Limits and Derivatives





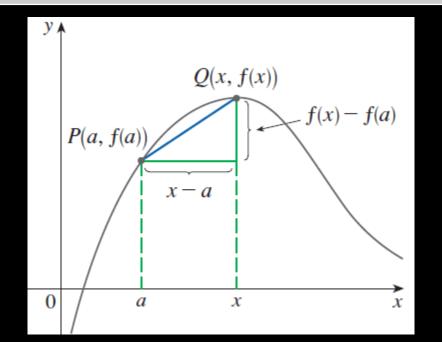
The Derivative as a Function

Tangent line

The **tangent line** to the curve y = f(x) at the point P(a, f(a)) is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



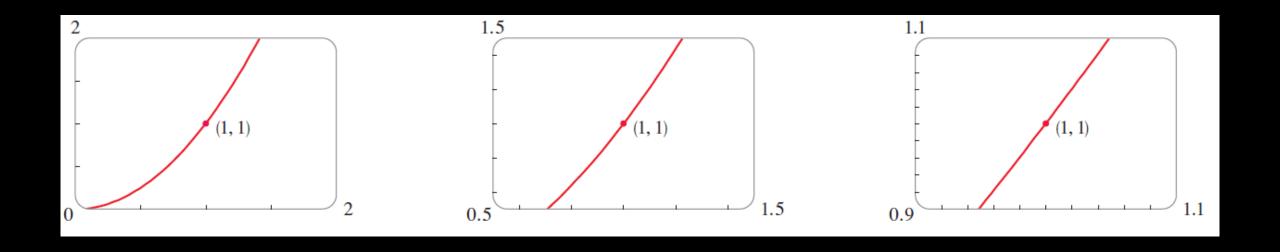
- Example 1: Find an equation of the tangent line to the parabola $y = x^2$ at the point P(1,1).
- Solution: Here we have a=1 and $f(x)=x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$
 we find that an equation of the tangent line at (1, 1) is
$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

$$y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$$

- The slope of the curve at a point.
 - The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line.



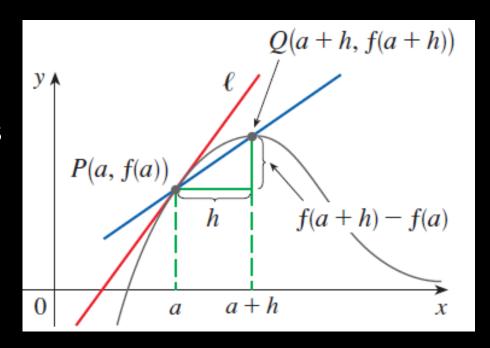
Another expression for the slope of a tangent line that is easier to use:

$$\circ$$
 Assuming $x = a + h$

$$m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$$

As x approaches a, h approaches 0 and so the expression for the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



• Example 2: Find an equation of the tangent line to the hyperbola y=3/x at the point (3,1).

SOLUTION Let f(x) = 3/x. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

• The equation of the tangent line to the hyperbola y = 3/x at (3,1):

The output is

$$y = 2 - \frac{x}{3}$$

Which is the same as

$$y - 1 = -\frac{1}{3}(x - 3)$$

```
setenv("PYTHON", "./venv/bin/python");
clc; clear;
pkg load symbolic;
SVMS X:
y = 3 / x; % Define the hyperbola equation
% Calculate the derivative
dy_dx = diff(y, x);
% Evaluate the derivative at x = 3
slope = double(subs(dy_dx, x, 3));
x1 = 3:
y1 = 1;
% Equation of the tangent line: y - y1 = slope * (x - x1)
% Rearranged to y = slope * (x - x1) + y1
tangent_line = slope * (x - x1) + y1;
disp(tangent_line);
```

This special limit is called the derivative.

Derivative

The derivative of a function f at a number a, denoted by

$$\dot{f} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the limit exists.

• If we write x = a + h, then we have h = x - a. Therefore, an equivalent way of stating the definition of the derivative is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

• Example 4: find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h-4) = -4$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(a+h)^2 - 8(a+h) + 9 \right] - \left[a^2 - 8a + 9 \right]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) = 2a - 8$$

```
setenv("PYTHON", "./venv/bin/python");
clc; clear;
pkg load symbolic;
syms x:
f = x^2 - 8*x + 9
dx_dy = diff(f, x);
% Evaluate the derivative at x = 2
slope = double(subs(dx_dy, x, 2));
disp(slope);
```

• Example 5: find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number

a (a > 0).

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} \cdot \frac{\sqrt{x}\sqrt{a}}{\sqrt{x}\sqrt{a}}$$

$$= \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} = \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} \cdot \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}}$$

$$= \lim_{x \to a} \frac{-(x - a)}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})}$$

$$= \frac{-1}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{-1}{a \cdot 2\sqrt{a}} = -\frac{1}{2a^{3/2}}$$

• The derivative of the function $f(x) = 1/\sqrt{x}$ at the number a (a > 0).

The relationship of "tangent line" to "derivative"

Tangent line equation ← slope = derivative of the function define curve

• Equation of the tangent line to the curve y=f(x) at the point (a,f(a)): $y-y_1=f'(a)(x-x_1)$

- 1. A curve has equation y = f(x).
- (a) Write an expression for the slope of the secant line through the points P(3, f(3)) and Q(x, f(x)).

(b) Write an expression for the slope of the tangent line at P.

(a) This is just the slope of the line through two points:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}$$

(b) This is the limit of the slope of the secant line PQ as Q approaches P

$$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

Find an equation of the tangent line to the curve at the given point

5)
$$y = 2x^2 - 5x + 1$$
, (3,4)

6)
$$y = x^2 - 2x^3$$
, $(1, -1)$

5)
$$y = 2x^2 - 5x + 1$$
, (3,4)
 $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 3} \frac{(2x^2 - 5x + 1) - 4}{x - 3} = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x + 1)(x - 3)}{x - 3}$
 $= \lim_{x \to 3} (2x + 1) = 2(3) + 1 = 7$
Tangent line: $y - 4 = 7(x - 3) \Leftrightarrow y - 4 = 7x - 21 \Leftrightarrow y = 7x - 17$
6) $y = x^2 - 2x^3$, (1,-1)

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h)^2 - 2(1+h)^3 \right] - (-1)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 2 - 6h - 6h^2 - 2h^3 + 1}{h} = \lim_{h \to 0} \frac{-2h^3 - 5h^2 - 4h}{h} = \lim_{h \to 0} \frac{-h(2h^2 + 5h + 4)}{h}$$

$$= \lim_{h \to 0} \left[-(2h^2 + 5h + 4) \right] = -4$$

Tangent line:
$$y - (-1) = -4(x - 1) \Leftrightarrow y + 1 = -4x + 4 \Leftrightarrow y = -4x + 3$$

```
5) y = 2x^2 - 5x + 1, (3,4)
```

```
setenv("PYTHON", "./venv/bin/python");
clc: clear: pkg load symbolic:
syms x;
y = 2*x^2 - 5*x + 1;
dx_dy = diff(y, x);
slope = double(subs(dx_dy, x, 3));
% Given point (3, 4)
x1 = 3;
v1 = 4;
tangent_line = slope * (x - x1) + y1;
disp(tangent_line);
```

6)
$$y = x^2 - 2x^3$$
, $(1, -1)$

Find f'(a) at the given number a.

19)
$$f(x) = \sqrt{4x + 1}$$
, $a = 6$

21)
$$f(x) = \frac{x^2}{x+6}$$
, $a = 3$

19)
$$f(x) = \sqrt{4x + 1}$$
, $a = 6$

$$f'(6) = \lim_{h \to 0} \frac{f(6+h) - f(6)}{h} = \lim_{h \to 0} \frac{\sqrt{4(6+h) + 1} - 5}{h} = \lim_{h \to 0} \frac{\sqrt{25 + 4h} - 5}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{25 + 4h} - 5\right)\left(\sqrt{25 + 4h} + 5\right)}{h\left(\sqrt{25 + 4h} + 5\right)} = \lim_{h \to 0} \frac{(25 + 4h) - 25}{h\left(\sqrt{25 + 4h} + 5\right)} = \lim_{h \to 0} \frac{4h}{h\left(\sqrt{25 + 4h} + 5\right)}$$

$$= \lim_{h \to 0} \frac{4}{\sqrt{25 + 4h} + 5} = \frac{4}{5 + 5} = \frac{2}{5}$$

21)
$$f(x) = \frac{x^2}{x+6}$$
, $a = 3$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\frac{x^2}{x + 6} - 1}{x - 3} = \lim_{x \to 3} \frac{\frac{x^2 - (x + 6)}{x + 6}}{x - 3} = \lim_{x \to 3} \frac{x^2 - x - 6}{(x + 6)(x - 3)}$$
$$= \lim_{x \to 3} \frac{(x + 2)(x - 3)}{(x + 6)(x - 3)} = \lim_{x \to 3} \frac{x + 2}{x + 6} = \frac{3 + 2}{3 + 6} = \frac{5}{9}$$

29. If $f(x) = 3x^2 - x^3$, find f'(1) and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

29. If $f(x) = 3x^2 - x^3$, find f'(1) and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h}$$

$$= \lim_{h \to 0} \frac{(3+6h+3h^2) - (1+3h+3h^2+h^3) - 2}{h} = \lim_{h \to 0} \frac{3h-h^3}{h} = \lim_{h \to 0} \frac{h(3-h^2)}{h}$$

$$= \lim_{h \to 0} (3-h^2) = 3 - 0 = 3$$

Tangent line: $y-2=3(x-1) \Leftrightarrow y-2=3x-3 \Leftrightarrow y=3x-1$

29. If $f(x) = 3x^2 - x^3$, find f'(1) and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

```
setenv("PYTHON", "./venv/bin/python");
clc: clear; pkg load symbolic;
syms x;
f = 3 * x^2 - x^3;
dx_dy = diff(f, x);
slope = double(subs(dx_dy, x, 1));
x1 = 1:
y1 = 2;
tangent_line = slope * (x - x1) + y1;
disp(tangent_line);
```

Content

Derivatives and Rates of Change



The Derivative as a Function

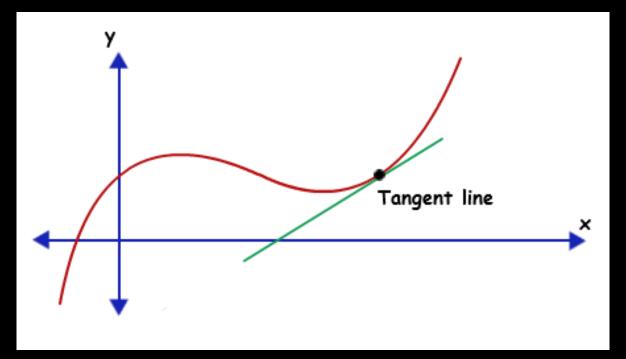
• We considered the derivative of a function f at a fixed number a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

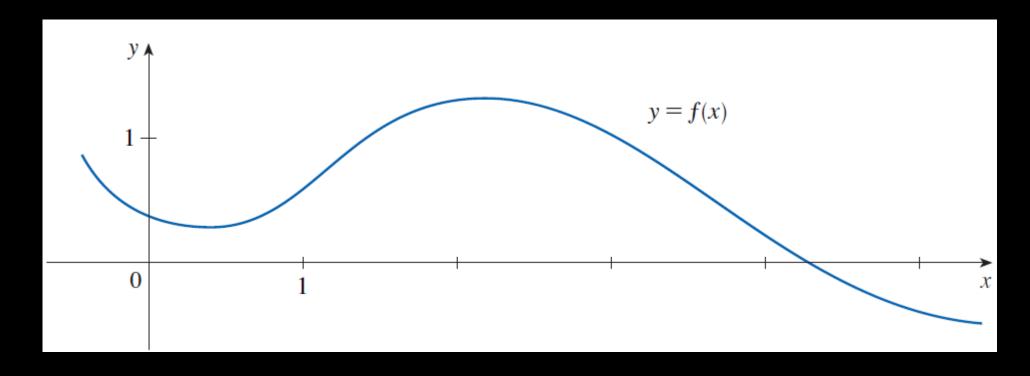
We assume the number a will vary, so the derivative becomes:

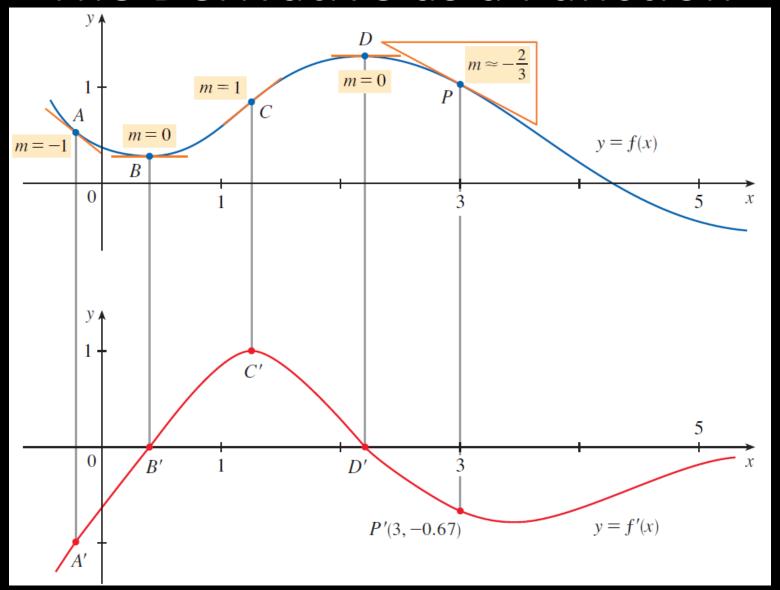
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Geometrically**, the derivative f'(x) is interpreted as the slope of the tangent line to the graph of f at the point (x, f(x)).
 - \circ The function f' is called the <u>derivative</u> of f because it has been <u>"derived"</u> from f by the limiting operation.



• Example 1: We can estimate the value of the derivative at any value of x by drawing the tangent at the point (x, f(x)) and estimating its slope.





Calculus is very important for computer science applications

https://medium.com/@18bhavyasharma/the-use-of-calculus-in-computer-science-a6917dbe33b9

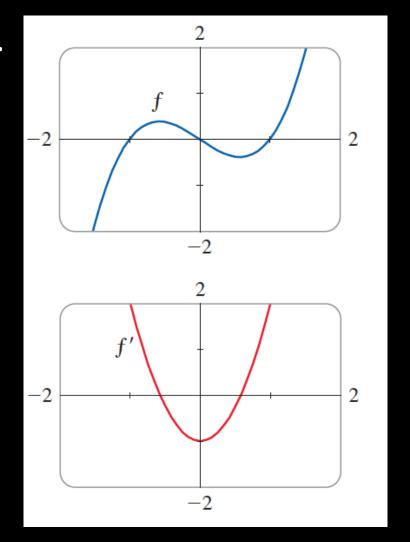
• EX 2: (a) If $f(x) = x^3 - x$, find a formula for f'(x). (b) Illustrate this formula by comparing the graphs of f and f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[(x+h)^3 - (x+h) \right] - \left[x^3 - x \right]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$



```
setenv("PYTHON", "./venv/bin/python");
clc: clear:
clf:
pkg load symbolic;
syms x;
f = x \wedge 3 - x;
dev_f = diff(f, x);
x_vals = linspace(-2, 2, 100);
% Substitute the valeue of x
% Convert symbolic expression to numeric
f_x = double(subs(f, x, x_vals));
dev_f_x = double(subs(dev_f, x, x_vals));
```

```
% Create a 2-row subplot
subplot(2, 1, 1);
plot(x_vals, f_x);
title('Function: f(x) = x^3 - x');
xlabel('x');
ylabel('f(x)');

subplot(2, 1, 2);
plot(x_vals, dev_f_x);
title('Derivative: f''(x) = 3x^2 - 1');
xlabel('x');
ylabel('f''(x)');
```

• Example 3: If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \qquad \text{(Rationalize the numerator.)}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that f'(x) exists if x > 0, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$.

$$\frac{\frac{a}{b} - \frac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

• Example 4: Find
$$f'$$
 if $f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{\frac{a}{b} - \frac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$= \lim_{h \to 0} \frac{\frac{1 - (x+h)}{2 + (x+h)} - \frac{1 - x}{2 + x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}$$

• Other notations for the derivative of y = f(x)

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Differentiability at a point

A function f is **differentiable at** a if f`(a) exits.

Differentiability on an interval

A function f is **differentiable on an open interval** (a,b) if it is differentiable at every number in the interval.

Theorem

If f is differentiable at a, then f is continuous at a.

• Notice that a continuous function does not mean that it is differentiable.

• Example 5: Where is the function f(x) = |x| differentiable?

- To answer this question, we have to investigate for
 - 1. x > 0
 - 2. x < 0
 - 3. x = 0

• For x > 0:

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

So, f is differentiable for any x > 0.

• For x < 0:

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

So, f is differentiable for any x < 0.

• For x = 0:

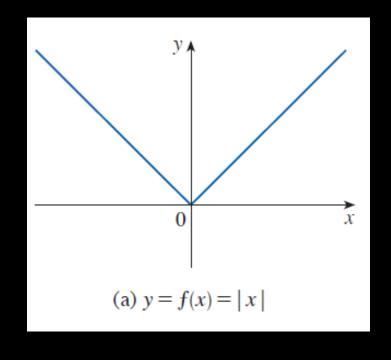
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

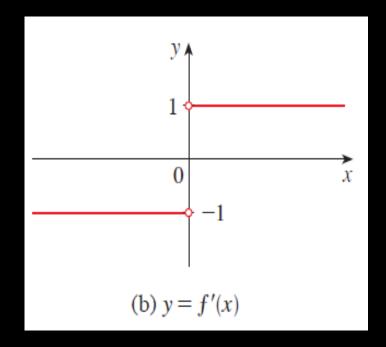
compute the left and right limits separately

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

: limits are different, f'(0) doesn't exist. Thus, f is differentiable at all x except 0.





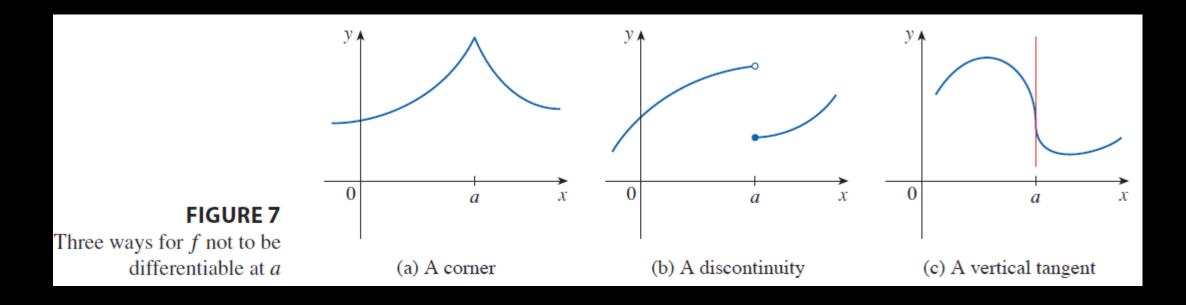
What is wrong with this code?

TASK: fix it

```
setenv("PYTHON", "./venv/bin/python");
clc; clear; clf; pkg load symbolic;
syms x;
f = abs(x);
dev_x = diff(f, x);
x_vals = linspace(-2, 2);
f_x = double(subs(f, x, x_vals));
dev_f_x = double(subs(dev_x, x, x_vals));
subplot(1, 1, 1);
plot(x_vals, f_x);
subplot(1, 2, 2);
plot(x_vals, dev_f_x);
```

• How can a function fail to be differentiable?

- How can a function fail to be differentiable?
- If the graph of f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there.
 - \circ [In trying to compute f'(x), we find that the left and right limits are different.]



 A higher derivative refers to the repeated process of taking derivatives of derivatives.

Second derivative:
$$\frac{d^2y}{dx^2}$$
, $f''(x)$, $f^{(2)}$

Third derivative:
$$\frac{d^3y}{dx^3}$$
, $f'''(x)$, $f^{(3)}$

$$n^{th}$$
 - order derivative: $\frac{d^n y}{dx^n}$, $f^{(n)}$

• Example 6: If $f(x) = x^3 - x$, find and interpret f''(x).

• The first derivative of $f'(x) = 3x^2 - 1$. So, the second derivative is

$$f''(x) = (f')'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

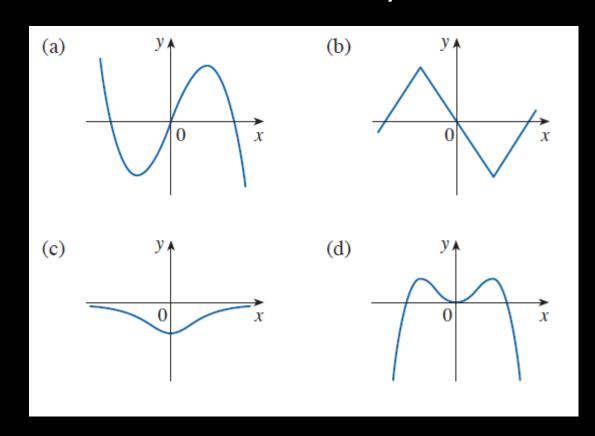
$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

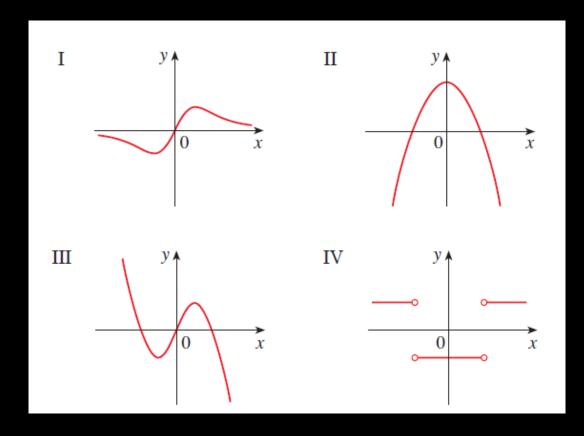
$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h) = 6x$$

- We can interpret a second derivative as a rate of change of a rate of change.
- The most familiar example of this is *acceleration*, which is the rate of change of velocity.

3. Match the graph of each function in (a) - (d) with the graph of its derivative in I–IV. Give reasons for your choices.





- (a) = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again.
- (b)= IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.
- (c)= I, since the slopes of the tangents to graph (c) are negative for x < 0 and positive for x > 0, as are the function values of graph I.
- (d)= III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21)
$$f(x) = 3x - 8$$
 22) $f(x) = mx + b$

27)
$$f(x) = \frac{1}{x^2 - 4}$$
 31) $f(x) = \frac{1}{\sqrt{1 + x}}$

21)
$$f(x) = 3x - 8$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h) - 8] - (3x - 8)}{h} = \lim_{h \to 0} \frac{3x + 3h - 8 - 3x + 8}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Domain of $f = \text{domain of } f' = \mathbb{R}$.

$$22) f(x) = mx + b$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[m(x+h) + b] - (mx+b)}{h} = \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h}$$
$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

Domain of $f = \text{domain of } f' = \mathbb{R}$.

27)
$$f(x) = \frac{1}{x^2 - 4}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2 - 4} - \frac{1}{x^2 - 4}}{h} = \lim_{h \to 0} \frac{\frac{(x^2 - 4) - [(x+h)^2 - 4]}{[(x+h)^2 - 4](x^2 - 4)}}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 - 4) - (x^2 + 2xh + h^2 - 4)}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \to 0} \frac{x^2 - 4 - x^2 - 2xh - h^2 + 4}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \to 0} \frac{-2xh - h^2}{h[(x+h)^2 - 4](x^2 - 4)}$$

$$= \lim_{h \to 0} \frac{h(-2x - h)}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \to 0} \frac{-2x - h}{[(x+h)^2 - 4](x^2 - 4)} = \frac{-2x}{(x^2 - 4)(x^2 - 4)} = -\frac{2x}{(x^2 - 4)^2}$$

Domain of $f = \text{Domain of } f' = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

$$31) f(x) = \frac{1}{\sqrt{1+x}}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{1 + (x+h)}} - \frac{1}{\sqrt{1 + x}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{1 + (x+h)}} - \frac{1}{\sqrt{1 + x}}}{h} \cdot \frac{\sqrt{1 + (x+h)}\sqrt{1 + x}}{\sqrt{1 + (x+h)}\sqrt{1 + x}}$$

$$= \lim_{h \to 0} \frac{\sqrt{1 + x} - \sqrt{1 + (x+h)}}{h\sqrt{1 + (x+h)}\sqrt{1 + x}} \cdot \frac{\sqrt{1 + x} + \sqrt{1 + (x+h)}}{\sqrt{1 + x} + \sqrt{1 + (x+h)}}$$

$$= \lim_{h \to 0} \frac{(1 + x) - [1 + (x+h)]}{h\sqrt{1 + (x+h)}\sqrt{1 + x}}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{1 + x + h}\sqrt{1 + x}} \frac{-1}{(\sqrt{1 + x} + \sqrt{1 + x + h})} = \lim_{h \to 0} \frac{-1}{\sqrt{1 + x + h}\sqrt{1 + x}} \frac{-1}{(\sqrt{1 + x} + \sqrt{1 + x})}$$

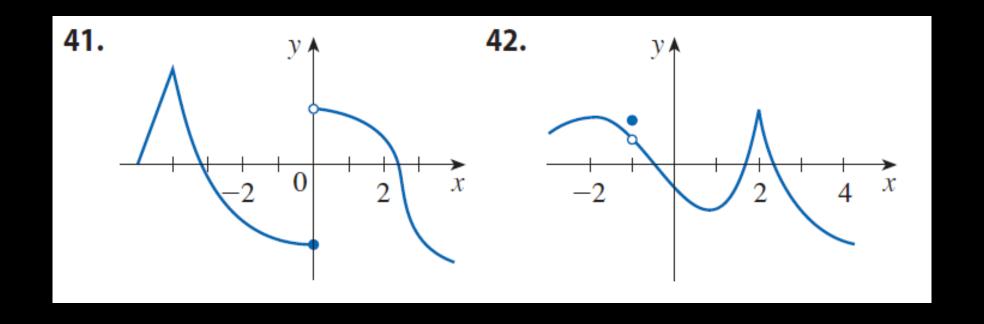
$$= \frac{-1}{\sqrt{1 + x}\sqrt{1 + x}} \frac{-1}{(\sqrt{1 + x} + \sqrt{1 + x})} = \frac{-1}{(1 + x)(2\sqrt{1 + x})} = -\frac{1}{2(1 + x)^{3/2}}$$

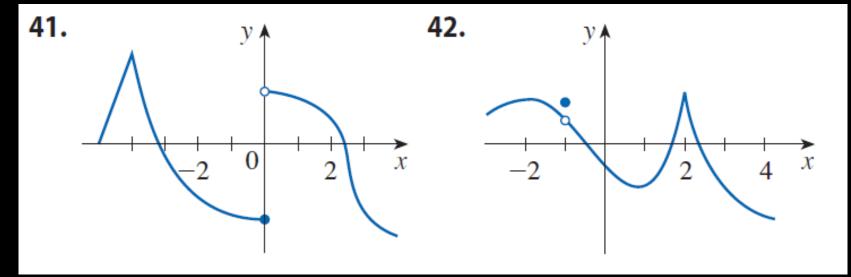
Domain of $f = \text{Domain of } f' = (-1, \infty)$.

```
setenv("PYTHON", "./venv/bin/python");
clc; clear; pkg load symbolic;
syms x;
syms m;
syms b;
% Define the functions
f1 = 3*x - 8:
f2 = m*x + b;
f3 = 1/(x^2 - 4);
f4 = 1/(sqrt(1 + x));
% Compute the derivatives
d_f1 = diff(f1, x);
d_f2 = diff(f2, x);
d_f3 = diff(f3, x);
d_f4 = diff(f4, x);
```

```
% Display the results with clear labels
disp('Derivative of f1 = 3x - 8:');
disp(d_f1);
disp(' '); % Add a blank line for separation
disp('Derivative of f2 = mx + b:');
disp(d_f2);
disp(' ');
disp('Derivative of f3 = 1/(x^2 - 4):');
disp(d_f3);
disp(' ');
disp('Derivative of f4 = 1/sqrt(1 + x):');
disp(d_f4);
disp(' ');
```

• The graph of f is given. State, with reasons, the numbers at which f is not differentiable.





- 41. f is not differentiable at x = -4, because the graph has a corner there, and at x = 0, because there is a discontinuity there.
- 42. f is not differentiable at x=-1, because there is a discontinuity there, and at x=2, because the graph has a corner there.

59. Show that the function f(x) = |x - 6| is not differentiable at 6. Find a formula for f' and sketch its graph.

$$f(x) = |x - 6| = \begin{cases} x - 6 & \text{if } (x - 6) \ge 0 \\ -(x - 6) & \text{if } (x - 6) < 0 \end{cases}$$

The right-hand limit is

$$\lim_{x \to 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^+} \frac{x - 6}{x - 6} = 1$$

The left-hand limit is

$$\lim_{x \to 6^{-}} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^{+}} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^{+}} \frac{-(x - 6)}{x - 6} = -1$$

Since the left and right limits are not equal, the limit does not exist and f is not differentiable at 6.

TASK

Section 2.8