Integrals

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The Area and Distance Problem
The Definite Integral
The Fundamental Theorem of Calculus
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The Area and Distance Problems

- The area problem: finding the area of the region S that lies under the curve $y = f(x)$ from a to b

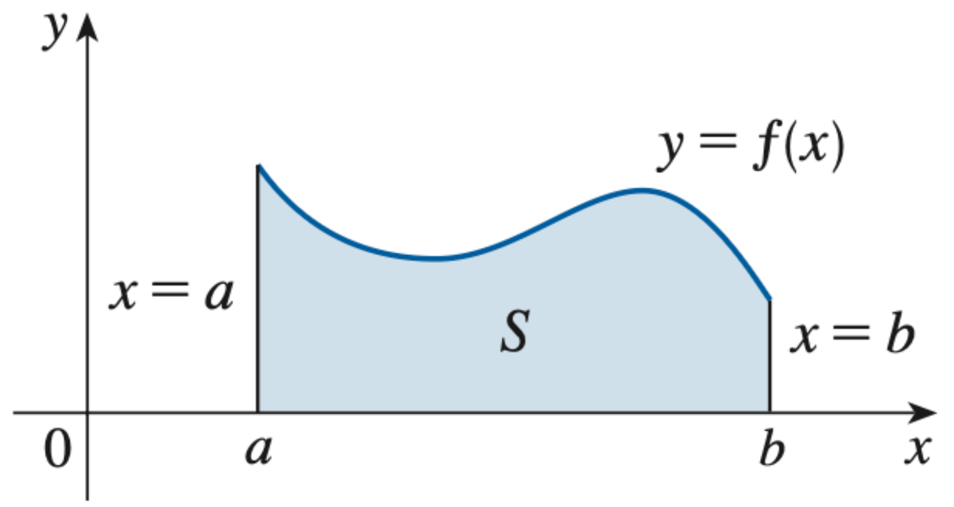
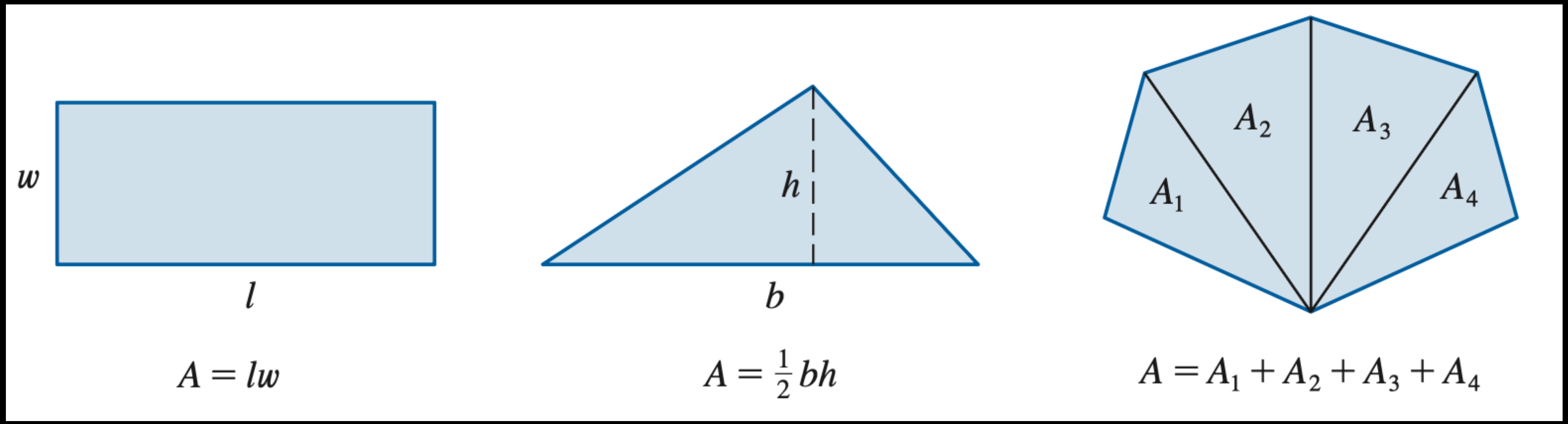


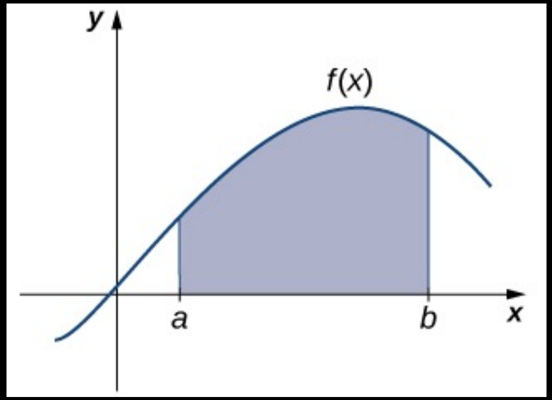
FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

The Area and Distance Problems

- It’s easy to compute the area of straight shapes



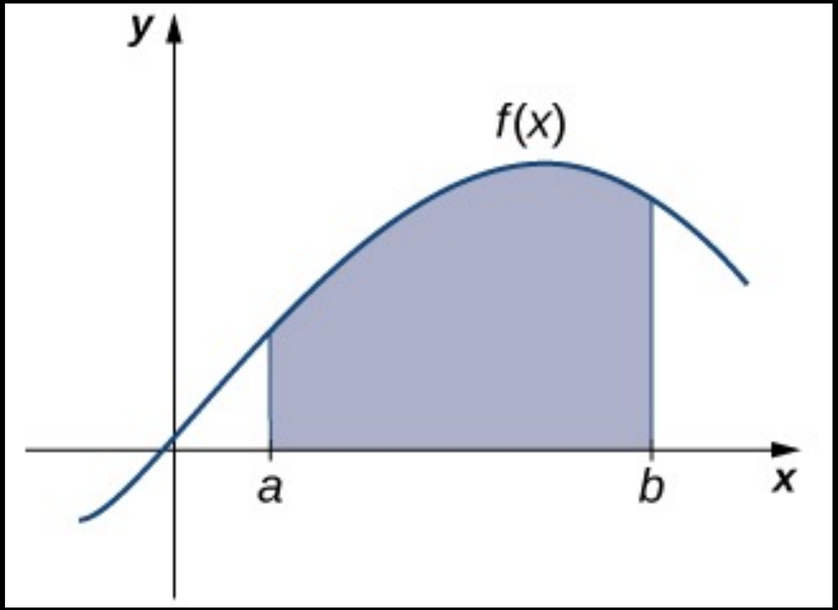
- It’s not easy to find the are of curved shapes

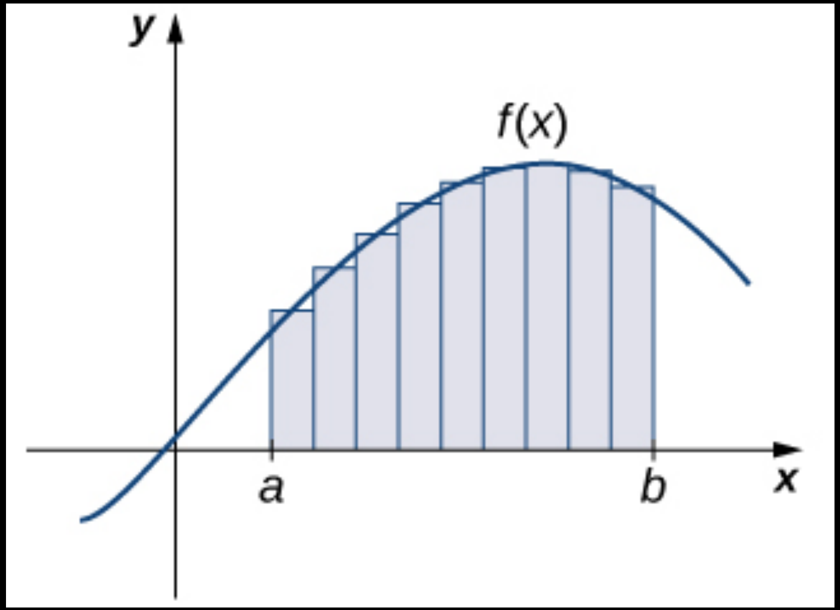


The Area and Distance Problems

- We follow the idea:

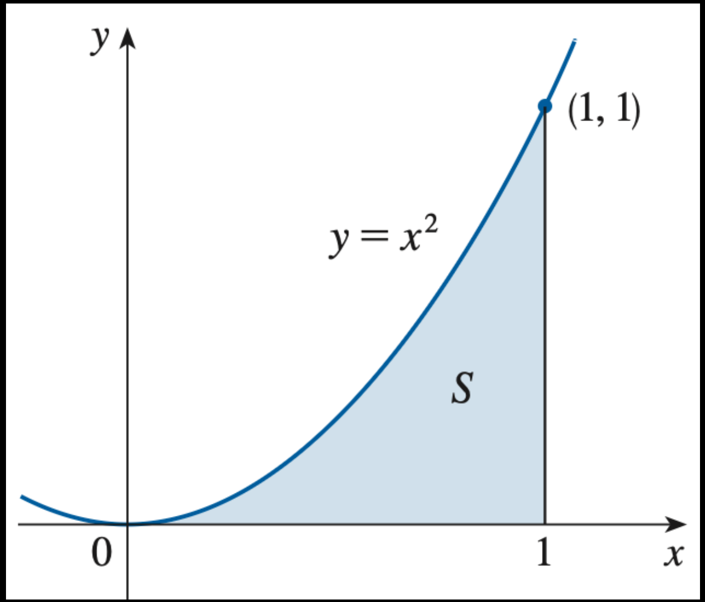
1. Approximate the area S by rectangles
2. Compute the limit of the sum of the areas of the approximating rectangles as we increase the number of rectangles





The Area and Distance Problems

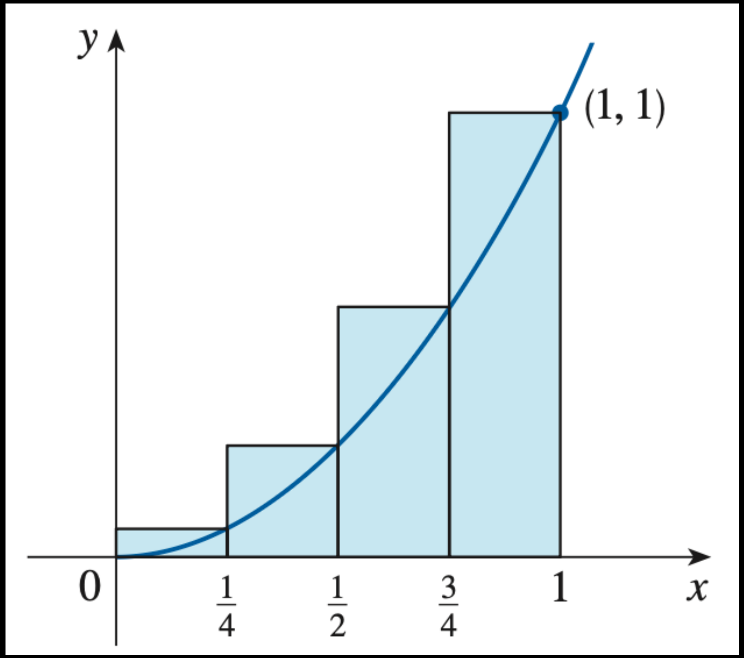
- Example: Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1



- Notice that the area of S is between 0 and 1, because S is contained in a square with side length 1

The Area and Distance Problems

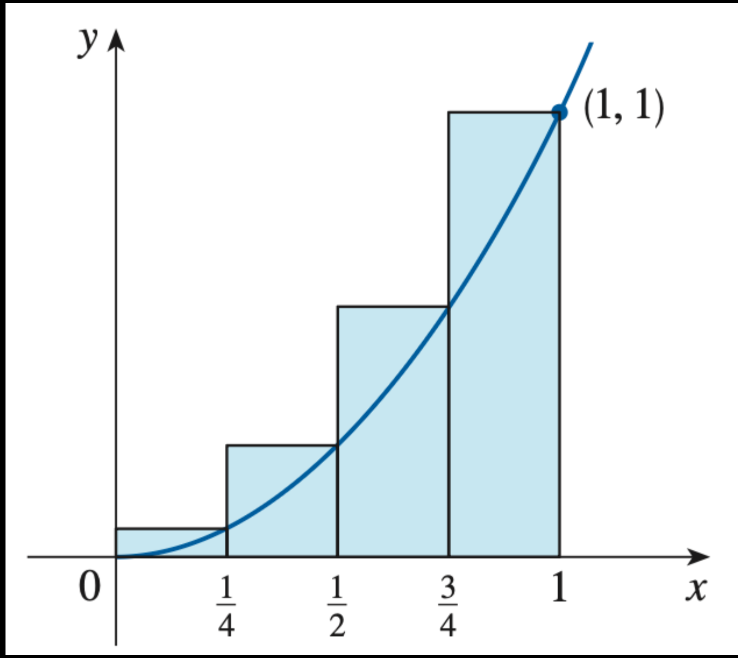
- Approximate each strip by a rectangle:
  - The bases are $[0, ¼], [¼, , 1/2], [1/2, ¾], [3/4, 1]$ → each rectangle has width = 1/4
  - The heights are $f(x) = x^2$



The Area and Distance Problems

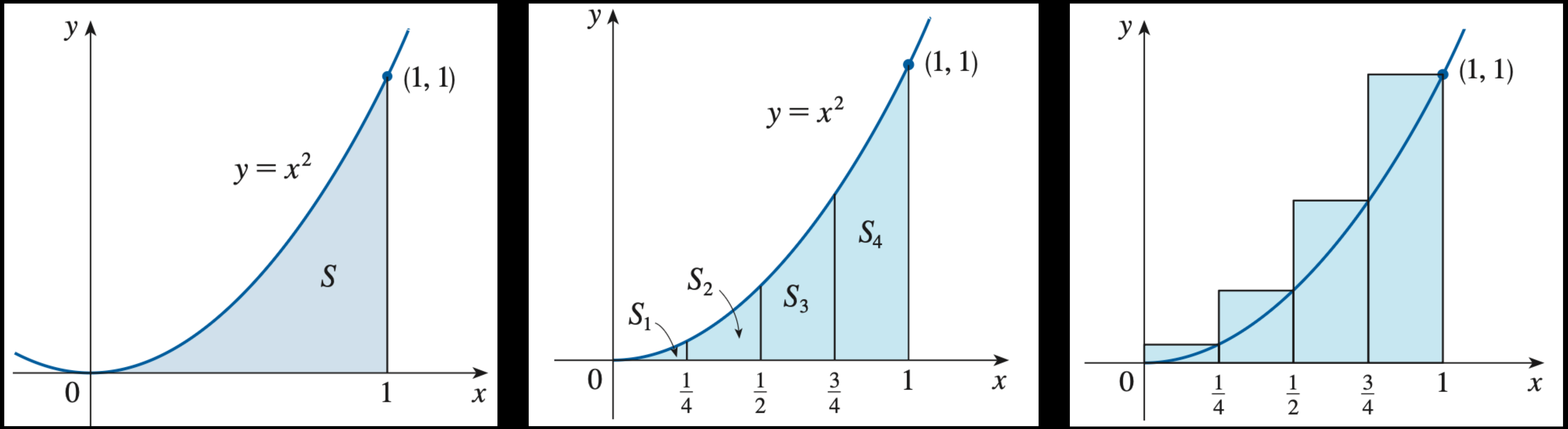
- So, the sum of the areas of the rectangles:

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} + 1^2 = 0.46875$$



The Area and Distance Problems

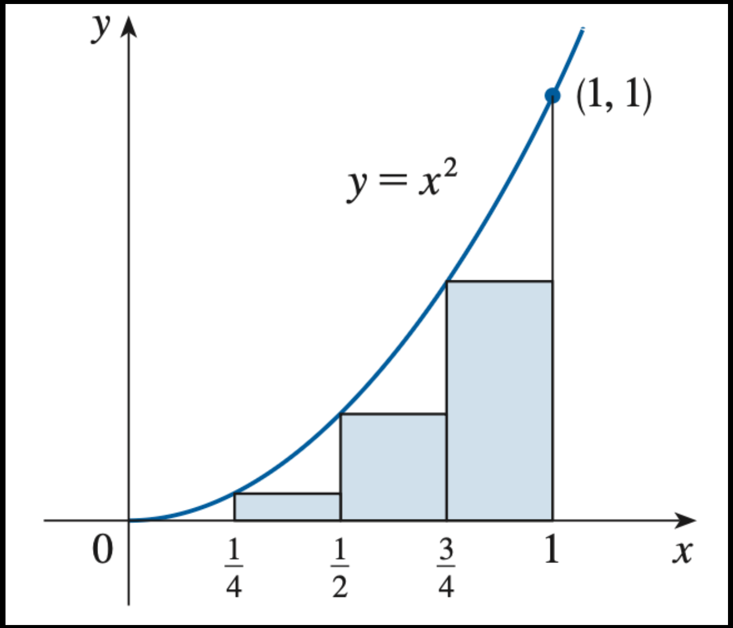
- The actual area A of S is less than computed are R_4
  - There are extra segments above the curve
  - We assumed the height to be equal to the right-point of the sub-interval



The Area and Distance Problems

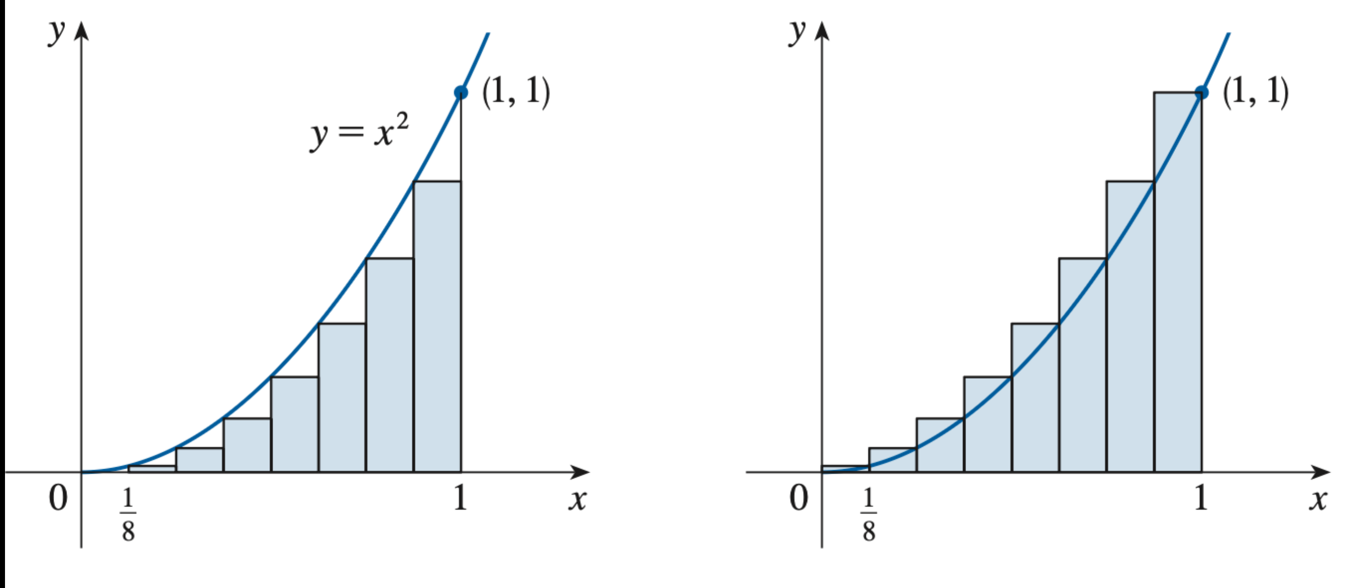
- We can use smaller rectangles
  - Assume the height to be equal to the left-point of the sub-interval

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} + \left(\frac{3}{4}\right)^2 = 0.21875$$



The Area and Distance Problems

- So, for the area of S, we have lower and upper estimates:
$$0.21875 \leq A \leq 0.46875$$
- Better accuracy → increase the number of rectangles under the curve
  - Thus, reducing each rectangle's width



(a) Using left endpoints (b) Using right endpoints

The Area and Distance Problems

- As we increase the number of rectangles, we get better estimates

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

- A good estimate would be $A \approx 0.3333$

The Area and Distance Problems

- We define the area A to be the limit of the sums of the areas of the approximating rectangles

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

The Area and Distance Problems

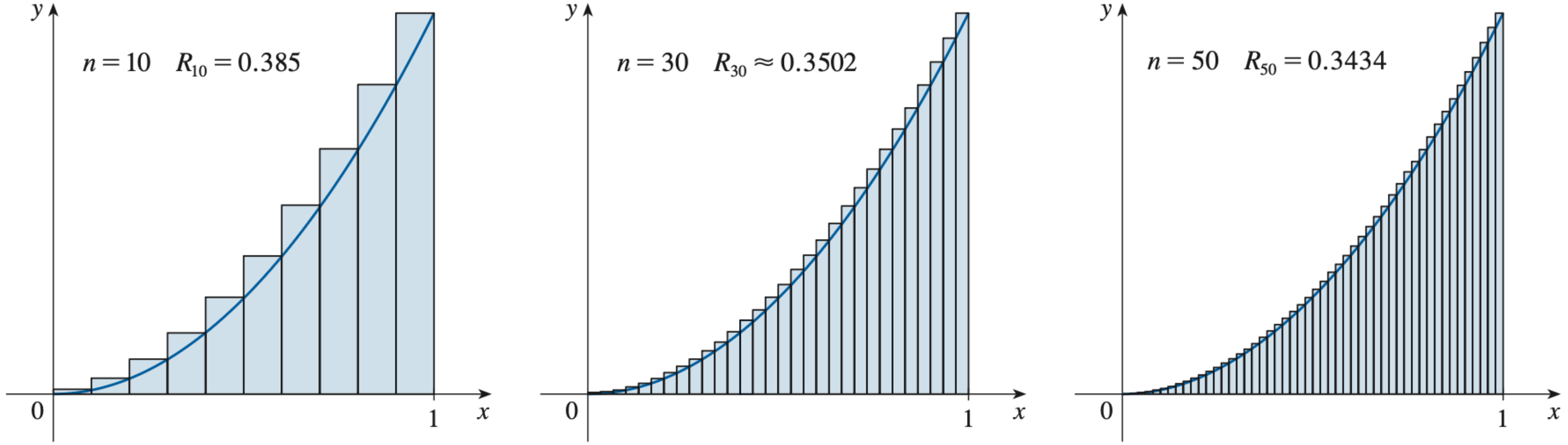


FIGURE 8 Right endpoints produce upper estimates because $f(x) = x^2$ is increasing.

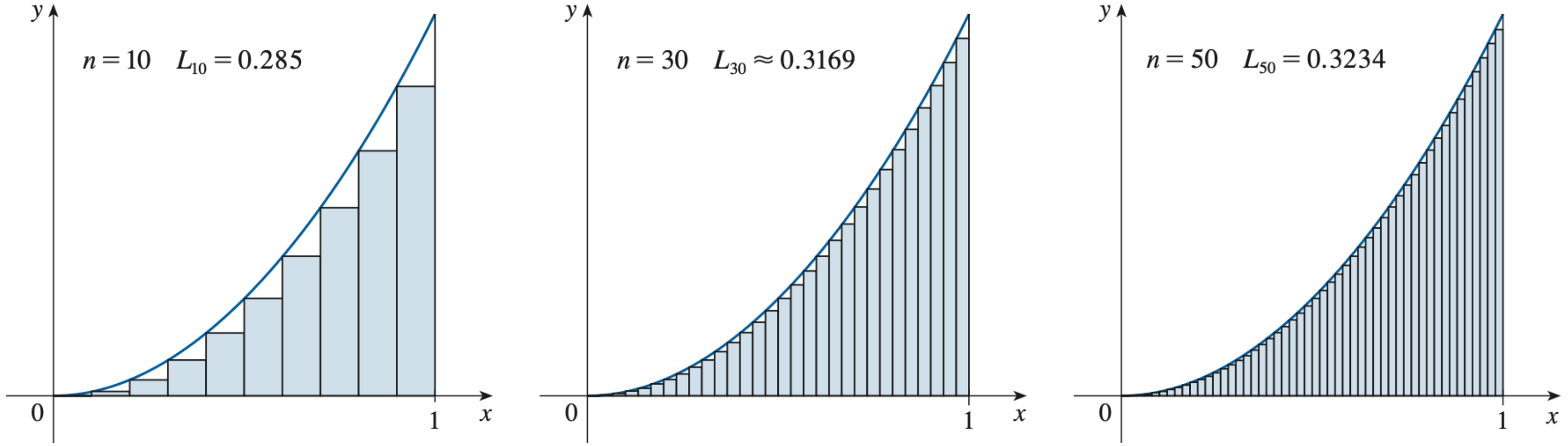


FIGURE 9 Left endpoints produce lower estimates because $f(x) = x^2$ is increasing.

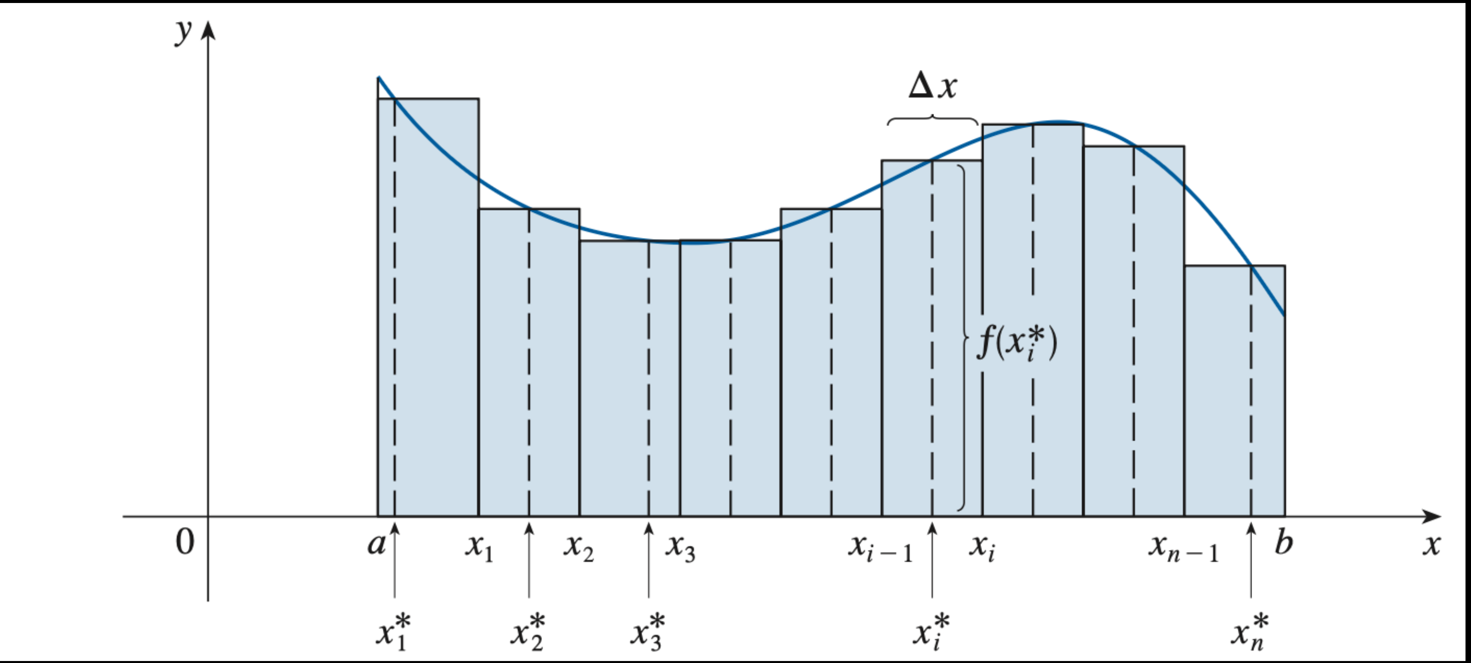
The Area and Distance Problems

Area
The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles: $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x]$$

The Area and Distance Problems

- The points x_i can be <u>left endpoints</u>, <u>right endpoints</u>, or <u>sample endpoints</u> within the sub-interval $[x_{i-1}, x_i]$

$$A = \lim_{n\to\infty}[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \ \ldots + f(x_n^*)\,\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



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The Definite Integral

Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a)$, $x_1, x_2, \dots, x_n(= b)$ be the endpoints of the subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$.
Then, the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

The Definite Integral

- Notes

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$	Integration
$\int$	Integral sign, it means the limit of the sums
$f(x)$	Integrand
$a,\ b$	Lower limit, upper limit
$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$	Riemann sum, means a sum of areas of approximating rectangles

The Definite Integral

- Riemann sum vs integration

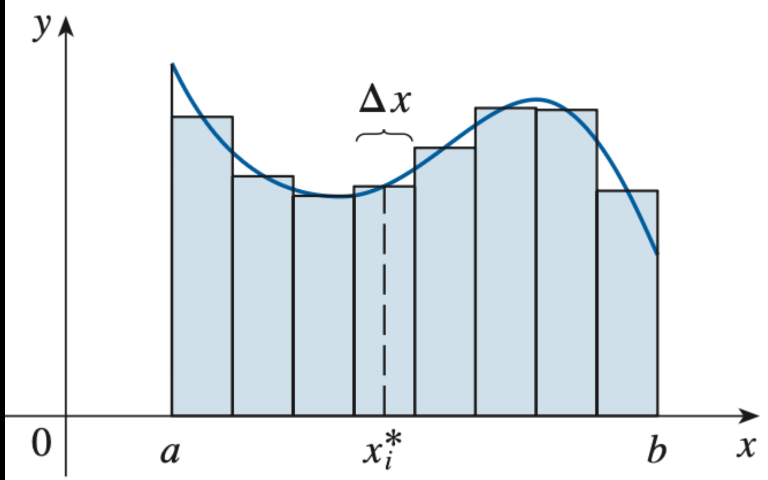


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

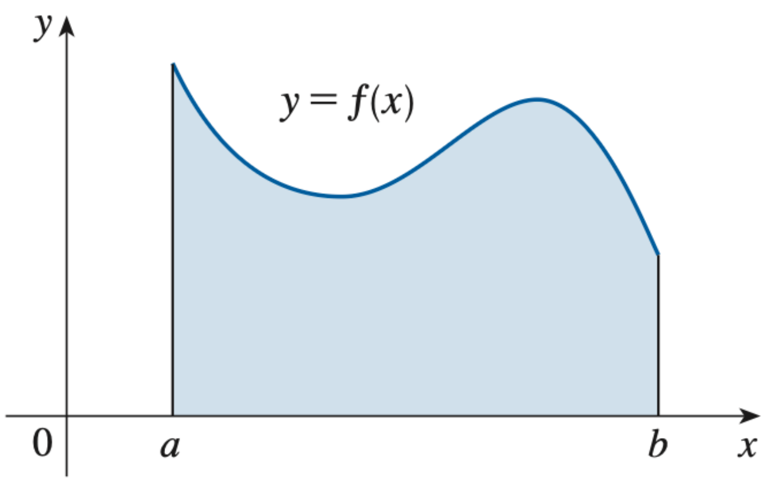


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

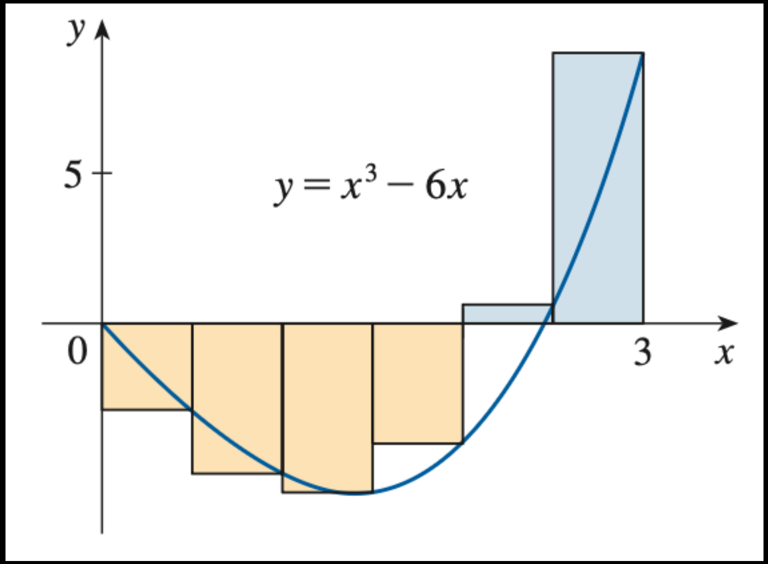
The Definite Integral

- Example: Evaluate the Riemann sum for $f(x) = x^3 - 6x, 0 \leq x \leq 3$, with $n = 6$ subintervals and taking the sample endpoints to be right endpoints.

The subinterval width $\Delta x = \frac{3-0}{6} = \frac{1}{2}$
The subintervals are: $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$
So, the right endpoints are $0.5, 1, 1.5, 2, 2.5, 3$
Riemann sum: $R_6 = \sum_{i=1}^{6} f(x_i)\Delta x$ $= f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x + f(3)\Delta x$
½$(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) = -3.9375$

The Definite Integral

- Notice that the sum is of the areas is -3.9375, which is not positive!
- This is because f is not a positive function
- The Riemann sum does not represent a sum of areas of rectangles
- But it represents the sum of the areas of the blue rectangles minus the sum of the areas of the gold rectangles



The Definite Integral

Theorem
If f is integrable on $[a, b]$, then $$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$ Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

The Definite Integral

- Rules and properties of sums

Sums of Powers

$\boxed{5}$ $$\sum_{i=1}^{n} 1 = n$$

$\boxed{6}$ $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$\boxed{7}$ $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{8}$ $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Properties of Sums

$\boxed{9}$ $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$\boxed{10}$ $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$\boxed{11}$ $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

The Definite Integral

- Example: evaluate $\int_0^3 (x^3 - 6x)\,dx$

$f(x) = x^3 - 6x$, $a = 0$, $b = 3, \Delta x = 3/n$
$x_0 = 0,\ x_1 = 0 + \frac{3}{n},$ $x_2 = 0 + 2\times\frac{3}{n} = \frac{6}{n},$ $x_3 = 0 +\ 3\times\frac{3}{n} = \frac{9}{n}$ $, \dots ,$ $\mathrm{x_i} = 0 + \mathrm{i}\left(\frac{3}{\mathrm{n}}\right)$

Thus

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

The Definite Integral

- Properties of the definite integral

$$\int_b^a f(x)dx = -\int_a^b f(x)\,dx$$

if $a = b$:

$$\int_a^b f(x)dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

The Definite Integral

- The next property tells us how to combine integrals of the same function over adjacent intervals.

$$\boxed{\int_a^c f(x)\,dx + \int_c^b f(x)dx = \int_a^b f(x)\,dx}$$

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

Exercises

1. Evaluate the Riemann sum for $f(x) = x - 1, -6 \leq x \leq 4$, with five subintervals, taking the sample points to be right endpoints.

Exercises

1. Evaluate the Riemann sum for $f(x) = x - 1, -6 \leq x \leq 4$, with five subintervals, taking the sample points to be right endpoints.

$f(x) = x^3 - 6x \ , a = -6 \ , \ \ b = 4, \Delta x = (4+6)/5 = 2$
$x_1 = -6 + 1(2) = -4, x_2 = -6 + 2(2) = -2,$ $x_3 = -6 + 3(2) = 0, x_4 = -6 + 4(2) = 2, x_5 = -6 + 5(2) = 4$
$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$
$= \Delta x \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right)$
$= 2\left(f(-4) + f(-2) + f(0) + f(2) + f(4)\right)$
$= 2\,(-5 - 3 - 1 + 2 + 3) = -10$

Exercises

2. If $f(x) = \cos x\,,\ \ 0 \le x \le 3\pi/4$ evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints.

Exercises

2. If $f(x) = \cos x\,, \quad 0 \le x \le 3\pi/4$ evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints.

$f(x) = \cos x, a = 0, b = 3\pi/4, n = 6$
$\Delta x = (3\pi/4)/6 = \pi/8,\ x{:}\,0,\ \frac{\pi}{8}, \frac{2\pi}{8}, \frac{3\pi}{8}, \frac{4\pi}{8}, \frac{5\pi}{8}$
$L_6 = \sum_{i=0}^{\frac{3\pi}{4}} f(x)\,\Delta x$
$= \Delta x\left(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right)$
$= \frac{\pi}{8}\left(\cos 0 + \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8} + \cos\frac{5\pi}{8}\right) = 1.033186$

Exercises

3. If $f(x) = x^2 - 4, \quad 0 \leq x \leq 3$ evaluate the Riemann sum with $n = 6$, taking the sample points to be midpoints.

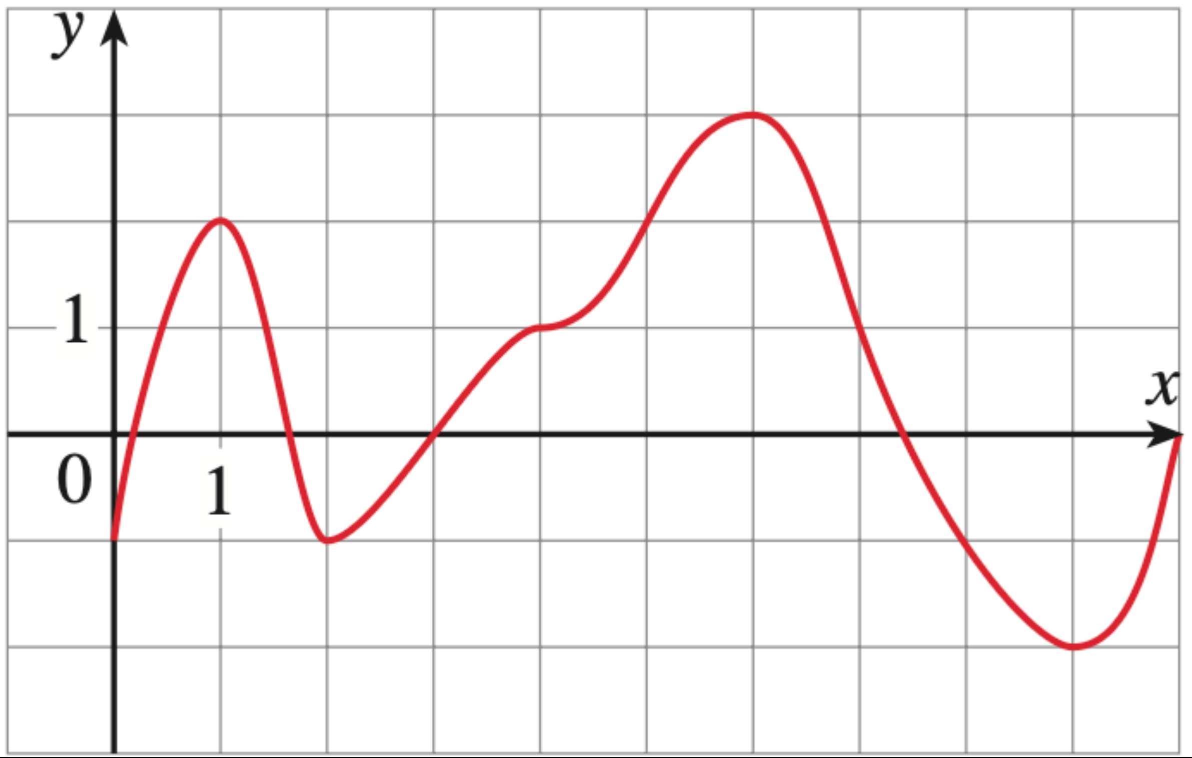
Exercises

3. If $f(x) = x^2 - 4, \quad 0 \leq x \leq 3$ evaluate the Riemann sum with $n = 6$, taking the sample points to be midpoints.

$f(x) = x^2 - 4,\ a = 0,\ b = 3,\ \ \mathrm{n} = 6,\ \ \Delta x = \frac{3}{6} = \frac{1}{2}$
The endpoints of the subintervals: $0, 0.5, 1, 1.5, 2, 2.5, 3$
$x_i^* = 1/2(x_{i-1} + x_i)$: $\frac{0+0.5}{2} = 0.25, \frac{0.5+1}{2} = 0.75, \frac{1+1.5}{2} = 1.25, \ \ \frac{1.5+2}{2} = 1.75, \frac{2+2.5}{2} = 2.25, \frac{2.5+3}{2} = 2.75$
$M_6 = \frac{1}{2}(f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75) = -\frac{49}{16}$

Exercises

5. The graph of a function f is given. Estimate $\int_0^{10} f(x)dx$ using five subintervals with (a) right subpoints, (b) left subpoints, (c) mid subpoints



Exercises

Subintervals: $[0, 2], [2, 4], [4, 6], [6, 8], [8, 10]$

Left subpoints: $0, 2, 4, 6, 8$

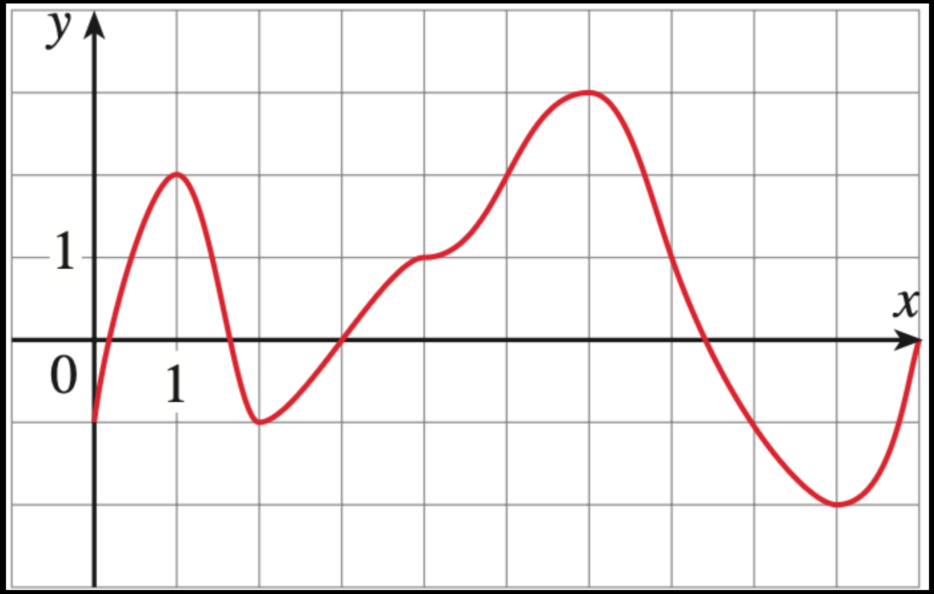
$$L_5 = \Delta x\big(f(0) + f(2) + f(4) + f(6) + f(8)\big)$$
$$= 2\,(-1 - 1 + 1 + 3 - 1) = 2$$

right subpoints: $2, 4, 6, 8, 10$

$$R_5 = \Delta x\big(f(2) + f(4) + f(6) + f(8) + f(10)\big)$$
$$= 2\,(-1 + 1 + 3 - 1 + 0) = 4$$

mid subpoints: $1, 3, 5, 7, 9$

$$M_5 = \Delta x\big(f(1) + f(3) + f(5) + f(7) + f(9)\big)$$
$$= 2\,(2 + 0 + 2 + 1 - 2) = 6$$



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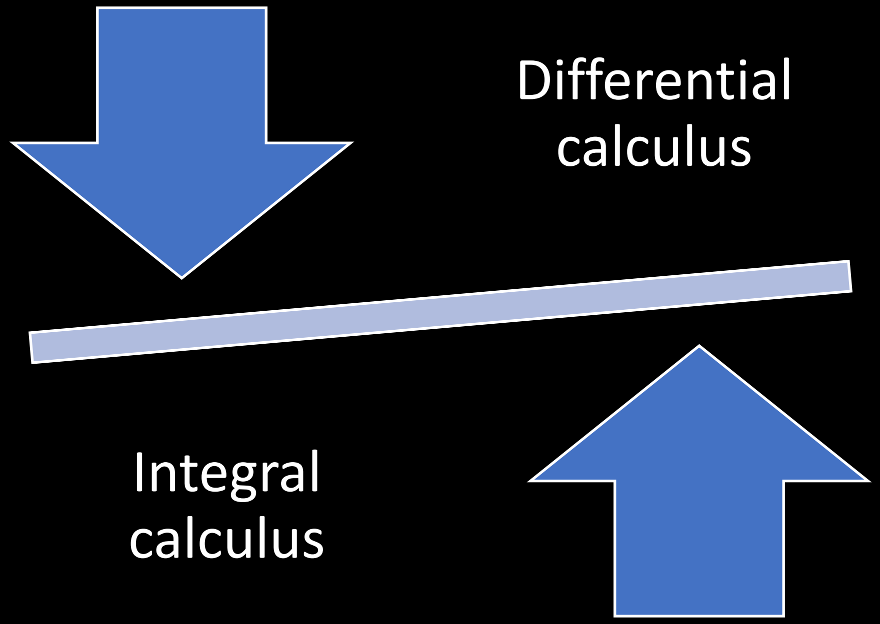
The Fundamental Theorem of Calculus

- It establishes a connection between the two branches of calculus:
  - **Differential calculus**: arose from the tangent problem
  - **Integral calculus**: arose from the area problem
  - Differentiation and integration are inverse processes



The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ 1. $g(x) = \int_a^x f(t)\,dt$, then $g`(x) = f(x)$ 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$, that is $F` = f$

1. The derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.
2. If we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)\,dx$ by subtracting the values of F at the endpoints of the interval $[a, b]$

The Fundamental Theorem of Calculus

- Example: Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\ dt$

Using FTC1, $\frac{d}{dx}\ g(x) = \frac{d}{dx}\left(\int_0^x \sqrt{1 + t^2}\right) = \sqrt{1 + x^2}$

The Fundamental Theorem of Calculus

- Example: Find $\frac{d}{dx}\left(\int_1^{x^4} \sec t \, dt\right)$

This is chain rule with FTC1

Let $u = x^4$

$$\frac{d}{dx}\left(\int_1^{x^4} \sec t \ \ dt\right) = \frac{d}{du}\left[\int_1^{u} \sec t \ \ dt\right]\frac{du}{dx}$$

$$= \sec u \frac{du}{dx} = \sec(x^4) \cdot 4x^3$$

The Fundamental Theorem of Calculus

- Example: Evaluate the integral $\int_1^3 e^x \, dx$

Using FTC2, we need to get the antiderivatives of $f(x) = e^x$

The antiderivative is $F(x) = e^x$

$\int_1^3 e^x \, dx = F(3) - F(1) = e^3 - e^1$

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Indefinite Integrals and the Net Change Theorem

- Indefinite integrals is when the integration is performed without intervals
- Indefinite integrals:

$$\int f(x)dx = F(x) \qquad \rightarrow \qquad F`(x) = f(x)$$

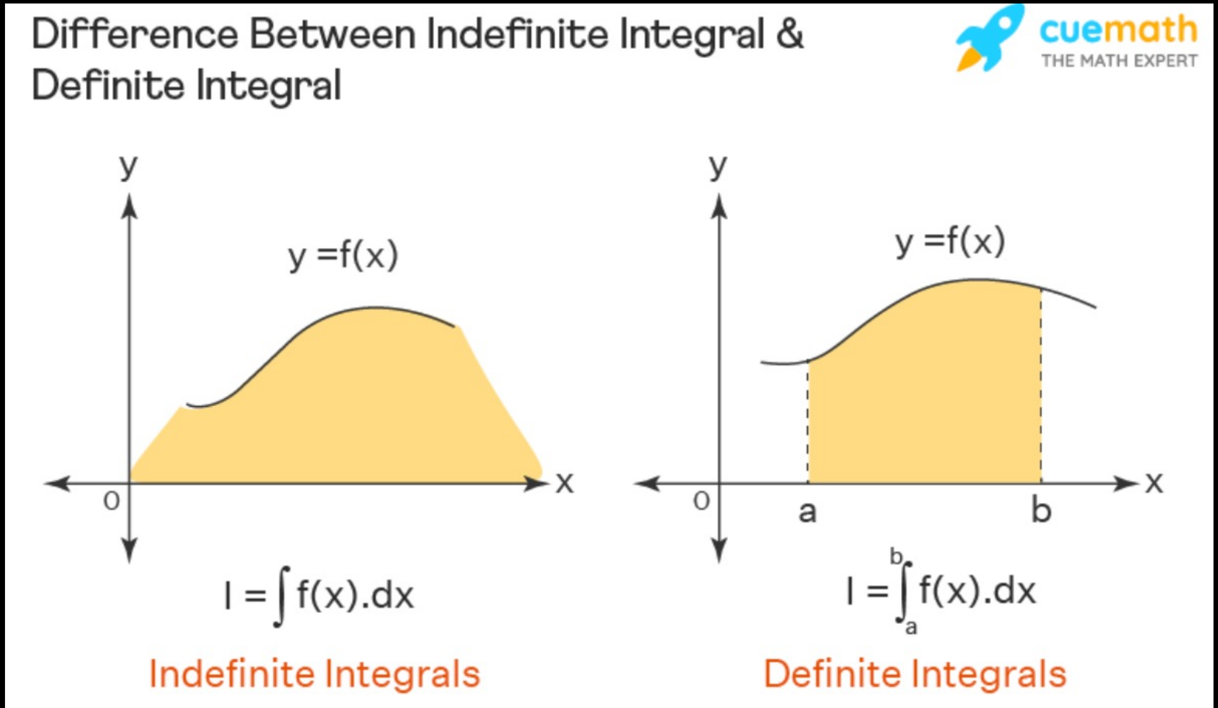
- For example:

$$\int x^2 dx = \frac{x^3}{3} + C \qquad , \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

C is an integration constant

Indefinite Integrals and the Net Change Theorem

- Distinguish between definite and indefinite integrals:
  - Definite integral: $\int_a^b f(x)\,dx$ → number
  - Indefinite integral: $\int f(x)\,dx$ → a function



Indefinite Integrals and the Net Change Theorem

- Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Indefinite Integrals and the Net Change Theorem

- Example: Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)dx$$

$$\int (10x^4 - 2\sec^2 x)dx$$

$$= \ 10\int (x^4)dx \ - 2\int \ \sec^2 x$$

$$= 10 \cdot \frac{x^5}{5} - 2 \cdot \tan x + C$$

Indefinite Integrals and the Net Change Theorem

- Example: Evaluate $\int \frac{cos\theta}{\sin^2\theta}\, d\theta$

The solution is not immediately apparent

$$\int \frac{cos\theta}{\sin^2\theta}\, d\theta$$

$$= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{cos\theta}{sin\,\theta}\right)\, d\theta$$

$$= \int \csc\theta\cot\theta\ \, d\theta = -\csc\theta + C$$

Indefinite Integrals and the Net Change Theorem

- Example: Evaluate $\int_0^3 (x^3 - 6x)dx$

Using FTC2 and the rules table

$$\int_0^3 (x^3 - 6x)dx$$

$$= \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3$$

$$F(3) - F(0) = \left(\frac{3^4}{4} - 6 \cdot \frac{3^2}{2}\right) - 0 = -6.75$$

Exercises

5–24 Find the general indefinite integral.

5. $\int (3x^2 + 4x + 1)dx$

6. $\int (5 + 2\sqrt{x})dx$

7. $\int (x + \cos x)dx$

8. $\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx$

17. $\int \left(e^x + \frac{1}{x}\right) dx$

18. $\int (2 + 3^x)dx$

Exercises

5. $\int (3x^2 + 4x + 1)dx$
6. $\int (5 + 2\sqrt{x})dx$
7. $\int (x + \cos x)dx$
8. $\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx$
17. $\int \left(e^x + \frac{1}{x}\right) dx$
18. $\int (2 + 3^x)dx$

$$\int (3x^2 + 4x + 1)dx = x^3 + 2x^2 + x + C$$

$$\int (5 + 2\sqrt{x})dx = \int \left(5 + 2x^{1/2}\right)dx = 5x + \frac{4}{3}x^{\frac{3}{2}} + C$$

$$\int (x + \cos x)dx = \frac{x^2}{2} + \sin x + C$$

$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx = \int \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right) dx = \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + C$$

$$\int \left(e^x + \frac{1}{x}\right) dx = e^x + \ln|x| + C$$

$$\int (2 + 3^x)dx = 2x + \frac{3^x}{\ln 3} + C$$

Exercises

27–54 Evaluate the definite integral.

27. $\int_{-2}^{3}(x^2-3)dx$

28. $\int_{1}^{2}(4x^3-3x^2+2x)\,dx$

29. $\int_{1}^{4}(8t^3\ -6t^{-2})dt$

30. $\int_{0}^{8}\left(\frac{1}{8}+\frac{1}{2}w+\frac{1}{3}w^{\frac{1}{3}}\right)dw$

31. $\int_{0}^{2}(2x-3)(4x^2+1)\,dx$

32. $\int_{1}^{2}\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx$

Exercises

27. $\int_{-2}^{3}(x^2 - 3)dx$

$$\int_{-2}^{3}(x^2 - 3)dx = \frac{x^3}{3} - 3x\Big]_{-2}^{3}$$

$$F(3) - F(-2) = \left(\frac{3^3}{3} - 3\times3\right) - \left(-\frac{2^3}{3} - 3\times-2\right) = -\frac{10}{3}$$

28. $\int_{1}^{2}(4x^3 - 3x^2 + 2x)\,dx$

$$\int_{1}^{2}(4x^3 - 3x^2 + 2x)\,dx = x^4 - x^3 + x^2]_1^2$$

$$F(2) - F(1) = (2^4 - 2^3 + 2^2) - (1^4 - 1^4 + 1^2) = 11$$

29. $\int_{1}^{4}(8t^3 - 6t^{-2})dt$

$$\int_{1}^{4}(8t^3 - 6t^{-2})dt = 2t^4 + 6t^{-1}]_1^4$$

$$F(4) - F(1) = \left(2\times4^4 + \frac{6}{4}\right) - \left(2\times1^4 + \frac{6}{1}\right) = 505.5$$

Exercises

30. $\int_0^8 \left(\frac{1}{8} + \frac{1}{2}w + \frac{1}{3}w^{\frac{1}{3}}\right) dw$

$$\int_0^8 \left(\frac{1}{8} + \frac{1}{2}w + \frac{1}{3}w^{\frac{1}{3}}\right) dw = \frac{1}{8}w + \frac{1}{4}w^2 + \frac{1}{4}w^{\frac{4}{3}}\Big]_0^8$$

$$F(8) - F(0) = \left(\frac{1}{8}\times 8 + \frac{1}{4}\times 8^2 + \frac{1}{4}\times 8^{\frac{4}{3}}\right) - 0 = 21$$

31. $\int_0^2 (2x-3)(4x^2+1)\, dx$

$$\int_0^2 (2x-3)(4x^2+1)\, dx = \int_0^2 8x^3 + 2x \; - 12x^2 - 3 \; dx$$

$$= \; 2x^4 + x^2 - 4x^3 - 3x]_0^2 = (32 - 32 + 4 - 6) - 0$$
$$= -2$$

32. $\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx$

$$\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx = \int_1^2 (x^{-2} - 4x^{-3}) dx = \frac{x^{-1}}{-1} - \frac{4}{-2}x^{-2}\Big]_1^2$$

$$F(2) - F(1) = \left(\frac{2^{-1}}{-1} - \frac{4}{-2}\times 2^{-2}\right) - \left(\frac{1^{-1}}{-1} - \frac{4}{-2}\times 1^{-2}\right) = -1$$

Exercises

27–54 Evaluate the definite integral.

42. $\int_0^1 (5x - 5^x)dx$

44. $\int_0^{\pi/4} (3e^x - 4\sec x \tan x)\, dx$

45. $\int_0^{\pi/4} \frac{(1+\cos^2\theta)}{\cos^2\theta} d\theta$

47. $\int_3^4 \sqrt{\frac{3}{x}}\, dx$

Exercises

42. $\int_0^1 (5x - 5^x)dx$

$\int_0^1 (5x - 5^x)dx = \left.\frac{5}{2}x^2 - \frac{5^x}{\ln 5}\right]_0^1$
$F(1) - F(0) = \left(\frac{5}{2}\times 1^2 - \frac{5^1}{\ln 5}\right) - \left(0 - \frac{1}{\ln 5}\right) = \frac{5}{2} - \frac{4}{\ln 5}$

44. $\int_0^{\pi/4} (3e^x - 4\sec x \tan x)\, dx$

$\int_0^{\pi/4} (3e^x - 4\sec x \tan x)\, dx = \left(3e^x - 4\sec x\right)\Big]_0^{\frac{\pi}{4}}$
$F\left(\frac{\pi}{4}\right) - F(0) = \left(3e^{\frac{\pi}{4}} - 4\sec\frac{\pi}{4}\right) - \left(3e^0 - 4\sec 0\right)$
$= 3e^{\frac{\pi}{4}} - 4\sqrt{2} + 1$

Exercises

45. $\int_0^{\pi/4} \frac{(1+\cos^2\theta)}{\cos^2\theta}\, d\theta$

$\int_0^{\pi/4} \frac{(1+\cos^2\theta)}{\cos^2\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{cos^2\,\theta}\, d\theta + \int_0^{\pi/4} \frac{\cos^2\theta}{cos^2\,\theta}\, d\theta$
$\int_0^{\pi/4} \sec^2\theta + 1\, d\theta = \tan\theta + \theta\Big]_0^{\frac{\pi}{4}}$
$F\left(\frac{\pi}{4}\right) - F(0) = \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (\tan 0 - 0) = 1 + \frac{\pi}{4}$

47. $\int_3^4 \sqrt{\frac{3}{x}}\, dx$

$\int_3^4 \sqrt{\frac{3}{x}}\, dx = \int_3^4 \sqrt{3}\cdot x^{-1/2} dx = \sqrt{3}\int_3^4 x^{-1/2} dx$
$\sqrt{3}\cdot\left(2x^{\frac{1}{2}}\right)_3^4 = 2\sqrt{3}\cdot\left(x^{\frac{1}{2}}\right)_3^4 = 2\sqrt{3}\cdot\left(2-\sqrt{3}\right) = 4\sqrt{3}-6$

The Substitution Rule

- Substitution rule in integration $\Leftrightarrow$ chain rule in differentiation

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)g`(x)\,dx = \int f(u)\ du$$

- Substitution rule says: it is permissible to operate with dx and du after integral signs as if they were differentials.

The Substitution Rule

- Example: Find $\int x^3 \cos(x^4+2)\, dx$

$\frac{d}{dx}(x^4+2) = 4x^3$, which, apart from the constant 4, appear in the integral

$\therefore$ we set $x^4+2=u$ and $x^3 dx = \frac{1}{4} du$

$$\begin{aligned}
\int x^3 \cos(x^4+2)\, dx &= \int \cos u \cdot \frac{1}{4} du \\
&= \frac{1}{4}\int \cos u \ \ du \\
&= \frac{1}{4}\sin u + C \\
&= \frac{1}{4}\sin(x^4+2) + C
\end{aligned}$$

The Substitution Rule

- Example: Evaluate $\int \sqrt{2x+1}\ dx$

Let $u = 2x + 1$
Then, $du = 2\ dx$
So, $dx = \frac{1}{2} du$
$\int \sqrt{2x+1}\ dx = \int \sqrt{u} \cdot \frac{1}{2} du$
$= \frac{1}{2} \int u^{\frac{1}{2}} \cdot du$
$= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$
$= \frac{1}{3} (2x+1)^{\frac{3}{2}} + C$

The Substitution Rule

- Example: Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

  Let $u = (3 - 5x)$

  Then, $du = -5dx$

  So, $dx = \frac{-1}{5} du$

  When $x = 1, u = -2$

  When $x = 2, u = -7$

$\int_1^2 \frac{dx}{(3-5x)^2}$	$= \int_{-2}^{-7} \frac{1}{(u)^2} \cdot -\frac{1}{5} du$
	$= -\frac{1}{5} \int_{-2}^{-7} \frac{1}{(u)^2} du$
	$= -\frac{1}{5} \int_{-2}^{-7} u^{-2} du$
	$= -\frac{1}{5} \cdot -\frac{1}{u} \Big]_{-2}^{-7}$
	$= -\frac{1}{5}\left(\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}$

Exercises

9–54 Evaluate the indefinite integral.

9. $\int x\sqrt{1-x^2}\ dx$

10. $\int (5-3x)^{10} dx$

11. $\int t^3 e^{-t^4}$

12. $\int \sin t\ \sqrt{1+\cos t}\ dt$

Exercises

9. $\int x\sqrt{1-x^2}\ dx$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so

$$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\,\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2} + C = -\tfrac{1}{3}(1-x^2)^{3/2} + C.$$

10. $\int (5-3x)^{10} dx$

Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int (5-3x)^{10}\,dx = \int u^{10}\left(-\tfrac{1}{3}\,du\right) = -\tfrac{1}{3}\cdot\tfrac{1}{11}u^{11} + C = -\tfrac{1}{33}(5-3x)^{11} + C.$$

11. $\int t^3 e^{-t^4}$

Let $u = -t^4$. Then $du = -4t^3\,dt$ and $t^3\,dt = -\frac{1}{4}\,du$, so $\int t^3 e^{-t^4}\,dt = \int e^u\left(-\frac{1}{4}\,du\right) = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{-t^4} + C.$

12. $\int \sin t\ \sqrt{1+\cos t}\ dt$

Let $u = 1 + \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$$\int \sin t\,\sqrt{1+\cos t}\,dt = \int \sqrt{u}\,(-du) = -\tfrac{2}{3}u^{3/2} + C = -\tfrac{2}{3}(1+\cos t)^{3/2} + C.$$

Exercises

59-80. Evaluate the definite integral.

59. $\int_0^1 \cos\left(\frac{\pi t}{2}\right) dt$

60. $\int_0^1 (3t-1)^{50} dt$

61. $\int_0^1 \sqrt[3]{1+7x}\ dx$

62. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\frac{1}{2}t\right)\ dt$

Exercises

59. $\int_0^1 \cos\left(\frac{\pi t}{2}\right) dt$

59. Let $u = \frac{\pi}{2}t$, so $du = \frac{\pi}{2}\,dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \frac{\pi}{2}$. Thus,

$$\int_0^1 \cos(\pi t/2)\,dt = \int_0^{\pi/2} \cos u\,\left(\frac{2}{\pi}\,du\right) = \frac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \frac{2}{\pi}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$$

60. $\int_0^1 (3t-1)^{50} dt$

60. Let $u = 3t - 1$, so $du = 3\,dt$. When $t = 0$, $u = -1$; when $t = 1$, $u = 2$. Thus,

$$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^{2} u^{50}\,\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{1}{51}u^{51}\right]_{-1}^{2} = \frac{1}{153}\left[2^{51} - (-1)^{51}\right] = \frac{1}{153}(2^{51}+1)$$

Exercises

61. $\int_0^1 \sqrt[3]{1+7x}\ dx$

61. Let $u = 1 + 7x$, so $du = 7\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 u^{1/3}\left(\tfrac{1}{7}\,du\right) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}\left(8^{4/3} - 1^{4/3}\right) = \tfrac{3}{28}(16-1) = \tfrac{45}{28}$$

62. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\frac{1}{2}t\right)\ dt$

62. Let $u = \frac{1}{2}t$, so $du = \frac{1}{2}\,dt$. When $t = \frac{\pi}{3}$, $u = \frac{\pi}{6}$; when $t = \frac{2\pi}{3}$, $u = \frac{\pi}{3}$. Thus,

$$\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right)dt = \int_{\pi/6}^{\pi/3} \csc^2 u\,(2\,du) = 2\Big[-\cot u\Big]_{\pi/6}^{\pi/3} = -2\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{6}\right)$$

$$= -2\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -2\left(\tfrac{1}{3}\sqrt{3} - \sqrt{3}\right) = \tfrac{4}{3}\sqrt{3}$$

Integration by Parts

- Integration by parts $\Leftrightarrow$ product rule for differentiation

$$\boxed{\int f(x)\, g`(x)\, dx = f(x)g(x) - \int g(x)f`(x)\, dx}$$

- Assume $u = f(x)$ and $v = g(x)$, then

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Integration by Parts

- Example: Find $\int x \sin x \, dx$

Let $x = f(x)$, $\sin x = g`(x)$

Then, $f`(x) = 1$, $g(x) = -\cos x$

So,

$\therefore \int x \sin x \, dx = f(x)g(x) - \int g(x)\, f`(x)\, dx$
$= x(-\cos x) - \int(-\cos x)dx$
$= -x\cos x + \sin x + C$

Integration by Parts

- Example: Evaluate $\int \ln x \, dx$.

Assume $u = \ln x$, $dv = dx$,

Then, $\mathrm{du} = \frac{1}{x} dx, v = x$

$\therefore \int \ln x \, dx = uv - \int v \, du$
$= x \ln x - \int x \cdot \frac{1}{x} dx$
$= x \ln x - x + C$

Integration by Parts

- Integration by Parts: Definite Integrals

$$\boxed{\int_a^b f(x)g`(x)dx = f(x)g(x)]_a^b - \int_a^b g(x)f`(x)dx}$$

Integration by Parts

- Example: Calculate $\int_0^1 \tan^{-1} x \, dx$

Let $u = \tan^{-1} x$, $dv = dx$
Then, $du = \frac{1}{1+x^2} dx$, $v = x$
$\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x \big]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$
$= 1 \tan^{-1} 1 \; - 0 \tan^{-1} 0 \; - \int_0^1 \frac{x}{1+x^2} dx$
$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$

To evaluate the second integral we use the substitution $t = 1 + x^2$
Then, $dt = 2x \, dx \rightarrow x \, dx = \frac{1}{2} \, dt$
When $x = 0, t = 0$; when $x = 1, t = 2$
$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} \ln\|t\| \Big]_1^2$
$= \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$

$$\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$

Exercises

5–42 Evaluate the integral

5. $\int te^{2t}\,dt$

6. $\int y\,e^{-y}\,dy$

7. $\int x \sin 10\,x\,dx$

8. $\int (\pi - x) \cos \pi x \; dx$

Exercises

5. $\int te^{2t}\,dt$

Let $u = t, dv = e^{2t}\,dt \quad \Rightarrow \quad du = dt, v = \frac{1}{2}e^{2t}.$

$\int te^{2t}\,dt = \frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C.$

6. $\int y\, e^{-y}\, dy$

Let $u = y, dv = e^{-y}\,dy \quad \Rightarrow \quad du = dy, v = -e^{-y}.$

$\int ye^{-y}\,dy = -ye^{-y} - \int -e^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy = -ye^{-y} - e^{-y} + C.$

7. $\int x \sin 10\, x\, dx$

Let $u = x, dv = \sin 10x\,dx \quad \Rightarrow \quad du = dx, v = -\frac{1}{10}\cos 10x.$

$$\int x\sin 10x\,dx = -\frac{1}{10}x\cos 10x - \int -\frac{1}{10}\cos 10x\,dx = -\frac{1}{10}x\cos 10x + \frac{1}{10}\int \cos 10x\,dx$$

$$= -\frac{1}{10}x\cos 10x + \frac{1}{100}\sin 10x + C$$

8. $\int (\pi - x)\cos \pi x\;\; dx$

Let $u = \pi - x, dv = \cos \pi x\,dx \quad \Rightarrow \quad du = -dx, v = \dfrac{1}{\pi}\sin \pi x.$

$$\int (\pi - x)\cos \pi x\,dx = \frac{1}{\pi}(\pi - x)\sin \pi x - \int -\frac{1}{\pi}\sin \pi x\,dx = \frac{1}{\pi}(\pi - x)\sin \pi x - \frac{1}{\pi^2}\cos \pi x + C.$$

Exercises

5–42 Evaluate the integral

29. $\int_0^1 x\,3^x\,dx$

30. $\int_0^1 \frac{(xe^x)}{1+x^2}\,dx$

31. $\int_0^2 y \sinh y\ \,dy$

32. $\int_1^2 w^2 \ln 2\,dw$

Exercises

29. $\int_0^1 x\, 3^x\, dx$

Let $u = x, dv = 3^x\,dx \quad \Rightarrow \quad du = dx, v = \dfrac{1}{\ln 3}3^x$. By (6),

$$\int_0^1 x3^x\,dx = \left[\frac{1}{\ln 3}\,x\,3^x\right]_0^1 - \frac{1}{\ln 3}\int_0^1 3^x\,dx = \left(\frac{3}{\ln 3} - 0\right) - \frac{1}{\ln 3}\left[\frac{1}{\ln 3}\,3^x\right]_0^1 = \frac{3}{\ln 3} - \frac{1}{(\ln 3)^2}(3-1)$$

$$= \frac{3}{\ln 3} - \frac{2}{(\ln 3)^2}$$

30. $\int_0^1 \frac{(xe^x)}{1+x^2}\, dx$

Let $u = xe^x, dv = \dfrac{1}{(1+x)^2}\,dx \quad \Rightarrow \quad du = (xe^x + e^x)\,dx = e^x(x+1)\,dx, v = -\dfrac{1}{1+x}$. By (6),

$$\int_0^1 \frac{xe^x}{(1+x)^2}\,dx = \left[-\frac{xe^x}{1+x}\right]_0^1 - \int_0^1 \left(-\frac{1}{1+x}\right)e^x(1+x)\,dx = \left(-\frac{e}{2}+0\right) + \int_0^1 e^x\,dx = -\frac{1}{2}e + \left[e^x\right]_0^1$$

$$= -\tfrac{1}{2}e + e - 1 = \tfrac{1}{2}e - 1$$

Exercises

31. $\int_0^2 y \sinh y \; dy$

Let $u = y$, $dv = \sinh y \, dy \quad \Rightarrow \quad du = dy$, $v = \cosh y$. By (6),

$$\int_0^2 y \sinh y \, dy = \Big[y \cosh y \Big]_0^2 - \int_0^2 \cosh y \, dy = 2\cosh 2 - 0 - \Big[\sinh y \Big]_0^2 = 2\cosh 2 - \sinh 2.$$

32. $\int_1^2 w^2 \ln 2 \, dw$

Let $u = \ln w$, $dv = w^2 \, dw \quad \Rightarrow \quad du = \dfrac{1}{w} \, dw$, $v = \frac{1}{3} w^3$. By (6),

$$\int_1^2 w^2 \ln w \, dw = \left[\tfrac{1}{3} w^3 \ln w \right]_1^2 - \int_1^2 \tfrac{1}{3} w^2 \, dw = \tfrac{8}{3} \ln 2 - 0 - \left[\tfrac{1}{9} w^3 \right]_1^2 = \tfrac{8}{3} \ln 2 - \left(\tfrac{8}{9} - \tfrac{1}{9} \right) = \tfrac{8}{3} \ln 2 - \tfrac{7}{9}.$$