CH 02: Basic Structures

Sets, Functions, Sequences, Sums, and Matrices

Content

CH 02
Sets
Set Operations
Functions
Sequences and Summations
Cardinality of Sets
Matrices

Sets

- **DEFINITION 1:** A set is an unordered collection of distinct objects, called elements or members of the set.
  - We write $a \in A$ to denote that a is an element of the set A.
  - The notation $a \notin A$ denotes that a is not an element of the set A.
- Roster method: the notation {a, b, c, d} represents the set with the four elements a, b, c, and d.
- Example: The set of vowels {a, e, i, o, u}
- Example: The set of odd positive integers less than 10: {1, 3, 5, 7, 9}

Sets

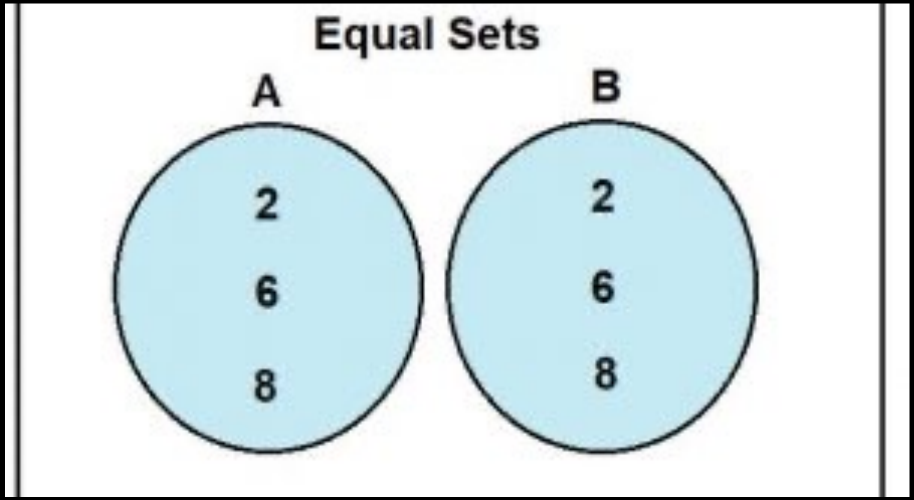
- Intervals: Sets of all the real numbers between two numbers a and b, with or without a and b.
    - $[a, b] = \{x \mid a \leq x \leq b\}$
    - $[a, b) = \{x \mid a \leq x < b\}$
    - $(a, b] = \{x \mid a < x \leq b\}$
    - $(a, b) = \{x \mid a < x < b\}$

Sets

- Important set names:
  - $N = \{0, 1, 2, 3, \ldots\}$, the set of all natural numbers
  - $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers
  - $Z^+ = \{1, 2, 3, \ldots\}$, the set of all positive integers
  - $Q = \{p / q \mid p \in Z, q \in Z, and\ q \neq 0\}$, the set of all rational numbers
  - R, the set of all real numbers
  - R^+, the set of all positive real numbers
  - C, the set of all complex numbers.

Sets

- **DEFINITION 2:** Two sets are equal if and only if they have the same elements.
- Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
- We write A = B if A and B are equal sets.

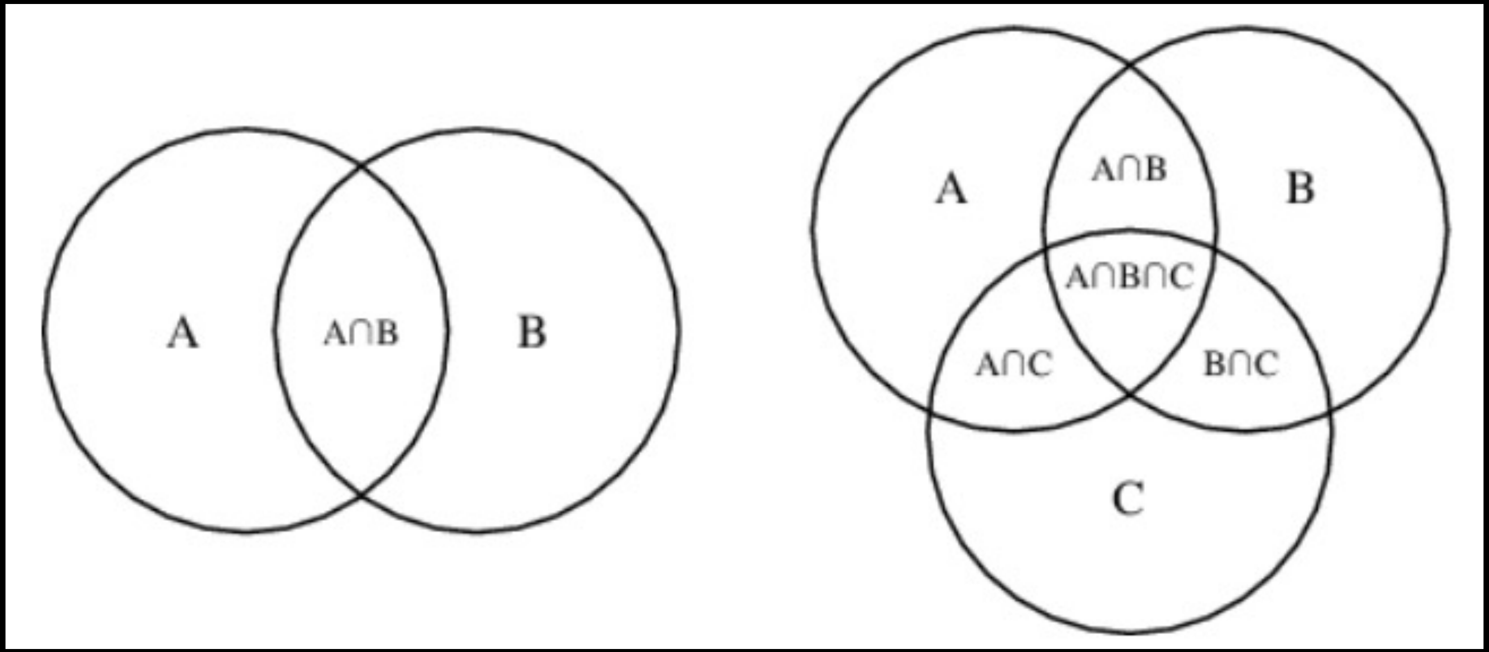


Sets

- The special set that has no elements is called the empty set, or null set, and is denoted by Ø.
    - The empty set can also be denoted by{ }
- A set with one element is called a singleton set.
    - Ø is not the same as {Ø}
    - {Ø} is a singleton set of an empty set (example, a folder with an empty folder)
    - Ø is an empty set (example, an empty folder)

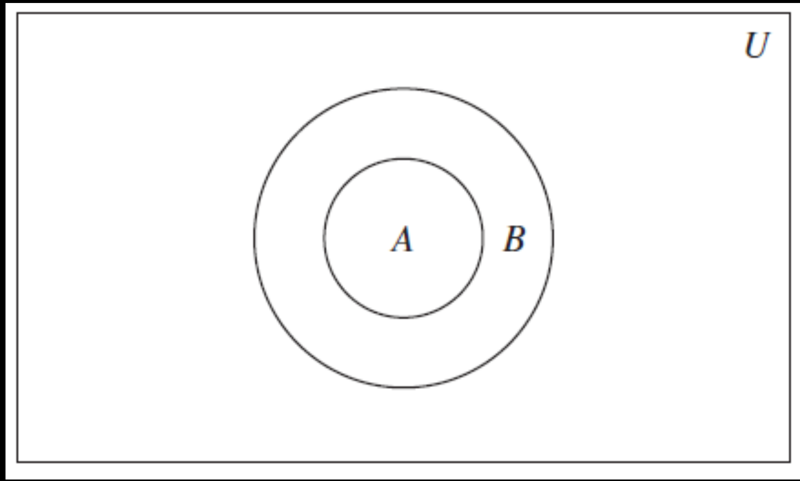
Sets

- Venn Diagram



Sets

- **DEFINITION 3:**
  The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B.
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- To stress that B is a superset of A, we use the equivalent notation $B \supseteq A$.
  - So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.



Sets

- **THEOREM 1:** For every set S,
  - $\emptyset \subseteq S$
  - $S \subseteq S$.
- To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

- Sets may have other sets as members.
  - For instance, $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Sets

- **DEFINITION 4:** The cardinality of a set S, $|S|$, is the number of distinct elements in S.

- Example: Let A be the set of odd positive integers less than 10. Then |A| = 5.

Sets

- **DEFINITION 6:**
  Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

- Example: What is the power set of the set $\{0, 1, 2\}$?
- Solution: $P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

Sets

- **DEFINITION 7:**
  The ordered tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,…, and a_n as its nth element.

Sets

- **DEFINITION 8:**
  Let A and B be sets. The cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$.

- Example: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
- Solution: $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Sets

- **DEFINITION 9:**
  The Cartesian product of the sets $A_1, A_2, \ldots, A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.
  - In other words, $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \, for \; i = 1, 2, \ldots, n\}$.

- Example: What is the Cartesian product A × B × C, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?
- Solution: $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

Exercises

9. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in R | \, x \text{ is an integer greater than } 1\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

Exercises

9. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in R | \, x \; is \; an \; integer \; greater \; than \; 1\}$ Yes

c) $\{2, \{2\}\}$ Yes

d) $\{\{2\}, \{\{2\}\}\}$ No

Exercises

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$

e) $\emptyset \subseteq \{x\}$

c) $\{x\} \in \{x\}$

Exercises

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$

True

e) $\emptyset \subseteq \{x\}$

True

c) $\{x\} \in \{x\}$

False

Exercises

21 - 22. What is the cardinality of each of these sets?

a) $\{a\}$ b) $\{\emptyset\}$ c) $\{\emptyset, \{\emptyset\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$

Exercises

21 - 22. What is the cardinality of each of these sets?

a) $\{a\}$

1

b) $\{\emptyset\}$

1

c) $\{\emptyset, \{\emptyset\}\}$

2

d) $\{a, \{a\}, \{a, \{a\}\}\}$

3

Exercises

23. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ b) $\{a, b\}$ c) $\{\emptyset, \{\emptyset\}\}$

Exercises

23. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$

$\boxed{\{\emptyset, \{a\}\}}$

b) $\{a, b\}$

$\boxed{\{\emptyset, \{a\}, \{b\}, \{a, b\}\}}$

c) $\{\emptyset, \{\emptyset\}\}$

$\boxed{\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}}$

Exercises

29. Let A = $\{a, b, c, d\}$ and $B = \{y, z\}$. Find

b) B × A.

Exercises

29. Let A = $\{a, b, c, d\}$ and $B = \{y, z\}$. Find

b) B × A.

$$\boxed{\begin{array}{l} \{ \\ (y, a), (y, b), (y, c), (y, d), \\ (z, a), (z, b), (z, c), (z, d) \\ \} \end{array}}$$

Exercises

34. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. b) $C \times B \times A$. c) $C \times A \times B$. d) $B \times B \times B$.

Exercises

34. Let A = {a, b, c}, B = {x, y}, and C = {0, 1}. Find

a) A × B × C. b) C × B × A. c) C × A × B. d) B × B × B.

a) $\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1),$
$(b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$

b) $\{(0, x, a), (0, x, b), (0, x, c), (0, y, a),$
$(0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$

c) $\{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x),$
$(0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$

d) $\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$

Exercises

35. Find A^2 if

a) A = {0, 1, 3}.

Exercises

35. Find A^2 if

a) A = {0, 1, 3}.

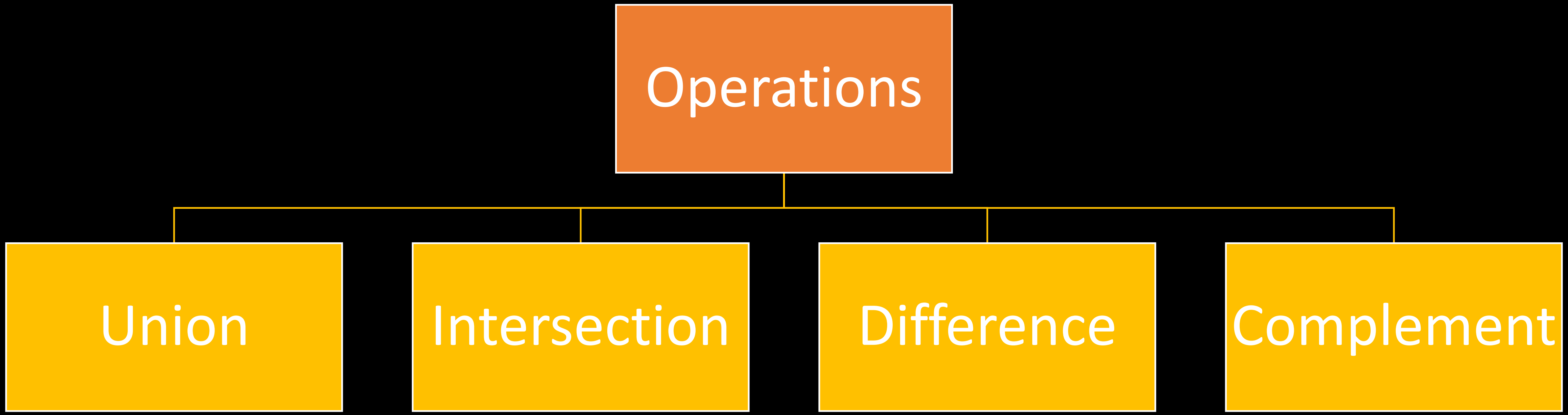
$$\boxed{\begin{array}{l}\{\\ (1,0),(1,1),(1,3),(0,0),(0,1),(0,3),\\ \\ (3,0),(3,1),(3,3)\\ \}\end{array}}$$

TASKS

Section 1
9 (e, b, f)
13 (a, e, c)
21 (a, d)
22 (b, c)
29 (a)
35 (b)

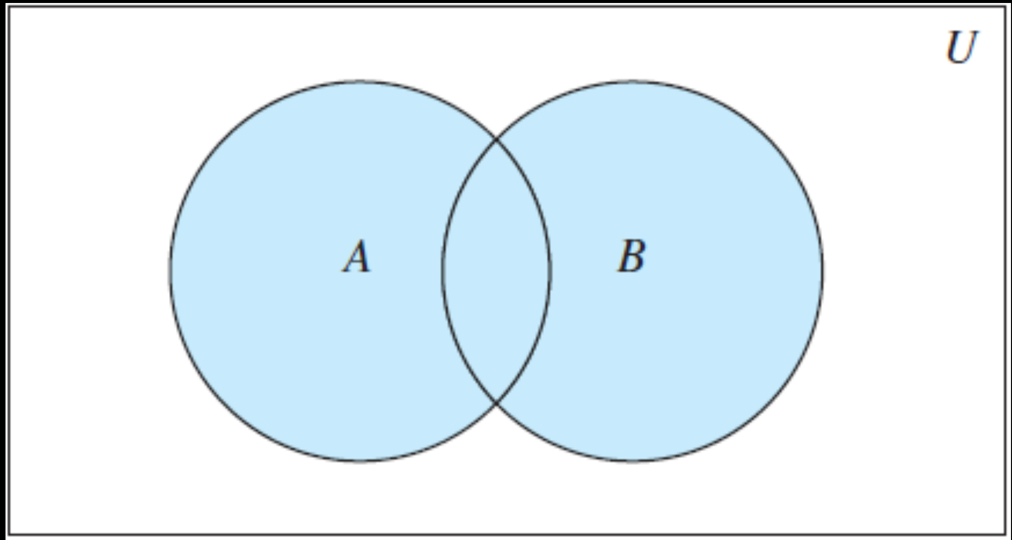
Content

Set Operations



Set Operations

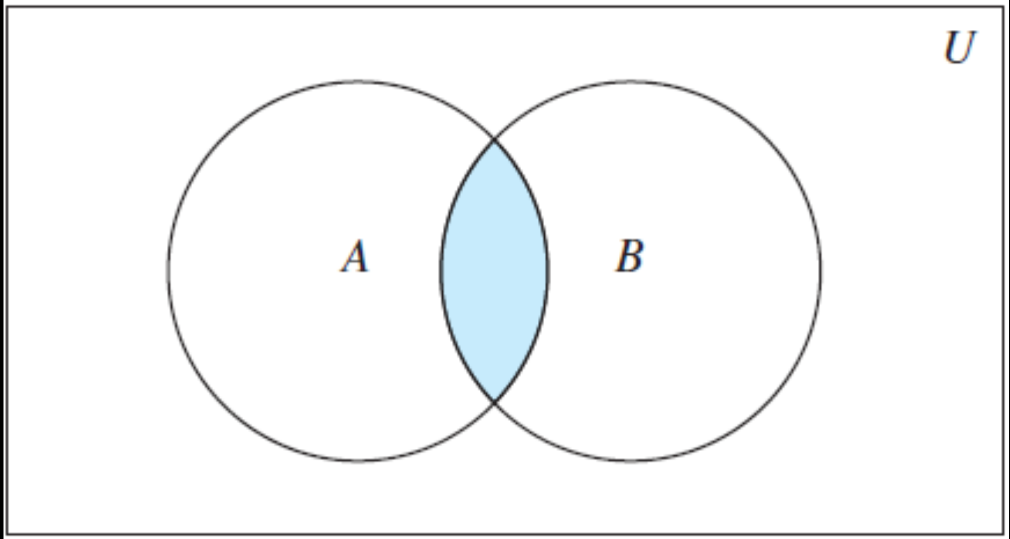
- Union:
  Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$.



$A \cup B$ is shaded.

Set Operations

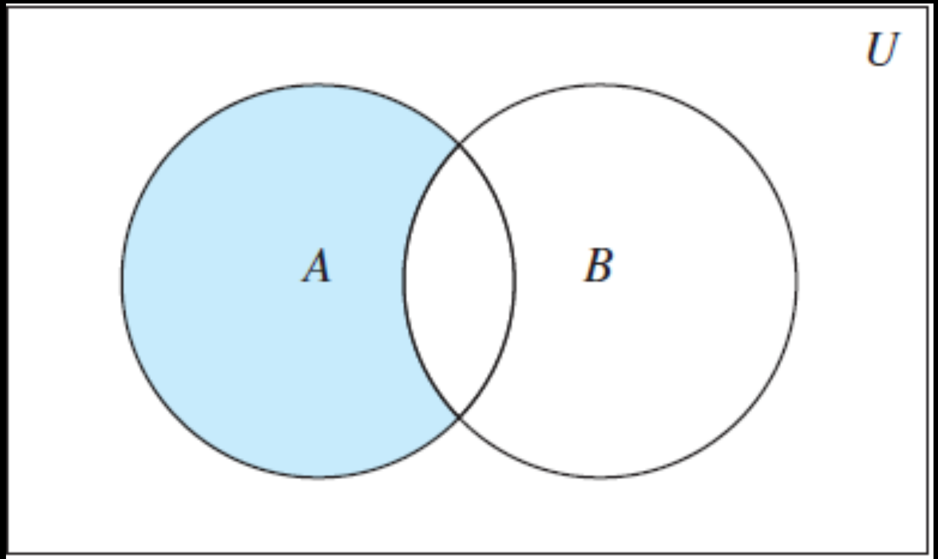
- Intersection:
  Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
  - $A \cap B = \{x \mid x \in A \land x \in B\}$.



$A \cap B$ is shaded.

Set Operations

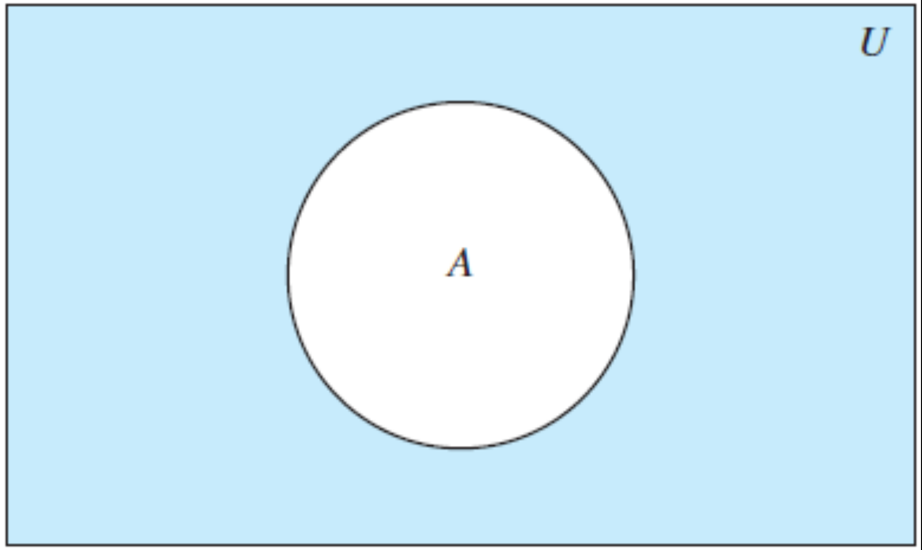
- Difference:
  Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.
  - The difference of A and B is also called the complement of B with respect to A..
  - $A - B = \{x \mid x \in A \wedge x \notin B\}$.



$A - B$ is shaded.

Set Operations

- Complement:
  Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.
  - Therefore, the complement of the set A is U − A.
  - $\bar{A} = \{x \in U \mid x \notin A\}$.



$\bar{A}$ is shaded.

Set Operations

- **DEFINITION 3:**
  Two sets are called disjoint if their intersection is the empty set.
- Example: Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8, 10}. Because A ∩ B = ∅, A and B are disjoint.

- To find the cardinality of a union of two finite sets A and B.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Set Operations

- Set Identities:

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Operations

- Use a membership table to show that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Set Operations

- Generalized Union:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- Generalized Intersection:

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Exercises

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.

Exercises

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.

a) $\{0,1,2,3,4,5,6\}$
b) $\{3\}$
c) $\{1, 2, 4,5\}$
d) $\{0, 6\}$

Exercises

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

Exercises

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

a)

$$x \in \overline{A \cap B \cap C} \equiv$$
$$x \notin A \cap B \cap C \equiv$$
$$x \notin A \wedge x \notin B \wedge x \notin C \equiv$$
$$x \in \bar{A} \vee x \in \bar{B} \vee x \in \bar{C} \equiv$$
$$x \in \bar{A} \cup \bar{B} \cup \bar{C}$$

Exercises

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

Exercises

53. Let $A_i = \{1, 2, 3, \ldots, i\}$ $for\ i = 1, 2, 3, \ldots$. Find

a)$\bigcup_{i=1}^{n} A_i$

Exercises

53. Let $A_i = \{1, 2, 3, \ldots, i\}\ for\ i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$

a) $\{1, 2, 3, \ldots, n\}$

Exercises

54. Let $A_i = \{\dots, -2, -1, 0, 1, \dots, i\}$. Find

b) $\bigcap_{i=1}^{n} A_i$

Exercises

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

b) $\bigcap_{i=1}^{n} A_i$

We note that these sets are increasing, that is, $A1 \sqsubseteq A2 \sqsubseteq A3$. The intersection is just the one with the smallest subscript.

$$A_1 = \{\ldots, -2, -1, 0, 1\}$$

TASKS

Section 2
4
53 (b)
54 (a)