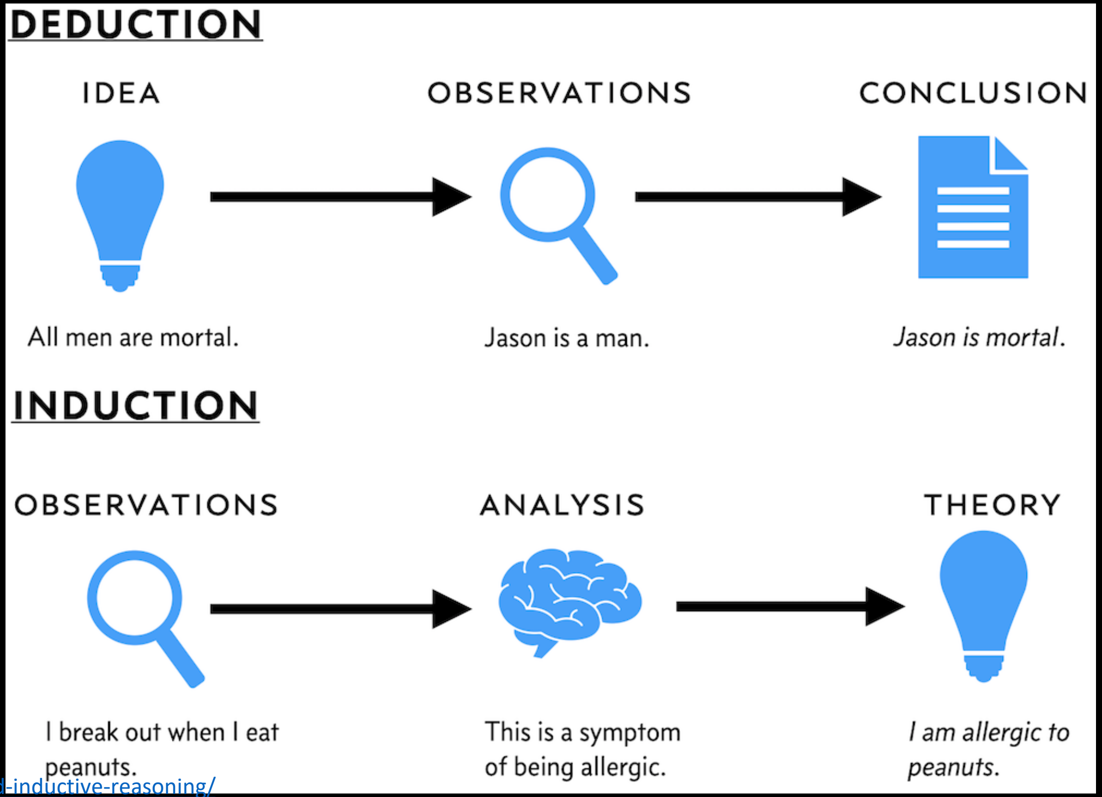
CH 05: Basic Structures

Sequences, Mathematical Induction, and Recursion

Content

Mathematical Induction I

- **Deduction:** Inferring a conclusion from general principals.
- **Induction:** Enunciating a general principal based on observations.



https://danielmiessler.com/p/the-difference-between-deductive-and-inductive-reasoning/

Mathematical Induction I

- **Example:** We have the following observations:

row 1	$1-\frac{1}{2}=\frac{1}{2}$
row 2	$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\frac{1}{3}$
row 3	$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{4}$

- Can we induce a general principal? Are there any patterns? Can we conclude what will be the result for the 4^{th} and 5^{th} row? Can we conclude what will be the result for the k^{th} row?

Mathematical Induction I

- **Example:** We have the following observations:

row 1	$1 - \boxed{\frac{1}{2}} = \boxed{\frac{1}{2}}$
row 2	$\left(1 - \frac{1}{2}\right)\left(1 - \boxed{\frac{1}{3}}\right) = \boxed{\frac{1}{3}}$
row 3	$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \boxed{\frac{1}{4}}\right) = \boxed{\frac{1}{4}}$

- The last term subtracted from 1 (e.g., ½, 1/3, ¼) is the result!

Mathematical Induction I

- So, for the 4th row we have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)=\frac{1}{5}$$

- For the 5th row, we have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)=\frac{1}{6}$$

$\therefore$ For the k^{th} row, we have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$$

Mathematical Induction I

- So, for the 4th row we have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)=\frac{1}{5}$$

- For the 5th row, we

This is induction.

$$\left(1 - \quad = \frac{1}{6}\right.$$

∴ For the k^{th} row, we have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$$

Mathematical Induction I

- **Example:** Use induction to prove that for every integer $n \geq 1$,

$$P(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- **Solution:** Incrementally show that $P(n)$ holds for the some integers, then conclude that it is valid for every integer n.
- $P(1) = \frac{1(1+1)}{2} = 1 \rightarrow P(1) = 1$
- $P(2) = \frac{2(2+1)}{2} = \frac{6}{2} = 3 \rightarrow P(2) = 1 + 2 = 3$
- $P(3) = \frac{3(3+1)}{2} = \frac{12}{2} = 6 \rightarrow P(3) = 1 + 2 + 3 = 6$
- $\therefore P(k) = \frac{k(k+1)}{2} = 1 + 2 + 3 + \cdots + k$ for any integer $k \geq 1$.

Mathematical Induction I

1. $\because P(k) = \frac{k(k+1)}{2}$
2. $\therefore p(k+1) \qquad = \frac{(k+1)(k+1+1)}{2} = \frac{k^2+3k+2}{2}$
3. $\therefore p(k+1) \qquad = p(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{k^2+k}{2} + \frac{2k}{2} + \frac{2}{2} = \frac{k^2+3k+2}{2}$$

Mathematical Induction I

1. $\because P(k) = \frac{k(k+1)}{2}$ ← Inductive hypothesis

2. $\therefore p(k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{k^2+3k+2}{2}$ ← Inductive step

3. $\therefore p(k+1) = p(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{k^2+k}{2} + \frac{2k}{2} + \frac{2}{2} = \frac{k^2+3k+2}{2}$$ ← Inductive step

$\therefore$ based on (1), (2), (3), $P(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ is correct for every $n \geq 1$.

Mathematical Induction I

- **Example:** Evaluate $2 + 4 + 6 + \cdots + 500$

- **Solution:**

$$2 + 4 + 6 + \cdots + 500$$

$$= 2(1 + 2 + 3 + \cdots + 250)$$

$$= 2 \times \frac{250 \times 251}{2} = 62{,}750$$

Mathematical Induction I

- **Example:** Evaluate $5+6+7+8+\cdots 50$

- **Solution:**

$$5+6+7+8+\cdots 50$$

$$=(1+2+3+4+\cdots+50)-(1+2+3+4)$$

$$=\frac{50\times 51}{2}-\frac{4\times 5}{2}=1{,}265$$

Mathematical Induction I

- **Example:** $1 + 2 + 3 + \ \ldots + (h - 1)$ for $h \geq 2$

- **Solution:**

$$1 + 2 + 3 + \ \ldots + (h - 1)$$

$$= \frac{(h-1)(h-1+1)}{2}$$

$$= \frac{(h-1)h}{2}$$

Mathematical Induction I

- Geometric sequence

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

For every integer $n \geq 0$ and every real number r except 1.

Mathematical Induction I

- **Example:** Assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a) $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b) $3^2 + 3^3 + 3^4 + \cdots + 3^m$

- **Solution:**

a) $1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3-1} = \frac{3^{m-1} - 1}{2}$

b) $3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2(1 + 3 + 3^2 + \cdots + 3^{m-2}) = 9 \cdot \frac{3^{m-1} - 1}{2}$

Exercises

- For each positive integer n, let $P(n)$ be the formula:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

a) Write $P(1)$. Is $P(1)$ true?

b) Write $P(k)$

c) Write $P(k+1)$

Exercises

- For each positive integer n, let $P(n)$ be the formula:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

a) Write $P(1)$. Is $P(1)$ true?

$P(1) = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6} = 1$. It is true because $1 = 1^2$

b) Write $P(k)$

$$P(k) = \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6}$$

c) Write $P(k+1)$

$$P(k) = \frac{(k+1) \cdot \left((k+1)+1\right) \cdot (2 \cdot (k+1)+1)}{6}$$

Exercises

- For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$

a) Write $P(2)$. Is $P(2)$ true?

b) Write $P(k)$

c) Write $P(k+1)$

Exercises

- For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$

a) Write $P(2)$. Is $P(2)$ true?

$P(2) = \frac{2\cdot(2-1)\cdot(2+1)}{3} = 2$. It is true because $2 = 1(1+1) = 2$. (note that the upper limit of the sum is $n-1$

b) Write $P(k)$

$$P(k) = \sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

c) Write $P(k+1)$

$$P(k) = \sum_{i=1}^{k+1-1} i(i+1) = \frac{(k+1)\big((k+1)-1\big)(k+1)+1)}{3} = \frac{(k+1)(k)(k+2)}{3}$$

Exercises

Prove each statement using mathematical induction.

6) For all integers $n \geq 1$, $2 + 4 + 6 + \cdots + 2n = n^2 + n$.

7) For all integers $n \geq 1$, $1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n-3)}{2}$

Exercises

6) For all integers $n \geq 1,\ 2 + 4 + 6 + \cdots + 2n = n^2 + n.$

$P(1) = 1^2 + 1 = 2$ → it is equal to the first number in the sequence, 2 ← base case
Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

$P(k) = 2 + 4 + 6 + \cdots + 2k = k^2 + k.$ ← inductive hypothesis
$P(k+1) = (k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + 3k + 2$ ← inductive step
$P(k+1) = P(k) + 2(k+1) = k^2 + k + 2k + 2 = k^2 + 3k + 2$ ← inductive step
Since both the base case and the inductive step have been proved, $P(n)$ is true for all integers $n \geq 1$.

Exercises

7) For all integers $n \geq 1, 1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n-3)}{2}$

$P(1) = \frac{1(5\cdot1-3)}{2} = 1$ → it is equal to the first term in the sequence. ← base case

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

$P(k) = 1 + 6 + 11 + 16 + \cdots + (5k - 4) = \frac{k(5k-3)}{2}$ ← inductive hypothesis

$P(k+1) = \frac{(k+1)(5(k+1)-3)}{2} = \frac{(k+1)(5k+2)}{2} = \frac{5k^2+2k+5k+2}{2} = \frac{(5k^2+7k+2)}{2}$ ← inductive step

$P(k+1) = P(k) + (5(k+1) - 4) = \frac{k(5k-3)}{2} + \frac{2(5k+1)}{2} = \frac{5k^2+7k+2}{2}$ ← inductive step

Since both the base case and the inductive step have been proved, $P(n)$ is true for all integers $n \geq 1$.

Exercises

Prove the following statement by mathematical induction.

10) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$.

Exercises

10) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$.

$P(1) = \frac{1(1+1)(2\cdot1+1)}{6} = 1$ → it is equal to the first term in the sequence. ← base case

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

$P(k) = 1 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ ← inductive hypothesis

$P(k+1) = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{2k^3+9k^2+13k+6}{6}$ ← inductive step

$P(k+1) = P(k) + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{2k^3+9k^2+13k+6}{6}$ ← inductive step

Exercises

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the following sums or to write them in closed form.

20) $4 + 8 + 12 + 16 + \cdots + 200$

21) $5 + 10 + 15 + 20 + \cdots + 300$

22) $3 + 4 + 5 + 6 + \cdots + 1000$

Exercises

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the following sums or to write them in closed form.

$$\boxed{\sum_{i=1}^{n} i = \frac{i(i+1)}{2}}$$

20) $4 + 8 + 12 + 16 + \cdots + 200$

$$\boxed{4 + 8 + 12 + 16 + \cdots + 200 = 4(1 + 2 + 3 + \cdots 50) = 4 \cdot \frac{50 \cdot 51}{2} = 5100}$$

21) $5 + 10 + 15 + 20 + \cdots + 300$

$$\boxed{5 + 10 + 15 + 20 + \cdots + 300 = 5(1 + 2 + 3 + \cdots + 60) = 5 \cdot \frac{60 \cdot 61}{2} = 9150}$$

22) $3 + 4 + 5 + 6 + \cdots + 1000$

$$\boxed{3 + 4 + 5 + 6 + \cdots + 1000 = (1 + 2 + 3 + \cdots + 1000) - (1 + 2) = \frac{1000 \cdot 1001}{2} - 3 = 500{,}497}$$

Exercises

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the following sums or to write them in closed form.

25) a. $1 + 2 + 2^2 + \cdots + 2^{25}$

b. $2 + 2^2 + 2^3 + \cdots + 2^{26}$

Exercises

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the following sums or to write them in closed form.

$$\boxed{\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}}$$

25) a. $1 + 2 + 2^2 + \cdots + 2^{25}$

$$\boxed{1 + 2 + 2^2 + \cdots + 2^{25} = \frac{2^{26}-1}{2-1} = 67{,}108{,}863}$$

b. $2 + 2^2 + 2^3 + \cdots + 2^{26}$

$$\boxed{2 + 2^2 + 2^3 + \cdots + 2^{26} = 2(1 + 2 + 2^2 + \cdots + 2^{25}) = 2 \cdot \frac{2^{26}-1}{2-1} = 2 \cdot 67{,}108{,}863 = 134{,}217{,}726}$$

TASK

Section 5.2
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Content

CH 05
Sequences
Mathematical Induction I
Mathematical Induction II
Strong Mathematical Induction and the Well-Ordering Principle
Application: Correctness of Algorithms
Defining Sequences Recursively
Solving Recurrence Relations by Iteration
Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients
General Recursive Definitions and Structural Induction

Mathematical Induction II

- This section shows how to use mathematical induction to prove additional kinds of statements:
  - Divisibility properties of the integers
  - Inequalities
  - Recursive definitions

Mathematical Induction II

- **Example:** Use mathematical induction to prove that for all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.
- **Solution:**
- Check the base case: $P(0) = 2^0 - 1 = 0$ → 0 is divisible by 3
- Inductive hypothesis: $P(k) = 2^{2k} - 1$ is divisible by 3
- Recall that an integer m is divisible by 3 if $m = 3r$ for any integer r.
- Inductive step: $P(k+1) = 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} \cdot 4 - 1$
- …

Mathematical Induction II

- …

$$
\begin{aligned}
P(k+1) = 2^{2(k+1)} - 1 = 2^{2k+2} - 1 &= 2^{2k} \cdot 4 - 1 \\
&= 2^{2k}(3+1) - 1 \\
&= 3 \cdot 2^{2k} + 2^{2k} - 1
\end{aligned}
$$

Mathematical Induction II

- …
- Inductive step: $P(k+1) = 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} \cdot 4 - 1$

$$
\begin{aligned}
P(k+1) = 2^{2(k+1)} - 1 = 2^{2k+2} - 1 &= 2^{2k} \cdot 4 - 1 \\
&= 2^{2k}(3+1) - 1 \\
&= \underbrace{3 \cdot 2^{2k}}_{\text{Divisible by 3 as } m = 3r} + \underbrace{2^{2k} - 1}_{\text{Divisible by 3 by the hypothesis}}
\end{aligned}
$$

Mathematical Induction II

$$\underbrace{2^{2(k+1)} - 1}_{\text{Divisible by 3}} = \underbrace{3 \cdot 2^{2k}}_{\text{Divisible by 3 as } m = 3r} + \underbrace{2^{2k} - 1}_{\text{Divisible by 3 by the hypothesis}}$$

- $\because$ Both terms of the right side are divisible by 3.
- $\therefore$ The left-hand side is also divisible by three. ← Inductive step

- Conclusion: For any integer $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Mathematical Induction II

- **Example:** Use induction to prove that for all integers $n \geq 3$, $2n + 1 < 2^n$.
- **Solution:**
- Basic case: $P(3) = 2 \cdot 3 + 1 = 7 < 2^3 = 8$

- Inductive hypothesis: $P(k) = 2k + 1 < 2^k$ for any integer $k \geq 3$

- Inductive step: $P(k+1) = 2(k+1) + 1 < 2^{k+1} \rightarrow 2k + 3 < 2^k \cdot 2$
- …

Mathematical Induction II

- Inductive step: $P(k+1) = 2(k+1) + 1 < 2^{k+1}$ → $2k + 3 < 2^k \cdot 2$
- $2k + 3 = 2k + 1 + 2$
- By inductive hypothesis: $2k + 1 < 2^k$
- By algebra: $2 < 2^k$ for $k \geq 2$
- Therefore,

$$2k + 3 = 2k + 1 + 2 \ < \ 2^k + 2^k = 2^k \cdot 2$$

By laws of exponents, this inequality holds for $k \geq 2$.

- Conclusion: $P(k+1) = 2(k+1) + 1 < 2^{k+1}$ for any $k \geq 3$.
- $\therefore$ For all integers $n \geq 3$, $2n + 1 < 2^n$.

Mathematical Induction II

- **Example:** Define a sequence $a_1, a_2, a_3, \ldots$ as follows.
  $a_1 = 2$, $a_k = 5a_{k-1}$ for all integers $k \geq 2$.

a) Write the first four terms of the sequence.

b) It is claimed that for each integer $n \geq 1$, the *n*th term of the sequence has the same value as that given by the formula $2 \cdot 5^{n-1}$. In other words, the claim is that the terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$. Prove that this is true.

Mathematical Induction II

- **Solution:**

a) $a_1 = 2,$
$a_2 = 5 \cdot a_1 = 5 \cdot 2 = 10,$
$a_3 = 5 \cdot a_2 = 5 \cdot 10 = 50,$
$a_4 = 5 \cdot a_3 = 5 \cdot 50 = 250$

b) Base case: $P(1) = 2 \cdot 5^{1-1} = 2$ → this is the first element in the sequence.

Inductive hypothesis: $P(k) = 2 \cdot 5^{k-1}$

- We must show that $P(k+1) = 2 \cdot 5^{k+1-1} \equiv a_{k+1} = 5 \cdot a_{k+1-1}$

Mathematical Induction II

- **Solution:**

$$P(k+1) = 2 \cdot 5^{k+1-1} = 2 \cdot 5^{k}$$

Also,

$$a_{k+1} = 5 \cdot a_{k+1-1} = 5 \cdot a_{k}$$

$$= 5 \cdot P(k) = 5 \cdot 2 \cdot 5^{k-1} = 2 \cdot 5^{k}$$

Mathematical Induction II

- **Solution:**

$$P(k+1) = 2 \cdot 5^{k+1-1} = 2 \cdot 5^{k}$$

Also,

$$a_{k+1} = 5 \cdot a_{k+1-1} = 5 \cdot a_k$$

$$= 5 \cdot P(k) = 5 \cdot 2 \cdot 5^{k-1} = 2 \cdot 5^{k}$$

- We conclude that $P(k+1) = a_{k+1} = 2 \cdot 5^{k}$
- $\therefore$ For any $n \geq 1$, $a_n = 2 \cdot 5^{n-1}$

Exercises

3) Observe that

$$\frac{1}{1\cdot 3} = \frac{1}{3}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{2}{5}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{3}{7}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} = \frac{4}{9}.$$

Guess a general formula and prove it by mathematical induction.

Exercises

- General formula: $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for any integer $n\geq1$

Proof by induction:

- Base case: $\frac{1}{2\cdot1+1}=\frac{1}{3}$ → equal to the same element in the sequence.
- Inductive hypothesis: $P(k)=\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ for any integer $k\geq1$
- Inductive step: $P(k+1)=\frac{k+1}{2k+3}$

Exercises

- The left-hand side:

$$P(k+1)$$

$$= P(k) + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

Exercises

- The left-hand side:

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k\,(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

Exercises

4. Observe that

$$
\begin{aligned}
1 &= 1, \\
1 - 4 &= -(1 + 2), \\
1 - 4 + 9 &= 1 + 2 + 3, \\
1 - 4 + 9 - 16 &= -(1 + 2 + 3 + 4), \\
1 - 4 + 9 - 16 + 25 &= 1 + 2 + 3 + 4 + 5.
\end{aligned}
$$

Guess a general formula and prove it by mathematical induction.

Exercises

The right-hand side is $1 + 2 + 3 + \cdots n$ but with alternating sign.

So, it's basically a summation where each term is multiplied by $(-1)^{n-1}$ for n ≥ 1. So, the **RHS** is $(-1)^{n-1} \sum_{i=1}^{n} i$

Similarly, the **LHS** $\sum_{i=1}^{n}(-1)^{i-1} i^2$.

Therefore, the general form of the equation is

$$1 - 4 + 9 - 16 + \cdots + (-1)^{n-1}\, n^2 = (-1)^{n-1}(1 + 2 + 3 + \cdots + n)$$

Or

$$\sum_{i=1}^{n} (-1)^{i-1} i^2 = (-1)^{n-1} \sum_{i=1}^{n} i = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

Exercises

Inductive step (LHS):

$$P(k+1) = 1 - 4 + 9 - 16 + \cdots + (-1)^{k-1}k^2 + (-1)^{k-1+1}(k+1)^2 \rightarrow P(k) + (-1)^k(k+1)^2 \rightarrow$$

$$\sum_{i=1}^{k}(-1)^{i-1}i^2 + (-1)^k(k+1)^2 \rightarrow (-1)^{k-1}\sum_{i=1}^{k} i + (-1)^k(k+1)^2 \rightarrow$$

$$(-1)^{k-1}\left(\frac{k(k+1)}{2} - \frac{2(k+1)^2}{2}\right) \rightarrow (-1)^{k-1} \cdot \frac{k^2 + k \; - (2k^2 + 4k + 2)}{2} \rightarrow (-1)^{k-1} \cdot \frac{(-k^2 - 3k - 2)}{2} \rightarrow$$

$$(-1)^k \cdot \frac{k^2 + 3k + 2}{2}$$

Exercises

Inductive step (RHS):

$$P_2(k+1) = (-1)^{k+1-1}\big(1+2+3+\cdots+k+(k+1)\big) \rightarrow$$

$$P_2(k+1) = (-1)^k \cdot \left(\sum_{i=1}^{k} i + (k+1)\right) \rightarrow$$

$$(-1)^k \cdot \left(\frac{k(k+1)}{2} + \frac{2(k+1)}{2}\right) = (-1)^k \cdot \frac{k^2+3k+2}{2}$$

Exercises

6) For each positive integer n, let $P(n)$ be the property $5^n - 1$ is divisible by 4.

a. Write $P(0)$. Is $P(0)$ true?

b. Write $P(k)$.

c. Write $P(k\ +1)$.

Exercises

6) For each positive integer n, let $P(n)$ be the property $5^n - 1$ is divisible by 4.

a. Write $P(0)$. Is $P(0)$ true?

$\boxed{P(0) = 5^0 - 1 = 0 \rightarrow \text{correct because } 0 \text{ is divisible by } 4}$

b. Write $P(k)$.

$\boxed{P(k) = 5^k - 1}$

c. Write $P(k \ + 1)$.

$\boxed{P(k+1) = 5^{k+1} - 1}$

Exercises

6) For each positive integer n, let $P(n)$ be the property $2^n < (n+1)!$.

a. Write $P(2)$. Is $P(2)$ true?

b. Write $P(k)$.

c. Write $P(k\ +1)$.

Exercises

6) For each positive integer n, let $P(n)$ be the property $2^n < (n+1)!$.

a. Write $P(2)$. Is $P(2)$ true?

$\boxed{P(2) = 2^2 = 4 < (2+1)! = 3 \cdot 2 \cdot 1 = 6 \rightarrow \text{True. Because } 4 < 6}$

b. Write $P(k)$.

$\boxed{P(k) = 2^k < (k+1)!}$

c. Write $P(k+1)$.

$\boxed{P(k+1) = 2^{k+1} < (k+2)!}$

Exercises

Prove each statement below by mathematical induction.

8) $5^n - 1$ is divisible by 4, for each integer $n \geq 0$.

9) $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

Exercises

8) $5^n - 1$ is divisible by 4, for each integer $n \geq 0$.

- Base case: $P(0) = 5^0 - 1 = 0$ which is divisible by 4.
- Inductive hypothesis: $P(k) = 5^k - 1$ is divisible by 4 for an integer $k \geq 0$.
- Inductive steps: $P(k+1) = 5^{k+1} - 1 \rightarrow 5^k \cdot 5 \; - 1 = 5^k \cdot (4+1) - 1 \rightarrow$ $5^k \cdot 4 + 5^k - 1$
- $5^k \cdot 4 + 5^k - 1 \equiv 5^k \cdot 4 + P(k)$
- $\because 5^k \cdot 4$ is a multiple of 4, then it is divisible by 4 and $P(k)$ is divisible by 4.
- $\therefore P(k+1)$ is divisible by 4.
- Conclusion: $5^n - 1$ is divisible by 4, for each integer $n \geq 0$.

Exercises

9) $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

- Base case: $P(0) = 7^0 - 1 = 1 - 1 = 0$ which is divisible by 6.
- Inductive hypothesis: $P(k) = 7^k - 1$ is divisible by 6 for an integer $k \geq 0$.
- Inductive steps: $P(k+1) = 7^{k+1} - 1 \rightarrow 7^k \cdot 7 \; - 1 = 7^k(6+1) - 1 \rightarrow$ $7^k \cdot 6 + 7^k - 1$
- $7^k \cdot 6$ is a multiple of 6, so it's divisible by 6 and $7^k - 1 = P(k)$ which is divisible by 6.
- $\therefore P(k+1)$ is divisible by 6
- Hence, $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

Exercises

Prove each statement below by mathematical induction.

19) $n^2 < 2^n$, for all integers $n \geq 5$.

20) $2^n < (n + 2)!$, for all integers $n \geq 0$.

Exercises

19) $n^2 < 2^n$, for all integers $n \geq 5$.

Base case: $P(5) = 5^2 = 25 < 2^5 = 32$

Inductive hypothesis: $P(k) = k^2 < 2^k$ for any $k \geq 5$

Inductive step: $P(k+1) = (k+1)^2 = k^2 + 2k + 1$

Inductive step: $P(k+1) = 2^{k+1} \rightarrow 2^k \cdot 2^1 \rightarrow 2^k \cdot 2$

By inductive hypothesis: $k^2 < 2^k$

By Algebra: $2k + 1 < 2^k$ for $k \geq 5$

$\therefore (k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1 < 2^k + 2^k < 2 \cdot 2^k = 2^{k+1}$

$\therefore n^2 < 2n$ for all integers $n \geq 5$

Exercises

20) $2^n < (n+2)!$, for all integers $n \geq 0$.

Base case: $P(0) = 2^0 = 1 < (0+2)! = 2$

Inductive hypothesis: $P(k) = 2^k < (k+2)!$ For all $k \geq 0$

Inductive step: $P(k+1) = 2^{k+1} = 2^k \cdot 2$

Inductive step: $P(k+1) = (k+3)! = (k+3) \cdot (k+2)!$

By inductive hypothesis: $2^k < (k+2)!$

By algebra: $2 < k+3$ for all integers $k \geq 0$

$\therefore 2^{k+1} = 2^k \cdot 2 < (k+2)! \cdot (k+3)$ for all $k \geq 0$

$\therefore 2^n < (n+2)!$ For all integer $n \geq 0$

Exercises

24) A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all integers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$.

Exercises

We want to prove that if $a_k = 7a_{k-1}$ for all integers $k \geq 2$ then $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$

Base case: $P(k = 1) = a_1 = 3 \equiv P(n = 1) = a_1 = 3$

Inductive hypothesis (RHS): $P(k) = a_k = 3 \cdot 7^{k-1}$ is true.

Inductive step (RHS): $P(k + 1) = a_{k+1} = 3 \cdot 7^k$

Inductive step (LHS): $P(k + 1) = a_{k+1} = 7a_k \rightarrow 7\left(3 \cdot 7^{k-1}\right) \rightarrow 3 \cdot 7^k$

$\because P(k + 1)_{RHS} = 3 \cdot 7^k \equiv P(k + 1)_{\mathrm{LHS}} = 3 \cdot 7^k$

$\therefore$ if $a_1 = 3$ and $a_k = 7a_{k-1}$ for all integers $k \geq 2$. Then, $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$ is correct.

TASK

Section 5.3
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