Math 1

CH 01: Logic and Proofs

QUIZ

- What is a proposition?
- What is a conjunction, disjunction?
- What is a conditional statement?
- What is a tautology, contradiction, logical equivalence?
- What are quantifiers?

What are Google search operators?

Content

The Foundations: Logic and Proofs
Propositional Logic
Applications of Propositional Logic
Propositional Equivalences
Predicates and Quantifiers
Nested Quantifiers
Rules of Inference
Introduction to Proofs
Proof Methods and Strategy

Nested Quantifiers

- Nested quantifiers, where one quantifier is within the scope of another, such as

$$\forall x \exists y(x + y = 0)$$

  is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Nested Quantifiers

- **EXAMPLE 1:**
  Assume that the domain for the variables x and y consists of all real numbers.
  The statement $\forall x \forall y(x + y = y + x)$ says that $x + y = y + x$ for all real numbers x and y.
- The statement $\forall x \exists y(x + y = 0)$ says that for every real number x there is a real number y such that $x + y = 0$.

Nested Quantifiers

- **EXAMPLE 2:**
  Translate into English the statement
  $$\forall x \forall y\big((x > 0) \land (y < 0) \rightarrow (xy < 0)\big)$$
  where the domain for both variables consists of all real numbers.

- **Solution:**
  "The product of a positive real number and a negative real number is always a negative real number."

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\forall x \forall y P(x, y)$
  - $\forall x \exists y P(x, y)$
  - $\exists x \forall y P(x, y)$
  - $\exists x \exists y P(x, y)$

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\boxed{\forall x \forall y P(x,y)}$
  - $\forall x \exists y P(x,y)$
  - $\exists x \forall y P(x,y)$
  - $\exists x \exists y P(x,y)$

$\boldsymbol{x_1}$	$\boldsymbol{y_1}$
$\boldsymbol{x_2}$	$\boldsymbol{y_2}$
.	.
.	.
.	.
$\boldsymbol{x_n}$	$\boldsymbol{y_n}$

For every x and for every y, $p(x, y)$ is true

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\forall x \forall y P(x, y)$
  - $\boxed{\forall x \exists y P(x, y)}$
  - $\exists x \forall y P(x, y)$
  - $\exists x \exists y P(x, y)$

$\boldsymbol{x_1}$
$\boldsymbol{x_2}$
.
.
.
$\boldsymbol{x_n}$

$\boldsymbol{y_1}$

For every x and for a y, $p(x, y)$ is true

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\forall x \forall y P(x, y)$
  - $\forall x \exists y P(x, y)$
  - $\boxed{\exists x \forall y P(x, y)}$
  - $\exists x \exists y P(x, y)$

$\boxed{\boldsymbol{x_1}}$

$$\boxed{\begin{matrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{y_n} \end{matrix}}$$

For an x and for every y, $p(x, y)$ is true

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\forall x \forall y P(x, y)$
  - $\forall x \exists y P(x, y)$
  - $\exists x \forall y P(x, y)$
  - $\boxed{\exists x \exists y P(x, y)}$

$\boxed{x_1}$ $\boxed{y_1}$ $\Rightarrow$ For an x and an y, $p(x, y)$ is true

Nested Quantifiers

- THINKING OF QUANTIFICATION AS LOOPS:
  - $\forall x \forall y P(x, y)$
  - $\forall x \exists y P(x, y)$
  - $\exists x \forall y P(x, y)$
  - $\exists x \exists y P(x, y)$

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Nested Quantifiers

- **EXAMPLE 4:**
  Let $Q(x, y)$ denote "$x + y = 0.$" What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

- **Solution:** $x + y = 0$

$\exists y \forall x Q(x, y)$	There is a real number y such that for every real number x, $Q(x, y)$.	False
$\forall x \exists y Q(x, y)$	For every real number x there is a real number y such that $Q(x, y)$.	True

Nested Quantifiers

- **EXAMPLE 6:**
  Translate the statement
  "$The\ sum\ of\ two\ positive\ integers\ is\ always\ positive$" into a logical expression.

- **Solution:** $\forall \boldsymbol{x} \forall \boldsymbol{y}\big((\boldsymbol{x} > \boldsymbol{0}) \wedge (\boldsymbol{y} > \boldsymbol{0}) \rightarrow (\boldsymbol{x} + \boldsymbol{y} > \boldsymbol{0})\big)$,

  or

  $\forall x \forall y (x + y > 0)$, where the domain for both variables consists of all positive integers

Nested Quantifiers

- **<u>EXAMPLE 9:</u>** Translate the statement
$$\forall x\left(C(x) \vee \exists y\big(C(y) \wedge F(x,y)\big)\right)$$
into English, where $C(x)$ is "$x\ has\ a\ computer$," $F(x,y)$ is "$x\ and\ y\ are\ friends$," and the domain for both x and y consists of all students in your school.

- ***<u>Solution:</u>*** The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, <u>every student in your school has a computer or has a friend who has a computer</u>.

Nested Quantifiers

- **EXAMPLE 14:**
  Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

- **Solution:** $\exists x \forall y (xy \neq 1)$.

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a) $\forall x \exists y (x < y)$
   b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$
   c) $\forall x \forall y \exists z (xy = z)$

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a) $\forall x \exists y (x < y)$
   b) $\forall x \forall y (((x \geq 0) \land (y \geq 0)) \rightarrow (xy \geq 0))$
   c) $\forall x \forall y \exists z (xy = z)$

a) For every real number x there exists a real number y such that x is less than y.
b) For every real number x and real number y, if x and y are both nonnegative, then their product is nonnegative.
c) For every real number x and real number y, there exists a real number z such that the product of x and y is equal to z

Exercises

15. Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.

c) Every student in this class has taken at least one computer science course.

Exercises

15. Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.

c) Every student in this class has taken at least one computer science course.

Let: $S(x)$: "x is a computer science student."
$C(y)$: "y is a course."
$D(y)$: "y is a course in discrete mathematics."
$T(x, y)$: "x has taken course y."

$$\forall x(S(x) \rightarrow \exists y(C(y) \wedge D(y) \wedge T(x, y)))$$

Exercises

15. Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.

c) Every student in this class has taken at least one computer science course.

Let: $S(x)$: $A(x)$: "x is a student in this class."
$C(y)$: "y is a course."
$C_{CS}(y)$: "y is a computer science course."
$T(x, y)$: "x has taken course y."

$$\forall x(A(x) \rightarrow \exists y(C(y) \land C_{CS}(y) \land T(x, y)))$$

Exercises

19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.

b) The difference of two positive integers is not necessarily positive.

d) The absolute value of the product of two integers is the product of their absolute values.

Exercises

19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.

b) The difference of two positive integers is not necessarily positive.

d) The absolute value of the product of two integers is the product of their absolute values.

b) $\neg\forall x\forall y\,((x > 0) \land (y > 0) \rightarrow (x - y > 0))$
d) $\forall x\forall y\,(|xy| = |x||y|)$

Exercises

24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x \forall y (x + y = y)$

Exercises

24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x \forall y(x + y = y)$

a) There is a real-number x, that is added to any real number y, does not change the value of y. (Addition identity – 0)

Exercises

26. Let $Q(x, y)$ be the statement “$x + y = x - y$.” If the domain for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$ d) $\exists x Q(x, 2)$ e) $\exists x \exists y Q(x, y)$
i) $\forall x \forall y Q(x, y)$ h) $\forall y \exists x Q(x, y)$

Exercises

26. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$ d) $\exists x Q(x, 2)$ e) $\exists x \exists y Q(x, y)$
i) $\forall x \forall y Q(x, y)$ h) $\forall y \exists x Q(x, y)$

a) This is false, since $1 + 1 \neq 1 - 1$.
d) This is false, since the equation $x + 2 = x - 2$ has no solution.
e) This is true, since we can take $x = y = 0$.
i) False
h) False

Exercises

28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\forall x \exists y (x^2 = y)$

Exercises

28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\forall x \exists y(x^2 = y)$

a) This is true, since for a given real x, $y = x^2$.
For example, if $x = 5$, then $y = 5^2 = 25$

Exercises

29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is $1, 2, or\ 3$ and y is $1, 2, or\ 3$.
Write out these propositions using disjunctions and conjunctions.

b) $\exists x \exists y P(x, y)$

d) $\forall y \exists x P(x, y)$

Exercises

29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is $1, 2, or\ 3$ and y is $1, 2, or\ 3$.
Write out these propositions using disjunctions and conjunctions.
b) $\exists x \exists y P(x, y)$ d) $\forall y \exists x P(x, y)$

b) $P(1,1) \vee P(1,2) \vee P(1,3) \vee P(2,1) \vee P(2,2) \vee P(2,3) \vee P(3,1) \vee P(3,2) \vee P(3,3)$

d) $(P(1,1) \vee P(2,1) \vee P(3,1)) \wedge (P(1,2) \vee P(2,2) \vee P(3,2)) \wedge (P(1,3) \vee P(2,3) \vee P(3,3))$

Exercises

46. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of

a) the positive real numbers.

b) the integers.

c) the nonzero real numbers.

Exercises

46. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of

a) the positive real numbers.

b) the integers.

c) the nonzero real numbers.

a) False, try $x = 1$ and $y = 0.5$
b) True, try $x = -1$ and $y = any\ integer$
c) True, try $x = -1$ and $y = any\ real\ number$

TASK

SECTION 1.5
2 (a)
19 (a, c)
26 (b, c, f, g)
28 (b)
29 (a, c)

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Rules of Inference

- **<u>Definition:</u>**
  Argument: is a sequence of propositions.
  - All but the final proposition in the argument are called *premises*
  - The final proposition is called the *conclusion*.
  - An argument is valid if the truth of all its premises implies that the conclusion is true.

$$\overbrace{\text{“}\underbrace{If\ you\ have\ a\ current\ password}_{\text{Premises}},\ \underbrace{then\ you\ can\ \log\ onto\ the\ network}_{\text{Conclusion}}.\text{”}}^{\text{Argument}}$$

Rules of Inference

- To determine if the argument is valid, we use rules of inference

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rules of Inference

"If it is raining, then I will drink tea" $p \to q$

"it is raining" p

"Therefore, I will drink tea" $\therefore q$

Modus ponens:

$$(\boldsymbol{p} \land (\boldsymbol{p} \to \boldsymbol{q})) \to \boldsymbol{q}$$

Rules of Inference

"If it is raining, then I will drink tea" $p \to q$

"I don't drink tea" $\neg q$

"Therefore, it is not raining" $\therefore \neg p$

Modus tollens:

$$(\neg \boldsymbol{q} \wedge (\boldsymbol{p} \to \boldsymbol{q})) \to \neg \boldsymbol{p}$$

Rules of Inference

"If it is raining, then I will drink tea" $p \rightarrow q$

"if I drink tea, then I will read a book" $q \rightarrow r$

"Therefore, if it rains, then I will read a book" $\therefore p \rightarrow r$

Hypothetical syllogism:
$\boldsymbol{((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)}$

Rules of Inference

"I will drink tea, or I will read a book" $p \vee q$

"I will not drink tea" $\neg p$

"Therefore, I will read a book" $\therefore q$

Disjunctive syllogism:

$((\boldsymbol{p} \vee \boldsymbol{q}) \wedge \neg \boldsymbol{p}) \rightarrow \boldsymbol{q}$

Rules of Inference

"I will drink tea" p

"Therefore, I will drink tea or I will read a book" $\therefore p \vee q$

Addition:
$\boldsymbol{p \rightarrow (p \vee q)}$

Rules of Inference

“I will drink tea and I will read a book” $p \wedge q$

“Therefore, I will drink tea” $\therefore p$

Simplification:

$(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \boldsymbol{p}$

Rules of Inference

“It is raining, or I will drink tea” $p \lor q$

“It is not raining, or I will read a book” $\neg p \lor r$

“Therefore, I will drink tea, or I will read a book” $\therefore q \lor r$

Resolution:
$((\boldsymbol{p} \lor \boldsymbol{q}) \land (\neg \boldsymbol{p} \lor \boldsymbol{r})) \rightarrow (\boldsymbol{q} \lor \boldsymbol{r})$

Rules of Inference

"I will drink tea" p

"I will read a book" q

"Therefore, I will drink tea and read a book" $\therefore p \wedge q$

Conjunction:

$$((\boldsymbol{p}) \wedge (\boldsymbol{q})) \rightarrow (\boldsymbol{p} \wedge \boldsymbol{q})$$

Rules of Inference

- Rules of Inference for Quantified Statements.

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Rules of Inference

- Universal modus ponens:

$$\begin{array}{l} \quad \forall x(P(x) \to Q(x)) \\ \quad P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore Q(a) \end{array}$$

- Universal modus tollens:

$$\begin{array}{l} \quad \forall x(P(x) \to Q(x)) \\ \quad \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore \neg P(a) \end{array}$$

Rules of Inference

Fallacies are rules of inference based on contingencies rather than tautologies.

Fallacies

- Affirming the conclusion.
- Denying the hypothesis.

Affirming the conclusion.

The proposition $((p \rightarrow q) \land q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true.

"If I have an iPhone, then I have a good camera"
"I have a good camera"
$\therefore$ "I have an iPhone"

Denying the hypothesis.

The proposition $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true.

"If I have an iPhone, then I have a good camera"
"I do not have an iPhone"
$\therefore$ "I do not have a good camera"

Exercises

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

$$\begin{array}{l} If\ George\ does\ not\ have\ eight\ legs, then\ he\ is\ not\ a\ spider. \\ George\ is\ a\ spider. \\ \hline \therefore\ George\ has\ eight\ legs. \end{array}$$

Exercises

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

$$\overbrace{\textit{If George does not have eight legs}}^{p}, \overbrace{\textit{then he is not a spider}}^{q}.$$

$$\textit{George is a spider.}\ \} \neg q$$

$$\therefore\ \underbrace{\textit{George has eight legs}}_{\neg p}.$$

This is modus tollens. We conclude that the conclusion of the argument (third statement) is true, given that the hypotheses (the first two statements) are true.

Exercises

3. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

Exercises

3. What rule of inference is used in each of these arguments?

$\overbrace{\text{Alice is a mathematics major.}}^{p}$ Therefore, Alice is either $\overbrace{\text{a mathematics major}}^{p}$ or $\underbrace{\text{a computer science major.}}_{q}$

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

Addition rule

Exercises

3. What rule of inference is used in each of these arguments?

$\overbrace{\text{If it is rainy,}}^{p}$ $\overbrace{\text{then the pool will be closed.}}^{q}$ $\overbrace{\text{It is rainy.}}^{p}$ $\overbrace{\text{Therefore, the pool is closed.}}^{q}$

$$
\begin{array}{l}
p \to q \\
p \\
\hline
\therefore q
\end{array}
$$

Modus ponens

Exercises

3. What rule of inference is used in each of these arguments?

$$\overbrace{\text{If I go swimming,}}^{p}\ \overbrace{\text{then I will stay in the sun too long.}}^{q}\ \overbrace{\text{If I stay in the sun too long,}}^{q}\ \underbrace{\text{then I will sunburn.}}_{r}\ \text{Therefore,}\ \underbrace{\text{if I go swimming,}}_{p}\ \underbrace{\text{then I will sunburn.}}_{r}$$

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hypothetical syllogism

Exercises

6. Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”

Exercises

“If it does not rain <u>or</u> if it is not foggy, <u>then</u> the sailing race will be held <u>and</u> the lifesaving demonstration will go on,”

“If the sailing race is held, <u>then</u> the trophy will be awarded,” <u>and</u>

“The trophy was <u>not</u> awarded”

imply the conclusion “It rained.”

Exercises

“$\overbrace{\text{If it does not rain}}^{p}$ <u>or</u> $\overbrace{\text{if it is not foggy}}^{q}$, then $\overbrace{\text{the sailing race will be held}}^{r}$ <u>and</u> $\underbrace{\text{the lifesaving demonstration will go on}}_{s}$,”

“$\overbrace{\text{If the sailing race is held}}^{r}$, then $\overbrace{\text{the trophy will be awarded}}^{f}$,” and

“$\overbrace{\text{The trophy was not awarded}}^{\neg f}$”

imply the conclusion “$\underbrace{\text{It rained}}_{\neg p}$.”

$$
\begin{array}{l}
(p \vee q) \rightarrow (r \wedge s) \\
r \rightarrow f \\
\neg f \\
\hline
\therefore \neg p
\end{array}
$$

Exercises

$(p \vee q) \rightarrow (r \wedge s)$

$r \rightarrow f$

$\neg f$

$\therefore \neg p$

$r \rightarrow f$

$\neg f$

$\therefore \neg r$

Modus tollens

Exercises

$$(p \vee q) \rightarrow (r \wedge s)$$
$$r \rightarrow f$$
$$\neg f$$

$$\therefore \neg p$$

$$r \rightarrow f$$
$$\neg f$$

$$\therefore \neg r$$

Modus tollens

Exercises

$$(p \vee q) \rightarrow (r \wedge s)$$

$$\neg r$$

$$\therefore \neg(p \vee q) \equiv \neg \mathrm{p} \wedge \neg q$$

Modus tollens

Exercises

$$(p \lor q) \rightarrow (r \land s)$$
$$\neg r$$

$$\therefore \neg(p \lor q) \equiv \neg p \land \neg q$$

Modus tollens

$$(p \lor q) \rightarrow (r \land s)$$
$$r \rightarrow f$$
$$\neg f$$

$$\therefore \neg p$$

p: "it does not rain "

$\neg p$: "it rained"

Exercises

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.

c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

Exercises

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.

$\overbrace{\underline{\text{All}}\text{ students in this class}}^{x}\ \overbrace{\text{understand logic}}^{P}$. $\overbrace{\text{Xavier}}^{c}$ $\underline{\text{is a student in this class}}$.
Therefore, $\underbrace{\text{Xavier understands logic}}_{P(c)}$.

$$\begin{array}{c} \forall x P(x) \\ \hline \therefore P(c) \end{array}$$

Universal instantiation

Exercises

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.

$\overbrace{\text{\underline{All} parrots}}^{\forall x P(x)}\ \overbrace{\text{like fruit}}^{L(x)}$. $\overbrace{\text{My pet bird}}^{c}\ \overbrace{\text{\underline{is not a parrot}}}^{\neg P(c)}$. Therefore, $\overbrace{\text{my pet bird does not like fruit}}^{\neg L(c)}$.

$$
\begin{array}{l}
\forall x\big(P(x) \rightarrow L(x)\big) \\
\neg P(c) \\
\hline
\therefore \neg \mathrm{L}(c)
\end{array}
$$

Invalid argument, fallacy of denying the hypothesis $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$

Exercises

19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

Exercises

19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

$$\forall n((n > 1) \rightarrow (n^2 > 1))$$
$$n^2 > 1$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$\therefore n > 1$$

Invalid argument, fallacy of affirming the conclusion. $((p \rightarrow q) \wedge q) \rightarrow p$

Try n be a negative number

TASK

SECTION 1.6
1
3 (b, d)
15 (b, d)
19 (b)

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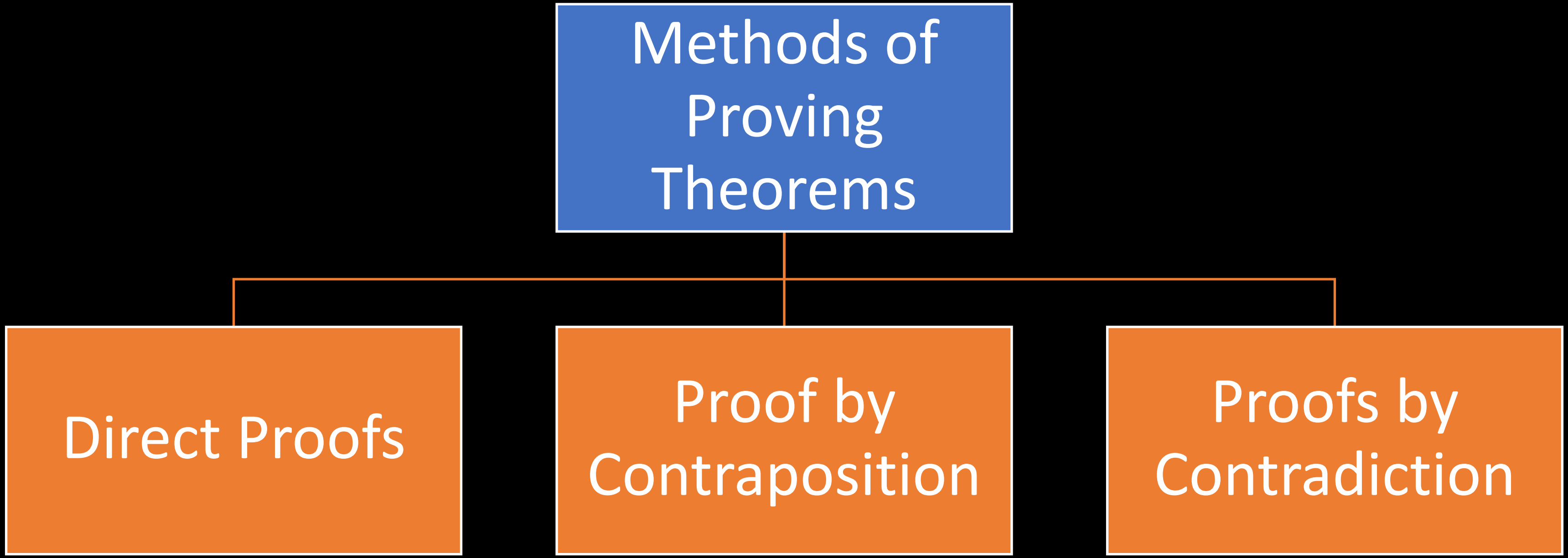
Introduction to Proofs

- A **theorem** is a statement that can be shown to be true.
- A **proof** is a valid argument that establishes the truth of a theorem.



Introduction to Proofs

- A **direct proof** of a conditional statement $p \rightarrow q$ is constructed by:
  1. Assume that p is true.
  2. Use p to show that q must be true.
- **EXAMPLE 1:** Give a direct proof of the theorem
  "If n is an odd integer, then n^2 is odd."

1. $\because n\ is\ odd$
2. $\therefore n = 2k + 1$, for some integer k
3. $\therefore n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
4. Suppose $2k^2 + 2k = x$
5. $\therefore n^2 = 2x + 1$, which has the same form of $n = 2k + 1$, which is an odd number.

Introduction to Proofs

- **Proofs by contraposition** (indirect proof) make use of the fact that the conditional statement $p \ \rightarrow \ q$ is equivalent to its contrapositive, $\neg q \ \rightarrow \ \neg p$.
  - The conditional statement $p \ \rightarrow \ q$ is proved by showing that its contrapositive, $\neg q \ \rightarrow \ \neg p$, is true.
- <u>**EXAMPLE 3:**</u> Prove that if n is an integer and $3n \ + \ 2$ is odd, then n is odd.

1. Suppose $p = "3n + 2 \ is \ odd"$, and q $= "n \ is \ odd“$
2. $\therefore$ we have p → q
3. To prove by contraposition, we need to prove $\neg q \rightarrow \neg p$ is true by direct proof
4. $\because \neg q$ means “n is even”
5. $\therefore n = 2k$
6. $\therefore q = 3 \ * (2k) + 2 = 6k + 2 = 2(k + 1)$
7. $\because$ any even number has the form of $2x$. Suppose that $k + 1 = x$.
8. $\therefore q \ = 2x$, which has the form of even, and therefore it is not odd.

Introduction to Proofs

- **Proof by contradiction** assumes the theorem is false, and then show that the assumption itself is false, and is therefore a contradiction.
  - Assume that p is true and q is false, then prove that $(p \ \wedge \ \neg q) \ \rightarrow \ \boldsymbol{F}$.

- **EXAMPLE 11:** Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Introduction to Proofs

1. Let $p =$ "$\sqrt{2}\ is\ irrational$". To prove by contradiction, we suppose $\neg p$ is true.
2. $\neg p$ means that $\sqrt{2}$ is rational
3. $\because \sqrt{2}$ is rational
4. $\therefore \sqrt{2} = \frac{a}{b}$, which is the same as $2 = a^2/b^2$
5. $\therefore a^2 = 2b^2$, this means that a^2 is even number because it has the form a $= \ 2k$
6. $\therefore (2k)^2 = 2b^2 \rightarrow 4k^2 = 2b^2 \rightarrow 2k^2 = b^2$
7. $\because b^2 = 2k^2$, then b is even.
8. $\because b$ and a are both even numbers
9. $\therefore a$ and b have a common factor of 2
10. $\therefore$ if we have $\frac{a}{2} \div \frac{b}{2} = \frac{c}{d}$, where c and d have no common factors
11. $\therefore \sqrt{2} = \frac{c}{d}$, which is false. (we assumed that $\sqrt{2} = \frac{a}{b}$)

EXAMPLE 16 What is wrong with this famous supposed "proof" that $1 = 2$?

"Proof": We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

We are proofing that a =b,
So, if we divide both sides by (a-b), which is 0, it will be an invalid division.

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Exercises

3. Show that the square of an even number is an even number using a direct proof.

Exercises

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We want to prove that: if n is even, then n^2 is also even

1. $\because n$ is even
2. $\therefore n = 2k$ for some integer k
3. $\therefore n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
4. Suppose $2k^2$ is x
5. $\therefore n^2 = 2x$, which has the form of an even number
6. $\therefore n^2$ is even

Exercises

6. Use a direct proof to show that the product of two odd numbers is odd.

Exercises

6. Use a direct proof to show that the product of two odd numbers is odd.

We want to prove that: if x is odd and y is odd, then xy is odd

1. $\because x$ is odd and y is odd
2. $\therefore x = 2k + 1$, and $y = 2n + 1$ for some integers k and n
3. $\therefore x * y = (2k + 1) * (2n + 1) = 4kn + 2k + 2n + 1 = 2(kn + k + n) + 1$
4. Suppose that $(kn + k + n)$ is some number b
5. $\therefore x * y = 2b + 1$, which has the form of an odd number
6. $\therefore x * y$ is odd

Exercises

7. Use a direct proof to show that every odd integer is the difference of two squares. [Hint: Find the difference of the squares of k + 1 and k where k is a positive integer.]

Exercises

7. Use a direct proof to show that every odd integer is the difference of two squares. [Hint: Find the difference of the squares of k + 1 and k where k is a positive integer.]

We want to prove that: if n is an odd number, then $n = (k+1)^2 - k^2$ for any positive integer k

1. Given that $n = (k+1)^2 - k^2$
2. $\therefore (k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1$
3. $\because n = 2k + 1$ has the form of an odd number
4. $\therefore n$ is odd

Exercises

17. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Exercises

17. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

To apply proof by contraposition, we need to prove $\neg q \rightarrow \neg p$ is true. So, we prove that $(x < 1)$ and $(y < 1) \rightarrow x + y < 2$

1. $\because x < 1$ and $y < 1$, then both numbers range from $-\infty$ to (not including) 1
2. $\therefore x + y$ will never reach 2 (try any number in the range)
3. $\therefore x + y < 2$

Exercises

18. Prove that if m and n are integers and mn is even, then m is even or n is even.

Exercises

18. Prove that if m and n are integers and mn is even, then m is even or n is even.

We will use contraposition proof. Prove that if m and n are both odd, then mn is odd

1. $\because m$ is odd, n is odd
2. $\therefore m = 2x + 1, n = 2y + 1$, where x and y are integers
3. $\therefore m * n = (2x + 1) * (2y + 1) = 4xy + 2x + 2y + 1 = 2(xy + x + y) + 1$
4. Suppose $xy + x + y$ is some integer k
5. $\therefore m * n = 2k + 1$, which is an odd number
6. $\because m * n$ is odd in case that m and n are odd
7. $\therefore$ if $m * n$ is even, then m or n (or both) is even

Exercises

28. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

Exercises

28. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

For "if and only if" statements, we need to prove 2 things:

- Prove that if n is even, then $7n + 4$ is even
- Prove that if $7n + 4$ is even, then n is even

So, for $p \leftrightarrow q$, we prove both:

- $p \rightarrow q$
- $q \rightarrow p$

Exercises

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For "if and only if" statements, we need to prove 2 things:

- Prove that if n is even, then $7n + 4$ is even
- Prove that if $7n + 4$ is even, then n is even

Use direct proof

1. $\because n$ is even
2. $\therefore n = 2k$, for some integer k
3. $\therefore 7n + 4 = 7(2k) + 4 = 14k + 4 = 2(7k + 2)$, has the form of even number
4. $\therefore 7n + 4$ is even ➔(1)

Exercises

28. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

For "if and only if" statements, we need to prove 2 things:

- Prove that if n is even, then $7n + 4$ is even
- Prove that if $7n + 4$ is even, then n is even

Use contraposition, prove that if n is odd, then $7n + 4$ is odd

1. $\because n$ is odd
2. $\therefore n = 2k + 1$ for some integer k
3. $\therefore 7n + 4 = 7 * (2k + 1) + 4 = 14k + 7 + 4 = 14k + 10 + 1 = 2(7k + 5) + 1$
4. Suppose that $7k + 5 = x$
5. $\therefore 7n + 4 = 2x + 1$, which has the form of odd number
6. $\therefore 7n + 4$ is odd in case that n is odd
7. $\therefore if\ 7n + 4$ is even, then n is even ➔ (2)

Exercises

28. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

For “if and only if” statements, we need to prove 2 things:

- Prove that if n is even, then $7n + 4$ is even
- Prove that if $7n + 4$ is even, then n is even

From (1) and (2) we conclude that n is even if and only if $7n + 4$ is even.

Exercises

40. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

Exercises

40. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

The statement says $\forall x(x = a^2 + b^2 + c^2)$ for any positive integer x and any integers a, b, c
We need to find an example that contradicts that rule.
Suppose $x = 7$, can we write 7 to be the sum of any three squared integers?
Suppose $a = 0, b = 1, c = 2$

$$a^2 + b^2 + c^2 = 0 + 1 + 4 = 5 \neq 7$$

Suppose any 3 numbers, square them, and sum them.
You will not get any combination that will result in 7.

Thus, our counter example is 7 cannot be written as the sum of three squares.

TASK

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Content

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