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The Computation of Common Lyapunov Functions via Nonsmooth Newton Method

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Abstract—**This paper studies common quadratic Lyapunov functions for a finite number of linear time invariant systems. We present Newton method which can find common quadratic Lyapunov functions for a number of stable linear systems, and a numerical example is given to illustrate our result.**

Keywords— Common quadratic Lyapunov function; Nonsmooth Newton method; Matrix functions; switched systems

I. INTRODUCTION

Recently, the problems of system stability and stabilization of switched systems have received an increased attention in control theory or control design [1-4]. Roughly speaking, a switched system is a system that combines continuous dynamical system with a logic based switching rule that determines mode switches in the system's operation at different points in time [5]. It is well-known that the stability of each of the constituent systems is not sufficient to ensure that the overall systems are stable for arbitrary switching signals [6]. For this reason, the problem of how to find a common Lyapunov function arises because it can assure the stability of a switched system. The most common types of common Lyapunov functions used in practice are quadratic, linear and piecewise-linear [7] . When the switched linear dynamical system is considered, the common Lyapunov function is always given by a quadratic form. The algorithms of finding common quadratic Lyapunov functions have been investigated in detail [8-10] .

In this paper, we shall give a novel approach to give common quadratic Lyapunov functions for a family of stable linear systems. In detail, the rest of the paper is organized as follows: in section II, the definition of common quadratic Lyapunov functions is introduced, and several notation that is useful in giving the key result of this paper. In section III, a new method to find common quadratic Lyapunov functions is proposed. Section IV concludes this paper.

II. COMMON QUADRATIC LYAPUNOV FUNCTIONS

Before proceeding, we introduce some notation and definition [11].

Let M_{mn} be the space of all $m\times n$ matrices.

O is the set which consists all $n\times n$ orthogonal matrices.

For X *and* $Y\in S^m$, we have

$$< X,Y >= X\bullet Y = Trace(XY) = \sum_{i,j=1}^{m} X_{ij}Y_{ij}$$

"T" represents the transpose of a matrix or a vectors.

$\|X\|$ is the norm of matrix $X : \|X\| =< X,X >^{1/2}$.

$\|a\|$ denotes the 2-norm of a vector a.

We can write $X \succ 0$ if X is a positive definite matrix and $X \succ= 0$ if X is a positive semidefinite matrix .

For $X \succ= 0$, we will denote theirs symmetric square root by $\sqrt{X}$ or $X^{1/2}$.

Definition 1 If a matrix $\mathrm{X}\in S^n$, then vec (X) is a vector whose entries come from the matrix X by stacking up all columns of X, from the 1st column to the nth column, on the top of each other. The operator "mat" is the inverse operator of "vec"; The operator svec (X) only fetches the lower half of the matrix X, including all the diagonal entries, and stacks them into a column vector in a similar way just like vec (X).

Definition 2 For any matrix $W\in S(n_1,\cdots,n_m)$, let Vec(W) be a vector as follows

$$Vec(W) := (vec(W_1)^T,\cdots,vec(W_m)^T)^T,$$

where $W_i, i=1,\cdots,m$ is the ith block of the matrix W. The operator Mat is the inverse operator of Vec; The operator Svec is defined by as following

$$Svec(W) := (svec(W_1)^T,\cdots,svec(W_m)^T)^T$$

and the operator Smat is the inverse of Svec.

The matrix-valued function

$$F : S(n_1,\cdots,n_m) \to S(n_1,\cdots,n_m)$$

under Svec turns out to be a vector valued function $f : R^{\bar{v}} \to R^{\bar{v}}$ is given by

$$f(x) := Svec[F(X)],$$

where $X = Smat(x)$.

A switched linear system is a dynamical system of the form

$$\dot{x} = A(t)x$$

where $A(t)$ can switched between several given finite collection of matrices $A_1, \cdots, A_k$ in $R^{n\times n}$.

Thus, a switched linear dynamical system works by switching between a finite number of linear time invariant dynamical systems $\Sigma_{A_i}: \dot{x} = A_i x$, $i = 1, \cdots, k$, which refers to as its modes.

Definition 3 The function $V(x) = x^T P x$ is a common quadratic Lyapunov function for the linear systems $\Sigma_{A_1}, \cdots, \Sigma_{A_k}$ if $P = P^T \succ 0$ and $A_i^T P + P A_i \prec 0$, for $i = 1, \cdots, k$.

III. MAIN RESULTS

In this part, we give the method for finding common Lyapunov functions in detail.

Let $S(n_1, \cdots, n_m)_+$ be the convex cone of all symmetric block-diagonal matrices which is positive semidefinite. Define the semidefinite projection matrix function $[\bullet]_+ : S(n_1, \cdots, n_m) \to S(n_1, \cdots, n_m)_+$ as a matrix valued function that satisfies

(1) the matrix $[X]_+ \in S(n_1, \cdots, n_m)_+$;

(2) $\|X - [X]_+\| \le \|X - Z\|$, all $Z \in S(n_1, \cdots, n_m)_+$.

It is well known that [11]

$$[X]_+ = p \bullet diag[\max(0, \lambda_1), \cdots, \max(0, \lambda_n)] \bullet p^T,$$

where the matrix $p \in O$ and $\lambda_1, \cdots, \lambda_n \in R$ satisfy $X = p\ M\ p^T$ and $M = diag[\lambda_1, \cdots, \lambda_n]$.

Using the above notation, we can rewrite the matrix inequations $A_i^T P + P A_i \prec 0$ as follows

$$[A^T P + PA\ ]_+ = 0, \tag{1}$$

where $A = \begin{bmatrix} A_1 & o & o \\ o & \ddots & o \\ o & o & A_k \end{bmatrix}$, $P = \begin{bmatrix} X_1 & o & o \\ o & \ddots & o \\ o & o & X_k \end{bmatrix}$ and o is zero matrix.

We can see that (1) are matrix-valued function, and the equations (1) can be solved by Newton method. Next, we introduce the concept of generalized derivative of a matrix-valued function F. The "generalized derivative" of the function F at X is a set given as follows

$$\partial \mathrm{F}_X = \mathrm{co}\{\partial_B F_X\}, where\ \partial_B F_X = \{\lim_{\substack{Z \to X \\ Z \in D_F}} F_Z{}'\},$$

where "co" represent the convex hull in the normal sense of convex analysis theory.

It is very easy to see that there is an isometry relationship between $\partial f(x)$ and ∂F_X. We can get

$$\partial F_X = \mathrm{Smat} \circ \partial f(x) \circ \mathrm{Svec}.$$

Now we will show how to compute (1). Note that $[X]_+ = [X + \sqrt{X^2}\,]/2$, where $\sqrt{X^2} = [X]_+ + [-X]_+$.If we can give the generalized derivative of $\sqrt{X^2}$, the generalized derivative of $[X]_+$ can also be given. The generalized derivative of $\sqrt{X^2}$ can be obtained as follows:

(1) for any $H \in S^n$, compute $vec(H)$

(2) compute

$$d = (I \otimes \sqrt{X^2} + \sqrt{X^2} \otimes I)^{-1} Y\ vec(H),$$

where $Y = (I \otimes X + X \otimes I)$.

(3) From $vecF_X{}'(H) = d$, we can get $F_X{}'$.

For matrix-valued equation $F(X) = 0$, the key step of generalized Newton method is the equation as follows

$$X_{k+1} = X_k - V_k^{-1} g(X_k). \tag{2}$$

Given a certain initial value X_0, we can repeat the process via (2) until the difference $|X_{k+1} - X_k|$ is small enough.

Finally, an example is presented to illustrate the above main results.

Example

Let

$$A_1 = \begin{bmatrix} -1.0 & -2.0 & -1.0 \\ 0.0 & -2.0 & -1.0 \\ 0.0 & 0.0 & -2.0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2.0 & -1.0 & -2.0 \\ 0.0 & -1.0 & -2.0 \\ 0.0 & 0.0 & -1.0 \end{bmatrix},$$

we can use the above method and get the a common Lyapunov function

$$V(x) = x^T Q x,$$

where $Q = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 2.0 \end{bmatrix}$.

IV. CONCLUSION

This paper present a novel method to find out a common quadratic Lyapunov function for a number of linear systems. We find common Lyapunov functions via solving some equations. Furthermore, this approach provides a numerical solution to Lyapunov function inequations in generally.

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