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Methods for Constructing Lyapunov Functions for A Class of Nonlinear Systems

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Abstract: In automatic control theory, the stability of differential equations can be determined with Lyapunov's second method, which is simple and concise. However, an approach to constructing Lyapunov functions for special scenarios is an issue that remains to be addressed. Lyapunov function s that are identified for special scenarios can confer many conveniences to engineering practice. In this study, we proved that, for linear time-invariant systems, Lyapunov functions can be obtained by analyzing the relationships among coefficients to enable a rapid evaluation of the system stability. We then generalized the conclusion on linear time-invariant systems to nonlinear systems by investigating the Lyapunov function for four types of nonlinear systems. It was determined that the methods developed in this study can be used to quickly find Lyapunov function for assessing the system stability in differential forms when undifferentiated variables vary in terms of odd powers. Lastly, we analyzed Lyapunov functions for nonlinear time-invariant and nonlinear time-variant systems-both types of systems containing function-type coefficients instead of constant coefficients-and used the Lyapunov functions to assess the system stability.

Keywords: system stability; Lyapunov's second method; construction methods for Lyapunov functions

1 Introduction

The Russian scientist Lyapunov was a significant contributor to the development of automatic control theory. Lyapunov's respective first and second methods are classic methods. Lyapunov's first method is only conceptual; it does not involve detailed calculation techniques and is therefore rarely used. His second method, on the other hand, involves detailed calculation methods and has thus been widely employed. By constructing Lyapunov functions and evaluating signs in the functions, one can directly assess the stability of a given system. This approach is only a sufficient condition rather than a necessary condition; No unified and generalized construction methods have been developed. Moreover, no general rule exists for function construction. Engineers usually search for Lyapunov functions from their personal experiences. In this study, we refined regular-form Lyapunov functions that are practically useful and can be employed as a reference and support in engineering practices.

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2 Fundamental Theorems and Definitions

2.1 Function Definition

Definition 1.Suppose there exists a function V, with a definition domain as region D that contains the origin (0,0). V is a continuous function in the definition domain. When $(x, y) = (0,0)$, we have $V = 0$. According to signs of the functions, we classify the function characteristics as follows[1-3]:

Provided that $(x, y) \neq (0,0)$, we have $V(x) \geq 0$, and V is now a positive definite function; if $V(x) \geq 0$, V is a positive semi-definite function.

Provided that $(x, y) \neq (0,0)$, we have $V(x) \leq 0$, and V is now a negative definite function; if $V(x) \leq 0$, V is a negative semi-definite function.

When V has a random sign in a region covering the origin, V is a sign-variable function. The following are the functional forms in question. The forms do not explicitly contain time t, as shown below:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_1, x_2) \end{aligned} \quad (1.1)$$

Definition 2.Provided that a function, V is continuous and differentiable in a neighborhood D, $V(x_1, x_2)$ has the following total derivative form:

$$\frac{\partial V(x_1,x_2)}{\partial x} f(x_1,x_2) + \frac{\partial V(x_1,x_2)}{\partial x} g(x_1,x_2)$$

(1.2)

2.2 Lyapunov's Second Method

Theorem1:Let differential equation (1.1) have an isolated singularity at the origin. Provided that there exists a continuous function V and its first-order partial derivative is continuous,and we have the following: If V is positive-definite and the total derivative form (1.2) is negative-definite in neighborhood D of the origin (0,0), the system is asymptotically stable with respect to the origin. If V is positive-definite and the total derivative form (1.2) is negative-semi-definite in neighborhood D of the origin (0,0), the system is stable with respect to the origin[4].

Theorem2:Let the differential equation (1.1) have an isolated singularity at the origin. Provided that there exists a continuous function V and its first-order partial derivative is continuous, we have the following: If there exists at least one point to allow V to be positive (negative)-definite and the total derivative form (1.2) to be positive (negative)-definite in neighborhood D of the origin (0,0), the system is unstable with respect to the origin. The above stability judgment theorems are only sufficient conditions rather than necessary conditions, and they provide forms for function V in certain circum -stances. If the conditions of the above theorems are met, relevant results will be obtained. If they are not met, function V must be replaced with a new one to re-judge the system stability.

As Lyapunov functions do not take a unified and generalized form, some important forms of the Lyapunov function are usually refined so that Lyapunov functions may be quickly found and used to judge system stability.

3 Construction of lyapunov function in different scenarios

3.1 Construction of Lyapunov Functions for Linear Time-invariant Systems

Judging the stability of a linear time-invariant system. The form of the system is given as follow[5].

$$\dot{x}_1 = ax_1 + bx_2 \quad (2.1)$$

$$\dot{x}_2 = cx_1 + dx_2 \quad (2.2)$$

If the above differential equation groups are stable, the coefficients of the equations should meet the conditions of $a<0$ and $d<0$, as proven below. That is, for a constant coefficient differential equation:

$$\dot{y} + py = q \quad (2.3)$$

If $q=0$, the equation is a homogeneous differential equation; if $q \neq 0$, the equation is an inhomogeneous differential equation. The solution of the differential equation is:

$$y = Ce^{-\int pdx} \quad (2.4)$$

Where C is a constant. Substituting differential equ -ation (2.1) into expression (2.4), we have a solution of the differential equation of the system:

$$x_1 = Ce^{\int adx_1} \quad (2.5)$$

If the system is required to be stable with respect to x_1, we should have $a<0$; if>0, the system would be divergent. For the same reason, we should have $d<0$. As shown in the above deduction process, for a differential equation containing differential variables on one side and other variables on the other side of an equality, the system would be stable only when the coefficients of the differential variables have opposite signs with respect to those before the differentiation. The above expressions are used to primarily judge the signs of a and d to provide a conclusion on the system stability. In the following, we continue to judge the signs of b and c to find a corresponding Lyapunov function[6].

1) If b and c have opposite signs, e.g., $c<0, b>0$, we have the following Lyapunov function[7-9]:

$$V(x_1 x_2) = \frac{-c}{2}x_1^2 + \frac{b}{2}x_2^2 \quad (2.6)$$

Proof: Define $V(x) = mx_1^2 + nx_2^2$, $m>0, n>0$, $V(0)=0$, and $V(x)$ is a positive definite function.

$$\dot{V}(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = 2max_1^2 + 2ndx_2^2 + (2mb + 2nc)x_1x_2$$

As $m>0, n>0$ and $a<0, d<0$, the first two terms in expression(2.7) are always less than zero. To allow expression (2.7) to always be less than zero, we may let:

$$2mb + 2nc = 0 \rightarrow mb = -nc \rightarrow \frac{m}{n} = -\frac{c}{b}$$

Therefore, let $m = \frac{-c}{2}, n = \frac{b}{2}$ ($c<0, b>0$).

Substituting the deduction result into expression (2.7), we obtain $\dot{V}(x) < 0$given that $V(x)>0, V(0)=0$, the above system is judged to be stable according to the judgment theorems of Lyapunov's second method.$V(x)$ can take a form as shown by expression (2.6). Similarly, provided that $c>0, b<0$, we have the following:

$$V(x) = \frac{c}{2}x_1^2 + \frac{-b}{2}x_2^2 \quad (2\text{-}8)$$

2) When b and c have the same signs, both b and c may be positive or negative. When both b and c are positive or negative, function $V(x)$ in expression(2.7), under the constraint of expression (2.8), can not always be less than zero. Therefore, $V(x)$ with the aforementioned method is invalid in the conditions of 2). In the following, we discuss the methods of selecting a Lyapunov function in two scenarios. If b and c are both negative, expression (2.7) can be rewritten as:

$$\dot{V}(x) = 2max_1^2 + 2ndx_2^2 + 2mbx_1x_2 + 2ncx_1x_2 \quad (2.9)$$

Let us consider a complete quadratic form:

$$(mx_1 + bx_2)^2 + (nx_1 + cx_2)^2 \quad (2.10)$$

Combining expressions (2.9) and (2.10), and decoupling expression (2.9), we have:

$$\dot{V}(x) = (2ma - m^2 - n^2)x_1^2 + (2nd - b^2 - c^2)x_2^2 + (mx_1 + bx_2)^2 + (nx_1 + cx_2)^2 \quad (2.11)$$

Here, we may consider selecting

$$V(x) = \frac{|c|}{2}x_1^2 + \frac{|b|}{2}x_2^2 \quad (2.12)$$

Substituting $V(x)$ into the form and using back asting, we judge what relationship should be fulfilled between b and c to achieve the above form of expression (2.12). When b and c are both positive, substituting expression (2.12) into expression (2.7) gives:

$$\dot{V}(x) = acx_1^2 + bdx_2^2 + (2bc)x_1x_2 \quad (2.13)$$

Completing the squares, we transform expression (2.13) into:

$$\dot{V}(x) = -(cx_1 - bx_2)^2 + c(a+c)x_1^2 + b(b+d)x_2^2 \quad (2.14)$$

(a) When b and c are both positive,the Lyapunov function can be written as:

$$V(x) = \frac{c}{2}x_1^2 + \frac{b}{2}x_2^2$$

(b) When b and c are both negative,Deriving from expression (2.14), we have:

$$\dot{V}(x) = -(-cx_1 + bx_2)^2 - c(a-c)x_1^2 - b(d-b)x_2^2 \quad (2.15)$$

we can construct a Lyapunov function:

$$V(x) = \frac{-c}{2}x_1^2 + \frac{-b}{2}x_2^2$$

3.2 Construction of Lyapunov Functions for Nonlinear Systems

Given a non-linear time-invariant system as follow to judge the stability of the system[10].

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -2x_1 - x_2 - x_2^3 \end{aligned} \quad (2.16)$$

The above equations have a Lyapunov function,$V(x) = 2x_1^2 + x_2^2$. It is evident that $V(x)$ is positive-definite. Given that $V(x) = -12x_1^2 - 2x_2^2 - 2x_2^4$ and $V(x)$ is negative-definite, the system is thereby stable. An approach to quickly finding Lyapunov functions suitable for the above non-linear time-invariant systems is needed. To this end, we first analyze the necessary conditions for judging the stability of a non-linear differential equation. For non-linear differential equations:

$$\dot{y} + p(x)y = q(x) \quad (2.17)$$

A solution of inhomogeneous differential equations is:

$$y = e^{\int -p(x)dx}[\int q(x)e^{\int p(x)dx}dx + C] \quad (2.18)$$

A solution to homogeneous differential equations is:

$$y = Ce^{\int -p(x)dx} \quad (2.19)$$

If $p(x) > 0$, the solution of homogeneous differential equations is convergent, and the solution is a general one. However, for inhomogeneous differential equations, the sign of $q(x)$ should also be considered. In conventional engineering practices, $q(x)$takes finite values in a certain range. Therefore, inhomogeneous equations are also convergent, with the sign of $p(x)$ playing a dominant role; i.e.$p(x) \geq 0$

According to the above deduction process, we construct the following differential equation groups, with $p(x)$ existing in the form of power functions. We determine the conditions that should be met for the equation groups to have a convergent solution:

$$\dot{x}_b = ax_a + mx_b + nx_b^\lambda = ax_a + (m + nx_b^{\lambda-1})x_b \quad (2.20)$$

Where λ is an integer, and $\lambda = 0,1,2,\cdots n$. The above equation has a solution as follows:

$$x_b = Ce^{-\int -[m+nx_b^\lambda]dx} = Ce^{\int [m+nx_b^\lambda]dx} \quad (2.21)$$

If the solution as shown in expression (2.21) is convergent, $m + nx_b^{\lambda-1}$ will be always less than zero. We can let $m < 0, n < 0$ and let $\lambda - 1$ be an even number, which means that λ is an odd number. This is a rule for growth transformation of non-linear functions.

Proof: If equation (2.20) does not involve nx_b^λ, the equation now can be re-written as:

$$\dot{x} = x_a + mx_b \quad (2.22)$$

Conducting a substitution according to expression (2.9), we have a solution of the equation:

$$x_b = Ce^{\int mdx} \quad (2.23)$$

When expression (2.23) is convergent, we have $m < 0$. Therefore give $m < 0$,and $m + nx_b^{\lambda-1}$ in the inhomogeneous equation is always less than zero, we may consider λ to be an odd number and $n \leq 0$. Accordingly, expression (2.20) is convergent, with $m + nx_b^{\lambda-1}$ referred to as a convergent nonlinear coefficient. This means that, when a differential term and the factor of the nonlinear functions of the variable are always negative, the solution of the equation is convergent. We provide rule for the equation transformation that the power of undifferentiated variables should be odd and determined that the factors of a differential variable corresponding to a differential term can be a superposition of even powers with negative coefficients.

3.3 Design Lyapunov Functions for Specific Nonlinear Systems

1) Design 1

If the additional nonlinear function is a nonlinear function of another variable in a differential form, such as[11]

$$\dot{x}_1 = ax_1 + bx_2 + f(x)x_2^{\lambda}$$
$$\dot{x}_2 = cx_1 + g(x)x_1^{\tau} + dx_2 \quad (2.24)$$

where λ and τ are positive integers,the forms can be detailed as:

$$\dot{x}_1 = ax_1 + b_{11}x_2 + b_{12}x_2^2 + \cdots + b_{1n}^{n} \quad (2.25)$$

$$\dot{x}_2 = c_{11}x_1 + c_{12}x_1^2 + \cdots c_{1n}x_1^{n} + dx_2 \quad (2.26)$$

If the corresponding coefficients in equations (2.25) and (2.26) opposeeach other, and if we assume $b_{1i} > 0$ and $c_{1i} > 0$, a Lyapunov function may take the following form:

$$V(x) = \frac{b_{11}}{2}x_2^2 + \frac{b_{12}}{3}x_2^3 + \cdots + \frac{b_{1n}}{n+1}x_2^{n+1} + \frac{|c_{11}|}{2}x_1^2 + \frac{|c_{12}|}{3}x_1^3 + \cdots + \frac{|c_{1n}|}{n+1}x_1^{n+1} \quad (2.27)$$

$$\dot{V}(x) = [c_{11}x_1 + c_{12}x_1^2 + \cdots c_{1n}x_1^{n} b_{11}x_2 + b_{12}x_2^2 + \cdots + b_{1n}^{n}] \times \begin{bmatrix} ax_1 + b_{11}x_2 + b_{12}x_2^2 + \cdots + b_{1n}^{n} \\ c_{11}x_1 + c_{12}x_1^2 + \cdots c_{1n}x_1^{n} + dx_2 \end{bmatrix}$$
(2.28)

In equation (2.28), $\dot{V}(x)$ contains coupled crossed terms with undeterminable signs. Therefore, a differential equation form conjectured from expression (2.24), and a form of $V(x)$, are inaccurate. Therefore, the conjectured forms of power functions for $f(x)$ and $g(x)$ in expression (2.24) are inaccurate.

2) Design 2

Consider the following differential equations[12]:

$$\dot{x}_1 = ax_1 + b_{11}x_2 + b_{13}x_2^3 + \cdots + b_{1n}x_2^{2n-1} \quad (2.29)$$

$$\dot{x}_2 = c_{11}x_2 + c_{12}x_1 + \cdots + c_{1n}x_1^{2n-1} + dx_2 \quad (2.30)$$

Compared with expression (2.24), the power functions for f(x) and g(x) are a sum of even powers. According to the conclusion about expression (2.21),we have $a < 0, d < 0$ We consider that the Lyapunov function may take the following form:

$$v(x) = \frac{I_{11}}{2}x_2^2 + \frac{I_{12}}{4}x_2^4 + \cdots + \frac{I_{1n}}{2n}x_2^{2n} + \frac{h_{11}}{2}x_1^2 + \frac{h_{12}}{4}x_1^4 + \cdots + \frac{h_{1n}}{2n}x_1^{2n} \quad (2.31)$$

where $h_{1i} > 0, I_{1j} > 0$ ($i, j = 1,2,3,\cdots n$).Consider the relationship among the corresponding coefficients:

$$\dot{v}(x) = [\frac{\partial V(x)}{\partial x_1} \quad \frac{\partial V(x)}{\partial x_2}] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad (2.32)$$

Let:

$$\tilde{H}(x) = h_{11}x_1 + h_{12}x_1^3 + h_{13}x_1^5 + \cdots + h_{1n}x_1^{2n-1}$$
$$\tilde{I}(x) = I_{11}x_2 + I_{12}x_2^3 + I_{13}x_2^5 + \cdots + I_{1n}x_2^{2n-1}$$
$$\tilde{B}(x) = b_{11}x_2 + b_{12}x_2^3 + b_{13}x_2^5 + \cdots + b_{1n}x_2^{2n-1}$$
$$\tilde{C}(x) = c_{11}x_1 + c_{12}x_1^3 + c_{13}x_1^5 + \cdots + c_{1n}x_1^{2n-1}$$

The above expressions can be simplified as:

$$\begin{bmatrix}\tilde{H}(x) & \tilde{I}(x)\end{bmatrix} \begin{bmatrix} ax_1 + \tilde{B}(x) \\ \tilde{C}(x) + dx_2 \end{bmatrix} = ax_1\tilde{H}(x) + \tilde{B}(x)\tilde{H}(x) + \tilde{C}(x)\tilde{I}(x) + dx_2\tilde{I}(x) \quad (2.33)$$

Since $v(x)$ is positive-definite, the coefficients of $\tilde{H}(x)$, namely $h_{12}, \cdots, h_{1i}$ are all positive. Similarly, the coefficients of $\tilde{I}(x)$ are all positive. Since $a < 0, d > 0$ and $x_1\tilde{H}(x) \geq 0$, we have $x_1\tilde{I}(x) \geq 0$, which means that the coefficients of even powers are not higher than zero. We now analyze the remaining cross terms (2.34):

$$\tilde{B}(x)\tilde{H}(x) + \tilde{C}(x)\tilde{I}(x) = \mathrm{I}+\mathrm{II}$$
$$= \left[b_{11}x_2 + b_{12}x_2^3 + \cdots + b_{1n}x_2^{2n-1}\right]\left[h_{11}x_1 + h_{12}x_1^3 + \cdots + h_{1n}x_1^{2n-1}\right]$$
$$+ \left[c_{11}x_1 + c_{12}x_1^3 + \cdots + c_{1n}x_1^{2n-1}\right]\left[I_{11}x_2 + I_{12}x_2^3 + \cdots + I_{1n}x_2^{2n-1}\right]$$

Given the positive and negative coefficients of the whole equation, the above expression can be written as (2.35):

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{n} h_{1i}b_{1j}x_1^{2i-1}x_2^{2j-1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{1i}I_{1j}x_1^{2i-1}x_2^{2j-1}\right] = 0$$

The results of the multiplication of cross terms can be written as:

$$\tilde{B}(x)\tilde{H}(x) + \tilde{C}(x)\tilde{I}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{1i}b_{1j}x_1^{2i-1}x_2^{2j-1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{1i}I_{1j}x_1^{2i-1}x_2^{2j-1} \quad (2.36)$$

We make a judgment according to the positive and negative values of c_{1i} and b_{1j}, Now:

$$\dot{v}(x) = ax_1\tilde{H}(x) + \tilde{B}(x)\tilde{H}(x) + \tilde{C}(x)\tilde{H}(x) + dx_2\tilde{I}(x) \quad (2.37)$$

Where we have $ax_1\tilde{H}(x) \geq 0$ and $dx_2\tilde{I}(x) \geq 0$. Here, $b_{1j} \leq 0$ and $c_{1j} \geq 0$, b_{1j} and c_{1j} are not all zeros. Alternatively $b_{1j} \leq 0$ and $c_{1j} \leq 0$ is all non-positive. b_{1j} and c_{1j} are not all zeros. The Lyapunov function can be directly written as:

$$v(x) = \frac{h_{11}}{2}x_1^2 + \frac{h_{12}}{4}x_1^4 + \cdots + \frac{h_{1n}}{2n}x_1^{2n} + \frac{I_{11}}{2}x_2^2 + \frac{I_{12}}{4}x_2^4 + \cdots + \frac{I_{1n}}{2n}x_2^{2n} \quad (2.38)$$

Namely,

$$v(x)=\frac{|c_{11}|}{2}x_1^2+\frac{|c_{12}|}{4}x_1^4+\cdots+\frac{|c_{1n}|}{2n}x_1^{2n}+\frac{|b_{11}|}{2}x_2^2+\frac{|b_{12}|}{4}x_2^4+\cdots+\frac{|b_{1n}|}{2n}x_2^{2n} \quad (2.39)$$

Conclusion: For differential equations, if a differential form of avariable contains another variable for which the powers are only odd powers, and given that the coefficients are all positive or all negative, we can obtain the above Lyapunov function.

3) Design 3

If a differential function contains even powers of other variables, such as[12]:

$$\dot{x}_1=ax_1+b_{11}x_2+b_{12}x_2^3+\cdots\cdots b_{1n}x_2^{2n-1}+x_2^2 \quad (2.40)$$

$$\dot{x}_2=c_{11}x_1+c_{12}x_1^3+\cdots\cdots c_{1n}x_2^{2n-1}+x_1^2 \quad (2.41)$$

we set a Lyapunov function in which only odd powers are integrated, and we set the coefficients before even powers are positive.

$$v(x)=\frac{h_{11}}{2}x_1^2+\frac{h_{12}}{4}x_1^4+\cdots+\frac{h_{1n}}{2n}x_1^{2n}+\frac{I_{11}}{2}x_2^2+\frac{I_{12}}{4}x_2^4+\cdots+\frac{I_{1n}}{2n}x_2^{2n} \quad (2.42)$$

$$\dot{V}(x)=\left[\frac{\partial V(x)}{\partial x_1}\quad\frac{\partial V(x)}{\partial x_2}\right]\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}$$
$$=ax_1\tilde{H}(x)+x_2^2\tilde{H}(x)+\tilde{B}(x)\tilde{H}(x)+\tilde{I}(x)\tilde{C}(x)+X_1^2\tilde{I}(x)+dx_2\tilde{I}(x)$$

(2.43)

According to the above analysis, we can obtain $ax_1\tilde{H}(x)\le 0$ $ax_2\tilde{I}(x)\le 0$ .The signs of two additional terms $x_2^2H^2(x)$ and $X_1^2I^2(x)$ cannot be determined. For example[13]:

$$\dot{x}_1=ax_1+x_2+x_2^3+x_2^2 \quad (2.44)$$

$$\dot{x}_2=-x_1-x_1^3+dx_2 \quad (2.45)$$

We take $\dot{v}(x)=\frac{1}{2}x_2^2+\frac{1}{4}x_2^4+\frac{1}{2}x_1^2+\frac{1}{4}x_1^4$

$$\dot{V}(x)=ax_1(x_1+x_1^3)+(x_2+x_2^3)(x_1+x_1^3)+x_2^2(x_1+x_1^3)$$
$$-(x_1+x_1^3)(x_2+x_2^3)+dx_2(x_2+x_2^3)$$

The sign of the above $\dot{V}(x)$ cannot be determined, and it is unable to use the above direct integral method to obtain a Lyapunov function.

4) Design 4

In differential equations, if an even power of a variable is the coefficient for another variable, and also the condition of $d(x)\ge 0$ is fulfilled, we may use the conclusion of expression (2.39) to design a Lyapunov function. For example[14]:

$$\dot{x}_1=x_2$$

$$\dot{x}_2=-x_1-x_1^2x_2$$

Solution: When a and c have opposite signs, a direct integral will generate a Lyapunov function.

$$V(x)=\frac{x_1^2}{2}+\frac{x_2^2}{2}$$

which is positive-definite, and $\dot{V}(x)=-x_1^2x_2^2$ is negative-definite. According to Lyapunov's second method, this system is stable. Therefore, we have:

$$\dot{x}_1=ax_1+b_{11}x_2+b_{12}x_2^3+\cdots\cdots b_{1n}x_2^{2n-1} \quad (2.46)$$

$$\dot{x}_2=c_{11}x_1+c_{12}x_1^3+\cdots\cdots c_{1n}x_2^{2n-1}+dx_2 \quad (2.47)$$

If b_{1i} is all positive and c_{1j} is all negative, or b_{1i} is all negative and c_{1j} is all positivethen a Lyapunov function can take the following form:

$$v(x)=\frac{|c_{11}|}{2}x_1^2+\frac{|c_{12}|}{4}x_1^4+\cdots+\frac{|c_{1n}|}{2n}x_1^{2n}+\frac{|b_{11}|}{2}x_2^2+\frac{|b_{12}|}{4}x_2^4+\cdots+\frac{|b_{1n}|}{2n}x_2^{2n} \quad (2.48)$$

For a differential equation containing even powers of a single variable, the above conclusion is inaccurate. However, the above conclusion can be generalized as:

$$\dot{x}_1=a(x_1,x_2)x_1+b_{11}x_2+b_{12}x_2^3+\cdots\cdots b_{1n}x_2^{2n-1} \quad (2.49)$$

$$\dot{x}_2=c_{11}x_1+c_{12}x_1^3+\cdots\cdots c_{1n}x_1^{2n-1}+d(x_1,x_2)x_2 \quad (2.50)$$

If the conditions of $a(x_1,x_2)\le 0$ and $d(x_1,x_2)\le 0$ are met, the conclusions about the above equations are still valid. The conclusion can be generalized for time-variant systems.If the equations take the forms such as

$$\dot{x}_1=A(x_1,x_2,t)x_1+b_{11}x_2+\cdots\cdots b_{1n}x_2^{2n-1} \quad (2.51)$$

$$\dot{x}_2=c_{11}x_1+\cdots\cdots c_{1n}x_1^{2n-1}+\dot{D}(x_1,x_2,t)x_2 \quad (2.52)$$

and the equations fulfill the condition that b_{1j} is all non-negative and c_{1i} is all non-positive, or vice versa ($i,j=1,2,\cdots\cdots n$), the conclusion about expression (2.39) remains valid.

4 CONCLUSION

In the practical engineering field, we can use the method given in this paper to judge the similar non-linear system directly, which saves a lot of time.

Conclusion one: For linear time-invariant systems,the form can be rewritten as:

$$\dot{x}_1=ax_1+bx_2$$

$$\dot{x}_2=cx_1+dx_2$$

For differential equations, the condition for convergent solutions is that, if the system status of x_1 is stable, we have $a<0$; if $a>0$ the system will be divergent.

Similarly, we have $d < 0$.

To satisfy the requirements for the coefficients:

(I) When b and c have opposite signs, for example $c < 0, b > 0$, the Lyapunov function will be

$$V(x) = -\frac{c}{2} x_1^2 + \frac{b}{2} x_2^2$$

(II) When b and c have the same signs. The conditions for decoupling are as follows:

a)When b and c are both negative, and if $a + c \le 0$ and $b + d \le 0$, the Lyapunov function can take the following form:

$$V(x) = \frac{c}{2} x_1^2 + \frac{b}{2} x_2^2$$

b) When b and c are both negative, and if $a \le c, d \le b$, we can construct a Lyapunov function as

$$V(x) = \frac{-c}{2} x_1^2 + \frac{-b}{2} x_2^2$$

Conclusion two: For nonlinear systems the form can be rewritten as:

$$\dot{x}_1 = ax_1 + bx_2 + f(x)x_2^\lambda$$

$$\dot{x}_2 = cx_1 + g(x)x_1^\tau + dx_2$$

Where λ and τ are both positive integers. Specifically, expressions have the following forms:

$$\dot{x}_1 = ax_1 + b_{11}x_2 + b_{12}x_2^2 + \cdots + b_{1n}^n$$

$$\dot{x}_2 = c_{11}x_1 + c_{12}x_1^2 + \cdots c_{1n}x_1^n + dx_2$$

the following form for the Lyapunov function：

$$V(x) = \frac{b_{11}}{2} x_2^2 + \frac{b_{12}}{3} x_2^3 + \cdots + \frac{b_{1n}}{n+1} x_2^{n+1}$$
$$+ \frac{|c_{11}|}{2} x_1^2 + \frac{|c_{12}|}{3} x_1^3 + \cdots + \frac{|c_{1n}|}{n+1} x_1^{n+1}$$

Conclusion three: We consider the following differential equations:

$$\dot{x}_1 = ax_1 + b_{11}x_2 + b_{13}x_2^3 + \cdots + b_{1n}x_2^{2n-1}$$

$$\dot{x}_2 = c_{11}x_2 + c_{12}x_1 + \cdots + c_{1n}x_1^{2n-1} + dx_2$$

$$\dot{x}_1 = a(x_1, x_2)x_1 + b_{11}x_2 + b_{12}x_2^3 + \cdots\cdots b_{1n}x_2^{2n-1}$$

$$\dot{x}_2 = c_{11}x_1 + c_{12}x_1^3 + \cdots\cdots c_{1n}x_1^{2n-1} + d(x_1, x_2)x_2$$

$$\dot{x}_1 = A(x_1, x_2, t)x_1 + b_{11}x_2 + \cdots\cdots b_{1n}x_2^{2n-1}$$

$$\dot{x}_2 = c_{11}x_1 + \cdots\cdots c_{1n}x_1^{2n-1} + \dot{D}(x_1, x_2, t)x_2$$

the Lyapunov function can be directly written as:

$$V(x) = \frac{|c_{11}|}{2} x_1^2 + \frac{|c_{12}|}{4} x_1^4 + \cdots + \frac{|c_{1n}|}{2n} x_1^{2n}$$
$$+ \frac{|b_{11}|}{2} x_2^2 + \frac{|b_{12}|}{4} x_2^4 + \cdots + \frac{|b_{1n}|}{2n} x_2^{2n}$$

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