Limits and Derivatives

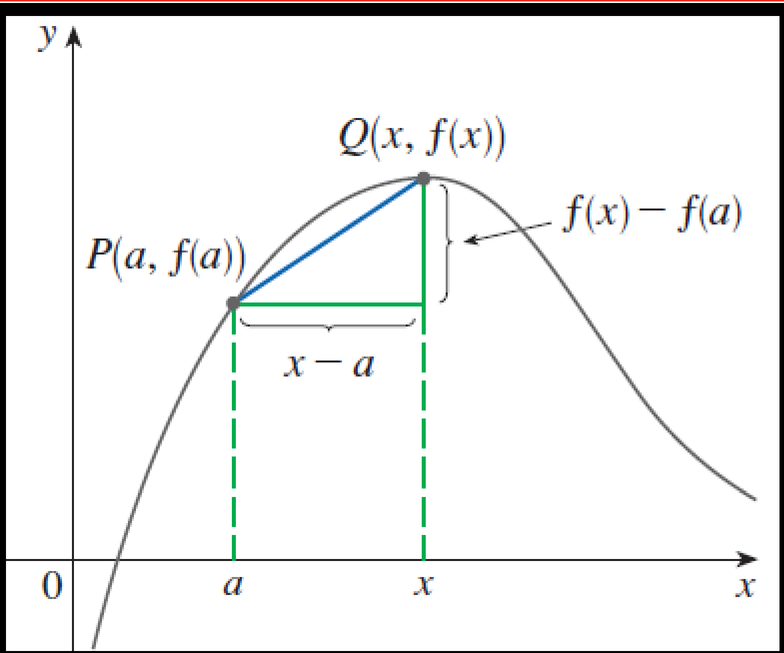
Content
Derivatives and Rates of Change
The Derivative as a Function

Derivatives and Rates of Change

$\boxed{1}$ **Definitio** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Derivatives and Rates of Change

- Example 1: Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.
- Solution: Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

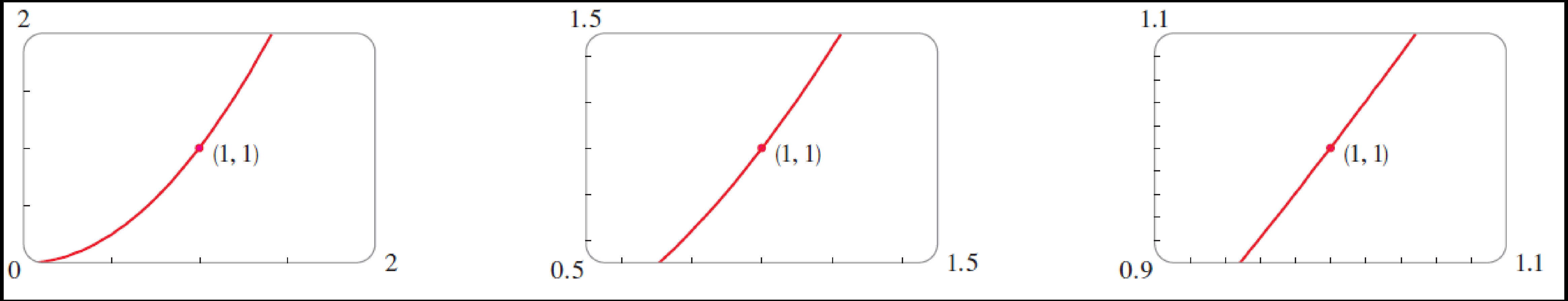
$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$$

Derivatives and Rates of Change

- We sometimes refer to the slope of the tangent line to a curve at a point as **the slope of the curve at the point**.
  - The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line.



Derivatives and Rates of Change

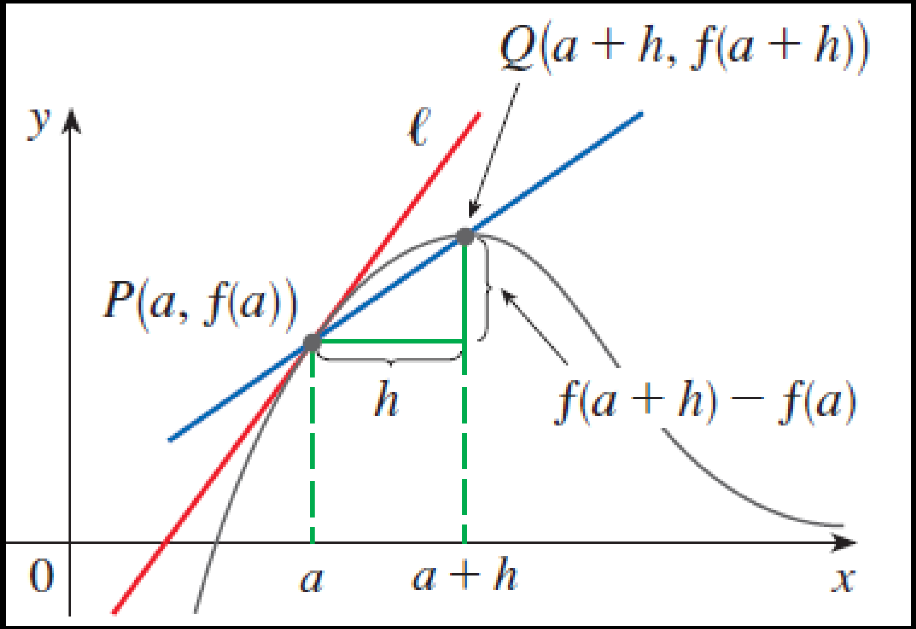
- Another expression for the slope of a tangent line that is easier to use.

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

Supposing $x = a + h$

As x approaches a, h approaches 0 and so the expression for the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Derivatives and Rates of Change

- Example 2: Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

SOLUTION Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $$x + 3y - 6 = 0$$

Derivatives and Rates of Change

- Limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering.

- Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives and Rates of Change

- This special limit is called the **derivative**.

4 Definitio The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

- If we write $x = a + h$, then we have $h = x - a$. Therefore, an equivalent way of stating the definition of the derivative is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivatives and Rates of Change

- Example 4: find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

Derivatives and Rates of Change

- (a) at the number 2

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4$$

Derivatives and Rates of Change

- (b) at the number a

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} (2a + h - 8) = 2a - 8
\end{aligned}
$$

notice that if we let $a = 2$, then $f'(2) = 2 * 2 - 8 = -4$.

Derivatives and Rates of Change

- Example 5: find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a $(a > 0)$.

Derivatives and Rates of Change

- Example 5: find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a $(a > 0)$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} \cdot \frac{\sqrt{x}\sqrt{a}}{\sqrt{x}\sqrt{a}}$$

$$= \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} = \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} \cdot \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}}$$

$$= \lim_{x \to a} \frac{-(x - a)}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})}$$

$$= \frac{-1}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{-1}{a \cdot 2\sqrt{a}} = -\frac{1}{2a^{3/2}}$$

Derivatives and Rates of Change

The relationship of “tangent line” to “derivative”

- The tangent line to $y = f(x)$ at $\big(a, f(a)\big)$ is the line whose slope is equal to the derivative of the function, which is $f'(a)$.
- we can write an equation of the tangent line to the curve $y = f(x)$ at the point $\big(a, f(a)\big)$:

$$y - y_1 = f'(a)(x - x_1)$$

Exercises

1. A curve has equation $y = f(x)$.

(a) Write an expression for the slope of the secant line through the points $P\big(3, f(3)\big)$ and $Q\big(x, f(x)\big)$.

(b) Write an expression for the slope of the tangent line at P.

Exercises

(a) This is just the slope of the line through two points:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}$$

(b) This is the limit of the slope of the secant line PQ as Q approaches P

$$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

Exercises

Find an equation of the tangent line to the curve at the given point

5) $y = 2x^2 - 5x + 1, \qquad (3, 4)$

6) $y = x^2 - 2x^3, \qquad (1, -1)$

Exercises

5) $y = 2x^2 - 5x + 1, \quad (3, 4)$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 3} \frac{(2x^2 - 5x + 1) - 4}{x - 3} = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x+1)(x-3)}{x - 3}$$

$$= \lim_{x \to 3} (2x + 1) = 2(3) + 1 = 7$$

Tangent line: $y - 4 = 7(x - 3) \quad \Leftrightarrow \quad y - 4 = 7x - 21 \quad \Leftrightarrow \quad y = 7x - 17$

6) $y = x^2 - 2x^3, \quad (1, -1)$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h)^2 - 2(1+h)^3\right] - (-1)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 2 - 6h - 6h^2 - 2h^3 + 1}{h} = \lim_{h \to 0} \frac{-2h^3 - 5h^2 - 4h}{h} = \lim_{h \to 0} \frac{-h(2h^2 + 5h + 4)}{h}$$

$$= \lim_{h \to 0} \left[-(2h^2 + 5h + 4)\right] = -4$$

Tangent line: $y - (-1) = -4(x - 1) \quad \Leftrightarrow \quad y + 1 = -4x + 4 \quad \Leftrightarrow \quad y = -4x + 3$

Exercises

Find $f'(a)$ at the given number a.

19) $f(x) = \sqrt{4x + 1}, \quad a = 6$

21) $f(x) = \frac{x^2}{x+6}, \qquad a = 3$

Exercises

19) $f(x) = \sqrt{4x+1}, \quad a = 6$

$$f'(6) = \lim_{h \to 0} \frac{f(6+h) - f(6)}{h} = \lim_{h \to 0} \frac{\sqrt{4(6+h)+1} - 5}{h} = \lim_{h \to 0} \frac{\sqrt{25+4h} - 5}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{25+4h} - 5\right)\left(\sqrt{25+4h} + 5\right)}{h\left(\sqrt{25+4h} + 5\right)} = \lim_{h \to 0} \frac{(25+4h) - 25}{h\left(\sqrt{25+4h} + 5\right)} = \lim_{h \to 0} \frac{4h}{h\left(\sqrt{25+4h} + 5\right)}$$

$$= \lim_{h \to 0} \frac{4}{\sqrt{25+4h} + 5} = \frac{4}{5+5} = \frac{2}{5}$$

21) $f(x) = \frac{x^2}{x+6}, \qquad a = 3$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x \to 3} \frac{\frac{x^2}{x+6} - 1}{x-3} = \lim_{x \to 3} \frac{\frac{x^2 - (x+6)}{x+6}}{x-3} = \lim_{x \to 3} \frac{x^2 - x - 6}{(x+6)(x-3)}$$

$$= \lim_{x \to 3} \frac{(x+2)(x-3)}{(x+6)(x-3)} = \lim_{x \to 3} \frac{x+2}{x+6} = \frac{3+2}{3+6} = \frac{5}{9}$$

Exercises

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

Exercises

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h}$$

$$= \lim_{h \to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{3h - h^3}{h} = \lim_{h \to 0} \frac{h(3 - h^2)}{h}$$

$$= \lim_{h \to 0} (3 - h^2) = 3 - 0 = 3$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

Exercises

30. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $g(x) = x^4 - 2$ at the point $(1, -1)$.

Exercises

30. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $g(x) = x^4 - 2$ at the point $(1, -1)$.

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to 1} \frac{(x^2+1)(x^2-1)}{x - 1}$$

$$= \lim_{x\to 1} \frac{(x^2+1)(x+1)(x-1)}{x - 1} = \lim_{x\to 1} [(x^2+1)(x+1)] = 2(2) = 4$$

Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

Content
Derivatives and Rates of Change
The Derivative as a Function

The Derivative as a Function

- In the preceding section we considered the derivative of a function f at a fixed number a:

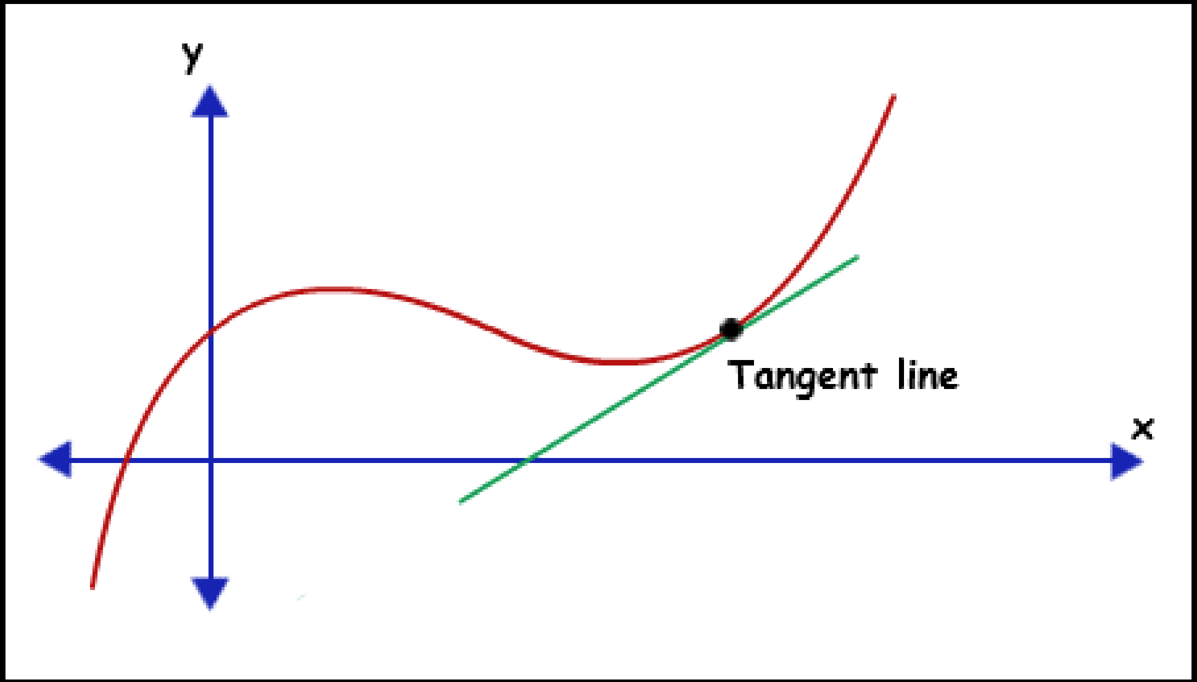
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We assume the number a will vary, so the derivative becomes:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

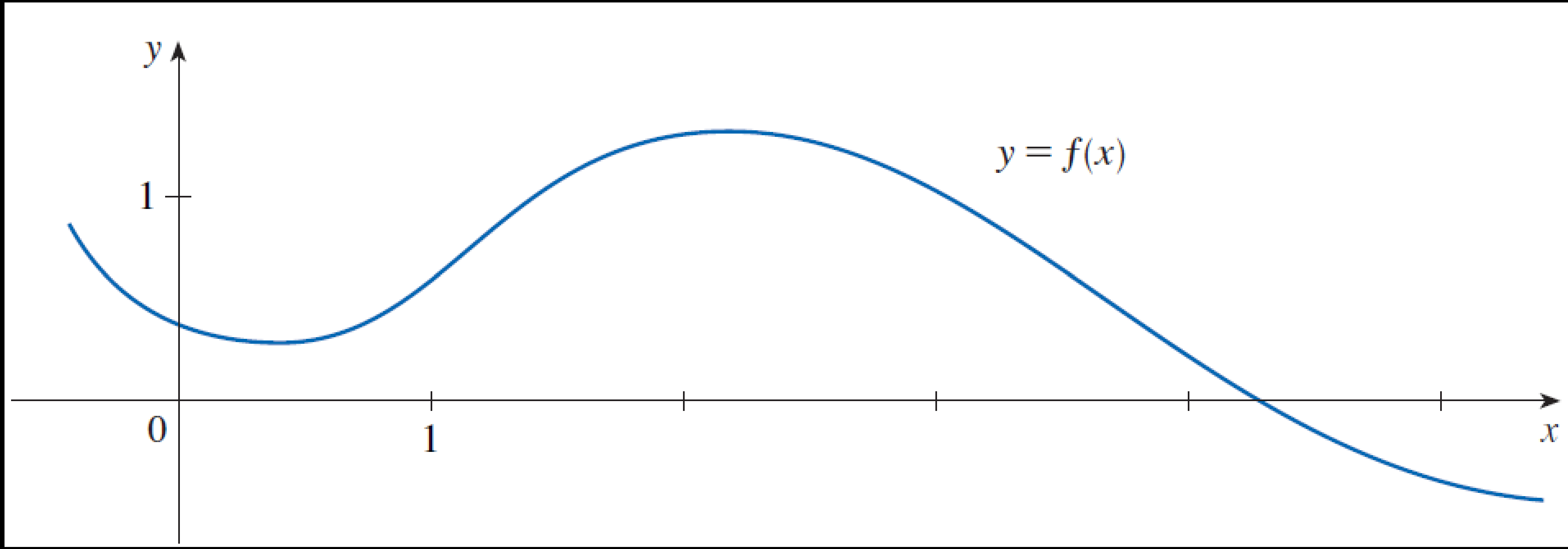
The Derivative as a Function

- **Geometrically**, the derivative $f'(x)$ is interpreted as the slope of the tangent line to the graph of f at the point $\big(x, f(x)\big)$.
  - The function f' is called the <u>derivative</u> of f because it has been <u>"derived"</u> from f by the limiting operation.

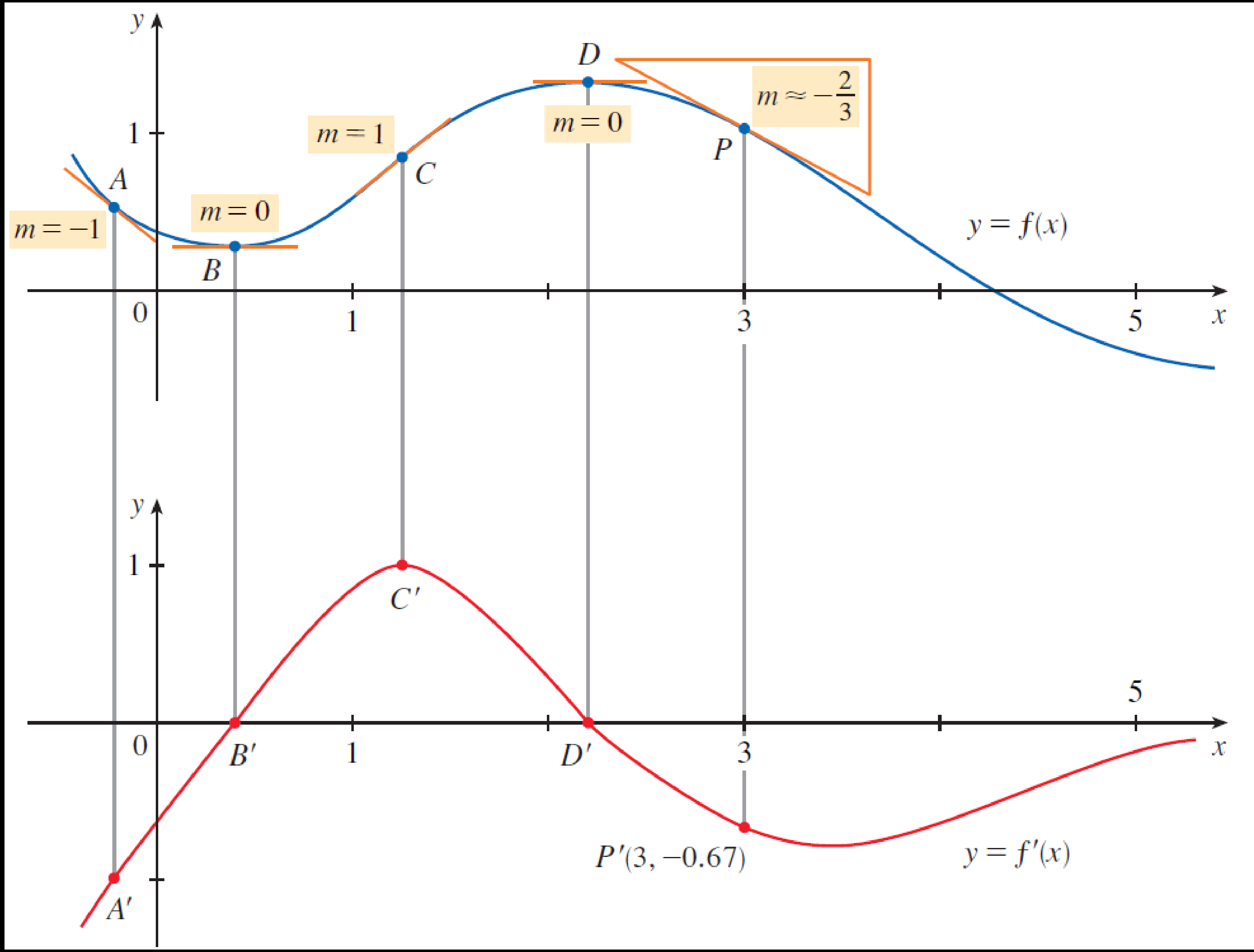


The Derivative as a Function

- Example 1: We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope.



The Derivative as a Function

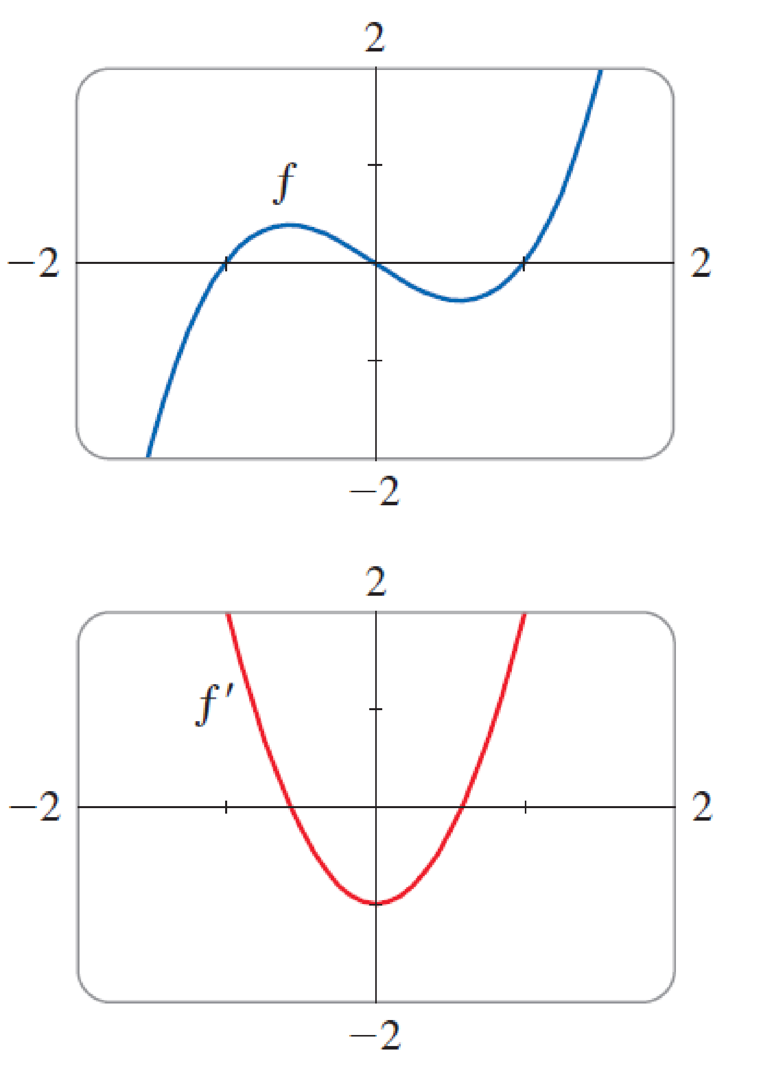


The Derivative as a Function

- Example 2:
  (a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
  (b) Illustrate this formula by comparing the graphs of f and f'.

The Derivative as a Function

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1
\end{aligned}
$$



The Derivative as a Function

- Example 3: If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

The Derivative as a Function

- Example 3: If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \qquad \text{(Rationalize the numerator.)}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$. ■

The Derivative as a Function

- Example 4: Find f' if $f(x) = \frac{1-x}{2+x}$

The Derivative as a Function

- Find f' if $f(x) = \frac{1-x}{2+x}$

$$\frac{\frac{a}{b} - \frac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h} \\
&= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)} \\
&= \lim_{h \to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)} \\
&= \lim_{h \to 0} \frac{-3h}{h(2+x+h)(2+x)} \\
&= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2} \qquad \blacksquare
\end{aligned}$$

The Derivative as a Function

- Other notations for the derivative of $y = f(x)$

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The Derivative as a Function

$\boxed{3}$ **Definitio** A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

$\boxed{4}$ **Theorem** If f is differentiable at a, then f is continuous at a.

- *Notice that a continuous function <u>does not</u> mean that it is differentiable.*

The Derivative as a Function

- Example 5: Where is the function $f(x) = |x|$ differentiable?

- To answer this question, we have to investigate for
    1. $x > 0$
    2. $x < 0$
    3. $x = 0$

The Derivative as a Function

- For $x > 0$:

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

So, f is differentiable for any $x > 0$.

The Derivative as a Function

- For $x < 0$:

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

So, f is differentiable for any $x < 0$.

The Derivative as a Function

- For $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

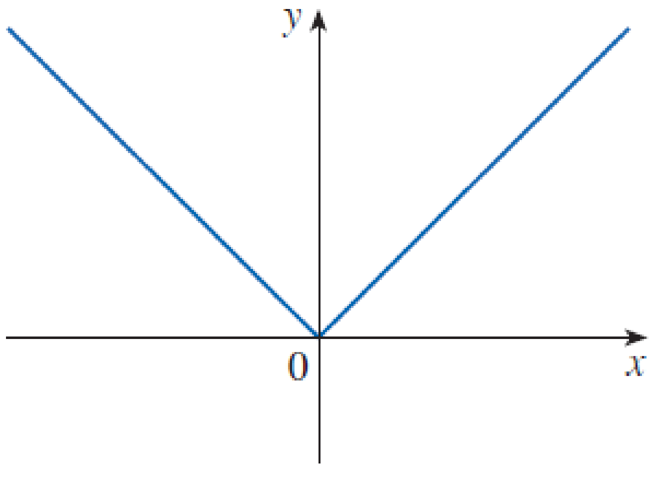
compute the left and right limits separately

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

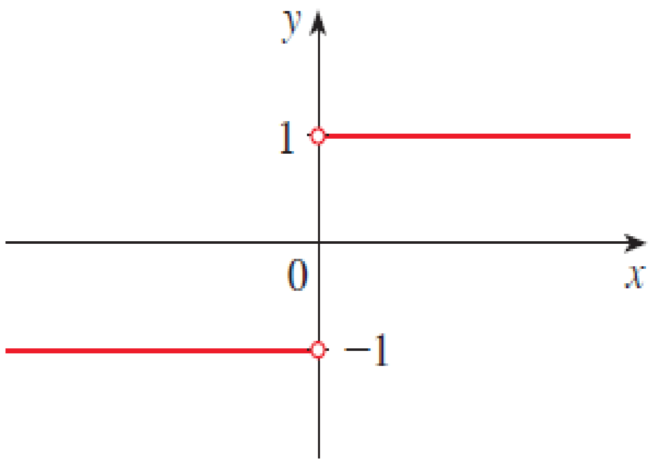
$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus, f is differentiable at all x except 0.

The Derivative as a Function



(a) $y = f(x) = |x|$



(b) $y = f'(x)$

The Derivative as a Function

- How Can a Function Fail To Be Differentiable?

The Derivative as a Function

- How Can a Function Fail To Be Differentiable?
- In general, if the graph of a function f has a “corner” or “kink” in it, then the graph of f has no tangent at this point and f is not differentiable there.
    - **[In trying to compute $f'(x)$, we find that the left and right limits are different.]**

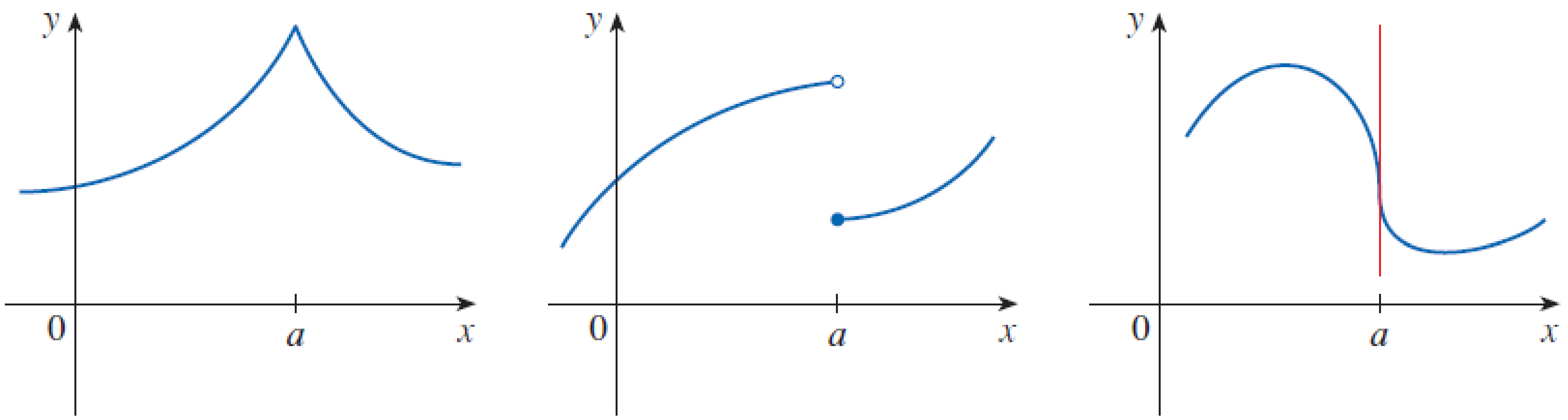


FIGURE 7
Three ways for f not to be differentiable at a

(a) A corner (b) A discontinuity (c) A vertical tangent

The Derivative as a Function

- A **higher derivative** refers to the repeated process of taking derivatives of derivatives.

Second derivative: $\frac{d^2y}{dx^2}, f''(x), f^{(2)}$

Third derivative: $\frac{d^3y}{dx^3}, f'''(x), f^{(3)}$

n^{th} - order derivative: $\frac{d^ny}{dx^n}, f^{(n)}$

The Derivative as a Function

- Example 6: If $f(x) = x^3 - x$, find and interpret $f''(x)$.

- We found the first derivative of $f'(x) = 3x^2 - 1$. So, the second derivative is

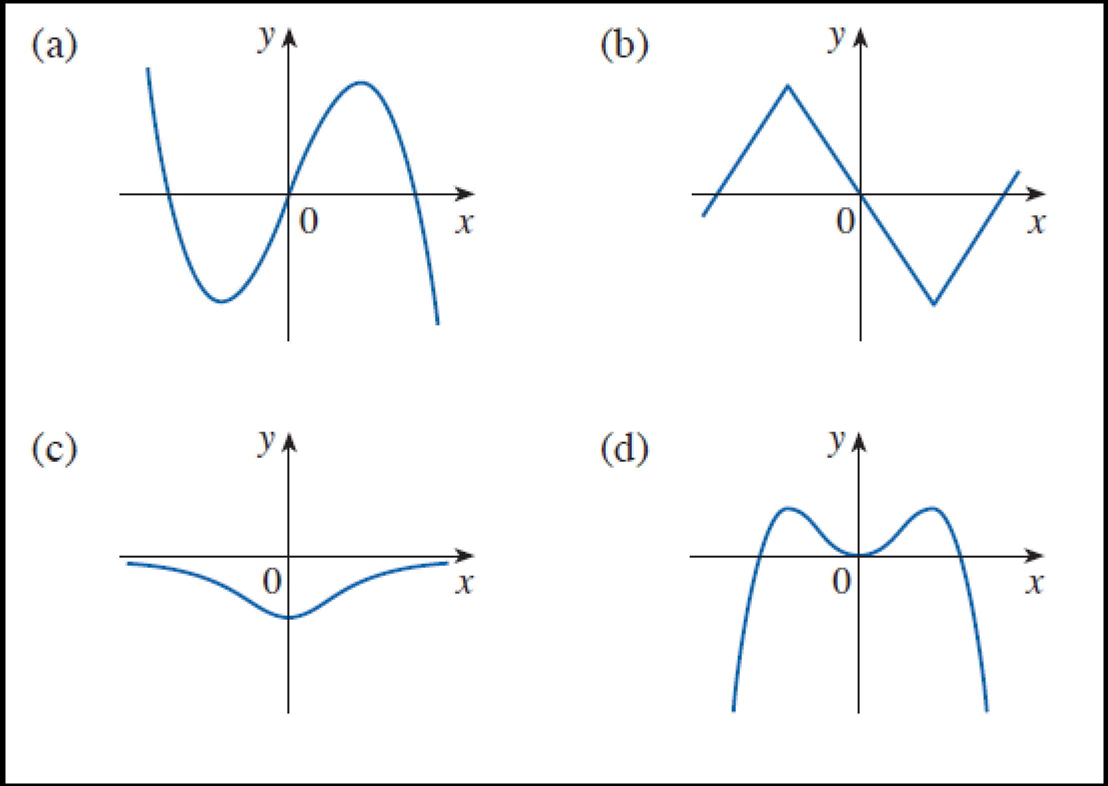
$$
\begin{aligned}
f''(x) = (f')'(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\
&= \lim_{h \to 0} (6x + 3h) = 6x
\end{aligned}
$$

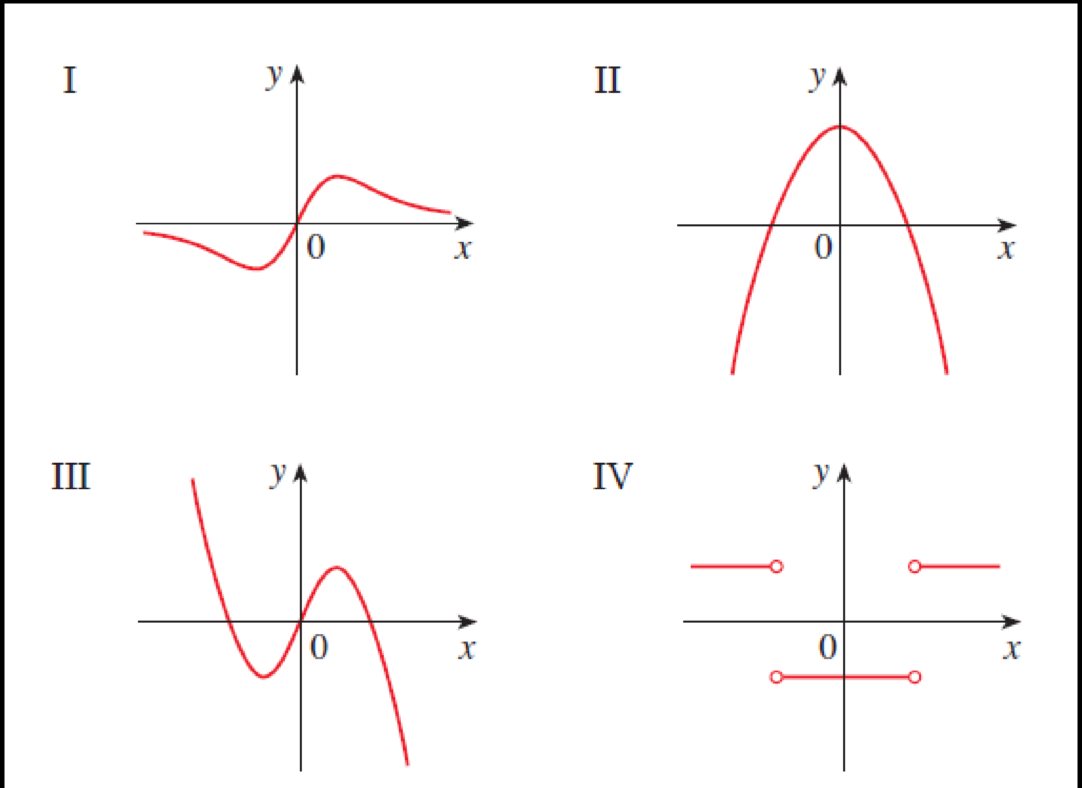
The Derivative as a Function

- In general, we can interpret a second derivative as a rate of change of a rate of change.
- The most familiar example of this is ***acceleration,*** which is the rate of change of velocity.

Exercises

3. Match the graph of each function in (a) – (d) with the graph of its derivative in I–IV. Give reasons for your choices.





Exercises

- (a) = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again.
- (b)= IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.
- (c)= I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.
- (d)= III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Exercises

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21) $f(x) = 3x - 8$

22) $f(x) = mx + b$

27) $f(x) = \frac{1}{x^2-4}$

31) $f(x) = \frac{1}{\sqrt{1+x}}$

Exercises

21) $f(x) = 3x - 8$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h) - 8] - (3x - 8)}{h} = \lim_{h\to 0} \frac{3x + 3h - 8 - 3x + 8}{h}$$

$$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

Domain of f = domain of $f' = \mathbb{R}$.

Exercises

22) $f(x) = mx + b$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[m(x+h)+b] - (mx+b)}{h} = \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

Domain of f = domain of $f' = \mathbb{R}$.

Exercises

27) $f(x)=\frac{1}{x^2-4}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2-4}-\frac{1}{x^2-4}}{h}=\lim_{h\to 0}\frac{\frac{(x^2-4)-\left[(x+h)^2-4\right]}{\left[(x+h)^2-4\right](x^2-4)}}{h}$$

$$=\lim_{h\to 0}\frac{(x^2-4)-\left(x^2+2xh+h^2-4\right)}{h\left[(x+h)^2-4\right](x^2-4)}=\lim_{h\to 0}\frac{x^2-4-x^2-2xh-h^2+4}{h[(x+h)^2-4](x^2-4)}=\lim_{h\to 0}\frac{-2xh-h^2}{h[(x+h)^2-4](x^2-4)}$$

$$=\lim_{h\to 0}\frac{h(-2x-h)}{h[(x+h)^2-4](x^2-4)}=\lim_{h\to 0}\frac{-2x-h}{[(x+h)^2-4](x^2-4)}=\frac{-2x}{(x^2-4)(x^2-4)}=-\frac{2x}{(x^2-4)^2}$$

Domain of f = Domain of $f'=(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

Exercises

31) $f(x)=\frac{1}{\sqrt{1+x}}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{1+(x+h)}}-\dfrac{1}{\sqrt{1+x}}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{1+(x+h)}}-\dfrac{1}{\sqrt{1+x}}}{h}\cdot\frac{\sqrt{1+(x+h)}\,\sqrt{1+x}}{\sqrt{1+(x+h)}\,\sqrt{1+x}}$$

$$=\lim_{h\to 0}\frac{\sqrt{1+x}-\sqrt{1+(x+h)}}{h\sqrt{1+(x+h)}\,\sqrt{1+x}}\cdot\frac{\sqrt{1+x}+\sqrt{1+(x+h)}}{\sqrt{1+x}+\sqrt{1+(x+h)}}$$

$$=\lim_{h\to 0}\frac{(1+x)-[1+(x+h)]}{h\sqrt{1+(x+h)}\,\sqrt{1+x}\left(\sqrt{1+x}+\sqrt{1+(x+h)}\right)}$$

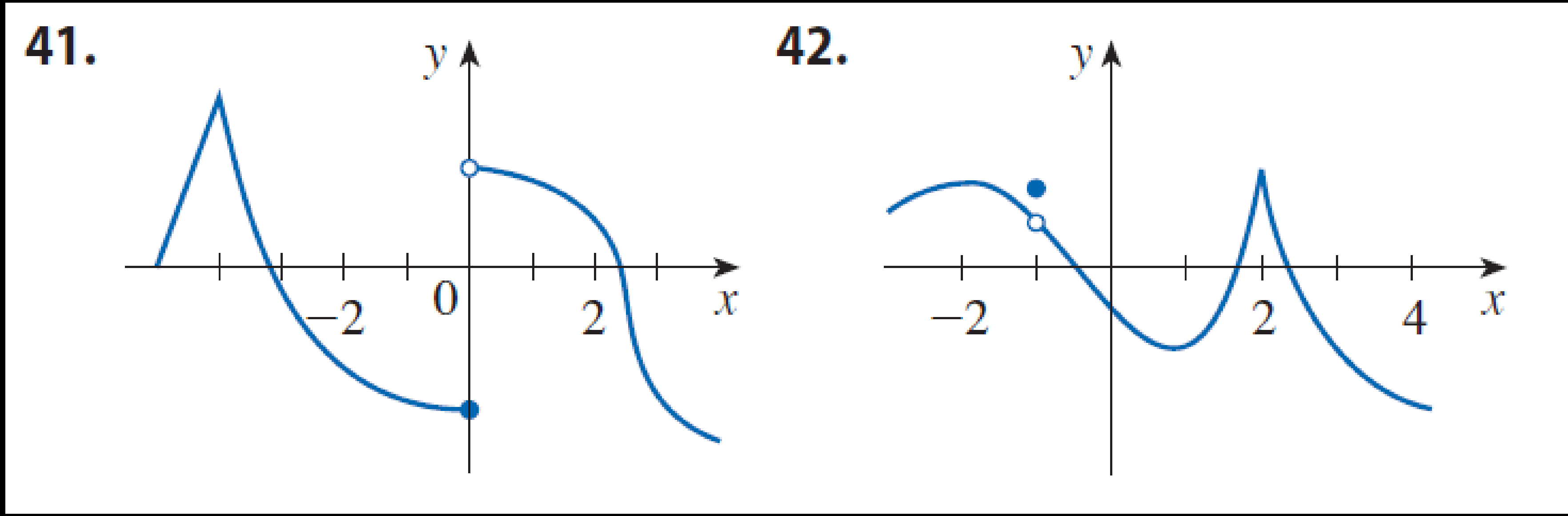
$$=\lim_{h\to 0}\frac{-h}{h\sqrt{1+x+h}\,\sqrt{1+x}\left(\sqrt{1+x}+\sqrt{1+x+h}\right)}=\lim_{h\to 0}\frac{-1}{\sqrt{1+x+h}\sqrt{1+x}\left(\sqrt{1+x}+\sqrt{1+x+h}\right)}$$

$$=\frac{-1}{\sqrt{1+x}\,\sqrt{1+x}\left(\sqrt{1+x}+\sqrt{1+x}\right)}=\frac{-1}{(1+x)\left(2\sqrt{1+x}\,\right)}=-\frac{1}{2(1+x)^{3/2}}$$

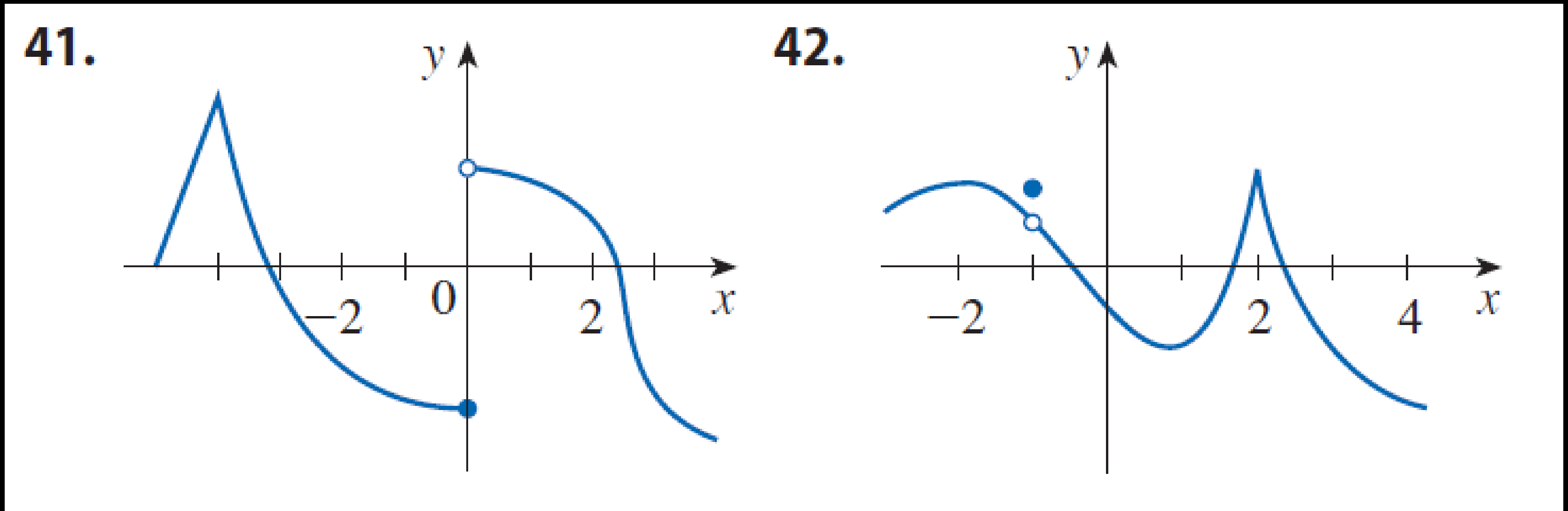
Domain of f = Domain of $f'=(-1,\infty)$.

Exercises

- The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



Exercises



41. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

42. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

Exercises

59. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

Exercises

$$f(x) = |x-6| = \begin{cases} x-6 & if\ (x-6) \geq 0 \\ -(x-6) & if\ (x-6) < 0 \end{cases}$$

The right-hand limit is

$$\lim_{x \to 6^+} \frac{f(x) - f(6)}{x-6} = \lim_{x \to 6^+} \frac{|x-6| - 0}{x-6} = \lim_{x \to 6^+} \frac{x-6}{x-6} = 1$$

The left-hand limit is

$$\lim_{x \to 6^-} \frac{f(x) - f(6)}{x-6} = \lim_{x \to 6^+} \frac{|x-6| - 0}{x-6} = \lim_{x \to 6^+} \frac{-(x-6)}{x-6} = -1$$

Since the left and right limits are not equal, the limit does not exist and f is not differentiable at 6.

TASK

Section 2.8
24
26
43
44

Project +