXUP(T,x)

- While $x \neq T.root$ and $x.color = black$
  - $\ldots$
- $x.color = black$

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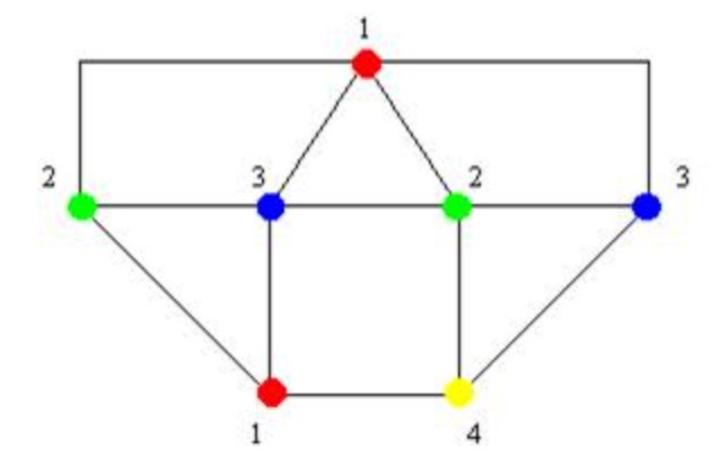
Complexity Classes

Complexity Classes

- **Types of problems**
  - **Tractable**: Problems that can be solved in polynomial time
  - **Intractable**: Problems that as they grow large, we are unable to solve them in reasonable time
  - **Decision**: A problem with a True/False answer
  - **Optimization**: A problem which asks, “What is the optimal solution to problem X?”

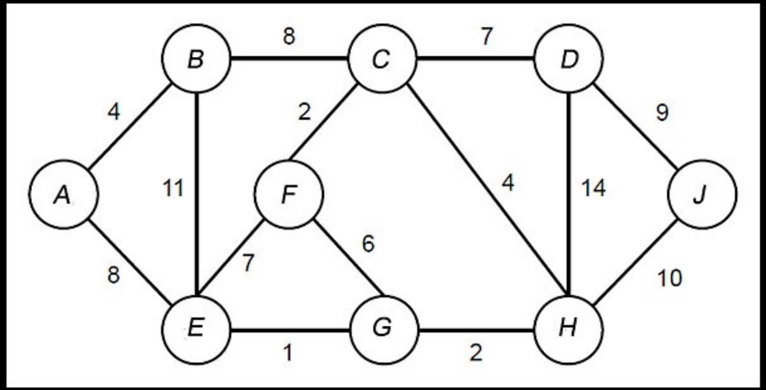
Complexity Classes

- Reasonable time = Polynomial time
  - On an input size n, the worst-case running time is $O(n^c)$ for some constant c.
  - Polynomial time: $O(n^2), O(n^3), O(1), O(n \lg n)$
  - Non-polynomial time: $O(2^n), O(n^n), O(n!)$
- Intractable problems: there is no efficient algorithm to solve.
  - Graph-coloring: coloring vertices such that no two adjacent vertices have the same color.



Complexity Classes

- Optimization problems: what is the best (optimal) solution?
  - Minimum spanning tree (MST): a spanning tree that has the minimum weight among all the possible spanning trees.

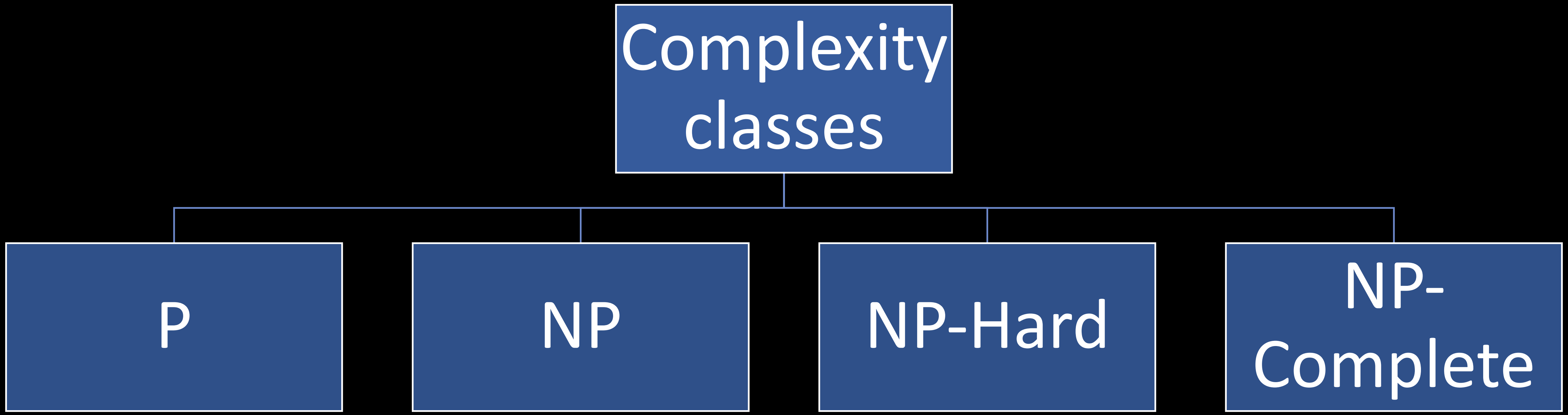


- Decision problems: Yes, or No?
  - Does a graph have an MST of weight $\leq w$?

Complexity Classes

- **Deterministic algorithms**: always compute the same answer and do the same sequence of computations.
  - **Predictability**: The same input will always result in the same output.
  - **Fixed Steps**: The algorithm follows a clear set of rules or steps.
  - **No Randomness**: There is no element of randomness or probability in the process.
- **Non-deterministic algorithms**: algorithms that “guess” the right solution.
  - **Multiple outcomes**: The same input might result in different outputs on different runs.
  - **Probabilistic Elements**: Often involves randomness or probability in decision-making.
  - **Parallel Path Exploration**: Can explore multiple solution paths simultaneously.

Complexity Classes



Complexity Classes

- **P (Polynomial Time)**: Problems **solvable** in polynomial time on deterministic algorithms.
    - Sorting algorithms.

- **NP (Non-deterministic Polynomial Time)**: Problems **verifiable** in polynomial time, solved using non-deterministic algorithms.
    - Hard to find an optimal solution.
    - Solutions can be verified in polynomial time.
    - Example: graph-coloring algorithms.

Complexity Classes

- Sometimes you can solve a problem by reducing it to a different problem. E.g., solving Problem B by solving Problem A.
  - If I can solve A in polynomial time, then I can construct a solution to B in polynomial time that is based on the solution of A.

- A problem is **NP-hard** if problems in NP are reducible to it.
  - Ex: Hamiltonian Cycle – a path in a graph that visits each vertex exactly once.
- Example: Travelling salesman problem can be reduced to Hamiltonian Cycle.

Complexity Classes

- **NP-Complete**: A problem that is NP-hard and NP.
- Open question: $P = NP$? → Is every problem whose solution can be verified quickly (in polynomial time) also solvable quickly (in polynomial time)?
- If $P = NP$, then all NP-complete problems would have efficient (polynomial-time) solutions
  - This has not been proven.
- Most computer scientists believe that $P \neq NP$.
  - This has not been proven.

Complexity Classes

- **P:** The easiest and can be solved efficiently.
- **NP:** Have a solution that can be verified quickly but finding it may be hard.
- **NP-complete:** The hardest problems in NP, and if one of them can be solved efficiently, so can all NP problems.
- **NP-hard** problems are at least as difficult as NP-complete problems, but they may not be decision problems and may not be in NP at all.