Functions and Models

Mathematical Models: A Catalog of Essential Functions

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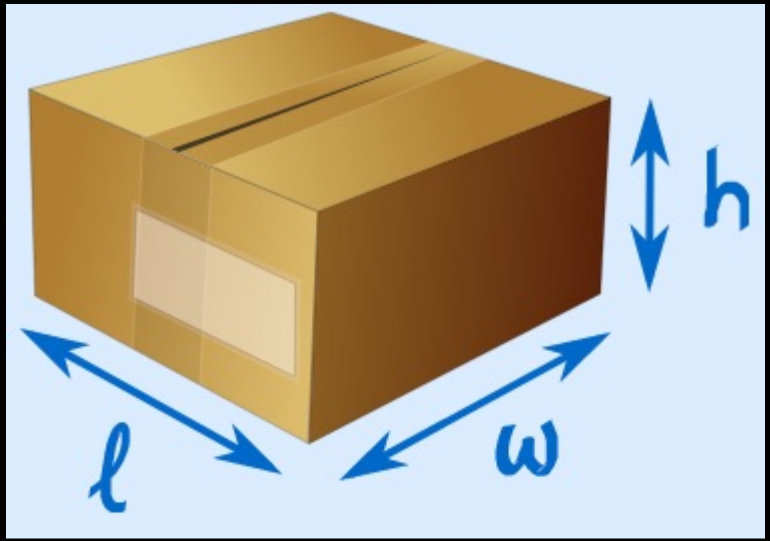
Introduction

- **Mathematical model:** a description (a function or an equation) of a real-world phenomenon
- Examples:
  - o the size of a population,
  - o the demand for a product,
  - o the speed of a falling object

- The purpose is to understand the phenomenon and to make predictions about future behaviour.

Introduction

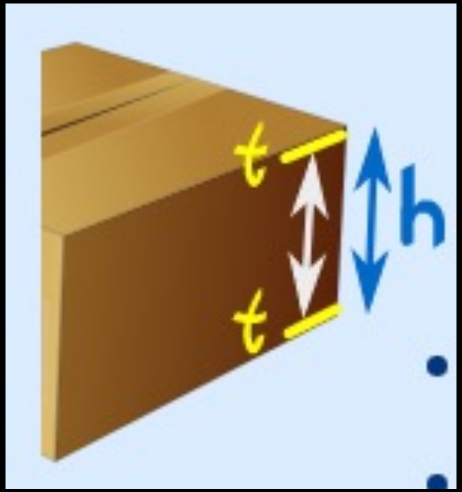
- Mathematics is used to "model", or represent, how the real-world works.



$$Volume = l \times w \times h$$

Introduction

- To be accurate about the volume of the box, we need to consider the thickness of the cardboard.
- The inside measurements need to be reduced by the thickness of each side:
  - The inside length is $l - 2t$
  - The inside width is $w - 2t$,
  - The inside height is $h - 2t$



$$Inside\ Volume\ =\ (l - 2t) \times (w - 2t) \times (h - 2t)$$

Introduction

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- The inside measurements need to be reduced by the thickness of each side:
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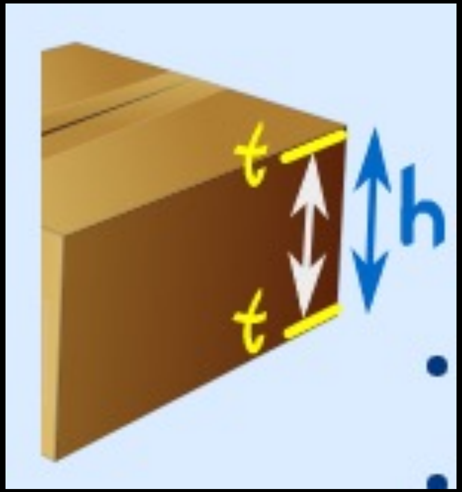
Now we have a **better** model.



$$Inside\ Volume = (l - 2t) \times (w - 2t) \times (h - 2t)$$

Introduction

- Mathematical models can also be used to forecast future behaviour.

Example: An ice cream company keeps track of how many ice creams get sold on different days.

By comparing this to the weather on each day they can make a mathematical model of **sales versus weather**.

They can then predict future sales based on the weather forecast, and decide how many ice creams they need to make ... ahead of time!

Introduction

- To model complex mathematical models, write them as computer programs

- More complex examples include:
  - Weather prediction
  - Economic Models (predicting interest rates, unemployment, etc)
  - Public health vs infectious diseases

Introduction

- How to formulate a mathematical model?

Introduction

- The mathematical modeling process:

1. Identify and name of the independent and dependent variables
    1. Apply laws and guides.
    2. Collect data and analyze the data in the form of a table to discover patterns.
2. From the numerical representation of the data, obtain a graphical representation of the data.
3. Apply mathematical rules to the model to draw conclusions.
4. Interpret the conclusions as information to explain the phenomenon.
5. Test our predictions by checking against new real data.
    1. Bad predictions → refine or build new model

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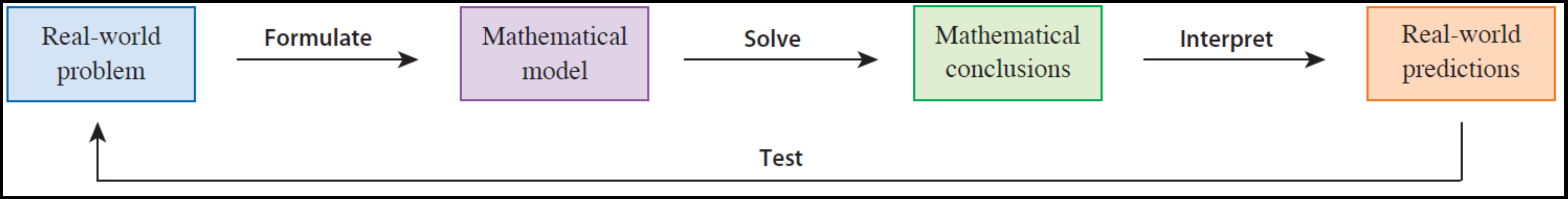
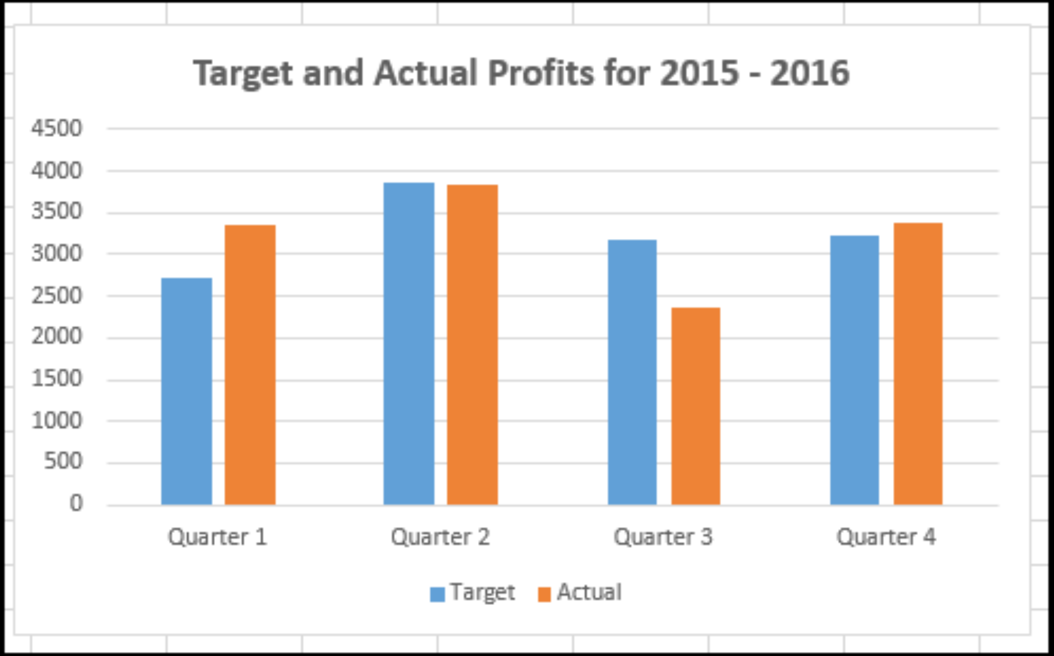


Introduction

- Graphical representation of the data:

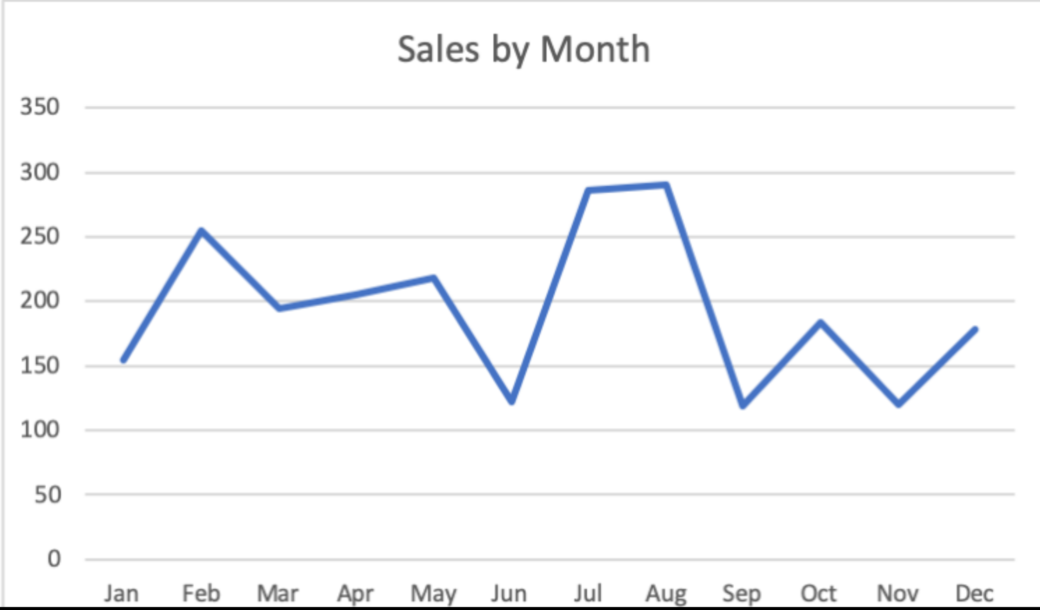
	Target	Actual
Quarter 1	2727	3358
Quarter 2	3860	3829
Quarter 3	3169	2374
Quarter 4	3222	3373



Introduction

- Graphical representation of the data:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Products	113824	188986	140691	167417	199789	122998	104406	162634	170378	171745	130481	158238
Services	320651	345882	282953	197752	385273	298660	156515	188593	374634	278056	208716	240923
Total revenue	434475	534868	423644	365169	585062	421658	260921	351227	545012	449801	339197	399161
Sales count	155	255	195	205	218	122	286	291	119	184	120	178



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Linear Models

- A **linear function** means that the graph of the function is a **line**.

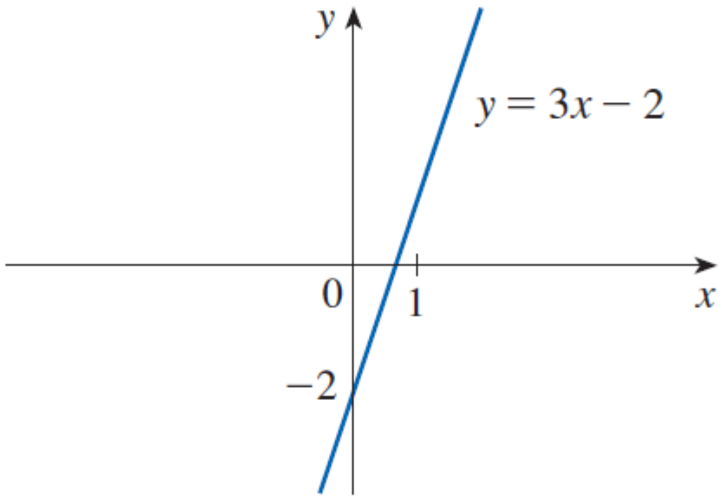
- Use the slope-intercept equation of a line to write a formula for the function

$$y = f(x) = mx + b$$

  - m is the slope of the line.
  - b is the y-intercept.

Linear Models

- Example: $f(x) = 3x - 2$



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

  - Whenever x increases by 0.1, the value of $f(x)$ increases by 0.3.
  - So, $f(x)$ increases three times as fast as x.
  - The **slope** of the graph, 3, can be interpreted as the **rate of change** of y with respect to x.

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Polynomials

- A function P is called a **polynomial** if
$$P(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$
  - n is a non-negative integer called **degree** of the polynomial.
  - The numbers a_0, a_1, … are called **coefficients** of the polynomial.

- Example:
$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

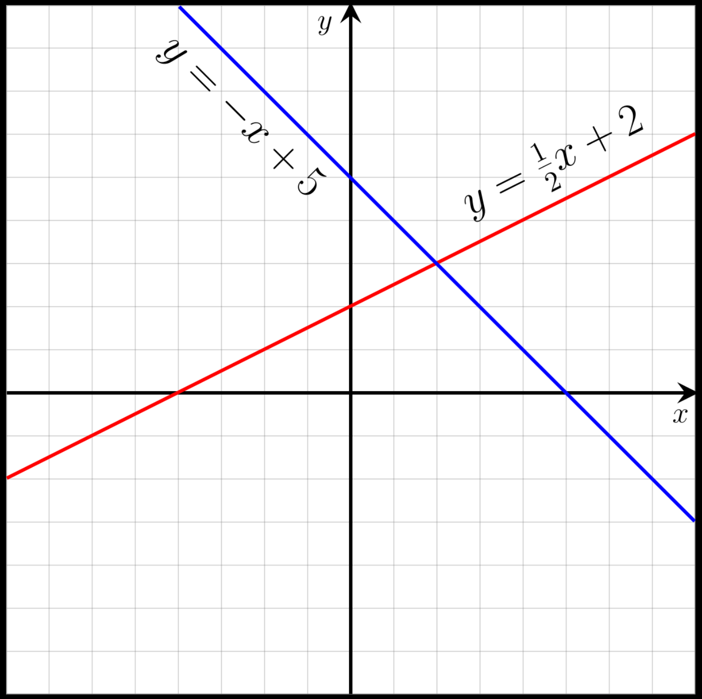
is a polynomial of degree 6.

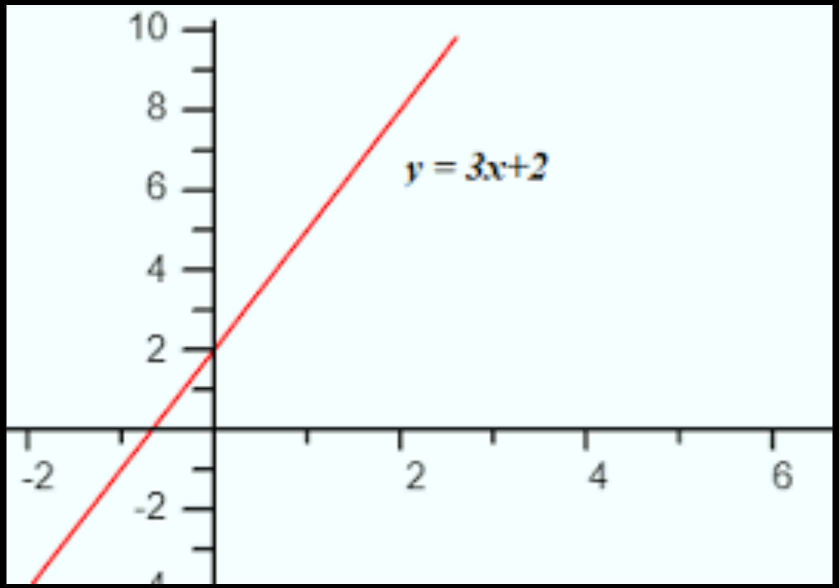
Polynomials

- A polynomial of degree 1:

$$P(x) = a_0 + a_1x^1 = mx + b$$

Which is a linear function.





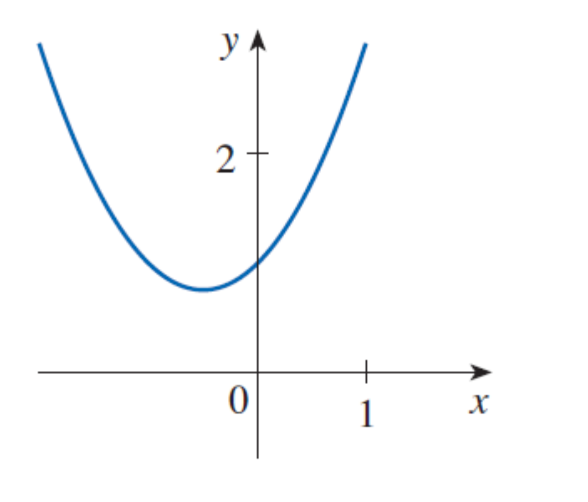
Polynomials

- A polynomial of degree 2:

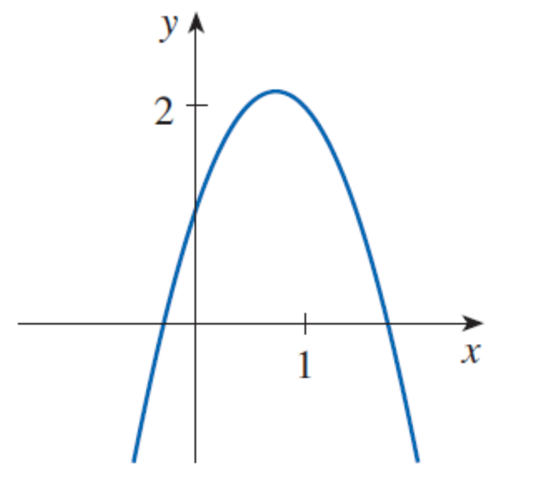
$$P(x) = ax^2 + bx + c$$

  Which is a quadratic function.

- Its graph is always a parabola.
  - The parabola opens upward if $a > 0$ and downward if $a < 0$



(a) $y = x^2 + x + 1$



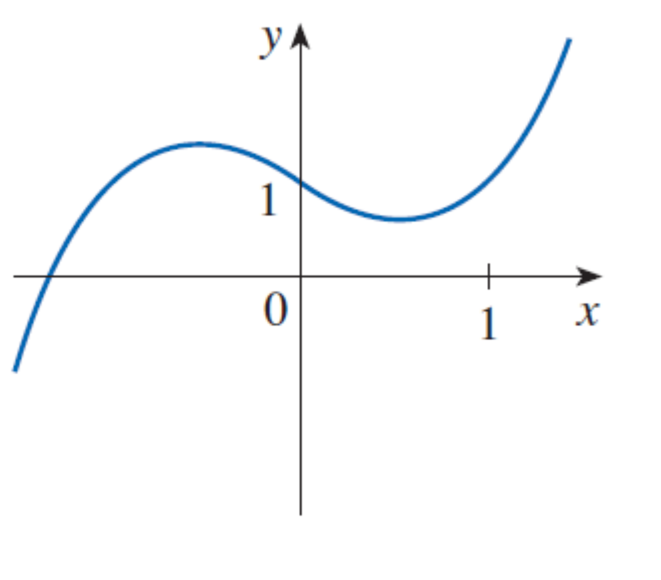
(b) $y = -2x^2 + 3x + 1$

Polynomials

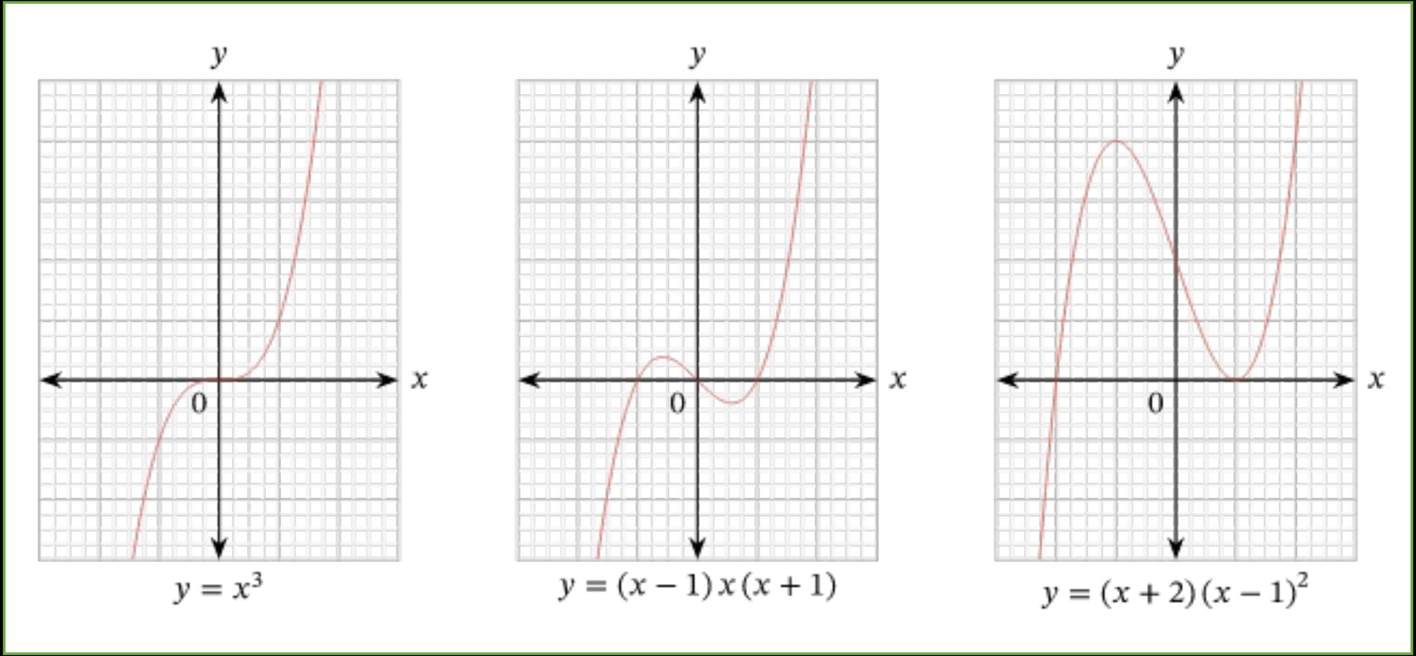
- A polynomial of degree 3:

$$P(x) = ax^3 + bx^2 + cx + d$$

Which is a cubic function.

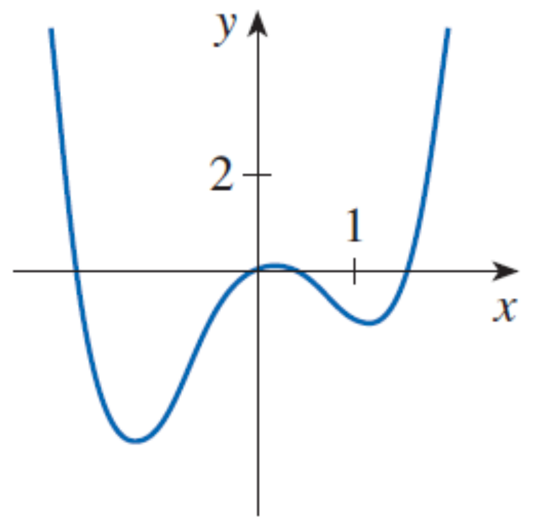


(a) $y = x^3 - x + 1$



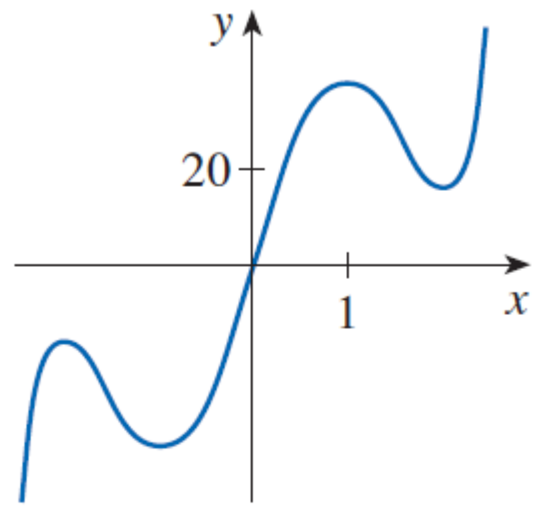
Polynomials

A polynomial of degree 4:



(b) $y = x^4 - 3x^2 + x$

A polynomial of degree 5:



(c) $y = 3x^5 - 25x^3 + 60x$

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Power Functions

- A function of the form $f(x) = x^a$, where a is a constant

- There are several cases:

$a = n$, where n is a positive integer
$a = 1/n$, where n is a positive integer
$a = -1$
$a = -2$

Power Functions

(1) $a = n$, where n is a positive integer.

- These are polynomials with one term.
- The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5

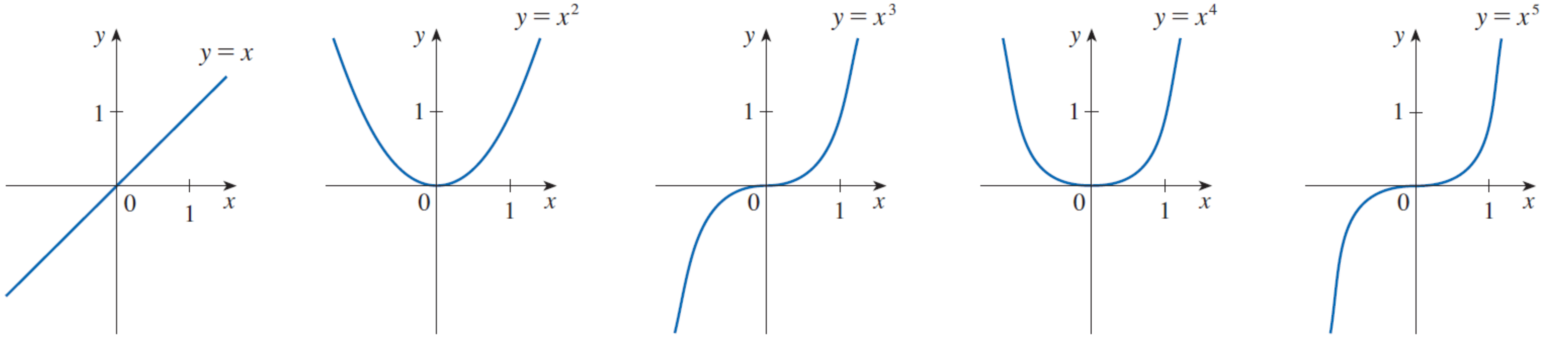
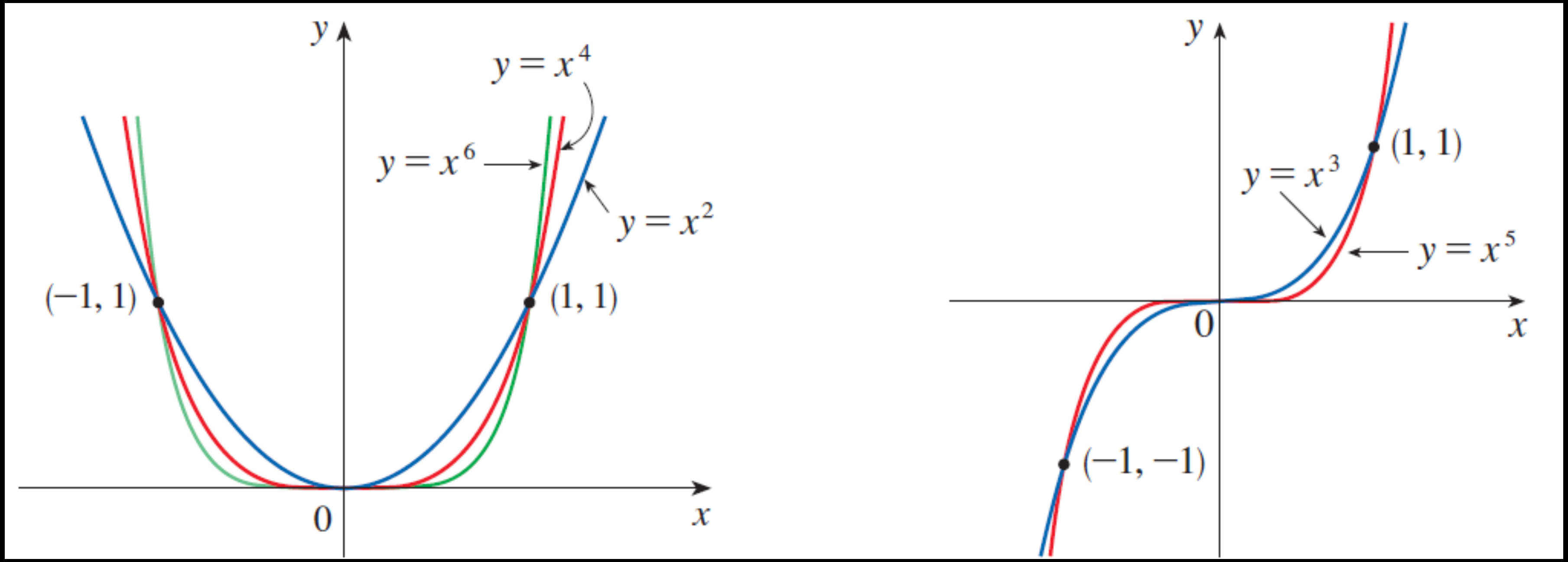


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Power Functions

- The shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
  - If n is even, then $f(x) = x^n$ is an even function and its graph is like that of $y = x^2$.
  - If n is odd, then $f(x) = x^n$ is an odd function and its graph is like that of $y = x^3$.

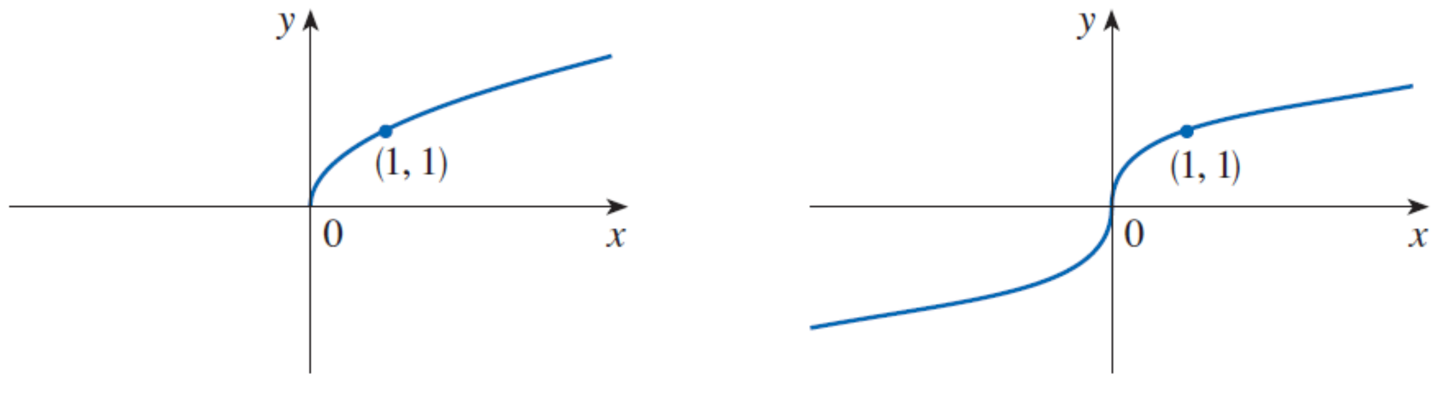


As n increases, the graph becomes flatter near 0 and steeper when |x| > 1.

Power Functions

(2) $a = 1/n$, where n is a positive integer

- The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a root function.
- If $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$
    - For other even values of n, the graph is like that of $f(x) = \sqrt{x}$.
- If $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$.
    - For other odd values of n, the graph is like that of $f(x) = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

Power Functions

(3) $a = -1$

- A reciprocal function $f(x) = x^{-1} = 1/x$

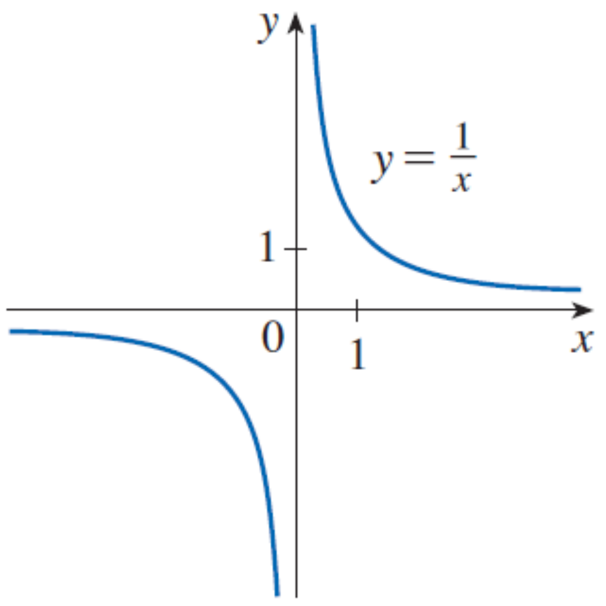


FIGURE 14
The reciprocal function

Power Functions

(4) $a = -2$

- A reciprocal function $f(x) = x^{-2} = 1/x^2$

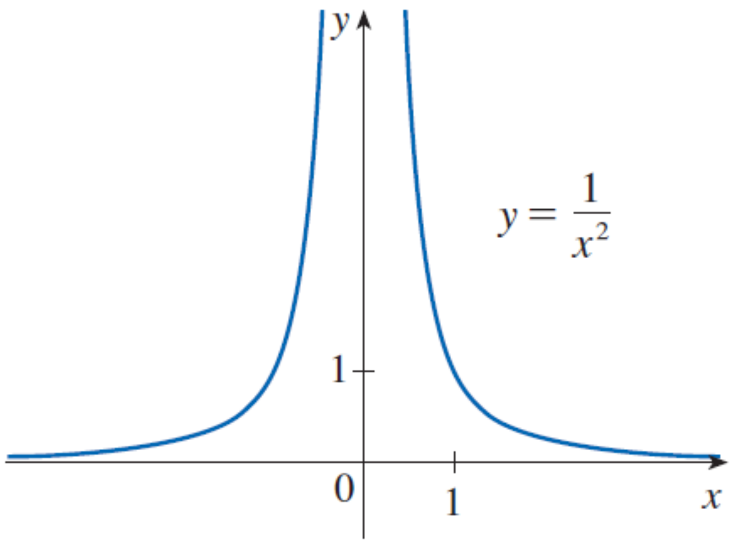


FIGURE 16
The reciprocal of the squaring function

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Rational Functions

- A **rational function** is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

  where P and Q are polynomials.

- The domain consists of all values of x such that $Q(x) \neq 0$.
- Notice that the function in the graph is not defined for $x = \pm 2$

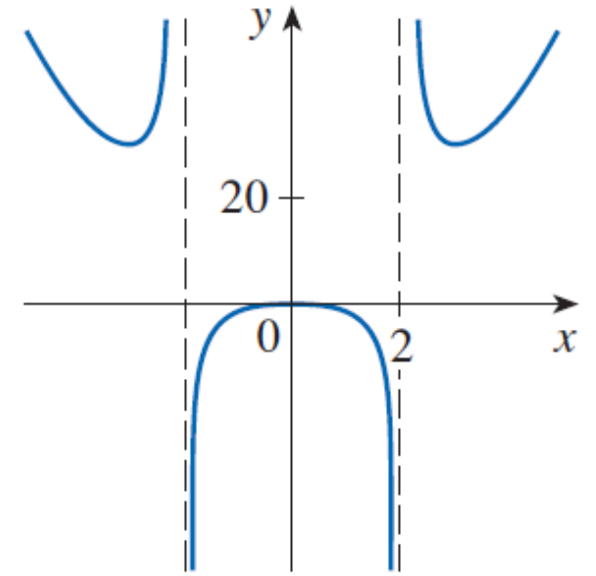

FIGURE 18

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

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Algebraic Functions

- **Algebraic function**: constructed using algebraic operations (+, -, *, /) starting with polynomials

- Examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

- Functions that are not algebraic are called **transcendental**

- Examples:

1. Trigonometric functions.
2. Exponential functions.
3. Logarithmic functions.

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Trigonometric Functions

- **Trigonometric functions**: relate an angle of a right-angled triangle to ratios of two side lengths.

sine

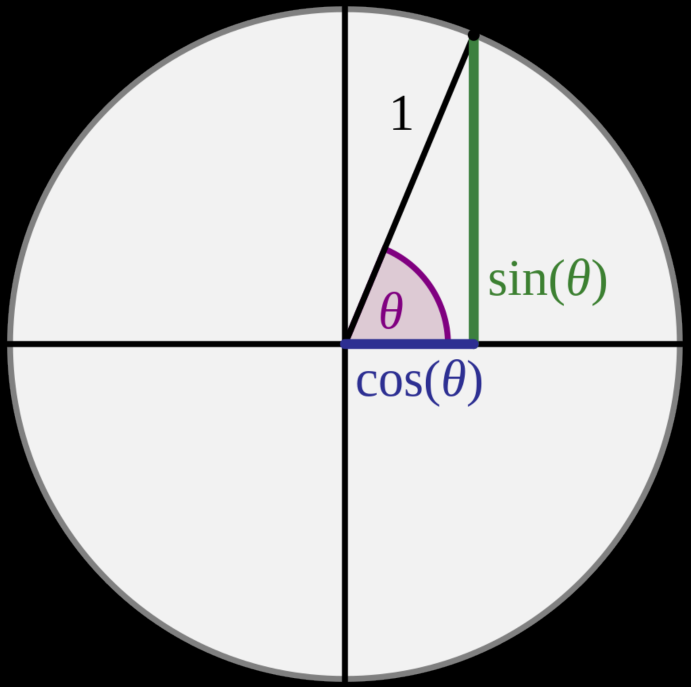
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Trigonometric Functions

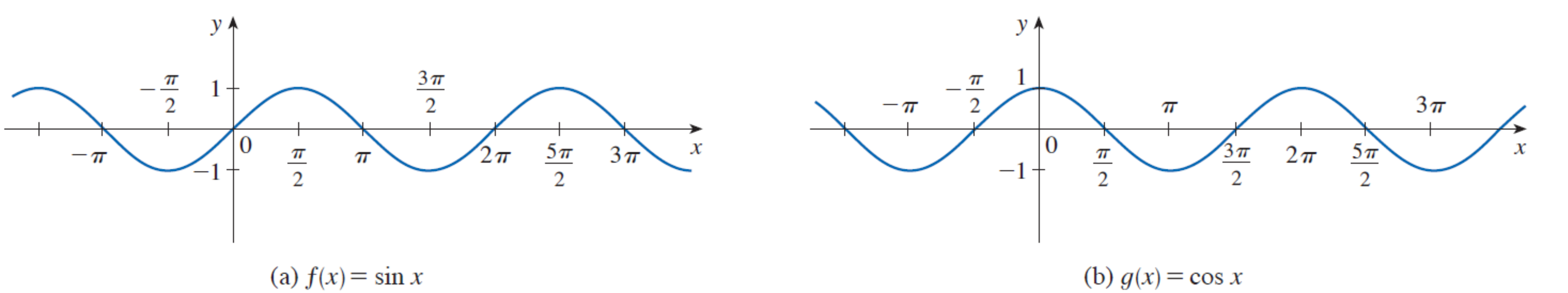
How trigonometric functions are originally computed without calculators?

Trigonometric Functions

- For both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$.

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

- Sine and cosine functions are periodic functions and have period 2π.



$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

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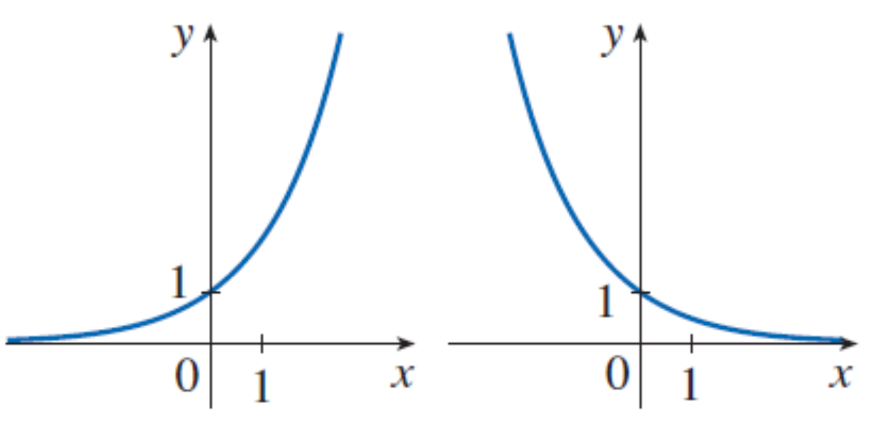
Exponential Functions

- **Exponential functions:** have the form $f(x) = b^x$, where the base b is a positive constant.



(a) $y = 2^x$ (b) $y = (0.5)^x$

- The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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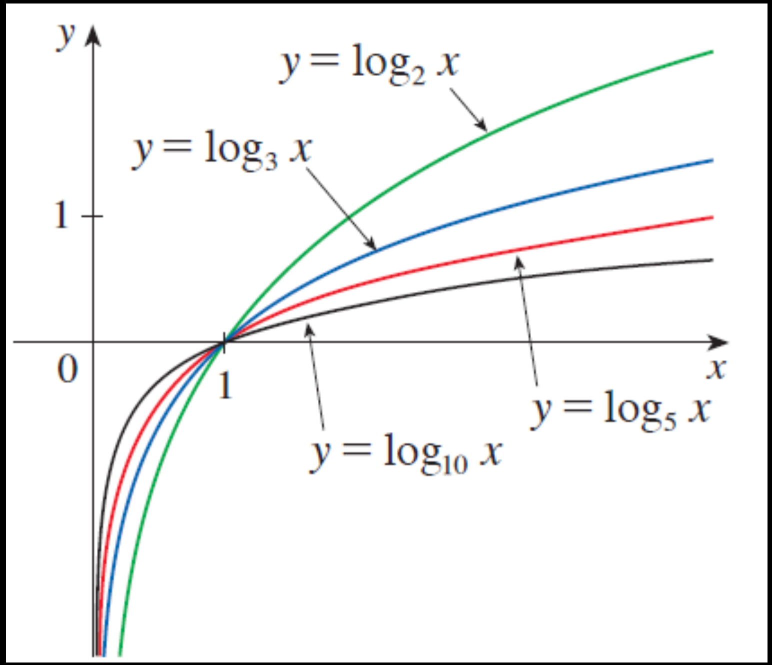
Logarithmic Functions

- **Logarithmic functions:** $f(x) = \log_b x$, where the base b is a positive constant.
  - The inverse of the exponential functions.
  - The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$ (c) $r(t) = t^{\sqrt{3}}$ (e) $y = \frac{(\sqrt{x})}{x^2+1}$

2. (a) $f(t) = \frac{3t^2+2}{t}$ (c) $s(t) = \sqrt{t+4}$ (e) $g(x) = \sqrt[3]{x}$

Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$

Polynomial of degree 3.
(This function is also an algebraic function.)

(c) $r(t) = t^{\sqrt{3}}$

Power

(e) $y = \dfrac{(\sqrt{x})}{x^2+1}$

Algebraic

2. (a) $f(t) = \dfrac{3t^2+2}{t}$

Rational function.
(This function is also an algebraic function.)

(c) $s(t) = \sqrt{t+4}$

Algebraic function.
It is a root of a polynomial.

(e) $g(x) = \sqrt[3]{x}$

Power function.
(This also an algebraic function
because it is a root of a polynomial.)

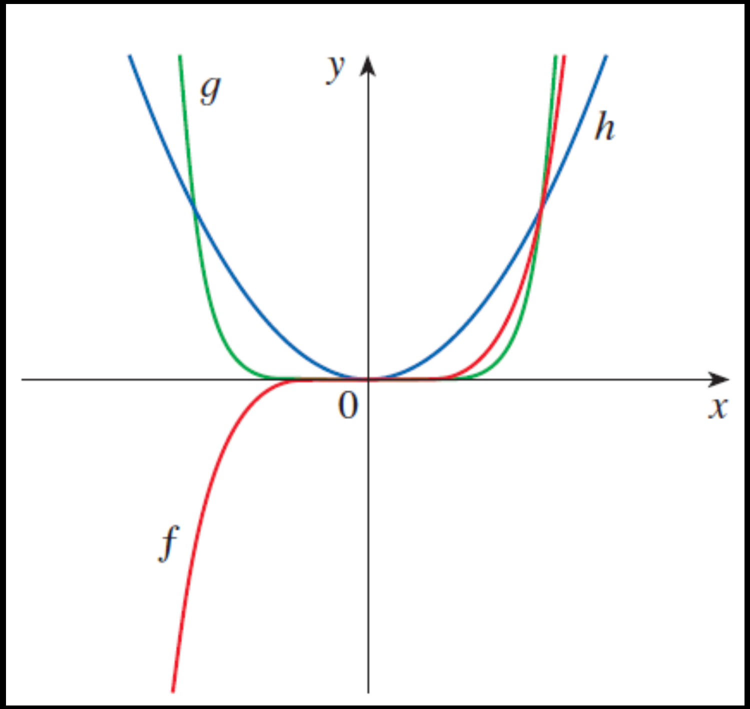
Exercises

3 Match each equation with its graph. Explain your choices. (Don’t use a computer or graphing calculator.)

(a) $y = x^2$

(b) $y = x^5$

(c) $y = x^8$



Exercises

3 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

(a) $y = x^2$

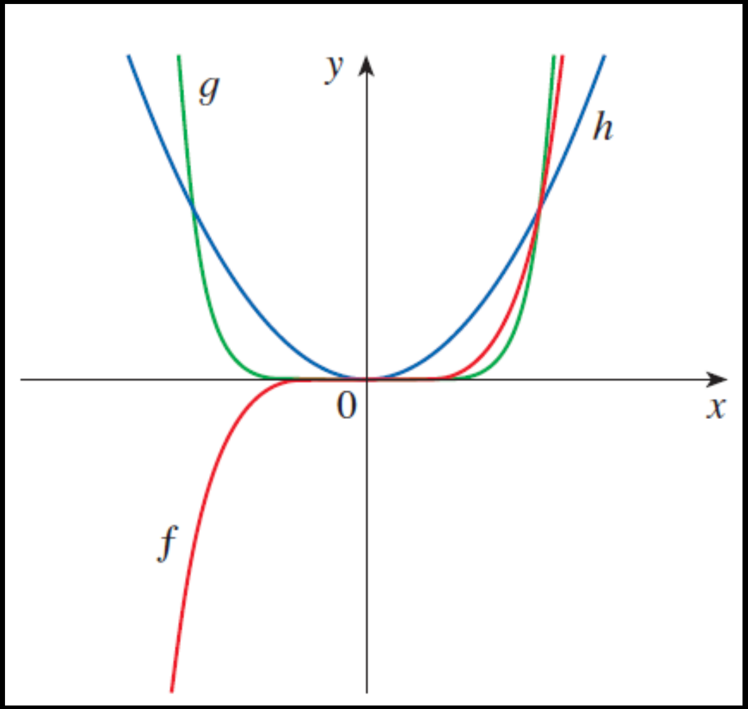
Graph (h), since it's an even function and symmetric about the y-axis

(b) $y = x^5$

Graph (f), since it's an odd function and symmetric about the origin

(c) $y = x^8$

Graph (g), since it's an even function and symmetric about the y-axis, and flatter than (h)



Exercises

5–6 Find the domain of the function

5) $f(x) = \frac{\cos x}{1 - \sin x}$

6) $\mathrm{g}(x) = \frac{1}{1 - \tan x}$

Exercises

5–6 Find the domain of the function.

5) $f(x) = \frac{\cos x}{1 - \sin x}$

The dominator cannot be 0, so $1 - \sin x \neq 0 \iff \sin x \neq 1 \iff x \neq \frac{\pi}{2} + 2n\pi$

Thus, the domain is $\{x \mid x \neq \frac{\pi}{2} + 2n\pi, n$ is an integer$\}$

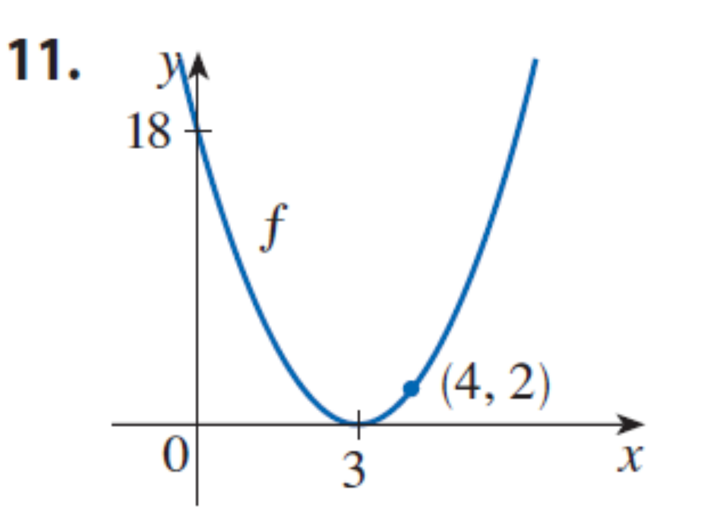
6) $g(x) = \frac{1}{1 - \tan x}$

The dominator cannot be 0, so $1 - \tan x \neq 0 \iff \tan x \neq 1 \iff x \neq \frac{\pi}{4} + 2n\pi$

Thus, the domain is $\{x \mid x \neq \frac{\pi}{4} + 2n\pi, n$ is an integer$\}$

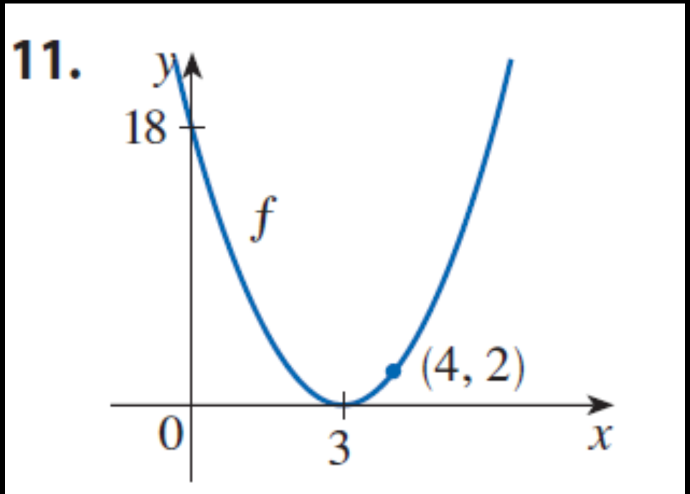
Exercises

11 Find a formula for the quadratic function whose graph is shown.



Exercises

11 Find a formula for the quadratic function whose graph is shown.



- The graph is a parabola, it is a quadratic function of the form $ax^2 + bx + c$.
- For $x = 0$ (y-intercept), $f(x) = 18. \therefore c = 18.$
- For $x = 3$, we have $f(x) = 0. \therefore 3^2a + 3b + 18 = 0 \Leftrightarrow 9a + 3b = -18 = 3a + b = -6$ → (1)
- For $x = 4$, we have $f(x) = 2. \therefore 4^2a + 4b + 18 = 2 \Leftrightarrow 16a + 4b = -18 + 2 = 4a + b = -4$ → (2)
- This is a system of two equations in the unknowns a and b and subtracting (1) from (2) gives $a = 2$.
- $\therefore 3(2) + b = -6 \Leftrightarrow b = -12$ so the formula is $f(x) = 2x^2 - 12x + 18$

Exercises

14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modelled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the earth's average surface temperature in 2100.

Exercises

14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modelled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the earth's average surface temperature in 2100.

(a) the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of 0.02 ◦C per year. The T-intercept is 8.50, which represents the average surface temperature in °C in the year 1900.
(b) $t = 2100 - 1900 = 200$ → $T = 0.02 * 200 + 8.50 = 12.50\ C$

Exercises

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a new-born?

Exercises

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a+1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a new-born?

(a) $D = 200$, so $c = 0.0417(200)(a+1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.
(b) For a new-born, a $= 0$, so $c = 8.34$ mg.

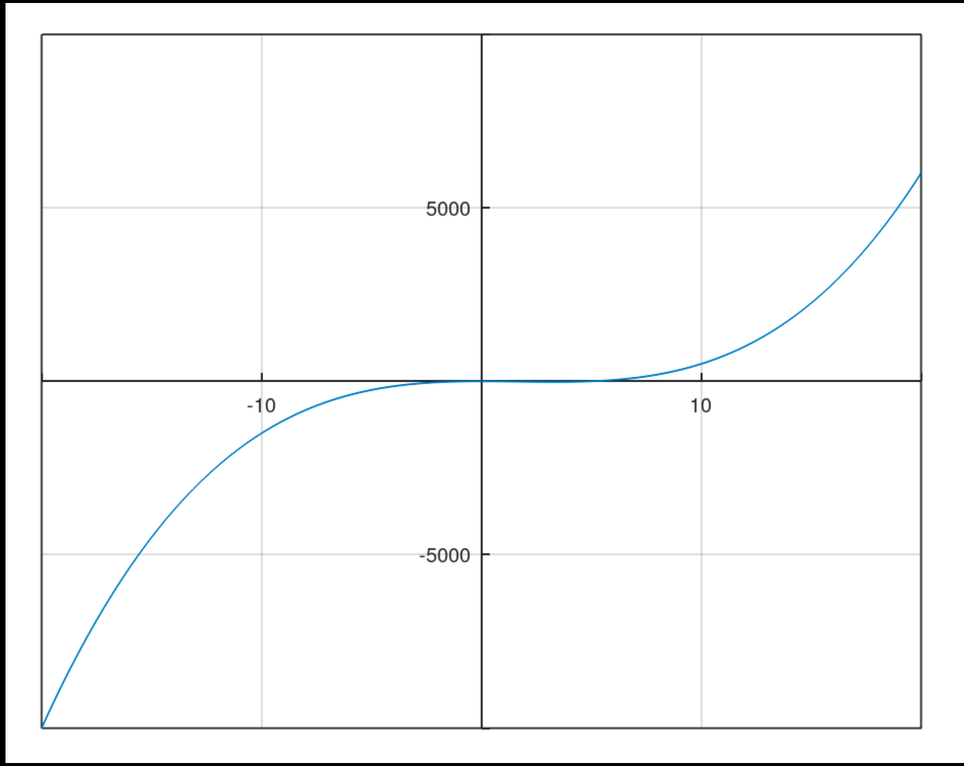
Exercises

10. Sketch several members of the family of polynomials $P(x) = x^3 - cx^2$. How does the graph change when c changes?

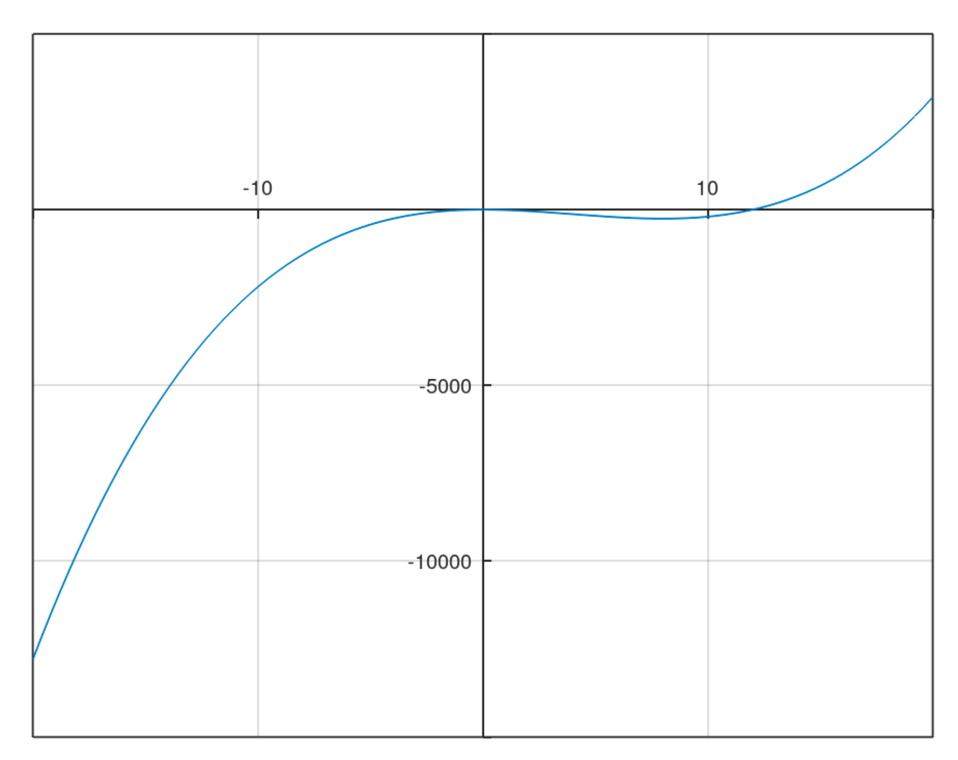
```
clc;
clear;

%% Change this...
c = 5;

f = @(x) x.^3 - c * x.^2;

x = linspace(-20, 20);

plot(x, f(x)); set(gca, "XAxisLocation", "origin", "YAxisLocation", "origin");
```

Exercises

$c = 5$



$c = 12$



Task (section 1.2)
1 (b, d, f)
2 (b, d, f)
4
12

Content
Introduction
Linear Models
Polynomials
Power Functions
Rational Functions
Algebraic Functions
Trigonometric Functions
Exponential Functions
Logarithmic Functions
Inverse Functions

Inverse Functions and Logarithms

- **Inverse functions**: functions that reverse each other
- Example: Table 1 shows the population of bacteria based on the hours passed. Table 2 shows the hours passed based on the population of the bacteria

Table 1 *N* as a function of *t*

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

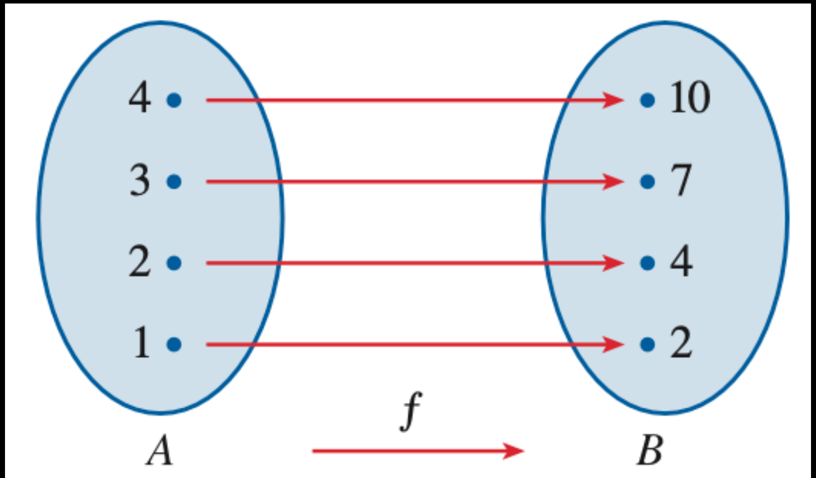
Table 2 *t* as a function of *N*

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

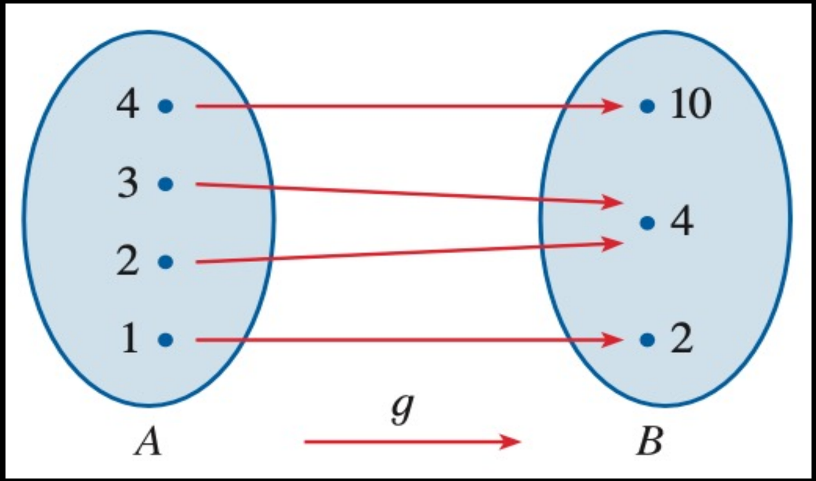
We say that Table 1 is the **inverse** of Table 2 and vice versa

Inverse Functions and Logarithms

- Not all functions can be invertible.
- Invertible functions must be **one-to-one**



f is one-to-one. It has an inverse
e.g., $f(4) = 10$, $f^{-1}(10) = 4$



g is **not** one-to-one. It has **no** inverse
e.g., $f(3) = 4$, $f^{-1}(4) = ?$

Inverse Functions and Logarithms

- Determine whether a function is one-to-one.

Horizontal Line Test
A function is one-to-one **if and only if** no horizontal line intersects its graph more than once.

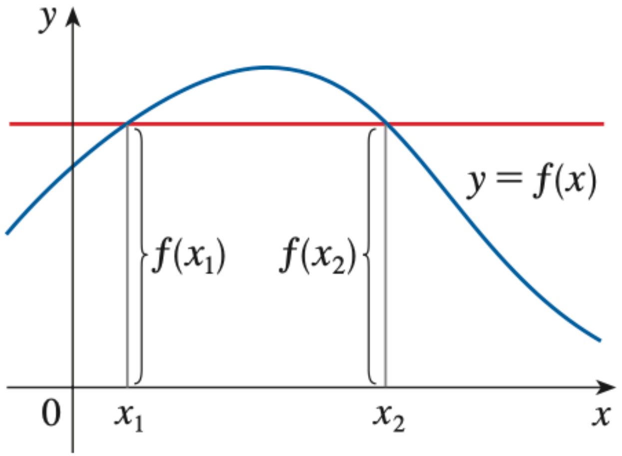
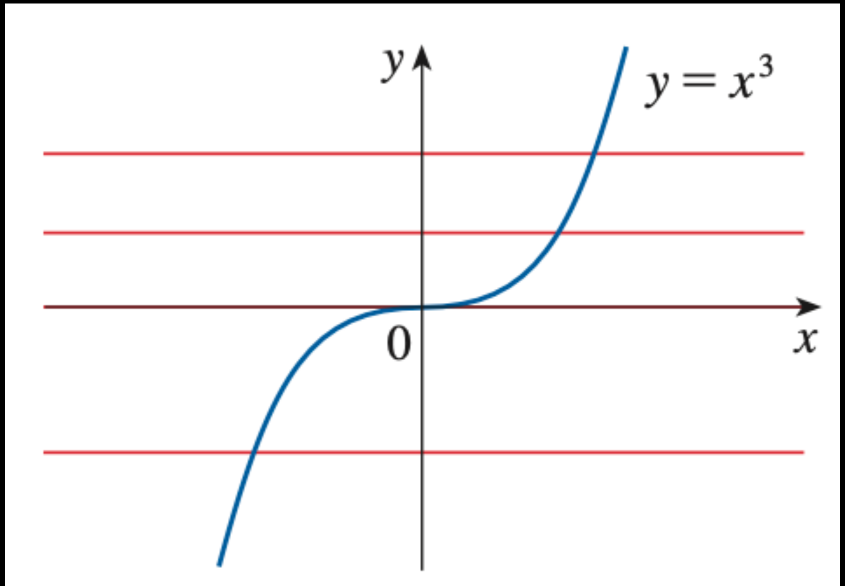


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

Inverse Functions and Logarithms

- Example: Is the function $f(x) = x^3$ one-to-one?
- Test some numbers x_1, x_2. If $x_1 \neq x_2$ gives $x_1^3 \neq x_2^3$, then it's one-to-one.
  - $3, 5;\ 2, -2$



Inverse Functions and Logarithms

- Example: Is the function $g(x) = x^2$ one-to-one?
- Not one-to-one. Try $x_1 = 1$ and $x_2 = -1$ → $g(1) = 1 = g(-1)$

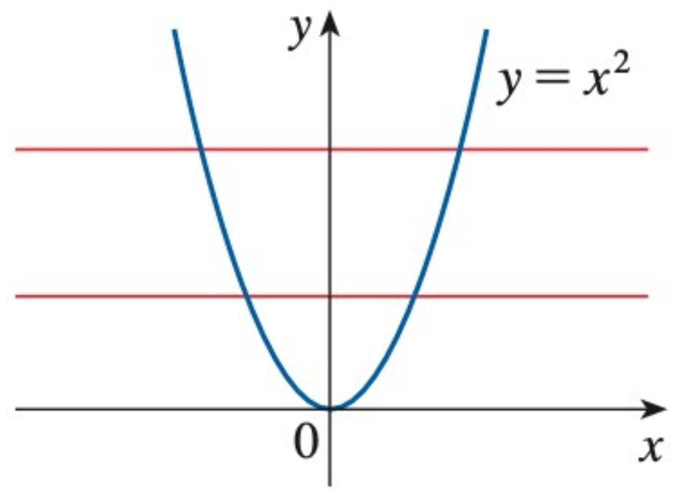
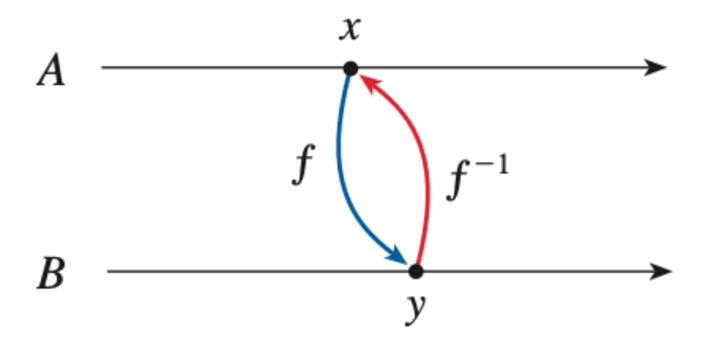


FIGURE 4
$g(x) = x^2$ is not one-to-one.

Inverse Functions and Logarithms

Inverse Functions
Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by $$f^{-1}(y) = x \iff f(x) = y$$

- Domain of f^{-1} = range of f
- Range of f^{-1} = domain of f



Inverse Functions and Logarithms

How to find the inverse of a one-to-one function

1. Write $y = f(x)$
2. Solve the equation for x in terms of y
3. Interchange the variables x, y, so the resulting equation is $y = f^{-1}(x)$

Inverse Functions and Logarithms

- Example: Find the inverse of $f(x) = x^3 + 2$

1. $y = x^3 + 2$
2. $x^3 = y - 2 \rightarrow x = \sqrt[3]{y - 2}$
3. $f^{-1}(x) = \sqrt[3]{x - 2}$

Inverse Functions and Logarithms

- If $b > 0$ and $b \neq 1$, we have the **exponential function** $f(x) = b^x$
- The inverse is the **logarithmic function** with base b

$$\log_b x = y \iff b^y = x$$

Laws of Logarithms
$\log_b(b^x) = x \quad \forall x \in R$
$b^{\log_b x} = x \quad \forall x > 0$
$\log_b(xy) = \log_b x + \log_b y$
$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
$\log_b(x^r) = r \log_b x \;\; \forall r \in R$

Inverse Functions and Logarithms

- A special logarithm is the **natural logarithm**: $log_e x = \ln x$
  - The e is Euler's constant, 2.718

$$\ln x = y \iff e^y = x$$

Laws of Natural Logarithms
$\ln(e^x) = x \quad x \in R$
$e^{\ln(x)} = x \quad x > 0$
$\ln e = 1$
$x^r = e^{r \ln x}$

Inverse Functions and Logarithms

- Example: solve the equation $e^{5-3x} = 10$

- Take the natural logarithm of both sides to cancel e

$$\ln e^{5-3x} = \ln 10 \quad \rightarrow \quad 5 - 3x = \ln 10$$

$$3x = 5 - \ln 10 \;\; \rightarrow \quad x = 1/3(5 - \ln 10)$$

$$x \cong 0.8991$$

Inverse Functions and Logarithms

- Logarithms with any base can be expressed in terms of the natural logarithm.

Change of Base Formula
For any positive number b $(b \neq 1)$, we have $$\log_b x = \frac{\ln x}{\ln b}$$

- Example: $\log_8 5 = \frac{\ln 5}{\ln 8} \cong 0.7739$

Inverse Functions and Logarithms

Trigonometric functions and their inverses

- $\sin(x) \iff \sin^{-1}(x) = \arcsin(x)$

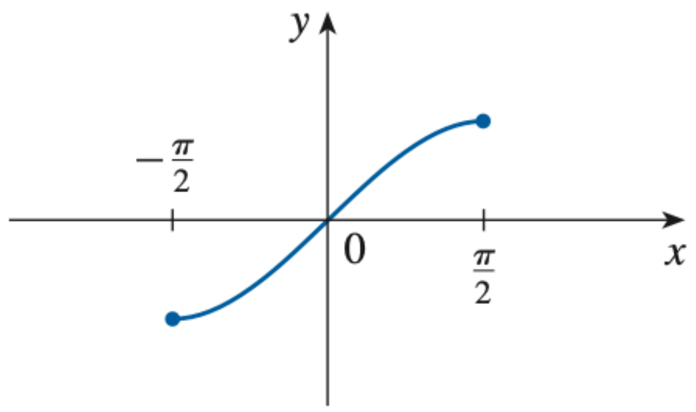


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

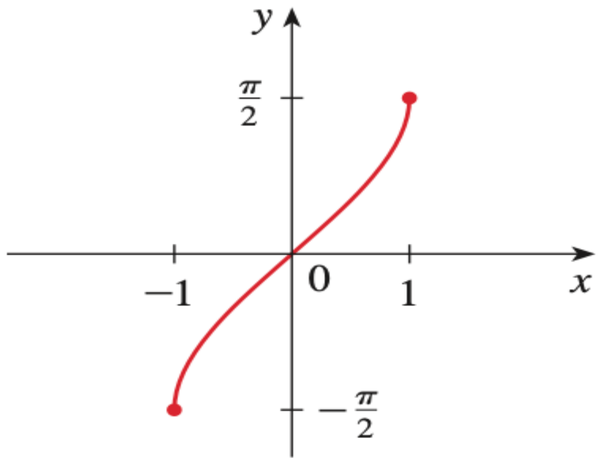
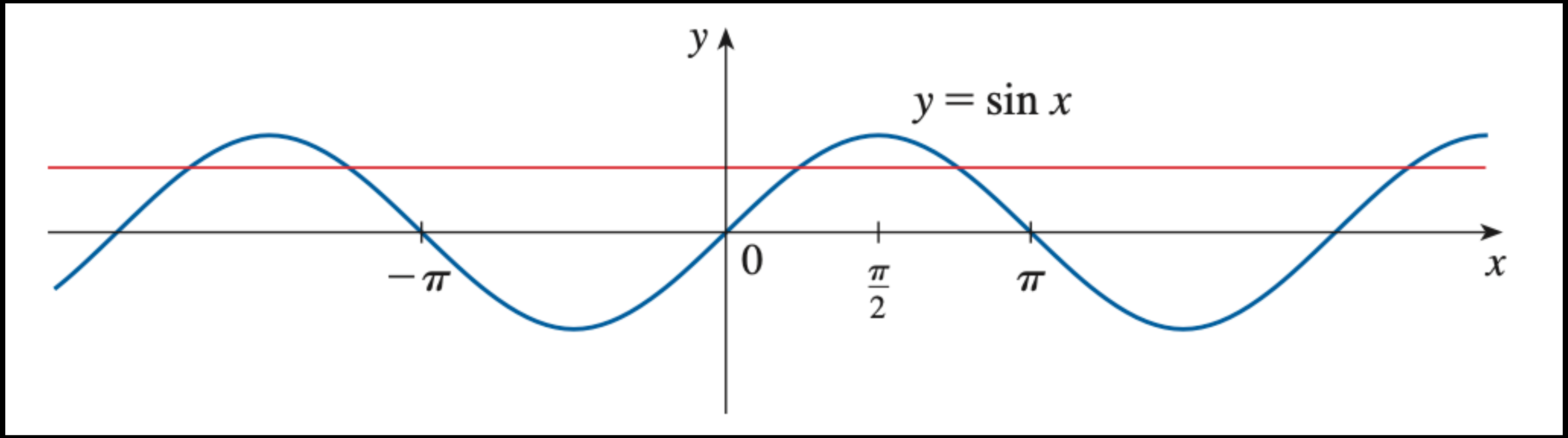


FIGURE 20
$y = \sin^{-1} x = \arcsin x$

Inverse Functions and Logarithms

Trigonometric functions and their inverses

- **Notice**: the sin function is one-to-one only when the domain is $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- sin doesn't pass the horizontal line test for all $x \in R$



Inverse Functions and Logarithms

Trigonometric functions and their inverses

- $\cos(x) \iff \cos^{-1}(x) = \arccos(x)$

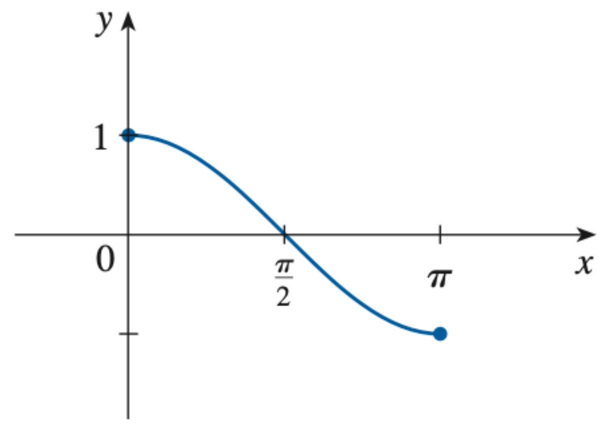


FIGURE 21
$y = \cos x, 0 \leqslant x \leqslant \pi$

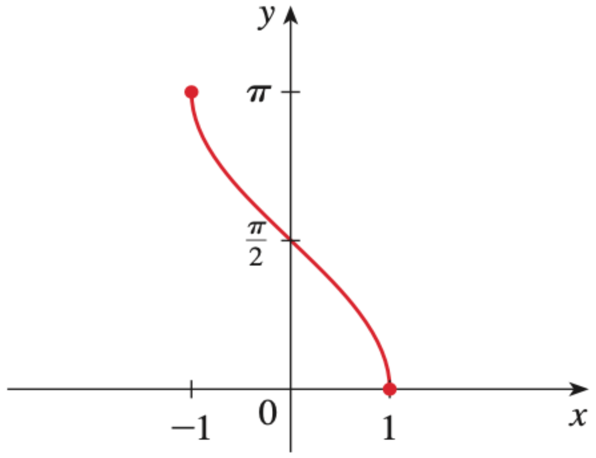


FIGURE 22
$y = \cos^{-1}x = \arccos x$

Inverse Functions and Logarithms

Trigonometric functions and their inverses

- $\tan(x) \iff \tan^{-1}(x) = \text{arc}tan(x)$

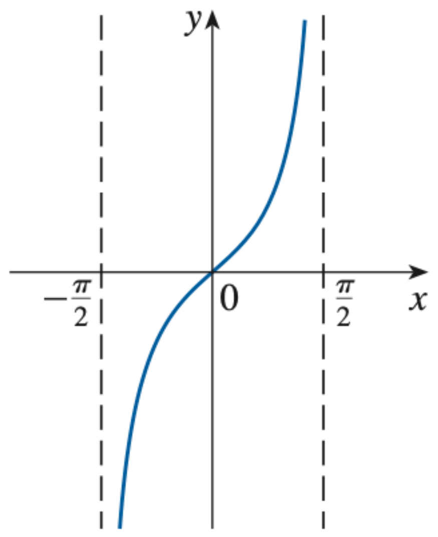
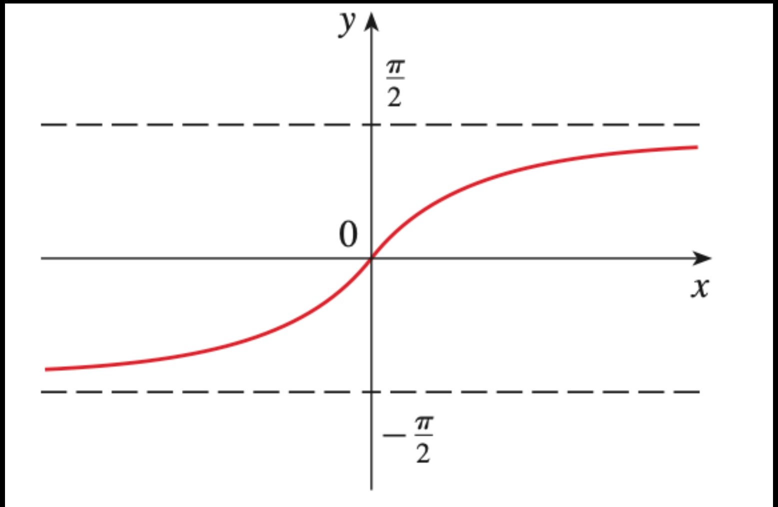


FIGURE 23
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



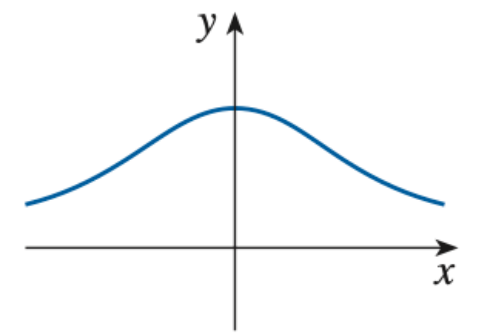
Exercises

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one

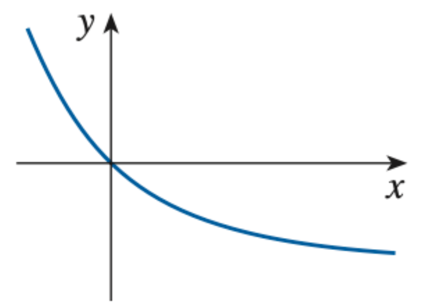
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

5.



7.



10. $f(x) = x^4 - 16$

11. $r(t) = t^3 + 4$

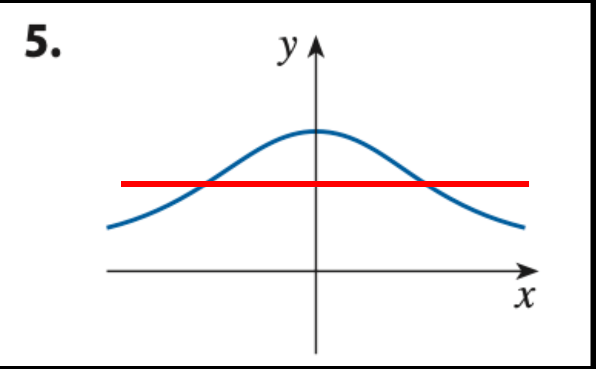
Exercises

3.	x	1	2	3	4	5	6
	$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

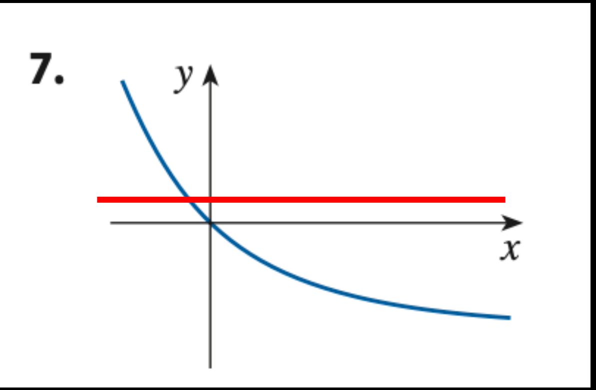
This is not one-to-one because $f(2) = f(6) = 2.0$

Exercises

This is not one-to-one because it doesn't pass the horizontal line test



This is one-to-one because it doesn't pass the horizontal line test



Exercises

10. $f(x) = x^4 - 16$

Try $x = 2, -2$: $f(2) = 2^4 - 16 = 0$, $f(-2) = (-2)^4 - 16 = 0$

Since $f(2) = f(-2) = 0$, the function is not one-to-one

11. $r(t) = t^3 + 4$

Solve it algebraically: assume we have $t_1 \neq t_2$, then we have $r(t_1) \neq r(t_2) \rightarrow$ $t_1^3 + 4 \neq t_2^3 + 4 \rightarrow t_1^3 \neq t_2^3 \rightarrow$ (take the cubic root) $t_1 \neq t_2$, which is valid.
Thus, it is one-to-one

Exercises

23–30 Find a formula for the inverse of the function.

25. $g(x) = 2 + \sqrt{x+1}$

26. $h(x) = \frac{6-3x}{5x+7}$

Exercises

25. $g(x) = 2 + \sqrt{x+1}$

$y = 2 + \sqrt{x+1} \rightarrow \sqrt{x+1} = y - 2 \rightarrow x + 1 = (y-2)^2 \rightarrow x = (y-2)^2 - 1$

$\therefore g^{-1}(x) = (x-2)^2 - 1$ for $x \geq 2$

26. $h(x) = \frac{6-3x}{5x+7}$

$y = \frac{6-3x}{5x+7} \rightarrow 6 - 3x = 5xy + 7y \rightarrow 5xy + 3x = 6 - 7y \rightarrow x(5y+3) = 6 - 7y \rightarrow$

$x = \frac{6-7y}{5y+3} \therefore h^{-1}(x) = \frac{6-7x}{5x+3}$

Exercises

25. $g(x) = 2 + \sqrt{x+1}$

26. $h(x) = \frac{6-3x}{5x+7}$

```
%% Setting python venv %%
setenv("PYTHON", "./venv/bin/python");
%%%%%%%%%%%%%%%%%%%%%%%%%
clc; clear;
pkg load symbolic
syms x y

g = 2 + sqrt(x+1)
%% Represent the equation f(x)= y, but we rearrange it as f(x)-y=0.
%% In other words, we find the value of x such that f(x) equals y.
%% Thus, we are solving x^2 + 2x - y = 0
inv_g = solve(g - y, x)
val = subs(g, 3)
% val = 4
inv_val = subs(inv_g, val)

%%%%%%%%%%%%%
h = (6-3*x)/(5*x+7)
inv_h = solve(h - y, x)
val = subs(h, 2)
%val = 0
inv_val = subs(inv_h, val)
```

Exercises

43-44 Use the laws of logarithms to expand each expression

(a) $\log_{10}(x^2 y^3 z)$

(b) $\ln\left(\frac{x^4}{\sqrt{x^2-4}}\right)$

Exercises

(a) $\log_{10}(x^2y^3z)$

$$\log_{10}(x^2y^3z) = \log_{10} x^2 + \log_{10} y^3 + \log_{10} z$$
$$= 2\log_{10} x + 3\log_{10} y + \log_{10} z$$

(b) $\ln\left(\frac{x^4}{\sqrt{x^2-4}}\right)$

$$\ln\left(\frac{x^4}{\sqrt{x^2-4}}\right) = \ln x^4 - \left(\ln(x^2-4)^{\frac{1}{2}}\right) = 4\ln x - \left(\frac{1}{2}\ln(x^2-4)\right)$$

$$= 4\ln x - 1/2(\ln[(x-2)(x+2)] = 4\ln x - 1/2(\ln(x-2) + \ln(x+2))$$

Exercises

Task (section 1.5)
4
6
8
9
23
24