Discrete Structures

CH 01: Logic and Proofs

Content

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Propositional Logic
Applications of Propositional Logic
Propositional Equivalences
Predicates and Quantifiers
Nested Quantifiers
Rules of Inference
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Propositional Equivalences

- **Definition 1:**

Tautology	A compound proposition that is always true
Contradiction	A compound proposition that is always false
Contingency	A compound proposition that is neither a tautology nor a contradiction

Propositional Equivalences

- **EXAMPLE 1:** Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$.
  - Because $p \vee \neg p$ is always true, it is a *tautology*.
  - Because $p \wedge \neg p$ is always false, it is a contradiction.

- **Solution:**

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Equivalences

- **Definition 2:**
  The notation $p \equiv q$ or $p \Leftrightarrow q$ means that p and q are *logically equivalent*.
  - The truth table of p is equivalent to the truth table of q

Propositional Equivalences

- **Example 2:** Show that $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.
  - This is De-Morgan law

TABLE 2 De Morgan's Laws.
$\neg(p \land q) \equiv \neg p \lor \neg q$
$\neg(p \lor q) \equiv \neg p \land \neg q$

- **Solution:**

TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Propositional Equivalences

- Table 6 contains some important equivalences.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Propositional Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Propositional Equivalences

- The notation

$$\bigvee_{j=1}^{n} p_j$$

means $p_1 \lor p_2 \lor \cdots \lor p_n$

- The notation

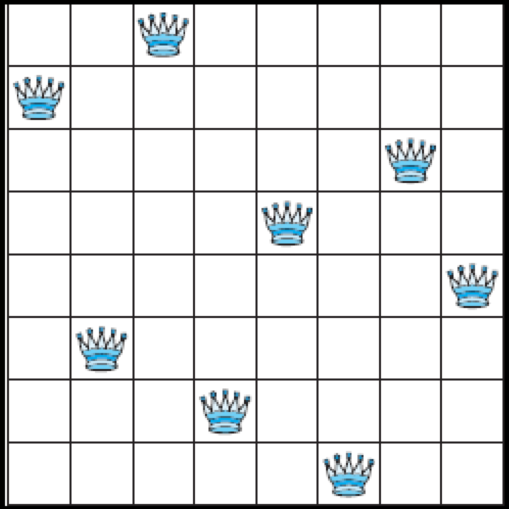
$$\bigwedge_{j=1}^{n} p_j$$

means $p_1 \land p_2 \land \cdots \land p_n$.

Propositional Equivalences

- **Satisfiability:**
  - A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that makes it true. $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$
  - A compound proposition is *unsatisfiable* when the proposition is false for all assignments of truth values to its variables. $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

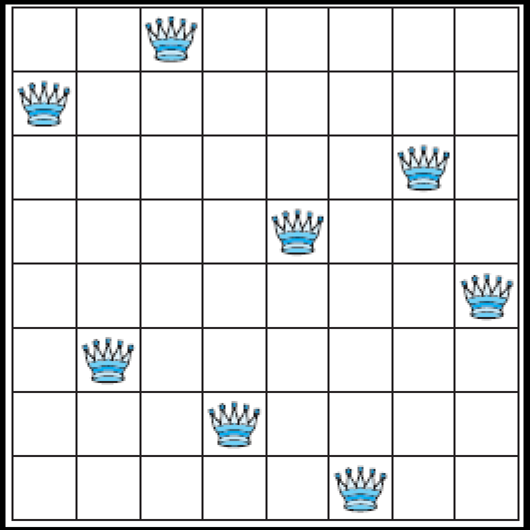
- **Applications of Satisfiability:**
  - The n-queens problem
  - Sudoku



	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

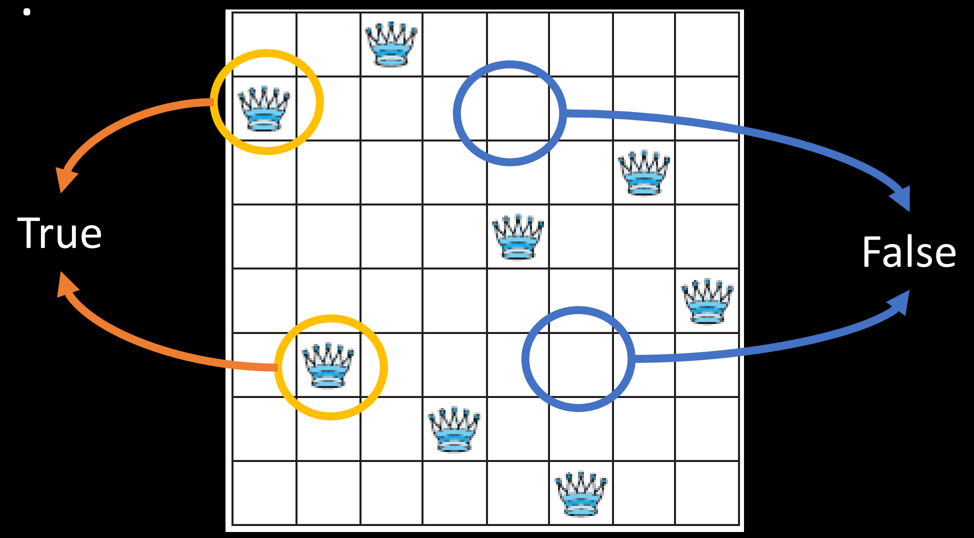
Propositional Equivalences

- **The n-queens problem:**
  The n-queens problem asks for a placement of n queens on an $n \times n$ chessboard so that no queen can attack another queen.
  - This means that no two queens can be placed in the same row, in the same column, or on the same diagonal.



Propositional Equivalences

- To model the n-queens problem as a satisfiability problem,
  - We introduce n^2 variables, $p(i,j)$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$.
  - For a given placement of a queens on the chessboard, $p(i,j)$ is true when there is a queen on the square in the ith row and jth column and is false otherwise.
  - squares (i,j) and (i',j') are on the same diagonal if either $i + i' = j + j'$ or $i - i' = j - j'$.



Propositional Equivalences

- To solve the n-queen problem, we assert the following rule:

$$Q = Q_1 \land Q_2 \land Q_3 \land Q_4 \land Q_5$$

Every row contains at least one queen.

There is at most one queen in each row

No column contains more than one queen

No diagonal contains two queens

No diagonal contains two queens

Propositional Equivalences

- To solve the n-queen problem, we assert the following rule:

$$Q = Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5$$

Every row contains at least one queen. → There is at most one queen in each row → No column contains more than one queen → No diagonal contains two queens → No diagonal contains two queens

$$Q_1 = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} p(i,j)$$

Propositional Equivalences

- To solve the n-queen problem, we assert the following rule:

$$Q = Q_1 \land Q_2 \land Q_3 \land Q_4 \land Q_5$$

Every row contains at least one queen. → There is at most one queen in each row → No column contains more than one queen → No diagonal contains two queens → No diagonal contains two queens

$$Q_2 = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n-1} \bigwedge_{k=j+1}^{n} (\neg p(i,j) \lor \neg p(k,j)).$$

Propositional Equivalences

- To solve the n-queen problem, we assert the following rule:

$$Q = Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5$$

- Every row contains at least one queen.
- There is at most one queen in each row
- No column contains more than one queen
- No diagonal contains two queens
- No diagonal contains two queens

$$Q_3 = \bigwedge_{j=1}^{n} \bigwedge_{i=1}^{n-1} \bigwedge_{k=i+1}^{n} (\neg p(i,j) \vee \neg p(k,j)).$$

Propositional Equivalences

- To solve the n-queen problem, we assert the following rule:

$$Q = Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5$$

Every row contains at least one queen. → There is at most one queen in each row → No column contains more than one queen → No diagonal contains two queens → No diagonal contains two queens

$$Q_4 = \bigwedge_{i=2}^{n} \bigwedge_{j=1}^{n-1} \bigwedge_{k=1}^{\min(i-1,n-j)} (\neg p(i,j) \vee \neg p(i-k, k+j))$$

$$Q_5 = \bigwedge_{i=1}^{n-1} \bigwedge_{j=1}^{n-1} \bigwedge_{k=1}^{\min(n-i,n-j)} (\neg p(i,j) \vee \neg p(i+k, j+k)).$$

Exercises

3. Use truth tables to verify the commutative laws

b) $p \land q \equiv q \land p$.

Exercises

3. Use truth tables to verify the commutative laws

b) $p \wedge q \equiv q \wedge p$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{q \wedge p}$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Exercises

6. Use a truth table to verify the first De Morgan law

$$\neg(p \land q) \equiv \neg p \lor \neg q.$$

Exercises

6. Use a truth table to verify the first De Morgan law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{\neg(p \wedge q)}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg q}$	$\boldsymbol{\neg p \vee \neg q}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Exercises

10. For each of these compound propositions, use the conditional-disjunction equivalence ($p \rightarrow q \ and \ \neg p \vee q$) to find an equivalent compound proposition that does not involve conditionals.

a) $\neg p \rightarrow \neg q$

b) $(p \vee q) \rightarrow \neg p$

c) $(p \rightarrow \neg q) \rightarrow (\neg p \rightarrow q)$

Exercises

- We apply the equivalence : $p \rightarrow q$ and $\neg p \vee q$ to the conditionals in the original statements.

a) $\neg p \rightarrow \neg q \equiv p \vee \neg q$

b) $(p \vee q) \rightarrow \neg p \equiv \neg(p \vee q) \vee \neg p$ by the conditional-disjunction equivalence

$\equiv (\neg p \wedge \neg q) \vee \neg p$ by the second De Morgan's law

$\equiv \neg p$ by the first absorption law

c) $(p \rightarrow \neg q) \rightarrow (\neg p \rightarrow q) \equiv \neg(p \rightarrow \neg q) \vee (\neg p \rightarrow q)$

$\equiv \neg(\neg p \vee \neg q) \vee (\neg\neg p \vee q)$

$\equiv (p \wedge q) \vee (p \vee q)$ by the double negation and De Morgan's laws

$\equiv (p \wedge q) \vee p \vee q$ by the associative law

$\equiv p \vee q$ by the absorption law

Exercises

11. Show that each of these conditional statements is a tautology by using truth tables.

b) $p \rightarrow (p \vee q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

f) $\neg(p \rightarrow q) \rightarrow \neg q$

Exercises

11. Show that each of these conditional statements is a tautology by using truth tables.

b) $p \rightarrow (p \vee q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

f) $\neg(p \rightarrow q) \rightarrow \neg q$

p	q	$p \vee q$	$p \rightarrow p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

p	q	$p \wedge q$	$p \rightarrow q$	$p \wedge q \rightarrow p \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$\neg(p \rightarrow q) \rightarrow \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Exercises

12. Show that each of these conditional statements is a tautology by using truth tables.

b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

Exercises

12. Show that each of these conditional statements is a tautology by using truth tables.

b) $[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \rightarrow q}$	$\boldsymbol{q \rightarrow r}$	$\boldsymbol{(p \rightarrow q) \land (q \rightarrow r)}$	$\boldsymbol{p \rightarrow r}$	$\boldsymbol{[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)}$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Exercises

19. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

Exercises

19. Determine whether $(\neg q \land (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

It is a tautology

q	p	$\neg q$	$p \rightarrow q$	$\neg q \land (p \rightarrow q)$	$\neg p$	$\neg q \land (p \rightarrow q) \rightarrow \neg p$
T	T	F	T	F	F	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Exercises

20. Show that $p \leftrightarrow q \; and \; (p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

Exercises

20. Show that $p \leftrightarrow q$ *and* $(p \land q) \lor (\neg p \land \neg q)$ are logically equivalent.

p	q	$p \land q$	$\neg p$	$\neg q$	$\neg p \land \neg q$	$(p \land q) \lor (\neg p \land \neg q)$	$p \leftrightarrow q$
T	T	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	F	T	T	T	T	T

Exercises

22. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

Exercises

22. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \rightarrow q}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg q}$	$\boldsymbol{\neg q \rightarrow \neg p}$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Exercises

36. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

Exercises

36. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

We just need to find an assignment of truth values that makes one of these propositions true and the other false.

We can let p be $true$ and the other two variables be $false$.
Then the first statement will be $F \rightarrow F$, which is $true$, but the second will be $F \wedge T$, which is $false$

Exercises

62. How many of the disjunctions $p \lor \neg q$, $\neg p \lor q$, $q \lor r$, $q \lor \neg r$, and $\neg q \lor \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

Exercises

62. How many of the disjunctions $p \lor \neg q$, $\neg p \lor q$, $q \lor r$, $q \lor \neg r$, and $\neg q \lor \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

If we want the first two of these to be $true$, then p and q must have the same truth value.

If q is $true$, then the third and fourth expressions will be $true$, and if r is $false$, the last expression will be $true$.

So, all five of these disjunctions will be $true$ if we set p and q to be $true$, and r to be $false$.

Exercises

65. Determine whether each of these compound propositions is satisfiable.

a) $(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$

Exercises

65. Determine whether each of these compound propositions is satisfiable.

a) $(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$

Satisfiable

p	q	$\neg q$	$p \lor \neg q$	$\neg p$	$\neg p \lor q$	$\neg p \lor \neg q$	$(\neg p \lor q) \land (\neg p \lor \neg q)$	$(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$
T	T	F	T	F	T	F	F	F
T	F	T	T	F	F	T	F	F
F	T	F	F	T	T	T	T	F
F	F	T	T	T	T	T	T	T

Exercises

67. Find the compound proposition Q constructed in Example 10 for the n-queens problem, and use it to find all the ways that n queens can be placed on an $n \times n$ chessboard, so that no queen can attack another when n is

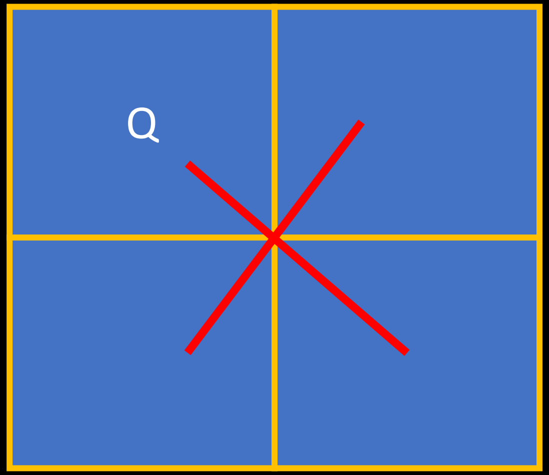
a) 2. b) 3. c) 4.

Exercises

67. Find the compound proposition Q constructed in Example 10 for the n-queens problem, and use it to find all the ways that n queens can be placed on an $n \times n$ chessboard, so that no queen can attack another when n is

a) 2.

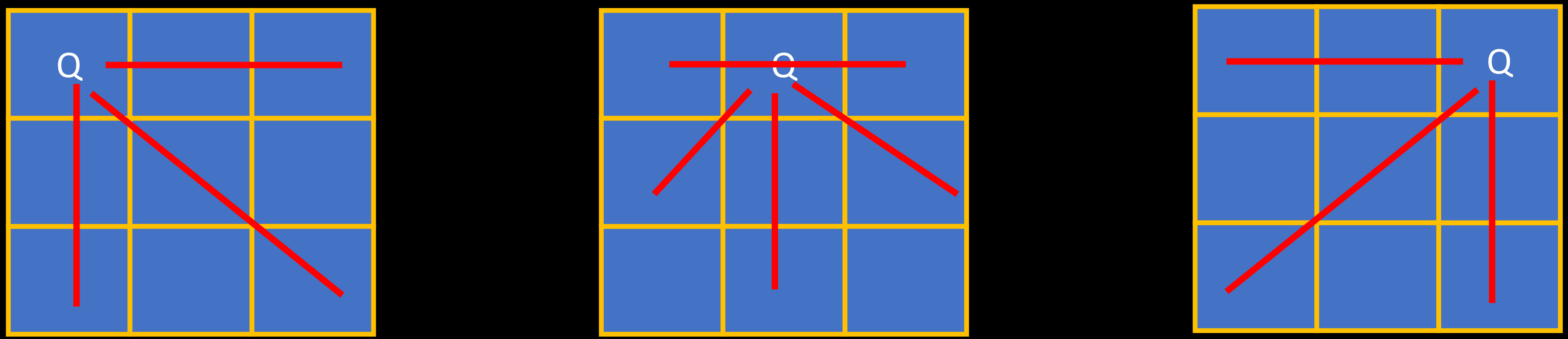
No Solution



Exercises

67. Find the compound proposition Q constructed in Example 10 for the n-queens problem, and use it to find all the ways that n queens can be placed on an $n \times n$ chessboard, so that no queen can attack another when n is

b) 3. No Solution



Exercises

67. Find the compound proposition Q constructed in Example 10 for the n-queens problem, and use it to find all the ways that n queens can be placed on an $n \times n$ chessboard, so that no queen can attack another when n is

c) 4. $(1, 2), (2, 4), (3, 1), (4, 3)$ or $(1, 3), (2, 1), (3, 4), (4, 2)$

	Q		
			Q
Q			
		Q	

		Q	
Q			
			Q
	Q		

TASKS

Section 1.3
3 (a)
9
11 (a, c, e)
12 (c, d)
18
21
28
35

Content

The Foundations: Logic and Proofs
Propositional Logic
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Predicates and Quantifiers

- Predicate logic is used to solve statements involving variables, such as
  - "x > 3,"   "x = y + 3,"   "x + y = z,"
  - "Computer x is under attack by an intruder,"
  - "Computer x is functioning properly,"

- The first part, the variable x, is the subject of the statement.
- The second part of the statement is called the predicate.
- The statement $P(x)$ is also said to be the value of the **propositional function** P at x.

Predicates and Quantifiers

- **EXAMPLE 1:**
  Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

- **Solution:**
  $P(4)$, which is the statement "4 > 3," is $true$.
  However, $P(2)$, which is the statement "2 > 3," is $false$.

Predicates and Quantifiers

- **EXAMPLE 3:**
  Let $Q(x, y)$ denote the statement "$x = y + 3$."
  What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

- **Solution:**
  To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$.
  Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is $false$.
  The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is $true$.

Predicates and Quantifiers

- Predicates are used to establish the correctness of computer programs
  - To show that computer programs always produce the desired output when given valid input.

- **Preconditions** are the statements that describe valid input.
- **Postconditions** are the conditions that the output should satisfy when the program has run.

Predicates and Quantifiers

- **<u>Quantification</u>** expresses the extent to which a predicate is true over a range of elements.

Quantifiers

- **Universal quantification**

  A predicate is $true$ for every element under consideration

- **Existential quantification**

  There is one or more element under consideration for which the predicate is $true$.

Predicates and Quantifiers

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Predicates and Quantifiers

- **EXAMPLE 8:**
  Let $P(x)$ be the statement "$x + 1 > x$."
  What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

- **Solution:**
  Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.

Predicates and Quantifiers

- **<u>EXAMPLE 13:</u>**
  Let $P(x)$ denote the statement "$x > 3$."
  What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

- **<u>Solution:</u>**
  Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Predicates and Quantifiers

- **EXAMPLE 15:**
  What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?
- **Solution:**
  The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4.
Because $P(4)$, which is the statement "$4^2 < 10$," is $false$, it follows that $\forall x P(x)$ is $false$.

Predicates and Quantifiers

- **EXAMPLE 16:**
  What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

- **Solution:**
  Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the same as the disjunction $P(1) \lor P(2) \lor P(3) \lor P(4)$.
  Because $P(4)$, which is the statement "$42 > 10$," is $true$, it follows that $\exists x P(x)$ is $true$.

Predicates and Quantifiers

- **EXAMPLE 17:**
  What do the statements $\forall x < 0\ (x^2 > 0)$, $\forall y \neq 0\ (y^3 \neq 0)$, and $\exists z > 0\ (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?
- **Solution:**

Statement	**Meaning**
$\forall x < 0\ (x^2 > 0)$	For every real number < 0, we have $x^2 > 0$
$\forall y \neq 0\ (y^3 \neq 0)$	For every real number y with $y \neq 0$, we have $y^3 \neq 0$.
$\exists z > 0\ (z^2 = 2)$	There exists a real number z with $z > 0$ such that $z^2 = 2$.

Predicates and Quantifiers

- Negation of universal quantifier:

$$\neg \forall x P(x) \;\equiv\; \exists x \, \neg P(x).$$

- Negation of existential quantifier:

$$\neg \exists x Q(x) \;\equiv\; \forall x \, \neg Q(x).$$

Predicates and Quantifiers

- **EXAMPLE 21:**
  What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

- **Solution:**

$$\neg \forall x(x^2 > x) \equiv \exists x \neg (x^2 > x) \equiv \exists x(x^2 \leq x)$$

$$\neg \exists x(x^2 = 2) \equiv \forall x \neg (x^2 = 2) \equiv \forall x(x^2 \neq 2)$$

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?

a) $P(0)$ b) $P(4)$ c) $P(6)$

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?

a) $P(0)$

$0 \leq 4$
True

b) $P(4)$

$4 \leq 4$
True

c) $P(6)$

$6 \leq 4$
False

Exercises

4. State the value of x after the statement $if\ P(x)\ then\ x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is

a) x = 0. b) x = 1. c) x = 2.

Exercises

4. State the value of x after the statement $if\ P(x)\ then\ x := 1$ is executed, where $P(x)$ is the statement “$x > 1$,” if the value of x when this statement is reached is

a) x = 0. b) x = 1. c) x = 2.

This is equivalent to:

If $(x > 1)$, then $x = 1$

a) Here x is still equal to 0, since the condition is $false$.
b) Here x is still equal to 1, since the condition is $false$.
c) This time x is equal to 1 at the end, since the condition is $true$, so the statement $x := 1$ is executed.

Exercises

5. Let $P(x)$ be the statement "x $spends$ $more$ $than$ $five$ $hours$ $every$ $weekday$ in $class$," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

Exercises

5. Let $P(x)$ be the statement "x $spends$ $more$ $than$ $five$ $hours$ $every$ $weekday$ in $class$," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

a) There is a student who spends more than 5 hours every weekday in class.
b) Every student spends more than 5 hours every weekday in class.
c) There is a student who does not spend more than 5 hours every weekday in class.
d) No student spends more than 5 hours every weekday in class.

Exercises

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ b) $P(1)$ c) $P(2)$ d) $P(-1)$ e) $\exists x P(x)$ f) $\forall x P(x)$

Exercises

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ b) $P(1)$ c) $P(2)$ d) $P(-1)$ e) $\exists x P(x)$ f) $\forall x P(x)$

	P(x)	**Result**
a	$P(0) = 0^2 = 0$	True
b	$P(1) = 1^2 = 1$	True
c	$P(2) = 2^2 = 4$	False
d	$P(-1) = -1^2 = 1$	False
e	$0 \ and \ 1$	True
f	Only $0 \ and \ 1$	False

Exercises

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$

c) $\exists n(n = -n)$

Exercises

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$

c) $\exists n(n = -n)$

a) For all numbers, $n; n + 1$ is always greater than n.
Try different set of numbers: $0, -1, -5, 10, 12$.
This is $True$

b) There exists a number, n, that is equal to its negative $(-n)$.
This is $True$ for $n = 0$

Exercises

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$ c) $\forall x((-x)^2 = x^2)$

Exercises

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$

c) $\forall x((-x)^2 = x^2)$

a) There exists a number, x, where the cube of x is -1.
This is $True$ for $x = -1$, such that $(-1)^3 = -1$

c) For all numbers x, the square of the negative of x is equal to the square of x.

$(-x)^2 = \left((-1)x\right)^2 = (-1)^2x^2 = x^2$

The predicate is $True$

Exercises

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers $0, 1, 2, 3, and\ 4$. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) ∃x¬P(x)

d) $\forall x \neg P(x)$ e) $\neg \exists x P(x)$ f) $\neg \forall x P(x)$

Exercises

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers $0, 1, 2, 3, and\ 4$. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) ∃x¬P(x)

d) $\forall x \neg P(x)$ e) $\neg \exists x P(x)$ f) $\neg \forall x P(x)$

a) $P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$

b) $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

c) $\neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

d) $\neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$

e) $\neg(P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4))$

f) $\neg(P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4))$

Exercises

20. Suppose that the domain of the propositional function $P(x)$ consists of $-5, -3, -1, 1, 3$, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\forall x\big((x \neq 1) \rightarrow P(x)\big)$

d) $\exists x((x \geq 0) \wedge P(x))$ e) $\exists x(\neg P(x)) \wedge \forall x((x < 0) \rightarrow P(x))$

Exercises

20. Suppose that the domain of the propositional function $P(x)$ consists of $-5, -3, -1, 1, 3$, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\forall x\big((x \neq 1) \rightarrow P(x)\big)$
d) $\exists x((x \geq 0) \land P(x))$ e) $\exists x(\neg P(x)) \land \forall x((x < 0) \rightarrow P(x))$

a) $P(-5) \lor P(-3) \lor P(-1) \lor P(1) \lor P(3) \lor P(5)$.
b) $P(-5) \land P(-3) \land P(-1) \land P(1) \land P(3) \land P(5)$
c) $P(-5) \land P(-3) \land P(-1) \land P(3) \land P(5)$.
d) $P(1) \lor P(3) \lor P(5)$.
e) $\big(\neg P(-5) \lor \neg P(-3) \lor \neg P(-1) \lor \neg P(1) \lor \neg P(3) \lor \neg P(5)\big)$
$\land (P(-1) \land P(-3) \land P(-5))$.

Exercises

36. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.

a) $\forall x(-2 < x < 3)$

b) $\forall x(0 \leq x < 5)$

c) $\exists x(-4 \leq x \leq 1)$

d) $\exists x(-5 < x < -1)$

Exercises

36. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.

a) $\forall x(-2 < x < 3)$

b) $\forall x(0 \leq x < 5)$

c) $\exists x(-4 \leq x \leq 1)$

d) $\exists x(-5 < x < -1)$

a) $\exists x((x \leq -2) \vee (x \geq 3))$
b) $\exists x((x < 0) \vee (x \geq 5))$
c) $\forall x((x < -4) \vee (x > 1))$
d) $\forall x((x \leq -5) \vee (x \geq -1))$

TASKS

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