A Mathematical Approach in Predicting a Solar Eclipse

Omar Khameis Ahmed Alzeyoudi

765612, AD387 Al-Ittihad, Mr.Salin Robert

1 Introduction.

An eclipse is an event that happens when one celestial object moves in the shadow of another celestial object. There are two types of eclipse: a solar eclipse which happens when the moon comes in between the sun and the earth blocking all sunlight from reaching the blue planet, and the other is lunar eclipse which happens when the earth passes between the sun and the moon. My goal in this project is to find a way to predict the next solar eclipse using calculus and the data about the celestial object needed for this even to occur, I also will be gathering data about celestial objects. Differential calculus will be crucial in this because I am observing at what rate are these celestial objects moving which will make me predict the location of these planets, but this is possible only for a two-body problem since we cannot solve three-body problems without simulation in physics. I will also use parametric equations using Cartesian coordinates to specify the exact locations of the moon, earth, and the sun. This topic has been famous for centuries and ways in which you can predict eclipses vary, a lot of work has been done some in which they can track all dates in which an eclipse occurred and then predict the next by the difference from each eclipse from the other and others use complex machinery that can solve hard mathematical problems in no time, but I don't have the acquired material for this therefore I will try to find a way in which I can predict the eclipse without the method of tracing but relating calculus to it.

2 History and Explanation

For a solar eclipse to occur one condition is required, which is the alignment of the sun, earth and moon. The geometry has to be understood for this event to take place, the moon has to be in between the sun and the earth unlike lunar eclipse which occurs when the earth has to be between the sun and the moon. This dynamic is possible to happen more than once a year "Solar eclipses occur two to five times a year, five being exceptional; there last were five in 1935" (Britannica, 2024).

2.1 Ancient Observation(Babylonian Astronomers, Chinese Astronomers, Greek Contributions):

The Babylonian prediction of eclipses was more than just an academic pursuit, instead they had a religious belief that the gods used eclipses as omens to predict the future for them and this reference hints out this statement "Lunar eclipses, in particular, were thought to portend the imminent death of a king"(Fred Espenak, 2023) and in king they meant a higher power in which it is god. This belief made them more focused on the scientific process in predicting an eclipse which led them to the discovery of what is called the Saros Cycle, which is essentially number of years between two eclipses. The saros cycle does not disprove that there are other eclipses between the intervals of the eclipses that follow this cycle, but it proves that there has to be an eclipse occurring after a certain number of years and process that is used to predict a lunar eclipse using the saros cycle is a bit different when it is used for the solar eclipse but they still share similarities. Babylonian clay tablets were used to have record of ancient eclipses "Babylonian Clay Tablets such as the one below, provide physical records of ancient eclipses viewed by humans, in this instance between 518 and 465 BCE"(NASA, 2017).

Figure 1: Babylonian Solar Eclipse Tablet listing eclipses between 518 and 465 BC).
Image Credit: NASA Sun Earth Day

The belief that an eclipse is used to foretell future wasn't only particularized for the Babylonian's but also the Chinese "In Ancient China, solar and lunar eclipses were regarded as heavenly signs that foretold the future of the Emperor."(NASA, 2017). This ideology wasn't the only one, instead in ancient China it was commonly held that solar eclipses occurred when a celestial dragon attached the devoured sun. Chinese eclipse records are some of the oldest in the world and go back more than 4,000 years, ultimately one would shout "the sun has been eaten". To frighten to what was believed a dragon they would bang drums and make loud noises which then became a tradition.

It is knows for Greeks that they were people of democracy, philosophy, poetry, medicine and also mathematics. The use of mathematics in Greek Era was so crucial in the history of predicting an eclipse, a mathematician astronomer Aristarchus of Samos who was the first person who made calculations of the Earth-Sun distance. In his book "On the Sizes and Distances of the Sun and Moon" he clearly mentions that the Sun was about 19 times as distant from the Earth as the Moon and hence 19 times the Moon's size, and this geometric observation led for our Era to have the chance to predict the eclipse mathematically. Another Greek scientist and mathematician named Milesians Thales, although his occupation was engineering but he also was the first ever natural philosopher. Thales was the first to predict an eclipse and that eclipse was called "eclipse of Thales", the surprising thing is that how thales predicted the eclipse was by using the Babylonian saros cycle "Thales may have used the Babylonian saros cycle which is used to predict eclipses."(Syracuse University ,2024), Babylonians as mentioned before were the ones who found the saros cycles but Greeks were the first to use it.

2.2 Medieval and Renaissance Advances(Islamic Golden Age, European Renaissance):

Al-Battani - Abu 'Abd Allah Muhammad ibn Jabir ibn Sinan al-Raqqi al-Harrani al-Sabi' al-Battani, was one of the most important astronomers of the ninth to the tenth century. He wrote *the book of the science of the ascensions of the signs of the zodiac in the spaces between the quadrants of the celestial sphere, a letter on the exact determination of the quantities of the astrological applications, commentary on Ptolemy's Tetrabilon,* and also *Astronomical treatise and tables.* He has a sea of observations and analysis in the field of astronomy and mathematics, he also discovered a new method and detailed it about the prediction of both solar and lunar eclipses "He also discovered a new method for determining the time of the vision of the new moon and made detailed study of solar and lunar eclipses, used as late as in the eighteenth century by Dunthorn, in his determination of the gradual change in lunar motion."(World Health Organization, 2013).

Al-Biruni, Abu Rayhan was an Iranian scholar and a genius polymath who has a reference in every available field of learning and was called *the teacher*. Al-Biruni calculated the radius and the circumference of the earth and was really precise in it "he arrived at a figure for the radius of the earth which has been equated to be 6,338 km, only 15 km from the estimate of today."(World Health Organization, 2013), the paper refers "he" as Al-Biruni. One of Al-Biruni proofs was the round shadow on the moon during lunar eclipse, which is an observation that is made more than once even by ancient philosopher.

One of the main historical figures that until this day his physics are used is Johannes Kepler, he found the three laws of planetary motion in his two works: *The New Astronomy(1609)* and *Harmonie of The World(1619).* His work are still considered an accurate description of the motion of any planet and any satellite and he also was the first to use the term "satellite". Kepler's work lead to the fact that the earth's orbit wasn't circular instead it was elliptical which made the model of predicting an eclipse nowadays more refined and accurate. Another mathematician in the European Renaissance is Edmond Halley, which was the first to publish a map predicting the time and path of a coming solar eclipse "The map he created shows England with a broad, gray band across it, with a darker patch within that shows how the moon's shadow would pass over the land" (Atlas Obscura ,2024), map is clear shown in Figure 2 below.

Figure 2: Halley's map of the eclipse of 1715. EDMOND HALLEY/PUBLIC DOMAIN sourced by University of Cambridge, Institute of Astronomy Library

2.3 Modern Era(19th and 20th Centuries, Current Methods):

Since the evolution of computers and hardware the life of humans became easy, and that is also true in predicting an eclipse. Nowadays with a push of a button from an astronomer a prediction is made, it sounds hard to create such machine but it is possible and that is what our modern era did. Satellites provide a continuous precise information on the position and movements of the Earth, Moon and Sun. The map in Figure 2 takes a lot of effort to be made whilst now there are maps that sync every second that describes the path of a solar eclipse.

Computers can solve complex mathematical models to stimulate the exact and precise refined orbits of the Earth, Moon and Sun which makes prediction more accurate, it also can process huge amounts of data quickly especially with the trending of AI and machine learning. It is mentioned before in this paper that the saros cycle is also a way to predict an eclipse, but using it manually is time and energy consuming that is why computers are such an important advent in our history that it made the calculations easier. Now we have a list of future eclipses that are thousands of years into the future "Using some 38,000 repeating mathematical terms, NASA can predict both solar and lunar eclipses for 1,000 years into the future. Beyond that, the Moon's wobble and Earth's changing rotation make eclipse prediction less accurate."(Sabine Bellstedt, 2023).

3 Link to Calculus.

3.1 Positions of the Earth, Moon and Sun

Parametric Equation can be used to create a formula for the positions of the Earth, Moon and Sun:

$$(x, y) = (f(t), g(t))$$

$$\boxed{Earth(t) = E(t) = (d_{SE}\cos(w_E t), d_{SE}\sin(w_E t))} \tag{1}$$

$$w_E = \frac{2\pi}{T_E} = 365.25\ days$$

$$\boxed{Moon(t) = M(t) = (d_{EM}\cos(w_M t), d_{EM}\sin(w_M t))} \tag{2}$$

$$w_M = \frac{2\pi}{T_M} = 27.3\ days$$

$$Sun(t) = S(t) = (0, 0) \quad taken\ as\ origin\ for\ earth\ orbit$$

After organizing the formulas and everything distances has to be calculated at first since angular velocities are already calculated. Now lets make an assumption about the forces acting on the earth, since it is orbiting the sun then two forces which are the Centrifugal and Centripetal forces has to balance out:

$$F_{Centrifugul} = F_{Centripetal}$$
$$\frac{GM_{sun}m_{earth}}{r^2} = \frac{mv_{earth}^2}{r}$$
$$\frac{\cancel{r}}{\cancel{m_{earth}}}\left(\frac{GM_{sun}\cancel{m_{earth}}}{r^{\cancel{2}}}\right) = \left(\frac{\cancel{m_{earth}}v^2}{r}\right)\frac{\cancel{r}}{\cancel{m_{earth}}}$$
$$\frac{GM_{sun}}{r} = v_{earth}^2$$
$$\boxed{r = \frac{GM_{sun}}{v_{earth}^2} = d_{SE}} \tag{3}$$

Equation 3 gives out the distance between the sun and the earth, certain conditions has to apply for this equation to work:

1. It has to be for a two body problem.
2. One body has to be taken as origin which therefore would be stationary.
3. The other body has to be orbiting the first body for centrifugal force to balance with centripetal force.

The mass (M) will be the mass of the stationary body and the velocity (v) would be for the body moving in orbit. In our situation were taking the sun as origin and earth orbiting the sun therefore Equation 3 can be used precisely to get the distance between both. It can also be used with the earth and moon taking the earth as origin and using the velocity of the moon to get the distance between the moon and the earth as shown in Equation 4 below.

$$\boxed{d_{EM} = \frac{GM_{earth}}{v_{moon}^2}} \tag{4}$$

$$d_{SE} = \frac{\left(6.6743\times10^{-11}m^3\mathrm{kg}^{-1}s^{-2}\right)\left(1.989\times10^{30}kg\right)}{\left(29784.8m/s\right)^2} = 1.49641\times10^{11} \approx 1.496\times10^{11}m$$

$$d_{EM} = \frac{\left(6.6743\times10^{-11}m^3\mathrm{kg}^{-1}s^{-2}\right)\left(5.972\times10^{24}kg\right)}{\left(1022\ m/s\right)^2} = 381613501 \approx 3.82\times10^{8}m$$

Now positions of the earth and moon as follow becomes:

$$E(t) = \left(\left(1.496\times10^{11}\right)\cos\left(365.25t\right), \left(1.496\times10^{11}\right)\sin\left(365.25t\right)\right)$$

$$M(t) = \left(\left(3.82\times10^{8}\right)\cos\left(27.3t\right), \left(3.82\times10^{8}\right)\sin\left(27.3t\right)\right)$$

3.2 Conditions for a solar eclipse:

Because a solar eclipse occur when the moon is between the sun and earth therefore the distance between the moon and the sun has to be smaller than the distance of earth and moon and larger than the distance between the earth and the moon:

$$d_{EM} < d_{SM} < d_{SE}$$

Once this condition is true there is a chance for a solar eclipse to occur, when analyzing the data there are a lot of position were this condition is true, so there has to be other laws to determine or at least maximize accuracy in the values we get. Therefore we have to take an account of the observation of the size of the moon and the sun in perspective of the earth, angular size of the earth changes the property of an eclipse either it becomes total, annular or partial. For a partial it becomes obvious since your observation becomes in two linear equations instead of one, but with annular and total we are only taking one axis.

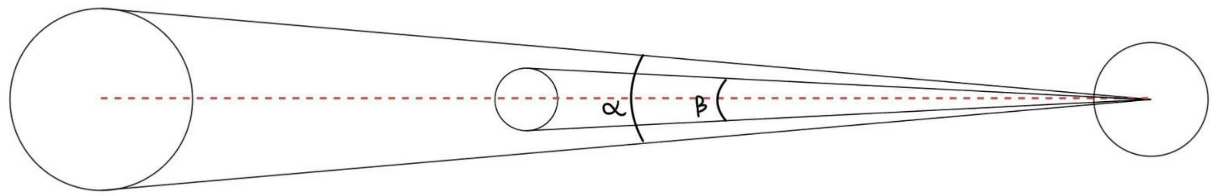

Figure 3: Not drawn to scale

in this the angle in perspective of the earth to the moon is β whilst to the sun is α, and for each variation of these numbers a type of eclipse occur

1. If $\beta \geq \alpha$, then a total eclipse occurred
2. If $\beta < \alpha$, then an annular eclipse occurred

For us to calculate the angles we have to use Equation 5 below:

$$\boxed{\delta = 2 \times \arctan \frac{d}{2D}} \tag{5}$$

Where d is the actual diameter of the object(Moon or Sun in our situation), and D distance from the observer to the object.

4 Discussion.

In this study, we explored the application of calculus in predicting a solar eclipse. Positions of the Earth, Moon and Sun were analyzed and condition were made for a solar eclipse to occur, once these conditions are true then a there is a high chance that an eclipse event will occur.

4.1 Keybindings of the research:

To predict the eclipse we first need to understand the dynamic of celestial objects that are needed for this event to happen, this means creating equations for the positions of the Earth, Moon and the Sun.

$$E(t) = ((1.496 \times 10^{11})\cos(365.25t), (1.496 \times 10^{11})\sin(365.25t))$$
$$M(t) = ((3.82 \times 10^{8})\cos(27.3t), (3.82 \times 10^{8})\sin(27.3t))$$
$$S(t) = (0, 0)$$

Also other conditions has to be met such as the apparent angular diameter specified in section 2.

1. If $\beta \geq \alpha$, then a total eclipse occurred
2. If $\beta < \alpha$, then an annular eclipse occurred

4.2 Predictions for the future:

The information that is in this research will help us to predict the future eclipses and their characteristics since the evolution of computers we can process and sync everything in a contemporary way. This process has to follow certain steps that aren't mentioned in this paper:

1. Get the positions of the Earth, Moon and Sun as a function of time.
2. Apply the conditions of the alignment of celestial objects and they're apparent angular diameter.
3. Use complex mathematical procedures to gather data of all stamps for all the celestial objects.
4. For each time step taken certain things has to be calculated
   (a) Forces acting on each body(such as gravitational).
   (b) Updating velocity values by using newton second law.
   (c) Updating position values by using equations of motion.
5. Refine the model you have with all the anomalies such as the eccentricity of the Earth.
6. Save the data and make them visually easy to understand.
7. Use the data to predict future solar eclipses
8. Refine it one last time using observational data of the past solar eclipses(Saros Cycle) and adjust the model to improve accuracy.

5 Conclusion:

Eclipses were a historical figure back then, they were thought a sign for foretelling future whilst others thought emphasized that a high power was trying to eat the sun such as a dragon. With all this weird beliefs we cannot deny that the tradition of people at that time created what it is now. The saros cycle were an ancient discovery lead by the Babylonians, Chinese traditions made data recording of the past eclipses an important even. Also all the contributions of the Islamic Golden Age and European Renaissance, and their contribution to the mathematical point of view in predicting eclipses. In this project I have learned that a solar eclipse is more than an event but it holds a historical story to it, and also the new mathematical and physical concepts taken such as Kepler three laws of planetary and Edmond Halley map of eclipse, I think they were brilliant on how it emphasizes history to it and the relation of mathematics in this.

References

[1] Espenak, F. (2023, November 20). How did the ancients predict eclipses? The Saros Cycle. SKY and TELESCOPE. https://skyandtelescope.org/astronomy-news/how-did-the-ancients-predicted-eclipses-the-saros-cycle/: :text=Babylonian

[2] Petruzzello, M. (2017, August 1). The Sun Was Eaten: 6 Ways Cultures Have Explained Eclipses. Encyclopedia Britannica. https://www.britannica.com/list/the-sun-was-eaten-6-ways-cultures-have-explained-eclipses

[3] NASA. (2017). Eclipse History. https://eclipse2017.nasa.gov/eclipse-history

[4] Syracuse University College of Arts and Sciences. (2024). The First Predicted Eclipse. https://artsandsciences.syracuse.edu/2024-eclipse/the-first-predicted-eclipse/: :text=Thales

[5] Bellstedt, S. (2023, April 19). Humans have been predicting eclipses for thousands of years, but it's harder than you might think. Astronomy Institution. https://www.astronomy.com/observing/humans-have-been-predicting-eclipses-for-thousands-of-years-but-its-harder-than-you-might-think/

[6] Anjum, S. (2013). Contribution of Muslim Astronomical Scientists to the World. Department of Islamic Studies, Aligarh Muslim University, Aligarh, India and the WHO. https://applications.emro.who.int/imemrf/Hamdard$_{M}ed/Hamdard_{M}ed_{2}013_{5}6_{42}4_{3}3.pdf$

[7] Laskow,S . (2024, March 11). Eclipse Maps Entered a Golden Age Thanks to Edmond Halley. Atlas Obscura. https://www.atlasobscura.com/articles/eclipse-maps-halley-18th-century-astronomy

Graphs, Figures, Tables and Equations

Figure 1: Babylonian Solar Eclipse Tablet listing eclipses between 518 and 465 BC).
Image Credit: NASA Sun Earth Day

Figure 2: Halley's map of the eclipse of 1715. EDMOND HALLEY/PUBLIC DOMAIN sourced by University of Cambridge, Institute of Astronomy Library

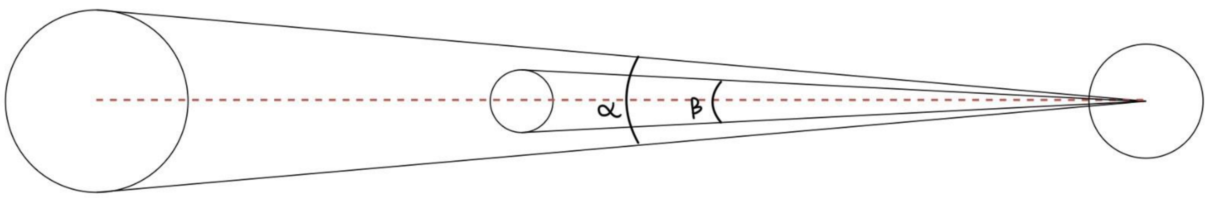

Figure 3: Not drawn to scale

$$\boxed{Earth(t) = E(t) = (d_{SE}\cos{(w_E t)}, d_{SE}\sin{(w_E t)})} \tag{1}$$

$$\boxed{Moon(t) = M(t) = (d_{EM}\cos{(w_M t)}, d_{EM}\sin{(w_M t)})} \tag{2}$$

$$\boxed{r = \frac{GM_{sun}}{v_{earth}^2} = d_{SE}} \tag{3}$$

$$\boxed{d_{EM} = \frac{GM_{earth}}{v_{moon}^2}} \tag{4}$$

$$\boxed{\delta = 2 \times \arctan\frac{d}{2D}} \tag{5}$$