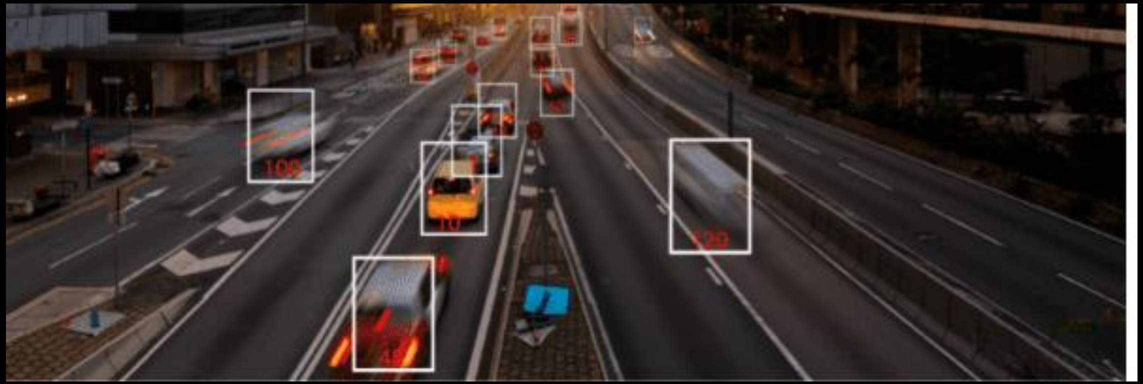
Functions and Models

Mathematical Models: A Catalog of Essential Functions

Overview

- The term machine learning was coined in 1959 by Arthur Samuel, an IBM employee and pioneer in the field of computer gaming and artificial intelligence.
- In 1960, scientists began experiments to recognize pattern in the sonar signals, electrocardiograms, and speech.
- Given a set of collected data, can we discover any patterns in this data to make predictions or make decisions in the future?



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Overview

- This chapter presents the fundamental concept behind various topics in the AI field.
  - Machine learning
  - Pattern recognition
  - Optimization
  - Computer vision
- Mainly, we are interested to make predictions about future data based on old data.
  - Predict stock prices.
  - Classifying a face whether smiling or not
  - Detect email spams.

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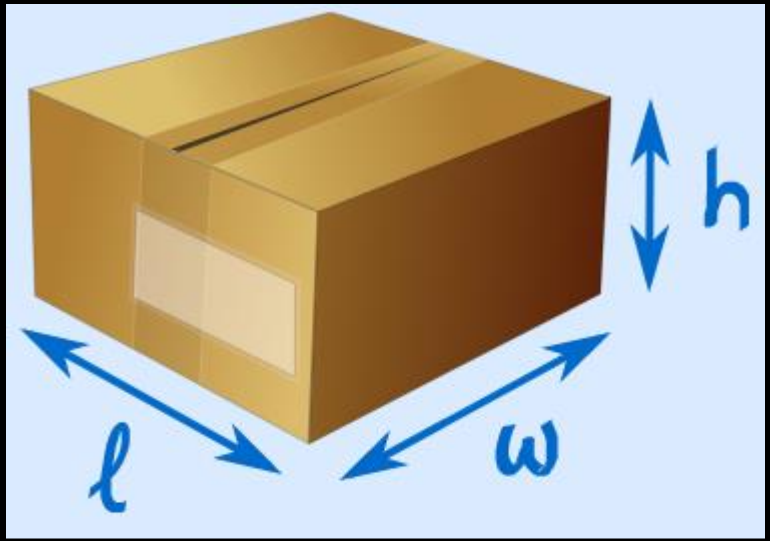
Introduction

- A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon
- Examples:
  - o the size of a population,
  - o the demand for a product,
  - o the speed of a falling object
- The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Introduction

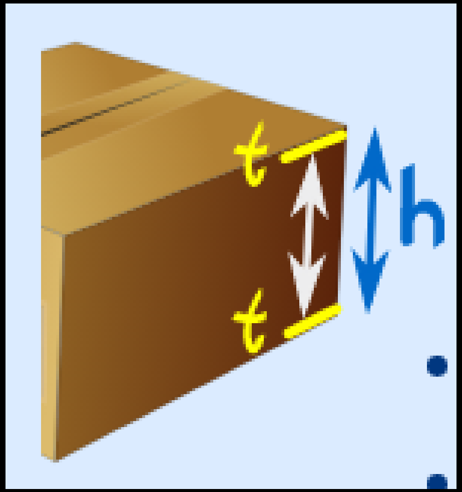
- Mathematics can be used to "model", or represent, how the real-world works.



$$Volume = l \times w \times h$$

Introduction

- To be accurate about the volume of the box, we need to consider the thickness of the cardboard.
- The inside measurements need to be reduced by the thickness of each side:
  - The inside length is $l - 2t$
  - The inside width is $w - 2t$,
  - The inside height is $h - 2t$



$$Inside\ Volume = (l - 2t) \times (w - 2t) \times (h - 2t)$$

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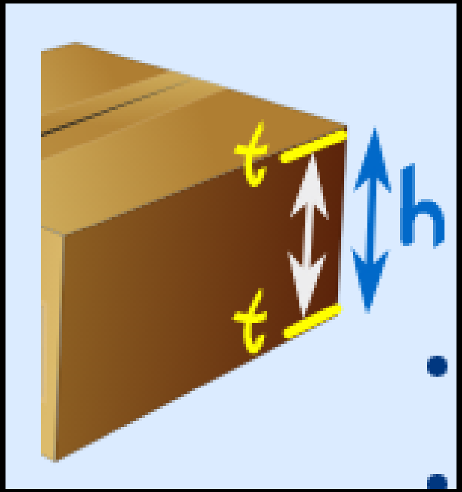
Now we have a **better** model.



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Introduction

- Mathematical models can also be used to forecast future behavior.

Example: An ice cream company keeps track of how many ice creams get sold on different days.

By comparing this to the weather on each day they can make a mathematical model of **sales versus weather**.

They can then predict future sales based on the weather forecast, and decide how many ice creams they need to make ... ahead of time!

Introduction

- Mathematical models can get very complex, and so the mathematical rules are often written into computer programs, to make a computer model.
- Demos:
  - https://www.mathsisfun.com/geometry/ellipse-reflect-model.html
  - https://www.mathsisfun.com/physics/pendulum.html
  - https://www.mathsisfun.com/physics/double-pendulum.html
- More complex examples include:
  - Weather prediction
  - Economic Models (predicting interest rates, unemployment, etc)
  - Public health vs infectious diseases

Introduction

- How to formulate a mathematical model?

Introduction

- The mathematical modeling process:

1. Identify and naming the independent and dependent variables
    1. Apply physical laws and guides.
    2. Collect data and analyze the data in the form of a table to discover patterns.
2. From the numerical representation of the data, obtain a graphical representation of the data.
3. Apply mathematical rules to the model to draw conclusions.
4. Take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions.
5. Test our predictions by checking against new real data.
    1. If the predictions don’t compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

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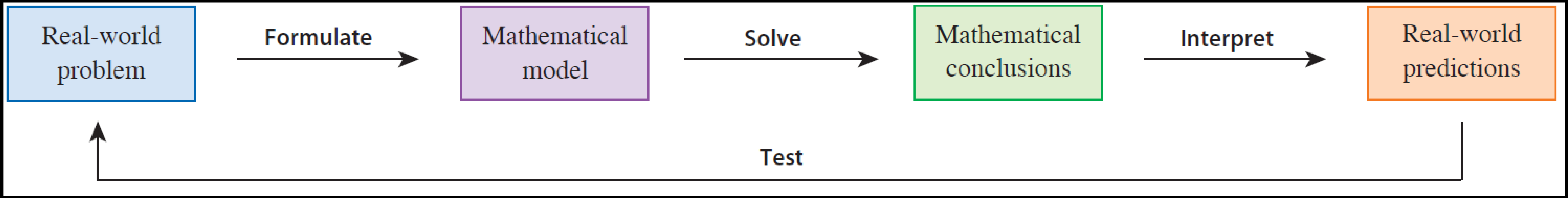
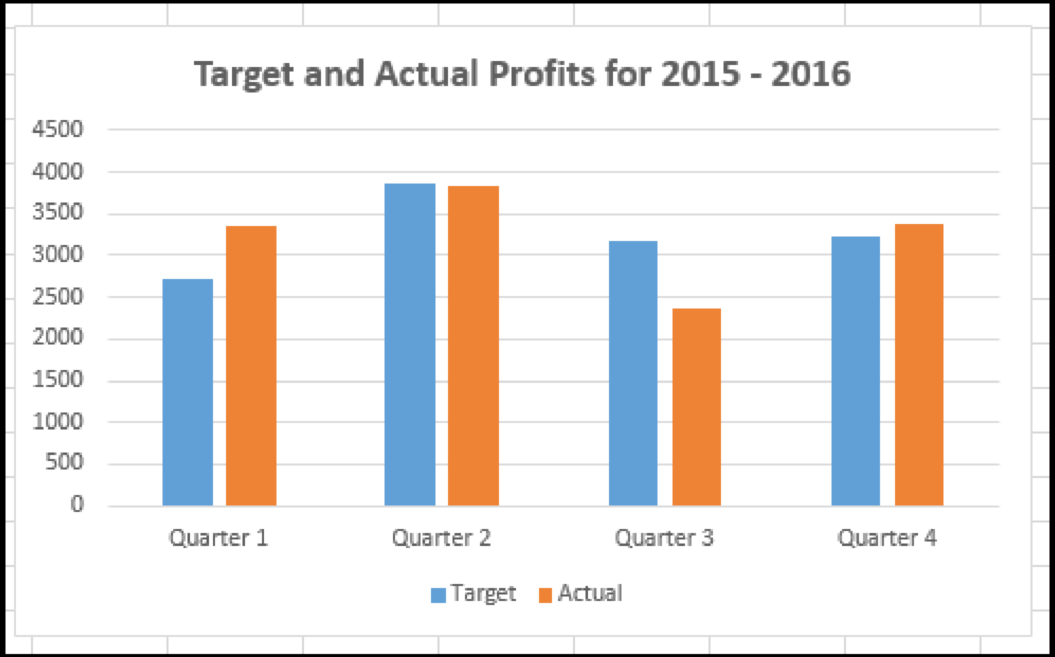


Introduction

- Graphical representation of the data:

	Target	Actual
Quarter 1	2727	3358
Quarter 2	3860	3829
Quarter 3	3169	2374
Quarter 4	3222	3373



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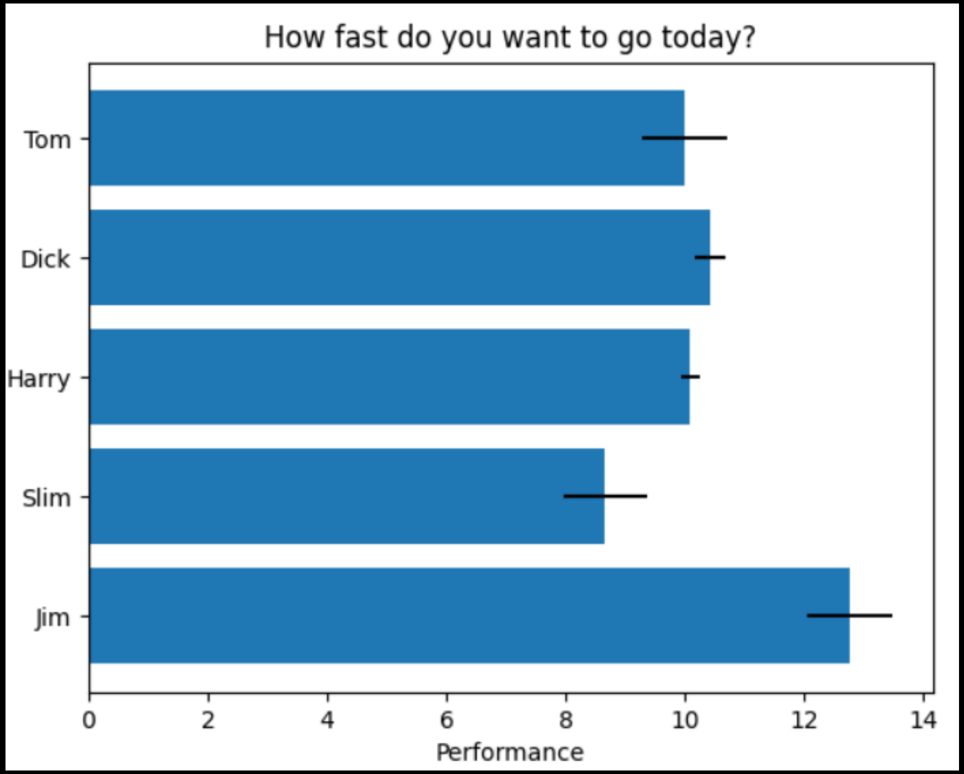

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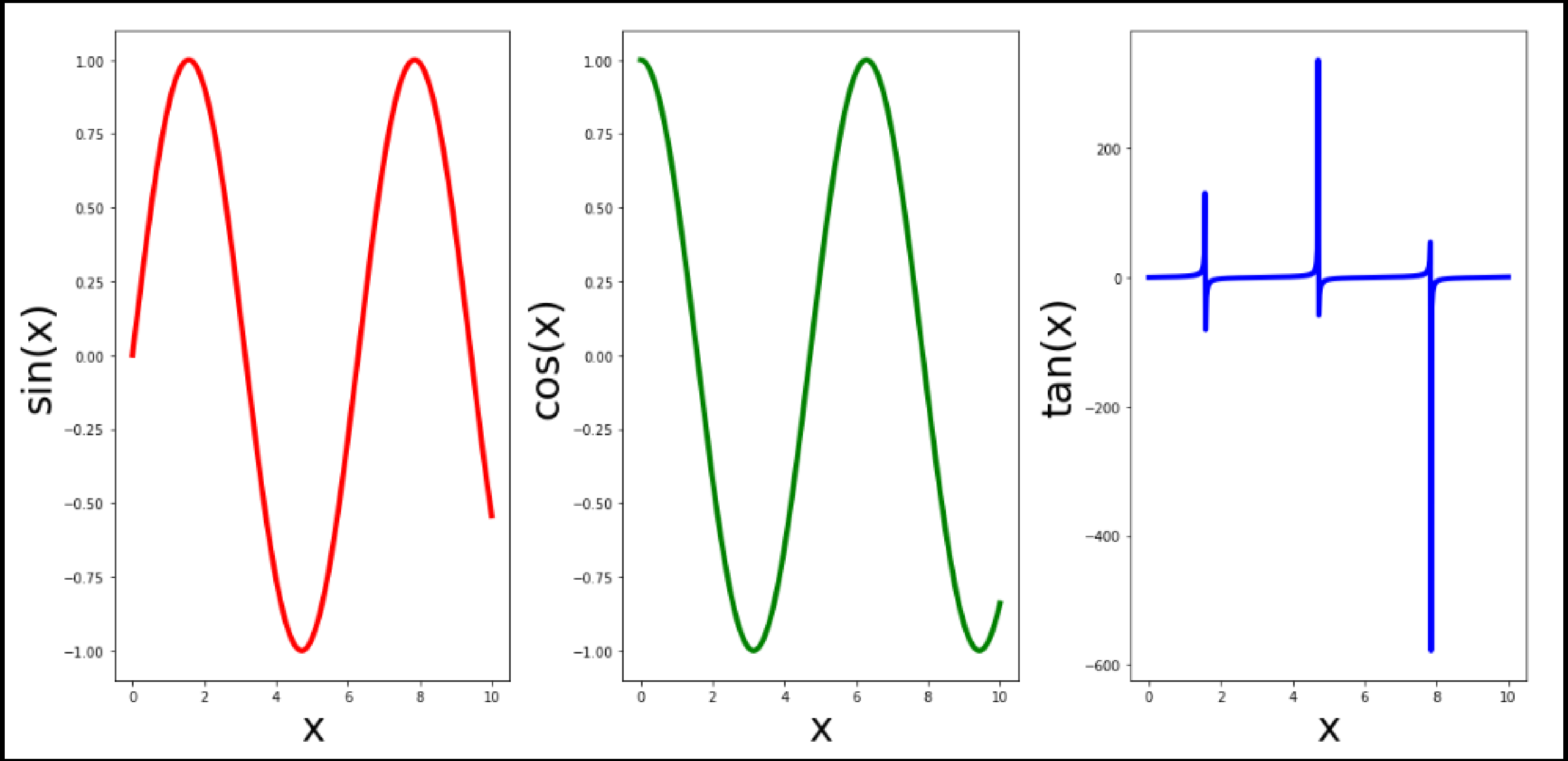
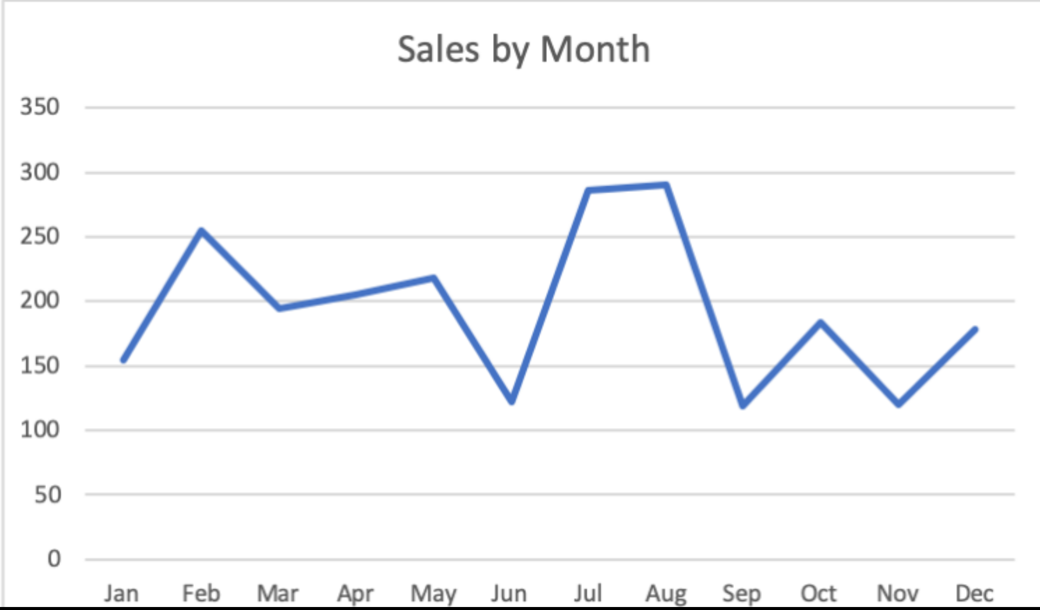


Introduction

- Graphical representation of the data:

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2	Products	113824	188986	140691	167417	199789	122998	104406	162634	170378	171745	130481	158238
3	Services	320651	345882	282953	197752	385273	298660	156515	188593	374634	278056	208716	240923
4	Total revenue	434475	534868	423644	365169	585062	421658	260921	351227	545012	449801	339197	399161
5	Sales count	155	255	195	205	218	122	286	291	119	184	120	178



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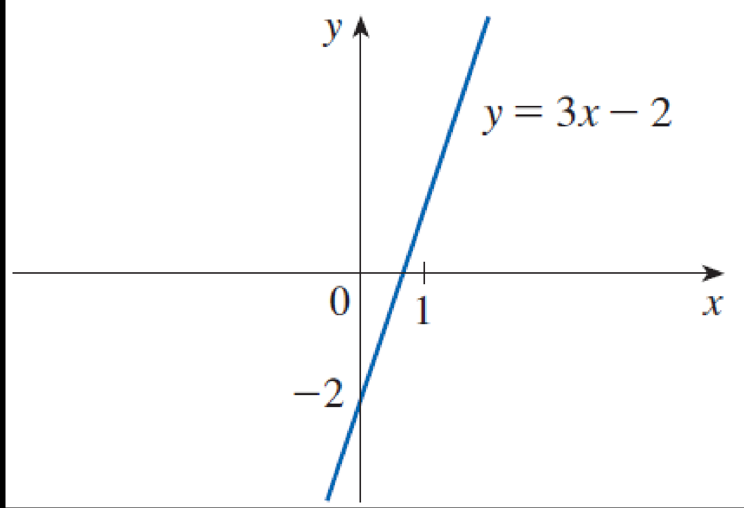
Linear Models

- When we say that y is a **linear function** of x, we mean that the graph of the function is a line.
  - We can use the slope-intercept form of the equation of a line to write a formula for the function
$$y = f(x) = mx + b$$
  - m is the slope of the line.
  - b is the y-intercept.

Linear Models

- Example: $f(x) = 3x - 2$



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

  - Whenever x increases by 0.1, the value of $f(x)$ increases by 0.3.
  - So, $f(x)$ increases three times as fast as x.
  - The slope of the graph, namely 3, can be interpreted as the rate of change of y with respect to x.

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Polynomials

- A function P is called a $polynomial$ if
$$P(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$
  - n is a non-negative integer called **degree** of the polynomial.
  - The numbers $a_0, a_1, \ldots$ are called **coefficients** of the polynomial.
- Example:
$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

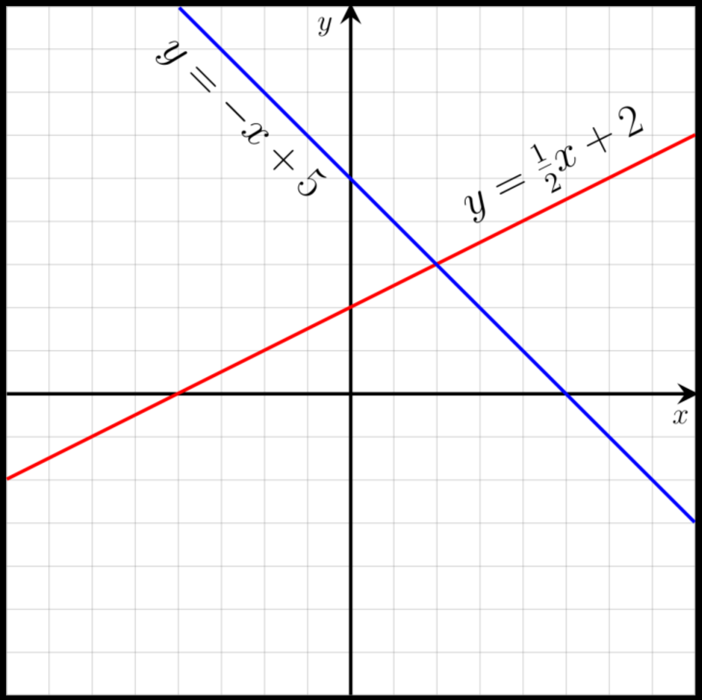
is a polynomial of degree 6.

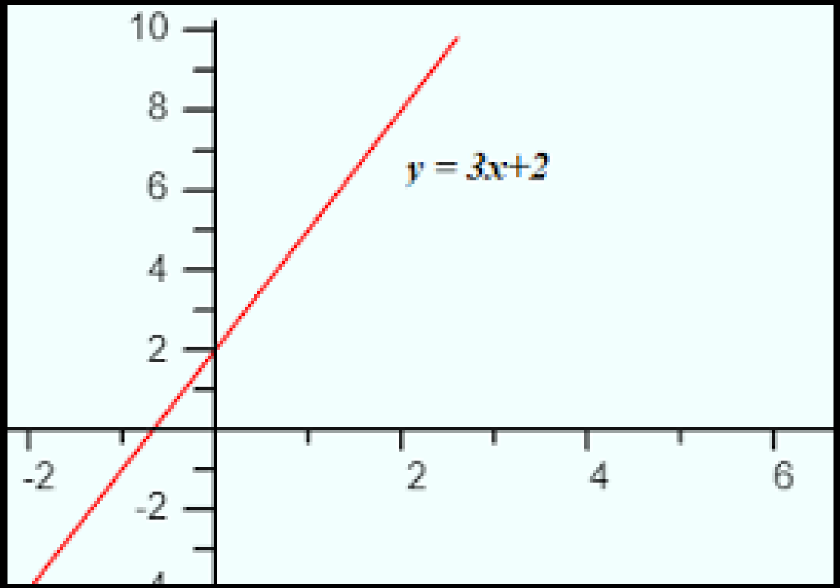
Polynomials

- A polynomial of degree 1:

$$P(x) = a_0 + a_1 x^1 = mx + b$$

Which is a linear function.





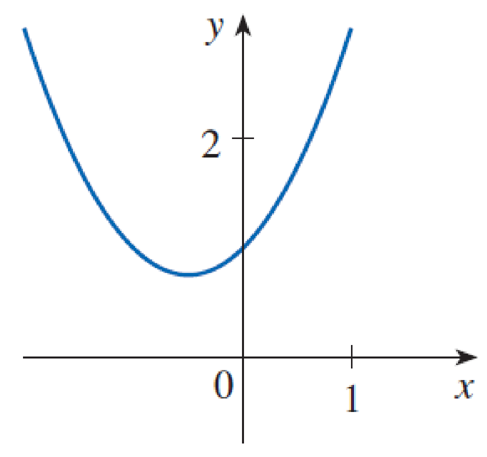
Polynomials

- A polynomial of degree 2:

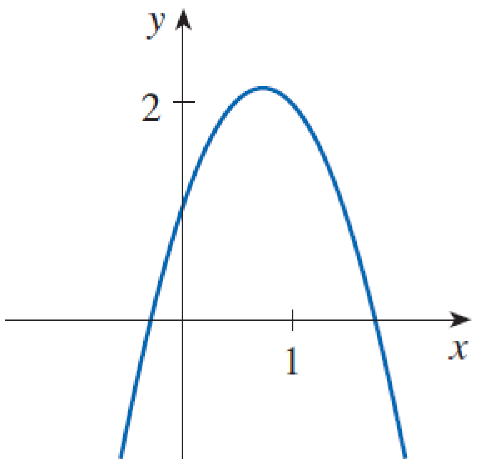
$$P(x) = ax^2 + bx + c$$

  Which is a quadratic function.

- Its graph is always a parabola.
  - The parabola opens upward if $a > 0$ and downward if $a < 0$



(a) $y = x^2 + x + 1$



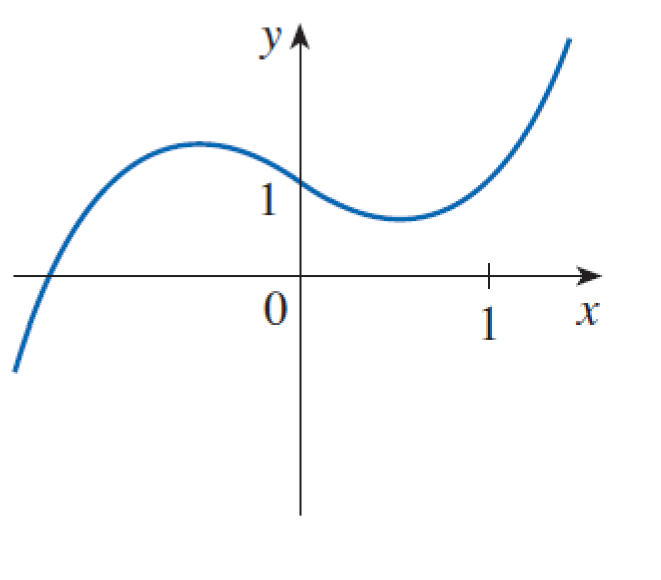
(b) $y = -2x^2 + 3x + 1$

Polynomials

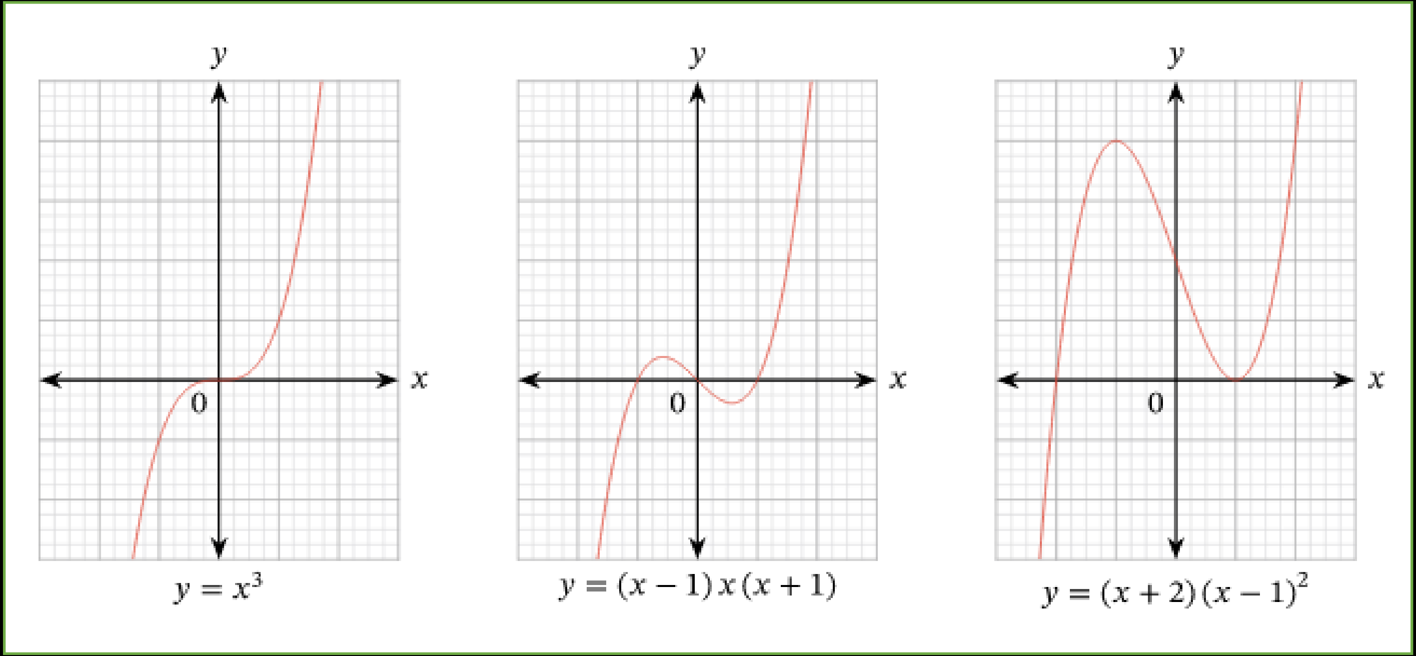
- A polynomial of degree 3:

$$P(x) = ax^3 + bx^2 + cx + d$$

Which is a cubic function.

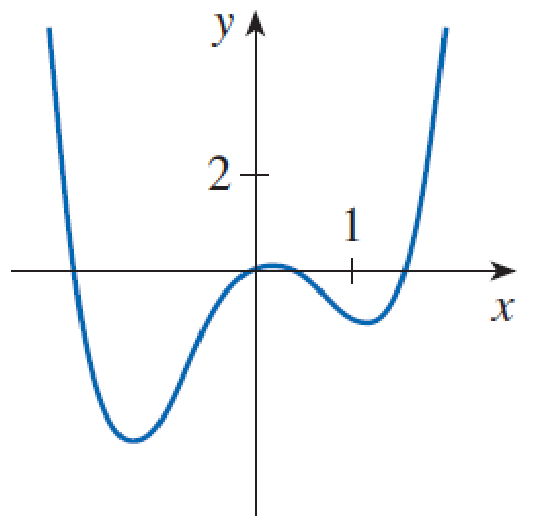


(a) $y = x^3 - x + 1$



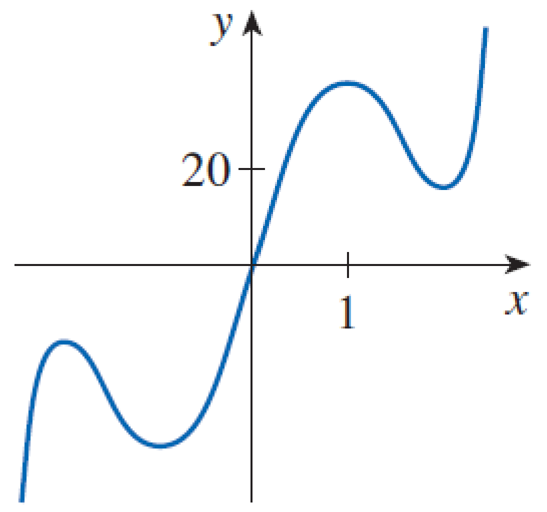
Polynomials

A polynomial of degree 4:



(b) $y = x^4 - 3x^2 + x$

A polynomial of degree 5:



(c) $y = 3x^5 - 25x^3 + 60x$

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Power Functions

- A function of the form $f(x) = x^a$, where a is a constant.
- There are several cases:

$a = n$, where n is a positive integer
$a = 1/n$, where n is a positive integer
$a = -1$
$a = -2$

Power Functions

(1) $a = n$, where n is a positive integer.

- These are polynomials with one term.
- The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4,$ and 5

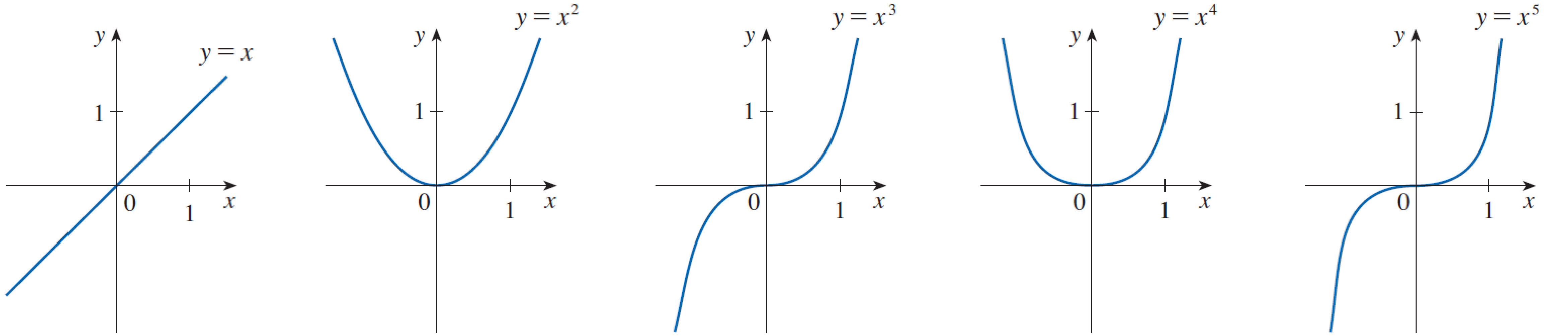
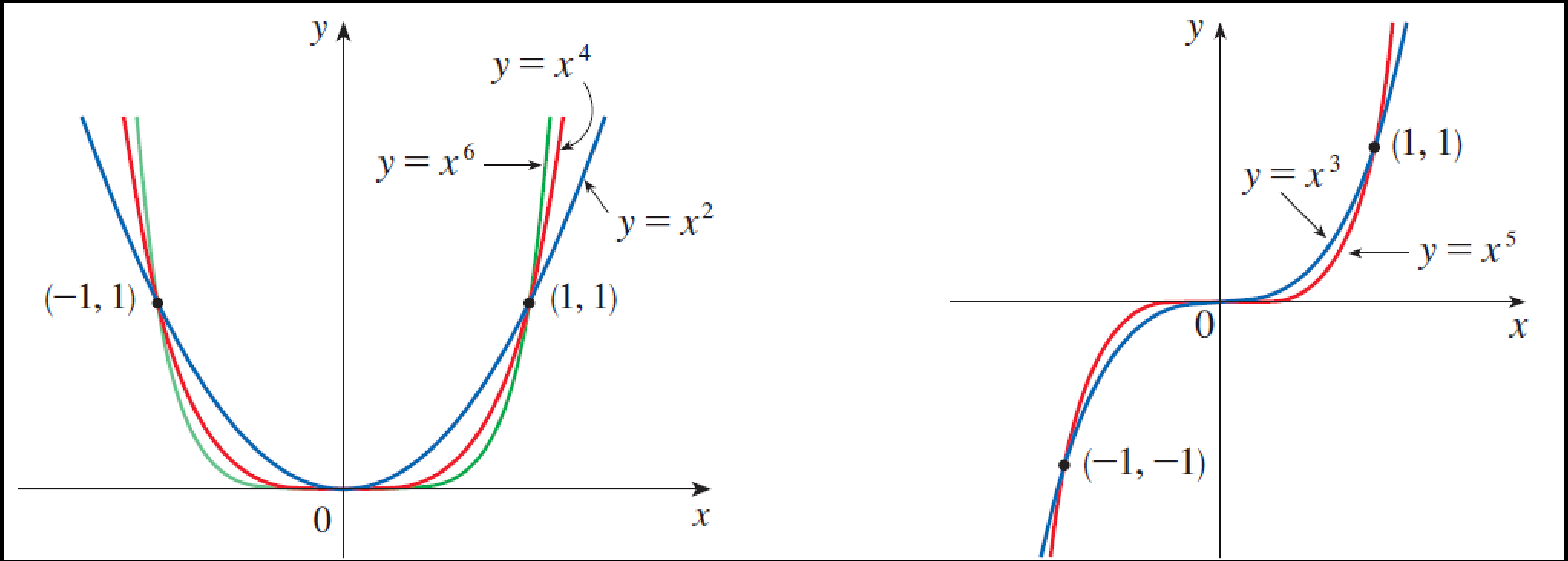


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Power Functions

- The shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
  - If n is even, then $f(x) = x^n$ is an even function and its graph is like that of $y = x^2$.
  - If n is odd, then $f(x) = x^n$ is an odd function and its graph is like that of $y = x^3$.

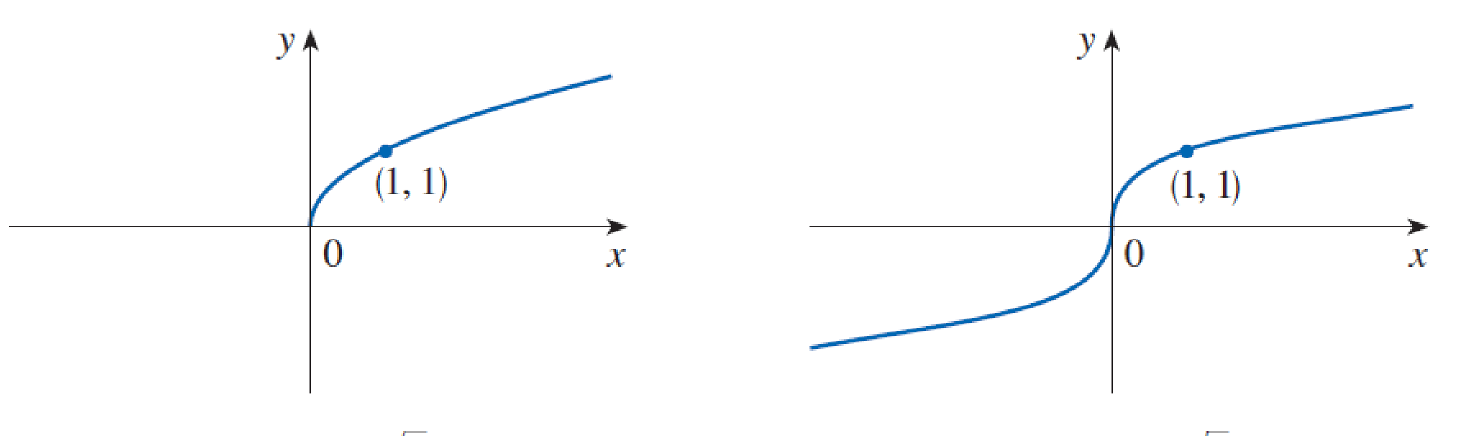


As n increases, the graph becomes flatter near 0 and steeper when |x| > 1.

Power Functions

(2) $a = 1/n$, where n is a positive integer

- The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a root function.
- If $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$
    - For other even values of n, the graph is like that of $f(x) = \sqrt{x}$.
- If $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$.
    - For other odd values of n, the graph is like that of $f(x) = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

Power Functions

(3) $a = -1$

- A reciprocal function $f(x) = x^{-1} = 1/x$

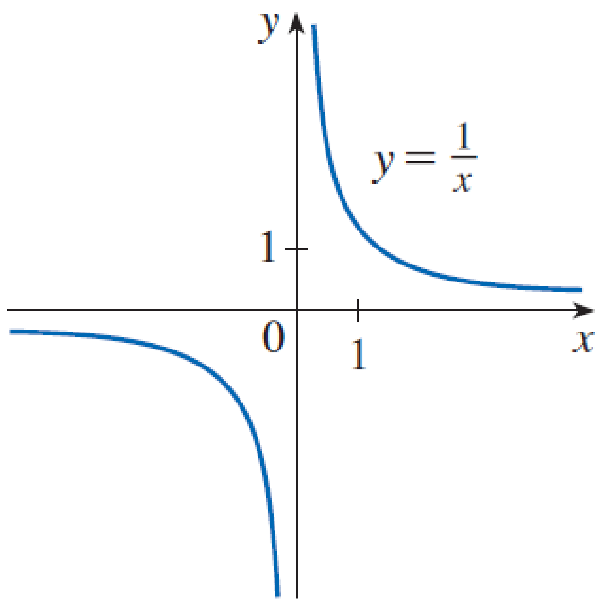


FIGURE 14
The reciprocal function

Power Functions

(4) $a = -2$

- A reciprocal function $f(x) = x^{-2} = 1/x^2$

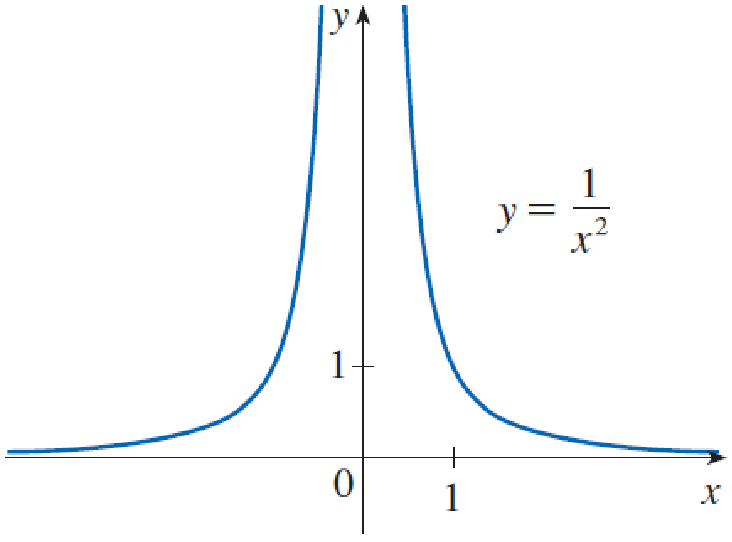


FIGURE 16
The reciprocal of the squaring function

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Rational Functions

- A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

  where P and Q are polynomials.

- The domain consists of all values of x such that $Q(x) \neq 0$.
- Notice that the function in the graph is not defined for $x = \pm 2$

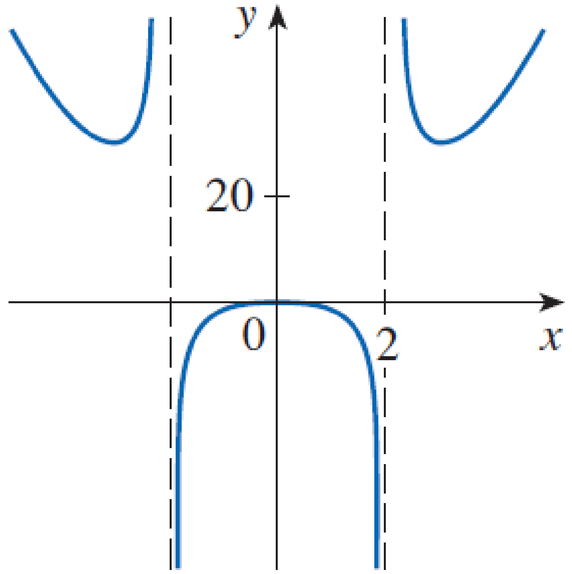

FIGURE 18

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

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Algebraic Functions

- An algebraic function can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.
- Examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

- Functions that are not algebraic are called **transcendental**.
- Examples:

1. Trigonometric functions.
2. Exponential functions.
3. Logarithmic functions.

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Trigonometric Functions

- Trigonometric functions are real functions which relate an angle of a right-angled triangle to ratios of two side lengths.

sine

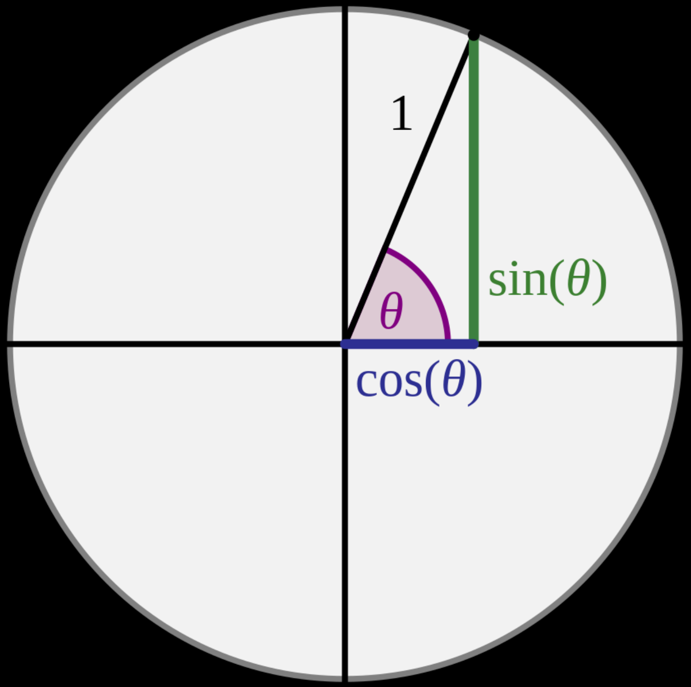
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Trigonometric Functions

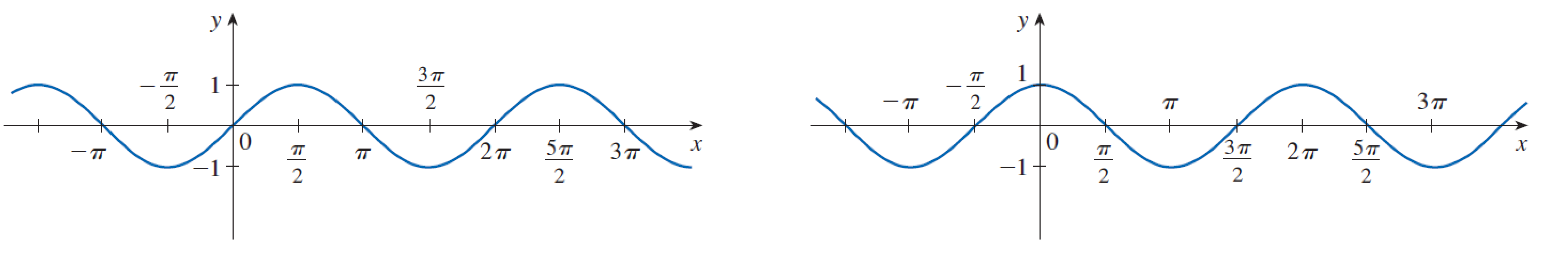
How trigonometric functions are originally computed without calculators?

Trigonometric Functions

- For both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$.

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

- Sine and cosine functions are periodic functions and have period 2π.



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

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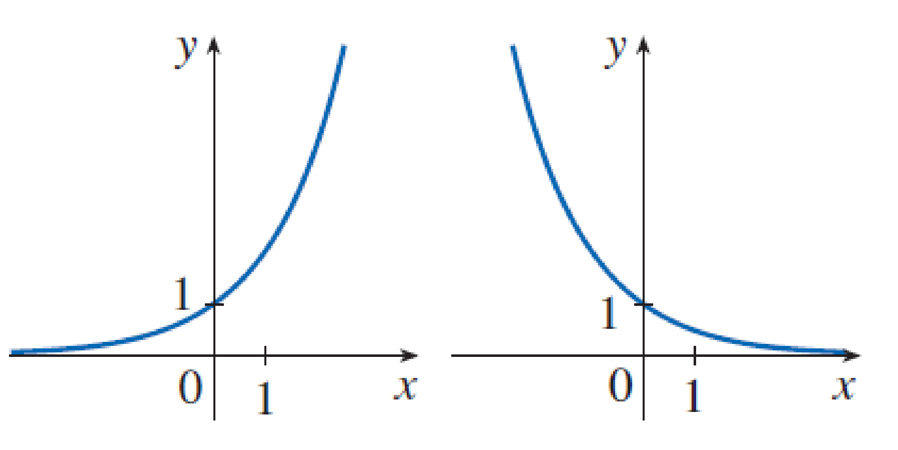
Exponential Functions

- The exponential functions are the functions of the form $f(x) = b^x$, where the base b is a positive constant.



(a) $y = 2^x$ (b) $y = (0.5)^x$

- The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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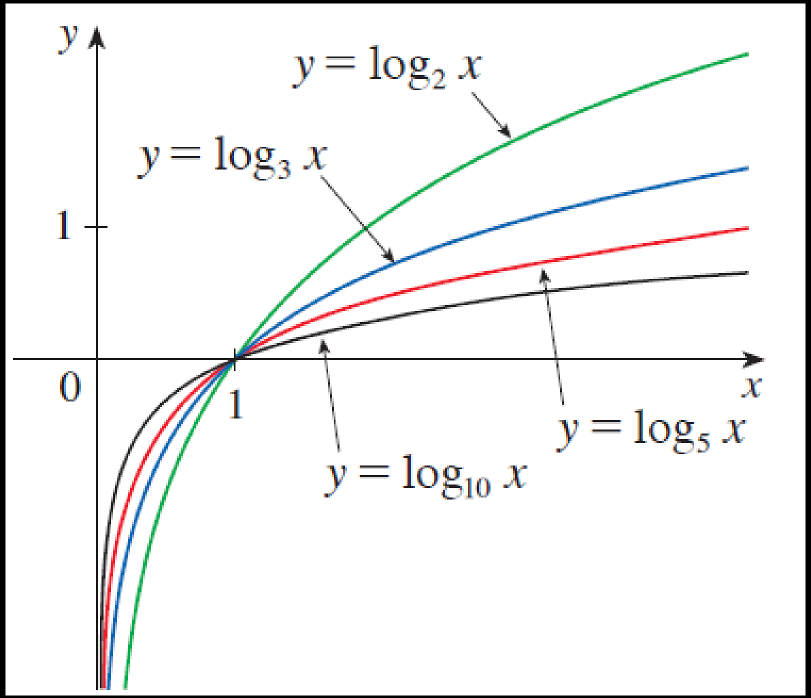
Logarithmic Functions

- The logarithmic functions $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions.
  - The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



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Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$ (c) $r(t) = t^{\sqrt{3}}$ (e) $y = \frac{(\sqrt{x})}{x^2+1}$

2. (a) $f(t) = \frac{3t^2+2}{t}$ (c) $s(t) = \sqrt{t+4}$ (e) $g(x) = \sqrt[3]{x}$

Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$

Polynomial of degree 3.
(This function is also an algebraic function.)

(c) $r(t) = t^{\sqrt{3}}$

Power

(e) $y = \dfrac{(\sqrt{x})}{x^2+1}$

Algebraic

2. (a) $f(t) = \dfrac{3t^2+2}{t}$

Rational function.
(This function is also an algebraic function.)

(c) $s(t) = \sqrt{t+4}$

Algebraic function.
It is a root of a polynomial.

(e) $g(x) = \sqrt[3]{x}$

Power function.
(This also an algebraic function
because it is a root of a polynomial.)

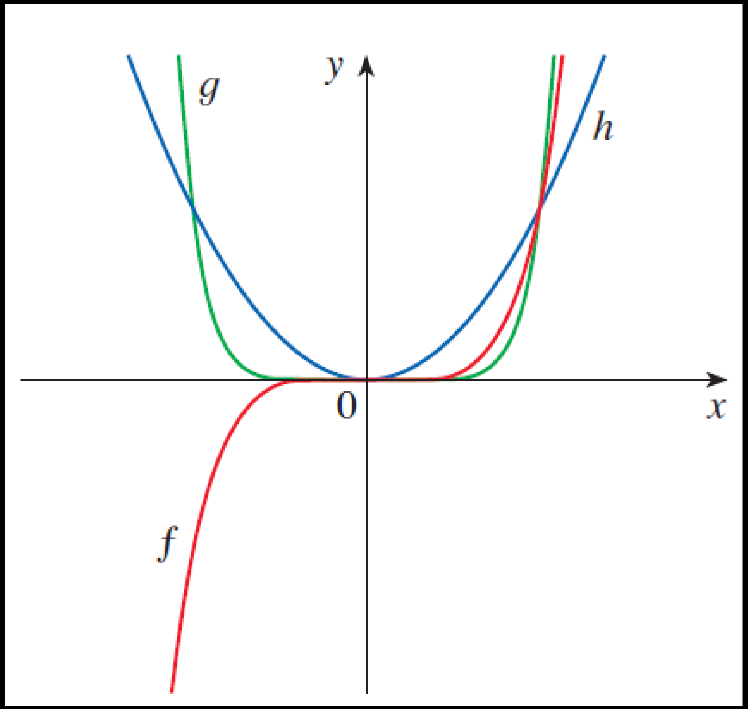
Exercises

3 Match each equation with its graph. Explain your choices. (Don’t use a computer or graphing calculator.)

(a) $y = x^2$

(b) $y = x^5$

(c) $y = x^8$



Exercises

3 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

(a) $y = x^2$

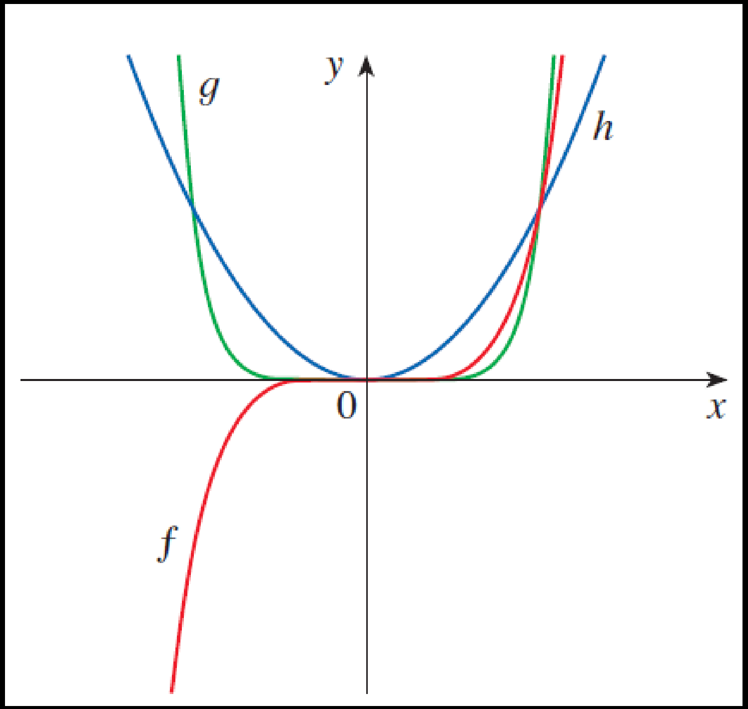
Graph (h), since it's an even function and symmetric about the y-axis

(b) $y = x^5$

Graph (f), since it's an odd function and symmetric about the origin

(c) $y = x^8$

Graph (g), since it's an even function and symmetric about the y-axis, and flatter than (h)



Exercises

5–6 Find the domain of the function.

5) $f(x) = \frac{\cos x}{1 - \sin x}$

6) $\mathrm{g}(x) = \frac{1}{1 - \tan x}$

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The dominator cannot be 0, so $1 - \sin x \neq 0 \iff \sin x \neq 1 \iff x \neq \frac{\pi}{2} + 2n\pi$

Thus, the domain is $\{x \mid x \neq \frac{\pi}{2} + 2n\pi, n$ is an integer$\}$

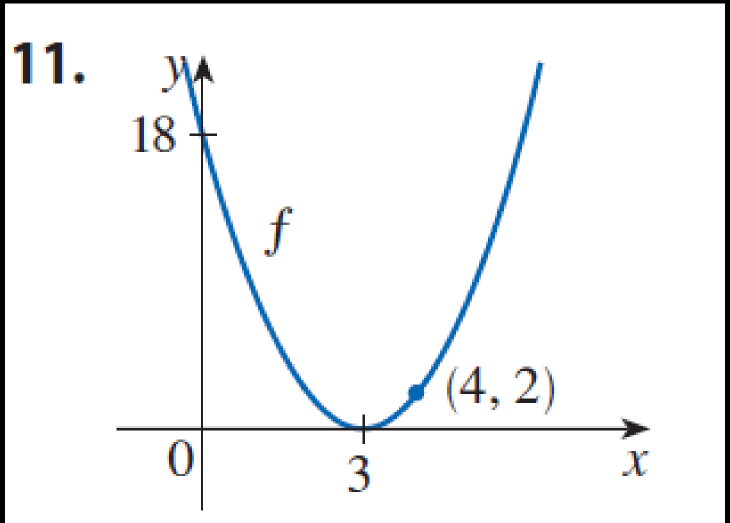
6) $g(x) = \frac{1}{1 - \tan x}$

The dominator cannot be 0, so $1 - \tan x \neq 0 \iff \tan x \neq 1 \iff x \neq \frac{\pi}{4} + 2n\pi$

Thus, the domain is $\{x \mid x \neq \frac{\pi}{4} + 2n\pi, n$ is an integer$\}$

Exercises

11 Find a formula for the quadratic function whose graph is shown.



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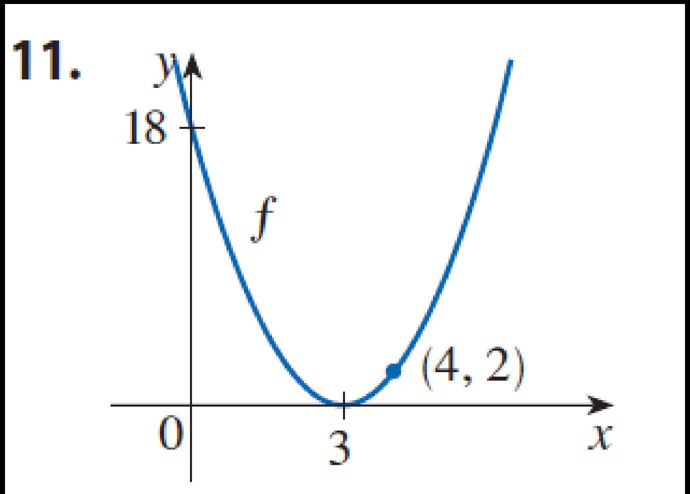

- The graph is a parabola, it is a quadratic function of the form $ax^2 + bx + c$.
- For $x = 0$ (y-intercept), $f(x) = 18$. $\therefore$ $c = 18$.
- For $x = 3$, we have $f(x) = 0$. $\therefore$ $3^2a + 3b + 18 = 0 \Leftrightarrow 9a + 3b = -18 = 3a + b = -6$ → (1)
- For $x = 4$, we have $f(x) = 2$. $\therefore$ $4^2a + 4b + 18 = 2 \Leftrightarrow 16a + 4b = -18 + 2 = 4a + b = -4$ → (2)
- This is a system of two equations in the unknowns a and b and subtracting (1) from (2) gives $a = 2$.
- $\therefore 3(2) + b = -6 \Leftrightarrow b = -12$ so the formula is $f(x) = 2x^2 - 12x + 18$

Exercises

14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the earth's average surface temperature in 2100.

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(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the earth's average surface temperature in 2100.

(a) the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of 0.02 ◦C per year. The T-intercept is 8.50, which represents the average surface temperature in °C in the year 1900.
(b) $t = 2100 - 1900 = 200$ → $T = 0.02 * 200 + 8.50 = 12.50\ C$

Exercises

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a new-born?

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(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a new-born?

(a) $D = 200$, so $c = 0.0417(200)(a + 1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.
(b) For a new-born, a $= 0$, so $c = 8.34$ mg.

Task
1 (b, d, f)
2 (b, d, f)
4
12

References

- Calculus: Early Transcendentals by James Stewart, 9th edition.
- https://www.mathsisfun.com/algebra/mathematical-models.html
- https://en.wikipedia.org/wiki/Machine_learning