CH 02: Basic Structures

Sets, Functions, Sequences, Sums, and Matrices

Content

CH 02
Sets
Set Operations
Functions
Sequences and Summations
Cardinality of Sets
Matrices

Functions

- **<u>Definition 1:</u>**
  Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
  - We write $f\ (a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
  - $f : A \to B$.
  - Functions are sometimes also called *mappings* or *transformations*.

Functions

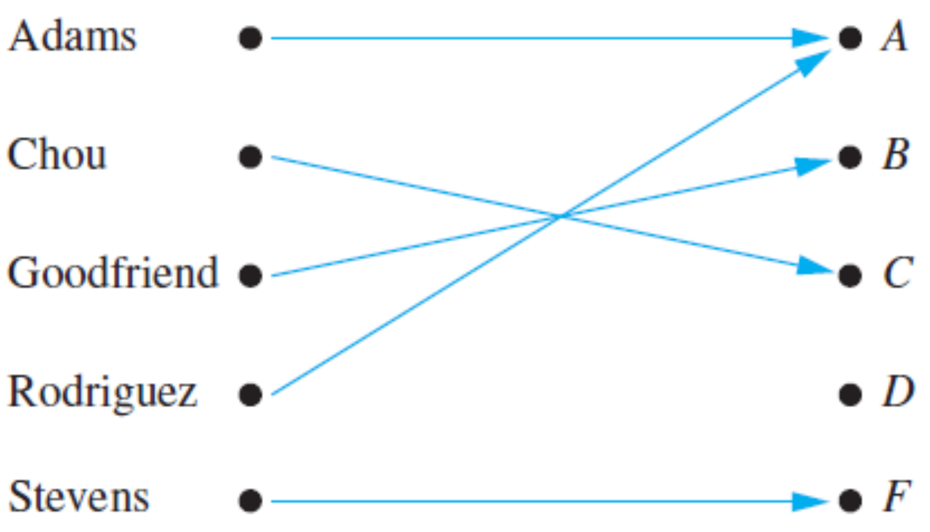


FIGURE 1 **Assignment of grades in a discrete mathematics class.**

Functions

- **Definition 2:**
  If f is a function from A to B
  - A is the domain of f
  - B is the codomain of f

  If $f(a) = b$
  - b is the image of a
  - a is a preimage of b

  The range, or image, of f is the set of all images of elements of A.

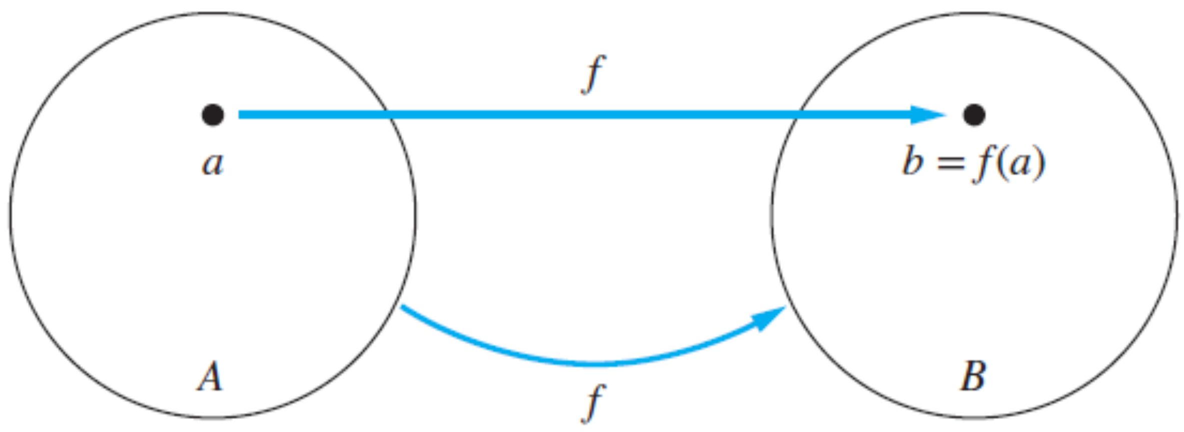


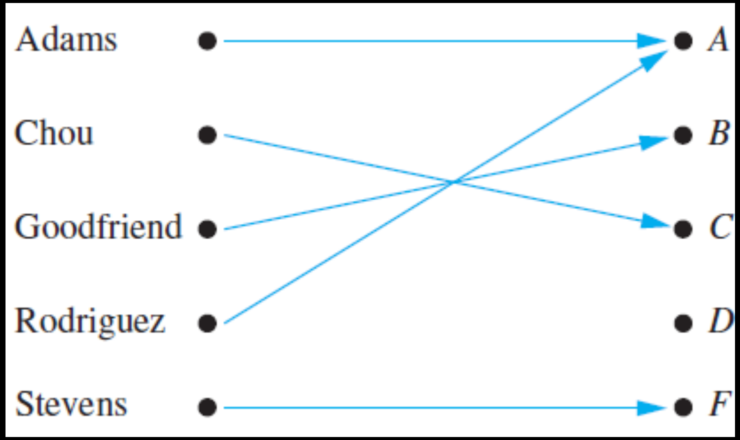
FIGURE 2 The function f maps A to B.

Functions

- **QUIZ:** What is the difference between the codomain and the range

Functions

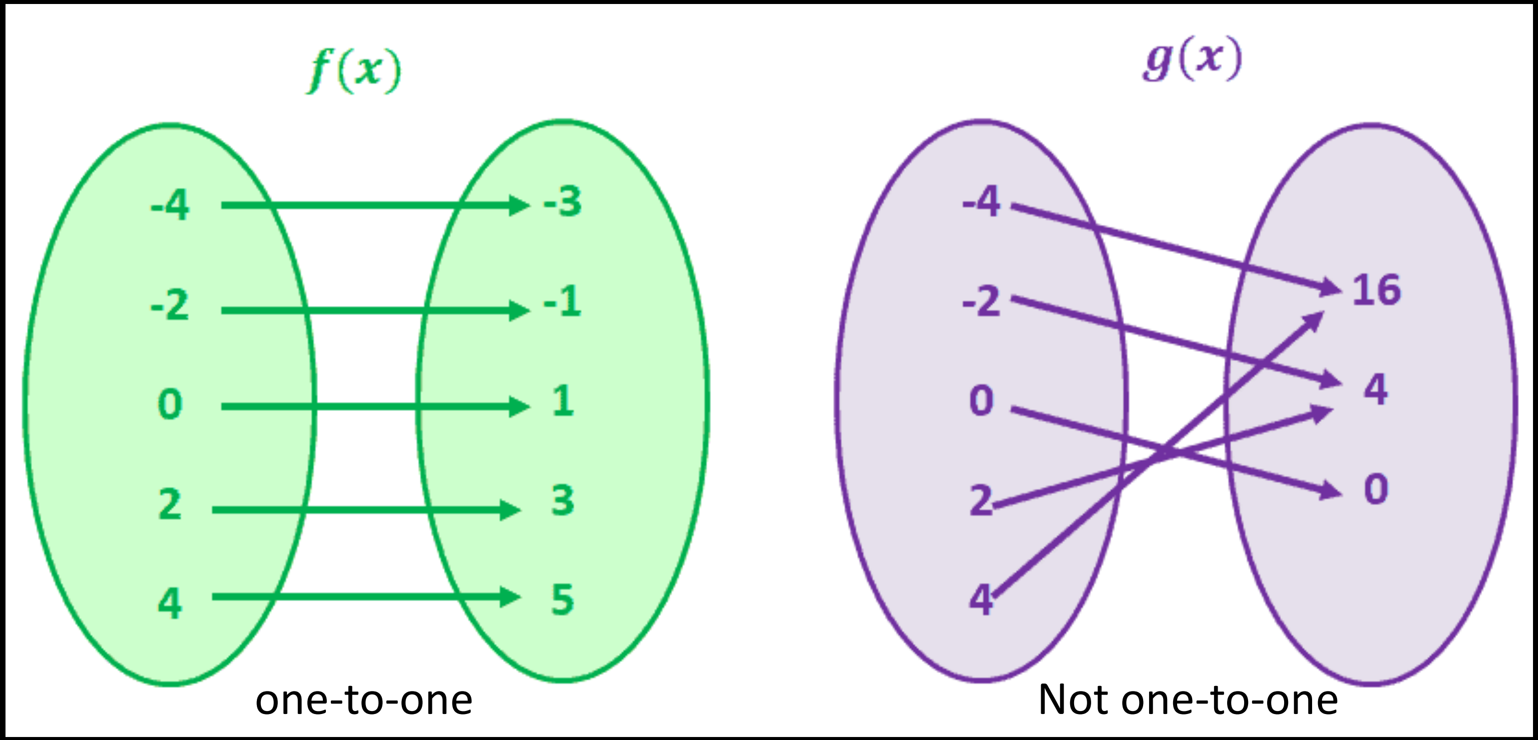
- **EXAMPLE 1:** What are the domain, codomain, and range of the function



- The domain is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- The codomain is the set {A, B, C, D, F}.
- The range is the set {A, B,C, F}, because each grade except D is assigned to some student.

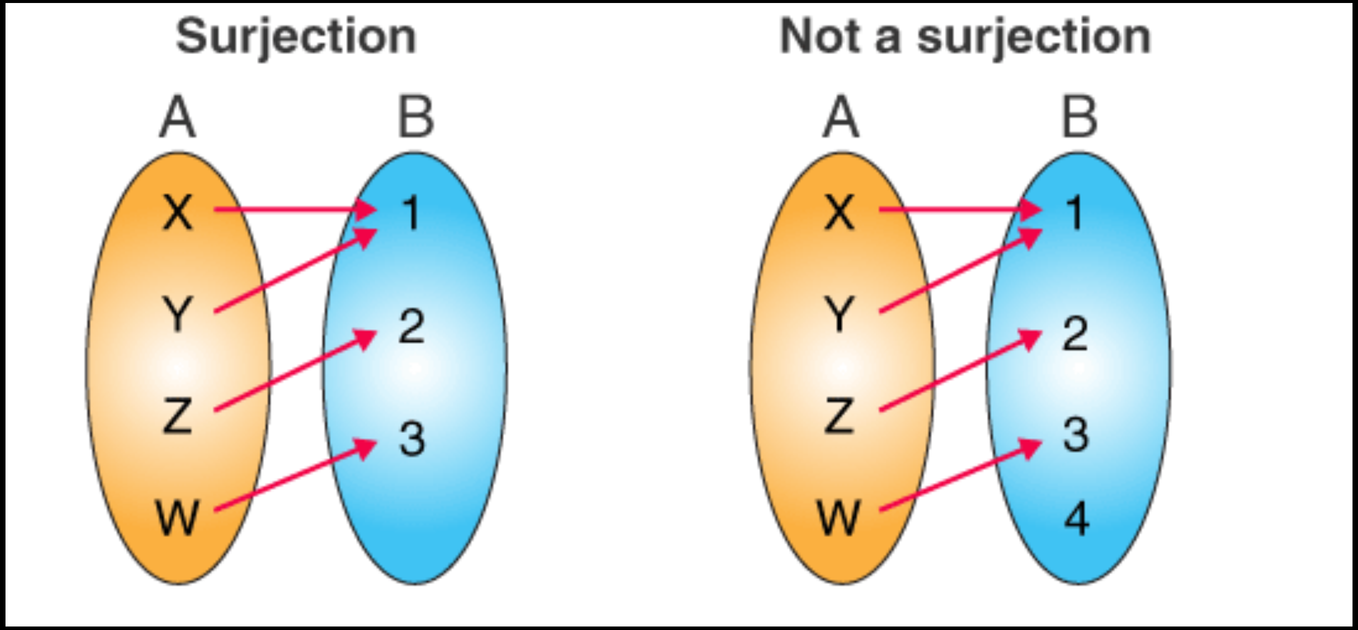
Functions

- **<u>Definition 5:</u>** A function f is said to be <u>one-to-one</u>, or an <u>injection</u>, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.



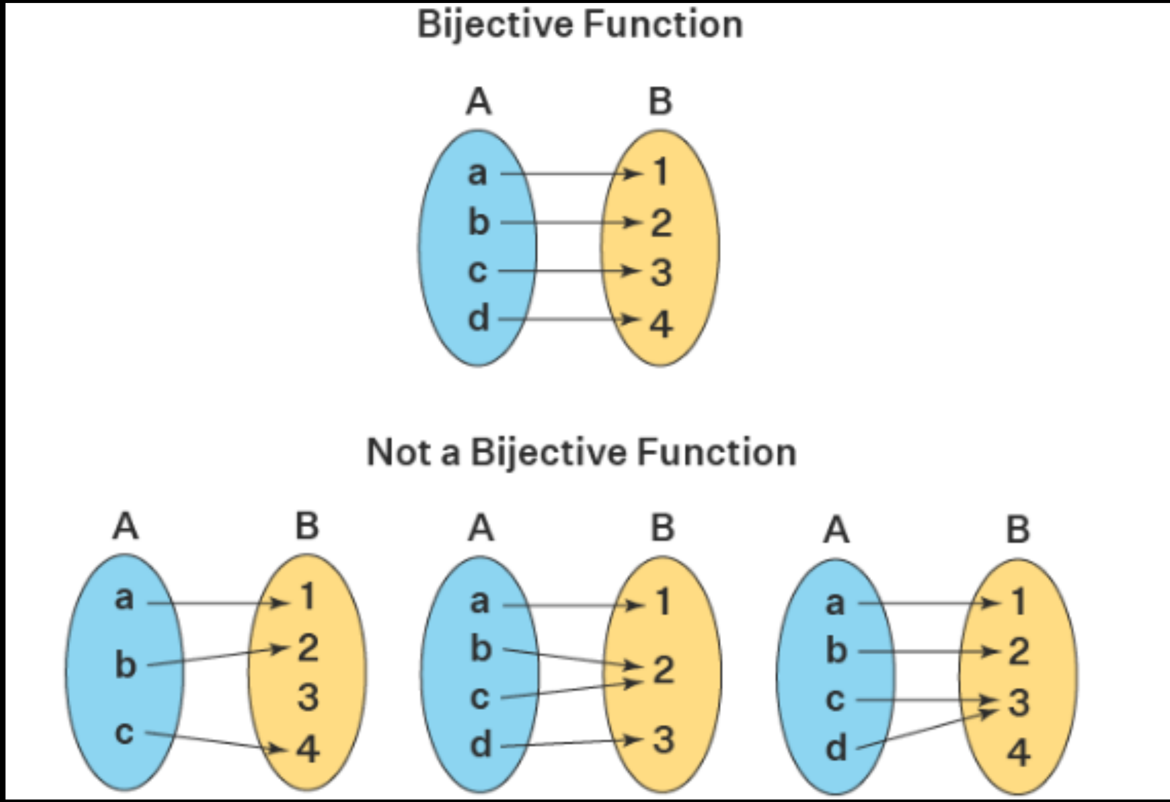
Functions

- **Definition 7:** A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.



Functions

- **Definition 8:** The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.



Functions

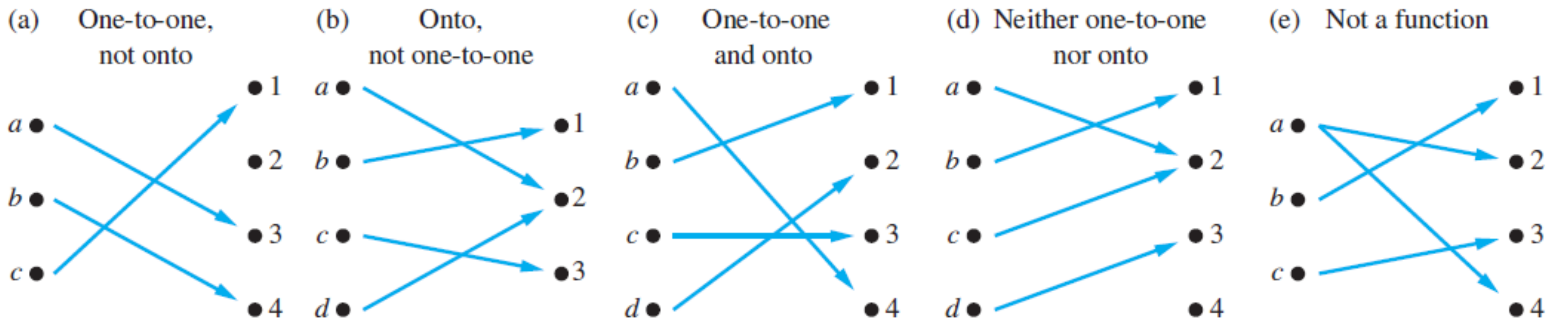


FIGURE 5 **Examples of different types of correspondences.**

Functions

- **Definition 9:** Inverse function $f^{-1}(b) = a$ when $f(a) = b$.

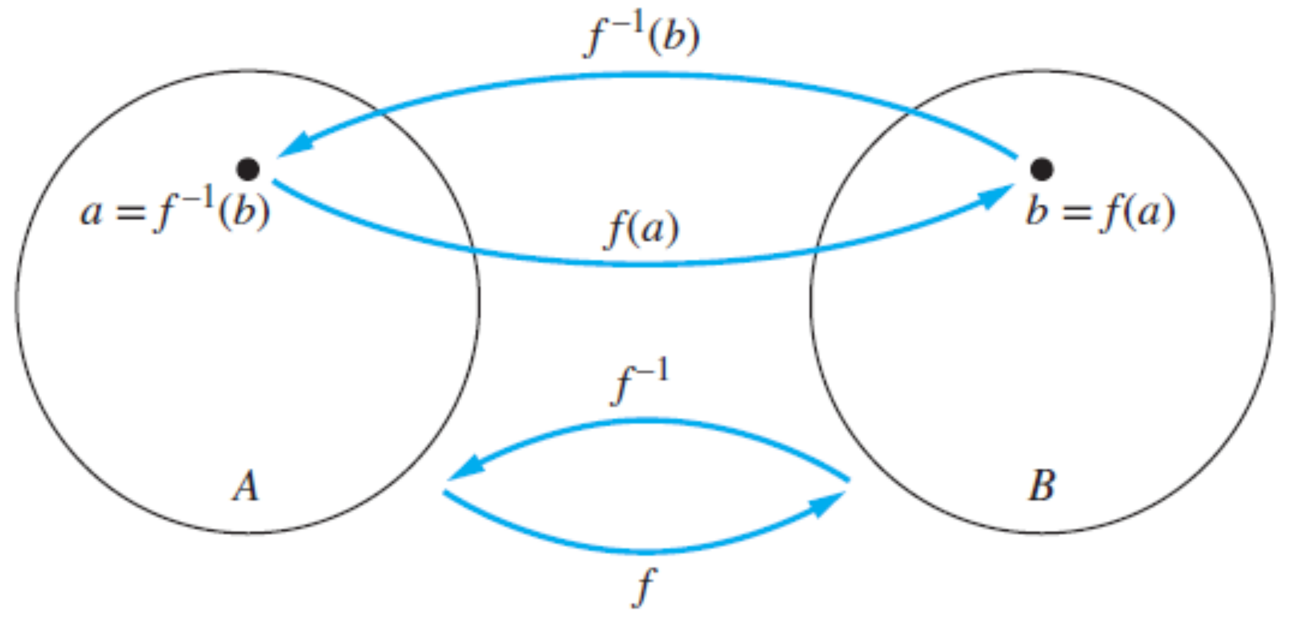


FIGURE 6 The function f^{-1} is the inverse of function f.

Functions

- **EXAMPLE 19:** Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2, f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

- **Solution:**
  The function f is invertible because it is a one-to-one correspondence.
  The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c, f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Functions

- **Definition 10:** Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, is the function from A to C defined by $(f \circ g)(a) = f(g(a))$.

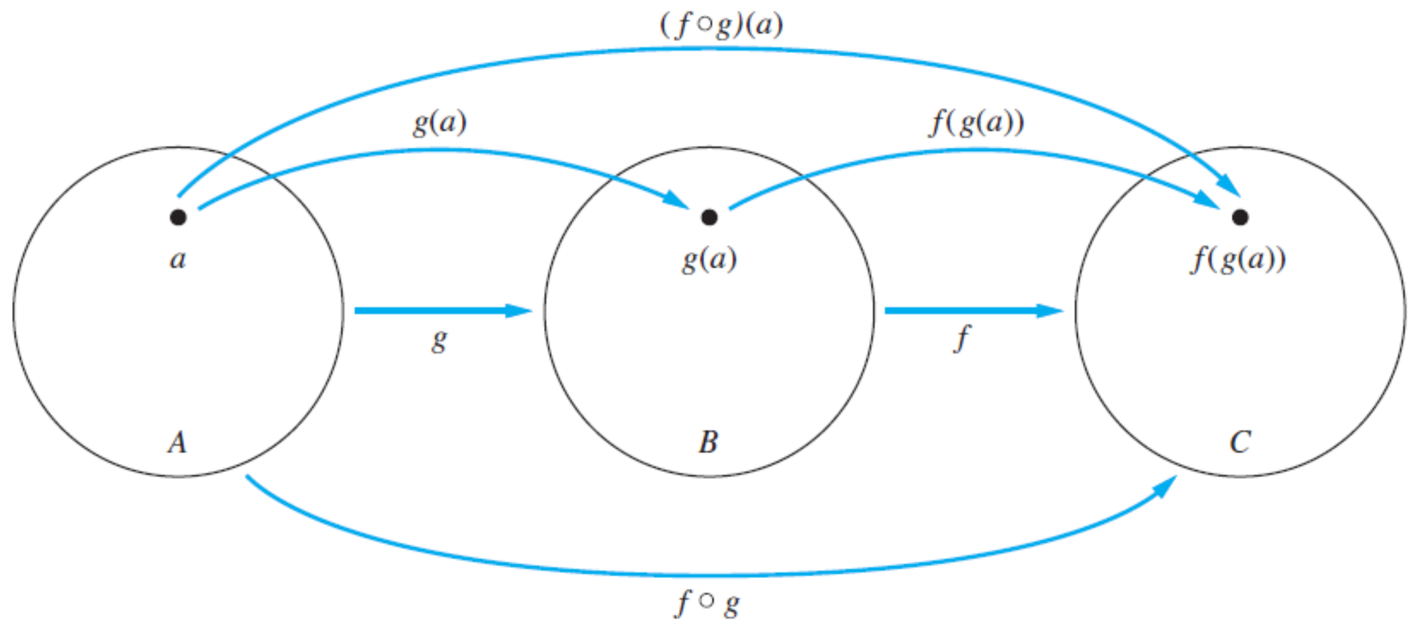


FIGURE 7 The composition of the functions f and g.

Functions

- **EXAMPLE 24:** Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

- **Solution:**

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Functions

- **Definition 12:**
  The floor function $\lfloor x \rfloor$ assigns to the real number x the largest integer that is less than or equal to x.
  The ceiling function $\lceil x \rceil$ assigns to the real number x the smallest integer that is greater than or equal to x.

- **Example 28:** $\left\lfloor \frac{1}{2} \right\rfloor = 0,\ \left\lceil \frac{1}{2} \right\rceil = 1,\ \left\lfloor -\frac{1}{2} \right\rfloor = -1,\ \left\lceil -\frac{1}{2} \right\rceil = 0,$

$$\lfloor 3.1 \rfloor = 3, \qquad \lceil 3.1 \rceil = 4, \qquad \lfloor 7 \rfloor = 7, \qquad \lceil 7 \rceil = 7.$$

Exercises

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each nonnegative integer its last digit

c) The function that assigns to a bit string the number of one bits in the string

Exercises

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each nonnegative integer its last digit

The domain is the set of nonnegative integers, and the range is the set of digits (0 to 9).

c) The function that assigns to a bit string the number of one bits in the string

The domain is the set of all bit strings, and the range is the set of nonnegative integers.

Exercises

8. Find these values.

d) $\lceil -0.1 \rceil$ e) $\lceil 2.99 \rceil$ g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

Exercises

8. Find these values.

d) $\lceil -0.1 \rceil$

$\boxed{0}$

e) $\lceil 2.99 \rceil$

$\boxed{3}$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$\boxed{1}$

Exercises

10. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one.

a) $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d$

b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$

Exercises

10. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one.

a) $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d$

One - to - one

b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$

Not one - to - one

Exercises

22. Determine whether each of these functions is a bijection from $\boldsymbol{R}$ to $\boldsymbol{R}$.

b) $f(x) = -3x^2 + 7$

d) $f(x) = x^5 + 1$

Exercises

22. Determine whether each of these functions is a bijection from $\boldsymbol{R}$ to $\boldsymbol{R}$.

To determine that a function is bijection, it must be <u>one-to-one</u> and <u>onto</u>

b) $f(x) = -3x^2 + 7$

This is not bijection because it is not one-to-one. Try x = -2 and x = 2.

d) $f(x) = x^5 + 1$

This is bijection because:

it is one-to-one. Try x = -2 and x = 2.

it is onto. Set any value to the expression $x^5 + 1$, you will get a real value for x.

Try $x^5 + 1 = 0$, then $x = -1$. Any value for $x = \sqrt[5]{x - 1}$

TASK

Section 2.3
4 (b, d)
8 (a, c, f)
10 (c)
22 (a, c)

Content

CH 02
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Functions
Sequences and Summations (CH 05.1 Sussana textbook)
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Sequences and Summations

- **Sequence:** A function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.
- It is represented as a set of elements written in a row.

$$a_m, a_{m+1}, a_{m+2}, \dots, a_n$$

  - Each individual element is called a **term**
  - the m is called **subscript** or **index**.

- If we write the sequence as $a_m, a_{m+1}, a_{m+2}, \dots$, then it is called **infinite sequence**

Sequences and Summations

- **Example:** Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$a_k = \frac{k}{k+1}$ for every integer $k \geq 1$

$b_i = \frac{i-1}{i}$ for every integer $i \geq 2$

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

Sequences and Summations

- **Example:** Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows:
  $c_j = (-1)^j$ for $j \geq 0$
- **Solution:**

This is an alternating sequence

- Even powers = 1
- Odd powers = -1

$$c_0 = (-1)^0 = 1$$
$$c_1 = (-1)^1 = -1$$
$$c_2 = (-1)^2 = 1$$
$$c_3 = (-1)^3 = -1$$
$$c_4 = (-1)^4 = 1$$
$$c_5 = (-1)^5 = -1$$

Sequences and Summations

- **Example:** Find an explicit formula for the following sequence:

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots$$

- **Solution:**

$$\begin{array}{cccccc}
\frac{1}{1^2}, & \frac{(-1)}{2^2}, & \frac{1}{3^2}, & \frac{(-1)}{4^2}, & \frac{1}{5^2}, & \frac{(-1)}{6^2}. \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6
\end{array}$$

This is can be represented as $a_k = \frac{\pm 1}{k^2} \rightarrow \frac{(-1)^{k+1}}{k^2}$ for $k \geq 1$ <u>or</u> $\frac{(-1)^k}{(k+1)^2}$ for $k \geq 0$

Sequences and Summations

- **Summations:** Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$. Compute the following:

a) $\sum_{k=1}^{5} a_k$ b) $\sum_{k=2}^{2} a_k$ c) $\sum_{k=1}^{2} a_{2k}$

Solution:

a) $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + (-1) + 0 + 1 + 2 = 0$

b) $\sum_{k=2}^{2} a_k = a_2 = -1$

c) $\sum_{k=1}^{2} a_{2k} = a_2 + a_4 = -1 + 1 = 0$

Sequences and Summations

- **Example:** Compute $\sum_{k=1}^{5} k^2$

- **Solution:**

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Sequences and Summations

- **Example:** Write $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ in expanded form

- **Solution:**

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Sequences and Summations

- **<u>Product:</u>** If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the product from k equals m to n of a_k, is the product of all the terms $a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_n$

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_n$$

Sequences and Summations

- **<u>Example:</u>** Compute the following products

a) $\prod_{k=1}^{5} k$ b) $\prod_{k=1}^{1} \frac{k}{k+1}$

- **<u>Solution:</u>**

a) $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b) $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

Sequences and Summations

- Properties of summations and products

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$

3. $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n}(a_k \cdot b_k)$

Sequences and Summations

- **Example:** Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expressions as a single summation or product:

a) $\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$ b) $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$

- **Solution:**

a. $$\begin{aligned}\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n} (k+1) + 2 \cdot \sum_{k=m}^{n} (k-1) \\ &= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2 \cdot (k-1) \\ &= \sum_{k=m}^{n} ((k+1) + 2 \cdot (k-1)) \\ &= \sum_{k=m}^{n} (3k-1)\end{aligned}$$

b. $$\begin{aligned}\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) &= \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right) \\ &= \prod_{k=m}^{n} (k+1) \cdot (k-1) \\ &= \prod_{k=m}^{n} (k^2 - 1)\end{aligned}$$

Sequences and Summations

- **Factorial:** The product of all consecutive integers up to a given integer.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

$$\begin{aligned}
&0! = 1 && 1! = 1 \\
&2! = 2 \cdot 1 = 2 && 3! = 3 \cdot 2 \cdot 1 = 6 \\
&4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 && 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
&6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 && 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040 \\
&8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && 9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&\quad = 40{,}320 && \quad = 362{,}880
\end{aligned}$$

- The values of $n!$ grow very rapidly.
  - $40! \cong 8.16 \times 10^{47}$, too large to be computed by basic computers

Sequences and Summations

- **Recursive factorial:** Given any nonnegative integer n,

$$n! = \begin{cases} 1 & if\ n = 0 \\ n \cdot (n-1)! & if\ n \geq 1 \end{cases}$$

Sequences and Summations

- **Example:** Simplify the following expressions

  a) $\frac{8!}{7!}$ b) $\frac{5!}{2!\cdot 3!}$

- **Solution:**
- a) $\frac{8!}{7!} = \frac{8\cdot 7!}{7!} = 8$
- b) $\frac{5!}{2!\cdot 3!} = \frac{(5\cdot 4\cdot 3!)}{2!\cdot 3!} = \frac{5\cdot 4}{2\cdot 1} = 10$

Sequences and Summations

- Factorial can be used to compute **combinations** or **choices**.
- n choose r: Represents the number of subsets of size r that can be chosen from a set with n elements. Assuming that $0 \leq r \leq n$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Sequences and Summations

- **Example:** Compute

a) $\binom{8}{5}$ b) $\binom{4}{0}$

- **Solution:**

a) $\binom{8}{5} = \frac{8!}{5!\cdot(8-5)!} = \frac{8*7*6*5*4*3*2*1}{(5*4*3*2*1)\cdot(3*2*1)} = 56$

b) $\binom{4}{0} = \frac{4!}{0!\cdot(4-0)!} = \frac{4!}{1\cdot 4!} = 1$

Sequences and Summations

- Sequences in programming are represented using a concept called **loops**.

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

- You can use sequences and loops to convert numbers from base10 to binary and vice versa.

Exercises

Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0

d) a_5

Exercises

Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0

$$\boxed{2 * (-3)^0 + 5^0 = 3}$$

d) a_5

$$\boxed{2 * (-3)^5 + 5^5 = 2639}$$

Exercises

Write the first four terms of the sequences defined by the formulas

1) $a_k = \frac{k}{10+k}$, for all integers $k \geq 1$

2) $b_j = \frac{5-j}{5+j}$, for all integers j ≥ 1

3) $c_i = \frac{(-1)^i}{3^i}$, for all integers i ≥ 0

Exercises

Write the first four terms of the sequences defined by the formulas

1) $a_k = \frac{k}{10+k}$, for all integers $k \geq 1$

$$\boxed{a_1 = \frac{1}{11}, a_2 = \frac{2}{12} = \frac{1}{6}, a_3 = \frac{3}{13}, a_4 = \frac{4}{14} = \frac{2}{7}}$$

2) $b_j = \frac{5-j}{5+j}$, for all integers j ≥ 1

$$\boxed{b_1 = \frac{4}{6} = \frac{2}{3}, b_2 = \frac{3}{7}, b_3 = \frac{2}{8} = \frac{1}{4}, b_4 = \frac{1}{9}}$$

3) $c_i = \frac{(-1)^i}{3^i}$, for all integers i ≥ 0

$$\boxed{c_0 = \frac{1}{1} = 1, c_1 = -\frac{1}{3}, c_2 = \frac{1}{9}, c_3 = -\frac{1}{27}}$$

Exercises

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms

10) $-1, 1, -1, 1, -1, 1$

11) $0, 1, -2, 3, -4, 5$

12) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

13) $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

Exercises

- Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms

10) $-1, 1, -1, 1, -1, 1$

$\boxed{a_n = (-1)^n \text{ where } n \text{ is an integer and } n \geq 1}$

11) $0, 1, -2, 3, -4, 5$

$\boxed{a_n = (n-1)(-1)^n \text{ where } n \text{ is an integer and } n \geq 1}$

12) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

$\boxed{a_n = \frac{n}{(n+1)^2} \text{ where } n \text{ is an integer and } n \geq 1}$

13) $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

$\boxed{a_n = \frac{1}{n} - \frac{1}{n+1} \text{ where } n \text{ is an integer and } n \geq 1}$

Exercises

Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a) $\sum_{i=0}^{6} a_i$

c) $\sum_{j=1}^{3} a_{2j}$

e) $\prod_{k=2}^{2} a_k$

Exercises

Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$.
Compute each of the summations and products below.

a) $\sum_{i=0}^{6} a_i$

c) $\sum_{j=1}^{3} a_{2j}$

e) $\prod_{k=2}^{2} a_k$

a) $2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$

c) $a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$

e) $a_2 = -2$

Exercises

Compute the summations and products

20) $\prod_{k=2}^{4} k^2$

23) $\Sigma_{i=1}^{1} i(i+1)$

Exercises

Compute the summations and products

20) $\prod_{k=2}^{4} k^2$

23) $\Sigma_{i=1}^{1} i(i+1)$

$\boxed{20)\ 2^2 \cdot 3^2 \cdot 4^2 = 576}$

$\boxed{23)\ 1(1+1) = 2}$

Exercises

Write the following using summation or product notation.

44) $(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

45) $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

Exercises

Write the following using summation or product notation.

44) $(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

$\boxed{44)\ \sum_{i=1}^{5}(-1)^{i+1}(i^3 - 1)}$

45) $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

$\boxed{45)\ \prod_{i=2}^{4}(i^2 - 1)}$

Exercises

Compute the following

68) $\frac{\left((n+1)!\right)^2}{(n!)^2}$

71) $\binom{5}{3}$

72) $\binom{7}{4}$

Exercises

Compute the following

68) $\frac{\left((n+1)!\right)^2}{(n!)^2}$

$$68)\ \frac{[(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1]^2}{[n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1]^2}=(n+1)^2$$

71) $\binom{5}{3}$

$$70)\ \frac{5!}{3!\cdot(5-3)!}=\frac{5*4*3*2*1}{(3*2*1)\cdot(2*1)}=10$$

72) $\binom{7}{4}$

$$72)\ \frac{7!}{4!\cdot(7-4)!}=\frac{7*6*5*4*3*2*1}{(4*3*2*1)\cdot(3*2*1)}=35$$

TASK

Section 5.1
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18 (b, d)
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Content

CH 02
Sets
Set Operations
Functions
Sequences and Summations
~~Cardinality of Sets~~
Matrices

Matrices

- **EXAMPLE 1:** matrix definition

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

Matrices

- **EXAMPLE 2:** matrix addition
  - Matrices of different sizes cannot be added, because such matrices will not both have entries in some of their positions.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}.$$

Matrices

- **EXAMPLE 4:** matrix multiplication
  - The number of elements in a row must be equal to the number of elements in a column

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}$$

$$\begin{bmatrix} \boxed{a_{11} \; a_{12} \; a_{13} \; a_{14}} \\ a_{21} \; a_{22} \; a_{23} \; a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

2 x 4 4 x 3 2 x 3

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Matrices

- **EXAMPLE 5:** transpose matrix

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Matrices

- **EXAMPLE 6:** symmetric matrix
  - The transpose of a square matrix is the same as the original matrix

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric.

Matrices

- **EXAMPLE 7:** Zero-one matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

  - The join operation is ∨

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

  - The meet operation is ∧

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Matrices

- **EXAMPLE 8:** Boolean product

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Matrices

- **<u>EXAMPLE 9:</u>** rth Boolean power

Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exercises

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$

a) What size is A?

b) What is the third column of A?

c) What is the second row of A?

d) What is the element of A in the (3, 2)th position?

e) What is A^t?

Exercises

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$

a) What size is A? $\boxed{3 \text{ X } 4}$

b) What is the third column of A? $\begin{matrix} 1 \\ 4 \\ 3 \end{matrix}$

c) What is the second row of A? $2 \quad 0 \quad 4 \quad 6$

d) What is the element of A in the (3, 2)th position? $\boxed{1}$

e) What is A^t? $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$

Exercises

2. Find A + B, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$.

Exercises

2. Find A + B, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}.$

$$\begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$$

Exercises

3. Find AB if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

Exercises

3. Find AB if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}.$

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} (2)\cdot(0)+(1)\cdot(1) & (2)\cdot(4)+(1)\cdot(3) \\ (3)\cdot(0)+(2)\cdot(1) & (3)\cdot(4)+(2)\cdot(3) \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (1)\cdot(3)+(-1)\cdot(1) & (1)\cdot(-2)+(-1)\cdot(0) & (1)\cdot(-1)+(-1)\cdot(2) \\ (0)\cdot(3)+(1)\cdot(1) & (0)\cdot(-2)+(1)\cdot(0) & (0)\cdot(-1)+(1)\cdot(2) \\ (2)\cdot(3)+(3)\cdot(1) & (2)\cdot(-2)+(3)\cdot(0) & (2)\cdot(-1)+(3)\cdot(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}$$

Exercises

29. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

a) Find $A^{[2]}$

Exercises

29. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

a) Find $A^{[2]}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

TASK

Section 2.6
2 (b)
3 (c)
29 (b)