Limits and Derivatives

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The Tangent and Velocity Problems
The Limit of a Function
Calculating Limits Using the Limit Laws
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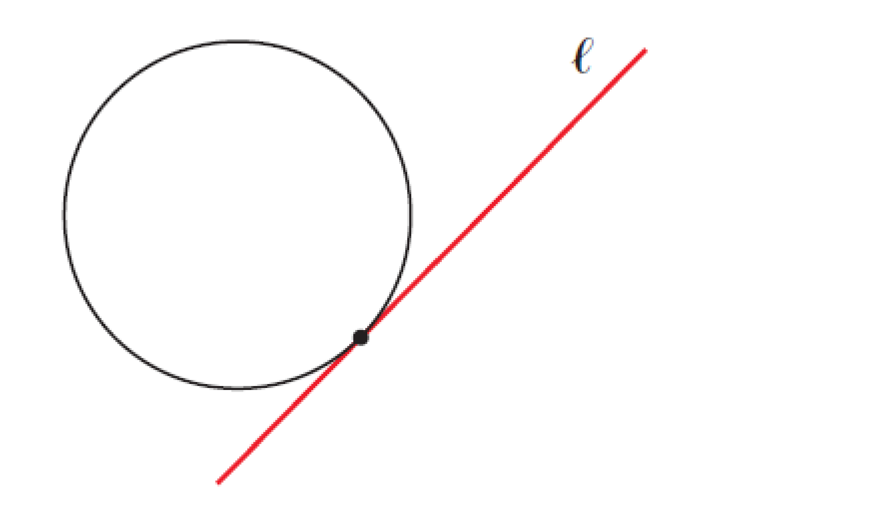
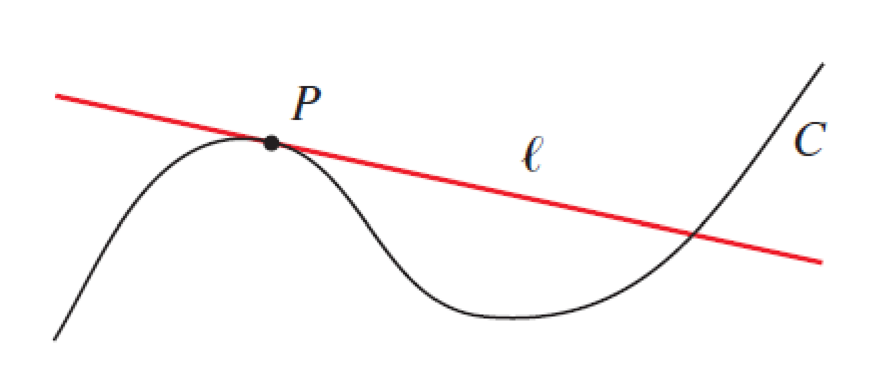
The Tangent and Velocity Problems

- A **tangent** is a line that touches the curve and follows the same direction as the curve at the point of contact.



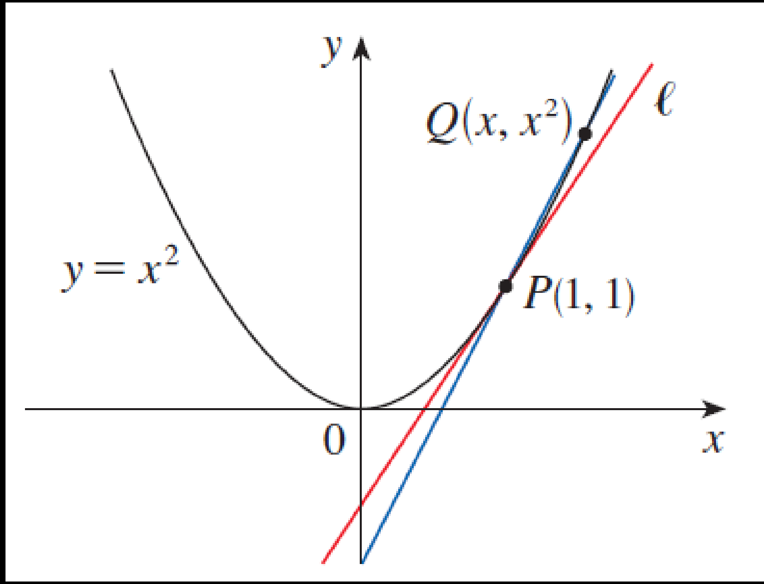
(a)



(b)

The Tangent and Velocity Problems

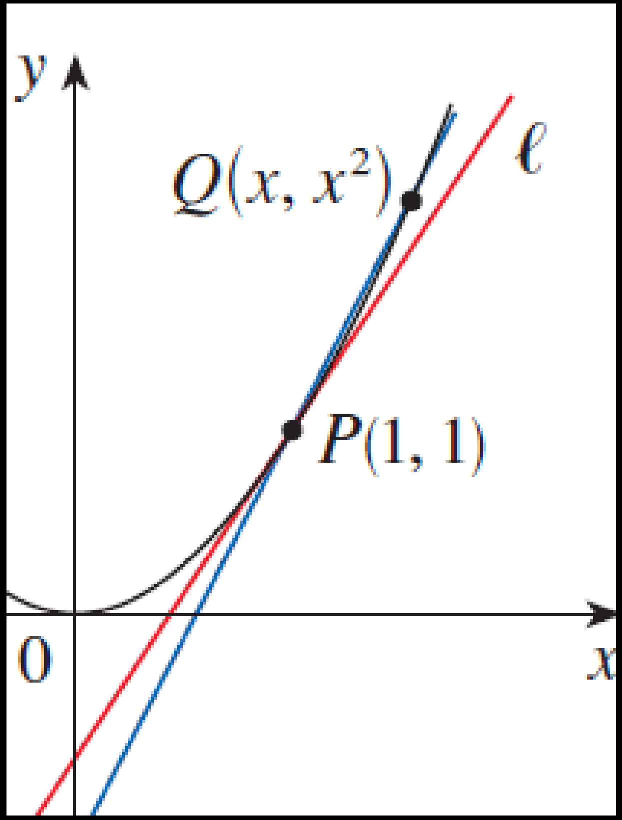
Example: Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.



- To find the equation of l, we to find the slope of l.
- To find the slope of l, we need two points. But we have only one point!

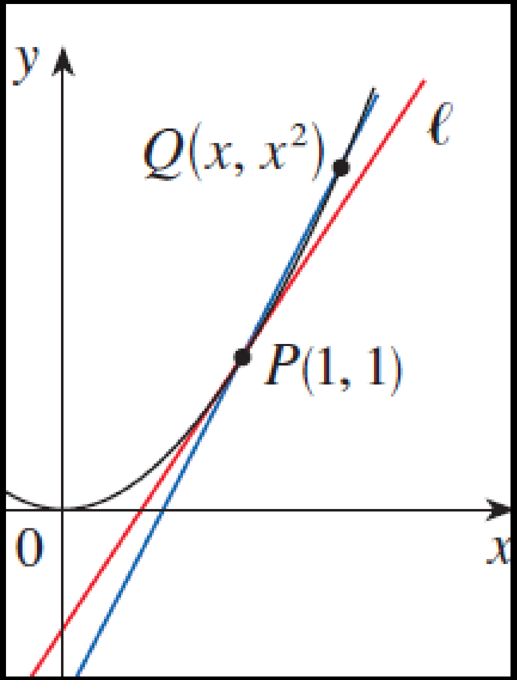
The Tangent and Velocity Problems

- A solution is to compute an approximation to m by choosing a nearby point $Q(x, x^2)$, then compute the slope m_{PQ} of the secant line PQ.



The Tangent and Velocity Problems

- A solution is to compute an approximation to m by choosing a nearby point $Q(x, x^2)$, then compute the slope m_{PQ} of the secant line PQ.
  1. We choose $x \neq 1$, $m_{PQ} = \frac{x^2-1}{x-1}$

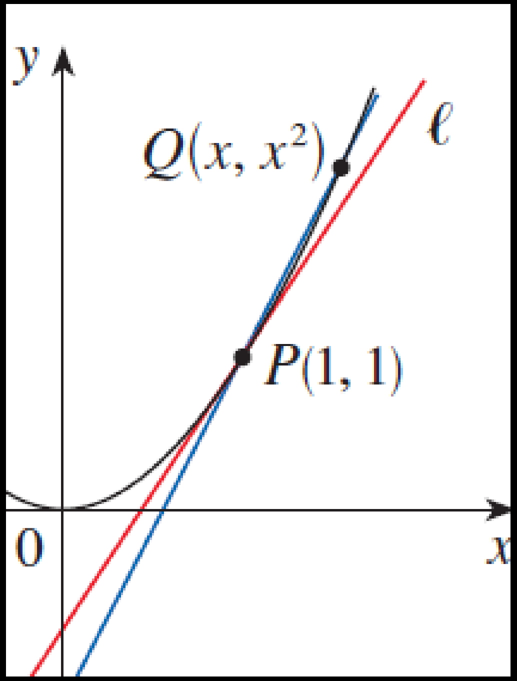


The Tangent and Velocity Problems

- A solution is to compute an approximation to m by choosing a nearby point $Q(x, x^2)$, then compute the slope m_{PQ} of the secant line PQ.
  1. We choose $x \neq 1,\ m_{PQ} = \frac{x^2-1}{x-1}$
  2. Trying different values for x close to 1, we get m_{PQ} close to 2.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001



The Tangent and Velocity Problems

- The closer Q to P, the closer x to 1.
- Hence, the closer m_{PQ} is to 2.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	m_{PQ}
2	3
1.5	2.5
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1.01	2.01
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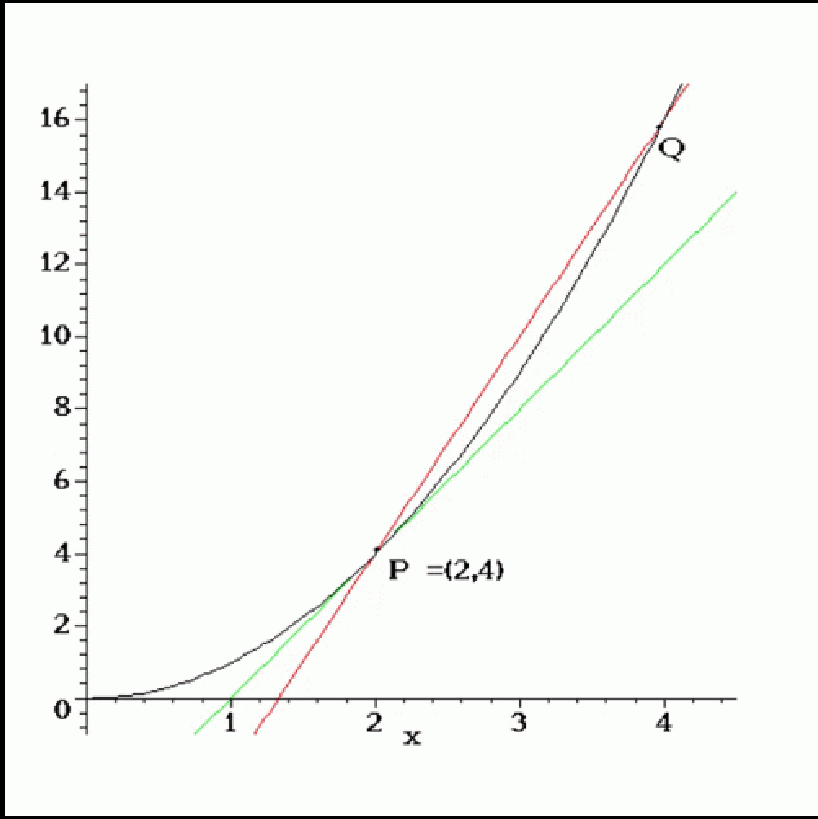


The Tangent and Velocity Problems

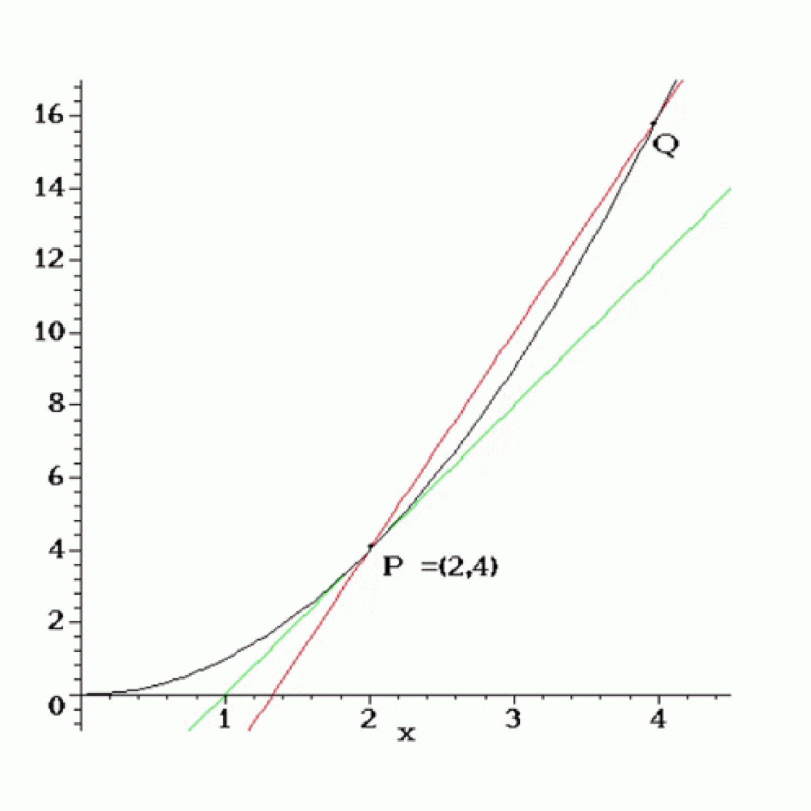
- The closer Q to P, the closer x to 1.
- Hence, the closer m_{PQ} is to 2.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

We express this process as *limits:*

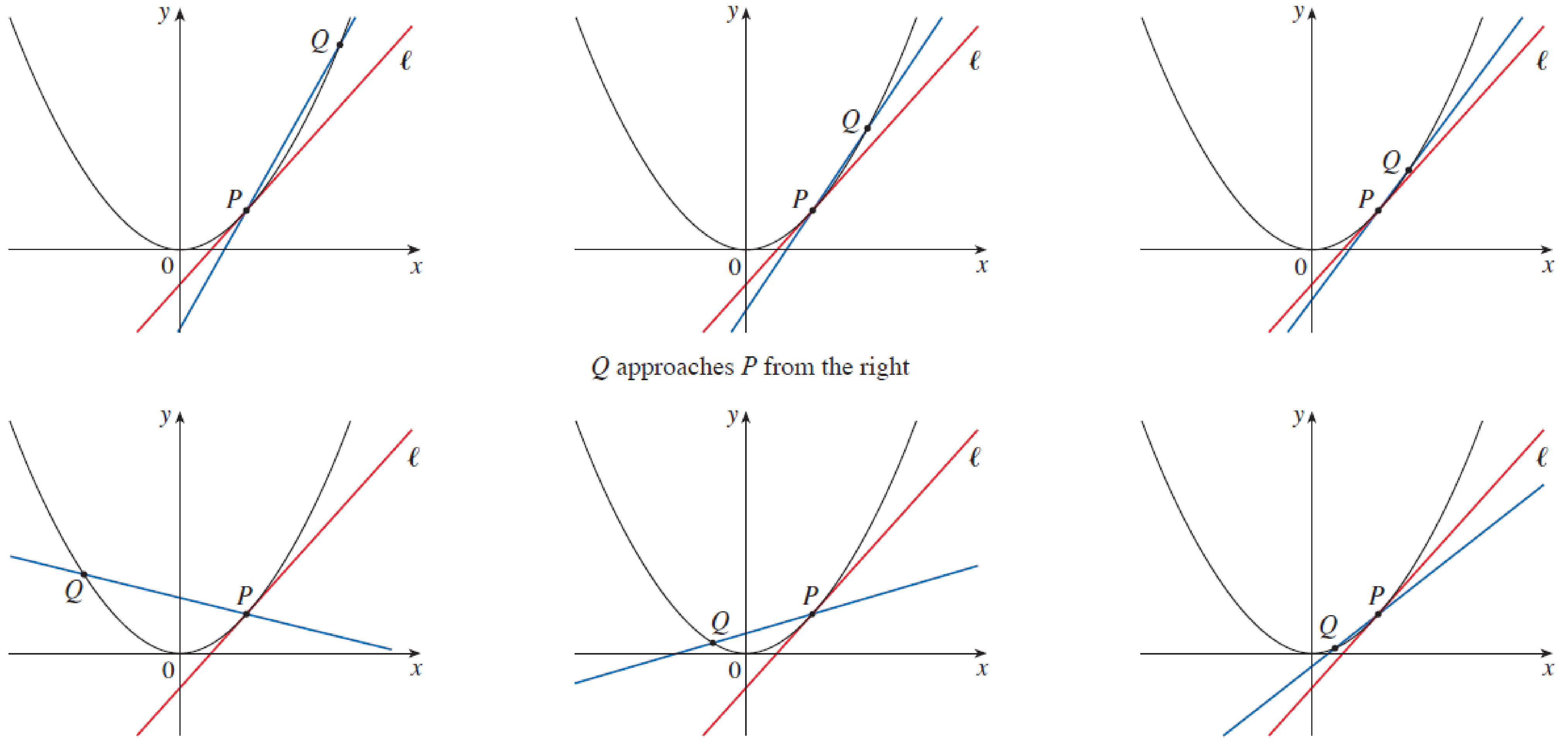
$$\lim_{Q \to P} m_{PQ} = m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



The Tangent and Velocity Problems

- Assuming that the slope is 2, we get the line equation using
$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$$

The Tangent and Velocity Problems



Q approaches P from the right

Q approaches P from the left

Exercises

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.

t (min)	5	10	15	20	25	30
V (L)	694	444	250	111	28	0

(a) If P is the point $(15, 250)$ on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of V to estimate the slope of the tangent line at . (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

Exercises

(a) Using $P(15,250)$,

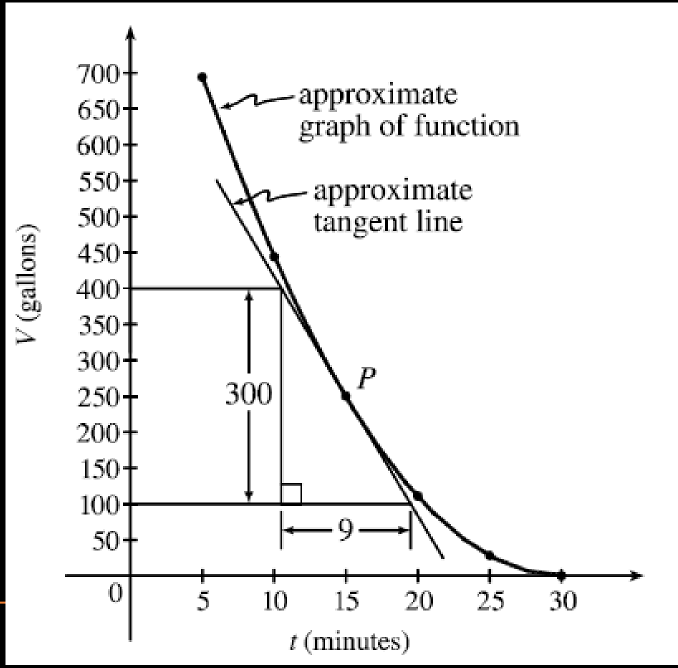
t	Q	Slope = m_{PQ}
5	(5, 694)	$\frac{694-250}{5-15} = -44.4$
10	(10, 444)	$\frac{444\ \ -250}{10-15} = -38.8$
20	(20, 111)	$\frac{(111-250)}{20-15} = -27.8$
25	(25, 28)	$\frac{28-250}{25-15} = -22.2$
30	(30, 0)	$\frac{0-250}{30-15} = 16.6$

Exercises

(b) Using the values of t that correspond to the points closest to P (t = 10 and t = 20), we have

$$\frac{(-38.8) + (-27.8)}{2} = -33.3$$

(c) we can estimate the slope of the tangent line at P to be $-\frac{300}{9} = -33.3$



Exercises

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.

(a) If Q is the point $(x, 1/(1 - x))$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5	(ii) 1.9	(iii) 1.99	(iv) 1.999
(v) 2.5	(vi) 2.1	(vii) 2.01	(viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

Exercises

(a) $\mathrm{P}(2,-1), \mathrm{Q}(\mathrm{x}, 1/(1-\mathrm{x}))$

(b) The slope appears to be 1.

(c) Using $\mathrm{m} = 1$, an equation of the tangent line to the curve at $\mathrm{P}\ (2,-1)$ is
$y-(-1) = 1(x-2) \Rightarrow y = x-3$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	(1.5, −2)	2
(ii)	1.9	(1.9, −1.111 111)	1.111 111
(iii)	1.99	(1.99, −1.010 101)	1.010 101
(iv)	1.999	(1.999, −1.001 001)	1.001 001
(v)	2.5	(2.5, −0.666 667)	0.666 667
(vi)	2.1	(2.1, −0.909 091)	0.909 091
(vii)	2.01	(2.01, −0.990 099)	0.990 099
(viii)	2.001	(2.001, −0.999 001)	0.999 001

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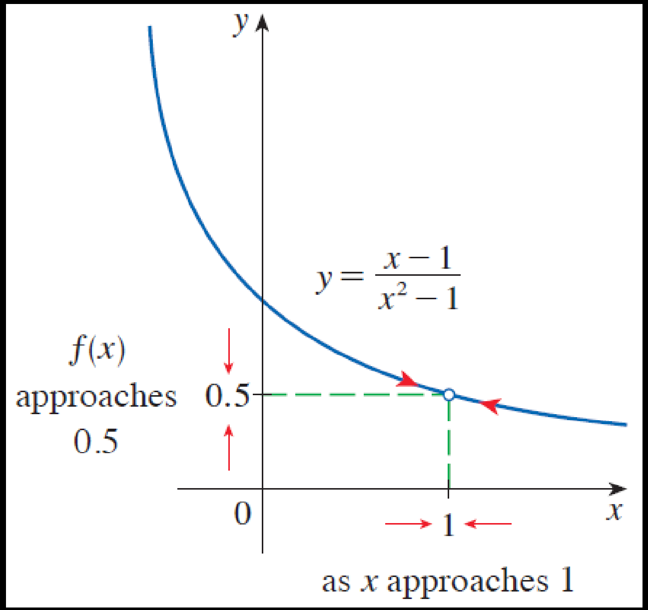
The Limit of a Function

- Given the equation:

$$\lim_{x \to a} f(x) = L$$

"The values of $f(x)$ is getting closer to the number L as x gets closer to the number a (from both sides of a) but $x \neq a$."



The Limit of a Function

- Example 1: Estimate the value of $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$
- Solution: The table lists values of the function for several values of t near 0.

So, $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$

t	$\frac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . .

The Limit of a Function

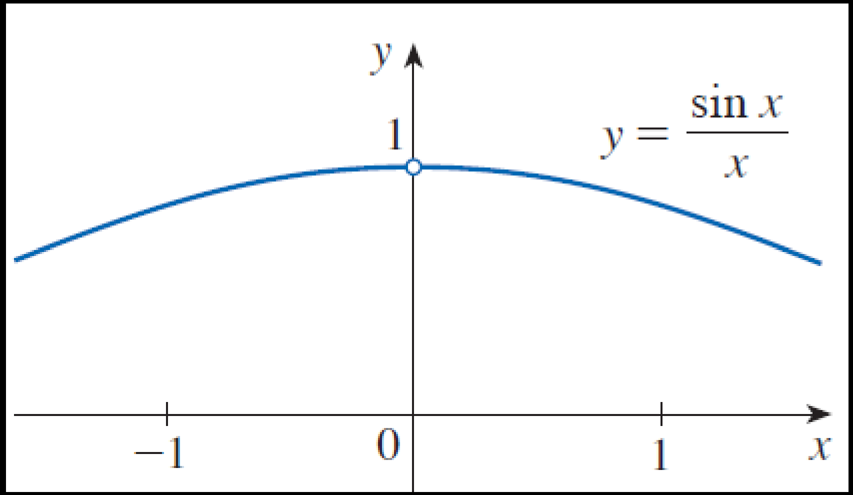
Read page 84 to 85 and page 88 to 89 for why calculators can give false results about limits.

The Limit of a Function

- Example 2: Guess the value of $\lim_{x \to 0} \frac{\sin(x)}{x}$ (x is the angle in *radian*)
- Solution: construct a table for values of x close to 0.

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

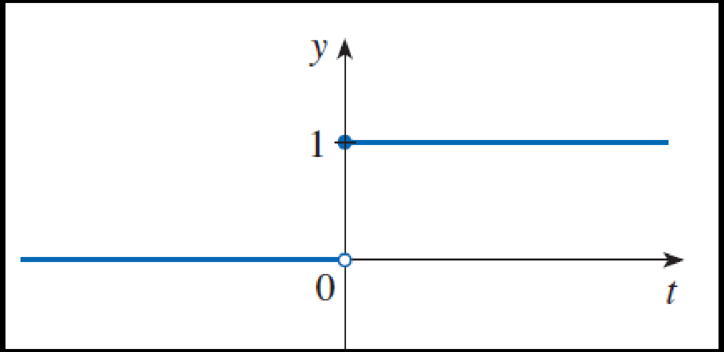
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$



The Limit of a Function

- Limits can be described for Heaviside functions

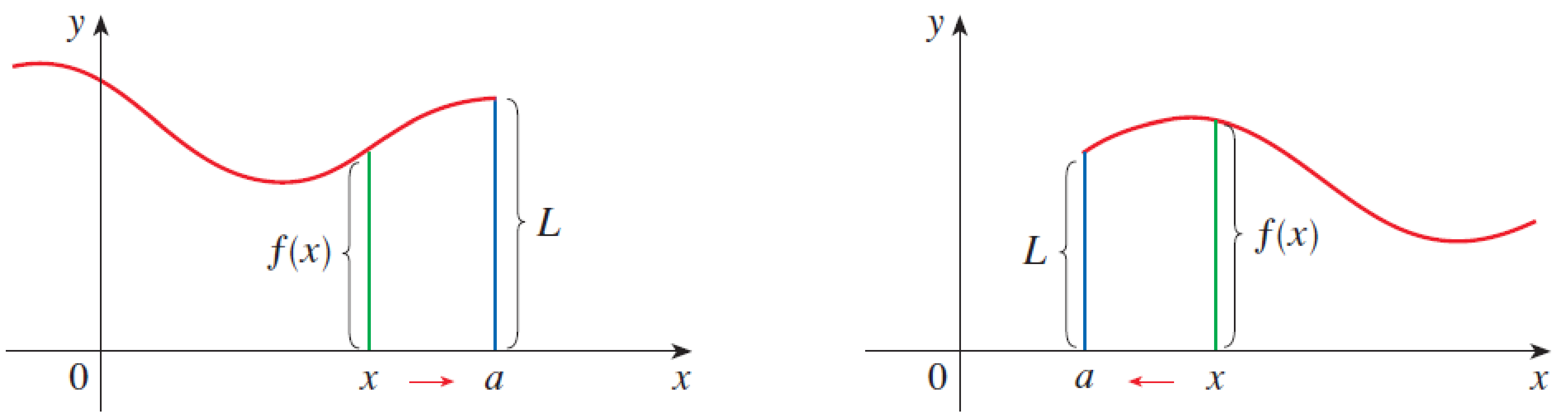
$$H(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ t \geq 0 \end{cases}$$



- There is no single number that $H(t)$ approaches as t approaches 0, so $\lim_{t\to 0} H(t)$ does not exist. However,
  - As t approaches 0 from the left, $H(t)$ approaches 0: $\lim_{t\to 0^+} H(t) = 1$
  - As t approaches 0 from the right, $H(t)$ approaches 1: $\lim_{t\to 0^-} H(t) = 0$

The Limit of a Function

- For instance, the notation $x \to 5^-$ means that we consider only $x < 5$, and $x \to 5^+$ means that we consider only $x > 5$.



(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

The Limit of a Function

- Example 4: Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$

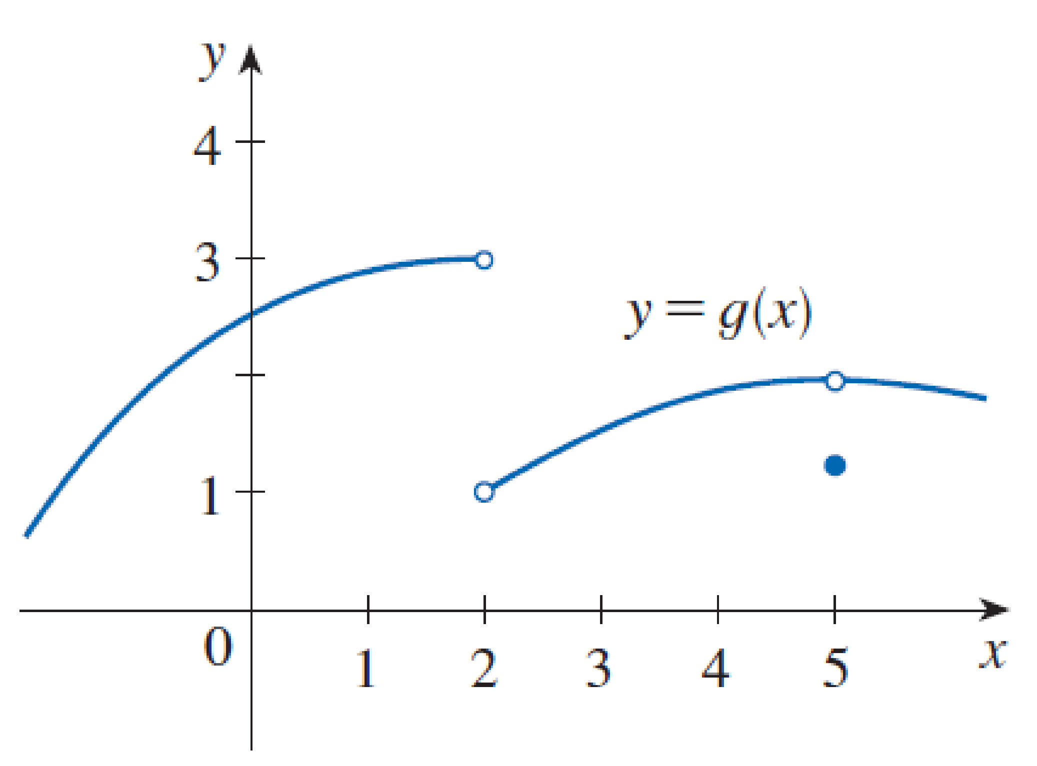
(b) $\lim_{x \to 2^+} g(x)$

(c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$

(e) $\lim_{x \to 5^+} g(x)$

(f) $\lim_{x \to 5} g(x)$



The Limit of a Function

- Example 4: Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ = 3

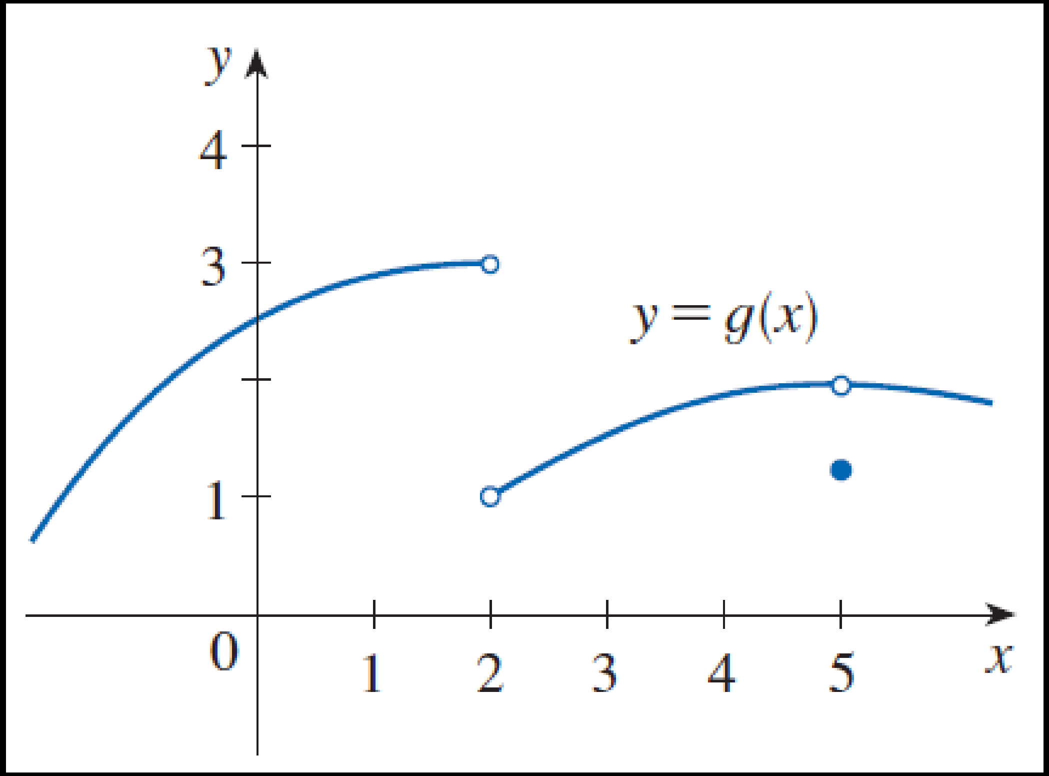
(b) $\lim_{x \to 2^+} g(x)$ = 1

(c) $\lim_{x \to 2} g(x)$ = doesn't exist

(d) $\lim_{x \to 5^-} g(x)$ = 2

(e) $\lim_{x \to 5^+} g(x)$ = 2

(f) $\lim_{x \to 5} g(x)$ = 2



Exercises

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$

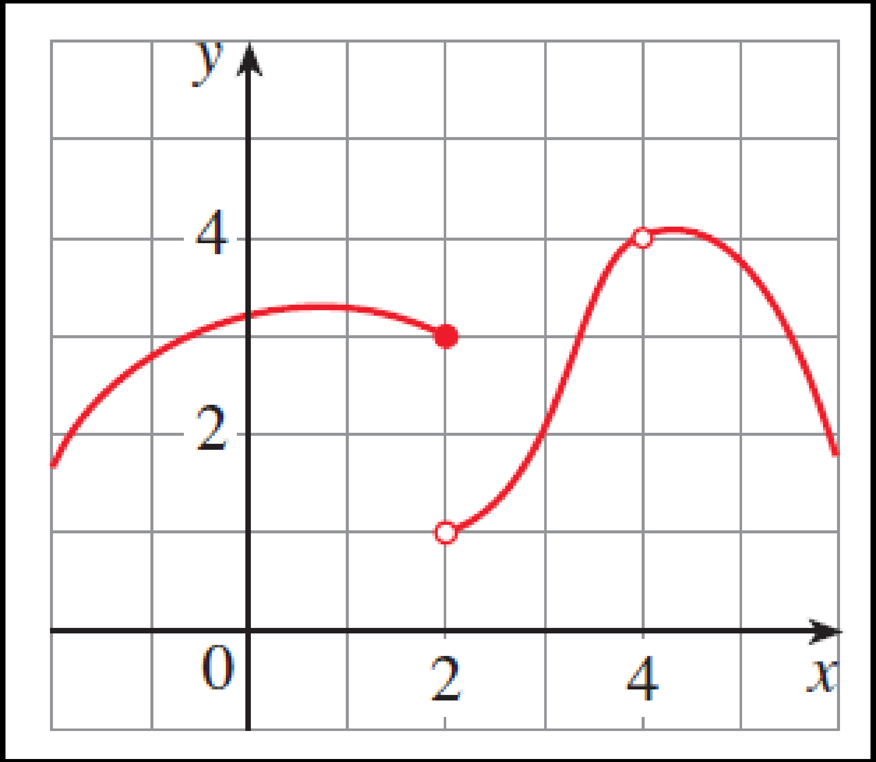
(b) $\lim_{(x \to 2^+)} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x \to 4} f(x)$

(f) $f(4)$



Exercises

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 2^-} f(x)$ $\boxed{3}$

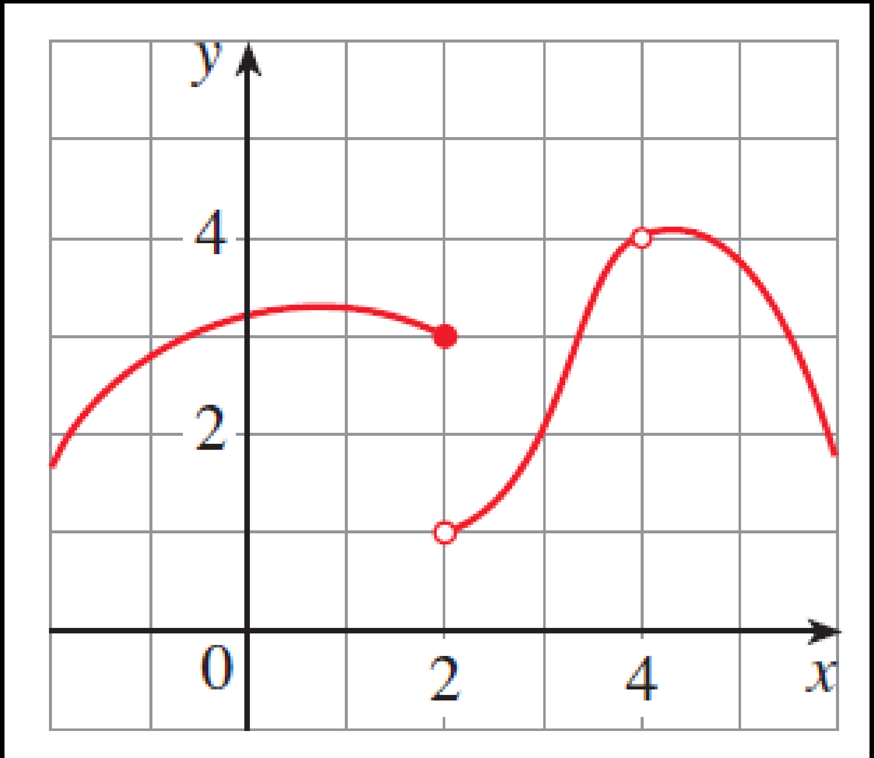
(b) $\lim_{(x\to 2^+)} f(x)$ $\boxed{1}$

(c) $\lim_{x\to 2} f(x)$ $\boxed{\text{Doesn't exist}}$

(d) $f(2)$ $\boxed{3}$

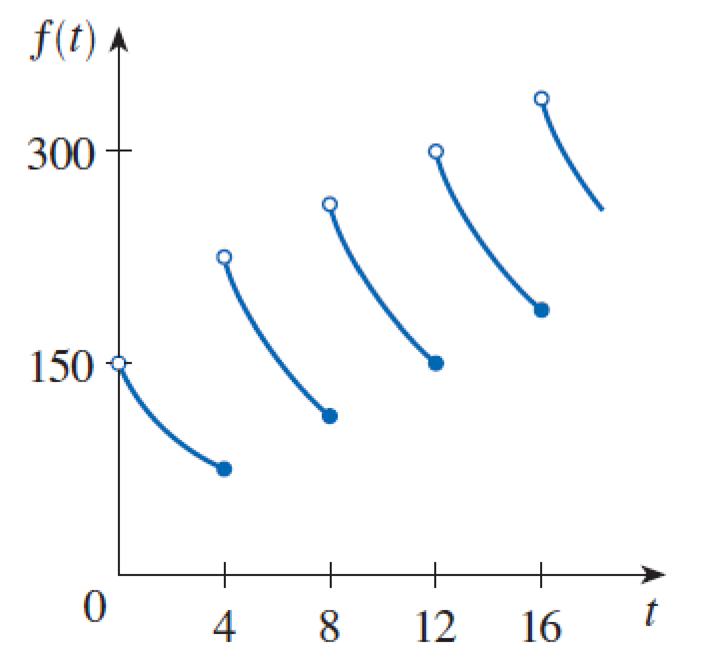
(e) $\lim_{x\to 4} f(x)$ $\boxed{4}$

(f) $f(4)$ $\boxed{\text{Doesn't exist}}$



Exercises

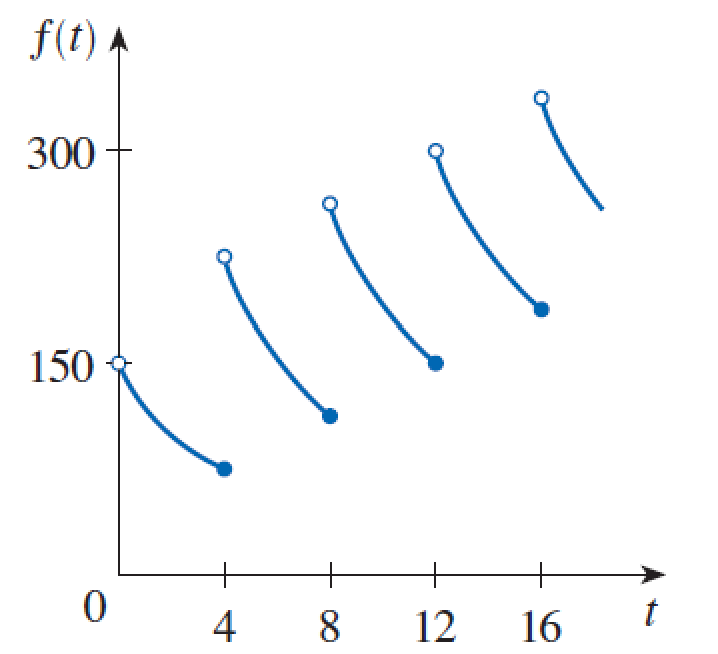
10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the Bloodstream after t hours. Find $\lim_{t \to 12^-} f(t)$ and $\lim_{t \to 12^+} f(t)$ and explain the significance of these one-sided limits.



Exercises

10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the Bloodstream after t hours. Find $\lim_{t\to 12^-} f(t)$ and $\lim_{t\to 12+} f(t)$ and explain the significance of these one-sided limits.

$$\lim_{t\to 12^-} f(t) = 150\, mg$$
$$\lim_{t\to 12+} f(t) = 300$$



These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h.

The lefthand limit represents the amount of the drug just before the fourth injection. The righthand limit represents the amount of the drug just after the fourth injection.

Exercises

Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19) $\lim\limits_{x \to 3} \frac{x^2 - 3x}{x^2 - 9}$ $\quad x = 3.1, 3.05, 3.01, 3.001, 3.001, 2.9, 2.95, 2.99, 2.999, 2.9999$

22) $\lim\limits_{h \to 0} \frac{(2+h)^5 - 32}{h}$ $\quad x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, 0.0001$

Exercises

19) $\lim_{x \to 3} \frac{x^2-3x}{x^2-9}$ $x = 3.1, 3.05, 3.01, 3.001, 3.001, 2.9, 2.95, 2.99, 2.999, 2.9999$

x	$f(x)$
3.1	0.508 197
3.05	0.504 132
3.01	0.500 832
3.001	0.500 083
3.0001	0.500 008

x	$f(x)$
2.9	0.491 525
2.95	0.495 798
2.99	0.499 165
2.999	0.499 917
2.9999	0.499 992

$$\lim_{x \to 3} \frac{x^2-3x}{x^2-9} = \frac{1}{2}$$

Exercises

22) $\lim_{h\to 0} \frac{(2+h)^5-32}{h}$ $\quad x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, 0.0001$

h	$f(h)$
0.5	131.312500
0.1	88.410100
0.01	80.804010
0.001	80.080040
0.0001	80.008000

h	$f(h)$
−0.5	48.812500
−0.1	72.390100
−0.01	79.203990
−0.001	79.920040
−0.0001	79.992000

$$\lim_{h\to 0} \frac{(2+h)^5-32}{h} = 80$$

Exercises

Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

25) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$

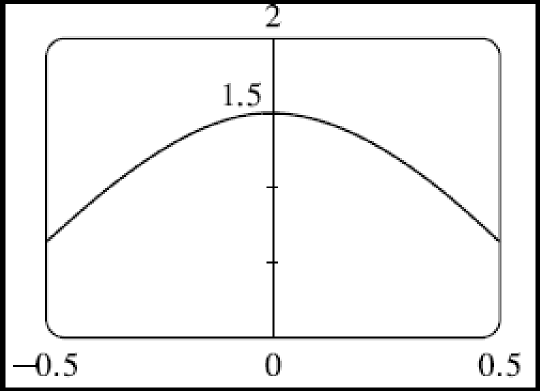
27) $\lim_{x \to 0^+} x^x$

Exercises

Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

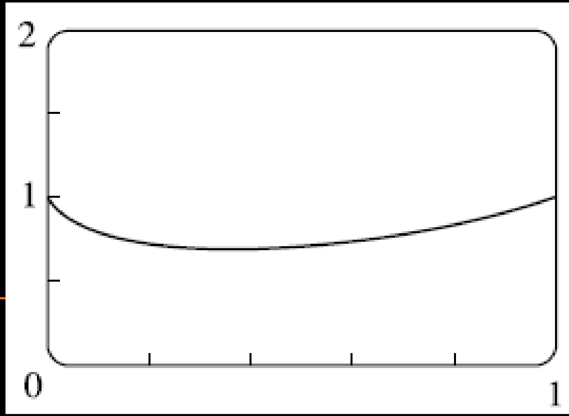
25) $\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan 2\theta} = 1.5$

θ	$f(\theta)$
± 0.1	1.457 847
± 0.01	1.499 575
± 0.001	1.499 996
± 0.0001	1.500 000



27) $\lim_{x\to 0^+} x^x = 1$

x	$f(x)$
0.1	0.794 328
0.01	0.954 993
0.001	0.993 116
0.0001	0.999 079



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Calculating Limits Using the Limit Laws

- Properties of limits

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

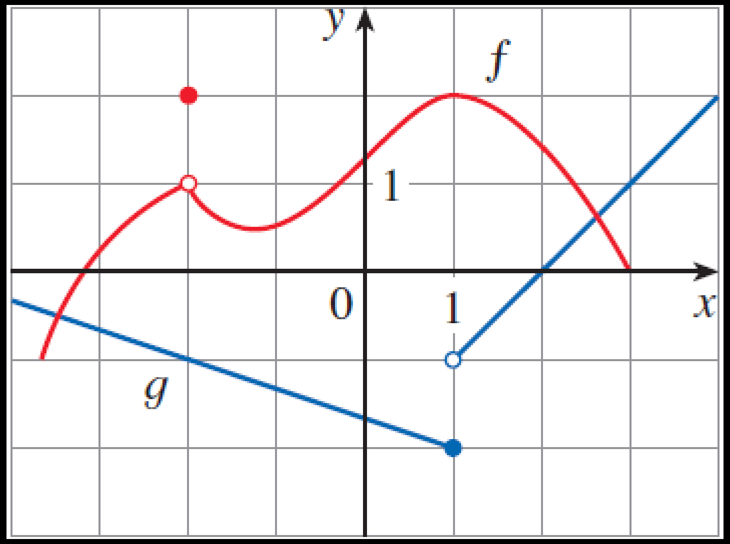
4. $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

- Example 1: Use the Limit Laws and the graphs of f and t to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$



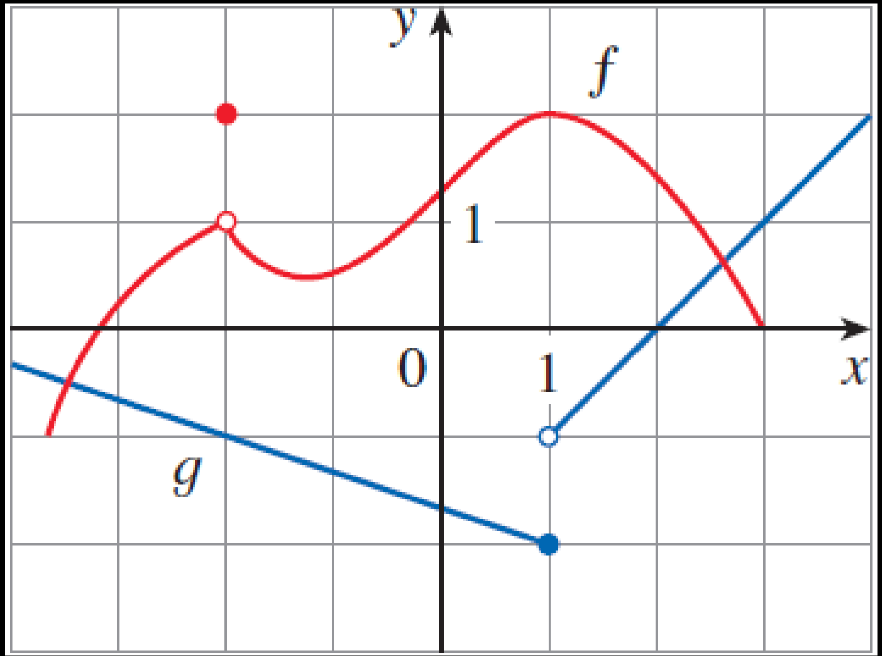
Calculating Limits Using the Limit Laws

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$

$$\lim_{x \to -2} f(x) = 1$$

$$\lim_{x \to -2} g(x) = -1$$

$\therefore \lim_{x \to -2} [f(x) + 5g(x)] = 1 + 5 * (-1) = -4$



Calculating Limits Using the Limit Laws

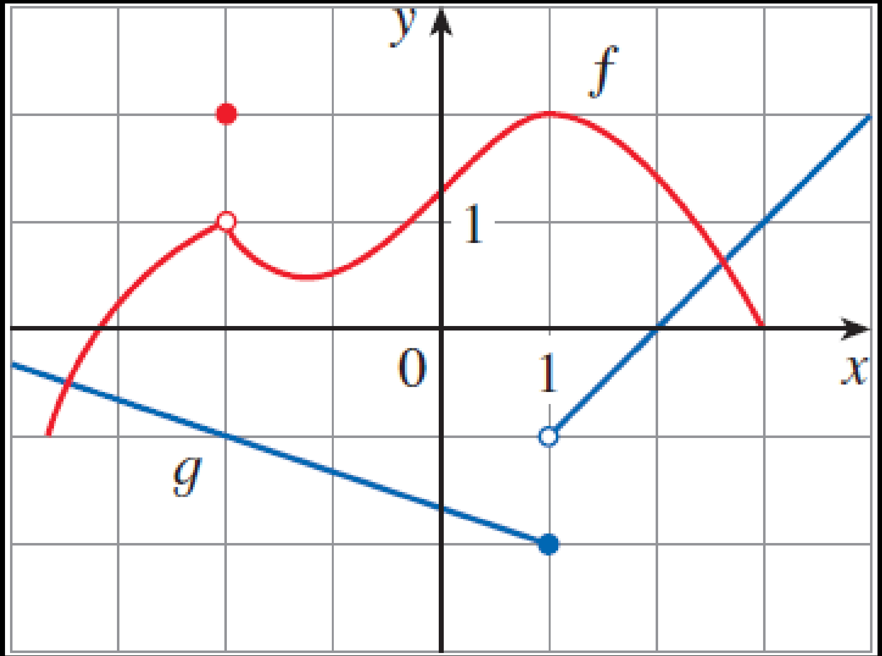
(b) $\lim_{x \to 1} [f(x)g(x)]$

$$\lim_{x \to 1} f(x) = 2$$

$$\lim_{x \to 1^-} g(x) = -2$$
$$\lim_{x \to 1^+} g(x) = -1$$

$$\therefore \lim_{x \to 1^-} [f(x)g(x)] = 2 * -2 = -4$$
$$\therefore \lim_{x \to 1^+} [f(x)g(x)] = 2 * -1 = -2$$

The left and right limits are not equal, so it doesn't exist.



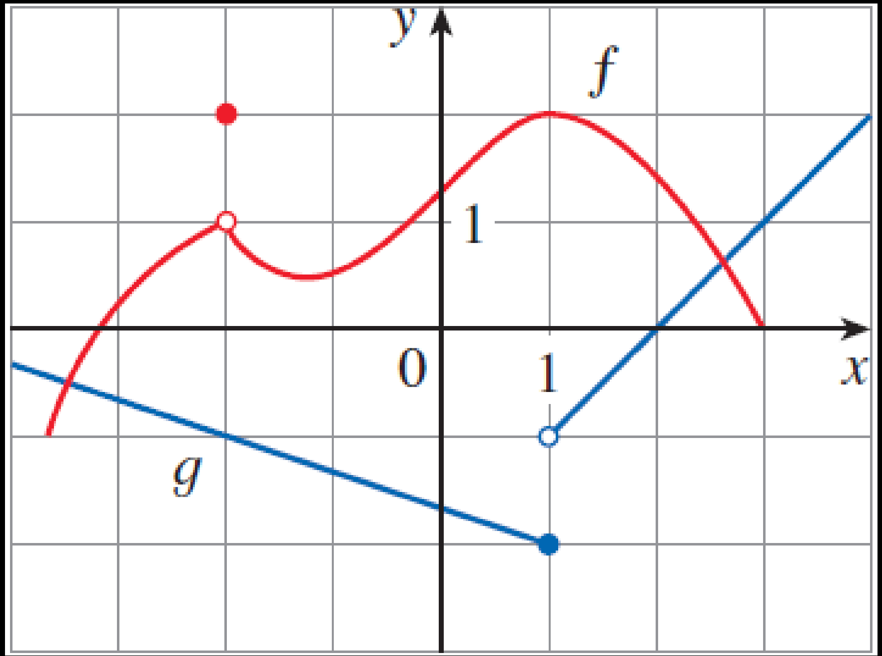
Calculating Limits Using the Limit Laws

(c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

$$\lim_{x \to 2} f(x) = 1.4$$

$$\lim_{x \to 2} g(x) = 0$$

The denominator approaches 0, so the limit doesn't exist.



Calculating Limits Using the Limit Laws

- Other properties

Power Law

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

Root Law

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

8. $\lim_{x\to a} c = c$

9. $\lim_{x\to a} x = a$

10. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

Calculating Limits Using the Limit Laws

- Limits can be evaluated by direct substitution

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Calculating Limits Using the Limit Laws

- Example 2: Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Calculating Limits Using the Limit Laws

- Example 2: Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

$$\lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4 =$$
$$2 * 5^2 - 3 * 5 + 4 = 39$$

$$\lim_{x \to 2} \frac{-2^3 + 2 * -2^2 - 1}{5 - 3 * -2} =$$
$$-\frac{1}{11}$$

Calculating Limits Using the Limit Laws

- Functions that can be solved by direct substitution are called **continuous at a**.
- However, not all limits can be evaluated initially by direct substitution.

Calculating Limits Using the Limit Laws

- Example 3:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

- We cannot find the limit by setting $x = 1$, because $f(1)$ isn't defined. Instead, we need to do some preliminary algebra.
  - *This is a <u>limit</u>, so we aren't concerned with $x = 1$, but an approximate value*

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = x + 1 = 1 + 1 = 2$$

Calculating Limits Using the Limit Laws

- Example 4: Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x+1 & if\ x \neq 1 \\ \pi & if\ x = 1 \end{cases}$$

The value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = 1 + 1 = 2$$

Calculating Limits Using the Limit Laws

- Some limits are best calculated by first finding the left- and right-hand limits.

$\boxed{1}$ **Theorem** $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Calculating Limits Using the Limit Laws

- Example 7: Show that $\lim_{x \to 0} |x| = 0$

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

$\therefore \lim_{x - 0^+} |x| = 0$ and $\lim_{x \to 0^-} |x| = 0$

The limit is defined from both sides, so $\lim_{x \to 0} |x| = 0$

Calculating Limits Using the Limit Laws

- Example 8: Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

$$\lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{-x}{x} = -1$$

The left and right limits aren't equal, so the limit does not exist.

Content
The Tangent and Velocity Problems
The Limit of a Function
Calculating Limits Using the Limit Laws
Continuity

Continuity

- A continuous process is one that takes place without interruption.

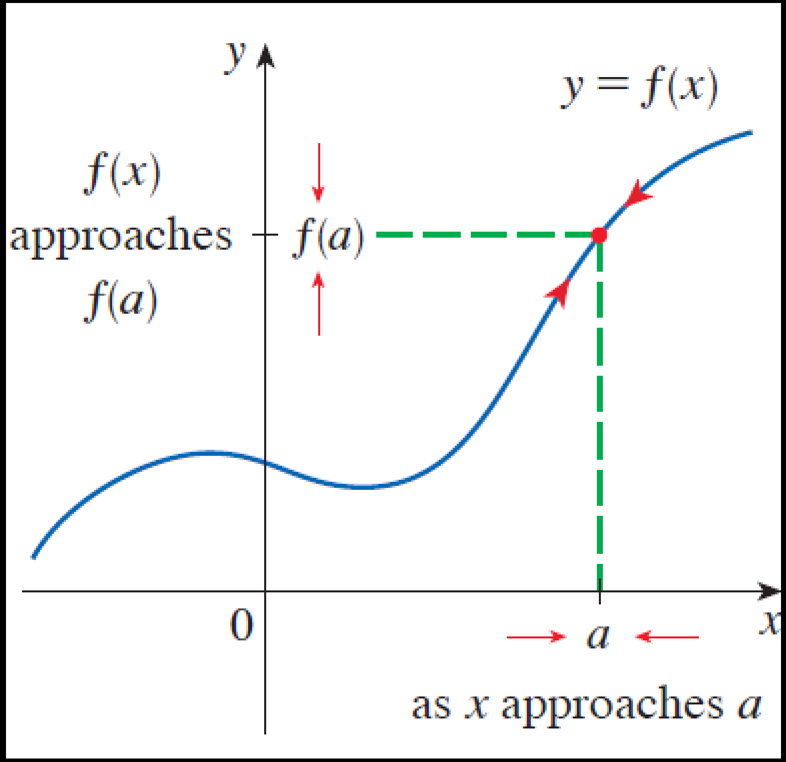
1 Definitio A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

- The definition implicitly requires three things if f is continuous at a:
  1. $f(a)$ is defined (that is, a is in the domain of f)
  2. $\lim_{x \to a} f(x)$ exists
  3. $\lim_{x - a} f(x) = f(a)$

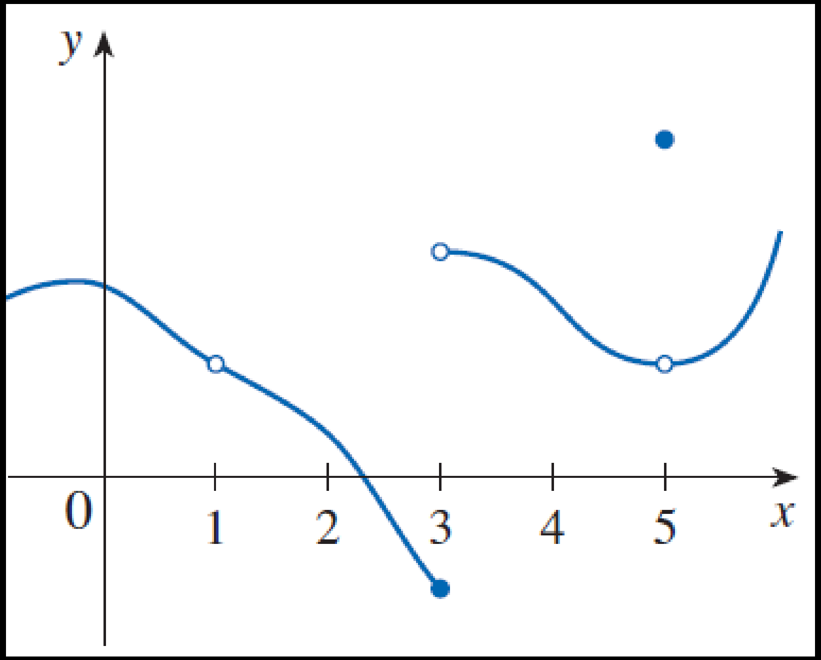
Continuity

- A continuous function f has the property that a small change in x produces only a small change in $f(x)$.



Continuity

- We say that f is **discontinuous** at a (or f has a discontinuity at a) if f is not continuous at a.



Continuity

- Example 2: Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$

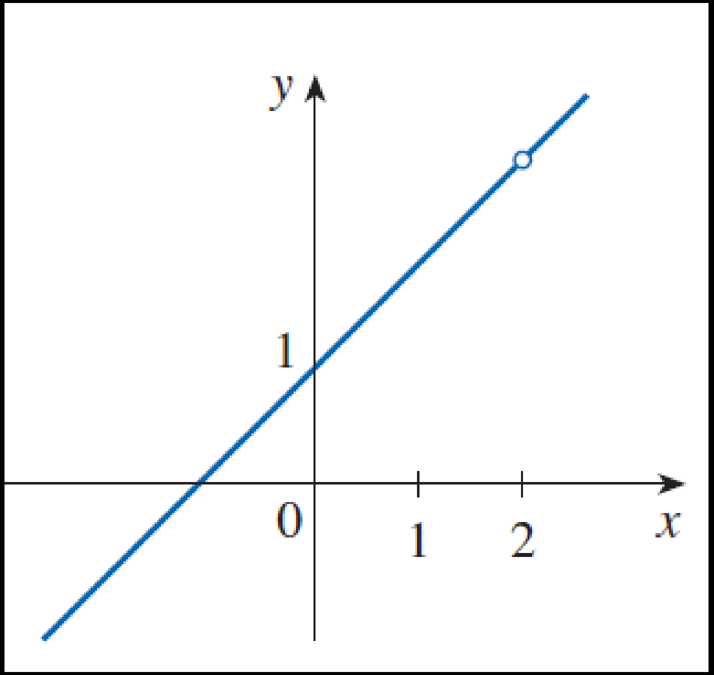
(c) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$

(d) $f(x) = \lfloor x \rfloor$

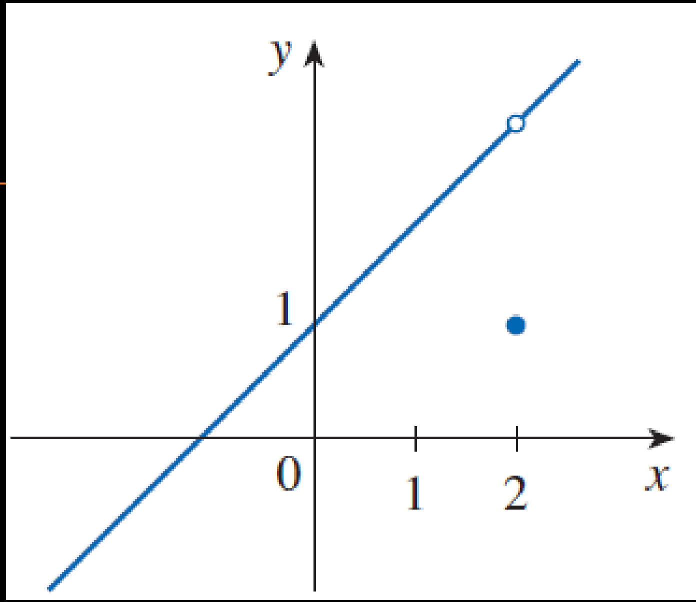
Continuity

(a) $f(x) = \frac{x^2 - x - 2}{x - 2}$

f is discontinuous at $x = 2$



Continuity



(b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$

Here $f(2) = 1$, but

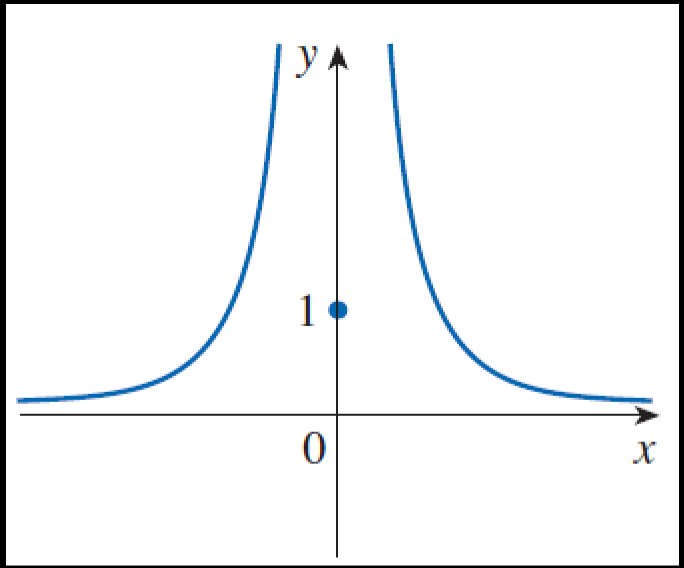
$$\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2} = 2 + 1 = 3$$

So, $f(2) \neq \lim_{x\to 2} f(x)$.

$\therefore f$ is not continuous at 2.

Continuity

(c) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$



Here $f(0) = 1$, but

$$\lim_{x \to 0} \frac{1}{x^2}$$

does not exist

So, f is discontinuous at 0.

Continuity

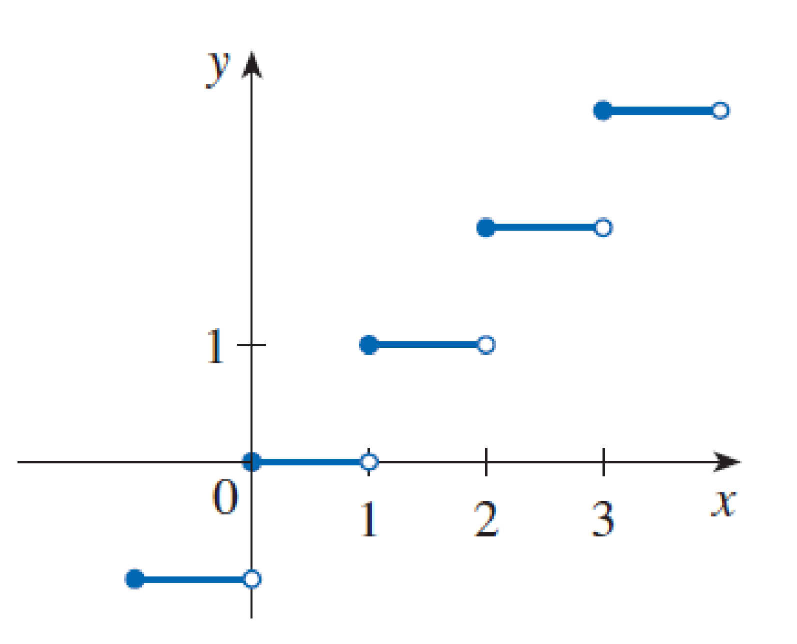
(d) $f(x) = \lfloor x \rfloor$, this is the greatest integer (or floor) function.

- The function rounds the real number down to the integer less than the number.
  - $\lfloor 1.15 \rfloor = 1$
  - $\lfloor 4.556 \rfloor = 4$
  - $\lfloor 50 \rfloor = 50$
  - $\lfloor -3.01 \rfloor = -4$

Continuity

(d) $f(x) = \lfloor x \rfloor$

The function has discontinuities at all of the integers because $\lim_{x \to n} f(x) = \lfloor x \rfloor$ does not exist if n is an integer.



Continuity

2 Definitio A function f is **continuous from the right at a number a** if

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x\to a^-} f(x) = f(a)$$

- Example 3: At each integer n, the function $f(x) = \lfloor x \rfloor$ is continuous from the right but discontinuous from the left because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} \lfloor x \rfloor = n = f(x)$$

but

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} \lfloor x \rfloor = n - 1 \neq f(x)$$

Continuity

3 **Definitio** A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

- Example 4: Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval$[-1, 1]$.

If $-1 < a < 1$, then using the limit laws:

$$\lim_{x \to a} f(x) = \lim_{x \to a} 1 - \sqrt{1 - x^2} = 1 - \lim_{x \to a} \sqrt{1 - x^2} = 1 - \sqrt{1 - a^2} = f(a)$$

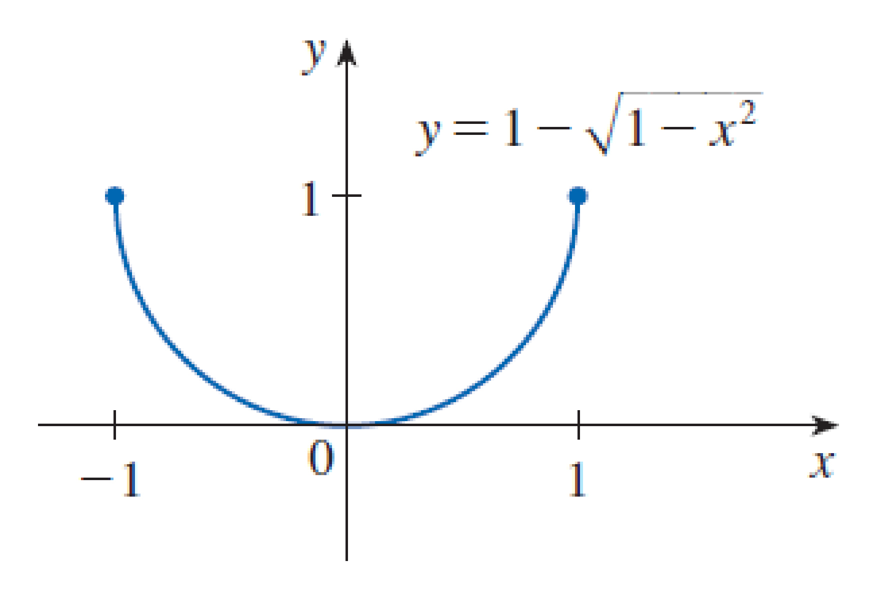
Continuity

- Thus, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1)$$

And

$$\lim_{x \to 1^-} f(x) = 1 = f(1)$$



Continuity

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

$\boxed{5}$ **Theorem**

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Continuity

- Example 5: Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Continuity

- Example 5: Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$

The function is rational, so it is continuous on the domain $\{x \mid x \neq \frac{5}{3}\}$.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2 * (-2)^2 - 1}{5 - 3 * -2} = -\frac{1}{11}$$

Continuity

- Most of the familiar functions are continuous at every number in their domains.

$\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Continuity

- Example 7: Evaluate $\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$

Continuity

- Example 7: Evaluate $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$

Based on Theorem 7, this function is continuous. Therefore,

$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \frac{\sin \pi}{2+\cos \pi} = \frac{0}{2-1} = 0$$

Continuity

- Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$.

$\boxed{8}$ **Theorem** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Continuity

- Example 8: Evaluate $\lim_{x \to 1} \arcsin(\frac{1-\sqrt{x}}{1-x})$

Continuity

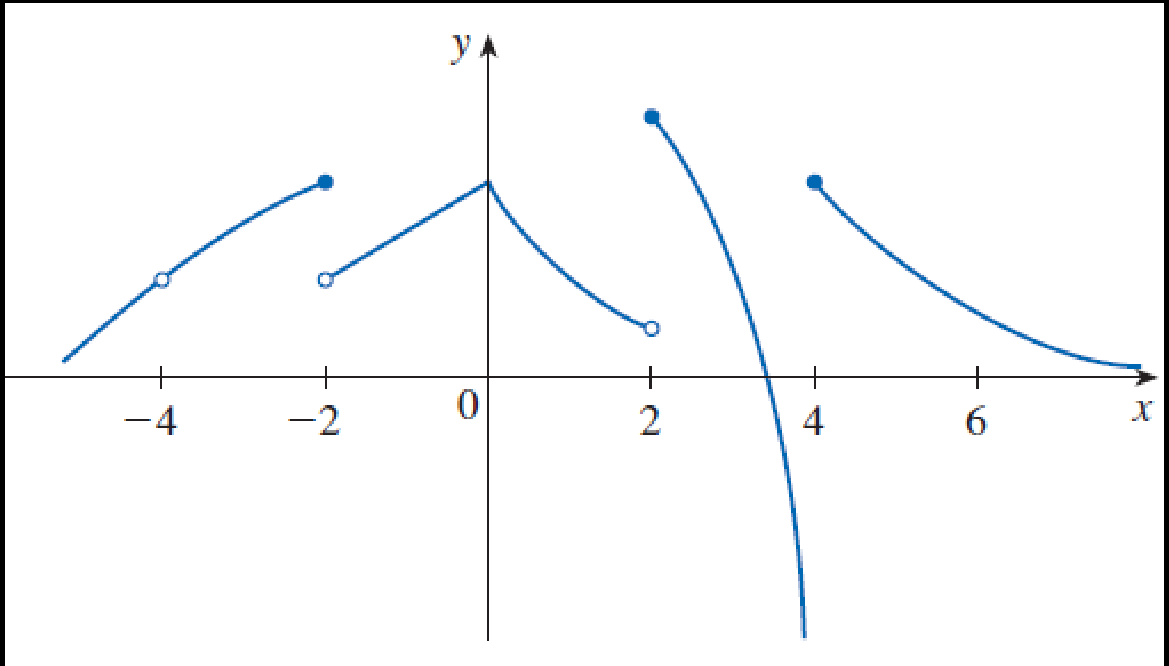
- Example 8: Evaluate $\lim_{x \to 1} \arcsin(\frac{1-\sqrt{x}}{1-x})$

$$\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right)$$
$$= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) = \arcsin\left(\lim_{x \to 1} \frac{1}{1+\sqrt{x}}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Exercises

3. (a) From the given graph of f, state the numbers at which f is discontinuous and explain why.

(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



Exercises

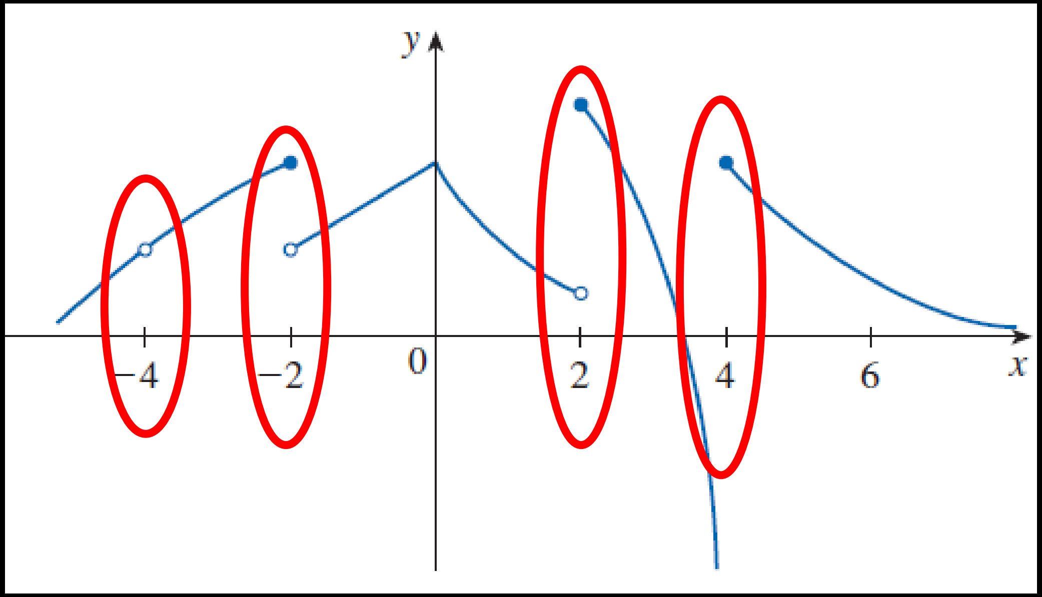
3. (a) From the given graph of f, state the numbers at which f is discontinuous and explain why.

At -4, because $f(-4)$ is undefined.

At -2, because the limit does not exist – left and right limits are different.

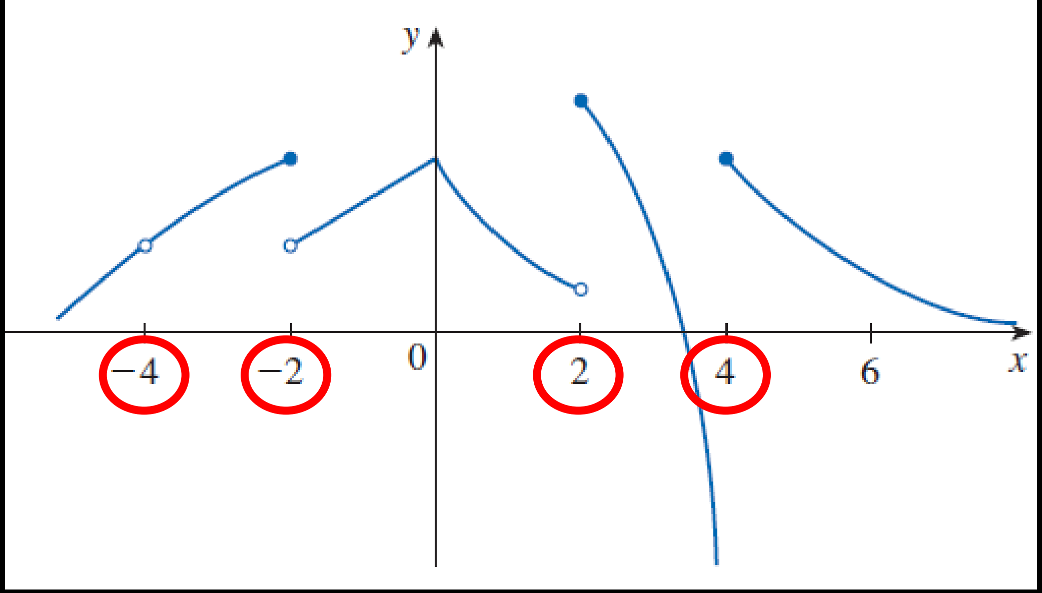
At 2, because the limit does not exist – left and right limits are different.

At 4, because the limit does not exist – left and right limits are different.



Exercises

(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



At -4, the function is not continuous from either side since $f(-4)$ is undefined.

At -2, the function is continuous from the left since $\lim_{x\to -2^-} f(x) = f(-2)$

At 2, the function is continuous from the right since $\lim_{x\to 2^+} f(x) = f(2)$

At 4, the function is continuous from the right since $\lim_{x\to 4^+} f(x) = f(4)$

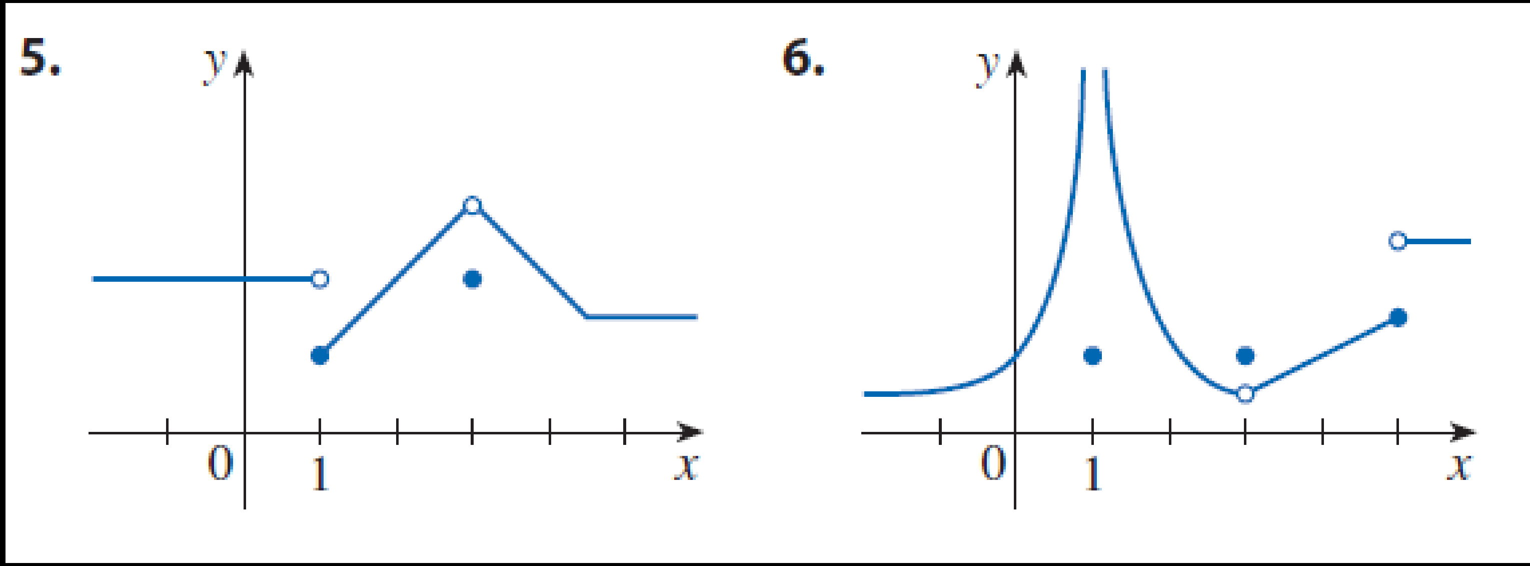
Exercises

5– 6 The graph of a function f is given.

(a) At what numbers a does $\lim_{x \to a} f(x)$ not exist?

(b) At what numbers a is f not continuous?

(c) At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?



Exercises

(a) At what numbers a does $\lim_{x\to a} f(x)$ not exist?

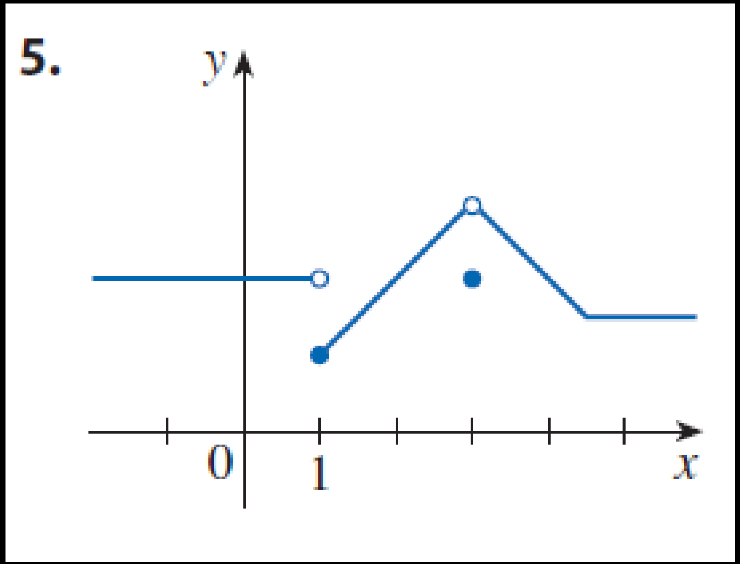
At $x = 1$ because the left and right limits are not the same.

(b) At what numbers a is f not continuous?

At $x = 1$ because the limit doesn't exist (part a).
At $x = 3$ because the $\lim_{x\to 3} f(x) \neq f(3)$

(c) At what numbers a does $\lim_{x\to a} f(x)$ exist but f is not continuous at a?

At $x = 3$, since the $\lim_{x\to 3} f(x) \neq f(3)$



Exercises

(a) At what numbers a does $\lim_{x \to a} f(x)$ not exist?

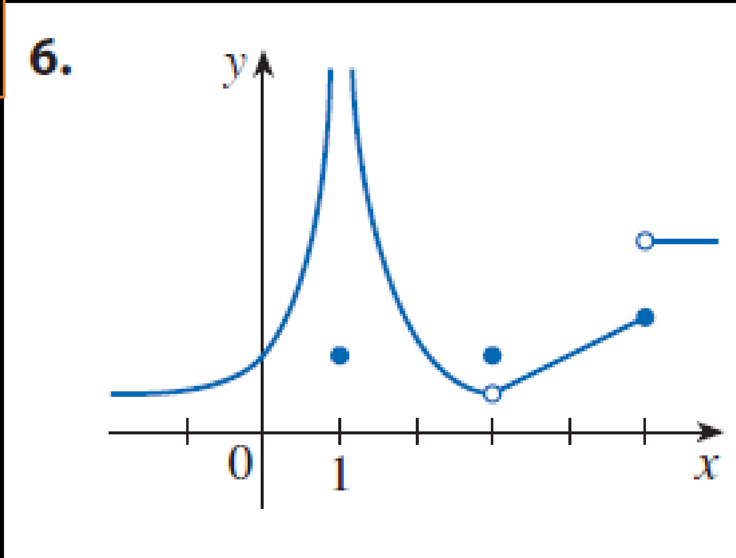
At $x = 1$ since the function increases without bound from the left and from the right. And at $x = 5$ since the left and right limits are not the same.

(b) At what numbers a is f not continuous?

At $x = 5, 1$ since the limit doesn't exist, and at $x = 3$, because the left and right limits aren't the same.

(c) At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

At $x = 3$, since the $\lim_{x \to 3} f(x) \neq f(3)$



Exercises

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(13) $f(x) = 3x^2 + (x+2)^5, \quad a = -1$

(14) $f(x) = \frac{t^2+5t}{2t+1}, \quad a = 2$

Exercises

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(13) $f(x) = 3x^2 + (x+2)^5, \quad a = -1$

$$\lim_{x\to -1}[3x^2 + (x+2)^5] = \lim_{x\to -1} 3x^2 + \lim_{x\to -1}(x+2)^5 =$$
$$3\lim_{x\to -1} x^2 + \lim_{x\to -1}(x+2)^5 = 3*(-1)^2 + (-1+2)^5 = 4 = f(-1)$$

The limit is continuous at $a = -1$

(14) $f(x) = \frac{t^2+5t}{2t+1}, \quad a = 2$

$$\lim_{x\to -2}\left[\frac{t^2+5t}{2t+1}\right] = \frac{\lim_{x\to 2} t^2 + 5t}{\lim_{x\to 2} 2t+1} = \frac{\lim_{x\to 2} t^2 + \lim_{x\to 2} 5t}{\lim_{x-2} 2t + \lim_{x\to 2} 1} = \frac{2^2 + 5*2}{2*2+1} = \frac{14}{5}$$

The limit is continuous at $a = 2$

Exercises

Use continuity to evaluate the limit.

(35) $\lim_{x \to 2} x\sqrt{20 - x^2}$

(36) $\lim_{\theta \to \frac{\pi}{2}} \sin(\tan(\cos(\theta)))$

Exercises

(35) $\lim_{x \to 2} x\sqrt{20\ - x^2}$

$$\boxed{\lim_{x \to 2} x\sqrt{20\ - x^2} = \ 2\ * \sqrt{20 - 2^2} = 8}$$

(36) $\lim_{\theta \to \frac{\pi}{2}} \sin(\tan(\cos(\theta)))$

$$\boxed{\lim_{\theta \to \frac{\pi}{2}} \sin(\tan(\cos(\theta))) = \sin\left(\tan\left(\cos\left(\frac{\pi}{2}\right)\right)\right) = \sin(\tan(0)) = \sin(0) = 0}$$

TASK

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