Functions and Models

Four Ways to Represent a Function

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Functions

- Functions arise whenever one quantity depends on another.

- Example:

The area A of a circle depends on the radius r of the circle.
The rule that connects r and A is given by the equation $A = \pi r^2$.
With each positive number r there is an associated value of A, and we say that A is a function of r.

Functions

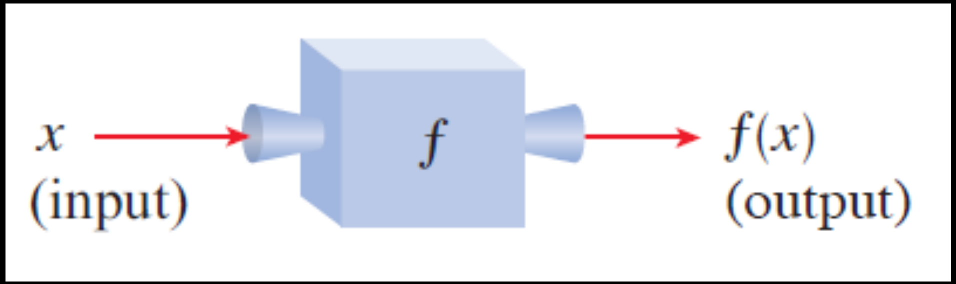
- If f represents the rule that connects A to r, then we express this in function notation as $A = f(r)$.

Function
A **function** f is a rule that assigns to each element x in a set D **exactly** one element, called $f(x)$, in a set E.

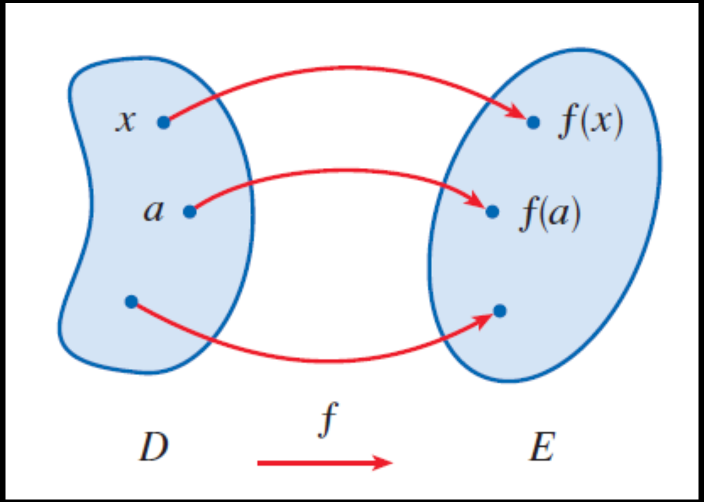
- The set D is called the **domain** of the function.
  - Any number in the domain is called an **independent variable**.
- The set E is called the **range** of the function.
  - Any number in the range is called a **dependent variable**.

Functions

- Think of a function as a machine:



- A function maps each element in the domain to **only** one element in the range.

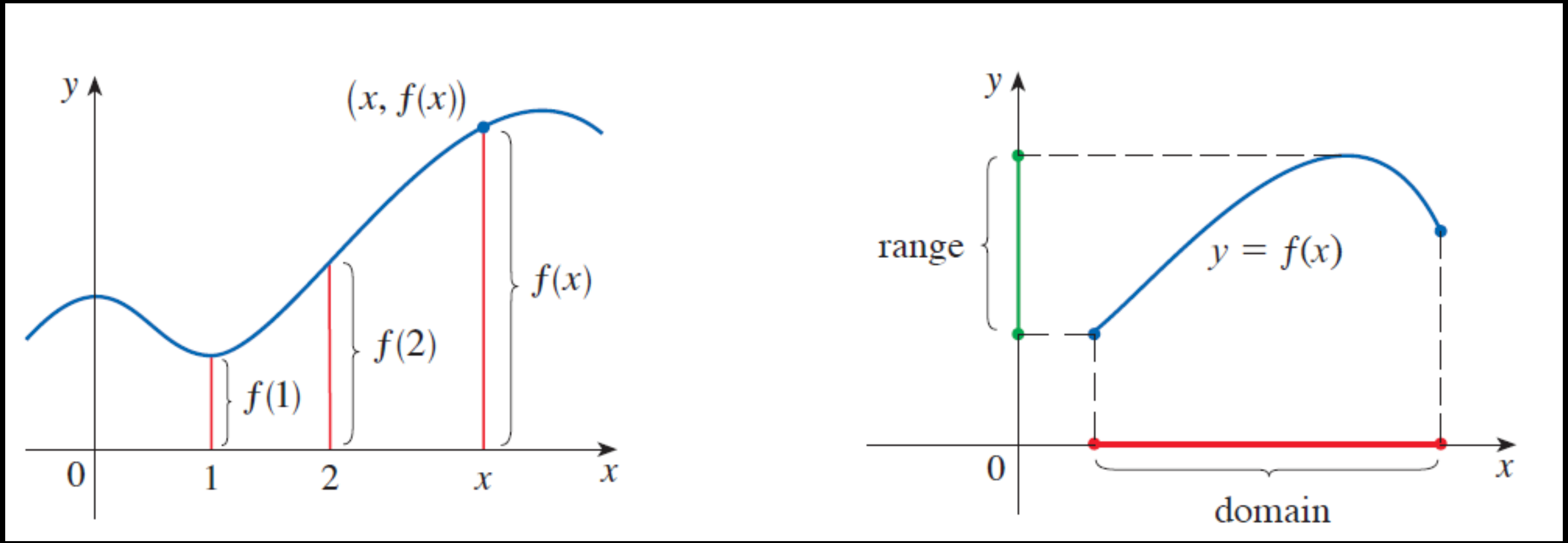


Functions

- A function can be represented as a set of ordered pairs.

$$\{(x, f(x)) \mid x \in D\}$$

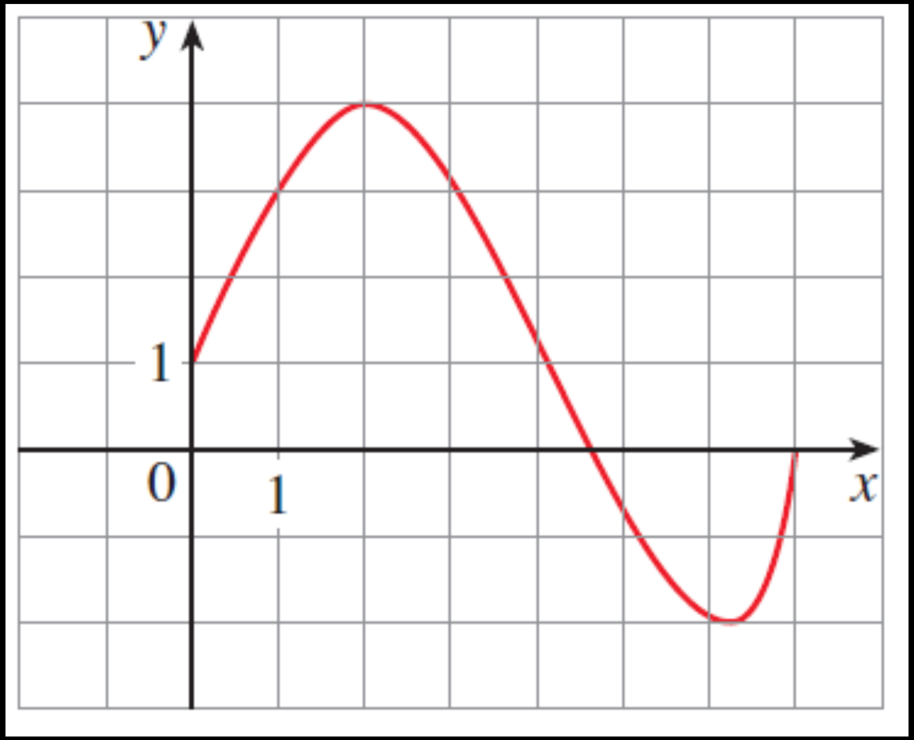
- Plotting a function is useful to visualize its behaviour or its 'life history'.
  - The <u>x-axis</u> tracks the values of the <u>domain</u>
  - The <u>y-axis</u> tracks the values of the <u>range</u>



Functions

Example

- The graph of a function f is shown.
  - Find the values of $f(1)$ and $f(5)$.
  - What are the domain and range of f?



Solution

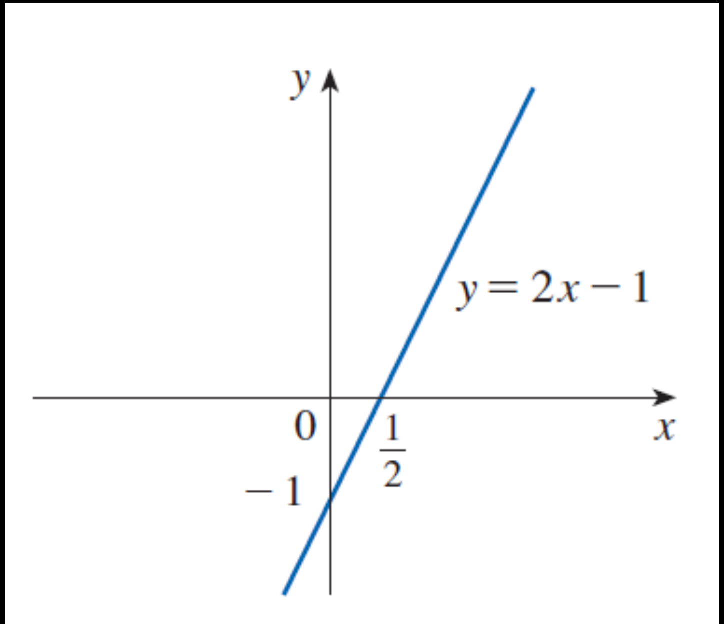
- $f(1) = 3,\ f(5) = -0.7$
- The domain is $[0, 7]$, the range is $[-2, 4]$

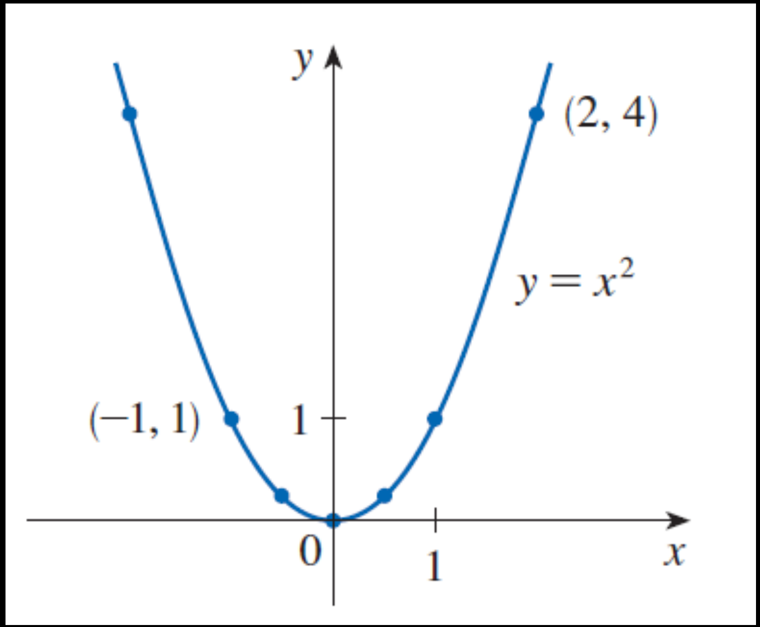
Functions

Example: Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

Solution: Substitute the x by a set of values and compute the output.





Functions

(a) $f(x) = 2x - 1$

```
clc;
clear;

% Define a linear function f(x) = 2x - 1
f = @(x) 2*x - 1;

% Create a vector of x values ranging from -10 to 10
x = linspace(-10, 10);

% Plot the function f(x) over the range of x
p = plot(x, f(x));
% Move the X and Y axes to the origin (0, 0)
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');
```

Functions

(b) $g(x) = x^2$

```
clc; clear;

% Define a linear function g(x) = x^2 % .^ means element wise
power, while ^ used for arrays.
g = @(x) x.^2;

% Create a vector of x values ranging from -10 to 10
x = linspace(-10, 10);

% Plot the function g(x) over the range of x
p = plot(x, g(x));
```

Functions

Example: If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h)-f(a)}{h}$

SOLUTION We first evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a + h) &= 2(a + h)^2 - 5(a + h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a + h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

Functions

Example: Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$ (b) $g(x) = \frac{1}{x^2 - x}$

Solution

(a) The square root of a negative number is undefined, the domain of x must confirm that $x + 2 \geq 0$, so $x \geq -2$
$\therefore x = [-2, \infty)$

(b) Since $g(x) = \frac{1}{x^2 - x} = \frac{1}{x\,(x-1)}$

$\because$ division by 0 is undefined, $\therefore x \neq 0$ or $x \neq 1$.

So, the domain of x is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

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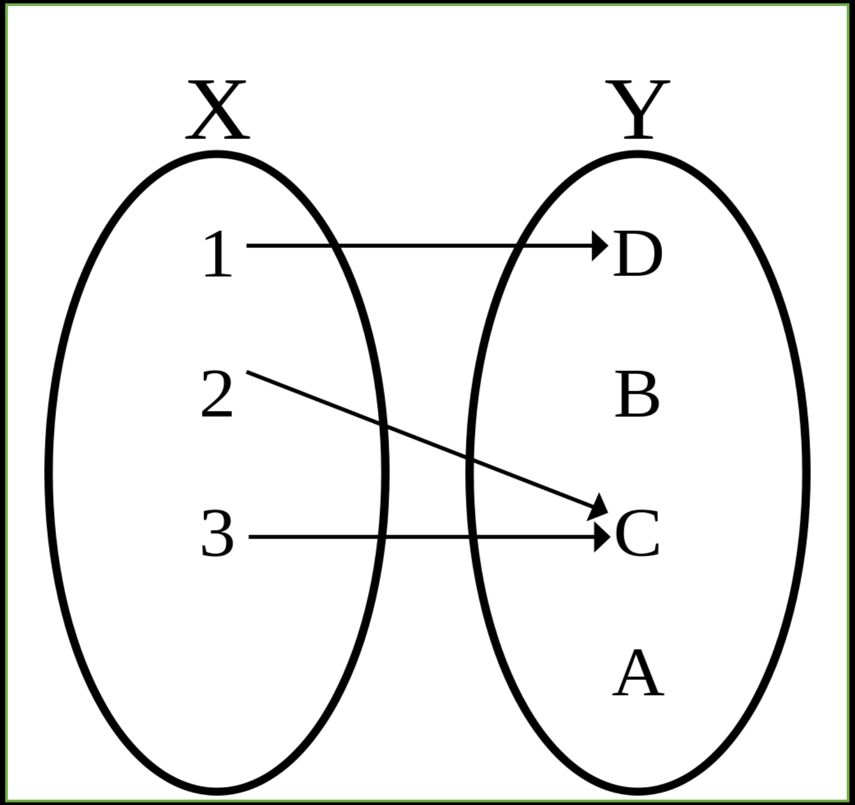
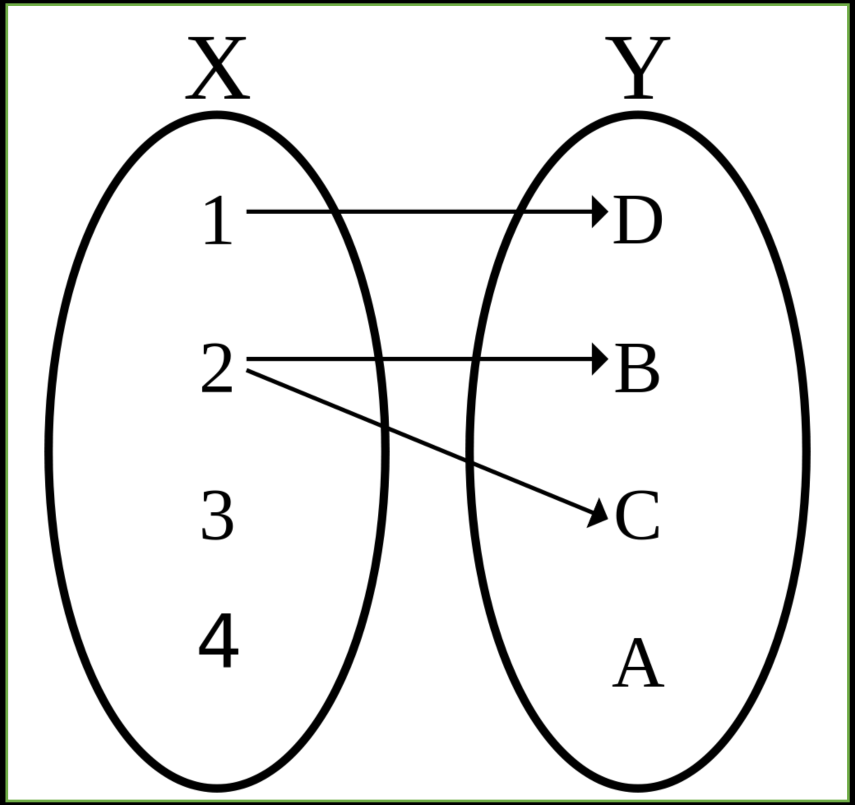
Which Rules Define Functions?

- Not every equation defines a function.
- The equation $y = x^2$ defines y as a function of x because the equation determines exactly one value of y for each value of x.
- The equation $y^2 = x$ does **not** define a function because some input values x correspond to more than one output y;
  - for instance, for the input $x = 4$ the equation gives the outputs $y = 2$ and $y = -2$.

Which Rules Define Functions?



This is a function.

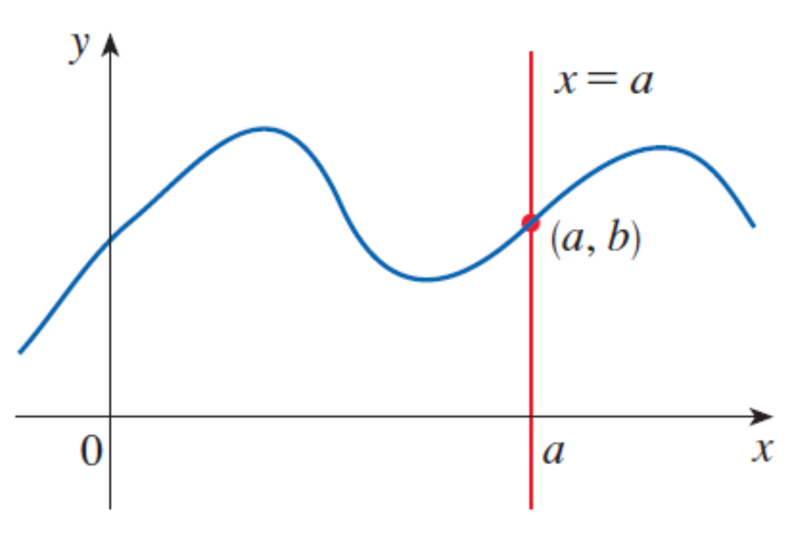


This is **not** a function.

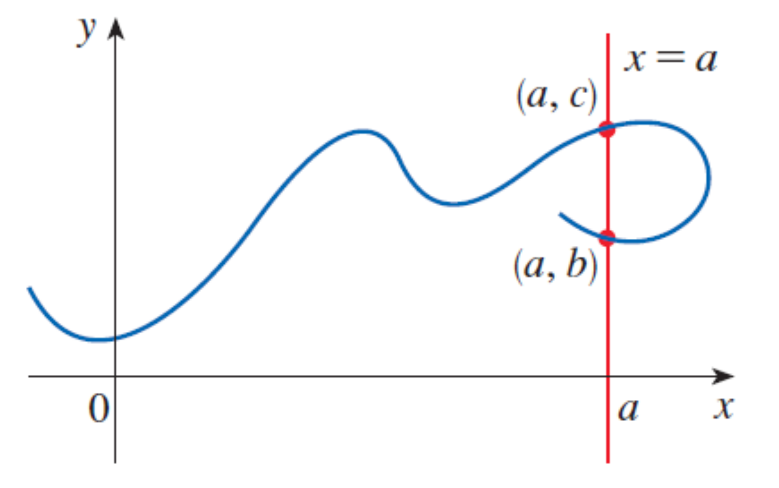
Which Rules Define Functions?

- For curves drawn in the xy-plane, we apply the **vertical line test**

The Vertical Line Test
A curve in the xy-plane is the graph of a function of x **if and only if** no vertical line intersects the curve more than once.



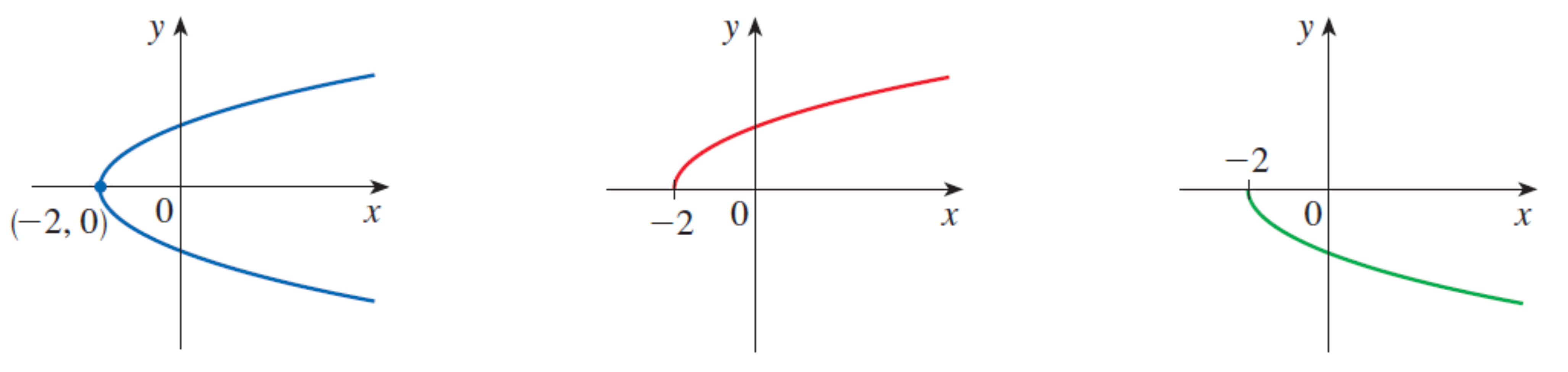
(a) This curve represents a function.



(b) This curve doesn’t represent a function.

Which Rules Define Functions?

- The parabola $x = y^2 - 2$ is not a function.
- Note that $x = y^2 - 2 \rightarrow y^2 = x + 2 \rightarrow y = \pm\sqrt{x+2}$
  - $y = \sqrt{x+2}$ is a function
  - $y = -\sqrt{x+2}$ is a function



(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

Which Rules Define Functions?

- $x = y^2 - 2, \quad y = \sqrt{x+2}, \qquad y = -\sqrt{x+2}$

```
clc; clear;

f1 = @(x) x.^2 - 2;
f2 = @(x) sqrt(x + 2);
f3 = @(x) -sqrt(x - 2);

x = linspace(-20, 20);

subplot(1, 3, 1);
plot(f1(x), x);
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin’);

subplot(1, 3, 2);
plot(f2(x), x);
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin’);

subplot(1, 3, 3);
plot(f3(x), x);
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');
```

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Piecewise Defined Functions

- **Piecewise Function:** a function defined by different formulas given a condition for their domain.

$$f(x) \begin{cases} x^2 & \text{if } x < 0 \\ 10 - x & \text{if } x >= 0 \text{ and } x < 10 \\ 2x + 3 & \text{if } x >= 10 \end{cases}$$

Piecewise Defined Functions

- **Example**: A function f is defined by

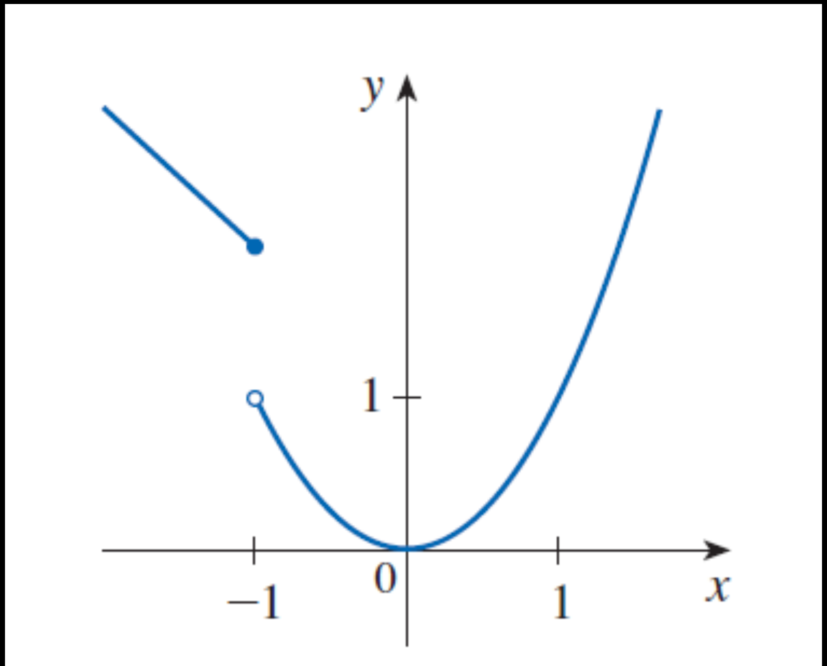
$$f(x) = \begin{cases} 1 - x & if\ x \leq -1 \\ x^2 & if\ x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution

$$\because -2 \leq -1 \ \rightarrow \ \therefore f(-2) = 1 - (-2) = 3$$
$$\because -1 \leq -1 \ \rightarrow \ \therefore f(-1) = 1 - (-1) = 2$$
$$\because 0 > -1 \ \rightarrow \ \therefore f(0) = 0^2 = 0$$

Piecewise Defined Functions



Piecewise Defined Functions

$$f(x) = \begin{cases} 1 - x & if\ x \leq -1 \\ x^2 & if\ x > -1 \end{cases}$$

```
clc; clear;

% Define the function f(x) that computes values based on the input x
function res = f(x)
  res = zeros(size(x)); % Initialize the result array to zeros

  % Define logical conditions for the two cases
  ind1 = x <= -1; % Condition for x values less than or equal to -1
  ind2 = x > -1;  % Condition for x values greater than -1

  % Assign values to res based on the conditions
  res(ind1) = 1 - x(ind1);  % For x <= -1, calculate 1 - x
  res(ind2) = x(ind2) .^ 2; % For x > -1, calculate x squared
endfunction

%% Alternative way to define the function using an anonymous function:
%% f = @(x) (x <= -1) .* (1 - x) + (x > -1) .* (x.^2);

x = linspace(-5, 5, 100);
plot(x, f(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');
```

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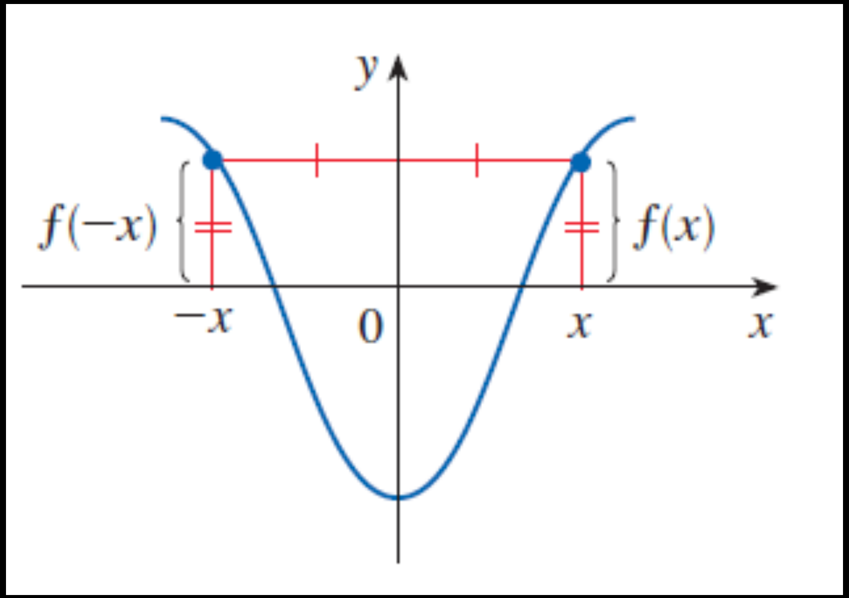
Even and Odd Functions

- **Even function** satisfies $f(-x) = f(x)$ for every number x in the domain.
- Example: $f(x) = x^2$ is an even function

$$f\,(-x) = (-x)^2 = x^2 = f(x)$$

- The graph of the function is symmetric with respect to the y-axis.

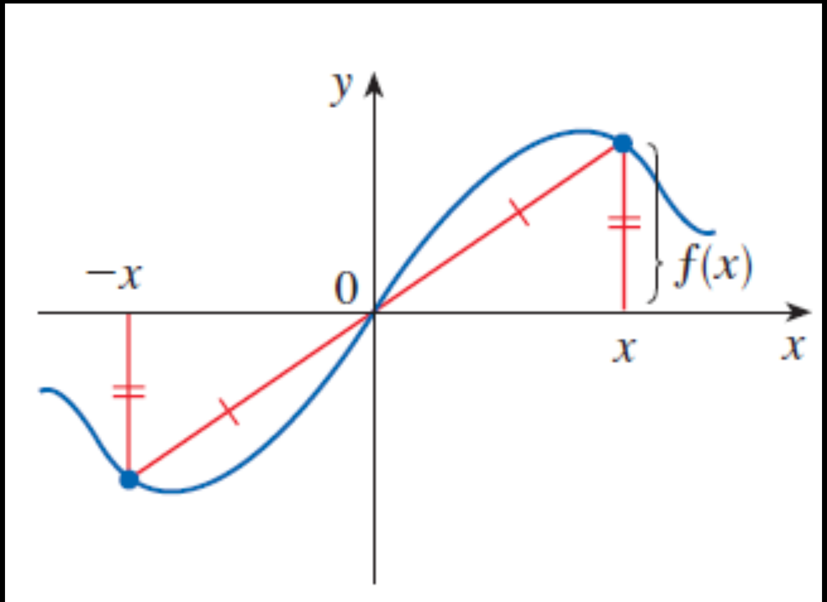


Even and Odd Functions

- **Odd function** satisfies $f(-x) = -f(x)$ for every number x in the domain.
- Example: $f(x) = x^3$ is an odd function

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

- The graph of the odd function is symmetric around the origin.



Even and Odd Functions

- $f(x) = x^3$
- $f(x) = x^2$

```
clc; clear;

%% create an even function that is x^2
f_even = @(x) x.^2

%% create an odd function that is x^3
f_odd = @(x) x.^3

x = linspace(-10, 10);

% plot the even function in the first subplot
subplot(1, 2, 1);
plot(x, f_even(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');

%% plot the odd function in the second subplot
subplot(1, 2, 2);
plot(x, f_odd(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');
```

Even and Odd Functions

- **Example**: Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Even and Odd Functions

- **Solution**:

(a) $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

Odd function

(b) $\mathrm{g}(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

Even function

(c) $\mathrm{h}(-x) = 2(-x) - (-x)^2 = -2x\ - x^2 = -(2x + x^2) \neq f(x) \neq -f(x)$

Neither is odd or even function

Even and Odd Functions

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

```
clc; clear;
f = @(x) x.^5 + x;
g = @(x) 1 - x.^4;
h = @(x) 2*x - x.^2;
x = linspace(-10, 10);

subplot(1, 3, 1);
plot(x, f(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');

subplot(1, 3, 2);
plot(x, g(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');

subplot(1, 3, 3);
plot(x, h(x));
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');
```

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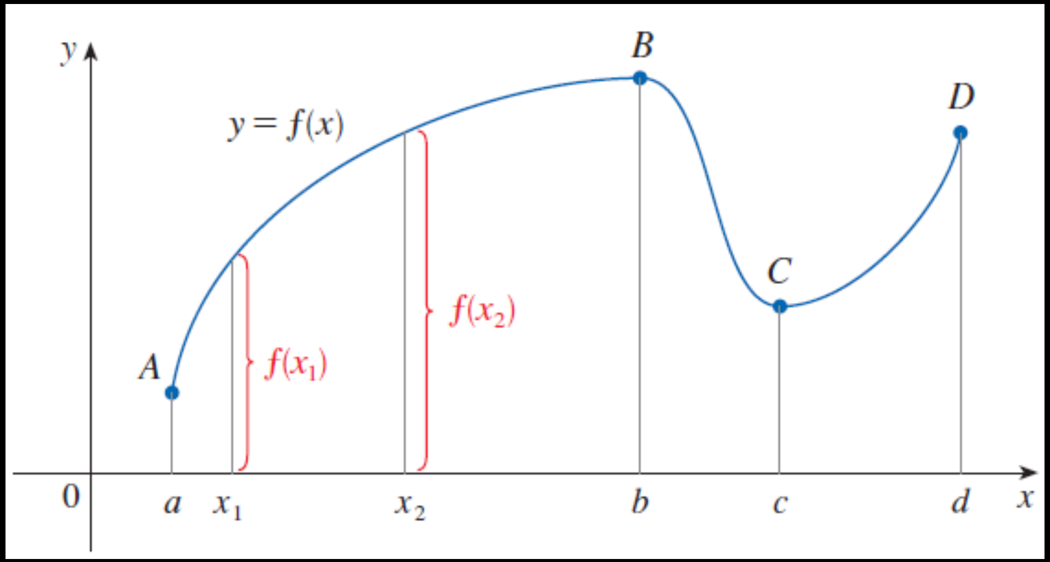
Increasing and Decreasing Functions

- The graph rises from A to B, falls from B to C, and rises again from C to D.



  - The function f is said to be increasing on the interval $[a, b]$,
  - decreasing on $[b, c]$,
  - and increasing again on $[c, d]$.

Increasing and Decreasing Functions

Increasing function
A function f is called **increasing** on an interval I if $f(x_1) < f(x_2) \qquad$ whenever $x_1 < x_2$ in I

Decreasing function
A function f is called **decreasing** on an interval I if $f(x_1) > f(x_2) \qquad$ whenever $x_1 > x_2$ in I

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Exercises

1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$ is it true that $f = g$?

Exercises

1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$ is it true that $f = g$?

True. Both functions give the same output values for every input value $x = u$, so f and g are equal.

```
clc; clear;

f = @(x) x + sqrt(2-x);
g = @(u) u + sqrt(2-u);

x = linspace(-10, 10);

f_res = f(x);
g_res = g(x);

isequal(f_res, g_res)
```

Exercises

2. If

$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

Is it true that $f = g$?

Exercises

2. If

$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

Is it true that $f = g$?

$\boxed{\text{False. The function } f(x) \text{ is undefined for } x = 1\text{, whereas } g(1) = 1.}$

Exercises

3. The graph of a function t is given.

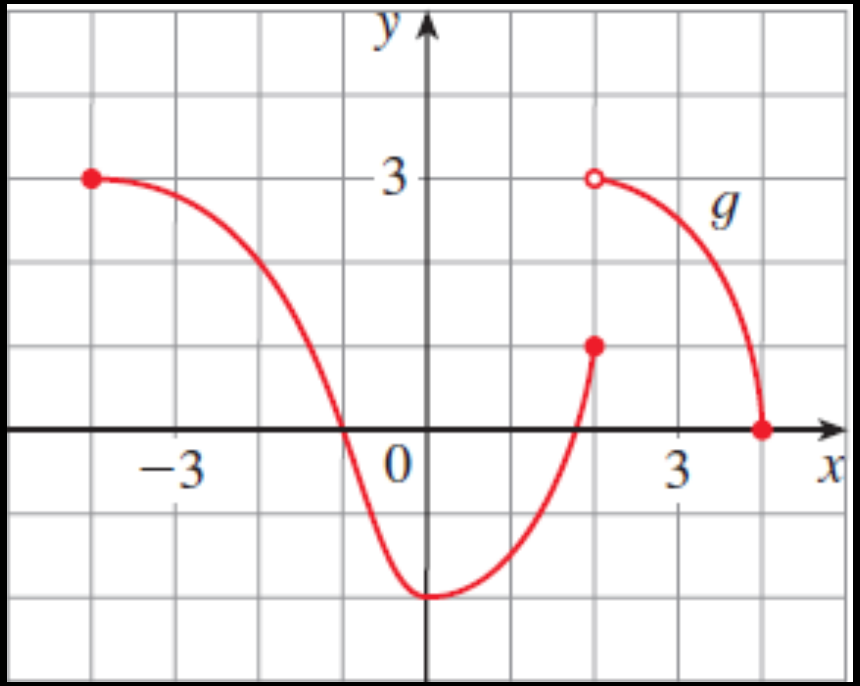
(a) State the values of $g(-2), g(0), g(2)$, and $g(3)$.

(b) For what value(s) of x is $g(x) = 3$?

(c) For what value(s) of x is $g(x) \leq 3$?

(d) State the domain and range of t.

(e) On what interval(s) is t increasing?



Exercises

3. The graph of a function t is given.

(a) State the values of $g(-2), g(0), g(2)$, and $g(3)$.

$g(-2) = 2, \quad g(0) = -2, \quad g(2) = 1, \quad g(3) = 2.5$

(b) For what value(s) of x is $g(x) = 3$?

For $x = -4$

(c) For what value(s) of x is $g(x) \leq 3$?

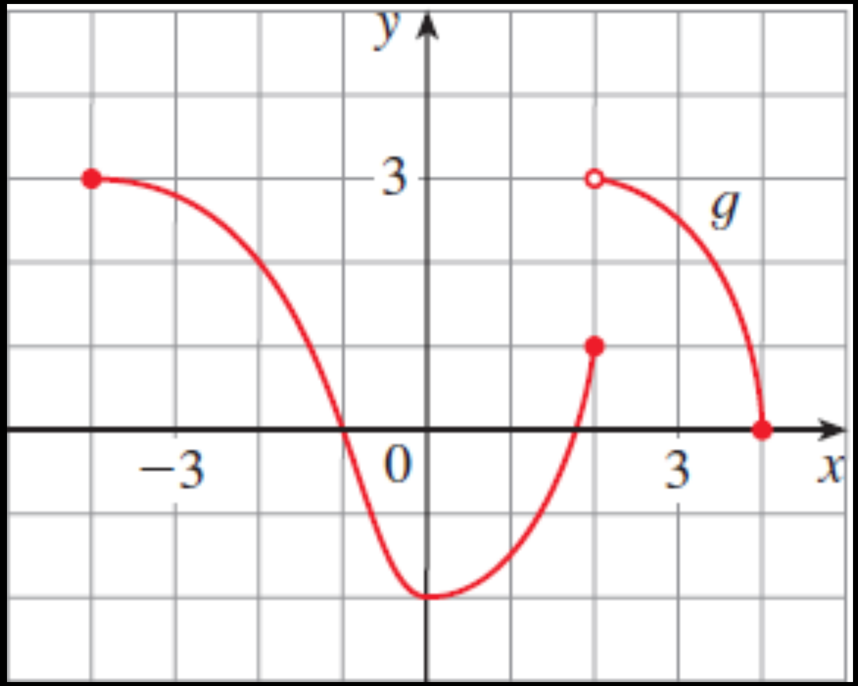
On the interval [-4, 4]

(d) State the domain and range of t.

Domain: [-4, 4]. Range: [-2, 3]

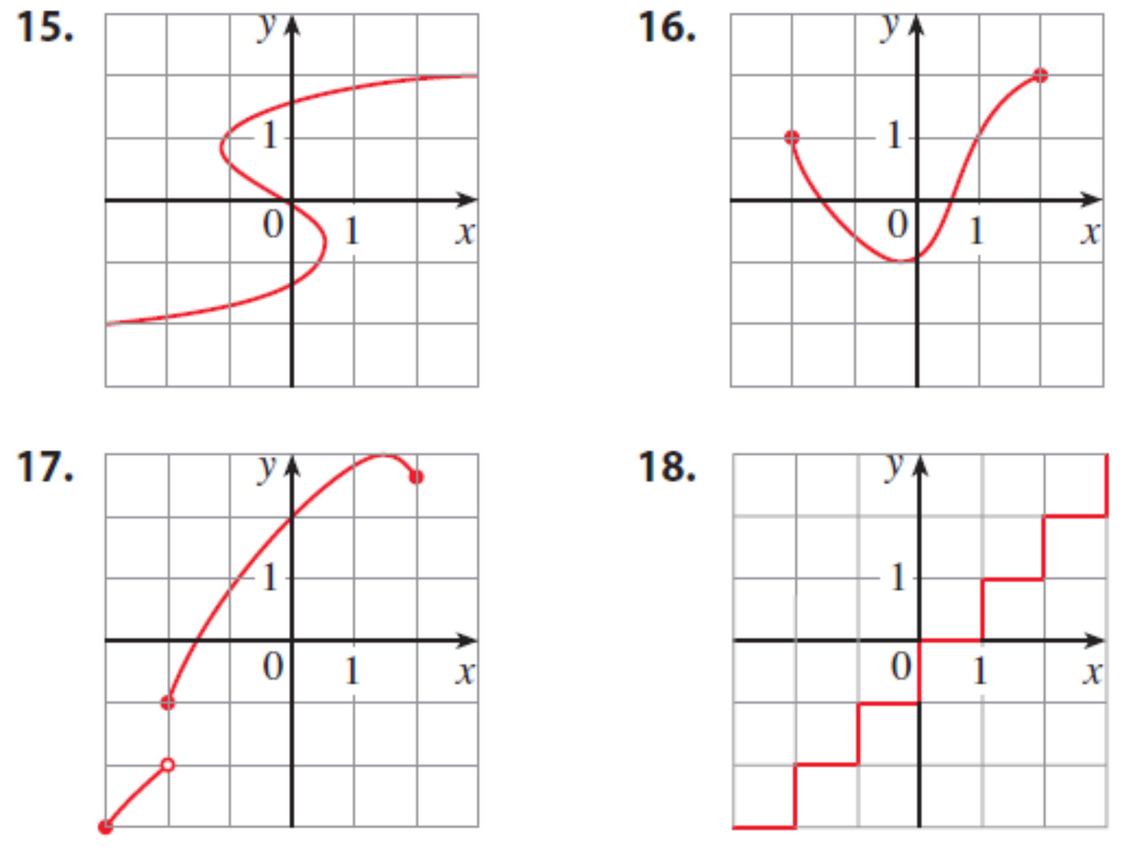
(e) On what interval(s) is t increasing?

[0, 2]



Exercises

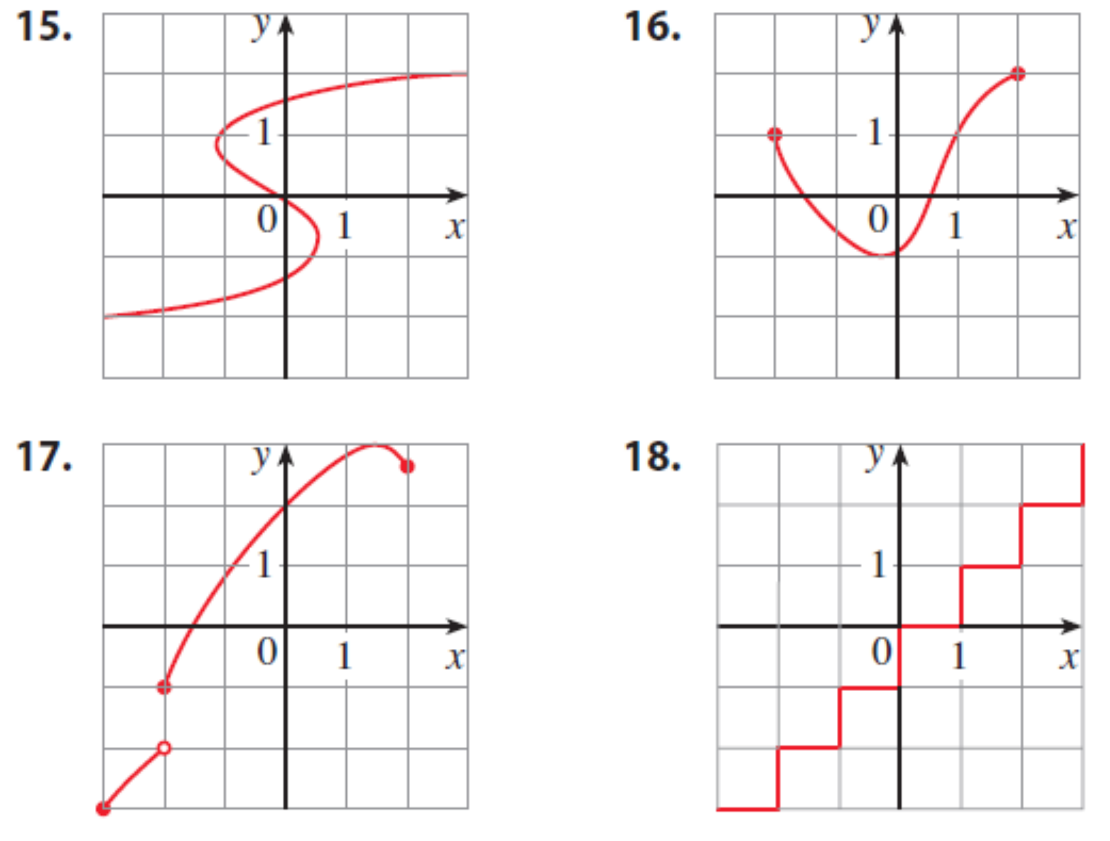
15–18 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.



Exercises

15–18 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

No, the curve is not the graph of a function because a vertical line intersects the curve more than once.



Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

No, the curve is not the graph of a function since for $x = 0, \pm 1$, and ± 2, there are infinitely many points on the curve.

Exercises

33. If $f(x) = 3x^2 - x + 2$, find $f(2), f(-2), f(a), f(-a), f(a+1)$, $2f(a), f(2a), f(a^2), [f(a)]^2$ and $f(a+h)$

Exercises

33. If $f(x) = 3x^2 - x + 2$, find $f(2), f(-2), f(a), f(-a), f(a+1)$, $2f(a), f(2a), f(a^2), [f(a)]^2$ and $f(a+h)$

$f(x) = 3x^2 - x + 2.$

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$

$f(a) = 3a^2 - a + 2.$

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4.$

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

Exercises

Find the domain of the function.

39. $f(x) = \frac{(x+4)}{x^2-9}$

40. $f(x) = \frac{x^2+1}{x^2+4x-21}$

Exercises

Find the domain of the function.

39. $f(x) = \frac{(x+4)}{x^2-9}$

$\boxed{\{x \in \mathbb{R} \mid x \neq -3, 3\}}$

40. $f(x) = \frac{x^2+1}{x^2+4x-21}$

The function is defined for all x, except for $x^2 + 4x - 21 = 0$.

$x^2 + 4x - 21 = 0 \Leftrightarrow (x+7)(x-3) \Leftrightarrow x = -7 \text{ or } x = 3$

Thus, the domain is $\{x \in \mathbb{R} \mid x \neq -7, 3\}$

Exercises

Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function.

49. $f(x) = \begin{cases} x^2 + 2 & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$

50. $f(x) = \begin{cases} 5 & if\ x < 2 \\ \frac{1}{2}x - 3 & if x \geq 2 \end{cases}$

Exercises

Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function.

49. $f(x) = \begin{cases} x^2 + 2 & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$

$$\boxed{f(-3) = (-3)^2 + 2 = 11.\ \ f(0) = 0.\ \ f(2) = 2}$$

50. $f(x) = \begin{cases} 5 & if\ x < 2 \\ \frac{1}{2}x - 3 & if x \geq 2 \end{cases}$

$$\boxed{f(-3) = 5.\ \ f(0) = 5.\ \ f(2) = -2}$$

Exercises

49. $f(x) = \begin{cases} x^2 + 2 & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$

50. $f(x) = \begin{cases} 5 & if\ x < 2 \\ \frac{1}{2}x - 3 & if\, x \geq 2 \end{cases}$

```
clc;  clear;
% Function definition for 'f', which takes input 'x'
function res = f(x)
  ind1 = x < 0;
  ind2 = x >= 0;

  res(ind1) = x(ind1) .^ 2 + 2;
  res(ind2) = x(ind2);
endfunction % End of function 'f'

% An anonymous function 'g' which takes 'x' as input
g = @(x) (x<2) .* 5 + (x>=2) .* (1/2 .* x - 3);

f(-3)
f(0)
f(2)
disp("----------");
disp(g(-3));
disp(g(0));
disp(g(2));
```

Exercises

Find a formula for the function whose graph is the given curve.

59. The line segment joining the points $(1, -3)$ and $(5,\ 7)$

60. The line segment joining the points $(-5, 10)$ and $(7, -10)$

61. The bottom half of the parabola $x + (y - 1)^2 = 0$

62. The top half of the circle $x^2 + (y - 2)^2 = 4$

Exercises

59. The line segment joining the points $(1, -3)$ and $(5,\ 7)$

1. Compute slope: $\frac{7+3}{5-1} = \frac{10}{4} = \frac{5}{2}$
2. Formulate the equation: $\left(y - (-3)\right) = \frac{5}{2}(x - 1)$
3. Compute the y-intercept:
   Thus, $y + 3 = \frac{5}{2}x - \frac{5}{2} \rightarrow y - \frac{5}{2}x = -\frac{5}{2} - 3$
4. Set $x = 0$, $y = -\frac{11}{2}$
5. Thus, the function is: $f(x) = \frac{5}{2}x - \frac{11}{2}$

Exercises

60. The line segment joining the points $(-5, 10)$ and $(7, -10)$

1. Compute the slope: $\frac{-10-10}{7+5} = -\frac{20}{12} = -\frac{5}{3}$
2. Formulate the equation: $(y - 10) = -\frac{5}{3}(x + 5)$
3. Compute the y-intercept:
   Thus, $y + \frac{5}{3}x = -\frac{25}{3} + 10$
4. Set $x = 0$, $y = \frac{5}{3}$
5. Thus, the function is: $f(x) = -\frac{5}{3}x + \frac{5}{3}$

Exercises

61. The bottom half of the parabola $x + (y-1)^2 = 0$

1. $(y-1)^2 = -x$
2. $y - 1 = \pm\sqrt{-x}$
3. $y = \pm\sqrt{-x} + 1$
4. We need the bottom half, which is the negative part: $y = -\sqrt{-x} + 1$

Exercises

62. The top half of the circle $x^2 + (y-2)^2 = 4$

1. $(y-2)^2 = 4 - x^2$
2. $y - 2 = \pm\sqrt{4 - x^2}$
3. $y = \pm\sqrt{4 - x^2} + 2$
4. We need the top half, which is the positive part: $y = \sqrt{4 - x^2} + 2$

Task
4
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