Functions and Models

Four Ways to Represent a Function

Content

Content
Functions
What Rules Define a Function?
Piecewise Defined Functions
Even and Odd Functions
Increasing and Decreasing Functions
Functions in Computer Science

Functions

- Functions arise whenever one quantity depends on another.
- Example:

The area A of a circle depends on the radius r of the circle.
The rule that connects r and A is given by the equation $A = \pi r^2$.
With each positive number r there is associated one value of A, and we say that A is a function of r.

Functions

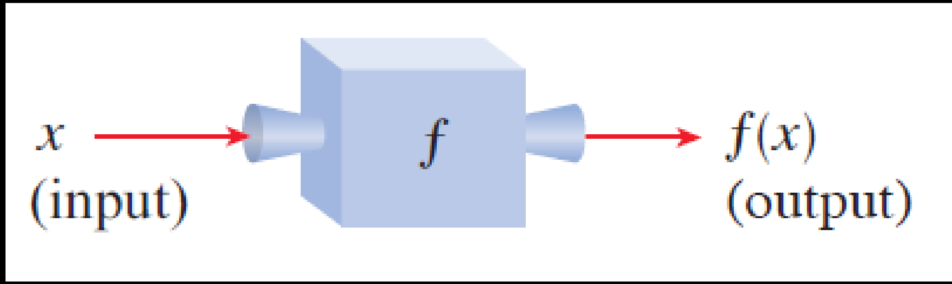
- If f represents the rule that connects A to r, then we express this in function notation as $A = f(r)$.

Function
A **function** f is a rule that assigns to each element x in a set D **exactly** one element, called $f(x)$, in a set E.

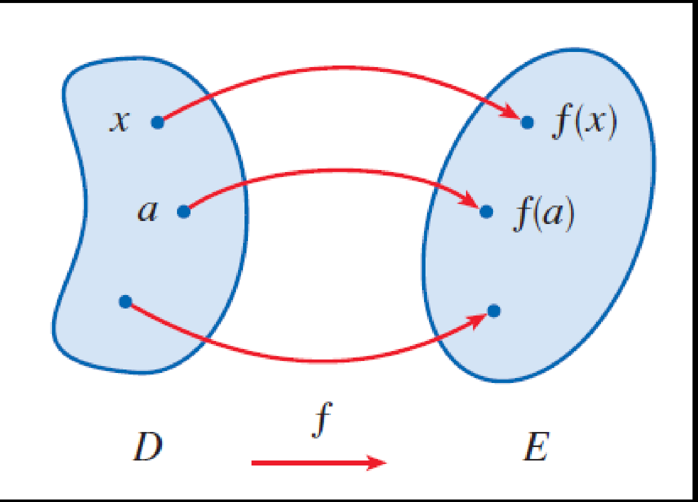
- The set D is called the domain of the function.
  - Any number in the domain is called an **independent variable**.
- The set E is called the range of the function.
  - Any number in the range is called a **dependent variable**.

Functions

- Think of a function as a machine:



- A function maps each element in the domain to **only** one element in the range.

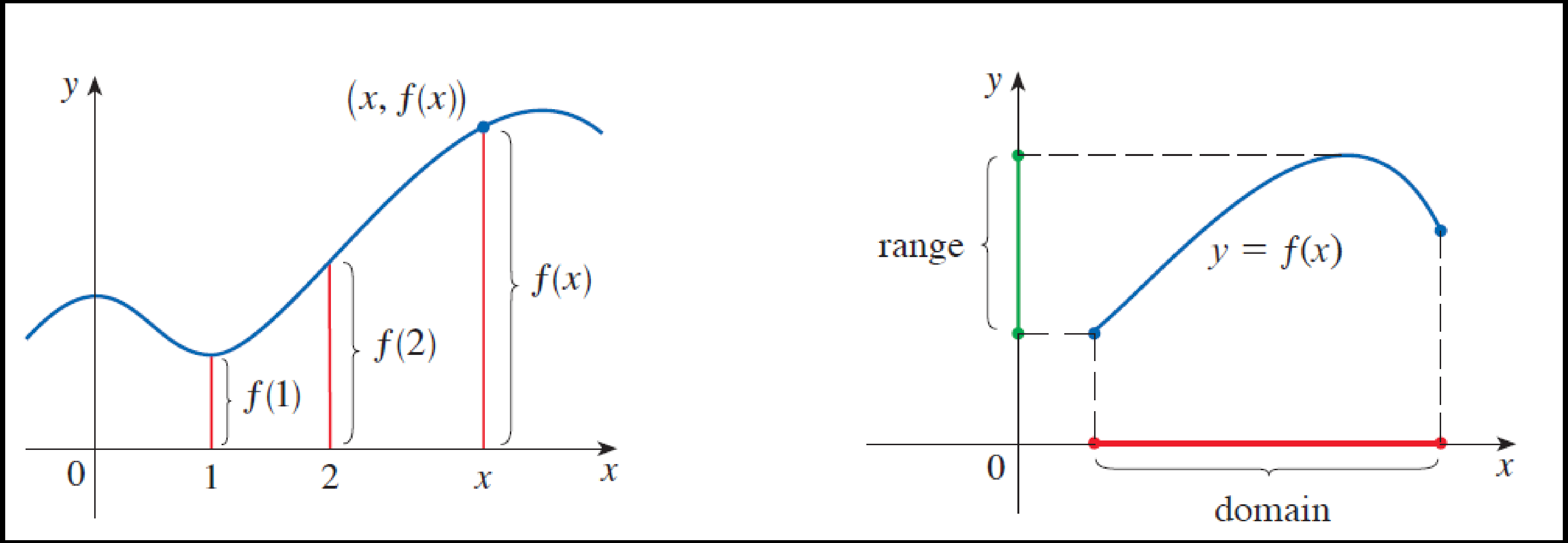


Functions

- A function can be represented as a set of ordered pairs.

$$\{(x, f(x)) \mid x \in D\}$$

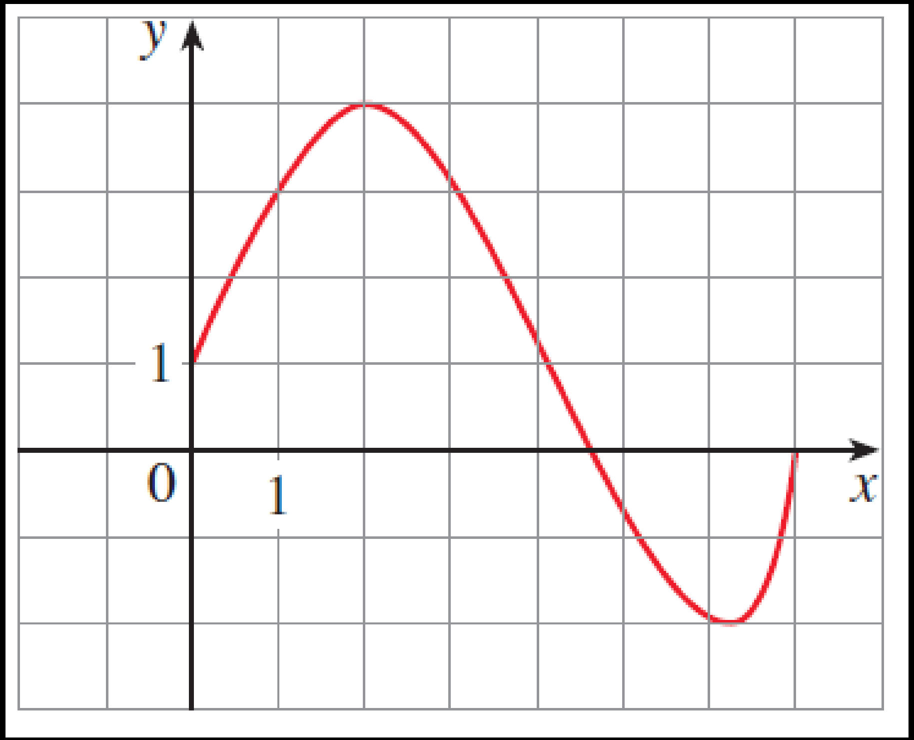
- Plotting a function is useful to visualize its behaviour or its 'life history'.
  - Plot the points on xy-axis, where the x-axis tracks the values of the domain, and the y-axis tracks the values of the range.



Functions

Example

- The graph of a function f is shown.
  - Find the values of $f(1)$ and $f(5)$.
  - What are the domain and range of f?



Solution

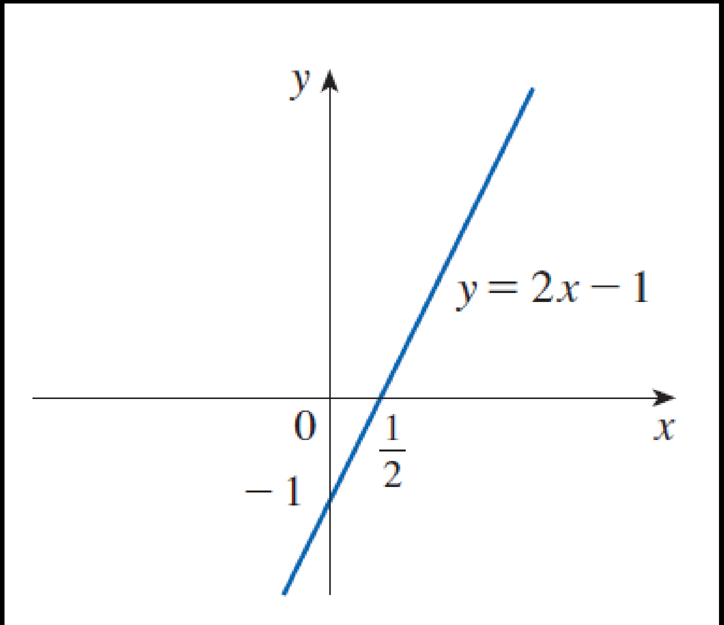
- $f(1) = 3, f(5) = -0.7$
- The domain is $[0, 7]$, the range is $[-2, 4]$

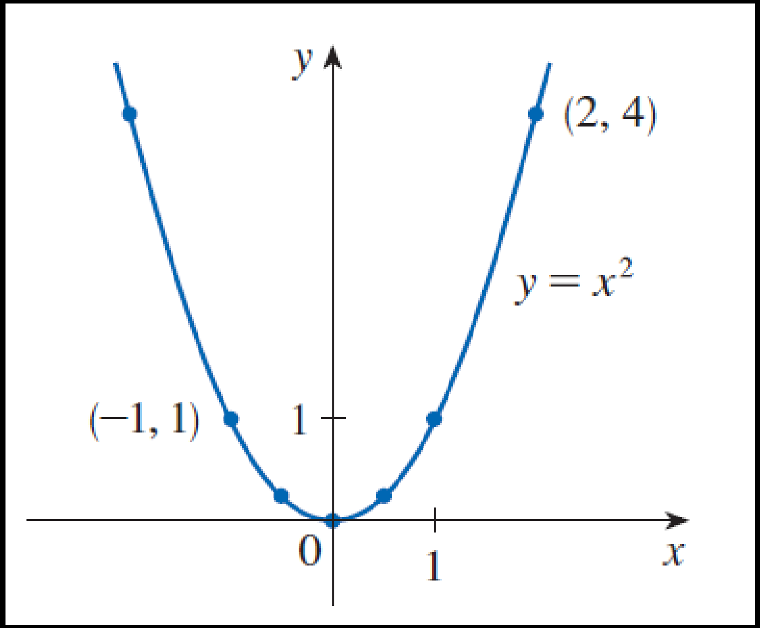
Functions

Example: Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

Solution: Substitute the x by a set of values and compute the output.





Functions

Example: If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h) - f(a)}{h}$

SOLUTION We first evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a + h) &= 2(a + h)^2 - 5(a + h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a + h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

Functions

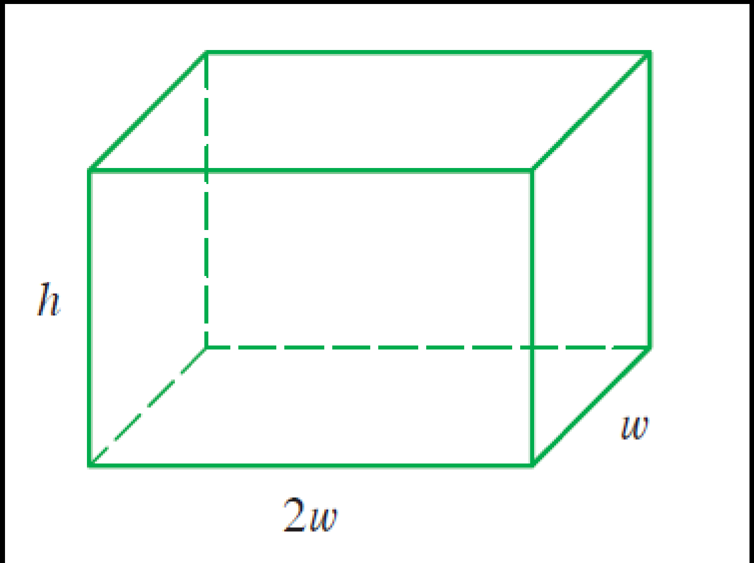
Example: A rectangular storage container with an open top has a volume of $10\ m^3$. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

Functions

Solution:

- The area of the base is $2w(w) = 2w^2$, so the cost, of the material for the base is $10(2w^2)$.
- Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$.
- The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$



Functions

- To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is $10\ m^3$. Thus

$$Volume = width\ * height\ * width$$
$$= \ w * 2w * h = 10$$

Which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = \boxed{20w^2 + \frac{180}{w}}$$

Functions

Example: Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$ (b) $g(x) = \frac{1}{x^2 - x}$

Solution

(a) Since the square root of a negative number is undefined, the domain of x must confirm that $x + 2 \geq 0$, so $x \geq -2$
$\therefore x = [-2, \infty)$

(b) Since $g(x) = \frac{1}{x^2 - x} = \frac{1}{x\,(x-1)}$,

$\because$ division by 0 is undefined, $\therefore x \neq 0$ or $x \neq 1$.

So, the domain of x is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Content

Content
Functions
What Rules Define a Function?
Piecewise Defined Functions
Even and Odd Functions
Increasing and Decreasing Functions
Functions in Computer Science

Which Rules Define Functions?

- Not every equation defines a function.

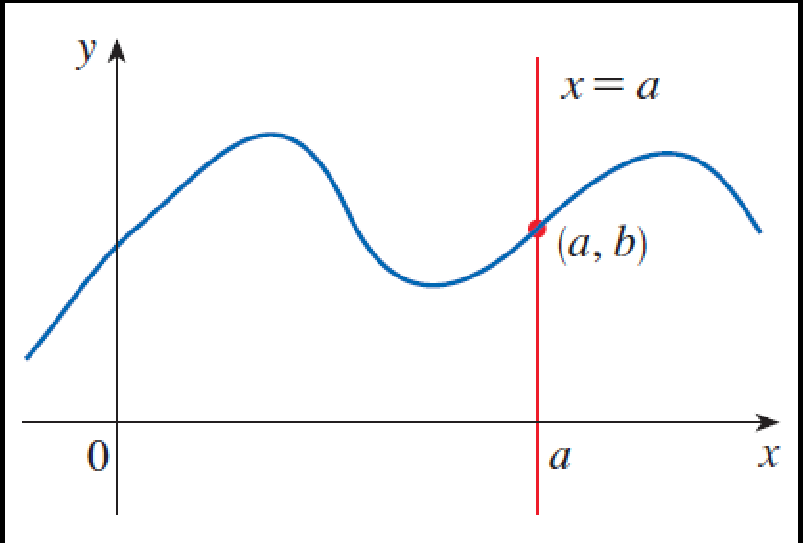
- The equation $y = x^2$ defines y as a function of x because the equation determines exactly one value of y for each value of x.

- The equation $y^2 = x$ does **not** define a function because some input values x correspond to more than one output y;
    - for instance, for the input $x = 4$ the equation gives the outputs $y = 2$ and $y = -2$.

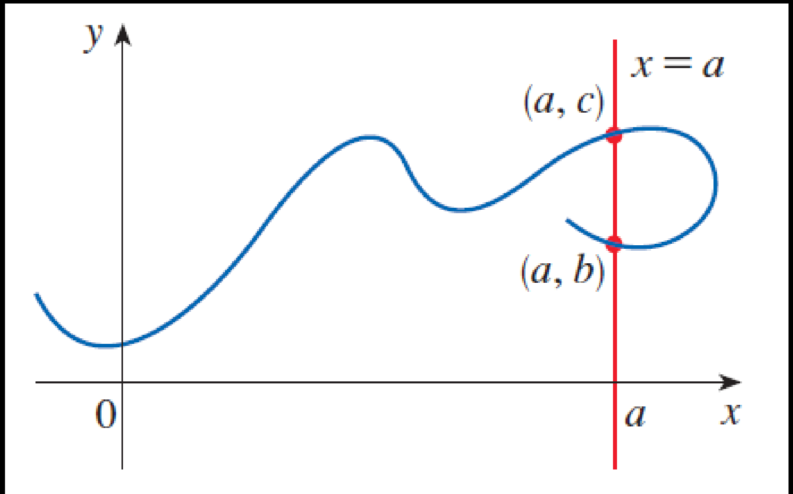
Which Rules Define Functions?

- For curves drawn in the xy-plane, we apply the **vertical line test**

The Vertical Line Test
A curve in the xy-plane is the graph of a function of x **if and only if** <u>no</u> vertical line intersects the curve more than once.



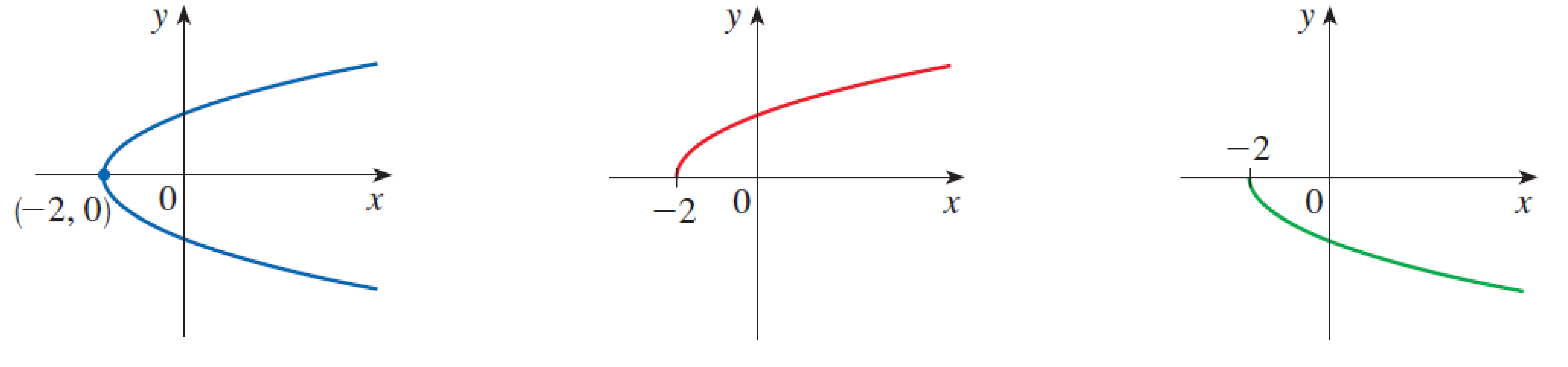
(a) This curve represents a function.



(b) This curve doesn't represent a function.

Which Rules Define Functions?

- The parabola $x = y^2 - 2$ is not a function.
- Note that $x = y^2 - 2 \rightarrow y^2 = x + 2 \rightarrow y = \pm\sqrt{x+2}$
  - $y = \sqrt{x+2}$ is a function
  - $y = -\sqrt{x+2}$ is a function



(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

Content

Content
Functions
What Rules Define a Function?
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Piecewise Defined Functions

- If a function is defined by different formulas givens a condition for their domain, it is called **piecewise function**

$$f(x) \begin{cases} x^2 & \text{if } x < 0 \\ 10 - x & \text{if } x >= 0 \text{ and } x < 10 \\ 2x + 3 & \text{if } x >= 10 \end{cases}$$

Piecewise Defined Functions

- **Example**: A function f is defined by

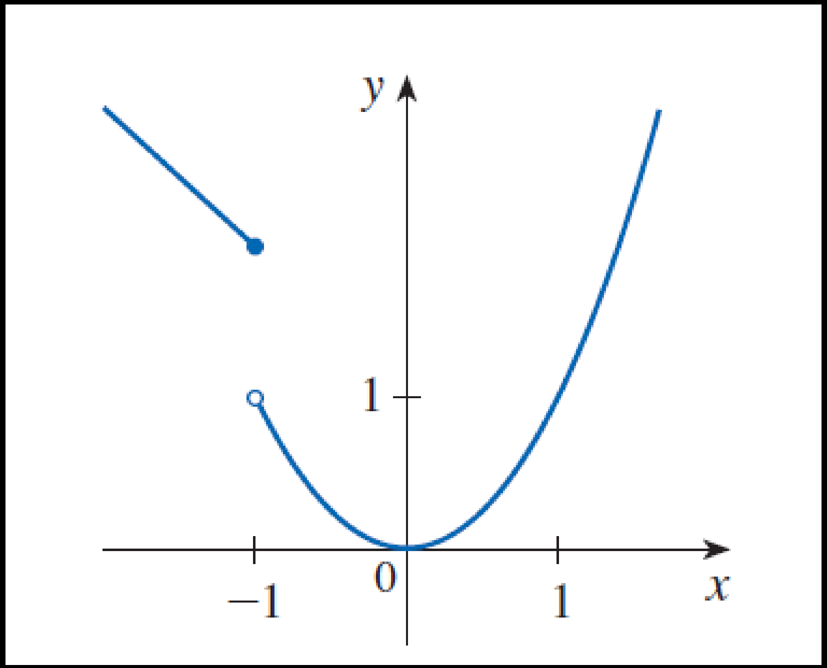
$$f(x) = \begin{cases} 1 - x & if\ x \le -1 \\ x^2 & if\ x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution

$$\because -2 \le -1, \therefore f(-2) = 1 - (-2) = 3$$
$$\because -1 \le -1, \therefore f(-1) = 1 - (-1) = 2$$
$$\because 0 > \ -1, \therefore f(0) = 0^2 = 0$$

Piecewise Defined Functions



Content

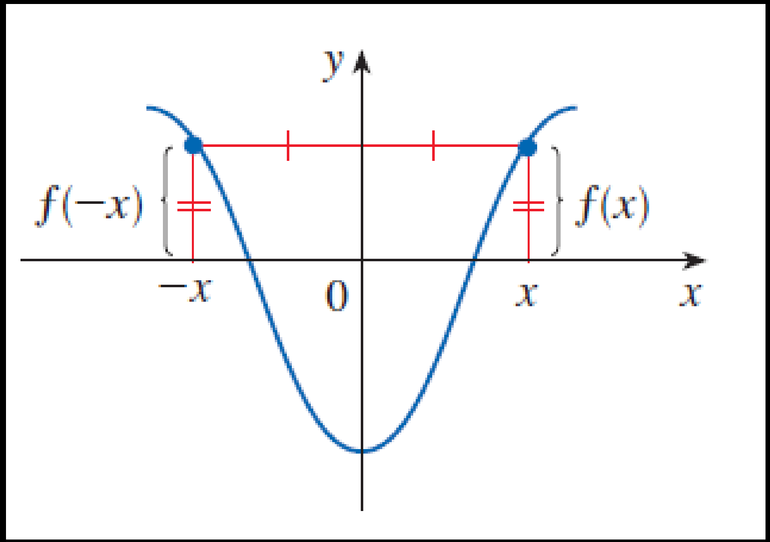
Content
Functions
What Rules Define a Function?
Piecewise Defined Functions
Even and Odd Functions
Increasing and Decreasing Functions
Functions in Computer Science

Even and Odd Functions

- **Even function** is a function f that satisfies $f(-x) = f(x)$ for every number x in the domain.
- Example: $f(x) = x^2$ is an even function

$$f\,(-x) = (-x)^2 = x^2 = f(x)$$

- The graph of the function is symmetric with respect to the y-axis.

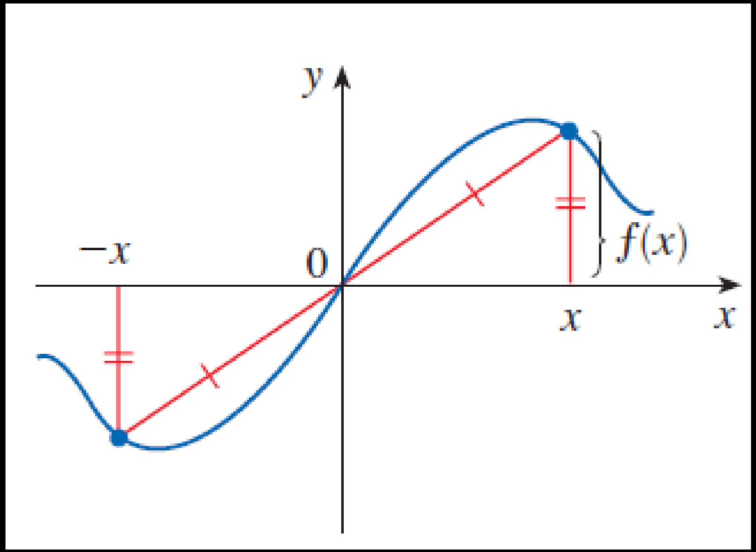


Even and Odd Functions

- **Odd function** is the function f that satisfies $f(-x) = -f(x)$ for every number x in the domain.
- Example: $f(x) = x^3$ is an odd function

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

- The graph of the odd function is symmetric around the origin.



Even and Odd Functions

- **Example**: Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Even and Odd Functions

- **Solution**:

(a) $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

Odd function

(b) $\mathrm{g}(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

Even function

(c) $\mathrm{h}(-x) = 2(-x) - (-x)^2 = -2x \; - x^2 = -(2x + x^2) \neq f(x) \neq -f(x)$

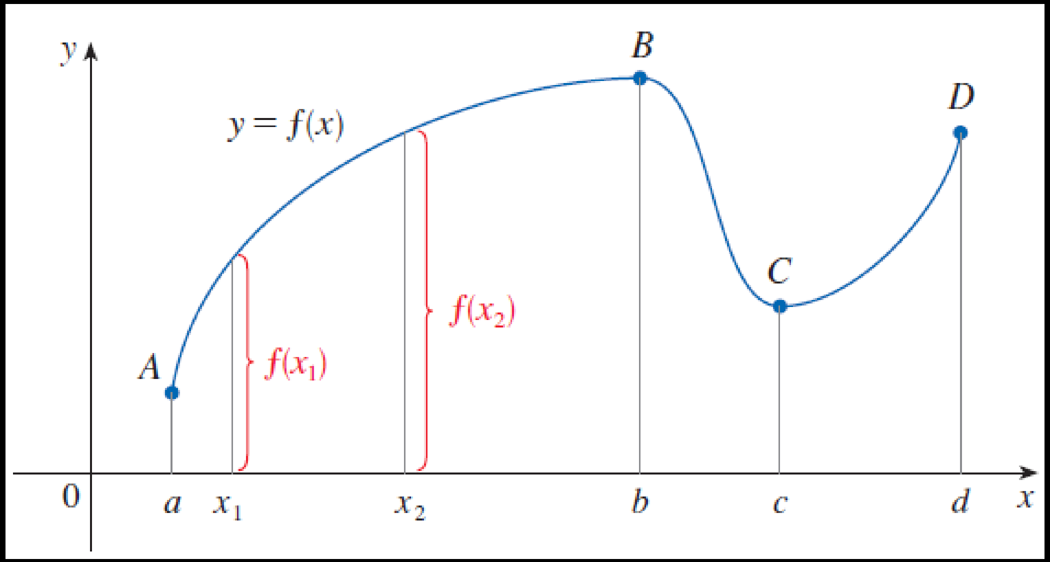
Neither is odd or even function

Content

Content
Functions
What Rules Define a Function?
Piecewise Defined Functions
Even and Odd Functions
Increasing and Decreasing Functions
Functions in Computer Science

Increasing and Decreasing Functions

- The graph rises from A to B, falls from B to C, and rises again from C to D.



  - The function f is said to be increasing on the interval $[a, b]$,
  - decreasing on $[b, c]$,
  - and increasing again on $[c, g]$.

Increasing and Decreasing Functions

Increasing function
A function f is called **increasing** on an interval I if $f(x_1) < f(x_2) \qquad$ whenever $x_1 < x_2$ in I

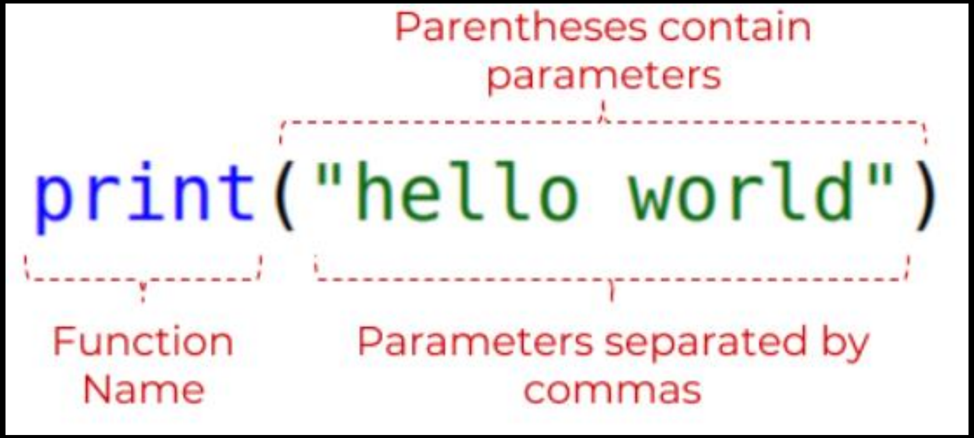
Decreasing function
A function f is called **decreasing** on an interval I if $f(x_1) > f(x_2) \qquad$ whenever $x_1 > x_2$ in I

Content

Content
Functions
What Rules Define a Function?
Piecewise Defined Functions
Even and Odd Functions
Increasing and Decreasing Functions
Functions in Computer Science

Functions in Computer Science

- In programming languages, a function is a block of code that executes when we call the function.



  - A function may take parameters (arguments) to process it and gives the corresponding output.
  - A function may or may not take any parameters.

Functions in Computer Science

- Example:

```
function average_grade( list_of_grades )
        …
end function


Midterm_grades = …
Print(“The average grade is”)
Print( average_grade(midterm_grades))
```

References

- Calculus by James Stewart
- https://www.cs.utah.edu/~germain/PPS/Topics/functions.html
- https://www.futurelearn.com/info/courses/programming-102-think-like-a-computer-scientist/0/steps/53095#:~:text=A%20function%20is%20simply%20a,which%20performs%20a%20particular%20task.